Kazhdan's Theorem on Arithmetic Varieties

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Let $X$ be a nonsingular algebraic variety over $C$ whose universal covering manifold $\tilde{X}$ is a symmetric Hermitian domain (i.e., a symmetric Hermitian space without Euclidean or compact factors). Then the group $\text{Aut}(\tilde{X})$ of automorphisms of $\tilde{X}$ (as a complex manifold) has only finitely many connected components, and its identity component $\text{Aut}(\tilde{X})^+$ is a real reductive Lie group with no compact factors. The variety $X$ will be said to be arithmetic if the group $\Gamma$ of covering transformations of $\tilde{X}$ over $X$ is torsion-free and is an arithmetic subgroup of $\text{Aut}(\tilde{X})$ in the sense that there is a linear algebraic group $G$ over $\mathbb{Q}$ and a surjective homomorphism $f: G(\mathbb{R})^+ \rightarrow \text{Aut}(\tilde{X})^+$ with compact kernel carrying $G(\mathbb{Z}) \times G(\mathbb{R})^+$ into a group commensurable with $\Gamma$.

The theorem in question is the following:

**Theorem 0.1.** If $X$ is an arithmetic variety, then, for all automorphisms $\sigma$ of $C$, $\sigma X$ is also an arithmetic variety.

The result was first stated, with some indications of a proof, by Kazhdan in his talk at the Nice Congress in 1970 [1, Tomoz, p.311-325], but, in fact, it is only recently that the proof has been completed.

In the case that $X$ is compact, a detailed proof of (0.1) was given by Kazhdan in his talk at the Budapest conference in 1971 (published 1975, [3]).
There is now an alternative approach to this case of the theorem: roughly speaking, the first part of Kozhukhov's paper, where the nondegeneracy of the Bergman metric in the universal covering manifold of $\mathbf{X}$ is established, can be replaced by an appeal to Yau's theorem on the existence of Kähler-Einstein metrics; the second, more technical part, can be replaced by an appeal to general results of Margulis (see [11], below).

Let $\mathbf{X} \to \mathbf{X}$ be a finite étale morphism. If $\mathbf{X}$ is arithmetic, then clearly so also is $\mathbf{X}$, and the converse assertion is also true provided the fundamental group $\pi_1(\mathbf{X})$ is torsion-free. From this it follows that it suffices to prove (0.1) under the hypothesis

(0.2) There exist an almost-$\mathcal{O}$-simple connected algebraic group $G_{\mathbb{Q}}$ over $\mathbb{Q}$, a surjective homomorphism $f: G_{\mathbb{R}} \to \text{Aut}(\mathbf{X})^\times$ with compact kernel $\Gamma$ and a congruence subgroup $\Gamma \subset G_{\mathbb{Q}}$ such that $\mathbf{X}$ is equal to $f(\Gamma) \backslash \mathbf{X}$.

With its unique algebraic structure $\mathbf{X}$ is a moduli variety for a class of abelian varieties with a family of Hodge cycles and a level structure. (See for example [5], §2.) In the case of $G_{\mathbb{Q}}$ one should also assume that the image of $\Gamma$ in $G_{\mathbb{Q}}$ is a congruence subgroup there.) For any automorphism $\sigma$ of $G_{\mathbb{Q}}$, the conjugate variety $\mathbf{X}$, will be a moduli variety for the conjugate class of abelian varieties with extra structure; and so in again an arithmetic variety. Thus (0.1) is true for arithmetic varieties of this type.

(Note that this argument makes use of Deligne's theorem [12].)
Hodge.)

In [4], Kaghden proves the following result: let $X$ be an even $(0,2)$, noncompact, and of type $E_6$, $E_7$, or mixed type $(D^N, D^N)$; then, if one assumes a certain result in the theory of group representations (Conjecture 3.17 below), (0.1) is true for $X$. A weak form of (3.17), sufficient for the purpose of proving (0.1), has recently been established by L. Clozel. Thus the last result of Kaghden, together with the proofs for complete varieties and moduli varieties, suffices to prove (0.1).

For his proof in these last cases, Kaghden shows, case by case, that $X$ contains an arithmetic subvariety $X'$ associated with a subgroup $G'$ of $G$ such that

$$\dim X > \dim X' > \dim (X^* - X)$$

where $X^*$ is the canonical (Baily-Borel) compactification of $X$. Because (0.1) is known for curves (they are all of type $A_1$), in proving (0.1) for $X$ he can assume by induction on the dimension that it holds for $X'$.

In these notes, I modify Kaghden's arguments to give a uniform proof that (0.1) holds for all $X$ (satisfying (0.2)) such that

$$(0.30) \quad \text{codim } (X^* - X) \geq 3,$$

$$(0.31) \quad G' \text{ contains a maximal torus } T' \text{ that splits over an imaginary quadratic extension of } F \text{ and is such that } T'(K) \text{ is compact.}$$

The proof does not use the classification and avoids treating the compact varieties separately, but it does assume (0.1) for the case of arithmetic varieties of type $A_1$ (for
these varieties it is easily proved using moduli varieties). It also, of course, makes use of Clozel’s result.

If $X$ satisfies (0.2). A construction of Piatetsky-Shapiro and Borovoi allows one to embed $X$ into an arithmetic variety $X'$ satisfying also (0.3). I do not know whether it is possible to prove directly that (0.1) for $X'$ implies (0.1) for $X$, but in any case one has the implications:

(0.1) for all arithmetic varieties satisfying (0.2) and (0.3)

$\Rightarrow$ Langlands’s conjecture [3, p.232-333] for all Shimura varieties

$\Rightarrow$ (0.1) for all arithmetic varieties.

(see [3] or [4, 3].)

The first section of these notes contains generalities on arithmetic varieties. Also a group $Q$ is defined that will play the role of $G(Q)/Z(Q)$ for the conjugate variety $\overline{X}$. Criteria are given in §2 for a variety to be arithmetic; in conjunction with Yau’s theorem, they suffice to show that $\overline{X}$ is arithmetic in the case that $X$ is compact. In §3 Clozel’s result is used to show that the Bergmann volume form on a certain covering manifold of $\overline{X}$ is not identically zero.

In the next section, the subbundles of the tangent bundle on $\overline{X}$ are studied and, finally, in §5 the main theorem is proved.
1. Generalities on arithmetic varieties, definition of $\tilde{X}$ and $\tilde{G}$

Let $G$ be a simply connected algebraic group over $Q$, and let $\tilde{X}$ be a symmetric Hermitian domain in which $G(\mathbb{R})$ acts in such a way that $\tilde{X} = G(\mathbb{R}) \backslash (\text{maximal compact subgroup})$. We always assume that $G$ has no $Q$-factor that is anisotropic over $\mathbb{R}$. For any compact open subgroup $K$ of $G(\mathbb{A}^f)$, $K \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$, and we let $\Gamma = \Gamma(K)$ be its image in $G^{\text{ad}}(\mathbb{Q})$. Clearly $\Gamma(K) = \Gamma(K) \cap G(\mathbb{Q})$, where $Z = Z(\mathbb{Q})$, and so we can always assume that $K \supseteq Z(\mathbb{Q})$.

Let $X_K$ or $X_{\Gamma}$ denote $\Gamma(K) \backslash \tilde{X}$ regarded as an algebraic variety. The strong approximation theorem shows that $G(\mathbb{Q})$ is dense in $G(\mathbb{A}^f)$, and it follows that

$$X_K(C) \triangleq \Gamma \backslash \tilde{X} = G(\mathbb{Q}) \backslash \tilde{X} \times G(\mathbb{A}^f) / K.$$

The actions on the right-hand term are

$$g(x, a)K = (gx, gak), \quad g \in G(\mathbb{Q}), \quad x \in \tilde{X}, \quad a \in G(\mathbb{A}^f), \quad k \in K.$$

Let $T$ denote the set of compact open subgroups $K$ of $G(\mathbb{A}^f)$ containing $Z(\mathbb{Q})$ and such that $\Gamma(K)$ is torsion-free. The groups $\Gamma(K)$, $K \in T$, have the following properties:

- Each $\Gamma(K)$ is an arithmetic group;
- $\bigcap_{K \in T} \Gamma(K) = \{1\}$; if $K_1, \ldots, K_n$ are in $T$, then $K = \bigcap K_i$ is in $T$ and $\Gamma(K) = \bigcap \Gamma(K_i)$;
- If $q \in G^{\text{ad}}(\mathbb{Q})$ and $K \in T$, then $q \Gamma(K) q^{-1} = \Gamma(qKq^{-1})$ with $qKq^{-1} \in T$. We shall often identify the elements of $T$ with their images in $G(\mathbb{A}^f) / Z(\mathbb{Q})$.

Consider the projective system $(X_K)_{K \in T}$ of algebraic varieties. There is a left action of $G(\mathbb{A}^f)$ on this system:

$$g : X_K \rightarrow X_{gKg^{-1}}, \quad [x,a] \mapsto [x, ag^{-1}], \quad x \in \tilde{X}, a, q \in G(\mathbb{A}^f).$$

Let $\tilde{X} = \varprojlim X_K$ — this is an irreducible scheme over $\mathbb{C}$ (not
Lemma 1.1. There are equalities (of sets)
\[ \hat{X}(C) = \text{lim}_{\alpha} X_{\alpha}(C) = G(\Phi) / \bar{X} \times G(M^f) = (G(\Phi) / Z(\Phi)) / \bar{X} \times (G(M^f) / Z(\Phi)). \]

Proof: Only the middle equality requires proof. Recall the following result [Bourbaki, Topologie Générale, III 7.2]: consider a pro-homogenous system \((G_\alpha)\) of topological groups acting continuously and compatibly on a topological space \(S\); then the canonical map \(S / \text{lim}_{\alpha} G_\alpha \to \text{lim}_{\alpha} S / G_\alpha\) is bijective if

(a) the isotropy group in \(G_\alpha\) of each \(s \in S\) is compact, and
(b) the orbit \(G_\alpha s\) of each \(s \in S\) is compact.

Apply this with \(S = G(\Phi) / \bar{X} \times G(M^f)\) and \((G_\alpha) = (K)\). Then (a) holds because each isotropy group is \(Z(\Phi)\), and (b) holds because \(K\) is compact. As \(\text{lim}_{\alpha} K = \Pi K = 1\), we have that
\[ G(\Phi) / \bar{X} \times G(M^f) \to \text{lim}_{\alpha} G(\Phi) / \bar{X} \times G(M^f) / K \]

is bijective, as required.

The action of \(G(M^f)\) on the system \((X_\alpha)\) defines an action of \(G(M^f)\) on \(\hat{X}\): for \(g \in G(M^f)\), \(g : \hat{X} \to \hat{X}\) where \(x, a \mapsto [x, a^g] = [x, a^g]. \) The knowledge of \(\hat{X}\) with this action is equivalent to the knowledge of the projective system \((X_\alpha)\) together with the action of \(G(M^f)\) on it; in particular \(X_\alpha = K / \hat{X} = K \backslash X\). We shall sometimes use \(\hat{X}\) to denote the projective system \((X_\alpha)\) rather than its limit. Note that the action of \(G(M^f) / Z(\Phi)\) on \(\hat{X}\) is effective.

