Shimura Variety—The family of quotients of a bounded symmetric domain X by the congruence subgroups of a fixed algebraic group G acting transitively on X. Examples include the family of elliptic modular curves, the family of Hilbert modular varieties corresponding to a fixed totally real field, and the family of Siegel modular varieties of a fixed dimension. The arithmetic properties of Shimura varieties were extensively studied by G. Shimura beginning in the early 1960s.

P. Deligne has given a definition according to which a Shimura variety is defined by a reductive algebraic group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphism $\mathbb{C}^\times \to G(\mathbb{R})$ satisfying certain axioms sufficient to ensure that X is a finite union of bounded symmetric domains [4]. The Shimura variety is then the family

$$\mathrm{Sh}_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

where $\mathbb{A}_f$ is the ring of finite adèles of $\mathbb{Q}$ and K runs through the compact open subgroups of $G(\mathbb{A}_f)$. Initially $\mathrm{Sh}_K(G,X)$ is only a complex manifold, but the theorem of Baily and Borel [2] endows it with a canonical structure of a quasi-projective algebraic variety. The Shimura varieties of Deligne's definition differ slightly from the earlier examples in that they are families of nonconnected varieties.

The *weight* of a Shimura variety is the common reciprocal of the restrictions of the maps $\mathbb{C}^\times \to G(\mathbb{R})$ in X to $\mathbb{R}^\times$. When the weight is defined over $\mathbb{Q}$, the Shimura variety may be a moduli variety for abelian varieties with Hodge class and level structures (Shimura varieties of Hodge type), or abelian motives with additional structure (Shimura varieties of abelian type) [4], [6]. A Shimura variety whose weight is not rational is not a moduli variety, and not every Shimura variety whose weight is rational is known to be a moduli variety.

The data (G,X) defining a Shimura variety determine a number field $E \subset \mathbb{C}$, called the *reflex field* for the Shimura variety, and every Shimura variety is known to have a *canonical model* over its reflex field that is characterized by the action of the absolute Galois group of E on certain special points of the Shimura variety [4], [5].

Holomorphic automorphic forms can be interpreted as the sections of certain vector bundles on Shimura varieties, called *automorphic vector bundles*, and the arithmetic properties of the automorphic forms are reflected in the arithmetic properties of the corresponding bundles.

The theorem of Baily and Borel provides a canonical compactification of a Shimura variety that is minimal in a certain sense, but which is usually highly singular. The theory of toroidal embeddings provides compactifications that are both projective and smooth, but which are not canonical [1].

The study of the boundaries of Shimura varieties suggests the definition of a more general object, that of a *mixed Shimura variety,* which plays the same role for Fourier-Jacobi series that a Shimura variety plays for holomorphic automorphic forms [5], [7].

Roughly speaking, the goal for the study of Shimura varieties is to generalize everything that is known about modular curves to all Shimura varieties. For example, R. Langlands has launched an ambitious program to identify the zeta function of a Shimura variety with an alternating product of automorphic L-functions [3].

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