[Work in progress]

On the Conjecture of Langlands and Rapoport

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FORENOTE (2007): The remarkable conjecture of Langlands and Rapoport (1987) gives a purely group-theoretic description of the points on a Shimura variety modulo a prime of good reduction. In an article in the proceedings of the 1991 Motives conference (Milne 1994, §4), I gave a heuristic derivation of the conjecture assuming a sufficiently good theory of motives in mixed characteristic. I wrote the present article in order to examine what was needed to turn the heuristic argument into a proof, and I distributed it to a few mathematicians (including Vasiu). Briefly, for Shimura varieties of Hodge type (i.e., those embeddable into Siegel modular varieties) I showed that the conjecture is a consquence of three statements:

(a) a good theory of rational Tate classes (see statements (a,b,c,d) in §3 below);
(b) existence of an isomorphism between integral étale and de Rham cohomology for an abelian scheme over the Witt vectors (see 0.1, 5.4 below);
(c) every point in $\mathrm{Sh}_p(\mathbb{F})$ lifts to a special point in $\mathrm{Sh}(\mathbb{Q}^{\mathrm{al}})$.

At the time I wrote the article, I erroneously believed that my work on Lefschetz classes etc. (Milne 1999a,b, 2002, 2005) implied (a). This work does show that (a) (and much more) follows from the Hodge conjecture for complex abelian varieties of CM-type, and in Milne 2007, I discuss some (apparently) much more accessible statements that imply (a). Also, it is possible to prove a variant of (a), which should suffice (in the presence of (b) and (c)) to prove a variant of the conjecture of Langlands and Rapoport, which becomes the true conjecture in the presence of (a).

The situation concerning (b) and (c) is better since Vasiu (2003b) and Kisin (2007) have announced proofs of (b), and Vasiu (2003a) has announced a proof of (c).

Finally, I mention that Pfau (1993, 1996b,a) has shown that the conjecture of Langlands and Rapoport for Shimura varieties of Hodge type implies that it holds for all Shimura varieties of abelian type (i.e., except for those defined by groups of type E_6, E_7, and mixed type D). Thus, the case of Shimura varieties of Hodge type is the crucial one.

It should be clear from what I have already written, that the present manuscript is only a rough working draft, and not a polished work — everything in it should be taken with a grain of salt.

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Introduction

In (Langlands and Rapoport 1987) there is stated a very remarkable conjecture that describes the points on a Shimura variety modulo a prime of good reduction purely in terms of the initial data defining the Shimura variety. In (Milne 1991; also Milne 1990b) I sketched a proof of the conjecture for the case of Siegel modular varieties, and in (Milne 1994b) I gave a *heuristic* derivation of the conjecture for Shimura varieties with rational weight based on the assumption of a sufficiently good theory of motives in mixed characteristic. The purpose of the present paper is to show that these heuristic arguments can be modified so as to give a proof of the conjecture for many Shimura varieties.

Let Sh_p be the quotient of a Shimura variety $\mathrm{Sh}(G,X)$ by a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. It is conjectured that Sh_p has good reduction at every prime v of the reflex field lying over p, and so from such a prime we obtain a set $\mathrm{Sh}_p(\mathbb{F})$ together with actions of $G(\mathbb{A}_f^p)$ and $\mathrm{Gal}(\mathbb{F}/k(v))$. Langlands and Rapoport construct a second set $\mathcal{L}(\mathbb{F})$ directly in terms of the initial data (G,X,K_p), and the conjecture asserts that there is a bijection $\mathcal{L}(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$ of $G(\mathbb{A}_f^p)\times\mathrm{Gal}(\mathbb{F}/k(v))$-sets. The main result of this article is that for Shimura varieties of Hodge type, i.e., sub Shimura varieties of Siegel modular varieties, there is a canonical equivariant map $\mathcal{L}^{\mathrm{lift}}(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$, where $\mathcal{L}^{\mathrm{lift}}(\mathbb{F})$ is the subset of "liftable" elements of $\mathcal{L}(\mathbb{F})$. The map is always injective. If the derived group of G is simply connected, then the map is defined on the whole of $\mathcal{L}(\mathbb{F})$, and we give a description of its image (see Section 6). The problem of proving that the map is surjective leads to the following conjecture:

CONJECTURE 0.1. Let A be an abelian scheme over the ring W of Witt vectors with entries in the algebraic closure of a finite field, and let K_0 be the field of fractions of W. Let $\mathfrak{s} = (s_i)_{i\in I}$ be a family of Hodge tensors on A including a polarization, and, for some fixed inclusion $\tau : W \hookrightarrow \mathbb{C}$, let G be the subgroup of $\mathrm{GL}(H^1((\tau A)(\mathbb{C}),\mathbb{Q}))$ fixing the s_i. Assume that G is reductive, and that the Zariski closure of G in $\mathrm{GL}(H^1(A_{/K_0},\mathbb{Z}_p))$ is hyperspecial. Then, for some faithfully flat $\mathbb{Z}_p$-algebra R, there exists an isomorphism of W-modules

$$R\otimes_{\mathbb{Z}_p} H^1(A_{/K_0},\mathbb{Z}_p) \to R\otimes_W H^1_{\mathrm{dR}}(A)$$

mapping the étale component of each s_i to the de Rham component (except possibly for some small p).

REMARK 0.2. (a) If there exists such an isomorphism for some faithfully flat $\mathbb{Z}_p$-algebra R, then there exists an isomorphism with $R = W$; moreover, there will then be an isomorphism of $\mathbb{Z}_p$-modules

$$H^1(A_{/K_0}, \mathbb{Z}_p) \to H^1_{\text{dR}}(A)^{f=1}$$

mapping the étale component of each s_i to the de Rham component. Here f is as in (Wintenberger 1984, p512).

(b) In (Fontaine and Messing 1987), it is shown that there is a canonical isomorphism

$$B_{\text{crys}} \otimes_{\mathbb{Q}_p} H^1(A_{/K_0}, \mathbb{Q}_p) \to B_{\text{crys}} \otimes_{K_0} H^1_{\text{dR}}(A_{/K_0}),$$

and (Blasius 1994) shows that it maps the étale component of each Hodge class to its de Rham class. One may hope (and the author did hope), that the map sends $A_{\text{crys}} \otimes_{\mathbb{Z}_p} H^1(A_{/K_0}, \mathbb{Z}_p)$ onto $A_{\text{crys}} \otimes_W H^1_{\text{dR}}(A)$, but Fontaine assures me that this is not so.

(c) For many pairs $(A, \mathfrak{s})$, for example, if $\mathfrak{s}$ consists only of a polarization and endomorphisms, the conjecture is known.

The map $\mathcal{L}^{\text{lift}}(\mathbb{F}) \to \text{Sh}_p(\mathbb{F})$ constructed is functorial in (G, X, K_p). In particular, whenever it is bijective, it implies the "refined version of the conjecture of Langlands and Rapoport" (Pfau 1993), and so the problem of extending the proof from Shimura varieties of Hodge type to all Shimura varieties of abelian type becomes a problem that can be stated purely in terms of the initial data used to define the Shimura variety and not involving the arithmetic of the Shimura variety. We discuss this in the last two sections of the paper.

Apart from the announcement (Milne 1991), the conjecture of Langlands and Rapoport has not previously been proved for any Shimura variety of dimension greater than zero, although weaker results have been obtained in special cases in (Ihara 1970; Shimura (unpublished); Milne 1979a, 1979b; Morita 1981; Zink 1983; Reimann and Zink 1991; Kottwitz 1992). In general, the proofs of these weaker results have been based on the Honda-Tate classification of the isogeny classes of abelian varieties over finite fields, whereas a proof of the conjecture of Langlands and Rapoport seems to require some understanding of the *category* of polarized abelian varieties up to isogeny.[1] The results of (Milne 1995b) play an important role in this paper.

Any proof (even statement) of the conjecture of Langlands and Rapoport requires the existence of a canonical *integral* model for the Shimura variety. A precise definition of such a model is given in (Milne 1992), and its existence is proved for all Shimura varieties of abelian type in (Vasiu 1995), using some recent results of Faltings (which the author has not seen).

Perhaps the most important motivation for the conjecture is the desire to understand the zeta functions of Shimura varieties. For an explanation of how the conjecture leads to an explicit formula for the number of points on the Shimura variety with coordinates in a finite field, and from there (following Kottwitz and Langlands) to an expression in terms of twisted orbital integrals for the trace of the twist of a Hecke operator by a power of the Frobenius endomorphism acting on the cohomology of a local system on the Shimura variety, see (Milne, 1992).

Warning: In general, the references are to the articles that will be most helpful to the reader; they should not be assumed to be the original sources.

[1] In fact, for a few Shimura varieties $\text{Sh}(G, X)$ such that G has particularly benign cohomology (there is no L-indistinguishability), it is possible to possible to prove the conjecture of Langlands and Rapoport using only Honda-Tate theory; see Reimann 1997.

Notations and Conventions

We define $\mathbb{C}$ to be the algebraic closure of $\mathbb{R}$ and $\mathbb{Q}^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The field with p elements is denoted by $\mathbb{F}_p$ and its algebraic closure is denoted by $\mathbb{F}$. The ring $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} | p \nmid n\}$. The ring of finite adèles is denoted by $\mathbb{A}_f$, and the ring finite adèles omitting the p-component is denoted by $\mathbb{A}_f^p$. Thus:

$$\mathbb{A}_f = \mathbb{Q} \otimes_{\mathbb{Z}} (\varprojlim_m \mathbb{Z}/m\mathbb{Z}), \quad \mathbb{A}_f^p = \mathbb{Q} \otimes_{\mathbb{Z}} (\varprojlim_{p \nmid m} \mathbb{Z}/m\mathbb{Z}).$$

The dual of an object X in some linear category is denoted by $X^\vee$. "Lattice" means "full lattice", i.e., a submodule generated by a basis for the vector space.

A bilinear form $\psi : V \times V \to R$ on a free finitely generated R-module V will be said to be *nondegenerate* if $\det(\psi) \neq 0$ and *perfect* if $\det(\psi) \in R^\times$. A *symplectic space* (V, ψ) over a ring R is a free finitely generated R-module V together with a perfect alternating form ψ.

A group scheme G over a scheme S will be said to be *reductive* if it is affine and smooth over S and its geometric fibres are connected and reductive (Demazure and Grothendieck 1970, XIX 2.7).

Except in §1, we use the following notations. For a commutative ring R, $\mathbf{Mod}_R$ denotes the category of finitely generated R-modules and $\mathbf{Mod}_R^{\mathrm{proj}}$ the category of finitely generated projective R-modules. When R is a field, we write $\mathbf{Vec}_R$ for $\mathbf{Mod}_R$. For an affine group (or monoid) scheme G over a ring R, $\mathbf{Rep}_R(G)$ denotes the category of representations of G on finitely generated projective R-modules. A representation will be denoted by $\xi : G \to \mathrm{GL}(V(\xi))$ or $\xi : G \to \mathrm{GL}(\Lambda(\xi))$, depending on whether R is a field or not, so that the forgetful fibre functor becomes $\xi \mapsto V(\xi)$ or $\xi \mapsto \Lambda(\xi)$.

For a scheme Z over a ring (or scheme) R and an R-algebra (or R-scheme) S, we denote the base change of Z to S by Z_S or $Z_{/S}$. For a group scheme G over a field k and a subfield k_0 of k, we let $(G)_{k/k_0}$ denote the group scheme over k_0 obtained from G by restriction of scalars. When k has infinite degree over k_0, we set $(\mathbb{G}_m)_{k/k_0} = \varprojlim (\mathbb{G}_m)_{k'/k_0}$ where k' runs over the finite extensions of k_0 contained in k. For the standard notations concerning $\mathbb{S} = (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$, see (Milne 1992, p159) for example.

The *flat topology* is that for which the covering families are the surjective families of flat affine maps (the "topologie fidèlement plate quasi-compacte" of (Demazure and Grothendieck 1970, IV.6.3)). All groupoids will be faithfully flat and affine. (A review of the theory of groupoids can be found in (Milne 1992, Appendix A).)

Our conventions concerning tensor categories follow those of (Deligne and Milne 1982). Thus, for R a commutative ring, an *R-linear tensor category* is an R-linear category $\mathbf{C}$ together with an R-bilinear functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and associativity and commutativity constraints satisfying certain axioms (including the existence of an identity object). An *R-linear tensor functor* from one such category $\mathbf{C}$ to a second $\mathbf{C}'$ is an R-linear functor $F : \mathbf{C} \to \mathbf{C}'$ together with isomorphisms $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ functorial in X and Y and compatible with the constraints. These notions correspond to those of a "$\otimes$-catégorie ACU R-linéaire" and a "$\otimes$-foncteur ACU R-linéaire" in (Saavedra 1972, p12, p39, p65). For k a field, the axioms for a k-linear Tannakian category include the condition that, for any identity object (U, u), the map $k \to \mathrm{End}(U)$ is an isomorphism. A functor from one R-linear tensor category to a second will always be an R-linear tensor functor, even when this is not explicitly stated. If ω is an R-linear tensor functor $\mathbf{C} \to \mathbf{Mod}_R$ and R' is an R-algebra, then $R' \otimes_R \omega$ may denote the functor

$$X \mapsto R' \otimes_R \omega(X) : \mathbf{C} \to \mathbf{Mod}_{R'}$$

or the functor

$$R' \otimes_R \mathbf{C} \to R' \otimes_R \mathbf{Mod}_R = \mathbf{Mod}_{R'}$$

which the previous functor induces.

For a field k , $\mathbf{AV}_{(p)}(k)$ denotes the category whose objects are the abelian varieties over k with $\mathrm{Mor}(A,B) = \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ as the morphisms, and $\mathbf{AV}(k)$ denotes the category with the same objects but with $\mathrm{Mor}(A,B) = \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a field k of characteristic zero, $\mathbf{Mot}(k)$ denotes the category of abelian motives over k defined using Hodge classes. To be precise, the objects of $\mathbf{Mot}(k)$ are triples $h(A,e,m)$ where A is an equidimensional variety each of whose connected components of positive dimension admits a structure of an abelian variety, e is an (absolute) Hodge class on A of degree $\dim A$ (and so defines endomorphisms of the cohomology groups of A) such that $e^2 = e$, and $m \in \mathbb{Z}$. For the structures that turn $\mathbf{Mot}(k)$ into a $\mathbb{Q}$-linear Tannakian category, see (Jannsen 1992) or (Scholl 1994). There is a fully faithful covariant functor

$$A \mapsto h_1(A) : \mathbf{AV}(k) \to \mathbf{Mot}(k).$$

There are the following functors on $\mathbf{Mot}(k)$:

ω_τ defined for each homomorphism $\tau : k \hookrightarrow \mathbb{C}$; for an abelian variety A,

$$\omega_\tau(h_1(A)) = H_1((\tau A)(\mathbb{C}), \mathbb{Q});$$

we sometimes write ω_B for ω_τ (B=Betti);

ω_f^p, ω_p defined for a fixed algebraic closure k^{al} of k; for an abelian variety A,

$$\omega_f^p(h_1(A)) = \mathbb{Q} \otimes_{\mathbb{Z}} \varprojlim_{p \nmid m} A_m(k^{\mathrm{al}})$$
$$\omega_p(h_1(A)) = \mathbb{Q} \otimes_{\mathbb{Z}} \varprojlim_{n} A_{p^n}(k^{\mathrm{al}});$$

$\omega_{\mathbf{dR}}$ the de Rham homology; for an abelian variety A,

$$\omega_{\mathrm{dR}}(h_1(A)) = H^1_{\mathrm{dR}}(A)^\vee, \quad H^1_{\mathrm{dR}}(A) = \mathbb{H}^1(A, \Omega^\bullet_{A/k}).$$

1 Tannakian Preliminaries

In this section only, $\mathbf{Mod}_R$ *denotes the category of all* R*-modules, and* $\mathbf{Rep}_R(G)$ *the category of representations of* G *on arbitrary* R*-modules.* The superscript "fg" denotes the subcategory of modules or representations finitely generated over R, and the superscript "proj" denotes the subcategory of modules or representations that are both finitely generated and projective over R. The forgetful functor on $\mathbf{Rep}_R(G)$ or one of its subcategories will be denoted by ω^G.