There is an action of \(G^{\text{st}}(\Phi)^*\) on \(\hat{X}\):
\[ \alpha [x, a] = [\alpha x, \alpha(a)] \quad \alpha \in G^{\text{st}}(\Phi)^*, \quad x \in \bar{X}, \quad a \in G(M^f). \]
The actions of $G(\Phi)$ on $\hat{X}$ defined by the maps $G(\Phi) \to G^0(\Phi)$ and $G(\Phi) \to G(\mathbb{A}^f)$ are equal:

$$[q_x, qa_q^{-1}] = [x, a_q^{-1}], \quad q \in G(\Phi), \quad x \in \hat{X}, \quad a \in G(\mathbb{A}^f).$$

Let $(X_d)$ be a projective system of smooth complex algebraic varieties such that the transition maps $X_\beta \to X_\alpha$, $\beta \geq \alpha$, are étale. Then $\hat{X} = \varprojlim X_d$ has a canonical structure as a complex manifold: a basis for the atlas on $\hat{X}$ is formed by pairs $(U, \phi)$ for which there exists an $x$ such that the projection $p_d: \hat{X} \to X_d$ is injective on $U$ and

$$(p_d(U), \phi \circ p_d^{-1})$$

is an open chart for $X_d$. We write $\hat{X}^an$ for $\hat{X}$ with this structure. Note that the topology on $\hat{X}^an$ is, in general, strictly finer than the projective limit of the topologies on the $X_d$. (However, the Zariski topology on $\hat{X}$ is the projective limit of the Zariski topologies on the $X_d$.)

In the situation considered above, if $G(\mathbb{A}^f)$ is given the discrete topology, then

$$\hat{X}^an = G(\Phi) \backslash \hat{X} \times G(\mathbb{A}^f).$$

Note that the map

$$\hat{X} \to \hat{X}^an, \quad x \mapsto [x, 1]$$

is injective (because $\cap \mathfrak{n}(K) = \{1\}$), and is an isomorphism of $\hat{X}$ (as a complex manifold) onto a connected component of $\hat{X}^an$. We shall use the following notations for the maps:

$$\hat{X} \xrightarrow{\iota} \hat{X}^an \xrightarrow{\rho} \hat{X} \xrightarrow{p_k} X_k \xrightarrow{q_k} X_k$$

If $x \in \hat{X}$, we often write $x_k$ and $\hat{x}$ for $p_k(x)$ and $\rho(x)$. 
Lemma 12. Let \( g \in G(M^c)/Z(\Phi) \); if \( g \) stabilizes \( \tilde{X} \subset \hat{X} \), then it belongs to \( G(\Phi)/Z(\Phi) \).

Let \( x \in \tilde{X} \). As \( g \) stabilizes \( \tilde{X} \), \( [x, g] = [x, 1] \) for some \( x' \in \tilde{X} \). This means that there exists a \( \tilde{g} \in G(\Phi)/Z(\Phi) \) such that \( (\tilde{g} \cdot x', \tilde{g}) = (x, g) \) as elements of \( \tilde{X} \times G(M^c)/Z(\Phi) \). In particular, \( g = \tilde{g} \in G(\Phi)/Z(\Phi) \).

Now for an automorphism \( \sigma \) of \( \hat{X} \). The discussion above shows that there is a canonical structure of a complex manifold on \( \tilde{X}^\sigma \).

Choose a connected component \( \tilde{X}_0 \) of \( (\sigma \hat{X})^0 \), and let \( \rho^0 \) and \( \rho_k^0 \) be the inclusion \( \tilde{X}_0 \hookrightarrow (\sigma \hat{X})^0 \) and the composite \( (\sigma q_k^0) \circ \rho^0 \) respectively: thus \( \tilde{X}_0 \xrightarrow{\rho^0} (\sigma \hat{X})^0 \xrightarrow{\sigma q_k^0} \hat{X}_k \).

The group \( G(M^c) \) continues to act on \( \tilde{X}^\sigma \), and this action is compatible with the complex structure on \( (\sigma \hat{X})^0 \). Define \( Q \in G(M^c)/Z(\Phi) \) to be the stabilizer of \( \tilde{X}^\sigma \). Note that \( \tilde{X}^\sigma \to \hat{X}^0 \) is a local isomorphism.

Let \( x \in \tilde{X}^\sigma \), and let \( M \) be the universal covering manifold \( \tilde{M} \times \hat{X}^0 \). For any \( m \in M \), there is a unique map \( M \to \tilde{X}^\sigma \) such that \( m \) maps to \( x \), and the composite \( M \to \tilde{X}^\sigma \to \tilde{X}^\sigma \) are all analytic. Clearly \( M \to \tilde{X}^\sigma \) is analytic, and so its image is contained in \( \tilde{X}^\sigma \). This shows that \( \tilde{X}^\sigma \to \hat{X}^0 \) is surjective and that \( \tilde{X}^\sigma \) is covering manifold of \( \hat{X}^0 \). In particular, \( \tilde{X}^\sigma \) is Zariski dense in \( \tilde{X}^\sigma \), and so the action of \( Q \) on \( \tilde{X}^\sigma \) is effective.

Lemma 13. For any \( K \in J \), the map

\[
(k, x) \mapsto kx : K \times \tilde{X}^\sigma \to (\sigma \hat{X})^0
\]
Proof: Let \( x \in \sigma \hat{X}^\infty \). By what we have just proved, there exists an \( \hat{x} \in \hat{X}^\infty \) such that \( \hat{x} \) and \( x \) have the same image in \( \sigma \hat{X}^\infty \). Let \( \hat{x} = \sigma(\hat{x}) \). Then \( (\sigma g_K)(\hat{x}) = (\sigma g_K)(x) \), and so \( g_K(\sigma^{-1} \hat{x}) = g_K(\sigma^{-1} x) \) in \( X^\infty_K = K \hat{X}^\infty \). Therefore there exists a \( k \in K \) such that \( k(\sigma^{-1} \hat{x}) = \sigma^{-1} x \), and so \( k \hat{x} = x \).

**Proposition 1.4.** For any \( K \in \mathcal{L} \), \( K' = G(M^f)/Z(\Phi) \). In particular, \( \Phi \) is dense in \( G(M^f)/Z(\Phi) \).

**Proof.** For any \( g \in G(M^f) \), \( g \tilde{X}^\infty \) is a connected component of \( \sigma \hat{X}^\infty \), and as the lemma shows there exists a \( k \in K \) such that \( kg \tilde{X}^\infty = \tilde{X}^\infty \). By definition, \( kg \in \Phi \), and therefore \( g \in k^{-1} \Phi \subset K' \).

**Proposition 1.5.** (a) The map

\[
[x, g] \mapsto gx: \Phi \tilde{X}^\infty \times G(M^f) \rightarrow \sigma \hat{X}^\infty
\]

is an isomorphism of complex manifolds \( (G(M^f) \) with the discrete topology \).

(b) For every \( K \), the map

\[
[x, g] \mapsto \sigma g_K : \Phi \tilde{X}^\infty_K \rightarrow \sigma K \hat{X}^\infty
\]

is an isomorphism of complex manifolds.

**Proof:** (a) The lemma shows that \( \tilde{X}^\infty \times G(M^f) \rightarrow \sigma \hat{X}^\infty \) is surjective, and it is obvious that the fibres of the map are the orbits of \( \Phi \).

(b) As \( \sigma X_K = K \hat{X} \). It follows from (a) that there is an isomorphism \( \Phi \tilde{X}^\infty \times G(M^f)/K \rightarrow \sigma \hat{X}^\infty \). The preceding proposition can be used to show that \( \Phi \tilde{X}^\infty \times G(M^f)/K \rightarrow \sigma \hat{X}^\infty \).

We write \( \Gamma(K) \) for \( \Phi \tilde{X}^\infty \). Thus the proposition above that

\[
\Gamma(K) \tilde{X}^\infty \rightarrow \sigma (\Gamma(K) \tilde{X})
\]
Proposition 1.6. For any $K \in \mathbb{L}$, $Q$ is contained in the commensurability group of $\Gamma^\sigma(K)$.

Proof: Let $g \in Q$; we have to show that $g \Gamma^\sigma(K)g^{-1}$ is commensurable with $\Gamma^\sigma(K)$. Note that $g \Gamma^\sigma(K)g^{-1} = \Gamma^\sigma(gKg^{-1})$. Let $K' = K \cap gKg^{-1}$.

The diagram of finite étale maps

$X' 
\downarrow
\downarrow

X_K 
\downarrow
\downarrow

X_gKg^{-1}$

gives rise to a similar diagram

$\sigma X_{K'} = \Gamma^\sigma(K') \backslash \hat{X}^\sigma$

$\sigma X_K = \Gamma^\sigma(K) \backslash \hat{X}^\sigma$

$\sigma X_{gKg^{-1}} = \Gamma^\sigma(gKg^{-1}) \backslash \hat{X}^\sigma$.

This shows that $\Gamma^\sigma(K')$ has finite index in both $\Gamma^\sigma(K)$ and $\Gamma^\sigma(gKg^{-1}) = g \Gamma^\sigma(K)g^{-1}$.

Remark 1.7. It follows from (1.6) that $G(\mathbb{A}^1)$ acts transitively on the space of connected components of $(\sigma X)^{an}$. Thus every connected component is of the form $g \hat{X}^\sigma$, $g \in G(\mathbb{A}^1)$, and has a stabilizer that is conjugate to $Q$.

Proposition 1.8. Let $Z$ be a nonempty $Q$-invariant subset of $\hat{X}^\sigma$. Assume that $Z$ is a complex analytic subset of an open submanifold of $\hat{X}^\sigma$ and that $p^\sigma_K(Z)$ is an algebraic subvariety of $\sigma X_K$ for some $K$ in $\mathbb{L}$; then $Z = \hat{X}^\sigma$.

Proof. The hypothesis implies that $\hat{Z} = (p^\sigma_K(Z))_{K \in \mathbb{L}}$ is a finitely algebraic subvariety of $\sigma \hat{X}$ and is $Q$-invariant. As $Q$ is dense in $G(\mathbb{A}^1)/Z(\Phi)$, $\hat{Z}$ is invariant under $G(\mathbb{A}^1)/Z(\Phi)$,
and so \( \sigma^{-1}(\overline{Z}) \) is a \( G(\mathbb{A}^g) / Z(\mathbb{Q}) \)-invariant sub scheme of \( \tilde{X} \). Let \( \tilde{Z} = \tilde{X} \cap \sigma^{-1}(\overline{Z}) \). Then \( \tilde{Z} \) is \( G(\mathbb{Q}) \)-invariant and, as \( G(\mathbb{Q}) \) is dense in \( G(\mathbb{R}) \) and \( G(\mathbb{R}) \) acts transitively, this shows that \( \tilde{Z} = \tilde{X} \).

For any \( K \) in \( S \), let \( X^*_K \) be the canonical (Baily-Borel) compactification of \( X_K \). The dimension of \( X^*_K - X_K \) is independent of \( K \), and we shall write it \( \dim(\mathcal{O}_X) \).

**Wolgang 1.9.** Let \( Z \) be a nonempty \( G \)-invariant analytic subset of \( X^* \) such that \( \dim Z > \dim(\mathcal{O}_X) \); then \( Z = \overline{X^*} \).

