What is a Motive?

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Abstract

Grothendieck introduced the notion of a "motif" in a letter to Serre in 1964. Later he wrote that, among the objects he had been privileged to discover, they were the most charged with mystery and formed perhaps the most powerful instrument of discovery.[1] In this talk, I shall explain what motives are, and why Grothendieck valued them so highly.

These are my notes for a "popular" talk in the 'What is ...?' seminar at the University of Michigan, Feb 3, 2009. [2]

Contents

1 Cohomology in topology

Let X be a compact manifold of dimension $2n$. There are attached to X cohomology groups

$$H^0(X, \mathbb{Q}), \ldots, H^{2n}(X, \mathbb{Q}),$$

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[1] Parmi toutes les chose mathématiques que j'avais eu le privilège de découvrir et d'amener au jour, cette réalité des motifs m'apparaît encore comme la plus fascinante, la plus chargée de mystère — au coeur même de l'identité profonde entre la "géométrie" et l' "arithmétique". Et le "yoga des motifs" ... est peut-être le plus puissant instrument de décourverte que j'aie dégagé dans cette première période de ma vie de mathématicien.

Grothendieck, Récoltes et Semailles, Introduction.

(Among all the mathematical things that I have had the privilege to discover and bring to the light of day, the reality of motives still appears to me the most fascinating, the most charged with mystery — at the heart even of the profound identity between "geometry" and "arithmetic". And the "yoga of motives" ... is perhaps the most powerful instrument of discovery found by me during the first period of my life as a mathematician.)

[2] Written *after* the lecture, of course. This is what I would have said if I'd been better organized and had had more time.

which are finite-dimensional $\mathbb{Q}$-vector spaces satisfying Poincaré duality (H^i is dual to H^{2n-i}), a Lefschetz fixed point formula, etc.. There are many different ways of defining them — singular cochains, Čech cohomology, derived functors — but the different methods all give exactly the same groups (provided they satisfy the Eilenberg-Steenrod axioms). When X is a complex analytic manifold, there are also the de Rham cohomology groups $H^i_{\mathrm{dR}}(X)$. These are vector spaces over $\mathbb{C}$, but they are not really new either because[3] $H^i_{\mathrm{dR}}(X) \simeq H^i(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ (however, when X is Kähler, the de Rham cohomology groups have a Hodge decomposition which provides additional information…).

2 Cohomology in algebraic geometry

Now consider a nonsingular projective algebraic variety X of dimension n over an algebraically closed field k. Thus X is defined by polynomials over k, and the conditions mean that, if $k = \mathbb{C}$, the points $X(\mathbb{C})$ of the variety form a compact manifold of dimension $2n$.

Weil's work on the numbers of points on algebraic varieties with coordinates in finite fields led him (in 1949) to make his famous "Weil" conjectures "concerning the number of solutions of equations over finite fields and their relation to the topological properties of the varieties defined by the corresponding equation over the field of complex numbers". In particular, he found that the numbers of points seemed to be controlled by the Betti numbers of a similar variety over $\mathbb{C}$. For example, for a curve C of genus g over the field $\mathbb{F}_p$ with p elements

$$\left||C(\mathbb{F}_p)| - p - 1\right| \leq 2gp^{\frac{1}{2}}, \quad g = \text{genus of } C,$$

and he was able to predict the Betti numbers of certain hypersurfaces over $\mathbb{C}$ by counting the numbers of points on a hypersurface of the same dimension and degree over $\mathbb{F}_p$ (his predictions were confirmed by Dolbeault). It was clear that most of the conjectures would follow from a cohomology theory for algebraic varieties with good properties ($\mathbb{Q}$ coefficients, correct Betti numbers, Poincaré duality theorem, Lefschetz fixed point theorem, …). In fact, as we shall see, no such cohomology theory exists with $\mathbb{Q}$ coefficients, but in the following years attempts were made to find a good cohomology theory with coefficients in some field of characteristic zero (not $\mathbb{Q}$). Eventually, in the 1960s Grothendieck defined étale cohomology and crystalline cohomology, and showed that the algebraically-defined de Rham cohomology has good properties in characteristic zero. The problem then became that we had too many good cohomology theories!

Besides the usual valuation on $\mathbb{Q}$, there is another valuation for each prime number ℓ defined by

$$|\ell^r \tfrac{m}{n}| = 1/\ell^r, \quad m, n \in \mathbb{Z} \text{ and not divisible by } \ell.$$

Each valuation makes $\mathbb{Q}$ into a metric space, and on completing it we obtain fields $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_5$, …, $\mathbb{R}$. For each prime ℓ distinct from the characteristic of k, étale cohomology gives cohomology groups

$$H^0(X,\mathbb{Q}_\ell), \ldots, H^{2n}(X,\mathbb{Q}_\ell)$$

which are finite-dimensional vector spaces[4] over $\mathbb{Q}_\ell$ and satisfy Poincaré duality, a Lefschetz fixed point formula, etc.. Also, there are de Rham groups $H^i_{\mathrm{dR}}(X)$, which are finite-dimensional vector spaces over k, and in characteristic $p \neq 0$, there are crystalline co-

[3] I use $\simeq$ to denote a *canonical* isomorphism.