Beyond the standard results on tensor categories over fields, we shall need the following theorem.

THEOREM 1.1. Let R be a regular ring of Krull dimension ≤ 1. Let $\mathbf{C}$ be an abelian R-linear tensor category, and let $\omega : \mathbf{C} \to \mathbf{Mod}_R^{\mathrm{fg}}$ be an exact faithful R-linear tensor functor. Let $\mathbf{C}_0$ be the full subcategory of $\mathbf{C}$ whose objects are those X for which $\omega(X)$ is projective, and assume that every object of $\mathbf{C}$ is a quotient of an object of $\mathbf{C}_0$.

(a) There exists a flat affine monoid G over R and an equivalence of R-linear tensor categories

$$\mathbf{C} \to \mathbf{Rep}_R^{\text{fg}}(G)$$

whose composite with ω^G is ω.

(b) The canonical morphisms (of functors of R-algebras)

$$G \to \underline{\text{End}}^{\otimes}(\omega) \to \underline{\text{End}}^{\otimes}(\omega|\mathbf{C}_0)$$

are isomorphisms.

(c) The monoid G is a group scheme if and only if $\mathbf{C}_0$ is rigid.

(d) Assume $\mathbf{C}_0$ is rigid. If $\omega' : \mathbf{C}_0 \to \mathbf{Mod}_R$ is an R-linear tensor functor with the properties,
   (i) ω' maps sequences in $\mathbf{C}_0$ that are exact as sequences in $\mathbf{C}$ to exact sequences in $\mathbf{Mod}_R$,
   (ii) $\omega'(\varphi)$ an isomorphism $\implies$ φ an isomorphism,
   then $\underline{\text{Hom}}_R^{\otimes}(\omega', \omega|_{\mathbf{C}_0})$ is a G-torsor for the flat topology.

By an abuse of language, a functor satisfying the conditions in (d) will said to be *exact*. Thus (see Lemma 1.3 below) an exact R-linear tensor functor $\mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$ is one that extends (essentially uniquely) to an exact faithful R-linear tensor functor on $\mathbf{Rep}_R(G)$ commuting with direct limits.

REMARK 1.2. (a) Under the tensor equivalence in (1.1a), $\mathbf{C}_0$ corresponds to $\mathbf{Rep}_R^{\text{proj}}(G)$.

(b) Let G be a flat affine monoid over a ring R. When R is regular of dimension ≤ 1, every object of $\mathbf{Rep}_R^{\text{fg}}(G)$ is a quotient of an object of $\mathbf{Rep}_R^{\text{proj}}(G)$ (Serre 1968, 2.2). Therefore the condition that every object of $\mathbf{C}$ be a quotient of an object of $\mathbf{C}_0$ is necessary for (a) to be true.

(c) Much of the theorem remains true (with the same proof) when R is not regular of dimension ≤ 1, but unless Serre's result holds for $\mathbf{Rep}_R^{\text{fg}}(G)$, it is vacuous.

(d) Let $\mathbf{C}_0$ be as in the statement of the theorem and rigid, and let $\omega : \mathbf{C}_0 \to \mathbf{Mod}_R$ be an R-linear tensor functor. Because ω is a tensor functor, it preserves duals (cf. Deligne 1990, p120). But every R-module admitting a dual is projective of finite-type (ibid. 2.6), and so ω takes values in $\mathbf{Mod}_R^{\text{proj}}$. It follows that if ω is left or right exact, then it is exact. Moreover, if $X \neq 0$, then $\text{ev} : X^\vee \otimes X \to \mathbb{1}$ is surjective, and so $\omega(X) \neq 0$; thus ω is also faithful.

PROOF. For the proofs of (a), (b), (c), see Saavedra (1972), II.4.1.

We prove (d). After (a) and (c), we can assume that $\mathbf{C} = \mathbf{Rep}^{\text{fg}}(G)$ where G is a flat affine group scheme over R, and we have to prove that, for any exact R-linear tensor functor $\omega : \mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$, the functor $\underline{\text{Hom}}_R^{\otimes}(\omega, \omega^G)$ of R-algebras is a G-torsor for the flat topology. Here ω^G denotes the forgetful functor on $\mathbf{Rep}_R^{\text{proj}}(G)$. Because $\mathbf{Rep}_R^{\text{proj}}(G)$ is rigid, $\underline{\text{Hom}}_R^{\otimes}(\omega, \omega^G) = \underline{\text{Isom}}_R^{\otimes}(\omega, \omega^G)$, which (b) shows to be a pseudo-torsor for G. It remains to prove that it is represented by a faithfully flat R-algebra.

According to the lemma below, ω extends to an exact faithful R-linear tensor functor $\omega_\infty : \mathbf{Rep}_R(G) \to \mathbf{Mod}_R$ commuting with direct limits. In this situation (Saavedra 1972, II.3.2.2) shows that $\underline{\text{Hom}}_R^{\otimes}(\omega_\infty, \omega^G)$ is represented by the R-algebra $\omega_\infty(B_d)$ where B is the affine algebra of G and B_d denotes the regular representation of G on B. Since

$$\underline{\text{Hom}}_R^{\otimes}(\omega_\infty, \omega^G) \xrightarrow{\approx} \underline{\text{Hom}}_R^{\otimes}(\omega, \omega^G),$$

this shows that the second functor is represented by the R-algebra $\varinjlim_{B_i} \omega(B_i)$, where the B_i run through the finitely generated submodules of B stable under the action of G. Each $\omega(B_i)$ is a

projective R-module, hence flat, and therefore their direct limit $\omega(B)$ is also flat. It is faithful because it represents a faithful functor. □

LEMMA 1.3. Let R be a regular ring of dimension ≤ 1. Every exact R-linear tensor functor $\omega : \mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$ extends to an exact faithful R-linear tensor functor $\mathbf{Rep}_R(G) \to \mathbf{Mod}_R$ commuting with direct limits.

PROOF. It follows from the result of Serre cited in Remark 1.2(a), that ω has (an essentially unique) extension to an exact functor $\mathbf{Rep}_R^{\text{fg}}(G) \to \mathbf{Mod}_R$, which is faithful because φ is an isomorphism if $\omega(\varphi)$ is an isomorphism. Now ω has a canonical extension to the Ind category of $\mathbf{Rep}_R^{\text{fg}}(G)$, but this can be identified with $\mathbf{Rep}_R(G)$. □

COROLLARY 1.4. Let G be a reductive group over a Henselian discrete valuation ring R whose residue field has dimension ≤ 1 (in the sense of Serre 1964). Every exact R-linear tensor functor $\omega : \mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$ is isomorphic to the forgetful functor.

PROOF. According to the theorem, $\underline{\text{Hom}}^{\otimes}(\omega, \omega^G)$ is a G-torsor, and therefore corresponds to an element of $H^1(R, G)$. Because G is of finite type, this fpqc-group can be interpreted as an fppf-group (Saavedra 1972, III.3.1.1.1), and because G is smooth

$$H^1(R, G) = H^1(k, G_{/k}) \qquad \text{(étale cohomology)}$$

where k is the residue field of R (Milne 1980, III.3.1). But $G_{/k}$ is connected, and so $H^1(k, G) = 0$ (Steinberg 1965). Thus $\underline{\text{Hom}}^{\otimes}(\omega, \omega^G)$ is the trivial G-torsor. □

EXAMPLE 1.5. Let G be an affine group scheme over $\mathbb{Q}$, and let G_p be a flat model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$ (by this we mean that G_p is a flat group scheme over $\mathbb{Z}_p$ equipped with an isomorphism $G_{p/\mathbb{Q}_p} \to G_{/\mathbb{Q}_p}$). Define $\mathbf{C}$ to be the category whose objects are the triples (V, Λ, φ) consisting of an object V of $\mathbf{Rep}_{\mathbb{Q}}^{\text{fg}}(G)$, an object Λ of $\mathbf{Rep}_{\mathbb{Z}_p}^{\text{fg}}(G_p)$, and an equivariant map $\varphi : \Lambda \to \mathbb{Q}_p \otimes_{\mathbb{Q}} V$ inducing an isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda \to \mathbb{Q}_p \otimes_{\mathbb{Q}} V$. Then $\mathbf{C}$ is an abelian category, and it becomes a $\mathbb{Z}_{(p)}$-linear tensor category when endowed with the obvious structures. Moreover

$$(V, \Lambda, \varphi) \mapsto \varphi^{-1}(V),$$

is an exact faithful $\mathbb{Z}_{(p)}$-linear tensor functor $\omega : \mathbf{C} \to \mathbf{Mod}_{\mathbb{Z}_{(p)}}$. The pair $(\mathbf{C}, \omega)$ satisfies the conditions of Theorem 1.1, and the flat affine group scheme $G_{(p)} = \underline{\text{Aut}}^{\otimes}(\omega)$ over $\mathbb{Z}_{(p)}$ is simultaneously a model of G and a model of G_p.

EXAMPLE 1.6. Suppose we are given a $\mathbb{Q}$-linear Tannakian category $\mathbf{C}$, a $\mathbb{Z}_p$-linear abelian tensor category $\mathbf{C}_p$, and an exact faithful $\mathbb{Q}_p$-linear tensor functor $I : \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbf{C} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbf{C}_p$. Define $\mathbf{C}_{(p)}$ to be the category whose objects are the triples (X, Λ, φ) with X an object $\mathbf{C}$, Λ an object of $\mathbf{C}_p$, and φ an isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda \to I(X)$. Then $\mathbf{C}$ is a $\mathbb{Z}_{(p)}$-linear tensor category.

PROPOSITION 1.7. Let G be a reductive group over a Henselian discrete valuation ring R whose residue field has dimension ≤ 1 (in the sense of Serre 1964), and let K be the field of fractions of R. Consider a fibre functor $\omega_0 : \mathbf{Rep}_K^{\text{fg}}(G_{/K}) \to \mathbf{Vec}_K$. The set of exact R-linear tensor functors $\omega : \mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$ such that $K \otimes \omega = \omega_0$ is in natural one-to-one correspondence with $\underline{\text{Hom}}^{\otimes}(\omega_0, \omega^G)/G(R)$; in particular, no such ω exists unless $\omega_0 \approx \omega^G$.

PROOF. Let ω be a functor $\mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$ such that $K \otimes \omega = \omega_0$. According to Corollary 1.4, $\omega \approx \omega^G$, and this implies that

$$\omega_0 = K \otimes \omega \approx K \otimes \omega^G = \omega^{G_K}.$$

Conversely, if there exists an isomorphism $\varphi : \omega_0 \to \omega^{G_K}$, then the choice of such an isomorphism determines a ω, namely, that mapped onto ω^G by φ. Clearly two isomorphisms determine the same ω if and only if they differ by an element of $G(R)$. □

Let G be an affine group scheme (flat) of finite type over a ring R, and let ξ be a representation of G on a finitely generated projective R-module $\Lambda(\xi)$. By a tensor on ξ we mean an element $t \in \Lambda(\xi)^{\otimes r} \otimes \Lambda(\xi)^{\vee \otimes s}$ fixed under the action of G. Note that t can regarded as a homomorphism $R \to \Lambda(\xi)^{\otimes r} \otimes \Lambda(\xi)^{\vee \otimes s}$ of G-modules, and it defines a tensor $\omega(t)$ on $\omega(\Lambda(\xi))$ for every R-linear tensor functor ω on $\mathbf{Rep}_R(G)$. A representation ξ_0 together with a family $(t_i)_{i \in I}$ of tensors is said to *defining* if, for all (flat) R-algebras S,

$$G(S) = \{g \in \mathrm{Aut}(S \otimes_R \Lambda(\xi_0)) \mid gt_i = t_i, \text{ for all } i \in I\}.$$

In particular, this implies that ξ_0 is faithful. For conditions under which a defining representation and tensors exist, see (Saavedra 1972, p151).

PROPOSITION 1.8. Let G be an affine group scheme flat over a Henselian discrete valuation ring R whose residue field has dimension ≤ 1, and assume that $(\xi_0, (t_i)_{i \in I})$ is defining for G.

(a) Consider a finitely generated projective R-module Λ and a family $(s_i)_{i \in I}$ of tensors for Λ. There exists an exact R-linear tensor functor $\omega : \mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$ such that $(\omega(\xi_0), (\omega(t_i)_{i \in I})) = (\Lambda, (s_i)_{i \in I})$ if and only if there exists an isomorphism $\Lambda \to \Lambda(\xi_0)$ mapping each s_i to t_i.

(b) For any exact R-linear tensor functor $\omega : \mathbf{Rep}_R^{\text{proj}}(G) \to \mathbf{Mod}_R$, the map $\alpha \mapsto \alpha(\xi_0)$ identifies $\mathrm{Hom}^{\otimes}(\omega, \omega^G)$ with the set of isomorphisms $\omega(\xi_0) \to \omega^G(\xi_0)$ mapping each $\omega(t_i)$ to t_i.

PROOF. (a) If ω exists, then according to Corollary 1.4 there exists an isomorphism $\omega \to \omega^G$, and so the condition is necessary. Conversely, suppose there is given an isomorphism $\varphi : \Lambda \to \Lambda(\xi_0)$ mapping each s_i to t_i. For any flat R-algebra S, $G(S)$ acts on $S \otimes_R \Lambda(\xi_0)$, and hence (via φ) on $S \otimes_R \Lambda$. The s_i are fixed under this last action, and so it defines a homomorphism $G(S) \to G(S)$, functorial in S, i.e., an automorphism of G as a group scheme. This automorphism defines a tensor functor with the correct property.

(b) Both sets are $G(R)$-torsors, and so any $G(R)$-equivariant map from one to the other is a bijection. □

REMARK 1.9. There are several variants of the above proposition.

(a) In the situation of the proposition, let H be a second affine group scheme flat over R, and let Λ be a representation of H and $(s_i)_{i \in I}$ a family of tensors for Λ (as a representation of H). If there exists an isomorphism $\Lambda \to \Lambda(\xi_0)$ mapping each s_i to t_i, then there is an exact R-linear tensor functor $\omega : \mathbf{Rep}^{\text{proj}}(G) \to \mathbf{Rep}^{\text{proj}}(H)$ such that $\omega(\xi_0) = \Lambda$. Indeed, as in the proof of the proposition, the isomorphism $\Lambda \to \Lambda(\xi_0)$ defines a homomorphism $H \to G$, and this induces ω.

(b) There is also a variant of the proposition in which the categories are not assumed to be neutral.

2 Statement of the Conjecture of Langlands and Rapoport

Let G be a reductive group over $\mathbb{Q}$ whose connected centre is split by a CM-field[2], let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ satisfying the axioms (Deligne 1979, 2.1.1.1—2.1.1.3), and let K_p be a hyperspecial subgroup of $G(\mathbb{Q}_p)$. According to (Tits 1979, 3.8.1), there is a reductive model G_p of G over $\mathbb{Z}_p$ such that $K_p = G_p(\mathbb{Z}_p)$. We sometimes omit the subscript on G_p and simply write $G(\mathbb{Z}_p)$, for example, for $G_p(\mathbb{Z}_p)$.

The Shimura variety defined by (G, X) has a canonical model $\mathrm{Sh}(G, X)$ over its reflex field $E(G, X)$, and we let $\mathrm{Sh}_p(G, X)$ denote the quotient $\mathrm{Sh}(G, X)/K_p$. It has complex points

$$\mathrm{Sh}_p(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z^p \cdot K_p = G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p)/Z^p$$

where Z is the centre of G, Z^p is the closure of $Z(\mathbb{Z}_{(p)}) =_{df} Z(\mathbb{Q}) \cap K_p$ in $Z(\mathbb{A}_f^p)$, and $G(\mathbb{Z}_{(p)}) = G(\mathbb{Q}) \cap K_p$ (Milne 1994b, 4.11).

