Study of an Isogeny Class

James Stuart Milne

Abstract

This is a translation of: Étude d’une class d’isogénie, in Variétés de Shimura et Fonctions L (Ed. L. Breen and J.P. Labesse), Publications Mathématiques de l’Université Paris 7 (1979), 73-81.

It is available at www.jmilne.org/math/.

Notations.

- $G$ is a group scheme over $\mathbb{Z}$ such that $G(R) = (\mathcal{O}_B^{\text{op}} \otimes R)^{\times}$ for any commutative ring $R$;

$V(R)$ is the $\mathcal{O}_B \otimes R$-module $\mathcal{O}_B \otimes R$.

- $A$ is the ring of adèles for $\mathbb{Q}$: $A = \mathbb{R} \times A_f = \mathbb{R} \times \mathbb{A}_f^p \times \mathbb{Q}_p^\times$, where

$A_f = \mathbb{Z}_f \otimes \mathbb{Q}$, $\mathbb{Z}_f = \lim \limits_{\leftarrow n} \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_p^f \times \mathbb{Z}_p$.

- $K$ is a sufficiently small open subgroup of $G(\mathbb{Z}_f)$.

- Two isomorphisms $T \xrightarrow{\varphi} V(\mathbb{Z}_f)$ are $K$-equivalent if there exists a $k \in K$ such that $\varphi = k \circ \varphi'$.

- For an abelian variety $A$, we set $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$, $A_m = \text{Ker}(m: A \to A)$, $A(p) = \lim \limits_{\leftarrow n} A_{p^n}$, $T_f A = \lim \limits_{\leftarrow (n,p)=1} A_n$ (i.e., the inverse system $(A_n)_n$ regarded as an object of the category of inverse systems), $T_f^p A = \lim \limits_{\leftarrow (n,p)=1} A_n$, and $V_f^p A = T_f^p A \otimes_{\mathbb{Z}} \mathbb{Q}$.

- $W$ is the ring of Witt vectors with components in $\overline{\mathbb{F}}_p$ and $W' = W \otimes_{\mathbb{Z}} \mathbb{Q}$; $DN$ is the Dieudonné module of the finite group scheme $N$, $DA = \lim \limits_{\leftarrow n} DA_{p^n}$, and $D'A = DA \otimes \mathbb{Q}$.

- $\Phi$ denotes the absolute Frobenius morphism of the Shimura variety over $\overline{\mathbb{F}}_p$ attached to the group $G$.
Let $E$ be a totally real number field of degree $d$ over $\mathbb{Q}$, $B$ a totally indefinite quaternion algebra over $E$, $\mathcal{O}_B$ a maximal order in $B$, and $p$ a prime number ($E$ is denoted $F$ in [3]). We assume that $p = \prod_{p|p} p$ in $E$ with the $p$ distinct, and that, if $E_p$ denotes the completion of $E$ at the prime $p$, then $B \otimes_E E_p$ is split; moreover, that $\mathcal{K} = \mathcal{K}^p \cdot G(\mathbb{Z}_p)$ where $\mathcal{K}^p = \mathcal{K} \cap G(\mathbb{A}_f^p)$. Fix an abelian variety $A$ over $\overline{\mathbb{F}}_p$ of dimension $2d$ and a homomorphism $i : \mathcal{O}_B \to \text{End}(A)$ such that $i(1) = 1$. We shall describe the set $Y_A$ of all isomorphism classes of triples $(A', \iota', \varphi)$ with $A'$ an abelian variety over $\overline{\mathbb{F}}_p$, $\iota'$ a homomorphism $\mathcal{O}_B \to \text{End}(A')$, and $\varphi$ a $K$-equivalence class of isomorphisms $\varphi : T_f^p A' \to V(\mathbb{Z}_f)$; we require that $(A', \iota')$ be isogenous to $(A, \iota)$ and that the tangent space $\mathfrak{t}_{A'}$ to $A'$ at the origin satisfy the following condition (see Exposé III §2):

\[(*)\] the subspaces of $\mathfrak{t}_{A'}$ defined by the idempotents $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of $\mathcal{O}_B \otimes \overline{\mathbb{F}}_p \cong M_2(\overline{\mathbb{F}}_p)$ are free $\mathcal{O}_E \otimes \overline{\mathbb{F}}_p$-modules of rank $1$.

We have seen in [1] that the set $K^S(\overline{\mathbb{F}}_p)$ of points with values in $\overline{\mathbb{F}}_p$ of the Shimura variety $K^S$ admits a description

$$K^S(\overline{\mathbb{F}}_p) = \bigsqcup A Y_A$$

with $A$ running over the set of isogeny classes of abelian varieties of the type under consideration. We fix from now on such an $A$ and put $Y = Y_A$. Let $\Phi$ denote the restriction to $Y$ of the Frobenius operator on the set $K^S(\overline{\mathbb{F}}_p)$.

According to [1], [3], we should distinguish the following two cases:

(NS) The commutant of $B$ in $\text{End}^0(A)$ is a totally imaginary field $E'$ of degree $2$ over $E$ which splits $B$; $A(p)$ is isogenous to a product $\prod_{p|p} A(p)$ with $A(p)$ a $p$-divisible group of height $2d_p = 2[E_p : \mathbb{Q}_p]$; if $p$ splits in $E'$ into $p = qq'$, then $A(p) \sim A(q) \times A(q')$ where $A(q)$ has slope $m_p/q$ and $A(q')$ has slope $(d_p - m_p')/d_p = m_p''/d_p$; otherwise $A(p)$ has slope $1/2$, and we put $1 m_p'' = d_p/2 = m_p''$.

(S) The commutant of $B$ in $\text{End}^0(A)$ is a quaternion algebra $B'$ over $E$; $A$ is isogenous to a power $A \sim A^*_{\text{sup}}$ of a supersingular elliptic curve $A_0$.