[4] There are also étale cohomology groups $H^i(X,\mathbb{Q}_p)$ for p the characteristic of k, but they are anomolous; for example, when E is a supersingular elliptic curve, $H^1(E,\mathbb{Q}_p) = 0$.

homology groups, which are finite-dimensional vector spaces over a field of characteristic zero (field of fractions of the ring of Witt vectors with coefficients in k).

These cohomology theories can't be the same, because they give vector spaces over very different fields. But they are not unrelated, because, for example, the trace of the map $\alpha^i \colon H^i(X) \to H^i(X)$ defined by a regular map $\alpha \colon X \to X$ is a *rational* number independent of the cohomology theory[5]. Thus, in many ways, they behave as if there were algebraically-defined cohomology groups $H^i(X, \mathbb{Q})$ such that $H^i(X, \mathbb{Q}_\ell) \simeq H^i(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ etc., but, in fact, there aren't.

3 Why is there no algebraic $\mathbb{Q}$-cohomology?

Why is there no algebraically-defined $\mathbb{Q}$-cohomology (functor from algebraic varieties to $\mathbb{Q}$-vector spaces) underlying the different cohomologies defined by Grothendieck?

FIRST EXPLANATION

Let X be nonsingular projective variety over an algebraically closed field k of characteristic zero (and not too big). When we choose an embedding $k \to \mathbb{C}$, we get a complex manifold $X(\mathbb{C})$ and it is known that

$$H^i(X, \mathbb{Q}_\ell) \simeq H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$$
$$H^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

In other words, each embedding $k \hookrightarrow \mathbb{C}$ does define a $\mathbb{Q}$-structure on the different cohomology groups. However, in general, different embeddings give very different $\mathbb{Q}$-structures.

To see this, note that because, because X is defined by finitely many polynomials having only finitely many coefficients, it has a model X_0 over a subfield k_0 of k such that k is an infinite Galois extension of k_0 — let $\Gamma = \mathrm{Gal}(k/k_0)$. The choice of the model defines an action of Γ on $H^i(X, \mathbb{Q}_\ell)$. If the different embeddings of k into $\mathbb{C}$ restricting to a fixed embedding of k_0 gave the same subspace $H^i(X(\mathbb{C}), \mathbb{Q})$ of $H^i(X, \mathbb{Q}_\ell)$, then the action of Γ on $H^i(X, \mathbb{Q}_\ell)$, would stabilize $H^i(X, \mathbb{Q})$. But infinite Galois groups are uncountable and $H^i(X, \mathbb{Q})$ is countable, and so this would imply that Γ acts through a finite quotient on $H^i(X, \mathbb{Q}_\ell)$. However, this is false in general.[6]

The same argument shows that an algebraically-defined cohomology that gave a $\mathbb{Q}$-structure on the $\mathbb{Q}_\ell$-cohomology would force Γ to act through a finite quotient, and so can't exist.

SECOND EXPLANATION

An elliptic curve E is a curve of genus 1 with a chosen point (the zero for the group structure). Over $\mathbb{C}$, $E(\mathbb{C})$ is isomorphic to the quotient of $\mathbb{C}$ by a lattice Λ (thus, topologically it is a torus). In particular $E(\mathbb{C})$ is a group, and the endomorphisms of E are the maps $z + \Lambda \mapsto \alpha z + \Lambda$ defined by a complex number α such that $\alpha\Lambda = \Lambda$. From this, it easy to see that $\mathrm{End}(E) \otimes \mathbb{Q}$ is either $\mathbb{Q}$ or a field K of degree 2 over $\mathbb{Q}$. The cohomology

[5]The proof of this in nonzero characteristic requires Deligne's results on the Weil conjectures (1973).

[6]Roughly speaking, the Tate conjecture says that, when k_0 is finitely generated over $\mathbb{Q}$, the image of the Galois group in $\mathrm{Aut}(H^i(X, \mathbb{Q}_\ell))$ is as large as possible subject to the constraints imposed by the existence of algebraic cycles.

group $H^1(X(\mathbb{C}),\mathbb{Q})$ has dimension 2 as a $\mathbb{Q}$-vector space, and so in the second case it has dimension 1 as a K-vector space.

In characteristic $p \neq 0$, there is a third possibility, namely, $\mathrm{End}(E) \otimes \mathbb{Q}$ can be a division algebra (noncommutative field) of degree 4 over $\mathbb{Q}$. The smallest $\mathbb{Q}$-vector space such a division algebra can act on has dimension 4.[7]

Thus there is *not* a $\mathbb{Q}$-cohomology theory underlying the different cohomology theories defined by Grothendieck, so how are we going to express the fact that, in many ways, they behave as if there were? Grothendieck's answer is the theory of motives. Before discussing it, I need to explain algebraic cycles.