**Proof:** Let \( d = \dim Z \), and let \( Z^{(d)} \) be the set of \( z \in Z \) such that \( Z \) has a component of dimension \( d \) at \( z \). Then \( Z^{(d)} \) is an analytic subset of \( Z \) [Narasimhan, p.67] and is \( G \)-invariant. Therefore we can assume \( Z = Z^{(d)} \). The image \( Z' \) of \( Z \) in \( X^*_K \) (any \( K \)) is analytic and such that \( Z' = Z'^{(d)} \). As \( \dim(X^*_K - X_K) < d \), the theorem of Remmert-Stein [loc.cit., p.123] shows that the closure \( \overline{Z'} \) of \( Z' \) in \( X^*_K \) is analytic, and Chow's theorem [loc. cit., p.125] shows that \( \overline{Z'} \) is algebraic. Therefore \( \overline{Z'} \cap X_K = Z' \) is algebraic, and the proposition applies.

A point \( \overline{x} \in \tilde{X} \) is said to be \emph{special} if there exists a torus \( T \subset G \) such that \( T \) is maximal and \( T(\mathbb{R}) \overline{x} = \tilde{x} \).

**Proposition 1.10.** Let \( x \in \tilde{X} \) be special, and let \( x^\sigma \in \tilde{X}^\sigma \) be any
Define such that $\hat{x}^\sigma = g(\sigma \hat{x})$ for some $g \in G(\Lambda^\sigma)$. Consider the $\sigma$-linear map of tangent spaces $\sigma: T_x(\tilde{X}) \rightarrow T_{\sigma x}(\tilde{X}^\sigma)$ defined by

$$
\begin{array}{c}
x \\
\downarrow \\
x_K
\end{array} \xrightarrow{\sigma} 
\begin{array}{c}
\hat{x} \\
\downarrow \\
\hat{x}_K
\end{array} \xrightarrow{g} 
\begin{array}{c}
\sigma \hat{x} \\
\downarrow \\
\sigma \hat{x}_K
\end{array} \xrightarrow{g^{-1}(X^\sigma)} 
\begin{array}{c}
x^\sigma \\
\downarrow \\
x^\sigma_K
\end{array}.
\end{array}
$$

Then there exists a homomorphism $j: T(\Phi) \rightarrow \Phi$ such that $j(T(\Phi))$ fixes $x$ and $\sigma$ commutes with the actions of $T(\Phi)$ on the two tangent spaces. The closure of $j(T(\Phi))$ in $\text{Aut}(T_{\sigma x}(\tilde{X}^\sigma))$ for the real topology contains an element inducing multiplication by $\Phi$.

**Proof:** Let $t \in T(\Phi) \subset G(\Lambda^\Phi)$, then $t \hat{x} = \hat{x}$ and so $t \sigma \hat{x} = \sigma \hat{x}$. As $\sigma \hat{x} = g^{-1} \hat{x}^\sigma$, this means that $(gtg^{-1}) \hat{x}^\sigma = \hat{x}^\sigma$. In particular, $gtg^{-1}$ acts on points of $\tilde{X}^\sigma$ into $\tilde{X}^\sigma$ and stabilizes $\tilde{X}^\sigma$. Thus $gtg^{-1} \in \Phi$, and we can define $j(t) = g tg^{-1}$. By construction $j(t)$ fixes $x^\sigma$, and it is routine to check that $\sigma$ carries the action of $t$ on $T_x(\tilde{X})$ into the action of $j(t)$ on $T_{\sigma x}(\tilde{X}^\sigma)$, the action of $t$ on $\tilde{X}$, $\tilde{X}_K$, and $\sigma X_K$ commute with $\tilde{X}_K$, $\sigma$, and $p_K$, and the action of $t$ on $\tilde{X}^\sigma$ is transformed by $g: \tilde{X}^\sigma \rightarrow \tilde{X}^\sigma$ into the action of $gtg^{-1}$. To prove the last assertion of the proposition, we need a lemma.

**Lemma 1.11.** Let $T$ be a torus over $\Phi$. For each automorphism $\sigma$ of $\Gamma$, there is a unique automorphism $t \rightarrow t^\sigma$ of $T(\pi)$ such that $X(t^\sigma) = \sigma X(t)$, all $t \in T(\pi), X \in X^*(T)$. The map $t \rightarrow t^\sigma$ is continuous and, if $T(\pi)$ is compact, takes
For $t \in T(\Phi)$, $\sigma(X(t)) = X(t^\sigma)$

**Proof.** There is an isomorphism

$$t \mapsto (X \mapsto X(t)) : T(\Phi) \to \text{Hom}(X^*(T), \mathbb{C}^*)$$

and we define $t \mapsto t^\sigma$ on $T(\Phi)$ to correspond to the map on $\text{Hom}(X^*(T), \mathbb{C}^*)$ induced by the $\mathbb{Z}$-linear map

$$\sigma : X^*(T) \to X^*(T).$$

Clearly $X(t^\sigma) = (\sigma X)(t)$, and $t \mapsto t^\sigma$ is the unique map with this property. The continuity is obvious. If $T(\Phi)$ is compact, it is the unique maximal compact subgroup of $T(\Phi)$, and so it is preserved by $t \mapsto t^\sigma$. For $t \in T(\Phi)$, $\sigma X(t) = (\sigma X)(t) = X(t^\sigma)$.

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**Proof of 1.10 continued:** The $\mathfrak{g}$-linear map $\alpha : T_x(X) \to T_{x^\sigma}(\bar{x}^\sigma)$ induces a $\mathbb{C}$-linear isomorphism $\beta : \sigma T_x(X) \to T_{x^\sigma}(\bar{x}^\sigma)$. Moreover,

$$\beta \circ \sigma(\rho_x(t)) = \rho_{x^\sigma}(\bar{j}(t)) \circ \beta, \quad t \in T(\Phi)$$

where $\rho_x$ and $\rho_{x^\sigma}$ denote the representations of $G$ and $G$ on the tangent spaces at $x$ and $x^\sigma$. Clearly $(\sigma \rho_x)(t) = \rho_x(t^\sigma)$, and so

$$\beta \circ \rho_x(t^\sigma) \circ \beta^{-1} = \rho_{x^\sigma}(\bar{j}(t)), \quad t \in T(\Phi).$$

As $T(\Phi)$ is dense in $T(\mathbb{R})$, it follows that for any $\gamma \in T(\mathbb{R})$ there exists a $\gamma'$ in the closure of $\rho_{\sigma}(\bar{j}(T(\Phi)))$ in $\text{Aut}(T_{x^\sigma}(\bar{x}^\sigma))$ such that $\gamma'$ acts as $\beta \circ \rho_x(t^\sigma) \circ \beta^{-1}$ on $T_{x^\sigma}(\bar{x}^\sigma)$. It is known (see, for example, [Helgason, VIII, 4.5]) that there is a $\gamma$ in $T(\mathbb{R})$ acting as $\bar{j}\bar{t}$ on $T_x(X)$; therefore

$$\gamma' = \beta \circ \rho_x(t) \circ \beta^{-1}$$

$$= \beta \circ (\text{multiplication by } \bar{j}\bar{t}) \circ \beta^{-1}$$

$$= \text{multiplication by } \bar{j}\bar{t}.$$
2. A criterion to be an arithmetic variety

Let \((M, g)\) be an oriented Riemannian manifold. Then there is a
unique volume element \(\mu\) on \(M\), having value +1 on any
orthonormal frame. In local coordinates

\[
g = \sum g_{ij} \, dx^i \otimes dx^j
\]

\[
\mu = \sqrt{\det(g_{ij})} \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n.
\]

Let \(\nabla\) be the connection defined by \(g\). The curvature tensor is
defined by

\[
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}
\]

In terms of local coordinates

\[
R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k} = \sum_k R_{kij} \frac{\partial}{\partial x^k}.
\]

The Riemann tensor \(r(X, Y)\) is determined in local coordinates
by

\[
r(X, Y) = \sum R_{ij} \, dx^i \otimes dx^j, \quad R_{ij} = \sum_k R^{k}_{iik}.
\]

Assume further that \(M\) is a complex manifold, with
multiplication by \(j\) being described by the tensor \(I\). Then
\(g\) is said to be Hermitian if \(g(JX, JY) = g(X, Y)\), all \(X, Y\).
In this case

\[
h(X, Y) \overset{df}{=} g(X, Y) + ig(X, JY)
\]

is a positive definite Hermitian form. The form
\[\bar{h}(X, Y) = g(X, JY)\] is skew-symmetric. It is called the
fundamental 2-form of the Hermitian metric \(h\), and if it is
closed, \(h\) is said to be Kählerian.

Lemma 2.1. Let \(M\) be a connected complex manifold of dimension \(n\) with
Kählerian Riemannian structure \(g\). If the associated
volume element is described in terms of local coordinates by
$\mu = k \left( \frac{i}{2} \right)^n dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n$

then the Ricci tensor is described by

$$r = \sum_{i \neq j} \frac{\partial^2 \log k}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j.$$  \hfill (2.1.1)

**Proof:** [Helgason, VIII 2.5]

In general, if $\mu = k \left( \frac{i}{2} \right)^n dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n$ is a volume element on a complex manifold, then the tensor defined by (2.1.1) will be called the Ricci tensor, $\text{Ric}(\mu)$, of $\mu$.

**Lemma 2.2** Let $\mathcal{M}$ be a connected complex manifold on which a group $G$ acts transitively, and let $\mu$ be a $G$-invariant volume element on $\mathcal{M}$ such that $g = \text{Ric}(\mu)$ is positive definite (and therefore is a Riemannian metric). Then $g$ is equal to the Ricci tensor (and therefore is an Einstein metric).

**Proof:** Let $\mu'$ be the volume form associated with $g$. Then $\mu' = f \mu$ for some positive function $f$ on $\mathcal{M}$. As $\mu$ is $G$-invariant, so also are $g$, $\mu'$, and $f$. Therefore $f$ is constant, and so the Ricci tensor of $g$ is $\text{Ric}(f \mu) = \text{Ric}(\mu) = g$.

**Theorem 2.3** Let $\mathcal{M}$ be a connected complex manifold on which a unimodular Lie group $G$ acts effectively and transitively. Assume there is a $G$-invariant volume element $\mu$ on $\mathcal{M}$ such that $\text{Ric}(\mu)$ is positive definite. Then $\mathcal{M}$ is a Hermitian symmetric domain, and $G \subseteq \text{Aut}(\mathcal{M})^+$. 
Proof: This is proved in [Koszul, Sāo Paulo notes 1959, p61]. Alternatively, one can combine the following results:

A complex homogeneous manifold with an invariant volume form whose Ricci tensor is positive definite is isomorphic to a homogeneous bounded domain [Piatetskii-Shapiro, Automorphic Functions ..., (1969), p48]

A connected unimodular Lie group acting effectively and transitively on a bounded domain is semisimple [Hano, Am. J. Math. 69 (1957) 385-400]


Theorem 2.4. Let $M$ be a connected complex manifold, and endow $\text{Aut}(M)$ with the compact-open topology. Let $\Gamma \subset \text{Aut}(M)$ be a subgroup of $\text{Aut}(M)$ with $\Gamma$ discrete and torsion free, and assume that the orbit of $\bar{Q}$ in $M$ is dense. Assume also that there is a $\bar{Q}$-invariant volume form $\mu$ on $M$ such that

(a) $\text{Ric}(\mu)$ is positive definite.
(b) $\int_M \mu < \infty$.