Let v be a prime of $E = E(G, X)$ dividing p, and let E_v be the completion of E at v. Assume that $\mathrm{Sh}_p(G, X)$ has a canonical integral model (in the sense of Milne 1992, 2.9) over the valuation ring $\mathcal{O}_v$ of E_v, and denote the model again by $\mathrm{Sh}_p(G, X)$, or just Sh_p. The conjecture of Langlands and Rapoport describes the set $\mathrm{Sh}_p(\mathbb{F})$ together with the actions of $G(\mathbb{A}_f^p)$ and $\mathrm{Gal}(\mathbb{F}/k(v))$ on it. Before stating the conjecture, we need to review some notations and constructions.

Choose an extension of v to a valuation on $\mathbb{Q}^{\mathrm{al}}$, and let $\mathbb{C}_p$ be the corresponding completion of $\mathbb{Q}^{\mathrm{al}}$; it is algebraically closed. Let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$, let W be the valuation ring of K_0, and let $\mathbb{F}$ be the residue field of W. The existence of the hyperspecial subgroup K_p implies that v is unramified over p, and so E_v is contained in K_0 and $\mathcal{O}_v$ and its residue field $k(v)$ are contained in W and $\mathbb{F}$. Moreover, $\mathbb{F}$ is an algebraic closure of $k(v)$ and W is the ring of Witt vectors with entries in $\mathbb{F}$. Thus:

$$\begin{array}{ccccccc}
\mathbb{Q}^{\mathrm{al}} & — & \mathbb{C}_p & & & & \\
| & & | & & & & \\
| & & K_0 & — & W & — & \mathbb{F} \\
| & & | & & | & & | \\
E & — & E_v & — & \mathcal{O}_v & — & k(v) \\
| & & | & & | & & | \\
\mathbb{Q} & — & \mathbb{Q}_p & — & \mathbb{Z}_p & — & \mathbb{F}_p
\end{array} \quad .$$

Let σ denote the automorphism $x \mapsto x^p$ of $\mathbb{F}/\mathbb{F}_p$ and its lifts to W and K_0.

By definition, $E = E(G, X)$ is the field of definition of the $G(\mathbb{C})$-conjugacy class M_X of cocharacters of $G_{\mathbb{C}}$ containing the μ_x for $x \in X$. In particular, M_X is defined over E_v. Let T_p be a maximal $\mathbb{Q}_p$-split subtorus of $G_{\mathbb{Q}_p}$ whose apartment contains the hyperspecial vertex fixed by K_p, and let T_0 be a maximal split subtorus of $G_{/K_0}$ containing T_p. We choose a cocharacter μ_0 of T_0 to represent M_X over K.

For each finite prime $\ell \neq p$, choose an extension of ℓ to a prime of $\mathbb{Q}^{\mathrm{al}}$, and hence an embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_\ell^{\mathrm{al}}$ of $\mathbb{Q}^{\mathrm{al}}$ into an algebraic closure of $\mathbb{Q}_\ell$. When $\mathfrak{G}$ is a $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid, $\mathfrak{G}(\ell)$ will denote

[2]This condition is unnecessary, but it makes some statements simpler, and the Shimura varieties that don't satisfy it are of no interest.

the base change of $\mathfrak{G}$ to a $\mathbb{Q}_\ell^{\text{al}}/\mathbb{Q}_\ell$-groupoid and $\mathfrak{G}(p)$ will denote its base change to a $\mathbb{C}_p/\mathbb{Q}_p$-groupoid.

We shall need to consider the following categories:

$\mathbf{V}_\infty$ the $\mathbb{R}$-linear Tannakian category whose objects are $\mathbb{Z}$-graded complex vector spaces with a semilinear endomorphism F such that $F^2 = (-1)^m$ on the direct summand of weight m;

$\mathbf{Vec}_{\mathbb{Q}_\ell}$ the $\mathbb{Q}_\ell$-linear Tannakian category of $\mathbb{Q}_\ell$-vector spaces;

$\mathbf{Isoc}(\mathbb{F})$ the $\mathbb{Q}_p$-linear Tannakian category of isocrystals over $\mathbb{F}$, i.e., finite-dimensional vector spaces over K_0 endowed with a σ-linear automorphism (usually denoted ϕ).

Each of these categories has a forgetful fibre functor (over $\mathbb{C}$, $\mathbb{Q}_\ell$, and K_0 respectively), and on replacing the last two with their base changes we obtain canonical fibre functors over $\mathbb{C}$, $\mathbb{Q}_\ell^{\text{al}}$, and $\mathbb{C}_p$. Let $\mathfrak{G}_\infty$, $\mathfrak{G}_\ell$, and $\mathfrak{G}_p$ be the corresponding groupoids.

Recall (Milne 1992, 3.27) that a pseudomotivic groupoid is a system $(\mathfrak{P}, (\zeta_\ell))$ consisting of a $\mathbb{Q}^{\text{al}}/\mathbb{Q}$-groupoid $\mathfrak{P}$ with kernel the Weil-number torus P, and homomorphisms $\mathfrak{G}_\ell \to \mathfrak{P}(\ell)$ for each ℓ (including p and ∞) that act on the kernels in a specified fashion; moreover, that there is a homomorphism $\varphi_S : \mathfrak{P} \to \mathfrak{G}_S$, well-defined up to isomorphism, from $\mathfrak{P}$ to the neutral $\mathbb{Q}^{\text{al}}/\mathbb{Q}$-groupoid defined by the Serre group S.

Assume initially that the weight of $\mathrm{Sh}(G, X)$ is defined over $\mathbb{Q}$. Then to each special point x of X there corresponds a homomorphism $\rho_x : S \to G$ uniquely determined by the condition that $(\rho_x)_{\mathbb{R}} \circ h_{\text{can}} = h_x$. On composing ρ_x with φ_S, we obtain a homomorphism $\varphi_x : \mathfrak{P} \to \mathfrak{G}_G$. Any homomorphism $\mathfrak{P} \to \mathfrak{G}_G$ isomorphic to such a homomorphism is said to be *special.*

Let φ be a homomorphism $\mathfrak{P} \to \mathfrak{G}_G$, and define

$$I(\varphi) = \{g \in G(\mathbb{Q}^{\text{al}}) \mid \text{ad}g \circ \varphi = \varphi\}.$$

For each ℓ, φ induces a homomorphism $\varphi(\ell) : \mathfrak{P}(\ell) \to \mathfrak{G}_G(\ell)$, and we set $\theta_\ell = \varphi(\ell) \circ \zeta_\ell$.

For each $\ell \neq p, \infty$, there is a homomorphism $\xi_\ell : \mathfrak{G}_\ell \to \mathfrak{G}_G$ that on points is the obvious section to

$$G(\mathbb{Q}_\ell^{\text{al}}) \rtimes \mathrm{Gal}(\mathbb{Q}_\ell^{\text{al}}/\mathbb{Q}_\ell) \longrightarrow \mathrm{Gal}(\mathbb{Q}_\ell^{\text{al}}/\mathbb{Q}_\ell).$$

Define

$$X_\ell(\varphi) = \{g \in G(\mathbb{Q}_\ell^{\text{al}}) \mid \text{ad}g \circ \xi_\ell = \theta_\ell\},$$

and let $X^p(\varphi)$ be the restricted topological product of the $X_\ell(\varphi)$. It is a $G(\mathbb{A}_f^p)$-torsor under the obvious right action.

A homomorphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ defines a σ-conjugacy class $[b(\theta)]$ in $G(K_0)$ (ibid. 3.36). Choose $b \in G(K_0)$ to represent $[b(\theta_p)]$ and define

$$X_p(\varphi) = \{g \in G(B)/G(W) \mid g^{-1} \cdot b \cdot \sigma g \in G(W) \cdot \mu_0(p^{-1}) \cdot G(W)\}.$$

Define $\Phi : X_p(\varphi) \to X_p(\varphi)$ by the rule:

$$\Phi g = b \cdot \sigma b \cdot \ldots \cdot \sigma^{[k(v):\mathbb{F}_p]} g, \quad g \in X_p(\varphi).$$

The action of this "Frobenius operator" extends to an action of $\mathrm{Gal}(\mathbb{F}/k(v))$ on X_p.

The group $I(\varphi)$ acts on both $X^p(\varphi)$ and $X_p(\varphi)$ on the left, and so we can define

$$\mathcal{L}(\varphi) = I(\varphi)\backslash X^p(\varphi) \times X_p(\varphi)/Z^p.$$

We let $G(\mathbb{A}_f^p)$ act on $\mathcal{L}(\varphi)$ through its action on $X^p(\varphi)$ and we let $\mathrm{Gal}(\mathbb{F}/k(v))$ act on it through its action on $X_p(\varphi)$.

If $\varphi' : \mathfrak{P} \to \mathfrak{G}_G$ is isomorphic to φ, then the choice of an isomorphism $\varphi \to \varphi'$ defines a bijection $\mathcal{L}(\varphi) \to \mathcal{L}(\varphi')$ of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\mathbb{F}/k(v))$-sets, which is independent of the choice of the isomorphism $\varphi \to \varphi'$ (ibid. 4.1). Thus $\mathcal{L}(\varphi)$ depends only on the isomorphism class of φ, and we can define

$$\mathcal{L}(G, X) = \coprod \mathcal{L}(\varphi)$$

where the disjoint union is over the set of isomorphism classes of special homomorphisms $\varphi : \mathfrak{P} \to \mathfrak{G}_S$.

When the weight of $\mathrm{Sh}(G, X)$ is not defined over $\mathbb{Q}$, the definition of $\mathcal{L}(G, X)$ is the same except that the pseudomotivic groupoid $\mathfrak{P}$ must be replaced with the quasimotivic groupoid

$$\mathfrak{Q} =_{df} \mathfrak{P} \times_S (\mathbb{G}_m)_{\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}};$$

see (Pfau 1993).

CONJECTURE 2.1 (LANGLANDS AND RAPOPORT). There exists a bijection

$$\mathcal{L}(G, X) \to \mathrm{Sh}_p(G, X)(\mathbb{F})$$

of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\mathbb{F}/k(v))$-sets.

REMARK 2.2. The conjecture is as stated in (Langlands and Rapoport 1987, 5.1e), except for the following modifications.

(a) The original conjecture asserts only that there exists a smooth model of $\mathrm{Sh}_p(G, X)$ over $\mathcal{O}_v$ for which there exists such a bijection, but does not attempt to characterize the model.
(b) The original conjecture applied only to Shimura varieties $\mathrm{Sh}_p(G, X)$ for which G^{der} is simply connected; this restriction was removed in (Milne 1992, §4).
(c) The quasimotivic groupoid constructed in the original paper did not have the properties claimed for it; this was corrected in (Pfau 1993).

We shall need a criterion from (Langlands and Rapoport 1987) for a homomorphism φ to be special.

The set of points of $\mathfrak{G}(\infty)$ can be identified with the real Weil group $W(\mathbb{C}/\mathbb{R})$, which is the extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W(\mathbb{C}/\mathbb{R}) \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

defined by the cocycle

$$d_{1,1} = d_{1,\iota} = d_{\iota,1} = 1, \quad d_{\iota,\iota} = -1, \quad \iota = \text{complex conjugation}.$$

Let s be the section $\sigma \mapsto (1, \sigma) : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to W(\mathbb{C}/\mathbb{R})$. For any $x \in X$, the formulas

$$\xi_x(z) = w_X(z) \rtimes \mathrm{id}, \quad (z \in \mathbb{C}^\times); \quad \xi_x(s(\iota)) = \mu_x(-1)^{-1} \rtimes \iota$$

determine a morphism $\xi_\infty : \mathfrak{G}(\infty) \to \mathfrak{G}_G$, whose isomorphism class is independent of the choice of x.

THEOREM 2.3. In the case that G^{der} is simply connected, a homomorphism $\varphi : \mathfrak{P} \to \mathfrak{G}_G$ is special if (and only if) it satisfies the following conditions:

(a) the homomorphism $\zeta_\infty \circ \varphi(\infty)$ is isomorphic to ξ_∞;
(b) for all $\ell \neq p, \infty$, the set $X_\ell(\varphi)$ is nonempty;
(c) the set $X_p(\varphi)$ is nonempty;
(d) for all abelian quotients $G \to T$ of G, the composite $\mathfrak{P} \xrightarrow{\varphi} \mathfrak{G}_G \to \mathfrak{G}_T$ is special.

PROOF. See (Langlands and Rapoport 1987, 5.3). □

3 A More Canonical Conjecture

In this section we state a conjecture that gives a description of $\mathrm{Sh}_p(\mathbb{F})$ intermediate between that provided by the original conjecture and the one arrived at by regarding $\mathrm{Sh}_p(G,X)$ as a moduli scheme for motives (Milne 1994b). First we need to review some results from (Milne 1995a,b).[3]

In those papers, the following are constructed:

(a) A canonical $\mathbb{Q}$-linear Tannakian category $\mathbf{PMot}(\mathbb{F})$ of "pseudomotives" over $\mathbb{F}$ with fundamental group the Weil-number torus P; $\mathbf{PMot}(\mathbb{F})$ is endowed with a polarization, a fibre functor $\omega_f^p : \mathbf{PMot}(\mathbb{F}) \to \mathbf{Mod}_{\mathbb{A}_f^p}$, and an exact tensor functor $\omega_{\text{crys}} : \mathbf{PMot}(\mathbb{F}) \to \mathbf{Isoc}(\mathbb{F})$.

(b) A canonical "reduction functor" $R : \mathbf{CM}(\mathbb{Q}^{\text{al}}) \to \mathbf{PMot}(\mathbb{F})$ from the category of CM-motives over $\mathbb{Q}^{\text{al}}$ to $\mathbf{PMot}(\mathbb{F})$; R is an exact $\mathbb{Q}$-linear tensor functor preserving polarizations and fibre functors.

(c) A Tannakian category $\mathbf{LMot}(\mathbb{F})$ of "Lefschetz motives" over $\mathbb{F}$, generated by abelian varieties and defined using the Lefschetz classes (those in the ring generated by divisor classes) as the correspondences; an "inclusion functor" $I : \mathbf{LMot}(\mathbb{F}) \to \mathbf{PMot}(\mathbb{F})$ which is an exact $\mathbb{Q}$-linear tensor functor preserving polarizations and fibre functors.

(d) A Tannakian category $\mathbf{LCM}(\mathbb{Q}^{\text{al}})$ of Lefschetz motives of CM-type over $\mathbb{Q}^{\text{al}}$, and a commutative diagram:

$$\begin{array}{ccc} \mathbf{LCM}(\mathbb{Q}^{\text{al}}) & \xrightarrow{I} & \mathbf{CM}(\mathbb{Q}^{\text{al}}) \\ \downarrow R & & \downarrow R \\ \mathbf{LMot}(\mathbb{F}) & \xrightarrow{I} & \mathbf{PMot}(\mathbb{F}). \end{array}$$

Let $\mathrm{Sh}_p(G,X)$ be as in the first paragraph of §2, and assume that it has a canonical integral model so that $\mathrm{Sh}_p(\mathbb{F})$ is defined. We assume initially that its weight is defined over $\mathbb{Q}$.

