28th March 1990.

Dear Deligne,

I am having trouble understanding the choice of sign in your definition of the canonical model in your Corvallis article (2.2.5). Perhaps you could help me?

Abelian varieties with complex multiplication.

Let E be a CM-algebra, and let (A,i) be an abelian variety over $\mathbb{C}$ with complex multiplication by E. By this I mean that i is an inclusion $E \hookrightarrow \mathrm{End}(A)\otimes\mathbb{Q}$ which makes $H_1(A,\mathbb{Q})$ into a free E-module of rank one. Let λ be a polarization of A compatible with i (i.e., such that the Rosati involution induces complex conjugation ι on E).

Let T be the torus over $\mathbb{Q}$ such that $T(\mathbb{Q}) = \{ a \in E^\times \mid a\cdot\iota a \in \mathbb{Q}^\times\}$. The action of E on $H_1(A,\mathbb{Q})$ defines a representation of T on $H_1(A,\mathbb{Q})$, and the Hodge structure on $H_1(A,\mathbb{Q})$ is defined by a cocharacter μ: $\mathbb{G}_m \longrightarrow T_{\mathbb{C}}$. Specifically, $\mu(z) \in T(\mathbb{C}) \subset GL(H_1(A,\mathbb{C}))$ acts as z on $H^{-1 0} = \mathrm{Tgt}_0(A)$ and as 1 on $H^{0 -1}$. Let E^* be the field of definition of μ, and define $N = N(T,\mu)$ to be the composite of

$$\mathrm{Res}_{E^*/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{\mathrm{Res}(\mu)} \mathrm{Res}_{E^*/\mathbb{Q}}(T_{E^*}) \xrightarrow{\mathrm{Norm}} T.$$

(Note: no inverse!)

If τ: $k \hookrightarrow K$ is an embedding of fields, and Y is a variety (or other algebro-geometric object) over k, then there is only one possible meaning for τY: apply τ to the equations defining Y.

Let $V_f(A) = (\prod T_\ell A)\otimes\mathbb{Q}$.

THEOREM 1. Let (A,i,λ) be a polarized abelian variety over $\mathbb{C}$ with complex multiplication by E; let $\tau \in \mathrm{Aut}(\mathbb{C}/E^*)$, and let $s \in \mathbb{A}^\times_{E^*}$ be such that $\mathrm{rec}(s) = \tau|E^{*ab}$; then there is a unique isogeny α: $A \longrightarrow \tau A$ such that:

(a) α is E-linear;

(b) $\alpha^*(\tau\lambda) = c\cdot\lambda$, $c \in \mathbb{Q}^\times$;

(c) $\alpha(N(s_f)\cdot x) = \tau x$, all $x \in V_f(A)$, $N = N(T,\mu)$, (or all $x \in A(\mathbb{C})_{\mathrm{tors}}$).

Here rec is the the reciprocity map of class field theory, according to your (current) normalization, i.e., uniformizing elements go to *geometric* Frobenius elements. For $s \in \mathbb{A}$, I write s_f for the projection of s into A_f.

REMARK: To verify the sign, I compare the statement with that in Shimura, Arithmetic Theory of Automorphic Functions, p 117. There he considers an elliptic curve A. In this case, E is a quadratic imaginary number field, with a fixed embedding $E \hookrightarrow \mathbb{C}$ defined by the action of E on $\mathrm{Tgt}_0(A)$ (the embedding is a CM-type for E). Moreover, $E^* = E$, T = "$E^\times$", and $N(T,\mu)$: $T \longrightarrow T$ is the identity map. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$, and choose an $s \in \mathbb{A}^\times_{f,E}$ such that $\mathrm{rec}(s) = \tau|E^{ab}$. (Thus, this s is the inverse of Shimura's — he follows Cassels-Fröhlich for the reciprocity map.) His theorem states that there are isomorphisms

$$\xi\colon \mathbb{C}/\mathfrak{a} \longrightarrow A, \quad \xi'\colon \mathbb{C}/s\mathfrak{a} \longrightarrow \tau A$$

such that $\tau\xi(y) = \xi'(s_f\cdot y)$ for $y \in E/\mathfrak{a}$. Define the isogeny α: $A \longrightarrow \tau A$ to be "$\xi'\circ\xi^{-1}$", and put $x = \xi(y)$. Then

$$\alpha(s_f\cdot x) = \xi'(s_f\cdot y) = \tau\xi(y) = \tau x,$$

as claimed.