Then $M$ is a Hermitian symmetric domain, and the closure $\bar{Q}$ of $Q$ in $\text{Aut}(M)$ is a semisimple Lie group whose identity component is $\text{Aut}(M)$. If moreover $Q$ is contained in the commensurability group of $\Gamma$, then $\Gamma$ is arithmetic and so $\Gamma \backslash M$ is an arithmetic variety.

Proof: Let $g = \text{Ric}(\mu)$. By assumption, $(M,g)$ is a Riemannian manifold, and so the group $\text{Is}(M,g)$ of its isometries is a
Lemma 2.5. Let $(M, g)$ be a Riemannian manifold. Then for all \( m \in M \), the map \( \text{Is}(M, g) \to M, \phi \mapsto \phi(m) \), is proper.

Proof. Let \( O(M) \) be the bundle of orthonormal frames over \( M \). Every automorphism \( \phi \) of \( M \) defines a compatible automorphism \( \tilde{\phi} \) of \( O(M) \). Let \( u \in O(M) \) and let \( m \) be its image in \( M \). The mapping \( \phi \mapsto \tilde{\phi}(u) \) embeds \( \text{Is}(M, g) \) as a closed submanifold of \( O(M) \) [ibid, p.413]. The projection \( O(M) \to M \) is clearly proper, and its restriction to \( \text{Is}(M, g) \) is the map \( \phi \mapsto \phi(m) \).

Let \( m \) be a point of \( M \) such that \( Gm \) is dense in \( M \). Then the lemma shows that \( Gm \) is closed and so equals \( M \). The orbit under \( G^+ \)
Lie groups [Montgomery-Zippin, p. 208]. As $\mu$ is $G$-invariant, $G$, and its closure $\overline{G}$, are contained in $\text{Is}(M,g)$. Write $G = \overline{G}$ when $G$ is a Lie group, and we are assuming that it acts transitively on $M$. The orbit under $G^+$ of $m$ of $M$ contains an open set in $M$, and so equals $M$ [Kobayashi-Nomizu, p. 177, Cor. 4.8]; thus $G^+$ also acts transitively on $M$.

The isometry group $K$ at the point $m$ is compact. Thus the fact that $M = G^+K$ carries a $G^+$-invariant measure such that $P\setminus M$ has finite volume is equivalent that $G^+$ carries left invariant measure relative to which $\Gamma \setminus G^+$ has finite volume. Hence, $\Gamma \subset G^+$ is a lattice in $G^+$, and so $G$ is unimodular (the image of the modular function $A_\mu$ in $\mathbb{R}$ is a subgroup with finite measure, and so is $[1,1]$ — see [Raghunathan, 193]). The preceding theorem now shows that $M$ is a Hermitian symmetric domain and $G^+ = \text{Aut}(M)^+$.

The final statement of the theorem is a consequence of the following theorem of Margulis: Let $G$ be a semisimple connected real Lie group with no compact factors, and let $\Gamma$ be a lattice in $G$; if the commensurability group of $\Gamma$ in $G$ is dense in $G$, then $\Gamma$ is an arithmetic subgroup of $G$ [Margulis, AMS Transl 109 (1977) 33-45, Thm 9].

Remark 2.5. In the statement of the theorem just cited, Margulis assumes that $\Gamma$ is irreducible, but this is unnecessary. If $\Gamma$ is reducible, then there exist connected normal subgroups $G_i$ of $G$ such that
\[ \Gamma = \prod \Gamma_i \text{ (almost direct product), } \Gamma_i = \Gamma \times G_i \text{ is an irreducible lattice in } G_i, \text{ and } \prod \Gamma_i \text{ is a subgroup of finite index in } \Gamma. \]

Clearly the commensurability group of \( \Gamma \) in \( G \) is the product of the commensurability groups of the \( \Gamma_i : \text{Comm} (\Gamma) = \prod \text{Comm} (\Gamma_i). \) Then if \( \text{Comm} (\Gamma) \) is dense in \( G \), each group \( \text{Comm} (\Gamma_i) \) is dense in \( G_i \), and Margulis's statement can be applied to show that each \( \Gamma_i \) is arithmetic.

\textbf{Corollary 2.6} Let \( X \) be an arithmetic variety as in \( \S 1 \), and suppose that there exists a \( \Omega \)-invariant volume element \( \mu \) on \( \widetilde{X}^\Omega \) satisfying the conditions (2.4a) and (2.4b). Assume that there exists a finite family of arithmetic subvarieties \( i_\alpha : X_\alpha \to X \) such that for some special point \( x \in X \), \( x_\alpha = i_\alpha^*(x) \) is special on \( X_\alpha \), all \( \alpha \). If each \( \sigma X_\alpha \) is arithmetic, and the sublattice of \( T_x(X) \) generated by the \( T_{x_\alpha}(X_\alpha) \) has dimension \( > \text{dim}(\partial X) \), then \( \sigma X \) is arithmetic.

\textbf{Proof:} Let \( \gamma = \text{Rec}(\mu) \), and let \( M \) be the closure of the orbit \( \gamma \tilde{x} \) for some \( \tilde{x} \in \widetilde{X}^\gamma \) lifting \( x \). Then \( (M, \gamma) \) is a Riemannian manifold, and the closure \( \overline{\Omega} \) of \( \gamma \) in \( Is(M, \gamma) \) acts transitively on \( M \) (by 2.5). Since \( \gamma \to \gamma \tilde{x}, \overline{\Omega} \to \widetilde{X}^{\overline{\Omega}} \), is a proper map, it is an embedding and \( M \) is a regular closed submanifold of \( \widetilde{X}^{\overline{\Omega}} \). By (1.13), \( M \) has a \( \overline{\Omega} \)-invariant complex structure — it is therefore a complex analytic subset of \( \widetilde{X}^{\overline{\Omega}} \). Since it contains \( \sigma X_\alpha \) for each \( \alpha \), it has dimension \( > \text{dim}(\partial X) \), and so (1.9) shows that \( M = \widetilde{X}^{\overline{\Omega}} \).

Now (2.5) can be applied.
Remark 2.7. It is now possible to complete the proof of Theorem 0.1 in the case that $X$ is compact. In this case, $\dim (TX) = -1$ and so the map $T \to X$ will serve for the family $(T \to X)$. Recall the following theorem of Yau [Comm. Pure Appl. Math.]: let $V$ be a smooth projective algebraic variety over $\mathbb{C}$; then there exists a unique Kähler metric on $\overline{V}$ such that $\text{Ric}(g) = q$. Apply this to $X$ and let $\mu$ be the inverse image on $\overline{X}$ of the volume element associated with $g$. The uniqueness of $g$ shows that $\mu$ is invariant under all automorphisms of $\overline{X}$. Since $\text{Ric}(g) = q$, it satisfies (12), and it satisfies (16) because $X$ is compact. Thus 2.5 applies.

In the remainder of this section we shall show that the condition (2.46) is automatically satisfied in the context of (2.6).

A complex manifold $\overline{M}$ will be said to be compactifiable if it can be embedded as an open dense subset of a compact analytic space $\overline{M}$ in such a way that $\overline{M} - M$ is an analytic subset. Horrocks' theorem [1] then shows that $\overline{M}$ can be chosen to be a manifold such that $\overline{M} - M$ is a divisor with normal crossings. Clearly, if $M = V^n$ with $V$ a quasi-projective algebraic variety, then $M$ is compactifiable (in fact, not even quasi-projectivity is necessary).

**Proposition 2.8.** Let $M$ be a compactifiable complex manifold of dimension $n$, and let $\mu$ be a volume form on $M$ such that

1. $\text{Ric}(\mu)$ is (locally) definite
2. $\text{Ric}(\mu)^n > \mu$

...
Then $S_M \mu < \infty$.

Let $D(a) = \{ z \in \mathbb{C} \mid |z| < a \}$
$D^*(a) = \{ z \in \mathbb{C} \mid 0 < |z| < a \}$.

Lemma 2.9, for $\delta < 1$,
$$\int_{D(\delta)} \frac{i}{\pi} \frac{dz \cdot d\bar{z}}{(1-|z|)^2} = \frac{2\delta^2}{1-\delta^2}$$
$$\int_{D^*(\delta)} \frac{i}{\pi} \frac{dz \cdot d\bar{z}}{|z|^2 (\log |z|)^2} = -1/\log \delta.$$

Proof: Put $z = te^{i\theta}, \bar{z} = te^{-i\theta}$. Then $dz \cdot d\bar{z} = 2i \pi dt \cdot d\theta.

Thus,
$$\int_{D(\delta)} \frac{i}{\pi} \frac{dz \cdot d\bar{z}}{(1-|z|)^2} = \int_0^\delta \frac{i}{\pi} \cdot 2\pi \cdot \frac{2it \cdot dt}{(1-t^2)^2} = \frac{2}{1-t^2} \bigg|_0^\delta = \frac{2\delta^2}{1-\delta^2}$$
$$\int_{D^*(\delta)} \frac{i}{\pi} \frac{dz \cdot d\bar{z}}{|z|^2 (\log |z|)^2} = -\int_0^\delta \frac{i}{\pi} \frac{2it \cdot dt}{t \cdot \log^2 t} = -\int_0^\delta \frac{1}{\log t} \bigg|_0^\delta$$
$$= -1/\log \delta.$$

Remark 2.10. Let $\mu_D = \frac{i}{\pi} \frac{dz \cdot d\bar{z}}{(1-|z|^2)^2}$ and $\mu_{D^*} = \frac{i}{\pi} \frac{dz \cdot d\bar{z}}{|z|^2 (\log |z|)^2}.$

Then $\mu_D$ is the Poincaré metric on the unit disc, and
$\text{Ric} (\mu_D) = \mu_D$. The inverse image of $\mu_{D^*}$ relative to the covering map $D \to D^*$ is $\mu_D$, and so $\text{Ric} (\mu_{D^*}) = \mu_D$ also. (Cf. [Griffiths, p.473]).