Let $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$ be an exact $\mathbb{Q}$-linear tensor functor such that $\omega_{\text{crys}} \circ \underline{M} \approx K_0 \otimes_{\mathbb{Q}} V$. (Recall that V denotes the forgetful functor $\xi \mapsto V(\xi)$ on $\mathbf{Rep}_{\mathbb{Q}}(G)$, and that $K_0 \otimes_{\mathbb{Q}} V$ denotes the functor $\xi \mapsto K_0 \otimes_{\mathbb{Q}} V(\xi)$.) A *p-integral structure* on $\underline{M}$ is an exact $\mathbb{Z}_p$-linear tensor functor $\underline{\Lambda} : \mathbf{Rep}_{\mathbb{Z}_p}(G_p) \to \mathbf{Mod}_W$ satisfying the following conditions:

(a) $K_0 \otimes_W \underline{\Lambda} = \omega_{\text{crys}} \circ \underline{M}$, i.e., the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{Rep}_{\mathbb{Q}_p}(G) & \xrightarrow{\mathbb{Q}_p \otimes \underline{M}} & \mathbb{Q}_p \otimes \mathbf{PMot}(\mathbb{F}) & \xrightarrow{\omega_{\text{crys}}} & \mathbf{Vec}_{K_0} \\ \uparrow & & & & \uparrow \\ \mathbf{Rep}_{\mathbb{Z}_p}(G) & & \xrightarrow{\quad\underline{\Lambda}\quad} & & \mathbf{Mod}_W. \end{array}$$

(b) there exists an isomorphism $\eta : W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}$ of tensor functors such that $K_0 \otimes_W \eta$ maps $\mu_0(p^{-1}) \cdot (W \otimes_{\mathbb{Z}_p} \Lambda(\xi))$ onto $\phi\underline{\Lambda}(\xi)$ for all ξ in $\mathbf{Rep}_{\mathbb{Z}_p}(G_p)$. Here Λ denotes the forgetful functor on $\mathbf{Rep}_{\mathbb{Z}_p}(G_p)$.

REMARK 3.1. Define a *filtered K_0-module* to be an isocrystal (N,ϕ) over $\mathbb{F}$ together with a finite exhaustive separated decreasing filtration on N, i.e., a family of subspaces

$$N = \mathrm{Filt}^{i_0}(N) \supset \cdots \supset \mathrm{Filt}^{i}(N) \supset \mathrm{Filt}^{i+1}(N) \supset \cdots \supset \mathrm{Filt}^{i_1}(N) = 0.$$

[3] See the forenote.

In Fontaine's terminology, a W-lattice Λ in N is *strongly divisible* if

$$\sum p^{-i}\phi(\text{Filt}^i N \cap \Lambda) = \Lambda,$$

and a filtered K_0-module admitting a strongly divisible lattice is said to be *weakly admissible.* If $\mu : \mathbb{G}_m \to \mathrm{GL}(\Lambda)$ splits the filtration on Λ, i.e.,

$$\text{Filt}^j\Lambda = \oplus_{i\geq j}\Lambda^i, \quad \Lambda^i = \{m \in \Lambda \mid \mu(x)m = x^i m, \text{ all } x \in K_0^\times\}.$$

then the condition to be strongly divisible is that $\phi\Lambda = \mu(p)\Lambda$. The cocharacter μ_0^{-1} in §2 defines a filtration on $\mathbb{Q} \otimes \Lambda(\xi)$ for all ξ, and μ_0 has been so chosen that μ_0^{-1} splits the filtration on $\Lambda(\xi)$ for all ξ. Thus the condition (b) for $\underline{\Lambda}$ to be a p-integral structure on $\underline{M}$ can be restated as:

there exists an isomorphism $\eta : W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}$ of tensor functors such that, for all $\xi \in \mathbf{Rep}_{\mathbb{Z}_p}(G)$, $\underline{\Lambda}(\xi)$ is strongly divisible for the filtration on $K_0 \otimes_W \underline{\Lambda}(\xi)$ defined (via $K_0 \otimes_W \eta$) by μ_0^{-1}.

Define $\Phi : X_p(\underline{M}) \to X_p(\underline{M})$ by the rule: for all ξ, $(\Phi\underline{\Lambda})(\xi) = \phi^{[k(v):\mathbb{F}_p]}(\underline{\Lambda}(\xi))$. The action of this "Frobenius operator" extends to an action of $\mathrm{Gal}(\mathbb{F}/k(v))$ on $X_p(\underline{M})$.

Define

$$\begin{aligned} I(\underline{M}) &= \mathrm{Aut}^\otimes(\underline{M}), \\ X^p(\underline{M}) &= \mathrm{Isom}^\otimes(\mathbb{A}_f^p \otimes V, \omega_f^p \circ \underline{M}), \\ X_p(\underline{M}) &= \{p\text{-integral structures on } \underline{M}\}. \end{aligned}$$

The group $I(\underline{M})$ acts on both $X^p(\underline{M})$ and $X_p(\underline{M})$ on the left, and so we can define

$$\mathcal{M}(\underline{M}) = I(\underline{M})\backslash X^p(\underline{M}) \times X_p(\underline{M})/Z^p.$$

We let $G(\mathbb{A}_f^p)$ act on $\mathcal{M}(\underline{M})$ through its action on $X^p(\underline{M})$, and we let $\mathrm{Gal}(\mathbb{F}/k(v))$ act on it through its action on $X_p(\underline{M})$.

If $\underline{M}$ is isomorphic to $\underline{M}'$, then the choice of an isomorphism $\underline{M} \to \underline{M}'$ defines a bijection $\mathcal{M}(\underline{M}) \to \mathcal{M}(\underline{M}')$ of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\mathbb{F}/k(v))$-sets, which is independent of the choice of the isomorphism. Thus $\mathcal{M}(\underline{M})$ depends only on the isomorphism class of $\underline{M}$.

A point x of X defines a tensor functor

$$H_x : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Hdg}_{\mathbb{Q}}, \quad (\xi, V(\xi)) \mapsto (V(\xi), \xi_{\mathbb{R}} \circ h_x).$$

When x is special, H_x takes values in the full subcategory of $\mathbf{Hdg}_{\mathbb{Q}}$ whose objects are the rational Hodge structures of CM-type. This subcategory is equivalent (via ω_B) with $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$. Fix a tensor inverse $\mathbf{Hdg}_{\mathbb{Q}} \to \mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ to ω_B. On composing H_x with it, we obtain a tensor functor $\underline{M}_x : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ together with an isomorphism $\omega_B \circ \underline{M}_x \approx H_x$. Any tensor functor $\underline{M}$ isomorphic to $R \circ \underline{M}_x$ for some special $x \in X$ will be called *special.*

LEMMA 3.2. For any special functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$, $\omega_{\mathrm{crys}} \circ \underline{M} \approx K_0 \otimes V$.

PROOF. Since the statement depends only on the isomorphism class of $\underline{M}$, we may assume that $\underline{M} = R \circ \underline{M}_x$ with x a special point of X.

The reduction functor $R : \mathbf{CM}(\mathbb{Q}^{\text{al}}) \to \mathbf{PMot}(\mathbb{F})$ has the property that $\mathbb{Q}^{\text{al}} \otimes_{K_0} (\omega_{\text{crys}} \circ R) = \omega_{\text{dR}}$. On composing both sides with $\underline{M}_x$, we find that

$$\mathbb{Q}^{\text{al}} \otimes_{K_0} (\omega_{\text{crys}} \circ \underline{M}) = \omega_{\text{dR}} \circ \underline{M}_x.$$

There is a comparison isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}^{\text{al}}} (\omega_{\text{dR}} \circ \underline{M}_x) = \mathbb{C} \otimes_{\mathbb{Q}} (\omega_B \circ \underline{M}_x),$$

and (from the definition of $\underline{M}_x$) there is given an isomorphism

$$\omega_B \circ \underline{M}_x \to V.$$

On combining these isomorphisms, we obtain an isomorphism of tensor functors

$$\mathbb{C} \otimes_{K_0} (\omega_{\text{crys}} \circ \underline{M}) \to \mathbb{C} \otimes_{\mathbb{Q}} V.$$

It remains to show that we can replace $\mathbb{C}$ with K_0 in this statement.

Consider the functor of K_0-algebras

$$F(R) = \underline{\text{Isom}}^{\otimes}(R \otimes_{K_0} (\omega_{\text{crys}} \circ \underline{M}), R \otimes_{\mathbb{Q}} V).$$

Since $G_{/K_0} = \underline{\text{Aut}}^{\otimes}(K_0 \otimes_{\mathbb{Q}} V)$, F is a pseudo-torsor for G_{K_0}, and, in fact, a torsor because $F(\mathbb{C})$ is nonempty. It therefore defines an element of $H^1(K_0, G)$. The field K_0 has dimension ≤ 1 (Serre 1964, II.3) and G is connected, and so $H^1(K_0, G) = 0$ (Steinberg 1965). Hence F is the trivial torsor: $F(K_0) \neq \emptyset$. □

The last lemma shows that $\mathcal{M}(\underline{M})$ is defined for any special homomorphism, and we know that it depends only on the isomorphism class of $\underline{M}$. Thus we can define

$$\mathcal{M}(G, X)(\mathbb{F}) = \coprod \mathcal{M}(\underline{M})$$

where $\underline{M}$ runs over the set of isomorphism classes of special homomorphisms $\mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$.

CONJECTURE 3.3. There exists a bijection

$$\mathcal{M}(G, X)(\mathbb{F}) \longrightarrow \text{Sh}_p(G, X)(\mathbb{F})$$

of $G(\mathbb{A}_f^p) \times \text{Gal}(\mathbb{F}/k(v))$-sets.

THEOREM 3.4. There exists a bijection

$$\mathcal{M}(G, X)(\mathbb{F}) \to \mathcal{L}(G, X)$$

of $G(\mathbb{A}_f^p) \times \text{Gal}(\mathbb{F}/k(v))$-sets.

PROOF. Let ω_∞, ω_ℓ, and ω_p be the fibre functors on the Tannakian categories $\mathbf{V}_\infty$, $\mathbf{Vec}_{\mathbb{Q}_\ell}$, and $\mathbf{Isoc}(\mathbb{F})$ whose groupoids are $\mathfrak{G}_\infty$, $\mathfrak{G}_\ell$, and $\mathfrak{G}_p$ (see the previous section). Choose a fibre functor $\bar{\omega}$ for $\mathbf{PMot}(\mathbb{F})$ over $\mathbb{Q}^{\text{al}}$ and isomorphisms

$$\mathbb{C} \otimes_{\mathbb{Q}^{\text{al}}} \bar{\omega} \to \omega_\infty, \quad \mathbb{Q}_\ell^{\text{al}} \otimes_{\mathbb{Q}^{\text{al}}} \bar{\omega} \to \omega_\ell, \quad \mathbb{C}_p \otimes_{\mathbb{Q}^{\text{al}}} \bar{\omega} \to \omega_p.$$

Then the system consisting of $\mathfrak{P} =_{df} \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\bar{\omega})$ together with the homomorphisms $\zeta_\ell : \mathfrak{G}_\ell \to \mathfrak{P}(\ell)$ provided by the isomorphisms is a pseudomotivic groupoid. A tensor functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$ induces a homomorphism $\varphi_{\underline{M}} : \mathfrak{P} \to \mathfrak{G}_G$, well-defined up to isomorphism, and the theory of Tannakian categories (Deligne 1990) shows that $\underline{M} \mapsto \varphi_{\underline{M}}$ defines a bijection between the set of isomorphism classes of tensor functors $\mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$ and the set isomorphism classes of homomorphisms $\mathfrak{P} \to \mathfrak{G}_G$. Clearly special functors correspond to special homomorphisms, and so it remains to show that, for each special functor $\underline{M}$, there is an equivariant bijection $\mathcal{M}(\underline{M}) \to \mathcal{L}(\varphi_{\underline{M}})$.

Again, it follows directly from the theory of Tannakian categories that $I(\underline{M}) = I(\varphi_{\underline{M}})$ and $X^p(\underline{M}) = X^p(\varphi_{\underline{M}})$, and so it remains to show that $X_p(\underline{M}) = X_p(\varphi_{\underline{M}})$.

Choose an isomorphism of tensor functors

$$\beta : \omega_{\text{crys}} \circ \underline{M} \to K_0 \otimes V.$$

There is a unique $b \in G(K_0)$ such that

$$\beta \circ \phi = b \circ \sigma \circ \beta$$

(equality of σ-linear isomorphisms of tensor functors $\omega_{\text{crys}} \circ \underline{M} \to K_0 \otimes V$). If β is replaced with $g \circ \beta$, $g \in G(K_0)$, then b is replaced by its σ-conjugate $gb(\sigma g)^{-1}$. On tracing through the definitions, one finds that b represents the σ-conjugacy class $[b(\varphi_{\underline{M}}(p) \circ \zeta_p)]$.

Let $\underline{\Lambda}$ be an exact $\mathbb{Z}_p$-linear tensor functor $\mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PMot}(\mathbb{F})$ such that $K_0 \otimes \underline{\Lambda} = K_0 \otimes_{\mathbb{Q}} V$. According to (1.4), there exists an isomorphism $\eta_p : W \otimes \Lambda \to \underline{\Lambda}$ of tensor functors. The composite $g = (K_0 \otimes_{\mathbb{Q}} \beta) \circ (K_0 \otimes_W \eta_p)$ lies in $G(K_0)$. If η_p is replaced by $\eta_p \circ w$ with $w \in G(W)$, then g is replaced by gw. The map $\underline{\Lambda} \mapsto gG(W)$ defines a bijection between set of $\underline{\Lambda}$'s and the set $G(K_0)/G(W)$, and one shows, as in (Milne 1994b, §4), that the $\underline{\Lambda}$'s that are p-integral structures on $\underline{M}$ correspond to the cosets $gG(W)$ such that $g^{-1} \cdot b \cdot \sigma g \in G(W) \cdot \mu_0(p)^{-1} \cdot G(W)$. □

COROLLARY 3.5. Conjecture 2.1 holds for $\mathrm{Sh}_p(G, X)$ if and only if Conjecture 3.3 does.

REMARK 3.6. The conditions (a)–(d) for φ to be special translate as follows.

(a) The polarization on $\mathbf{PMot}(\mathbb{F})$ defines a tensor functor $\mathbb{R} \otimes_{\mathbb{Q}} \mathbf{PMot}(\mathbb{F}) \to \mathbf{V}_\infty$ (Deligne and Milne 1982, 5.20); the composite of this with $\mathbb{R} \otimes \underline{M}$ should be isomorphic to the functor defined by the homomorphism $\mathfrak{G}_\infty \to \mathfrak{G}_G$. [[Make this more explicit.]]
(b) For all $\ell \neq p, \infty$, there exists an isomorphism $\mathbb{Q}_\ell \otimes V \to \omega_\ell(\underline{M})$;
(c) The set $X_p(\underline{M})$ is nonempty.
(d) The restriction of $\underline{M}$ to the full subcategory $\mathbf{Rep}_{\mathbb{Q}}(G^{\text{ab}})$ of $\mathbf{Rep}_{\mathbb{Q}}(G)$ is isomorphic to $H_x : \mathbf{Rep}_{\mathbb{Q}}(G^{\text{ab}}) \to \mathbf{PMot}(\mathbb{F})$ for one (hence all) $x \in X$.

REMARK 3.7. The above discussion can be extended to Shimura varieties whose weight is not defined over $\mathbb{Q}$ by replacing $\mathbf{PMot}(\mathbb{F})$ with the category of quasimotives over $\mathbb{F}$.

Shimura varieties of Hodge type

DEFINITION 3.8. A Shimura variety $\mathrm{Sh}(G, X)$ is of *Hodge type* if there is a symplectic space (V, ψ) over $\mathbb{Q}$ and an injective homomorphism $G \hookrightarrow \mathrm{GSp}(V, \psi)$ such that

(a) for each $x \in X$, the Hodge structure (V, h_x) has type $\{(-1, 0), (0, -1)\}$;
(b) for each $x \in X$, either $+2\pi i\psi$ or $-2\pi i\psi$ is a polarization for (V, h_x).

We say that $\mathrm{Sh}_p(G,X)$ is of *Hodge type* if $\mathrm{Sh}(G,X)$ is of Hodge type and it is possible to choose the homomorphism $G \hookrightarrow \mathrm{GSp}(V,\psi)$ so that, in addition,

(c) there is a $\mathbb{Z}_p$-lattice $V(\mathbb{Z}_p)$ in V such that ψ restricts to a perfect pairing $V(\mathbb{Z}_p)\times V(\mathbb{Z}_p) \to \mathbb{Z}_p$ and G_p is the Zariski closure of G in $\mathrm{GSp}(V(\mathbb{Z}_p),\psi)$.