Moduli.

Let E be a CM-algebra, and define T as before. Let V be a free E-module V of rank one, and let (V,h,ψ) be a polarized Hodge structure on V (as a $\mathbb{Q}$-vector space) of type $\{(-1,0),(0,-1)\}$ such that $h(\mathbb{C}^\times) \subset T(\mathbb{R})$ and $\psi(a\cdot x,y) = \psi(x,\iota a\cdot y)$ for $a \in E$. Write μ for the cocharacter of T corresponding to h.

Consider quadruples (A,i,λ,η) where (A,i,λ) is a polarized abelian variety with complex multiplication by E for which there exists an E-linear isomorphism

$$(H_1(A,\mathbb{Q}),h_A,\psi_\lambda) \approx (V,h,\psi)$$

of polarized Hodge structure, and η is an $E\otimes\mathbb{A}_f$-linear isomorphism $V(\mathbb{A}_f) \longrightarrow V_f(A)$ making ψ correspond to ψ_λ. (In the first $\approx$, ψ and ψ_λ only have to correspond up to an element of $\mathbb{Q}^\times$, and in the second up to an element of $\mathbb{A}_f^\times$.) Two such quadruples (A,i,λ,η) and (A',i',λ',η') are said to be isogenous if there is an E-linear isogeny $A \longrightarrow A'$ carrying λ into λ' (up to $\mathbb{Q}^\times$) and such that $V_f(\alpha)\circ\eta = \eta'$. Write $[A,i,\lambda,\eta]$ for the isogeny class of (A,i,λ,η). Define $\tau(A,i,\lambda)$ to be the obvious triple, and define $\tau\eta$ to be the composite

$$V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{\tau} V_f(\tau A).$$

THEOREM 2. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E^*)$, and let $s \in \mathbb{A}^\times_{E^*}$ be such that $\mathrm{rec}(s) = \tau|E^{*ab}$; then

$$\tau[A,i,\lambda,\eta] = [A,i,\lambda,\eta\circ N(s_f)], \quad N = N(T,\mu).$$

PROOF: The isogeny α: $A \longrightarrow \tau A$ given by theorem 1 defines an isogeny

$$(A,i,\lambda,\eta\circ N(s_f)) \longrightarrow \tau(A,i,\lambda,\eta).$$

VARIANT: Consider quadruples (A,i,λ,k) with k and an isomorphism $V(A) \longrightarrow V(\mathbb{A}_f)$. Then

$$\tau[A,i,\lambda,k] = [A,i,\lambda,N(s_f)^{-1}\circ k].$$

This follows immediately from the preceding formula because $(A,i,\lambda,\eta) \mapsto (A,i,\lambda,\eta^{-1})$ is bijection from the first set of quadruples to second, commuting with the action of Aut($\mathbb{C}$). This formula can be rewritten:

$$\mathrm{rec}(s)\ [A,i,\lambda,k] = [A,i,\lambda,N(s_f)^{-1}\circ k],\ \ N = N(T,\mu).$$

I compare this with the "Théorème de Shimura-Taniyama" 4.19 in your Bourbaki talk #389, 1971. There $\varphi(s)$ is the inverse of rec(s) (your talk predates Antwerp). For a given T, h, μ, the map r(T,h) in your Bourbaki talk is $N(T,\mu)^{-1}$, i.e., $r(T,h)(x) = N(T,\mu)(x)^{-1}$ (see the definition of r(T,h) on p140 of your talk), but the μ of your Bourbaki talk also the reciprocal of my μ — thus we actually have $r(T,h) = N(T,\mu)$, $h \leftrightarrow \mu^{-1}$. Thus formula (4.19.1) of your talk can be written (with the current conventions)

$$\mathrm{rec}(s^{-1})\ [A,i,\lambda,k] = [A,i,\lambda,N(s_f)\circ k],$$

which agrees with the above formula.