Lemma 2.11. If $\mu$ be a volume element on $D(1)^n \times D^*(1)^{n-1}$
satisfying the estimates (2.9.1). Then
\[ \mu \leq \mu_0^* \times \mu_0^{\text{ext}}. \]

**Proof.** If \( r = n \), this is precisely Ahlfors's lemma [Griffiths, Thm. 2.21].

Consider the covering map
\[ D(1)^n \to D(1)^r \times D(1)^{n-r}. \]
Then \( \phi^* \mu \) satisfies (2.8.1) and so \( \phi^* \mu \leq \mu_0^* = \phi^* (\mu_0^* \times \mu_0^{\text{ext}}) \).
This implies that \( \mu \leq \mu_0^* \times \mu_0^{\text{ext}}. \)

**Lemma 2.12.** Let \( U = D(1)^n \) and let \( U' = D(1)^r \times D(1)^{n-r} \). Then there exists an open neighborhood \( V \) of \( U - U' \) in \( U \) such that
\[ \int_V \mu_0^* \times \mu_0^{\text{ext}} < \infty. \]

**Proof:** A stronger statement is proved in (3.11) below.

**Proof of 2.8.** Embed \( M \) in a compact manifold \( \overline{M} \) in such a way that \( N \subseteq \overline{M} - M \) is a divisor with normal crossing. Then there is a finite family of open subsets \( U_i \) of \( \overline{M} \) such that \( N \subseteq U \cup U_i \) and, for each \( i \), the pair \( (U_i, U_i \cap M) \) is isomorphic to \( (D(1)^n, D(1)^r \times D(1)^{n-r}) \). For each \( i \), choose \( V_i \subseteq U_i \) to correspond to an open subset of \( D(1)^n \) satisfying the conditions of (2.12). Then the complement \( C \) of \( UV_i \) in \( \overline{M} \) is compact, and it is contained in \( M \). Thus,
\[ \int_M \mu \leq \int_C \mu + \sum \int_{V_i \cap M} \mu \leq \int_C \mu + \sum \int_{V_i \cap M} \mu_0^* \times \mu_0^{\text{ext}} < \infty. \]

**Corollary 2.13.** Let \( X \) be an arithmetic variety (as in 3.1), and let \( U \) be an open submanifold of \( X^\circ \) such that \( X - U \) is an analytic subset of \( X^\circ \).
Assume that there is a \( G \)-invariant volume element on \( U \) such that \( \text{Ric}(\mu) > 0 \). Suppose further that there exists a finite
family of arithmetic subvarieties $i_{x_0} : X_{x_0} \to X$ such that for some special point $x$ of $X^\circ$, in the image of $U \to X$, $i_{x_0}^*(x)$ is special on $X_{x_0}$ for all $x$. If each $\sigma X_{x_0}$ is arithmetic, then $\sigma X$ is arithmetic. 

**Proof.** Let $x \in U$ such that $x \in \sigma X$, and let $g = \text{Ric}(\mu)$. It is a Riemannian metric on $U$. Define $M$ to be the closure of $U$ of the orbit $\mathcal{O}$. As in the proof of (2.6), $M = \overline{Gx}$, where $\overline{G}$ is the closure of $G$ in $Is(U, g)$, and it is a complex analytic submanifold of $U$. As it contains the $x_{x_0}$ for each $x_0$, it is also closed in $U$, and so $M = U$.

We know that $\mu$ is $M$-invariant, and we now check that it satisfies (2.46) and we now check that it satisfies (2.46). Suppose $\text{Ric}(\mu) = c\mu$ at some point of $\overline{Gx}$. Then $c > 0$, and because $\overline{G}$ acts transitively on $M$ and $\mu$ is $G$-invariant, we must have $\text{Ric}(\mu) = c\mu$ holding on all of $M$. Since $\text{Ric}(c\mu) = \text{Ric}(\mu)$, this shows that $c\mu$ satisfies the estimates (2.8.1). Therefore so also does the volume element induced by $c\mu$ on $\overline{G} \setminus M$, and so Proposition 2.8 implies that $\int_{\overline{G} \setminus M} c\mu < \infty$.

We can now apply (2.4) to $M, \mathcal{O}, \Gamma$, and we find that $\Gamma \setminus M$ is an arithmetic variety. In particular, it is an open algebraic subvariety of $\sigma X$, and so (1.8) implies that $M = \overline{\sigma X}$. We conclude that $\sigma X = \Gamma \setminus M$, and so $\sigma X$ is an arithmetic variety.
3. The Bergmann metric on $\tilde{X}^n$

Let $M$ be a complex manifold of dimension $n$, and let
\[ \mathcal{H}(M) = \{ \omega \in \Gamma(M, \mathcal{L}^2_m) \mid |\int_M \omega \wedge \overline{\omega} | < \infty \} \]
Then $\mathcal{H}(M)$ is a separable Hilbert space with inner product
\[ (\omega, \overline{\omega}) = i^{\frac{n^2}{2}} \int_M \omega \wedge \overline{\omega}. \]
Let $\omega_1, \omega_2, \ldots$ be an orthonormal basis for $\mathcal{H}(M)$, and let
\[ \mu_M = \sum \omega_i \wedge \overline{\omega}_i. \]
This is a non-negative $C^\infty$ 2n-form on $M$, called the Bergmann volume form. When it has no zeros it is a volume element. Note that
\[ \int_M \mu_M = \text{dim} \mathcal{H}(M). \]

**Proposition 3.1** (a) Let $m \in M$; then
\[ \mu_M(m) = \sup_{\omega \in \mathcal{H}(M) \atop (\omega, \omega) = 1} (\omega, \overline{\omega})(m) \]
(b) All automorphisms of $M$ leave $\mu_M$ invariant.
(c) If $M'$ is a connected closed submanifold of $M$, then $\mu_M|_{M'} = c \mu_{M'}$ where $c$ is a function on $M'$, $0 \leq c \leq 1$; moreover, if $M - M'$ is a complex analytic subvariety of dimension $\leq n-1$, then $\mathcal{H}(M) \rightarrow \mathcal{H}(M')$ is bijective and so $\mu_M|_{M'} = \mu_{M'}$.
(a) If $M_1$ and $M_2$ are complex manifolds of dimensions $n_1$ and $n_2$, then
\[ \mu_{M_1} \wedge \mu_{M_2} = (-1)^{n_1 n_2} \mu_{M_1 \wedge M_2}. \]

**Proof:** [Kobayashi, 2.1, 2.2, 2.3, 2.4, 2.5]

Consider the condition:

(3.2) For every $m \in M$, there exists an $\omega \in \mathcal{H}(M)$ such that $\omega(m) \neq 0$. 

Trans. AMS 92 (1959)
When this condition holds, $\mu_m$ is a volume element, and we let $h_m = \text{Ric}(\mu_m)$ be the associated Hermitian tensor.

Consider the condition:

\[ (3.3) \text{ for all } m \in M \text{ and } Z \in T_m(M), \text{ there exists an } \omega = f d\zeta_1 \wedge \cdots \wedge d\zeta_p \in \mathcal{H}(M) \text{ such that } f(z) = 0, \ Z(f) \neq 0. \]

**Proposition 3.6.** The form $h_m$ is invariant under all automorphisms of $M$ and is positive semi-definite. It is positive definite if and only if (3.3) holds.

**Proof.** [Ibid., 3.5]

**Remark 3.5.** Let $\mathcal{P}(\mathcal{H}(M)^\vee)$ be the (possibly infinite dimensional) projective space of lines in the dual Hilbert space to $\mathcal{H}(M)$, and assume that $M$ satisfies (3.2). Then there is a canonical map $j : M \to \mathcal{P}(\mathcal{H}(M)^\vee)$ such that, if $\omega$ is nowhere zero on $U \subset M$, then $j(m), m \in U,$ is the class of the line $\omega \to (\omega/\omega_0)(m)$. It is possible to regard $\mathcal{P}(\mathcal{H}(M)^\vee)$ as an infinite dimensional complex manifold, and the usual construction in the finite dimensional case generalizes to give a complete Kähler metric on $\mathcal{P}(\mathcal{H}(M)^\vee)$. The Bergmann Hermitian form $h_m^\prime$, on $M$ is the inverse image under $j$ of this canonical metric on $\mathcal{P}(\mathcal{H}(M)^\vee)$. The map $j$ is an immersion, and $h_m$ is a metric, if and only if (3.3) holds. [See Kobayashi, §7.85 for this.]

**Remark 3.6.** Let $X$ be an arithmetic variety and assume there exists a family $(X_\infty \to X)$ as in (2.6). Then it follows from (2.6) and (2.12) that $\delta^1 X$ is an arithmetic variety if
the Bergman form on $\mathcal{X}^\circ$ is a metric; conversely, if $\mathcal{X}$ is an arithmetic variety, then $\mathcal{X}^\circ$ is a bounded domain and so Bergman's original theorem says that it has a nondegenerate Bergman metric.

The main result of this section is a first step towards proving that $\mathcal{X}^\circ$ has nondegenerate Bergman metric.

**Theorem 3.7** With the notations of §1, $H(\mathcal{X}^\circ) \neq \emptyset$.

**Remark 3.8** Let $M$ be a compactifiable complex manifold. For all $\varepsilon > 0$, there exists an open subset $U_\varepsilon$ in $M$ with compact complement such that, for any étale covering $\varphi : N \to M$, $|\int_{\varphi^{-1}(U_\varepsilon)} \mu_N| < \varepsilon$ deg $\varphi$ where $\mu_N$ is the Bergmann volume form on $N$.

The proof is based on the following elementary result.

**Lemma 3.9.** Let $\varphi_m : D^\times(1) \to D^\times(1)$ be the map $z \mapsto z^m$. For all $\varepsilon > 0$, there exists a $\delta$ such that

$$|\int_{\varphi_m^{-1}(D^\times(\delta))} \mu_{D^\times(1)}| < \varepsilon \text{ for all } m.$$

**Proof:** The Bergman metric on $D(1)$ is the Poincaré metric, and so from (6.1c) and (2.9) we find that

$$|\int_{\varphi_m^{-1}(D^\times(\delta))} \mu_{D^\times(1)}| = \int_{D^\times(\delta)^m} \frac{dz \wedge d\overline{z}}{(1-|z|^2)^2} = \frac{2}{(1/\delta)^{2m} - 1}$$

As $\log x$ for all $x > 1$, we see that
\[
\frac{2}{(1/8)^{2/m}} < \frac{2}{\log (1/8)^{2/m}} = \frac{m}{\log (1/8)}
\]
from which the sublemma is obvious.

Remark 3.10. Note that the notation in this section conflicts with that in (2.10) — \( \frac{i}{\pi} \frac{dz \wedge \bar{z}}{1z^{1/k}(\log |z|)^{2/k}} \) is not the Bergmann volume element on \( D^*(1) \). Denote this element by \( V_D^*(1) \). Then Allfor's lemma applied on the covering space \( D'(1) \) of \( D^*(1) \) shows that \( V_D^*(1) \geq V_{D'}^*(1) \) (ch. 2.11). Thus, we find again that
\[
\int_{D'(1)} V_{D'}^*(1) = \int_{D^*(1)} V_D^*(1) = \frac{1}{\log (1/8)^m} = \frac{m}{\log (1/8)}.
\]

Lemma 3.11. Let \( U = D(1)^n \) and \( U' = D(1)^r \times D^*(1)^{n-r} \). For all \( \varepsilon > 0 \), there exists an open neighbourhood \( V \) of \( U - U' \) on \( U \) such that for any étale covering \( \varphi : N \to U' \), \( \varphi^{-1}(V) \cap N \) has measure \( \varepsilon \) deg \( \varphi \).