If $\mathrm{Sh}(G,X)$ is of Hodge type, then the choice of a symplectic embedding defines a homomorphism $t : G \to \mathbb{G}_m$ such that $t \circ w_X = -2$. Thus $\mathbf{Rep}(G)$ is endowed with the structure of a Tate triple, and it is a matter of indifference whether we work with tensors of the form $V^{\otimes 2m} \to \mathbb{Q}(m)$ or $\mathbb{Q} \to V^{\otimes r} \otimes V^{\vee\otimes s}$.

Choose a symplectic representation $G \hookrightarrow \mathrm{GSp}(V,\psi)$ and a lattice $V(\mathbb{Z}_p)$ in $V(\mathbb{Q}_p)$ satisfying the conditions (3.8a,b,c). Choose also a defining set of tensors $\mathfrak{t} = (t_i)_{i\in I}$ for G in $\mathrm{GL}(V)$ with $t_{i_0} = \psi$.

For an abelian variety A over $\mathbb{F}$, $h_1(A)$ is an object of the Tannakian category $\mathbf{PMot}(\mathbb{F})$, and we define a *tensor* on $h_1(A)$ to be a morphism $\mathbb{1} \to h_1(A)^{\otimes r} \otimes h_1(A)^{\vee\otimes s}$ in $\mathbf{PMot}(\mathbb{F})$ (equivalently, a morphism $h_1(A)^{\otimes 2m} \to \mathbb{Q}(m)$). Consider pairs $M = (A,\mathfrak{s})$ where A is an abelian variety over $\mathbb{F}$ and $\mathfrak{s} = (s_i)_{i\in I}$ is a family of tensors on $h_1(A)$ indexed by I.

For each special point $x \in X$, we get an abelian variety A_x over $\mathbb{Q}^{\mathrm{al}}$ of CM-type and a family of Hodge tensors $\mathfrak{s}_x$ on A_x indexed by I. The reduction functor $R : \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{PMot}(\mathbb{F})$ maps $M_x = (A_x,\mathfrak{s}_x)$ to a pair $(A,\mathfrak{s})$ as in the last paragraph, and any pair isomorphic to such a pair is said to be *special*.

A *p-integral structure* on $M = (A,\mathfrak{s})$ is a lattice Λ in $\omega_{\mathrm{crys}}(A)$ for which there exists an isomorphism $V(W) \to \Lambda$ sending each t_i to s_i and such that $\mu_0(p^{-1})V(W)$ maps onto $\phi\Lambda$.

Let $M = (A,\mathfrak{s})$ be a special pair. Define

$$\begin{aligned} I(M) &= \{\alpha \in \mathrm{Aut}(A) \mid \alpha(s_i) = s_i \text{ for all } i \in I\} \\ X^p(M) &= \{\eta^p : V(\mathbb{A}_f^p) \xrightarrow{\approx} \omega_f^p(A) \mid \eta^p(t_i) = s_i \quad \forall i \in I\} \\ X_p(M) &= \{p\text{-integral structures on } M\}. \end{aligned}$$

Then $I(M)$ acts on $X^p(M)$ and $X_p(M)$ on the left, $G(\mathbb{A}_f^p)$ acts on $X^p(M)$ on the right, and $\mathrm{Gal}(\mathbb{F}/k(v))$ acts on $X_p(M)$. We define

$$\mathcal{M}(M) = I(M)\backslash X^p(M) \times X_p(M),$$

regarded as a $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\mathbb{F}/k(v))$-set. An isomorphism $M \to M'$ induces a bijection $\mathcal{M}(M) \to \mathcal{M}(M')$ of $G(\mathbb{A}_f^p)$-sets which is independent of the isomorphism. Therefore $\mathcal{M}(M)$ depends only on the isomorphism class of M, and we can define

$$\mathcal{M}(G,X)(\mathbb{F}) = \coprod_M \mathcal{M}(M)$$

where M runs over the isomorphism classes of admissible pairs M.

PROPOSITION 3.9. This definition of $\mathcal{M}(G,X)(\mathbb{F})$ agrees with that preceding Conjecture 3.3.

PROOF. Let ξ_0 be the symplectic representation of G fixed above. The functor h_1 realizes $\mathbf{AV}(\mathbb{F})$ as a full subcategory of $\mathbf{PMot}(\mathbb{F})$, and the functor $\underline{M} \mapsto \underline{M}(\xi_0)$ is fully faithful and takes values in $\mathbf{AV}(\mathbb{F})$. The rest of the proof is straighforward. □

The proposition, while a fairly immediate consequence of the results of Milne (1995b), represents a major step towards our understanding of the conjecture of Langlands and Rapoport for Shimura varieties of Hodge type, because it allows us to replace the rather mysterious homomorphisms of groupoids $\mathfrak{P} \to \mathfrak{G}_G$ with the more accessible pairs $(A, \mathfrak{s})$ consisting of an abelian variety with tensors.

4 The Etale Description of the Points

Let $\mathrm{Sh}_p(G,X)$ be as in the first paragraph of §2. We assume that $\mathrm{Sh}(G,X)$ is of abelian type, that its weight is defined over $\mathbb{Q}$, and that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$. The last condition holds if and only if the largest split subtorus of $Z_{\mathbb{R}}$ is split over $\mathbb{Q}$, i.e., if $(Z_{\text{split}})_{\mathbb{R}} = (Z_{\mathbb{R}})_{\text{split}}$. It implies that

$$\mathrm{Sh}_p(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_p = G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p).$$

Let k be a field containing $E = E(G,X)$, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$ for some algebraic closure k^{al} of k. An *étale p-integral structure* on a functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(k)$ is an exact $\mathbb{Z}_p$-linear tensor functor $\underline{\Lambda}_p : \mathbf{Rep}_{\mathbb{Z}_p}(G_p) \to \mathbf{Rep}_{\mathbb{Z}_p}(\Gamma)$ such that $\omega_p \circ \underline{M} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \underline{\Lambda}_p$, i.e., such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{Rep}_{\mathbb{Q}_p}(G) & \xrightarrow{\mathbb{Q}_p \otimes \underline{M}} & \mathbb{Q}_p \otimes \mathbf{Mot}(k) & \xrightarrow{\omega_p} & \mathbf{Rep}_{\mathbb{Q}_p}(\Gamma) \\ \uparrow & & & & \uparrow \\ \mathbf{Rep}_{\mathbb{Z}_p}(G) & & \xrightarrow{\qquad\underline{\Lambda}_p\qquad} & & \mathbf{Rep}_{\mathbb{Z}_p}(\Gamma) \end{array}$$

LEMMA 4.1. For any exact $\mathbb{Z}_p$-linear tensor functor $\underline{\Lambda}_p : \mathbf{Rep}_{\mathbb{Z}_p}(G_p) \to \mathbf{Rep}_{\mathbb{Z}_p}(\Gamma)$ there exists an isomorphism of $\mathbb{Z}_p$-linear tensor functors $\Lambda \to \omega_{\text{forget}} \circ \underline{\Lambda}_p$.

PROOF. Apply Corollary 1.4. □

DEFINITION 4.2. Let k be a field containing E, and let $\tau : k \to \mathbb{C}$ be an E-homomorphism. Write ω_τ for the composite of the base change functor $\mathbf{Mot}(k) \to \mathbf{Mot}(\mathbb{C})$ with the Betti fibre functor ω_B. A $\mathbb{Q}$-linear tensor functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(k)$ will be said to be *admissible with respect to* τ if it satisfies the following conditions:

(a) there exists an isomorphism $\omega_\tau \circ \underline{M} \to V$ (uniquely determined up to an element of $G(\mathbb{Q})$);
(b) the isomorphism in (a) and the Hodge structure on $\omega_\tau \circ \underline{M}$ define a $G(\mathbb{Q})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$; this should be contained in X;
(c) the action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$ on $\omega_f^p \circ \underline{M}$ is trivial;
(d) there exists an étale p-integral structure on $\underline{M}$.

LEMMA 4.3. If $\underline{M}$ is admissible with respect to one E-homomorphism $k \to \mathbb{C}$, then it is admissible with respect to all.

PROOF. The proof is similar to that of (Milne 1994b, 3.29). □

An $\underline{M}$ as in the lemma will simply be called *admissible.*

DEFINITION 4.4. For any field k containing $E(G,X)$ and admitting a complex embedding, define $\mathcal{A}_p(k)$ to be the set of triples $(\underline{M}, \eta^p, \underline{\Lambda}_p)$ consisting of

- an admissible $\mathbb{Q}$-linear tensor functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(k)$;
- an isomorphism of tensor functors $\eta^p : \mathbb{A}_f^p \otimes V \to \omega_f^p \circ \underline{M}$; and
- an étale p-integral structure $\underline{\Lambda}_p$ on $\underline{M}$.

An isomorphism from one such triple $(\underline{M}, \eta^p, \underline{\Lambda}_p)$ to a second $(\underline{M}', \eta^{p\prime}, \underline{\Lambda}'_p)$ is an isomorphism $\underline{M} \to \underline{M}'$ of $\mathbb{Q}$-linear tensor functors mapping η^p to $\eta^{p\prime}$ and Λ_p to Λ'_p. The group $G(\mathbb{A}_f^p)$ acts on $\mathcal{A}_p(k)$ on the right according to the rule:

$$(\underline{M}, \eta^p, \underline{\Lambda}_p)g = (\underline{M}, \eta^p \circ g, \underline{\Lambda}_p), \quad g \in G(\mathbb{A}_f^p).$$

Because of Lemma 4.3, $\mathcal{A}_p(k)$ is a functor on the category whose objects are the fields containing $E(G, X)$ and admitting a complex embedding and whose morphisms are E-algebra homomorphisms.

Let $(\underline{M}, \eta^p, \underline{\Lambda}_p) \in \mathcal{A}_p(\mathbb{C})$, and choose isomorphisms $\beta : \omega_B \circ \underline{M} \to H_x$ (some $x \in X$) and $\eta_p : \Lambda \to \underline{\Lambda}$. The composite

$$\mathbb{A}_f^p \otimes V \xrightarrow{\eta^p} \omega_f^p \circ \underline{M} \xrightarrow{\mathbb{A}_f^p \otimes \beta} \mathbb{A}_f^p \otimes V$$

is an element $g^p \in \underline{\mathrm{Aut}}^{\otimes}(\mathbb{A}_f^p \otimes V) = G(\mathbb{A}_f^p)$, and the composite

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} V \xrightarrow{\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \eta_p} \omega_p \otimes \underline{M} \xrightarrow{\mathbb{Q}_p \otimes_{\mathbb{Q}} \beta} \mathbb{Q}_p \otimes V$$

is an element $g_p \in \underline{\mathrm{Aut}}^{\otimes}(\mathbb{Q}_p \otimes V) = G(\mathbb{Q}_p)$. The class of (x, g^p, g_p) in $\mathrm{Sh}_p(G, X)(\mathbb{C})$ is independent of the choice of β and η_p, and the map $(\underline{M}, \eta^p, \Lambda_p) \mapsto [x, g^p, g_p]$ defines a bijection

$$\mathcal{A}_p(\mathbb{C})/\approx \to \mathrm{Sh}_p(G, X)(\mathbb{C}).$$

THEOREM 4.5. There is a unique family of bijections

$$\alpha(k) : \mathcal{A}_p(k)/\approx \to \mathrm{Sh}_p(k) \qquad k \text{ a field, } k \supset E,$$

functorial in k (considered as an E-algebra), and such that $\alpha(\mathbb{C})$ is the map defined above.

PROOF. The proof is similar to that of (Milne 1994b, 3.31). Alternatively, it can be deduced from that theorem by choosing a defining representation and tensors. □

Let k be a field containing E and admitting a complex embedding. For any admissible $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(k)$, define

$$\begin{aligned} I(\underline{M}) &= \mathrm{Aut}^{\otimes}(\underline{M}), \\ X^p(\underline{M}) &= \mathrm{Isom}^{\otimes}(\mathbb{A}_f^p \otimes V, \omega_f^p(\underline{M})), \\ X_p(\underline{M}) &= \{\text{étale } p\text{-integral structures on } \underline{M}\}. \end{aligned}$$

Then $I(\underline{M})$ acts on $X^p(\underline{M})$ and $X_p(\underline{M})$ on the left, and $G(\mathbb{A}_f^p)$ acts on $X^p(\underline{M})$ on the right. We define

$$\mathcal{M}(\underline{M}) = I(\underline{M})\backslash X^p(\underline{M}) \times X_p(\underline{M}),$$

regarded as a $G(\mathbb{A}_f^p)$-set. An isomorphism $\underline{M} \to \underline{M}'$ induces a bijection $\mathcal{M}(\underline{M}) \to \mathcal{M}(\underline{M}')$ of $G(\mathbb{A}_f^p)$-sets which is independent of the isomorphism. Therefore $\mathcal{M}(\underline{M})$ depends only on the isomorphism class of $\underline{M}$, and we can define

$$\mathcal{M}(G, X)(k) = \coprod_{\underline{M}} \mathcal{M}(\underline{M})$$

where $\underline{M}$ runs over the isomorphism classes of admissible $\underline{M}$'s.

COROLLARY 4.6. There is a canonical bijection

$$\mathcal{M}(G,X)(k) \to \mathrm{Sh}_p(G,X)(k)$$

of $G(\mathbb{A}_f^p)$-sets, functorial in k (considered as an E-algebra).

PROOF. This is a restatement of Theorem 4.5. □

REMARK 4.7. Define $\mathbf{Mot}_{(p)}(k)$ to be the category whose objects are the triples (M,Λ,φ) with M in $\mathbf{Mot}(k)$, Λ in $\mathbf{Rep}_{\mathbb{Z}_p}(\Gamma)$, and $\varphi : \Lambda \to \omega_p(M)$ a homomorphism inducing an isomorphism $\mathbb{Q}_p \otimes \Lambda \to \omega_p(M)$. It is a $\mathbb{Z}_{(p)}$-linear tensor category. We say that an exact $\mathbb{Z}_{(p)}$-linear tensor functor $\underline{N} : \mathbf{Rep}_{\mathbb{Z}_{(p)}}(G) \to \mathbf{Mot}_{(p)}(k)$ is *admissible* if $\mathrm{Gal}(k^{\mathrm{al}}/k)$ acts trivially on $\omega_f^p \circ \underline{N}$ and there exists an isomorphism $\omega_B \circ \underline{N} \to \Lambda$ sending $h_{\underline{N}}$ to h_x for some $x \in X$. With the analogous definitions,

$$\mathrm{Sh}_p(k) = \coprod I(\underline{N})\backslash X^p(\underline{N})$$

where $\underline{N}$ runs over the isomorphism classes of admissible functors $\mathbf{Rep}_{\mathbb{Z}_{(p)}}(G) \to \mathbf{Mot}_{(p)}(k)$.

REMARK 4.8. (a) Without the condition that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$, the moduli problem will not be fine. However, the description of $\mathrm{Sh}_p(k)$ given in Theorem 4.5 will still be valid, with appropriate changes to take account of the fact that $Z^p \neq Z(\mathbb{Z}_{(p)})$, provided k is algebraically closed or (perhaps) a field of dimension ≤ 1 in the sense of (Serre 1962), for example, if it is algebraically closed or is Henselian with respect to a discrete valuation whose residue field is algebraically closed.

(b) For an algebraically closed field k of characteristic zero, define the category $\mathbf{QMot}(k)$ of quasi-motives to be the "largest" Tannakian category fitting into the diagram:

$$\begin{array}{ccc} \mathbf{QMot}(k) & \longleftarrow & \mathbf{Rep}_{\mathbb{Q}}(\mathbb{G}_m)_{\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}} \\ \uparrow & & \uparrow \\ \mathbf{Mot}(k) & \longleftarrow & \mathbf{CM}(k). \end{array}$$

Thus, relative to the Betti fibre functors,

$$G^{\mathbf{QMot}(k)} = G^{\mathbf{Mot}(k)} \times_S (\mathbb{G}_m)_{\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}}.$$

The description of $\mathrm{Sh}_p(k)$, k algebraically closed, given in Theorem 4.5 remains valid for Shimura varieties whose weight is not defined over $\mathbb{Q}$ provided one replaces $\mathbf{Mot}(k)$ with $\mathbf{QMot}(k)$.