Shimura Varieties.

Consider Shimura variety Sh(G,X) of Hodge type. This means that there is an embedding $(G,X) \hookrightarrow (GSp,\mathcal{X})$ where $(GSp,\mathcal{X})$ is the pair corresponding to a symplectic space (V,ψ) as in your Corvallis talk 1.3.1. Let $\mathfrak{s} = (s_i)$ be a family of tensors such that G is the subgroup of GSp "fixing" the tensors.

Let $\mathcal{A}(G,X,V)$ be the set of isogeny classes of triples (A,t,η) where A is an abelian variety over $\mathbb{C}$, t is a set of Hodge cycles on A, and η is a level structure $V(\mathbb{A}_f) \longrightarrow V_f(A)$ satisfying the obvious conditions. There is a canonical right action of $G(\mathbb{A}_f)$ on $\mathcal{A}(G,X,V)$, namely,

$$[A,t,\eta]\cdot a = [A,t,\eta\circ a],\ \ a \in G(\mathbb{A}_f).$$

LEMMA: There is a canonical $G(\mathbb{A}_f)$-equivariant bijection $\mathcal{A}(G,X,V) \longrightarrow Sh(G,X)$.

PROOF: If (A,t,η) represents an element of $\mathcal{A}(G,X,V)$, then there exists an isomorphism

$$\beta\colon H_1(A,\mathbb{Q}) \longrightarrow V$$

carrying elements of t to corresponding elements of $\mathfrak{s}$ (up to the correct scalar) and such that $h_A \leftrightarrow h_x$ for some $x \in X$. Define g to be the composite

$$V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{\beta} V(\mathbb{A}_f),$$

and map (A,t,η) to [x,g]. A different choice of β replaces (x,g) with $q\cdot(x,g)$, $q \in G(\mathbb{Q})$.

There is a natural action of $\mathrm{Aut}(\mathbb{C}/E^*(G,X))$ on $\mathcal{A}(G,X,V)$, namely, $\tau[A,t,\eta] = [\tau A,\tau t,\tau\eta]$ with $\tau\eta\colon V(\mathbb{A}_f) \longrightarrow V_f(\tau A)$ the map $x \mapsto \tau(\eta(x))$, which we can transfer to Sh(G,X). In the case of (T,h) considered above, the action is

$$\mathrm{rec}(s)\ [x,a] = [x,\ a\cdot N(s_f)].$$

Thus, if x is a special point of X, we should have

$$\mathrm{rec}(s)\,[x,1] = [x,\, N(s_f)].$$

Unfortunately, the definition on p269 of Corvallis article (as I understand it) has $N(s_f)$ replaced with its reciprocal.

COMMENTS: (a) Changing the identification Sh(G,X) with a moduli set doesn't seem to affect the sign, provided the identification is $G(\mathbb{A}_f)$-equivariant — both acting on the right. (At least, taking $\mathcal{X}$ to consist of Hodge structures of type $\{(1,0),(0,1)\}$, or taking level structures to be maps $V(A) \longrightarrow V(\mathbb{A}_f)$, doesn't seem to make any difference.)

(b) The formulas defining the canonical models are precisely the same in your Bourbaki talk and your Corvallis talk. However, there appear to be three changes of sign:

(i) the reciprocity map;

(ii) the maps h and μ associated with a Hodge structure;

(iii) the action of $G(\mathbb{A}_f)$ has been changed from the left to the right.

Yours sincerely

Jim

J.S. Milne

april 19, 1990

Dear Milne,

Mea culpa, mea maxima culpa.

My sign is wrong, and your explanation at the end of your letter plausible: I could not count up to three. Anyway, the case of level structures on $\mathcal{B}_m$ is convincing enough.

My apologies, and thank you for warning me

P. Deligne

P. DELIGNE