Proof. The fundamental group of \( U' \) is \( \mathbb{Z}^{n-r} \), and so every étale covering of it is of the form
\[
N = D(1)^r \times D^*(1)^{n-r} \to D(1)^r \times D^*(1)^{n-r}, (z_1, \ldots, z_r, z_{r+1}, \ldots) \to (z_1, \ldots, z_r, z_{r+1}, \ldots).
\]

The proof is by means of visualization. Consider, for example, the case \( n=2, U' = D^*(1)^2 \). Then
\[
\begin{align*}
U - U' & \quad V \quad D(1)^2 \times D(1)^2
\end{align*}
\]

(0) Look at \( V \cap D(1)^2 \times D(1)^2 \) — apply (3.9)
(1) Each other connected component \( V_0 \) of \( V \) is simply connected. Thus
\[
\varphi^{-1}(V_0) = \text{disjoint union of degree(\varphi)} \text{ cover of } V_0
\]
Hence
\[
\int_{
\varphi^{-1}(V_0)} \mu_N \leq \int_{
\varphi^{-1}(V_0)} \mu_{(\varphi(V))} = \text{deg}(\varphi) \int_{V_0} \mu_{V_0},
\]
and so one only has to arrange things so that \( \int_{V_0} \mu_{V_0} < \varepsilon/100 \). (12.11, 12.41, 12.60)
Proof of 3.8. Embed $M$ in a contractible manifold $\tilde{M}$ in such a way that $\tilde{M} - M$ is a divisor with normal crossings. Then there is a finite family of open subsets $(U_i)_{i \in S}$ of $\tilde{M}$ such that $\tilde{M} - M \subset U \cup U_i$ and, for each $i$, the pair $(U_i, U_i \cap M)$ is isomorphic to $(D(1)^n, D(1) \times D^*(1)^{n-r})$. For each $i$, choose a neighbourhood $V_i$ of $U_i - U_i \cap M$, as in the sublemma. Then the complement of $U \equiv U \cup V_i$ in $\tilde{M}$ is contractible, and it is contained in $M$. Moreover, for any étale covering $\varphi : N \to M$,

$$\left| \sum_{\varphi^{-1}(U)} M_N \right| \leq \sum_{i=1}^{S} \left| \sum_{\varphi^{-1}(V_i)} M_N \right| \leq s \left( \frac{\varepsilon}{s} \deg \varphi \right) = \varepsilon \deg \varphi.$$

The next result gives a criterion for showing $H(\tilde{M})$ non-zero.

Proposition 3.12. Let $M$ be a contractible manifold of dimension $n$, and let $\rho : \tilde{M} \to M$ be an infinite Galois covering with Galois group $\Gamma$. Assume there is a sequence of normal subgroups of finite index in $\Gamma$,

$$\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \cdots > \Gamma_i > \cdots > \Gamma_{13}$$

such that $\cap \Gamma_i = \Gamma_{13}$. Let $M_i = \Gamma_i \setminus \tilde{M}$ and let $h_i = \dim H(M_i)$. Then

(a) $h_i < \infty$;

(b) $\{ h_i / (\Gamma : \Gamma_i) \}$ is bounded;

(c) if the sequence $h_i / (\Gamma : \Gamma_i)$ does not tend to zero, then $H(\tilde{M}) \neq 0$.

Remark 3.13. In general, if $\varphi : Y \to X$ is an étale covering,
one can show that
\[(\deg \varphi) \varphi^*(\mu_X) \geq \mu_Y \geq (\deg \varphi) \varphi^*(\mu_X)\].

Hence
\[(\deg \varphi)^2 \sum_x \mu_x \geq \sum_y \mu_y \geq \sum_x \mu_x\]

For example, if \(Y\) is a disjoint union of copies of \(X\), then \(\mu_Y = \varphi^*(\mu_X)\) and \(\sum_y \mu_y = \deg \varphi \sum_x \mu_x\).

In the situation of the proposition, one has the trivial estimate
\[(\Gamma : \Gamma_i) h_0 \geq \frac{h_i}{(\Gamma : \Gamma_i)} \geq \frac{h_0}{(\Gamma : \Gamma_i)},\]

with \(h_i = (\Gamma : \Gamma_i)\), so the case that \(\tilde{M}\) is a trivial covering of \(M\).

Proof of 3.12. As the Bergman volume form \(\mu_M\) on \(M\) is \(\Gamma\)-invariant, it induces a form \(\mu_i\) on \(M_i\) such that \(\varphi_i^* \mu_i = \mu_{M_i}\), where \(\varphi_i\) is the covering map \(M_i \to M\). Clearly,
\[\sum_{M_i} \mu_i = \sum_{(\Gamma : \Gamma_i)} \mu_{M_i} = h_i / (\Gamma : \Gamma_i).\]

According to (3.8), there exists an open subset \(U = U^c\) with compact complement, \(C = M - U\), such that
\[|S_{\varphi^{-1}(U)} \mu_N| < \deg \varphi\]

for any finite covering \(\varphi : N \to M\). Because \(C\) is compact, there exists a finite set \((U_r)_{r \in \mathbb{R}}\) of open subsets of \(M\) such that (a) there exist isomorphisms \(\varphi_r : U_r \cong B(1)^r\) (Hilbert manifolds);
(b) \(\bigcup_{r=1}^3 \varphi_r^{-1}(B(1)^r) = C\).

Because \(\varphi_r(U_r)\) is a disjoint union of copies of \(U_r\), we have
\[\varphi_i(U_r) = M_i^{-1}(U_r) \geq M_{M_i} \quad \text{and} \quad \mu_i(U_r) \geq \mu_i(U_{1/2} R).\]
\[ \mu_{ir} = \varphi_r^* (\mu_{r0}^n) \quad (\text{see 3.1d}), \] We conclude that
\[ \sum C \mu_i = s \sum \varphi_r^* (\mu_{r0}^n) = s (\sum (\mu_{r0}^n))^n = B < \infty, \]
and that
\[ h_i / (\Gamma : \Gamma_i) = \sum M \mu_i = \sum C \mu_i + \sum A \mu_i \leq B + 1. \]

Then prove both (a) and (b).

Now assume that the sequence \( h_i / (\Gamma : \Gamma_i) \) does not tend to zero.

By passing to a subsequence, we can in fact assume that
for some \( a > 0 \), \( h_i / (\Gamma : \Gamma_i) \geq a \) all \( i \). Choose \( U \subseteq M \) as in (3.8) with \( \varepsilon = a/12 \), and let \( C = M - U \); then
\[ \sum C \mu_i = \sum M \mu_i - \sum A \mu_i = h_i / (\Gamma : \Gamma_i) - \sum \varphi_r^n (U) \mu_i \geq a - \varepsilon = a. \]

Let \( \nu = \sum \varphi_r^* (\mu_{r0}^n) \). We showed above that, on \( U \),
\[ \mu_i \leq \mu_{ir} = \varphi_r^* (\mu_{r0}^n), \quad \text{and so on} \ C, \mu_i \leq \nu. \]

**Lemma 3.14.** There exists an \( x_0 \in C \) such that a subsequence of
\( (\mu_i / \nu)(x_0) \) converges to some \( b > 0 \).

**Proof:** Suppose \( \sup_{x \in C} (\mu_i / \nu)(x) \to 0 \); then \( \sum C \mu_i \to 0 \), which contradicts the assertion above that \( \sum C \mu_i = a/12 \). Hence
the sequence \( \sup_{x \in C} (\mu_i / \nu)(x) \) does not tend to zero, and so we can choose a subsequence of \( i \) so that \( \sup_{x \in C} (\mu_i / \nu)(x) \to b > 0 \).

Choose \( x_i \) such that \( |(\mu_i / \nu)(x_i) - \sup_{x \in C} (\mu_i / \nu)(x)| < 1/2i \).
Then \( (\mu_i / \nu)(x_i) \to b \). Now take a subsequence of the \( x_i \) converging to a limit \( x_0 \). Then clearly, \( (\mu_i / \nu)(x_0) \to b. \)

**Lemma 3.15.** Let \( x_0 \in M \). Then there exists a sequence of open neighbourhoods
... \subset N_i \subset N_{i+1} \subset \cdots$$ of \vec{x}_0$ such that $$\tilde{M} = \cup N_i$$ and the restriction of $$p_i : \tilde{M} \rightarrow M_i$$ to $N_i$ is an isomorphism of $N_i$ with a dense open subset of $M_i$.  

**Proof:** Choose a Riemannian metric $\bar{\rho}$ on $M$, and let $\tilde{\rho}$ be the metric it induces on $\tilde{\mathbb{R}}^n$. Define $$N_i = \{ \tilde{x} \in \tilde{M} \mid \bar{\rho}(\tilde{x}, \tilde{x}_0) < \bar{\rho}(\tilde{y}_0, \tilde{x}), \forall \tilde{y} \in \tilde{N}_i, \tilde{y} \neq \tilde{x} \}$$

Check that these sets have the right properties (cf. Kaz I, p.167).

We now prove (c) of the proposition. Let $x_0$ be as in (3.4), and let $\tilde{x}_0 \in \tilde{M}$ with $\bar{\rho}$ it. Then, from (3.1a) we know that there exists an $i \in \ell(M_i)$ such that $(\omega_i, \omega_i) = 1$ and $$\langle \omega_i, \omega_i \rangle (p_i(x_0)) \geq \frac{1}{2} \mu_{M_i}(p_i(x_0)) \quad (3.15.1)$$

**Lemma 3.16.** There exists a compact neighbourhood $\tilde{U}$ of $\tilde{x}_0$ such that for some $c > 0$, $$S_{p_i(U)} \omega_i \cdot \omega_i > c, \quad \forall i.$$  

**Proof:** Obvious from (3.14) and (3.15.1). (Cf. Kaz II p.153)

Now let $\tilde{\mathcal{E}}$ be the Hellwardt space of measurable square-integrable sections $\gamma$ of $\tilde{S} \tilde{M}$. For each $i$, define $\eta_i \in \tilde{\mathcal{E}}$ by

$$\left\{ \begin{array}{l} \eta_i | N_i = p_i^*(\omega_i) \\ \eta_i | \tilde{M} - N_i = 0 \end{array} \right.$$  

As $(\eta_i | \eta_i) = 1$, there exists a weakly convergent subsequence of the $\eta_i$ tending to $\eta \in \tilde{\mathcal{E}}$. Now $\eta$ is holomorphic on $\cup N_i = \tilde{M}$, and $$\int_{\tilde{M}} |\eta|^2 < 1/2,$$ and so $\eta \neq 0$. Thus $\tilde{\mathcal{E}}(\tilde{M}) \neq 0$.  


Conjecture 3.17. Let $\Gamma$ be a semisimple real Lie group, $\Gamma$ an arithmetic subgroup of $\Gamma$, and

$$
\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_i > \cdots \rightarrow \Gamma_{\infty}
$$

a sequence of normal subgroups of $\Gamma$ of finite index such that

$$\bigcap_{i=1}^{\infty} \Gamma_i = \Gamma_{\infty}.$$ 

Let $\rho_i$ be an irreducible cuspidal representation of $\Gamma_i$ and define $h_i(W) = \dim \left( \text{Hom}_{\Gamma_i} (W, L^2 (\Gamma_i \backslash \Gamma) ) \right)$. Then $h_i(W)$ does not tend to zero as $i \rightarrow \infty$.