Shimura varieties of Hodge type

Choose a symplectic representation $G \hookrightarrow \mathrm{GSp}(V,\psi)$ and a lattice $V(\mathbb{Z}_p)$ in $V(\mathbb{Q}_p)$ satisfying the conditions (3.8a,b,c). Choose also a defining set of tensors $\mathfrak{t} = (t_i)_{i\in I}$ for G in $\mathrm{GL}(V)$ with $t_{i_0} = \psi$. Using the dictionary provided by Proposition 1.8, we obtain the following translation of the above.

Let k be a field containing E, and let τ be an E-homomorphism $k \to \mathbb{C}$. A pair $M = (A,\mathfrak{s})$ consisting of an abelian variety A over k together with a family $\mathfrak{s} = (s_i)_{i\in I}$ of Hodge tensors is *admissible respect to* τ if it satisfies the following conditions:

(a) there exists an isomorphism $\omega_\tau(A) \to V(\mathbb{Q})$ mapping each t_i to s_i;

(b) under the isomorphism in (a), h_A corresponds to h_x for some $x \in X$;
(c) the action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$ on $\omega_f^p(A)$ is trivial;
(d) there exists a Γ-stable lattice Λ in $\omega_p(A)$ and an isomorphism $V(\mathbb{Z}_p) \to \Lambda$ sending each t_i to s_i.

If $M = (A, \mathfrak{s})$ is admissible with respect to one E-homomorphism $E \to \mathbb{C}$, then it admissible with respect to all, and so we can drop the τ from the terminology.

Let $M = (A, \mathfrak{s})$ be admissible. We define a p-integral structure on M to be a lattice Λ in $\omega_p(M)$, stable under Γ, for which there exists an isomorphism $V(\mathbb{Z}_p) \to \Lambda$ sending each t_i to s_i. Define

$$\begin{aligned} I(M) &= \{\alpha \in \mathrm{Aut}(A) \mid \alpha(s_i) = s_i \quad \forall i \in I\} \\ X^p(M) &= \{\eta^p : V(\mathbb{A}_f^p) \xrightarrow{\approx} \omega_f^p(A) \mid \eta^p(t_i) = s_i \quad \forall i \in I\} \\ X_p(M) &= \{p\text{-integral structures on } M\}. \end{aligned}$$

Then $I(M)$ acts on $X^p(M)$ and $X_p(M)$ on the left, and $G(\mathbb{A}_f^p)$ acts on $X^p(M)$ on the right. We define

$$\mathcal{M}(M) = I(M)\backslash X^p(M) \times X_p(M),$$

regarded as a $G(\mathbb{A}_f^p)$-set. As before, $\mathcal{M}(M)$ depends only on the isomorphism class of M, and we can define

$$\mathcal{M}(G, X)(k) = \coprod_M \mathcal{M}(M)$$

where M runs over the isomorphism classes of admissible pairs M. Corollary 4.6 shows that there is a canonical bijection

$$\mathcal{M}(G, X)(k) \to \mathrm{Sh}_p(G, X)(k)$$

of $G(\mathbb{A}_f^p)$-sets, functorial in k (considered as an E-algebra).

REMARK 4.9. Let A be an abelian variety regarded as an object of $\mathbf{AV}_{(p)}(k)$, and let $A_{\mathbb{Q}}$ be the same abelian variety regarded as an object of $\mathbf{AV}(k)$. Now $\omega_p(A)$ and $\omega_B(A)$ are lattices in $\omega_p(A_{\mathbb{Q}})$ and $\omega_B(A_{\mathbb{Q}})$ respectively. A pair $N = (A, \mathfrak{s})$ consisting of an $A \in \mathrm{ob}(\mathbf{AV}_{(p)}(k))$ and a family $\mathfrak{s} = (s_i)_{i \in I}$ of tensors on A (meaning on $h_1(A_{\mathbb{Q}})$) is said to an *integral admissible pair* if it satisfies the following condition:

(*) there exists an isomorphism $\omega_B(A) \to V(\mathbb{Z}_{(p)})$ of $\mathbb{Z}_{(p)}$-modules carrying each s_i to t_i and h_A to h_x for some $x \in X$.

The map $(A, \mathfrak{s}) \mapsto (A_{\mathbb{Q}}, \mathfrak{s}, \omega_p(A))$ defines an equivalence between the category of integral admissible pairs and the category of triples $(A_{\mathbb{Q}}, \mathfrak{s}, \Lambda)$ consisting of an admissible pair $(A_{\mathbb{Q}}, \mathfrak{s})$ and an étale p-integral Λ structure on $A_{\mathbb{Q}}$. Indeed, it is clear that this is a fully faithful functor, and so it remains to show that it is essentially surjective. Consider a triple $(A_{\mathbb{Q}}, \mathfrak{s}, \Lambda)$. There exists an A' in $\mathbf{AV}_{(p)}(K_0)$ for which there is an isomorphism $A' \to A$ sending $\omega_p(A')$ onto Λ. We may replace $A_{\mathbb{Q}}$ with $A'_{\mathbb{Q}}$, and hence assume that, if A denotes $A_{\mathbb{Q}}$ regarded as an object of $\mathbf{AV}_{(p)}(K_0)$, then $\omega_p(A) = \Lambda$. We are given isomorphisms

$$\beta : \omega_B(A_{\mathbb{Q}}) \to V(\mathbb{Q}), \quad \beta_p : \omega_p(A) \to V(\mathbb{Z}_p)$$

with certain properties. The maps $\mathbb{Q}_p \otimes_{\mathbb{Q}} \beta$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \beta_p$ differ by an element g of $G(\mathbb{Q}_p)$. But by (Milne 1995b, 4.9), $G(\mathbb{Q}_p) = G(\mathbb{Q}) \cdot G_p(\mathbb{Z}_p)$, and so $g = q \cdot z$. When we replace β with $q \circ \beta$ and β_p with $z^{-1} \circ \beta_p$ we find that $\mathbb{Q}_p \otimes_{\mathbb{Q}} \beta$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \beta_p$ now agree. This means that β maps $\omega_B(A)$ onto $V(\mathbb{Z}_{(p)})$, and it satisfies (*).

5 Crystalline p-integral Structures

Let $\mathrm{Sh}_p(G,X)$ be as in the first paragraph of §4. We extend the homomorphism $\mathbb{Q}^{\mathrm{al}} \to \mathbb{C}_p$ (see §2) to an isomorphism $\mathbb{C} \to \mathbb{C}_p$.

As we explained in (Milne 1994b), because étale p-integral structures do not reduce well, to pass from the points on the Shimura variety with coordinates in K_0 to the those with coordinates in $\mathbb{F}$, we need to replace étale p-integral structures with crystalline p-integral structures. After stating a conjecture that would make this possible in the general case, we discuss the case of Shimura varieties of Hodge type.

LEMMA 5.1. For any admissible tensor functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(K_0)$, there exists an isomorphism

$$K_0 \otimes V \to \omega_{\mathrm{dR}} \circ \underline{M}$$

carrying the filtration defined by μ_0^{-1} into the Hodge filtration.

PROOF. For each K_0-algebra R, define $F(R)$ to be the set of isomorphisms of R-linear tensor functors $R \otimes_{\mathbb{Q}} V \to R \otimes (\omega_{\mathrm{dR}} \circ \underline{M})$ carrying $\mathrm{Filt}(\mu_0^{-1})$ into the Hodge filtration. Then F is a pseudo-torsor for the subgroup P of G_{K_0} respecting the filtration defined by μ_0^{-1} on each representation of G. This group P is a parabolic subgroup of G (Saavedra 1972, IV.2.2.5), and hence is connected (Borel 1991, p155, 11.16). Once we show $F(\mathbb{C}) \neq \emptyset$, so that F is a torsor, it will follow from (Steinberg 1965) that $F(K_0) \neq \emptyset$.

There is a canonical comparison isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} (\omega_B \circ \underline{M}) \to \mathbb{C} \otimes_{K_0} (\omega_{\mathrm{dR}} \circ \underline{M})$$

which carries the Hodge filtration on the left to the de Rham filtration on the right. By assumption, for some $x \in X$, there exists an isomorphism

$$\omega_B \circ \underline{M} \approx H_x$$

preserving Hodge structures. On combining these isomorphisms, we obtain an isomorphism

$$\mathbb{C} \otimes H_x \approx \mathbb{C} \otimes_{K_0} (\omega_{\mathrm{dR}} \circ \underline{M})$$

carrying $\mathrm{Filt}(\mu_x^{-1})$ to the Hodge filtration. But μ_0 and μ_x are in the same $G(\mathbb{C})$-conjugacy class, and so there exists an isomorphism of tensor functors

$$\mathbb{C} \otimes V \approx \mathbb{C} \otimes H_x$$

carrying $\mathrm{Filt}(\mu_0^{-1})$ to $\mathrm{Filt}(\mu_x^{-1})$. □

Let $\mathbf{MF}_{K_0}$ denote the category of weakly admissible filtered K_0-modules (see 3.1), and we let $\mathbf{MF}_W$ denote the category $\underline{MF}_{tf}$ of (Fontaine 1983). Thus an object of $\mathbf{MF}_W$ is a finitely generated W-module M together with a descending filtration $\mathrm{Filt}^i M$ and σ-linear maps $\varphi^i : \mathrm{Filt}^i M \to M$ satisfying certain conditions (ibid. 2.1). To give an object M of $\mathbf{MF}_W$ that is a free W-module is the same as to give a filtered K_0-module N together with a strongly divisible lattice in N.

The functor $\omega_{\mathrm{dR}} : \mathbf{Mot}(K_0) \to \mathbf{Mod}_{K_0}$ has a canonical factorization into

$$\mathbf{Mot}(K_0) \xrightarrow{\omega_{\mathrm{crys}}} \mathbf{MF}_{K_0} \xrightarrow{\text{forget}} \mathbf{Mod}_{K_0};$$

in fact, it factors through the category of admissible filtered modules (Fontaine 1983).

A *crystalline p-integral structure* on a functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(K_0)$ is an exact $\mathbb{Z}_p$-linear tensor functor $\underline{\Lambda}_{\text{crys}} : \mathbf{Rep}_{\mathbb{Z}_p}(G_p) \to \mathbf{MF}_W$ such that $K_0 \otimes_W \underline{\Lambda}_{\text{crys}} = \omega_{\text{dR}} \circ \underline{M}$, i.e., such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{Rep}_{K_0}(G) & \xrightarrow{K_0 \otimes \underline{M}} & K_0 \otimes \mathbf{Mot}(K_0) & \xrightarrow{\omega_{\text{crys}}} & \mathbf{MF}_{K_0} \\ \uparrow & & & & \uparrow \\ \mathbf{Rep}_{\mathbb{Z}_p}(G) & & \xrightarrow{\quad\underline{\Lambda}_{\text{crys}}\quad} & & \mathbf{MF}_W \end{array}$$

Recall (Wintenberger 1984) that for any filtered module M, there is a canonical splitting μ_W of the filtration on M, and that μ_W splits the filtration on any strongly divisible submodule of M.

LEMMA 5.2. Let $\underline{\Lambda}_{\text{crys}}$ be a p-integral crystalline structure on an admissible tensor functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(K_0)$. Then there exists an isomorphism of tensor functors $W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}_{\text{crys}}$ carrying μ_0^{-1} into μ_W.

PROOF. It follows from Corollary 1.4 that there exists an isomorphism $\alpha : W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}_{\text{crys}}$, uniquely determined up to composition with an element of $G_p(W)$. Let μ' be the cocharacter of $G_{p/W}$ mapped by α to μ_W. We have to show that μ' is $G_p(W)$-conjugate to μ_0^{-1}.

According to Lemma 5.1, there exists an isomorphism $\beta : K_0 \otimes_{\mathbb{Z}_p} \Lambda \to K_0 \otimes_W \underline{\Lambda}_{\text{crys}}$ carrying $\text{Filt}(\mu_0^{-1})$ into $\text{Filt}(\mu_W)$. After possibly replacing β with its composite with an element of (a unipotent subgroup) of $G(K_0)$, we may assume that β maps μ_0^{-1} to μ_W. Since β can differ from $K_0 \otimes_W \alpha$ only by an element of $G(K_0)$, this shows that μ' is $G(K_0)$-conjugate to μ_0^{-1}.

Let T' be a maximal (split) torus of $G_{p/W}$ containing the image of μ'. From its definition, we know μ_0 factors through a specific torus $T \subset G_{p/W}$. According to (Demazure and Grothendieck 1970, XII.7.1), T' and T will be conjugate locally for the étale topology on $\operatorname{Spec} W$, which in our case means that they are conjugate by an element of $G_p(W)$. We may therefore suppose that they both factor through T. But two characters of T are $G(K_0)$-conjugate if and only if they are conjugate by an element of the Weyl group—see for example (Milne 1992, 1.7)—and, because the hyperspecial point fixed by $G_p(W)$ lies in the apartment corresponding T, $G_p(W)$ contains a set of representatives for the Weyl group. Thus μ' is conjugate to μ_0^{-1} by an element of $G_p(W)$. □

Let $\underline{N} : \mathbf{Rep}_{(p)}(G) \to \mathbf{Mot}_{(p)}(K_0)$ be an admissible functor, and let π_1 and π_2 be the fundamental groups of the two categories in the sense of (Deligne 1990). Then $\underline{N}$ defines a morphism $\pi_2 \to \underline{N}(\pi_1)$, and we let $\mathbf{Mot}_{(p)}^{\underline{N}}(K_0)$ denote the category of objects of $\mathbf{Mot}_{(p)}(K_0)$ endowed with an action of $\underline{N}(\pi_1)$ extending the natural action of π_2. The functor $\underline{N}$ defines an equivalence of tensor categories $\mathbf{Rep}_{(p)}(G) \to \mathbf{Mot}_{(p)}^{\underline{N}}(K_0)$ (ibid. 8.17).

CONJECTURE 5.3. For any admissible functor $\underline{N}$ and sufficiently large prime p (depending only on G) there exists an exact $\mathbb{Z}_{(p)}$-linear tensor functor $\omega : \mathbf{Mot}_{(p)}^{\underline{N}}(K_0) \to \mathbf{MF}_W$ such that $K_0 \otimes_W \omega = \omega_{\text{crys}}$.

The functor ω should also be compatible with the functor V_{cris} of (Fontaine 1990, p305).

For an admissible functor $\underline{M}$, we let $X_{\text{crys}}(\underline{M})$ be the subset of $X_p(\underline{M})$ consisting of the étale p-integral structures $\underline{\Lambda}$ on $\underline{M}$ for which $\underline{N} = (\underline{M}, \underline{\Lambda})$ satisfies the conjecture, and we let

$$\mathcal{M}_{\text{crys}}(\underline{M}) = I(\underline{M}) \backslash X^p(\underline{M}) \times X_{\text{crys}}(\underline{M}).$$

Thus $\mathcal{M}_{\text{crys}}(\underline{M}) \subset \mathcal{M}(\underline{M})$, and the conjecture says that the two should be equal except possibly for some small p.

Shimura varieties of Hodge type

Now assume that $\mathrm{Sh}_p(G,X)$ is of Hodge type, and choose a symplectic representation $G \hookrightarrow \mathrm{GSp}(V,\psi)$, a lattice $V(\mathbb{Z}_p)$ in $V(\mathbb{Q}_p)$, and a defining family of tensors $\mathfrak{t} = (t_i)_{i\in I}$ as in Section 4. Let $M = (A,\mathfrak{s})$ be an admissible pair. A *crystalline p-integral structure* on A is a strongly divisible lattice $\Lambda \subset \omega_{\text{crys}}(A)$ for which there exists an isomorphism $V(W) \to \Lambda$ mapping each t_i to s_i. The proof of Lemma 5.2 then shows that the isomorphism can be chosen so that $\mathrm{Filt}(\mu_0^{-1})$ maps to the Hodge filtration.