4 Algebraic cycles

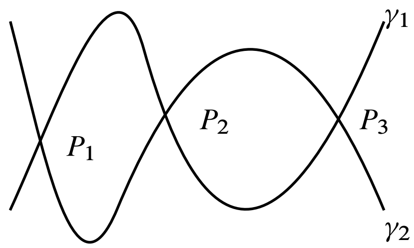
Definitions

Let X be a nonsingular projective variety of dimension n over a field k (not necessarily algebraically closed). A prime cycle on X is a closed algebraic subvariety Z on V that can not be written as a union of two proper closed algebraic subvarieties. Its codimension is $n - \dim Z$. If Z_1 and Z_2 are prime cycles, then

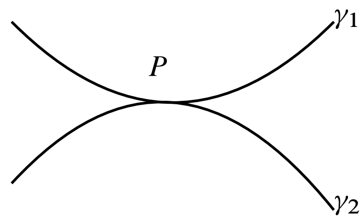
$$\mathrm{codim}(Z_1 \cap Z_2) \leq \mathrm{codim}(Z_1) + \mathrm{codim}(Z_2),$$

and when equality holds we say that Z_1 and Z_2 intersect properly.

The group $C^r(X)$ of algebraic cycles of codimension r on X is the free abelian group generated by the prime cycles of codimension r. Two algebraic cycles γ_1 and γ_2 are said to intersect properly if every prime cycle in γ_1 intersects properly with every prime cycle in γ_2, in which case their intersection product $\gamma_1 \cdot \gamma_2$ is well-defined — it is a cycle of codimension $\mathrm{codim}\, Z_1 + \mathrm{codim}\, Z_2$. For example:



$\gamma_1 \cdot \gamma_2 = P_1 + P_2 + P_3$



$\gamma_1 \cdot \gamma_2 = 2P$

In this way, we get a partially defined map

$$C^r(X) \times C^s(X) - - > C^{r+s}(X).$$

In order to get a map defined on the whole of a set, we need to be able to move cycles. Two cycles γ_0 and γ_1 on X are rationally equivalent[8] if there exists an algebraic cycle γ on

[7] The alert reader will ask how $\mathrm{End}(E)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$ can act on the two-dimensional $\mathbb{Q}_\ell$-vector space $H^1(E,\mathbb{Q}_\ell)$. It can because $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \approx M_2(\mathbb{Q}_\ell)$ when $\ell \neq p$. The ring $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ *may* be a division algebra, but this is OK because the coefficient field for crystalline cohomology is not $\mathbb{Q}_p$ unless $k = \mathbb{F}_p$, in which case the third possibility doesn't occur. Thus there is no contradiction (but only just).

[8] This is the algebraic analogue of homotopy equivalence.

$X \times \mathbb{P}^1$ such that γ_0 is the fibre of γ over 0 and γ_1 is the fibre of γ over 1. This generates an equivalence relation, and we let $C^r_{\mathrm{rat}}(X)$ denote the quotient group. It can be shown that the intersection product defines a bi-additive map[9]

$$C^r_{\mathrm{rat}}(X) \times C^s_{\mathrm{rat}}(X) \to C^{r+s}_{\mathrm{rat}}(X). \tag{1}$$

Let $C^*_{\mathrm{rat}}(X) = \bigoplus_r C^r_{\mathrm{rat}}(X)$. This is a $\mathbb{Q}$-algebra, called the Chow ring of X.

Rational equivalence is the finest equivalence relation on algebraic cycles giving a well defined map (1) on equivalence classes. The coarsest such equivalence relation is numerical equivalence: two algebraic cycles γ and γ' are numerically equivalent if $\gamma \cdot \delta = \gamma' \cdot \delta$ for all algebraic cycles δ of complementary dimension. The numerical equivalence classes of algebraic cycles form a ring $C^*_{\mathrm{num}} = \bigoplus_r C^r_{\mathrm{num}}(X)$ which is a quotient of the Chow ring.

For example, a prime cycle of codimension 1 on $\mathbb{P}^2$ is a curve defined by an irreducible homogeneous polynomial $P(X_0, X_1, X_2)$. The prime cycles defined by two polynomials are rationally equivalent if and only if the polynomials have the same degree. The group $C^1_{\mathrm{rat}}(\mathbb{P}^2) \simeq \mathbb{Z}$ (generated by any line in $\mathbb{P}^2$).