Kaz II

In [5, p156], Kazhdan calls this "Theorem A", which we will prove in another paper." At present the statement still seems to be unproven, but Chooz has recently shown the following weaker result which is sufficient for our purposes: let $\Gamma$ be a simply connected semisimple algebraic group over $\mathbb{Q}_p$, and let $p_0$ be a prime such that $G(\mathbb{Q}_{p_0})$ has a supercuspidal representation $\pi_{p_0}$ (e.g., take $p_0$ to be any prime such that $G$ is split over $\mathbb{Q}_{p_0}$). Let $K_{p_0}$ be a compact open subgroup of $G(\mathbb{Q}_{p_0})$ such that $L(\pi_{p_0}, K_{p_0}) \neq 0$; let $K_0 > K_{p_0}$ be a sequence of compact open subgroups of $G(\mathbb{Q}_p)$ such that

1) $(K_i)_{p_0} = K_{p_0};$

2) there exists a finite set $S$ of primes such that $(K_i)_p$ is maximal for $p \in S;$

3) $\bigcap_{i=1}^{\infty} K_i = \{1\};$

then for any irreducible cuspidal representation $W$ of $G(\mathbb{R})$, there exists a constant $\alpha > 0$ such that

$$
\frac{h_i(W)}{(\rho_0 : \rho_i)} > \alpha, \quad \rho_i = G(\mathbb{Q}) \cap K_i
$$

Here

$\rho_0$ Hecke alg.
Now return to the situation in the statement of (3.7). The $W \overset{df}{=} \tilde{H}(\tilde{X})$ is in a natural way, a cuspidal representation of $G(\mathbb{R})$.

Lemma 3.19. There is an isomorphism

$$\text{Hom}_{G(\mathbb{R})}(W, L^2(\Gamma \backslash X)) \cong \tilde{H}(X).$$


Lemma 3.20. The dimensions of $\tilde{H}(X)$ and $\tilde{H}(\sigma X)$ are equal.

Proof. Let $\tilde{X}$ be a smooth variety containing $X$ as a dense open subvariety. Then (3.12) shows that $\tilde{H}(X) \overset{\cong}{\rightarrow} \tilde{H}(\tilde{X})$, and $\tilde{H}(\tilde{X}) = \Gamma(\tilde{X}, L^2(\tilde{X}, \mathbb{R})_0)$. The lemma is now obvious.

We are now ready to prove (3.7). Choose a family $K_i$ satisfying the conditions of Clozel's theorem. Then the theorem and (3.19) show that for some $a > 0$, dist $\tilde{H}(X_i) / (\Gamma_0 : \Gamma_i) > a$ all $i$, where $X_i = \Gamma(K_i) \backslash \tilde{X}$. Now (3.20) shows that dist $\tilde{H}(\sigma X_i) / (\Gamma_0^\sigma : \Gamma_i^\sigma) > a$, where $\Gamma_i = \Gamma^\sigma(K_i)$, and (3.12) implies that $\tilde{H}(\tilde{X}) \neq 0$. 

4. Subbundles of $T(\widetilde{X})$

Let $G$ be a simply connected reductive algebraic group over $\mathbb{Q}$, and let $\widetilde{X}$ be a symmetric Hermitian domain on which $G(\mathbb{R})$ acts in such a way that $\widetilde{X} = G(\mathbb{R})/(\text{maximal compact subgroup})$.

Let
\[
\widetilde{X} = X_1 \times \cdots \times X_s, \quad X_i \text{ irreducible symmetric Hermitian domain}
\]

Choose a point $x = (x_1, \ldots, x_s) \in \widetilde{X}$. Then
\[
G_{1x} = G_1 \times \cdots \times G_s \times K_x, \quad G_i \text{ noncompact}, \quad G_i(\mathbb{R})x = G_i(\mathbb{R})x_i = X_i.
\]

Let
\[
X_i' = \{x_1, \ldots, x_i \times \cdots \times x_s \} \subset \widetilde{X}.
\]
Then $T(X_i') \subset T(\widetilde{X})$, and $T(\widetilde{X}) = \bigoplus_{i=1}^{s} T(X_i')$.

For $I \subset \{1, \ldots, s\}$, define
\[
T^I(\widetilde{X}) = \bigoplus_{i \in I} T(X_i') = \{v \in T(\widetilde{X}) \mid v \in \bigoplus T_x(X_i')\}.
\]
Then $T^I(\widetilde{X})$ is a subbundle of $T(\widetilde{X})$, stable under $G(\mathbb{R})$.

**Lemma 4.1.** Every $G(\mathbb{Q})$-stable complex subbundle of $T(\widetilde{X})$ is of the form $T^I(\widetilde{X})$ for some $I \subset \{1, \ldots, s\}$.

**Proof.** Let $W$ be such a subbundle. As $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ (real approximation theorem) and $G(\mathbb{R})$ acts transitively on $\widetilde{X}$, it will suffice to show that $W_x = T^I(\widetilde{X})x$, some $I$, for a fixed $x \in \widetilde{X}$. Let $K_x$ be the isotropy group of $x$. Then $W_x$ is a $K_x$-stable subbundle of $T(\widetilde{X})$.

With the usual notations, let
\[
\text{Lie}(G) = \mathfrak{g} = \mathbb{R} + \mathfrak{g'}, \quad \mathfrak{r} = \text{Lie}(K_x), \quad \gamma = T_x(\widetilde{X})
\]
\[
\text{Lie}(G_i) = \mathfrak{g}_i = \mathfrak{r}_i + \mathfrak{z}_i, \quad \gamma_i = T_{x_i}(\widetilde{X}_i)
\]
Then $\mathfrak{r} = \text{Lie}(K_0) \oplus \mathfrak{r} \oplus \cdots \oplus \mathfrak{r}_s$ and $\gamma = \bigoplus \gamma_i$. Note that
\( W_x \) is a \( \mathbb{R} \)-stable subspace of \( x \). Almost by definition of what it means for \( x \) to be irreducible [Helgason, VIII.5, p.377], the action of \( \mathbb{R} \) on \( x \) is irreducible. Therefore \( W_x \cong \oplus_{i \in I} x_i \), some \( I \).

**Lemma 4.2.** Let \( G \) be a simply connected, almost simple, algebraic group over a number field \( F \). Let \( S_\infty \) be the set of infinite places of \( F \), and let \( v \in S_\infty \) be such that \( G(F_v) \) is not compact. Then for any congruence group \( \Gamma \subset G(F) \), \( G(F_v) \Gamma \) is dense in \( \prod_{v \in S_\infty} G(F_v) \).

**Proof:** For some compact open subgroup \( K \) of \( G(A_F^e) \), \( \Gamma = K \cap G(F) \). The strong approximation theorem shows that \( G(F_v) \cap G(F) \) is dense in \( G(A_F^e) = \prod_{v \in S_\infty} G(F_v) \times G(A_F^e) \) and so, for any open subset \( U \) of \( \prod_{v \in S_\infty} G(F_v) \), there exist elements \( \alpha \in G(F_v) \) and \( \beta \in G(F) \) such that \( \alpha \beta \in U \times K \). Clearly \( \beta \in \Gamma \), and \( \alpha \beta \in U \); thus \( G(F_v) \Gamma \) is dense in \( \prod_{v \in S_\infty} G(F_v) \).

**Remark 4.3.** Let \( G' = \text{Res}_{F/\mathbb{Q}} G \) with \( G \) and \( F \) as in the Lemma. Then the conclusion of the lemma can be restated as follows: let \( G' \) be a noncompact factor of the Lie group \( G'(\mathbb{R}) \), and let \( \Gamma \) be a congruence subgroup of \( G'(\mathbb{Q}) \); then \( G' \Gamma \) is dense in \( G'(\mathbb{R}) \).

Since \( T^2(\mathbb{R}) \) is stable under \( \Gamma \), it defines a subbundle \( T^2(X) \) of \( T(X) \). By construction, \( T^2(\mathbb{R}) \) is involutive, and so \( T^2(X) \) is an involutive subbundle of \( T(X) \).
Proposition 4.4. Assume \( \mathcal{G} \) is almost simple over \( \mathcal{Q} \). If the foliation defined by \( T^I(X) \) on \( X \) has a closed leaf, then 
\[ I = \emptyset \quad \text{or} \quad I = \{i, \ldots, s\} \]

Proof: Let \( Z \) be the closed leaf, and let \( \bar{Z} = \pi^{-1}(Z) \). Then \( \bar{Z} \) is a closed submanifold of \( \bar{X} \) stable under \( \pi \) and all \( G_i(\mathbb{R}) \) for \( i \in I \). If \( I \neq \emptyset \), then (4.8) shows that \( \pi^*G_i(\mathbb{R}) \) is dense in \( G(\mathbb{R}) \), which acts transitively on \( \bar{X} \). Therefore \( \bar{Z} = \bar{X} \) and \( Z = X \), whence \( T^I(X) = T(X) \).
5 Completion of the proof

Let $X, \tilde{X}, G, \ldots$ be as in §1 and assume $\text{codim} (\partial X) \geq 3$. Recall the notations,

\[
\begin{align*}
\tilde{X} & \to \hat{X}^o \to \hat{X} \\
\sigma \hat{X} & \to (\sigma \hat{X})^o \to \sigma \hat{X} \\
X_K & \to X_K \to X_K = X \\
(\sigma X)^o & \to \sigma X
\end{align*}
\]

Let $\mu$ be the Bergmann volume form on $\hat{X}^o$, and let $Z_0$ be the set on which $\mu$ is zero. As we saw in §3, $Z_0$ is a proper subset of $\hat{X}^o$. Clearly it is a complex analytic subset, $G$-invariant. The complement $\bar{U}_0$ of $Z_0$ in $\hat{X}^o$ is also $G$-invariant, and (see 3.5) there is a map

\[\tilde{\sigma} : \bar{U}_0 \to \text{IP}(\mathcal{H}(\tilde{X}^o)^\vee)\]

such that the Bergmann Hermitian form on $\bar{U}_0$ is the inverse image of the canonical metric on $\text{IP}(\mathcal{H}(\tilde{X}^o)^\vee)$. Note that $\tilde{\sigma}$ acts on $\text{IP}(\mathcal{H}(\tilde{X}^o)^\vee)$ through its action on $\tilde{X}^o$, and that $\tilde{\sigma}$ is $G$-equivariant map. Define

\[\tilde{Z}_1 = \{ z \in \bar{U}_0 | \text{rank} (d\tilde{\sigma})_z < \max_x \text{rank} (d\tilde{\gamma})_x \} \]

and $\tilde{Z} = \bar{Z}_0 \cup \tilde{Z}_1$ — this is again a $G$-invariant complex analytic subset of $\tilde{X}^o$ not equal to $\tilde{X}_J$, and so (1.9) shows that $\text{codim} (\tilde{Z}) > \text{codim} (\partial X) \geq 3$. The complement $\tilde{U} = \tilde{X}^o - \tilde{Z}$ of $\tilde{Z}$ is also $G$-invariant. We have a diagram

\[T(\tilde{U}) \to \text{IP}(\tilde{U}) \to T(\text{IP}) \]

Define

\[\tilde{W} = \text{Ker} ( T(\tilde{U}) \to \gamma^* T(\text{IP})) \]

Because it has constant rank, $\tilde{W}$ is a subbundle of $T(\tilde{U})$. It is $G$-invariant.
As in \( \delta \), let \( \tilde{X} = X_t \times \cdots \times X_s \), and, for each \( I \subseteq \{1, \ldots, s\} \), define \( T^I(\tilde{X}) \subseteq T(X) \) and \( T^I(X) \subseteq T(X) \). Recall the following result: Let \( \tilde{V} \) be a complete algebraic variety over \( \mathbb{C} \), and let \( V \subset \tilde{V} \) be a nonsingular subvariety such that \( \tilde{V} - V \) has codimension \( \geq 3 \); then \( F \mapsto F^v \) induces an equivalence between the category of coherent locally free algebraic sheaves on \( \tilde{V} \) and that of coherent locally free analytic sheaves on \( V^an \) (see, for example, [Hartshorne, p222-223]). Thus \( T^I(X) \) is an algebraic subbundle of \( T(X) \), and \( \sigma T^I(X) \subseteq \sigma T(X) = T(\sigma X) \) is defined. Let \( T^I(\tilde{X}_c) \) equal to the inverse image of \( \sigma T^I(X) \) in \( \tilde{X}_c \).