Let $M = (A_{\mathbb{Q}},\mathfrak{s})$ be admissible, and let Λ be an étale p-integral structure on M. According to Remark 4.9, $(A_{\mathbb{Q}},\mathfrak{s},\Lambda)$ defines an integral admissible pair $N = (A,\mathfrak{s})$ with A an object of $\mathbf{AV}_{(p)}(K_0)$. The Néron criterion shows that A has good reduction, and so can be regarded as an abelian scheme over W. Thus $\omega_{\text{dR}}(A)$ is a W-module—denote it as $\omega_{\text{crys}}(A)$ when considered a lattice in $\omega_{\text{crys}}(A_{\mathbb{Q}})$. According to (Fontaine 1983, p91), $\omega_{\text{crys}}(A)$ is a strongly divisible lattice in $\omega_{\text{crys}}(A_{\mathbb{Q}})$.

CONJECTURE 5.4. For all integral admissible pairs $N = (A,\mathfrak{f})$, $\omega_{\text{crys}}(A)$ is a crystalline p-integral structure on $(A_{\mathbb{Q}},\mathfrak{f})$, i.e., there exists an isomorphism of W-modules $V(W) \to \omega_{\text{crys}}(B)$ mapping each t_i to s_i, except possibly for some small p.

REMARK 5.5. The conjecture is true for the Siegel modular variety or, more generally, a Shimura variety of PEL-type. Note that in order to prove it for $N = (A,\mathfrak{s})$, it suffices to show that, for some ring R faithfully flat over W, there is an isomorphism

$$V(R) \to R \otimes_W \omega_{\text{crys}}(A)$$

mapping each t_i to s_i (same proof as Corollary 1.4), or even that for some such R, there is an isomorphism

$$R \otimes_{\mathbb{Z}_p} V_{\text{cris}}(\omega_{\text{crys}}(A)) \to R \otimes_W \omega_{\text{crys}}(A)$$

mapping each t_i to s_i. I had hoped that, in analogy with the published proofs of similar theorems, the proposition stated in (Fontaine 1990, p305) was derived by showing that the base change map

$$A_{\text{crys}} \otimes_{\mathbb{Z}_p} V_{\text{cris}}(D) \to A_{\text{crys}} \otimes_W D,$$

is an isomorphism, but Fontaine assures me that this is not the case.

For an admissible pair $M = (A_{\mathbb{Q}},\mathfrak{s})$, we let $X_{\text{crys}}(M)$ be the subset of $X_p(M)$ consisting of étale p-integral structures Λ on M for which the corresponding pair $(A,\mathfrak{s})$ satisfies the conjecture, and we let

$$\mathcal{M}_{\text{crys}}(\mathcal{M}) = I(M)\backslash X^p(M) \times X_{\text{crys}}(M) \subset \mathcal{M}(M).$$

6 The Conjecture of Langlands and Rapoport

Let $\mathrm{Sh}_p(G,X)$ be as in the first paragraph of Section 4. Throughout this section, K will be a finite extension of K_0 contained in $\mathbb{C}_p$, and $\mathcal{O}_K$ will be its ring of integers.

Let s be a Hodge class on an abelian variety over K, and assume that A has good reduction to A_0 over $\mathbb{F}$. We say that s has *good reduction* if there exists a tensor s_0 on $h_1(A)$ in $\mathbf{PMot}(\mathbb{F})$ such that $s_0(\ell) = s(\ell)$ for all ℓ (including p). Then s_0 is uniquely determined by s, and is called the *reduction* of s. An abelian motive $M = h(A,e,m)$ has *good reduction* if both A and e have good reduction, to A_0 and e_0 say, in which case we set $M_0 = h(A_0,e_0,m)$. Potential good reduction can be defined similarly.

DEFINITION 6.1. An exact $\mathbb{Q}$-linear tensor functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(K)$ has *good reduction* if $M(\xi)$ has good reduction for all ξ in $\mathbf{Rep}_{\mathbb{Q}}(G)$ and there exists a functor $\underline{M}_0 : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$ such that $M(\xi)_0 = M_0(\xi)$ for all ξ.

Clearly $\underline{M}_0$ is uniquely determined by $\underline{M}$ (if it exists).

PROPOSITION 6.2. If $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{PMot}(\mathbb{F})$ has good reduction to $\underline{M}_0$, then there is a canonical $G(\mathbb{A}_f^p)$-equivariant map $\mathcal{M}_{\text{crys}}(\underline{M}) \to \mathcal{M}(\underline{M}_0)$.

PROOF. Since $\omega_f^p \circ \underline{M} = \omega_f^p \circ \underline{M}_0$, it is clear that $X^p(\underline{M}) = X^p(\underline{M}_0)$. Similarly, $\omega_{\text{crys}}(\underline{M}) = \omega_{\text{crys}}(\underline{M}_0)$, and so it follows from the definitions that there is a map $X_{\text{crys}}(\underline{M}) \to X_p(\underline{M}_0)$. Finally, an automorphism of $\underline{M}$ induces an automorphism of $\underline{M}_0$. □

EXAMPLE 6.3. (a) For any special $x \in X$, there exists a functor $\underline{M}_x : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{CM}(\mathbb{Q}^{\text{al}})$, which will be defined over some $K \subset \mathbb{Q}^{\text{al}}$ finite over K_0. Call a functor $\underline{M} : \mathbf{Rep}_{\mathbb{Q}}(G) \to \mathbf{Mot}(K)$ *special* if it becomes isomorphic to such an $\underline{M}_x$ over some (possibly larger) K. Every special functor has good reduction: if $\underline{M} \approx \underline{M}_x$, then $\underline{M}_0 = R \circ \underline{M}_x$.

(b) If $\underline{M}$ factors through $\mathbf{LMot}(K)$, then $\underline{M}$ has good reduction.

(c) Suppose $\mathrm{Sh}_p(G, X)$ is of Hodge type, and let $\underline{M}$ correspond to $(A, \mathfrak{s})$. Then $\underline{M}$ has good reduction if and only if each s_i in $\mathfrak{s}$ has good reduction.

(d) Consider an admissible pair $N = (A, \mathfrak{s})$. Suppose that there exists an admissible pair $N' = (A', \mathfrak{s}')$ for which there exists an isomorphism $A_0 \to A_0'$ carrying $s_i(\ell)$ to $s_i'(\ell)$ for all ℓ (including $\ell = p$). If N' has good reduction, then so also does N.

PROPOSITION 6.4. Every $\underline{M}_0$ arising by reduction from an admissible $\underline{M}$ is special in each of the following two cases:

(a) the derived group of G is simply connected;

(b) every $\underline{M}_0$ arising by reduction from an admissible $\underline{M}$ also arises by reduction from a special admissible $\underline{M}$ (not necessarily defined over K_0).

PROOF. (a) It is possible to check the criterion (3.6).

(b) This is obvious. □

Define

$$\mathcal{M}(G, X)(W) = \coprod \mathcal{M}_{\text{crys}}(\underline{M})$$

where $\underline{M}$ runs over the isomorphism classes of admissible $\underline{M}$'s having good reduction.

COROLLARY 6.5. If $\mathrm{Sh}_p(G, X)$ satisfies at least one of the two conditions in 6.4, then there is a canonical map $\mathcal{M}(G, X)(W) \to \mathcal{M}(G, X)(\mathbb{F})$ of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(K_0/E_v)$-sets.

We now suppose that $\mathrm{Sh}_p(G, X)$ has a canonical integral model. In particular, this means that there is a surjective reduction map

$$\mathrm{Sh}_p(K_0) = \mathrm{Sh}_p(W) \longrightarrow \mathrm{Sh}_p(\mathbb{F})$$

of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(K_0/E_v)$-sets. When $\mathrm{Sh}_p(G, X)$ satisfies the at least one of the two conditions in 6.4, then we get a diagram

$$\begin{array}{ccc} \mathcal{M}(W) \subset \mathcal{M}(K_0) & \xrightarrow{\approx} & \mathrm{Sh}_p(G,X) = \mathrm{Sh}_p(W) \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathbb{F}) & & \mathrm{Sh}_p(\mathbb{F}) \end{array}$$

in which the second vertical map is surjective. We would like to prove that the upper map $\mathcal{M}(W) \to \mathrm{Sh}_p(W)$ induces an isomorphism $\mathcal{M}(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$.

The Siegel modular variety

In this case, the canonical integral model is the moduli variety constructed by Mumford, and it is now straightforward to verify that the map $\mathcal{M}(W) \to \mathrm{Sh}_p(W)$ induces an isomorphism $\mathcal{M}(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$, i.e., that the conjecture of Langlands and Rapoport holds for $\mathrm{Sh}_p(G,X)$.

The conjecture for a subvariety

We now consider the case that there is an inclusion $(G,X,K_p) \hookrightarrow (G',X',K'_p)$, inducing an inclusion $\mathrm{Sh}_p(G,X) \subset \mathrm{Sh}_p(G',X')$, and that the conjecture is known for $\mathrm{Sh}_p(G',X')$. More explicitly, we assume that $\mathrm{Sh}_p(G',X')$ has a canonical integral model, and that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}'(W) & \xrightarrow{\approx} & \mathrm{Sh}'_p(W) \\ \downarrow & & \downarrow \\ \mathcal{M}'(\mathbb{F}) & \xrightarrow{\approx} & \mathrm{Sh}'_p(\mathbb{F}) \end{array}$$

of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\mathbb{F}/k(v))$-sets. Here, and throughout this subsection, we abbreviate $\mathcal{M}(G',X')$ and $\mathrm{Sh}_p(G',X')$ to $\mathcal{M}'$ and Sh'_p.

Let $\mathrm{Sh}_p(G,X)$ denote the integral closure of $\mathrm{Sh}_p(G,X)_{/E_v}$ in $\mathrm{Sh}_p(G',X')_{/\mathcal{O}_v}$. We obtain a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sh}_p(W) & \hookrightarrow & \mathrm{Sh}'_p(W) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_p(\mathbb{F}) & \hookrightarrow & \mathrm{Sh}'_p(\mathbb{F}). \end{array}$$

Moreover, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sh}_p(W) & \hookrightarrow & \mathrm{Sh}'_p(W) \\ \uparrow \text{ inj} & & \uparrow \approx \\ \mathcal{M}(W) & \to & \mathcal{M}'(W). \end{array}$$

The map $\mathcal{M}(W) \to \mathrm{Sh}_p(W)$ is injective, and so therefore is $\mathcal{M}(W) \to \mathcal{M}'(W)$.

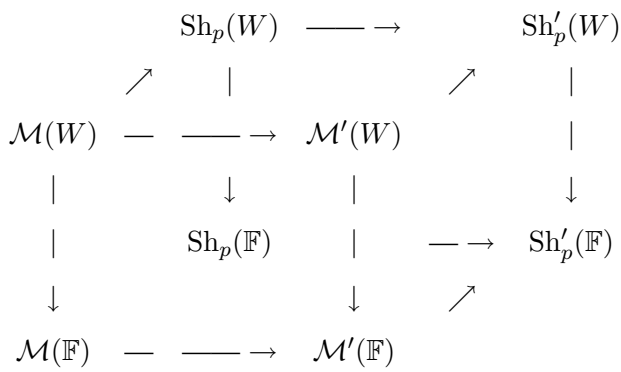
Finally, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{\text{crys}}(W) & \to & \mathcal{M}'(W) \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathbb{F}) & \to & \mathcal{M}'(\mathbb{F}). \end{array}$$

LEMMA 6.6. The canonical map $\mathcal{M}(\mathbb{F}) \to \mathcal{M}'(\mathbb{F})$ is injective.

PROOF. Straightforward. □

The above four commutative squares form the four complete sides of the following cube:



PROPOSITION 6.7. Let $\mathcal{M}^{\text{lift}}(\mathbb{F})$ be the image of $\mathcal{M}(W) \to \mathcal{M}(\mathbb{F})$. Then there is a unique map $\mathcal{M}^{\text{lift}}(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$ making the bottom face (and hence the whole cube) commute. It is injective, and its image is the image of $\mathcal{M}(W) \to \mathrm{Sh}_p(\mathbb{F})$.

PROOF. Elementary. □

Shimura varieties of Hodge type

Let $\mathrm{Sh}_p(G, X)$ be of Hodge type, and choose a symplectic embedding $G \hookrightarrow \mathrm{GSp}(V, \psi)$, a lattice $V(\mathbb{Z}_p)$, and a defining family of tensors $\mathfrak{t} = (t_i)_{i \in I}$ as before. It is known (Vasiu 1995) that $\mathrm{Sh}_p(G, X)$ has a canonical integral model.

6.8. The points of $\mathcal{M}(\mathbb{F})$ are in natural one-to-one correspondence with the isomorphism classes of triples $(A, \mathfrak{s}, \eta^p)$ with A in $\mathbf{AV}_{(p)}(\mathbb{F})$, $\mathfrak{s}$ a family of tensors on $h_1(A)$ indexed by I, and η^p an isomorphism $(V(\mathbb{A}_f^p), \mathfrak{t}) \to (\omega_f^p(A), \mathfrak{s})$ satisfying the following conditions:

(a) for some special point $x \in X$, $(A_{\mathbb{Q}}, \mathfrak{s}) \approx (A_x, \mathfrak{s}_x)_0$; (when G^{der} is simply connected, this is equivalent to $(A_{\mathbb{Q}}, \mathfrak{s})$ satisfying the conditions (3.6));
(b) there exists an isomorphism $(V(W), \mathfrak{t}) \to (\omega_{\text{crys}}(A), \mathfrak{s})$ such that $\mu_0(p^{-1})V(W)$ maps onto $\phi\omega_{\text{crys}}(A)$.

6.9. The points of $\mathrm{Sh}_p(\mathbb{F})$ are in natural one-to-one correspondence with the isomorphism classes of triples $(A, \mathfrak{s}, \eta^p)$, as in the above paragraph, satisfying the following condition: there exists an abelian scheme $\widetilde{A}$ over W and a family $\widetilde{\mathfrak{s}}$ of Hodge tensors on $\widetilde{A}$ indexed by I such that

(a) the pair $(\widetilde{A}, \widetilde{\mathfrak{s}})$ has good reduction, and $(\widetilde{A}, \widetilde{\mathfrak{s}})_0 \approx (A, \mathfrak{s})$;
(b) there is an isomorphism $(\omega_B(\widetilde{A}), \widetilde{\mathfrak{s}}) \to (V(\mathbb{Z}_{(p)}), \mathfrak{s})$ sending $h_{\widetilde{A}}$ onto h_x for some $x \in X$.

Identify both $\mathcal{M}(\mathbb{F})$ and $\mathrm{Sh}_p(\mathbb{F})$ with subsets of $\mathrm{Sh}_p(G(\psi), X(\psi))(\mathbb{F})$, and let $\mathcal{M}^{\text{lift}}(\mathbb{F}) = \mathcal{M}(\mathbb{F}) \cap \mathrm{Sh}_p(\mathbb{F})$.

PROPOSITION 6.10. There is a canonical injection from the subset $\mathcal{M}^{\text{lift}}$ of $\mathcal{M}(\mathbb{F})$ to $\mathrm{Sh}_p(\mathbb{F})$ with image the set of points represented by a triple $(A, \mathfrak{s}, \eta^p)$ satisfying the conditions:

(a) for some special point $x \in X$, $(A_{\mathbb{Q}}, \mathfrak{s}) \approx (A_x, \mathfrak{s}_x)_0$;
(b) there exists an isomorphism $(V(W), \mathfrak{t}) \to (\omega_{\text{crys}}(A), \mathfrak{s})$ such that $\mu_0(p^{-1})V(W)$ maps onto $\phi\omega_{\text{crys}}(A)$.

PROPOSITION 6.11. Assume G^{der} is simply connected. Then $\mathcal{M}^{\text{lift}}(\mathbb{F}) = \mathcal{M}(\mathbb{F})$.