A prime cycle of codimension 1 in $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve defined by an irreducible polynomial $P(X_0, X_1; Y_0, Y_1)$ separately homogeneous in each pair of symbols (X_0, X_1) and (Y_0, Y_1). The rational equivalence class of the cycle is determined by the pair of degrees. The group $Z^1_{\mathrm{rat}}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z} \times \mathbb{Z}$ (basis $e_0 = \mathbb{P}^1 \times \{0\}$ and $e_2 = \{0\} \times \mathbb{P}^1$, and the diagonal $\Delta_{\mathbb{P}^1} \sim_{\mathrm{rat}} e_0 + e_2$).

I write $C^*_\sim(X)_\mathbb{Q}$ for $C^*_\sim(X) \otimes_\mathbb{Z} \mathbb{Q}$ — here and elsewhere $\sim = \mathrm{rat}$ or num.

Cycle maps

For all the cohomology theories we are interested in, there is a cycle class map

$$\mathrm{cl}\colon C^*_{\mathrm{rat}}(X)_\mathbb{Q} \to H^*(X) \stackrel{\text{def}}{=} \bigoplus\nolimits_r H^r(X)$$

that doubles degrees and sends intersection products to cup products.

Correspondences

We are only interested cohomology theories that are contravariant functors, i.e., such that a regular map $f\colon Y \to X$ defines homomorphisms $H^i(f)\colon H^i(X) \to H^i(Y)$. However, this is a weak condition, because there are typically not many regular maps from one algebraic variety to a second. Instead, we should allow "many-valued maps", or, more precisely, correspondences.

The group of correspondences of degree r from X to Y is defined to be

$$\mathrm{Corr}^r(X, Y) = C^{\dim X + r}(X \times Y).$$

For example, the graph Γ_f of a regular map $f\colon Y \to X$ lies in $C^{\dim X}(Y \times X)$; its transpose Γ^t_f lies in $C^{\dim X}(X \times Y) = \mathrm{Corr}^0(X, Y)$. In other words, a regular map from Y to X defines a correspondence of degree zero from X to Y.[10]

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[9] In particular, any two algebraic cycles γ_1 and γ_2 are rationally equivalent to algebaic cycles γ'_1 and γ'_2 that intersect properly, and the rational equivalence class of $\gamma'_1 \cdot \gamma'_2$ is independent of the choice of γ'_1 and γ'_2.

[10] The switching of directions is unfortunate, but we have to do it somewhere, and I'm following Grothendieck and most subsequent authors.

A correspondence γ of degree 0 from X to Y defines a homomorphism $H^*(X) \to H^*(Y)$, namely,

$$x \mapsto q_*(p^*x \cup \mathrm{cl}(\gamma)).$$

Here p and q are the projection maps

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y.$$

The map on cohomology defined by the correspondence Γ_f^t is the same as that defined by f.

We use the notations:

$$\mathrm{Corr}^r_\sim(X,Y) = C_\sim^{\dim X + r}(X \times Y), \quad \mathrm{Corr}^r_\sim(X,Y)_\mathbb{Q} = \mathrm{Corr}^r_\sim(X,Y) \otimes_\mathbb{Z} \mathbb{Q}.$$

5 Definition of motives

Grothendieck's idea was that there should be a universal cohomology theory taking values in a $\mathbb{Q}$-category of motives $\mathcal{M}(k)$.[11]

- ◇ Thus, $\mathcal{M}(k)$ should be a category like the category $\mathsf{Vec}_\mathbb{Q}$ of finite-dimensional $\mathbb{Q}$-vector spaces (but not too like). Specifically:
  - Homs should be $\mathbb{Q}$-vector spaces (preferably finite-dimensional);
  - $\mathcal{M}(k)$ should be an abelian category;
  - even better, $\mathcal{M}(k)$ should be a tannakian category over $\mathbb{Q}$ (see below).
- ◇ There should be a universal cohomology theory

  $$X \rightsquigarrow hX\colon \text{(nonsingular projective varieties)} \to \mathcal{M}(k).$$

  Specifically:
  - each variety X should define a motive hX, and each correspondence of degree zero from X to Y should define a homomorphism $hX \to hY$ (in particular, a regular map $Y \to X$ defines a homomorphism $hX \to hY$).
  - every good[12] cohomology theory should factor uniquely through $X \rightsquigarrow hX$.

FIRST ATTEMPT

We can simply define $\mathcal{M}_\sim(k)$ to be the category with one object hX for each nonsingular projective variety X over k, and with the morphisms defined by

$$\mathrm{Hom}(hX, hY) = \mathrm{Corr}^0_\sim(X,Y)_\mathbb{Q}.$$

[11] J'appelle "motif" sur k quelque chose comme un group de cohomologie ℓ-adique d'un schéma algébrique sur k, mais considéré comme indépendent de ℓ, et avec sa structure "entière", ou disons pour l'instant "sur $\mathbb{Q}$", déduire de la théorie des cycles algébriques. La triste vérité, c'est que pour le moment je ne sais pas définir la catégorie abélienne des motifs, bien que je commence à avoir un yoga assez précis sur cette catégorie...

Grothendieck, letter to Serre, 16.8.1964.