**Lemma 5.1.** For some \( I \subseteq \{1, 2, \ldots, s\} \), \( \tilde{W} = T^I(\tilde{X}_c) \cap \tilde{U} \).

**Proof:** Since both \( \tilde{U} \) and \( \tilde{W} \) are \( \sigma \)-invariant and \( \sigma^c \in \mathbb{G} \), we can pass to the quotient and obtain \( U = \tilde{U} \cap \sigma \mathcal{O}X \) and \( \tilde{W} = \tilde{X} \cap \sigma \tilde{W} \) a subbundle of \( T(U) \). As codim \( (\sigma X - U) \geq 3 \), the result recalled above shows that \( \tilde{W} \) is an algebraic subbundle of \( T(U) \). Let \( j \) be the inclusion \( U \hookrightarrow \sigma X \). Regard \( \tilde{W} \) as a sheaf on \( U \), and form \( j^* \tilde{W} \) — this is a coherent algebraic sheaf on \( \sigma X \). Let \( Y \) be the subset of \( \sigma X \) where \( j^* \tilde{W} \) is not locally free. Then \( Y \) is an algebraic subset of \( \sigma X \) and its inverse image on \( \tilde{X}_c \) is \( \sigma \)-invariant (as the set where \( j, \tilde{W} \) is not locally free). Therefore (1.5) shows that \( Y \) is empty, and so \( j^* \tilde{W} \) is locally free. A similar argument applied to the support of the kernel of \( j_* \tilde{W} \rightarrow T(\sigma X) \) shows that \( j_* \tilde{W} \) is a locally free subsheaf of \( T(\sigma X) \). The fact recalled above shows that it
is algebraic. Similar arguments apply to each variety in the projective system \( \sigma \hat{X} \), and so we obtain an algebraic subbundle \( \hat{W} \subset T(\hat{X}) \) invariant under \( \sigma \), and therefore under \( G(A^\lambda) \).

Hence \( \sigma^{-1} \hat{W} \) is a subbundle of \( T(\hat{X}) \) invariant under \( G(A^\lambda) \), and \((\sigma^{-1} \hat{W})^{\sigma} \mid \hat{X} = G(1^n)\)-invariant. Now (4.1) shows that \((\sigma^{-1} \hat{W})^{\sigma} \mid \hat{X} = T^I(\hat{X})\) for some \( I \) and, because \( G(A^\lambda) \) acts transitively on the connected components of \( \hat{X}^{\sigma} \), this implies that \( \sigma^{-1} \hat{W} = T^I(\hat{X}) \). This completes the proof.

The condition that a subbundle of the tangent bundle to an algebraic variety be involutive is algebraic. Thus \( T^I(\hat{X}^\sigma) \) is involutive.

**Lemma 5.2**: The foliation of \( U = \Gamma^{\sigma} \setminus \tilde{U} \) defined by \( W = \Gamma^{\sigma} \setminus \tilde{W} \) has a closed leaf.

**Proof**: The reader is invited to check for himself the proof of Kazhdan (Kaz II, p. 159-158; essentially, the leaves of the foliation are the equivalence classes for the relation defined on p. 153). Alternatively, it may be possible to give a proof along the following lines: as we observed above, there is a \( G \)-equivariant map \( \tilde{U} \to \text{IP}(\text{SL}(\hat{X}^\sigma)^{\lambda}) \); if we can pass to the quotient by \( \Gamma^{\sigma} \), we get a map \( U = \Gamma^{\sigma} \setminus \tilde{U} \to \Gamma^{\sigma} \setminus \text{IP}(\text{SL}(\hat{X}^\sigma)^{\lambda}) \) whose fibres are the leaves of the foliation.
We now assume also the condition (0.26). There $G = \text{Res}_{\mathbb{F}/\mathbb{Q}} G'$ with $G'$ absolutely simple and $\mathbb{F}$ totally real, and for some special point $\tilde{x} \in \tilde{X}$, the maximal torus $T \subset G$ fixing $\tilde{x}$ is of the form $\text{Res}_{\mathbb{F}/\mathbb{Q}} T'$ where $T'$ splits over a quadratic imaginary extension $L$ of $\mathbb{F}$.

As $T_L$ is split, we can write
\[ g_L^* = t_L^* \oplus \bigoplus_{\alpha \in R} (g_L^*) \alpha, \quad g_L^* = \text{Lie}(G_L^*), \quad t_L^* = \text{Lie}(T_L^*), \]
where $R$ is the set of roots $R(G_L^*, T_L^*)$. For each $\alpha \in R$, let
\[ b_\alpha^* = t_L^* \oplus (g_L^*) \alpha \oplus (g_L^*) \alpha. \]

This is defined over $\mathbb{F}$, and we let $H_{\mathbb{F}}'$ be the corresponding connected subgroup of $G'$. Let $H_\mathbb{Q} = \text{Res}_{\mathbb{F}/\mathbb{Q}} H_{\mathbb{F}}'$ — it is a reductive group of type $A_1$ and $H_{\mathbb{Q}}$ is $\mathbb{Q}$-simple.

Let $\tilde{X}_\alpha$ be the orbit of $\tilde{x}$ under the action of $H_\mathbb{Q}(\mathbb{Q})$. Then $\tilde{X}_\alpha$ is a Hermitian symmetric domain (possibly consisting of one element) with $\text{Aut}(\tilde{X}_\alpha)^+ = (H_\mathbb{Q}(\mathbb{Q})/(\text{max'l compact normal subgroup}))^+$.

It follows from [Deligne, p.1153] that the projective system $A$ arithmetic varieties $\tilde{X}_\alpha$ embeds into $\hat{X}$. We assume Theorem 0.1 for the families $\tilde{X}_\alpha$; in particular $\tilde{X}_\alpha$ is the family associated with a $\Phi$-group $\Phi_\alpha$.

**Lemma 5.3.** The tangent space $T_{\tilde{x}}(\tilde{X})$ is generated by the subspaces $T_{\tilde{x}}(\tilde{X}_\alpha)$. (Note, $\tilde{x}$ is as defined above.)

**Proof:** Easy.

**Lemma 5.4.** Let $\tilde{x}^\sigma$ be any point of $\tilde{X}^\sigma$ such that $\tilde{x}^\sigma = g(\sigma \tilde{x})$ for some $g \in G(\mathbb{A}^\sigma)$, where $\tilde{x}$ is the image of $\tilde{x}$ (see above) in $\tilde{X}$. Then $\tilde{x}^\sigma \in \tilde{U}$.

**Proof:** Otherwise $\tilde{Z} = g(\tilde{x}^\sigma) \subset H^\sigma(\mathbb{Q})\tilde{x}^\sigma$, and so $\tilde{Z} = \tilde{X}^\sigma_{\alpha}$ for all $\alpha$. This implies that $\dim \tilde{Z} = \dim \tilde{x}^\sigma$, which contradicts
an earlier statement.

Let \( J = \{ \sigma : F \rightarrow 1R | G_\sigma \text{ is not compact} \} \), where \( G_\sigma = G \otimes E_{\mathfrak{a}, \sigma} 1R \). Then \( J \) indexes the irreducible components of \( \tilde{X}_\sigma \), and we shall use it for this purpose rather than \( \{1, \ldots, n\} \).

Thus \( I \) is now a subset of \( J \). Let \( J_\phi = \{ \sigma : F \rightarrow 1R | H^0_{\phi, \sigma} \text{ is not compact} \} \).

Lemma 5.5. Either \( J_\phi \cap I = \emptyset \) or \( J_\phi \subseteq I \).

Proof. The set \( \tilde{Z} \cap \tilde{X}_\phi \) is stable under \( \xi \cap H^1(\mathcal{M}) = H^1(\Phi)/Z_\phi(\Phi) \) and so is either empty or all of \( \tilde{X}_\phi \). The last lemma shows that it is not equal to \( \tilde{X}_\phi \), and so we have that \( \tilde{X}_\phi \subseteq \tilde{U} \).

From (3.1), (5.2), and 6.4 we know that \( W/\sigma X_\phi \) is 0 or \( T(\sigma X_\phi) \) (recall that we are assuming \( \sigma X_\phi \) is arithmetic). Then \( \sigma^{-1}W/\sigma X_\phi \) is 0 or \( T(X_\phi) \). But (by 6.1), \( \sigma^{-1}W = T(\mathcal{X}) \) for some \( I \subseteq J \). We conclude that \( I = \mathcal{J}_\phi \) (and \( \sigma^{-1}W/\sigma X_\phi = T(X_\phi) \)) or \( I \cap J_\phi = \emptyset \) (and \( \sigma^{-1}W/\sigma X_\phi = 0 \)).

Note that it is not possible for \( I \) to contain all \( J_\phi \), for then \( I = J \) and the Bergmann-Hermitian form is identically zero (cf. p. 51). In the other hand, if \( I \cap J_\phi = \emptyset \) for all \( \phi \), then \( I = \emptyset \), \( W = 0 \), and we can apply (2.13) to complete the proof of (0.1).

Thus it remains to show that the hypothesis that \( I \cap J_\phi = \emptyset \) for some, but not all, \( \phi \) leads to a contradiction.
Lemma 5.6. Let $\frac{g'}{I} = \sum_{j \in I} \frac{g'_j}{J_j}$, then $\frac{g'}{I}$ is a sub-algebra of $G'$.

Proof: Let $G' = \mathbb{K} \oplus Y'$ as usual. Then

$$ I \times I \subseteq \forall x \in I, \quad \frac{g'_j}{J_j} < \mathbb{K} $$

As $\mathbb{K}$ is a subalgebra of $G'$, the result is now obvious.

Let $H'_I$ be the subgroup of $G'$ corresponding to $\frac{g'}{I}$.

Then: for $x \in I$, $H'_I \equiv$ connected ($\Rightarrow$ $H'_I$ reductive)

for $x \in I$, $H'_I ( IR ) \tilde{x}_x = ( \tilde{x}_x )$ for all $x$, and as $H'_I ( IR ) \tilde{x}_x = \tilde{x}_x$ (add [Kobayashi- Nom, I, p 178, 4.5] if this is not obvious).

(We are writing $\tilde{x}_x = ( \ldots , \tilde{x}_x , \ldots )$)

This shows that $H'_I \equiv G'$ (see 2.3), and as $H'_I = G'$.

This is the contradiction that proves the theorem.