The de Rham–Witt and $\mathbb{Z}_p$-cohomologies of an algebraic variety

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Received 31 August 2004; accepted 14 January 2005

Communicated by Johan De Jong
To Mike Artin on the occasion of his 70th birthday.
Available online 7 April 2005

Abstract

We prove that, for a smooth complete variety $X$ over a perfect field,

$$H^i(X, \mathbb{Z}_p(r)) \cong \text{Hom}_{\mathcal{D}_c^b(R)}(\mathbb{1}, R\Gamma(W\Omega^{\bullet}_X)(r)[i]).$$

where $H^i(X, \mathbb{Z}_p(r)) = \lim \leftarrow \mathbb{H}^{i-r}(X_{et}, v_n(r))$ (Amer. J. Math. 108 (2) (1986) 297–360), $W\Omega^{\bullet}_X$ is the de Rham–Witt complex on $X$ (Ann. Scient. Ec. Num. Sup. 12 (1979b) 501–661), and $\mathcal{D}_c^b(R)$ is the triangulated category of coherent complexes over the Raynaud ring (Inst. Hautes. Etudes. Sci. Publ. Math. 57 (1983) 73–212).

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Keywords: Crystalline cohomology; de Rham–Witt complex; Triangulated category

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$^1$ Partially supported by GRB (University of Maryland) and IHES.

0001-8708/$-$ see front matter © 2005 Published by Elsevier Inc.
doi:10.1016/j.aim.2005.01.007

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1. Introduction

According to the standard philosophy (cf. [2, 3.1]), a cohomology theory \( X \mapsto H^i(X, r) \) on the algebraic varieties over a fixed field \( k \) should arise from a functor \( R\Gamma \) taking values in a triangulated category \( D \) equipped with a \( t \)-structure and a Tate twist \( D \mapsto D(r) \) (a self-equivalence). The heart \( D^\geq \) of \( D \) should be stable under the Tate twist and have a tensor structure; in particular, there should be an essentially unique identity object \( \mathbb{1} \) in \( D^\geq \) such that \( \mathbb{1} \otimes D \cong D \cong D \otimes \mathbb{1} \) for all objects in \( D^\geq \). The cohomology theory should satisfy

\[
H^i(X, r) \cong \text{Hom}_D(\mathbb{1}, R\Gamma(X)(r)[i]).
\]  

(1)

For example, motivic cohomology \( H^i_{\text{mot}}(X, \mathbb{Q}(r)) \) should arise in this way from a functor to a category \( D \) whose heart is the category of mixed motives \( k \). Absolute \( \ell \)-adic étale cohomology \( H^i_{\text{et}}(X, \mathbb{Z}_\ell(r)) \), \( \ell \neq \text{char}(k) \), arises in this way from a functor to a category \( D \) whose heart is the category of continuous representations of \( \text{Gal}(\overline{k}/k) \) on finitely generated \( \mathbb{Z}_\ell \)-modules [5]. When \( k \) is algebraically closed, \( H^i_{\text{et}}(X, \mathbb{Z}_\ell(r)) \) becomes the familiar group \( \lim_{\rightarrow} H^i_{\text{et}}(X, \mu_{p^i}) \) and lies in \( D^\geq \); moreover, in this case, (1) simplifies to

\[
H^i(X, r) \cong H^i(R\Gamma(X)(r)).
\]

(2)

Now let \( k \) be a perfect field of characteristic \( p \neq 0 \), and let \( W \) be the ring of Witt vectors over \( k \). For a smooth complete variety \( X \) over \( k \), let \( W\Omega^\bullet_X \) denote the de Rham–Witt complex of Bloch–Deligne–Illusie (see [10]). Regard \( \Gamma = \Gamma(X, -) \) as a functor from sheaves of \( W \)-modules on \( X \) to \( W \)-modules. Then

\[
H^i_{\text{cris}}(X/W) \cong H^i(R\Gamma(W\Omega^\bullet_X))
\]

[9, 3.4.3], where \( H^i_{\text{cris}}(X/W) \) is the crystalline cohomology of \( X \) [1]. In other words, \( X \mapsto H^i_{\text{cris}}(X/W) \) arises as in (2) from the functor \( X \mapsto R\Gamma(W\Omega^\bullet_X) \) with values in \( D^+(W) \).