(I call a "motif" over k something like an ℓ-adic cohomology group of an algebraic scheme over k, but considered as independent of ℓ, and with its "integral" structure, or, let us say for the moment "$\mathbb{Q}$" structure, deduced from the theory of algebraic cycles. The sad truth is that for the moment I do not know how to define the abelian category of motives, even though I am beginning to have a rather precise yoga on this category.)

[12] The technical term is Weil cohomology theory.

Correspondences compose, and so this is a category. However, it is clearly deficient. For example, an endomorphism e of a $\mathbb{Q}$-vector space V such that $e^2 = e$ decomposes the vector space into its 0 and 1 eigenspaces

$$V = \mathrm{Ker}(e) \oplus eV,$$

and if (W, f) is a second such pair, then

$$\mathrm{Hom}_{\mathbb{Q}\text{-linear}}(eV, fW) \simeq f \circ \mathrm{Hom}_{\mathbb{Q}\text{-linear}}(V, W) \circ e \qquad (\text{inside } \mathrm{Hom}_{\mathbb{Q}\text{-linear}}(V, W)).$$

A similar statement holds in any abelian category, and so, if we want $\mathcal{M}_{\sim}(k)$ to be abelian, we should at least add the images of idempotents in

$$\mathrm{End}(hX) \stackrel{\text{def}}{=} \mathrm{Corr}^{\mathbf{0}}_{\sim}(X, X)_{\mathbb{Q}} \stackrel{\text{def}}{=} C^{\dim X}_{\sim}(X \times X).$$

SECOND ATTEMPT

We now define $\mathcal{M}_{\sim}(k)$ to be the category with one object $h(X, e)$ for each pair with X as before and e an idempotent in the ring $\mathrm{Corr}^{\mathbf{0}}_{\sim}(X, X)$. Morphisms are defined by

$$\mathrm{Hom}(h(X, e), h(Y, f)) = f \circ \mathrm{Corr}^{\mathbf{0}}_{\sim}(X, Y)_{\mathbb{Q}} \circ e$$

(subset of $\mathrm{Corr}^{\mathbf{0}}_{\sim}(X, Y)_{\mathbb{Q}}$). That's it! This is the category of effective motives for rational or numerical equivalence depending on the choice of $\sim$, which we should denote $\mathcal{M}^{\mathrm{eff}}_{\sim}(k)$. It contains the preceding category as the full subcategory of objects $h(X, \Delta_X)$.

For example, the discussion earlier shows that $\mathrm{Corr}^{\mathbf{0}}_{\mathrm{rat}}(\mathbb{P}^1, \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$ with $e_0 \stackrel{\text{def}}{=} (1, 0)$ represented by $\mathbb{P}^1 \times \{0\}$ and $e_2 \stackrel{\text{def}}{=} (0, 1)$ represented by $\{0\} \times \mathbb{P}^1$. Correspondingly,

$$h(\mathbb{P}^1, \Delta_{\mathbb{P}^1}) = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1$$

with $h^i\mathbb{P}^1 = h(\mathbb{P}^1, e_i)$ (this is true both in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(k)$ and in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{num}}(k)$). We write $\mathbb{1} = h^0\mathbb{P}^1$ and $\mathbb{L} = h^2\mathbb{P}^1$.

For some purposes, the category of effective motives is the most useful,[13] but generally one would prefer a category in which objects have duals. This can be achieved quite easily by inverting the Lefschetz motive $\mathbb{L}$.

THIRD ATTEMPT

The objects of $\mathcal{M}_{\sim}(k)$ are now triples $h(X, e, m)$ with X and e as before, and with $m \in \mathbb{Z}$. Morphisms are defined by

$$\mathrm{Hom}(h(X, e, m), h(Y, f, n)) = f \circ \mathrm{Corr}^{n-m}_{\sim}(X, Y)_{\mathbb{Q}} \circ e.$$

This is the category of motives over k. It contains the preceding category as the full subcategory of objects $h(X, e, 0)$.

Sometimes $\mathcal{M}_{\mathrm{rat}}(k)$ is called the category of Chow motives and $\mathcal{M}_{\mathrm{num}}(k)$ the category of Grothendieck (or numerical) motives.

[13]For example, in investigating a category of effective motives over finite fields with $\mathbb{Z}$ (rather than $\mathbb{Q}$) coefficients, Niranjan Ramachandran and I have discovered a beautiful relation between the orders of Exts in the category and special values of zeta functions. The relation disappears when one passes to the full category of motives.