PROOF. The choice of an η_p defines a lifting $\widetilde{A}$ of A (Norman 1981, p433), which we want to show satisfies the conditions of (6.9). Note that for each i, we obtain a tensor s_i in $\omega_f^p(\widetilde{A})$, $\omega_{\text{crys}}(\widetilde{A})$, and $\bar{\omega}(A)$. For each $\mathbb{Q}$-algebra R, let $F(R)$ be the set of isomorphisms $\omega_B(\widetilde{A}) \to V(\mathbb{Q})$ sending each s_i to t_i. Then F is a G-torsor, and hence defines an element of $H^1(\mathbb{Q}, G)$. This element maps to zero in $H^1(\mathbb{Q}, G^{\text{ab}})$, essentially because we know the conjecture for Shimura varieties of dimension zero, and it also maps to zero in $H^1(\mathbb{Q}_\ell, G)$ for each ℓ (including p and ∞). Because G^{der} satisfies the Hasse principle, this implies that the element is zero in $H^1(\mathbb{Q}, G)$, and therefore there is a $\beta \in F(\mathbb{Q})$. Now each of the s_i on $\widetilde{A}$ is rational and in the zeroth level of the Hodge filtration, and so is a Hodge class. Finally Condition (3.6a) implies that the homomorphism $\beta : \omega_B(\widetilde{A}) \to V(\mathbb{Q})$ just constructed satisfies condition (4.2b). □

COROLLARY 6.12. Assume G^{der} is simply connected. The conjecture of Langlands and Rapoport is true for $\text{Sh}_p(G, X)$ if every admissible pair $(A, \mathfrak{s})$ satisfies the following conditions:

(a) the elements of $\mathfrak{s}$ have good reduction (for example, if $(A_{\mathbb{Q}}, \mathfrak{s})_0 \approx (A_x, \mathfrak{s}_x)_0$ for some special $x \in X$, or $\text{Sh}_p(G, X)$ is of Lefschetz type);
(b) there exists an isomorphism $(V(W), \mathfrak{t}) \to (\omega_{\text{crys}}(A), \mathfrak{s})$.

REMARK 6.13. The maps $\mathcal{L}(\mathbb{F}) \to \mathcal{M}(\mathbb{F}) \to \text{Sh}_p(\mathbb{F})$ we have defined are very natural, and their composite is surjective if and only if every p-integral admissible pair $(A, \mathfrak{s})$ satisfies the conditions in the corollary. Conceivably, the conjecture could still be correct without these conditions holding, but this appears highly improbable.

7 Shimura Varieties of Dimension 0

As was pointed out in (Pink 1989), one should consider a slightly more general notion than that defined above, namely, a Shimura variety should be defined by

(a) a connected reductive group G over $\mathbb{Q}$ whose connected centre is split by a CM-field;
(b) a continuous left homogeneous space X for $G(\mathbb{R})$;
(c) a continuous $G(\mathbb{R})$-equivariant map $x \mapsto h_x : X \to \text{Hom}(\mathbb{S}, G_{\mathbb{R}})$ with finite fibres such that each h_x satisfies Deligne's axioms.

In this paper, we shall allow this extra generality only in the case of a Shimura variety of dimension zero. Thus, a *Shimura variety of dimension zero* is defined by a torus T over $\mathbb{Q}$ that is split by a CM-field, a finite (discrete) set X on which $T(\mathbb{R})$ acts transitively, $T(\mathbb{R}) \times X \to X$, and a homomorphism $h : \mathbb{S} \to T_{\mathbb{R}}$. The Shimura variety then is the profinite variety with complex points

$$\text{Sh}(T, X, h)(\mathbb{C}) = T(\mathbb{Q})^- \backslash X \times G(\mathbb{A}_f).$$

It has a canonical model over the field of definition $E(T, h)$ of μ_h.

EXAMPLE 7.1. (a) Consider (G, X, h) with $G = \mathbb{G}_m$, X equal to the set of isomorphisms $\mathbb{Z} \to \mathbb{Z}(1)$, and $h(z) = z\bar{z}$. Then $\text{Sh}(\mathbb{G}_m, X, h) \approx \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$.

(b) Consider the pair $(G(\psi), X(\psi))$ defining the Siegel modular variety, and let $c : G(\psi) \to \mathbb{G}_m$ be the usual homomorphism. Define $X(\psi) \to X$ to send x to the unique λ such that $\lambda \circ \psi$ is a polarization of $V(\psi)$. Then we get a map $\mathrm{Sh}(G(\psi), X(\psi)) \to \mathrm{Sh}(\mathbb{G}_m, X, h)$ that induces an isomorphism

$$\pi_0(\mathrm{Sh}(G(\psi), X(\psi))) \to \mathrm{Sh}(\mathbb{G}_m, X, h).$$

(c) The last example generalizes. Consider a pair (G, X) defining a Shimura variety, and let $c : G \to G^{\mathrm{ab}}$ be the maximal commutative quotient of G. The homomorphism $h_X = c \circ h_x$ is independent of $x \in X$. There is a triple $(G^{\mathrm{ab}}, X^{\mathrm{ab}}, h_X)$ defining a Shimura variety of dimension zero, and a canonical equivariant map

$$\pi_0(\mathrm{Sh}(G, X)) \to \mathrm{Sh}(G^{\mathrm{ab}}, X^{\mathrm{ab}},$$

which is an isomorphism if G^{der} is simply connected.

[[Now state and prove a "Langlands-Rapoport" conjecture for Shimura varieties of dimension zero.]]

8 The Functorial Form of the Conjecture

In this section, all reductive groups will have simply connected derived groups.

We now consider triples (G, X, K_p) where G is a reductive group over $\mathbb{Q}$, X is a homogeneous $G(\mathbb{R})$-set equipped with a homomorphism $x \mapsto h_x : X \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, and K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$ (which should be thought of as a $\mathbb{Z}_p$-model of $G_{\mathbb{Q}_p}$). We assume that either (G, X) defines a Shimura variety of dimension zero in the sense of the last section or a Shimura variety in the usual sense. The set of such triples forms a category with the obvious notion of morphism, and $(G, X, K_p) \mapsto \mathrm{Sh}_p(G, X)$ is a functor on this category.

CONJECTURE 8.1 (FUNCTORIAL FORM OF THE CONJECTURE). There is an isomorphism $\mathcal{L}(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$ of functors from the category of triples (G, X, K_p) to the category of $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\mathbb{F}/k(v))$-sets.

[[Since $k(v)$ depends on (G, X), this has to be explained.]]

In particular, this implies that, for a given triple (G, X, K_p), the map $\mathcal{L}(G, X, K_p)(\mathbb{F}) \to \mathrm{Sh}_p(G, X)(\mathbb{F})$ is equivariant for the action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$. Together with the results of the last section, this shows that the "functorial form of the conjecture" implies the "refined form of the conjecture" in (Pfau 1993).

Consider a morphism $(G, X, K_p) \to (G', X', K_p')$ inducing an isomorphism $G^{\mathrm{der}} \to G'^{\mathrm{der}}$. Then the diagram

$$\begin{array}{ccc} \mathrm{Sh}_p(G', X')(\mathbb{F}) & \leftarrow & \mathrm{Sh}_p(G, X)(\mathbb{F}) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_p(G'^{\mathrm{ab}}, X'^{\mathrm{ab}}) & \leftarrow & \mathrm{Sh}_p(G^{\mathrm{ab}}, X^{\mathrm{ab}}) \end{array}$$

is cartesian, i.e., it realizes $\mathrm{Sh}_p(G, X)(\mathbb{F})$ as a fiber product.

CONJECTURE 8.2. For any morphism $(G, X, K_p) \to (G', X', K'_p)$ inducing an isomorphism $G^{\text{der}} \to G'^{\text{der}}$, the diagram

$$\begin{array}{ccc} \mathcal{L}(G', X') & \leftarrow & \mathcal{L}(G, X) \\ \downarrow & & \downarrow \\ \mathcal{L}(G'^{\text{ab}}, X'^{\text{ab}}) & \leftarrow & \mathcal{L}(G^{\text{ab}}, X^{\text{ab}}) \end{array}$$

is cartesian.

Once Conjecture 8.2 is acquired, standard arguments (Milne 1990a, II.9) together with (Milne 1992, 4.19) show that, in order to prove the functorial form of the conjecture of Langlands and Rapoport for all Shimura varieties of type $(A, B, C, D^{\mathbb{R}})$ it suffices to prove it for certain simple Shimura varieties of Hodge type.

REMARK 8.3. In (Pfau 1993), Conjecture 8.2 is proved when the kernel of $G \to G'$ has trivial H^1. Pfau has announced a proof of the general case of the conjecture in an e-mail message whose TEX-code the author has not been able to compile, and which he has therefore not been able to read.[4]

Bibliography

Blasius, D., A p-adic property of Hodge classes on abelian varieties, Proc. Symp. Pure Math. 55, Part 2, **1994**.

Borel, A., Linear Algebraic Groups, **1991**.

Deligne, P., and Milne, J., Tannakian categories, in Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math, 900, Springer, **1982**, 101–228.

Deligne, P., Variétés de Shimura: Interprétation modulaire, et techniques de construction de modèles canoniques, Proc. Symp. Pure Math., 33, **1979**, part 2, 247–290.

Deligne, P., Catégories tannakiennes, in The Grothendieck Festschrift, II, **1990**, 111–195.

Demazure, M., and Grothendieck, A., **1970**, *Schémas en Groupes,* Lecture Notes in Math., Springer, Heidelberg, 151, 152, 153.

Fontaine, J-M., Cohomologie de De Rham, cohomologie cristalline et representations p-adiques, in Algebraic Geometry, Lecture Notes in Math. 1016, Springer, **1983**, 86–108.

Fontaine, J-M., Représentations p-adiques des corps locaux, in The Grothendieck Festschrift, II, **1990**, 249–309.

Fontaine, J-M., and Messing, W., p-adic periods and p-adic etale cohomology, [Arcata85] **1987**, 179–207.

Kottwitz, R., Points on some Shimura varieties over finite fields, Journal AMS, 5, **1992**, 373-444.

Ihara, Y., Non-abelian class fields over function fields in special cases, Actes Congr. Internat. Math., vol 1, Nice, **1970**, pp 381–389.

Jannsen, U., Motives, numerical equivalence, and semisimplicity, Invent. Math., 107, **1992**, 447–452.

Langlands, R., and Rapoport, M., Shimuravarietäten und Gerben, J. reine angew. Math. 378, **1987**, 113–220.

[4] See Pfau 1996a

Milne, J., Points on Shimura varieties mod p, Proc. Symp. Pure Math. 33, **1979a**, part 2, 165–184.

Milne, J., Etude d'une class d'isogénie, in Variéties de Shimura et Fonctions L, Publications Mathétiques de l'Université Paris 7, **1979b**, 73–81.

Milne, J., Canonical models of (mixed) Shimura varieties and automorphic vector bundles, in Automorphic Forms, Shimura Varieties, and L-functions, Perspectives in Math. 10, **1990a**, pp283–414.

Milne, J., The conjecture of Langlands and Rapoport for Siegel modular varieties, manuscript (March 13, **1990b**), 60pp.

Milne, J., 1991, The conjecture of Langlands and Rapoport for Siegel modular varieties, Bull. A.M.S. 24, **1991**, 335–341.

Milne, J., The points on a Shimura variety modulo a prime of good reduction, in The Zeta Function of Picard Modular Surfaces, (Langlands and Ramakrishnan Eds.), Les Publications CRM, Montréal, **1992**, 153-255.

Milne, J., Motives over finite fields, in Motives (Eds. U. Jannsen, S. Kleiman, J.-P. Serre), Proc. Symp. Pure Math., AMS, 55, Part 1, pp401–459, **1994a**.

Milne, J., Shimura varieties and motives, in Motives (Eds. U. Jannsen, S. Kleiman, J.-P. Serre), Proc. Symp. Pure Math., AMS, 55, Part 2, pp447–523, **1994b**.

Milne, J., Lefschetz motives. Incomplete preliminary draft, June 24, **1995a**, 28pp.[5]

Milne, J., Hodge and Tate classes, and abelian motives over finite fields. Incomplete preliminary draft, June 3, **1995b**.[6]

Morita, Y., Reduction modulo $\mathfrak{P}$ of Shimura curves, Hokkaido Math. J., 10, **1981**, 209–238.

Norman, P., Lifting abelian varieties, Invent. Math., 64, **1981**, 431–443.

Pfau, M., The reduction of connected Shimura varieties at of good reduction, Thesis, University of Michigan, **1993**, and subsequent papers.[7]

Pink, R., Arithmetical compactification of mixed Shimura varieties, Dissertation, Bonn **1989**.

Reimann, H., and Zink, T., The good reduction of Shimura varieties associated to quaternion algebras over a totally real field, Preprint **1991**.

Saaavedra Rivano, N., Catégories Tannakiennes, Lecture Notes in Math., Vol 265, Springer, Heidelberg, **1972**.

Scholl, A., Classical Motives, Proc. Symp. Pure Math. 55, Part 1, **1994**, 163–187.

Serre, J.-P., Cohomologie Galoisienne, **1964**.

Serre, J.-P., Groupes de Grothendieck des schémas en groupes réductifs déployés, Publ. Math. I.H.E.S., 34, **1968**, 37–52.

Steinberg, R., Regular elements of semi-simple algebraic groups, Publ. Math. I.H.E.S, 25, **1965**, 201–312.

Tits, J., Reductive groups over local fields, Proc. Symp. Pure Math. 33, Part 1, **1979**, 29–69.

Vasiu, A., Integral canonical models of Shimura varieties of preabelian type, manuscript, August **1995**.[8]

[5] See Milne 1999a,b.

[6] See the forenote.

[7] See Pfau 1996b,a.

[8] See Vasiu 1999.

Wintenberger, Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux, Ann. of Math., 119, **1984**, 511-548.

Zink, T., Isogenieklassen von Punkten von Shimuramannigfaltigkeiten mit werten in einem endlichen Korper, Math. Nach. 112, **1983**, 103–124.

Additional references (2007)

KISIN, M. 2007. Integral canonical models of shimura varieties. Talk at Journées Arithmétiques, Edinburgh, July 2007.

MILNE, J. S. 1994. Shimura varieties and motives, pp. 447–523. *In* Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI.

MILNE, J. S. 1999a. Lefschetz classes on abelian varieties. *Duke Math. J.* 96:639–675.

MILNE, J. S. 1999b. Lefschetz motives and the Tate conjecture. *Compositio Math.* 117:45–76.

MILNE, J. S. 2002. Polarizations and Grothendieck's standard conjectures. *Ann. of Math. (2)* 155:599–610.

MILNE, J. S. 2005. Quotients of tannakian categories and rational Tate classes. Preprint, available at www.jmilne.org/math/; also arXiv:math.CT/0508479.

MILNE, J. S. 2007. Rational Tate classes on abelian varieties. July 20, 2007, v1, available at jmilne.org and the arXive.

PFAU, M. 1993. The reduction of connected Shimura varieties at primes of good reduction. PhD thesis, University of Michigan.

PFAU, M. 1996a. A conjecture about the reduction of connected Shimura varieties at good primes. Submitted to the Ann. of Math. in November, 1996; 37 pages.

PFAU, M. 1996b. The conjecture of Langlands and Rapoport for certain Shimura varieties of non-rational weight. *J. Reine Angew. Math.* 471:165–199.

REIMANN, H. 1997. The semi-simple zeta function of quaternionic Shimura varieties, volume 1657 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

VASIU, A. 1999. Integral canonical models of Shimura varieties of preabelian type. *Asian J. Math.* 3:401–518.

VASIU, A. 2003a. CM-lifts of isogeny classes of Shimura F-crystals over finite fields. Preprint arXiv:math/0304128v1 [math.NT].

VASIU, A. 2003b. A motivic conjecture of Milne. Preprint arXiv:math/0308202v2 [math.NT].