Let \( R \) be the Raynaud ring, let \( D(X, R) \) be the derived category of the category of sheaves of graded \( R \)-modules on \( X \), and let \( D(R) \) be the derived category of the category of graded \( R \)-modules \([11, 2.1]\). Then \( \Gamma \) derives to a functor

\[
R\Gamma : D(X, R) \to D(R).
\]

When we regard \( W\Omega^\bullet_X \) as a sheaf of graded \( R \)-modules on \( X \), \( R\Gamma(W\Omega^\bullet_X) \) lies in the full subcategory \( D_c^b(R) \) of \( D(R) \) consisting of coherent complexes [12, II 2.2], which Ekedahl has shown to be a triangulated subcategory with \( t \)-structure [11, 2.4.8]. In this
note, we define a Tate twist \( (r) \) on \( \mathbb{D}^b_c(R) \) and prove that

\[
H^i(X, \mathbb{Z}_p(r)) \cong \text{Hom}_{\mathbb{D}^b_c(R)}(1, R\Gamma(W\Omega^*_X)(r)[i]).
\]

Here \( H^i(X, \mathbb{Z}_p(r)) = \lim_{\to n} H^i_{et}(X, \mathcal{V}_n(r)) \) with \( \mathcal{V}_n(r) \) the additive subsheaf of \( W_n\Omega^*_X \) locally generated for the étale topology by the logarithmic differentials [14, §1], and 1 is the identity object for the tensor structure on graded \( R \)-modules defined by Ekedahl [11, 2.6.1]. In other words, \( X \mapsto H^i(X, \mathbb{Z}_p(r)) \) arises as in (1) from the functor \( X \mapsto R\Gamma(W\Omega^*_X) \) with values in \( \mathbb{D}^b_c(R) \).

This result is used in the construction of the triangulated category of integral motives in [16].

It is a pleasure for us to be able to contribute to this volume: the \( \mathbb{Z}_p \)-cohomology was introduced (in primitive form) by the first author in an article whose main purpose was to prove a conjecture of Artin, and, for the second author, Artin's famous 18.701-2 course was his first introduction to real mathematics.

2. The Tate twist

According to the standard philosophy, the Tate twist on motives should be \( N \mapsto N(r) = N \otimes \mathbb{T}^\otimes r \) with \( \mathbb{T} \) dual to \( \mathbb{L} \) and \( \mathbb{L} \) defined by \( R\mathbb{h}(\mathbb{P}^1) = \mathbb{L} \oplus \mathbb{L}[-2] \).

The Raynaud ring is the graded \( W \)-algebra \( R = R^0 \oplus R^1 \) generated by \( F \) and \( V \) in degree 0 and \( d \) in degree 1, subject to the relations \( FV = p = VF, Fa = \sigma a \cdot F, aV = V \cdot a, ad = da (a \in W), d^2 = 0, \) and \( FdV = d \); in particular, \( R^0 \) is the Dieudonné ring \( W_\sigma[F, V] \) [11, 2.1]. A graded \( R \)-module is nothing more than a complex

\[
M^* = (\cdots \rightarrow M^i \xrightarrow{d} M^{i+1} \rightarrow \cdots)
\]

of \( W \)-modules whose components \( M^i \) are modules over \( R^0 \) and whose differentials \( d \) satisfy \( FdV = d \). We define \( T \) to be the functor of graded \( R \)-modules such that \( (TM)^i = M^{i+1} \) and \( T(d) = -d \). It is exact and defines a self-equivalence \( T : \mathbb{D}_c^b(R) \to \mathbb{D}_c^b(R) \).

The identity object for Ekedahl's tensor structure on the graded \( R \)-modules is the graded \( R \)-module

\[
1 = (W, F = \sigma, V = p\sigma^{-1})
\]

concentrated in degree zero [11, 2.6.1.3]. It is equal to the module \( E_{0/1} = d_f R^0/(F - 1) \) of Ekedahl [3, p. 66].

There is a canonical homomorphism

\[
1 \oplus T^{-1}(1)[-1] \to R\Gamma(W\Omega^{*,*}_{\mathbb{P}^1})
\]
(in $D^b_c(R)$), which is an isomorphism because it is on $W_1\Omega^*_p = \Omega^*_p$ and we can apply Ekedahl’s “Nakayama lemma” [11, 2.3.7]. See [8, I 4.1.11, p. 21], for a more general statement. This suggests our definition of the Tate twist $r$ (for $r \geq 0$), namely, we set

$$M(r) = T^r(M)[-r]$$

for $M$ in $D^b_c(R)$.