6 What is known about $\mathcal{M}_\sim(k)$ and $X \rightsquigarrow hX$

KNOWN PROPERTIES OF THE CATEGORY $\mathcal{M}_\sim(k)$

- ◇ The Hom sets are $\mathbb{Q}$-vector space, which are finite-dimensional if $\sim$=num (but not usually otherwise).
- ◇ Direct sums of motives exist, so $\mathcal{M}_\sim(k)$ is an additive category. For example,

$$(X, e, m) \oplus (Y, f, m) = (X \sqcup Y, e \oplus f, m).$$

- ◇ An idempotent f in the endomorphism ring of a motive M decomposes the motive into a direct sum of the kernel and image of e. In fact, if $M = (X, e, m)$, then

$$M = (X, e - efe, m) \oplus (X, efe, m).$$

  Thus $\mathcal{M}_\sim(k)$ is pseudo-abelian.
- ◇ The category $\mathcal{M}_{\mathrm{num}}(k)$ is abelian and semisimple, but $\mathcal{M}_\sim(k)$ is not abelian, except possibly when k is algebraic over a finite field.[14]
- ◇ There is a good tensor product structure on $\mathcal{M}_\sim(k)$, which is defined by

$$h(X, e, m) \otimes h(Y, f, n) = h(X \times Y, e \times f, m + n).$$

  Let $hX = h(X, \Delta_X, 0)$; then $hX \otimes hY = h(X \times Y)$, and so the Künneth formula holds for $X \rightsquigarrow hX$.
- ◇ The above statements hold also for effective motives, but in $\mathcal{M}_\sim(k)$ there are duals. This means that for each motive M there is a dual motive $M^\vee$ and an "evaluation map" ev: $M^\vee \otimes M \to \mathbb{1}$ having a certain universal property. For example,

$$h(X, e, m)^\vee = h(X, e^t, \dim X - m)$$

  if X is irreducible.

I should stress that, although $\mathcal{M}_{\mathrm{rat}}(k)$ is not abelian, it is still a very important category. In particular, it contains more information than $\mathcal{M}_{\mathrm{num}}(k)$.

IS $X \rightsquigarrow hX$ A UNIVERSAL COHOMOLOGY THEORY?

Certainly, the functor $X \rightsquigarrow hX$ sending a X to its Chow motive is universal. This is almost a tautology: good cohomology theories are those that factor through $\mathcal{M}_{\mathrm{rat}}(k)$.

With $\mathcal{M}_{\mathrm{num}}(k)$ there is a problem: a correspondence numerically equivalent to zero will define the zero map on motives, but we don't know in general that it defines the zero map on cohomology. In order for a good cohomology theory to factor through $\mathcal{M}_{\mathrm{num}}(k)$, it must satisfy the following conjecture:

CONJECTURE D. The cohomology class of an algebraic cycle is zero if the cycle is numerically equivalent to zero.

In other words, if $\mathrm{cl}(\gamma) \neq 0$ then γ is not numerically equivalent to zero. Taking account of Poincaré duality, we can restate this as follows: if there exists a cohomology class γ' such that $\mathrm{cl}(\gamma) \cup \gamma' \neq 0$, then there exists an algebraic cycle γ'' such that $\gamma \cdot \gamma'' \neq 0$. Thus,

[14] There is a folklore conjecture that the natural functor $\mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{M}_{\mathrm{num}}(k)$ is an equivalence of categories when k is algebraic over a finite field.

the conjecture is an existence statement for algebraic cycles. Unfortunately, we have no method for proving the existence of algebraic cycles. More specifically, if we expect that a cohomology class is algebraic, i.e., the class of an algebraic cycle, we have no way of going about proving that it is. This is a major problem, perhaps the major problem, in arithmetic geometry and in algebraic geometry.

Conjecture D is known for abelian varieties in characteristic zero, and it is implied by the Hodge conjecture.

WHAT IS A TANNAKIAN CATEGORY?

By an affine group, I mean a matrix group (possibly infinite dimensional)[15]. For such a group G over $\mathbb{Q}$, the category $\mathsf{Rep}_{\mathbb{Q}}(G)$ of representations of G on finite-dimensional $\mathbb{Q}$-vector spaces is an abelian category with tensor products and duals, and the forgetful functor is an exact faithful functor from $\mathsf{Rep}_{\mathbb{Q}}(G)$ to $\mathsf{Vec}_{\mathbb{Q}}$ preserving tensor products.

A neutral tannakian category T over $\mathbb{Q}$ is an abelian category with tensor products and duals for which there exist exact faithful functors to $\mathsf{Vec}_{\mathbb{Q}}$ preserving tensor products; the tensor automorphisms of such a functor ω form an affine group G, and the choice of such functor ω determines an equivalence of categories $\mathsf{T} \to \mathsf{Rep}_{\mathbb{Q}}(G)$. Thus, a neutral tannakian category is an abstract version of the category of representations of an affine group that has no distinguished "forgetful" functor (just as a vector space is an abstract version of k^n that has no distinguished basis).

A tannakian category T over $\mathbb{Q}$ (not necessarily neutral) is an abelian category with tensor products and duals for which there exists an exact faithful tensor functor to the category of vector spaces over some field of characteristic zero (not necessarily $\mathbb{Q}$); we also require that $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$; the choice of such a functor defines an equivalence of T with the category of representations of an affine groupoid.

IS $\mathcal{M}_{\mathrm{NUM}}(k)$ TANNAKIAN?

No, it isn't. In an abelian category T with tensor products and duals it possible to define the trace of an endomorphism. This is preserved by any exact faithful tensor functor $\omega\colon \mathsf{T} \to \mathsf{Vec}_{\mathbb{Q}}$, and so, for the identity map u of an object M,

$$\mathrm{Tr}(u|M) = \mathrm{Tr}(\omega(u)|\omega(M)) = \dim_{\mathbb{Q}} \omega(M) \in \mathbb{N}.$$

For the identity map u of a variety X, $\mathrm{Tr}(u|hX)$ turns out to be the Euler-Poincaré characteristic of X (alternating sum of the Betti numbers). For example, if X is a curve of genus g, then

$$\mathrm{Tr}(u|h(X)) = \dim H^0 - \dim H^1 + \dim H^2 = 2 - 2g,$$

which may be negative. This proves that there does not exist an exact faithful tensor functor $\omega\colon \mathcal{M}_{\mathrm{num}}(k) \to \mathsf{Vec}_{\mathbb{Q}}$.

To fix this we have to change the inner workings of the tensor product structure. When we write

$$hX = h^0X \oplus h^1X \oplus \cdots \oplus h^{2n}X, \tag{2}$$

[15] More precisely, an affine group is an affine group scheme over a field (not necessarily of finite type). Such a group is an inverse limit of affine algebraic group schemes, each of which can be realized as a subgroup of some GL_n.

and change the sign of the "canonical" isomorphism

$$h^i X \otimes h^j X \simeq h^j X \otimes h^i X$$

for ij odd, then $\mathrm{Tr}(u|h(X))$ becomes the sum of the Betti numbers of X rather than the alternating sum. If we can do this for all X, then $\mathcal{M}(k)$ becomes a Tannakian category (neutral if k has characteristic zero, but not otherwise).

However, in general, we don't know that we can write hX in the form (2). For that, we need the following conjecture.

CONJECTURE C. In the ring $\mathrm{End}(hX) = C^{\dim X}_{\mathrm{num}}(X \times X)$, the diagonal Δ_X has a canonical decomposition into a sum of orthogonal idempotents

$$\Delta_X = \pi_0 + \cdots + \pi_{2n}. \tag{3}$$

Such an expression defines a decomposition of hX as in (2) with $h^i X = h(X, \pi_i, 0)$, and this decomposition should have the property that it is mapped to the decomposition

$$H^*(X) = H^0(X) \oplus H^1(X) \oplus \cdots \oplus H^{2n}(X)$$

for any good cohomology theory for which Conjecture D holds.

Again the conjecture is an existence statement for algebraic cycles, and hence is hard. It is known for all nonsingular projective varieties over finite fields (here one constructs the π^i using the Frobenius map) and for abelian varieties in characteristic zero (by definition abelian varieties have a group structure, which is commutative, and the maps $n\colon A \to A$, $n \in \mathbb{Z}$, allow one to construct the π^i).

Until Conjectures C and D are proved, Grothendieck's dream remains unfulfilled.

ASIDE 6.1. Murre (1993) conjectured that a decomposition (3) exists even in $C^{\dim X}_{\mathrm{rat}}(X \times X)$. It has been shown that his conjecture is equivalent to the existence of an interesting filtration on the Chow groups, which had been conjectured by Beilinson and Bloch.

ASIDE 6.2. Grothendieck (1968) described the 'theory of motives' as "a systematic theory of 'arithmetic properties' of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence."

7 Motives have zeta functions

Let X be a projective nonsingular variety over $\mathbb{Q}$. Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (field with p-elements), and fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. For each n, there is exactly one subfield $\mathbb{F}_{p^n}$ of $\mathbb{F}$ of with p^n elements, and $\mathbb{F}_{p^m} \subset \mathbb{F}_{p^n}$ if and only if $m|n$.

Scale the polynomials defining X so that they have integer coefficients, and let $X(\mathbb{F}_{p^n})$ denote the set of zeros of the polynomials in $\mathbb{F}_{p^n}$. This set is finite, and the zeta function of X at p is defined by

$$\log Z_p(X,t) = \sum |X(\mathbb{F}_{p^m})| \frac{t^m}{m}.$$

It is known that $Z_p(X,t)$ is a quotient of polynomials with integer coefficients, and we set

$$\zeta(X,s) = \prod\nolimits_{p \text{ good}} Z_p(X, p^{-s}).$$

For example, let $X = \mathbb{P}^0 = \text{point}$. Then $|X(\mathbb{F}_{p^m})| = 1$ for all p and m, and so

$$\log Z_p(X,t) = \sum \frac{t^m}{m} = \log \frac{1}{1-t};$$

thus

$$\zeta(X,s) = \prod\nolimits_p \frac{1}{1-p^{-s}},$$

which is the Riemann zeta function $\zeta(s)$.