Ekedahl has defined a nonstandard $t$-structure on $D^b_c(R)$ the objects of whose heart $\Delta$ are called diagonal complexes [11, 6.4]. It will be important for our future work to note that $T = T(\mathbb{1})[-1]$ is a diagonal complex: the sum of its module degree $(-1)$ and complex degree $(+1)$ is zero. The Tate twist is an exact functor which defines a self-equivalence of $D^b_c(R)$ preserving $\Delta$.

3. Theorem and corollaries

Regard $W\Omega^*_X$ as a sheaf of graded $R$-modules on $X$, and write $R\Gamma$ for the functor $D(X, R) \to D(R)$ defined by $\Gamma(X, -)$. As we noted above, $R\Gamma(W\Omega^*_X)$ lies in $D^b_c(R)$.

**Theorem.** For any smooth complete variety $X$ over a perfect field $k$ of characteristic $p \neq 0$, there is a canonical isomorphism

$$H^i(X, \mathbb{Z}_p(r)) \cong \text{Hom}_{D^b_c(R)}(\mathbb{1}, R\Gamma(W\Omega^*_X)(r)[i]).$$

**Proof.** For a graded $R$-module $M^*$,

$$\text{Hom}(\mathbb{1}, M^*) = \text{Ker}(1 - F: M^0 \to M^0).$$

To obtain a similar expression in $D^b(R)$ we argue as in Ekedahl [3, p. 90]. Let $\hat{R}$ denote the completion $\lim R/(V^n R + dV^n R)$ of $R$ [3, p. 60]. Then right multiplication by $1 - F$ is injective, and $\mathbb{1} \cong \hat{R}^0/\hat{R}^0(1 - F)$. As $F$ is topologically nilpotent on $\hat{R}^1$, this shows that the sequence

$$0 \longrightarrow \hat{R} \overset{(1-F)}{\longrightarrow} \hat{R} \longrightarrow \mathbb{1} \longrightarrow 0,$$

is exact. Thus, for a complex of graded $R$-modules $M$ in $D^b(R)$,

$$\text{Hom}_{D(R)}(\mathbb{1}, M) \overset{[7,10,9]}{\cong} H^0(R\text{Hom}(\mathbb{1}, M)) \overset{(3)}{=} H^0(R\text{Hom}(\hat{R} \overset{(1-F)}{\longrightarrow} \hat{R}, M)).$$
If $M$ is complete in the sense of Illusie 1983, 2.4, then $\text{RHom}(\hat{R}, M) \cong R\text{Hom}(R, M)$ [3, 5.9.3ii, p. 78], and so

$$\text{Hom}_{D(R)}(\mathbb{I}, M) \cong H^0(\text{Hom}(R \xrightarrow{(1-F)} \hat{R}, M))$$

$$\cong H^0(\text{Hom}(R, M) \xrightarrow{1-F} \text{Hom}(R, M)).$$

(4)

Following Illusie [11, 2.1], we shall view a complex of graded $R$-modules as a bicomplex $M^{**}$ in which the first index corresponds to the $R$-grading; thus the $j$th row $M^{i,j}$ of the bicomplex is the $R$-module $(\cdots \to M^{i,j} \to M^{i+1,j} \to \cdots)$, and the $i$th column $M^{i,*}$ is a complex of (ungraded) $R^0$-modules. The $j$th-cohomology $H^j(M^{**})$ of $M^{**}$ is the graded $R$-module

$$(\cdots \to H^j(M^{i,*}) \to H^j(M^{i+1,*}) \to \cdots).$$

Now, $\text{Hom}(R, M^{**}) = M^{0,*}$, and so

$$H^0(\text{Hom}(R, M^{**}(r)[i])) = H^{i-r}(M^{r,*}).$$

(5)

The complex of graded $R$-modules $R\Gamma(W\Omega^\bullet_X)$ is complete [11, 2.4, Example (b), p. 33], and so (4) gives an isomorphism

$$\text{Hom}_{D(R)}(\mathbb{I}, R\Gamma(W\Omega^\bullet_X)(r)[i])$$

$$\cong H^0(\text{Hom}(R, R\Gamma(W\Omega^\bullet_X)(r)[i]) \xrightarrow{1-F} \text{Hom}(R, R\Gamma(W\Omega^\bullet_X)(r)[i])).$$

(6)

The $j$th-cohomology of $R\Gamma(W\Omega^\bullet_X)$ is obviously

$$H^j(R\Gamma(W\Omega^\bullet_X)) = (\cdots \to H^j(X, W\Omega^j_X) \to H^j(X, W\Omega^{j+1}_X) \to \cdots)$$