As our next example, let $X = \mathbb{P}^1$. Then $|X(\mathbb{F}_{p^m})| = 1 + p^m$, and so

$$\log Z_p(X,t) = \sum (1+p^m)\frac{t^m}{m} = \log \frac{1}{(1-t)(1-pt)};$$

thus

$$\zeta(X,s) = \zeta(s)\zeta(s-1).$$

Finally, let X be an elliptic curve E over $\mathbb{Q}$. Then

$$|E(\mathbb{F}_{p^m})| = 1 - (a_p^m + \bar{a}_p^m) + p^m$$

where a_p is a complex number of absolute value $p^{\frac{1}{2}}$. Therefore

$$\zeta(E,s) = \frac{\zeta(s)\zeta(s-1)}{L(E,s)}$$

where

$$L(E,s) = \prod\nolimits_p \frac{1}{(1-a_p p^{-s})(1-\bar{a}_p p^{-s})}.$$

This definition of the zeta function of variety doesn't extend to motives, but there is another cohomological definition that does.

For example, we saw that

$$h(\mathbb{P}^1) = h^0(\mathbb{P}^1) \oplus h^2(\mathbb{P}^1),$$

and one shows that

$$\begin{aligned} \zeta(h^0(\mathbb{P}^1)) &= \zeta(s) \\ \zeta(h^1(\mathbb{P}^1)) &= \zeta(s-1). \end{aligned}$$

For an elliptic curve E,

$$h(E) = h^0(E) \oplus h^1(E) \oplus h^2(E)$$

with

$$\zeta(h^1(E)) = L(E,s)^{-1}.$$

Thus, attached to every motive there is a zeta function $\zeta(M,s)$, which is a function of the complex variable s, having, conjecturally at least, many good properties. The functions that arise in this way are called motivic L-functions. On the other hand, there is an entirely different method of constructing functions $L(s)$ from modular forms, automorphic forms, or, most generally, from automorphic representations — these are called automorphic L-functions. Their definition does not involve algebraic geometry. The following is a fundamental guiding principle in the Langlands program[16].

[16] By which I mean Langlands's program, not the geometric analogue, which appears to lack arithmetic interest.

BIG MODULARITY CONJECTURE. Every motivic L-function is an alternating product of automorphic L-functions.

Let E be an elliptic curve over $\mathbb{Q}$. The (little) modularity conjecture says that $\zeta(h^1 E, s)$ is the Mellin transform of a modular form. The proof of this by Wiles (et al.) was the main step in the proof of Fermat's last theorem.

8 The conjecture of Birch and Swinnerton-Dyer, and some mysterious squares

Let E be an elliptic curve over $\mathbb{Q}$. Beginning about 1960, Birch and Swinnerton-Dyer used one of the early computers (EDSAC 2) to study $L(E, s)$ near $s = 1$. These computations led to their famous conjecture: let $L(E, 1)^*$ denote first nonzero coefficient in the expansion of $L(E, s)$ as a power series in $s - 1$; the conjecture states that

$$L(E, 1)^* = \{\text{terms understood}\}\,\{\text{mysterious term}\}\,.$$

The mysterious term is conjectured to be the order of the Tate-Shafarevich group of E, which (if finite) is known to be a square.

About the same time, they studied

$$L_3(E, s) = \prod\nolimits_p \frac{1}{(1 - a_p^3 p^{-s})(1 - \bar{a}_p^3 p^{-s})}$$

near $s = 2$, and they found (computationally) that

$$L_3(E, 1)^* = \{\text{terms understood}\}\{\text{mysterious square}\}.$$

The mysterious square can be quite large, for example 2401. What is it?

As we noted above, $L(E, s) = \zeta(h^1(E), s)$. We can regard the conjecture of Birch and Swinnerton-Dyer as a statement about the motive $h^1(E)$. The conjecture has been extended to all motives over $\mathbb{Q}$. One can show that

$$h^1(E) \otimes h^1(E) \otimes h^1(E) = 3h^1(E, \Delta_E, -1) \oplus M$$

for a certain motive M, and that

$$\zeta(M, s) = L_3(E, s).$$

Thus, the mysterious square is conjecturally the "Tate-Shafarevich group" of the motive M.

[To be continued (maybe).]

9 Final note

Strictly, $\mathcal{M}(k)$ should be called the category of *pure* motives. It is attached to the category nonsingular projective varieties over k. Grothendieck also envisaged a category *mixed* motives attached to the category of *all* varieties over k. It should no longer be semisimple, but each mixed motive should have a filtration whose quotients are pure motives. There is at present no direct definition of the category of mixed motives, not even conjectural, but several mathematicians have constructed triangulated categories that are candidates to be its derived category; it remains to define a t-structure on one of these categories whose heart is the category of mixed motives itself.