[11, 2.2.1], and so (5) allows us to rewrite (6) as

$$\text{Hom}_{D(R)}(\mathbb{I}, R\Gamma(W\Omega^\bullet_X)(r)[i]) \cong H^{i-r}(R\Gamma(W\Omega^r_X) \xrightarrow{1-F} R\Gamma(W\Omega^r_X)).$$

This gives an exact sequence

$$\cdots \to \text{Hom}(\mathbb{I}, R\Gamma(W\Omega^\bullet_X)(r)[i]) \to H^{i-r}(X, W\Omega^r_X) \xrightarrow{1-F} H^{i-r}(X, W\Omega^r_X) \to \cdots$$

(7)

On the other hand, there is an exact sequence [10, I 5.7.2]

$$0 \to \nu_r \cdot (r) \to W_r \Omega^r_X \xrightarrow{1-F} W_r \Omega^r_X \to 0$$
of prosheaves on $X_{\text{et}}$, which gives rise to an exact sequence

$$
\cdots \to H^i(X, \mathbb{Z}_p(r)) \to H^{i-r}(X, W_\bullet \Omega_X^r) \to H^{i-r}(X, W_\bullet \Omega_X^r) \to \cdots
$$

(8)

[14, 1.10]. Here $v_\bullet(r)$ denotes the projective system $(v_n(r))_{n \geq 0}$, and $H^i(X, W_\bullet \Omega_X^r) = \lim_{\leftarrow n} H^i(X, W_n \Omega_X^r)$ (étale or Zariski cohomology— they are the same).

Since $H^r(X, W \Omega_X^r) \cong H^r(X, W_\bullet \Omega_X^r)$ [9, 3.4.2, p. 101], the sequences (7) and (8) will imply the theorem once we check that there is a suitable map from one sequence to the other, but the right hand square in

$$
\begin{array}{ccc}
W_\bullet \Omega_X^r & \xrightarrow{1-F} & W_\bullet \Omega_X^r \\
\downarrow & & \downarrow \\
W \Omega_X^r & \xrightarrow{1-F} & W \Omega_X^r
\end{array}
\quad
\begin{array}{ccc}
R \Gamma W_\bullet \Omega_X^r & \xrightarrow{1-F} & R \Gamma W_\bullet \Omega_X^r \\
\downarrow & & \downarrow \\
R \Gamma W \Omega_X^r & \xrightarrow{1-F} & R \Gamma W \Omega_X^r
\end{array}
$$


gives rise to such a map. □

As in Milne [14, p. 309], we let $H^i(X, (\mathbb{Z}/p^n\mathbb{Z})(r)) = H^i_{\text{et}}(X, v_n(r)).$

**Corollary 1.** There is a canonical isomorphism

$$
H^i(X, (\mathbb{Z}/p^n\mathbb{Z})(r)) \cong \text{Hom}_{D^b_c(R)}(\mathbb{I}, R \Gamma W_n \Omega_X^r(r)[i]).
$$

**Proof.** The canonical map $v_\bullet(r)/p^nu_\bullet(r) \to v_n(r)$ is an isomorphism [10, I 5.7.5, p. 598], and the canonical map $W \Omega_X^r/p^nu \Omega_X^r \to W_n \Omega_X^r$ is a quasi-isomorphism [10, I 3.17.3, p. 577]. The corollary now follows from the theorem by an obvious five-lemma argument. □

Lichtenbaum [13] conjectures the existence of a complex $\mathcal{Z}(r)$ on $X_{\text{et}}$ satisfying certain axioms and sets $H^i_{\text{mot}}(X, r) = H^i_{\text{et}}(X, \mathcal{Z}(r))$. Milne [15, p. 68] adds the "Kummer $p$-sequence" axiom that there be an exact triangle

$$
\mathcal{Z}(r) \xrightarrow{p^u} \mathcal{Z}(r) \to v_n(r)[-r] \to \mathcal{Z}(r)[1].
$$

Geisser and Levine [6, Theorem 8.5] show that the higher cycle complex of Bloch (on $X_{\text{et}}$) satisfies this last axiom, and so we have the following result.

**Corollary 2.** Let $\mathcal{Z}(r)$ be the higher cycle complex of Bloch on $X_{\text{et}}$. Then there is a canonical isomorphism

$$
H^i_{\text{et}}(X, \mathcal{Z}(r)) \xrightarrow{p^u} \mathcal{Z}(r) \cong \text{Hom}_{D^b_c(R)}(\mathbb{I}, R \Gamma W_n \Omega_X^r(r)[i]).
$$
Acknowledgments

We thank P. Deligne for pointing out a misstatement in the introduction to the original version.

References