**Lemma 1.** Let $T \subset T_f A$ be such that $T_f A/T$ is finite; then there exists a unique isogeny $\alpha : A' \to A$ such that the image of $T_f \alpha$ is $T$.

**Proof.** Since $T_f A/T$ is finite, the cokernel $N$ of $T/nT \to T_f A/nT_f A$ is independent of $n$ for $n$ sufficiently large. Choose such an $n$, and let $\psi$ be the surjective map $A_n = T_f A/nT_f A \to N$. In order for $\alpha : A' \to A$ to be an isogeny with $T_f \alpha(T_f A') = T$, it is necessary and sufficient that $\text{Ker}(\alpha) = N$ and that $\psi$ be the map $A_n \to N$ defined by the snake lemma starting from the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & N & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow 0 & & \downarrow n & & \downarrow n & & \downarrow & & \downarrow 0 \\
0 & \longrightarrow & N & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0.
\end{array}
$$

The integers $d_p$, $m_p'$, and $m_p''$ are denoted $d_v$, $m_v'$, and $m_v''$ in [1], and will be denoted $d_v$, $m_v'$, and $m_v''$ respectively in Exposé VI where $v$ is the place of $E$ associated with the ideal $p$. 
Recall that for an abelian variety $A$, $\text{Ext}^r(A, \mathbb{G}_m) = 0$ for $r \neq 1$ and $\text{Ext}^1(A, \mathbb{G}_m)$ is the abelian variety $A^\vee$ dual to $A$; moreover, $A^\vee \cong A$. Thus to give $\alpha$ amounts to giving $\alpha^\vee: A^\vee \to A^\vee$ such that $\ker(\alpha^\vee) = N^\vee$ (where $N^\vee$ denotes the Cartier dual of $N$) and $N^\vee \to A^\vee_\text{irr}$ is $\varphi^\vee$. We must take $A^\vee = A^\vee / N^\vee$. 

Since $V_f^p A$ is free of rank one over $B \otimes \mathbb{Z}_p$, $T_f^p A$ contains a lattice isomorphic to $V(\mathbb{Z}_p^\sigma)$, and we can choose the initial pair $(A, i)$ such that there exists an isomorphism $\varphi: T_f^p A \to V(\mathbb{Z}_p^\sigma)$. Let $A(\infty) = \cup_n A_n$. Denote by $V_f A$ the projective system

$$
\lim \lim \quad A(\infty)^{(n)} = \cdots \leftarrow A(\infty)^{(m)} \leftarrow A(\infty)^{(mn)} \leftarrow \cdots
$$

where $A(\infty)^{(n)} = A(\infty)$ for all $n$. We have $T_f A \subset V_f A$. (Over $\mathbb{C}$, $T_f A$ can be identified with $H_1(A, \mathbb{Z}) \otimes \hat{\mathbb{Z}}$ and $V_f A$ with $T_f A \otimes_\mathbb{Z} \mathbb{Q}$). A lattice $\Lambda$ in $V_f A$ is a subobject $\lim \lim \Lambda^{(n)}$ such that

- $m\Lambda^{(mn)} = \Lambda^{(n)}$ for all $m$ and $n$, and
- $m_0\Lambda$ is contained in $T_f A$ for some $m_0$ and defines a finite quotient $T_f A/m_0\Lambda$.

We can write $V_f A = V_f^p A \times V_p A$ with $V_p A = \lim \lim A(p)^{(n)}$, and then a lattice $\Lambda$ decomposes into a product $\Lambda = \Lambda^p \times \Lambda_p$ with $\Lambda^p = \Lambda \cap V_f^p A$ and $\Lambda_p = \Lambda \cap V_p A$. Let $X$ be the set of all pairs $(\Lambda, \psi)$ with $\Lambda$ a lattice in $V_f A$ and $\psi$ a $K$-equivalence class of isomorphisms $\psi: \Lambda^p \to V(\mathbb{Z}_p^\sigma)$ which satisfies the following conditions:

(a) $\Lambda$ is stable under the obvious action of $\mathcal{O}_B$ on $V_f A$;
(b) if $D(\Lambda/p\Lambda)$ is the Dieudonné module of the finite group $\Lambda/p\Lambda$, then $D(\Lambda/p\Lambda)/FD(\Lambda/p\Lambda)$ satisfies the condition $(\ast)$.

When $\alpha$ is an element of $\text{End}^0(A)$ such that $m\alpha \in \text{End}(A)$, we define $V_f \alpha: V_f A \to V_f A$ to be the family of mappings $\left\{ A(\infty)^{(mn)} \overset{m\alpha}{\to} A(\infty)^{(n)} \right\}$. Correspondingly, there is an action of $\text{End}^0(A)$ on $X$ defined by:

$$
\alpha(\Lambda, \psi) = ((V_f \alpha)\Lambda, \psi \circ V_f(\alpha)^{-1}).
$$

**Lemma 2.** There exists a canonical bijection

$$
\text{End}^0(A) \setminus X \to Y.
$$

**Proof.** Let $(\Lambda, \psi) \in X$ be such that $m\Lambda \subset T_f A$. Choose $(A', i', \varphi)$ such that there exists an isogeny $\alpha: A' \to A$ with $T_f \alpha(T_f A') = m\Lambda$, $\alpha \circ i'(b) = i(b)$ for $b \in \mathcal{O}_B$, and $\varphi = \frac{1}{m}\psi \circ (T_f \alpha)$. Since $t_{A'} \cong (D(\Lambda/p\Lambda)/FD(\Lambda/p\Lambda))^\vee$ (see [3]), $t_{A'}$ satisfies the condition $(\ast)$. If $(\Lambda, \psi)$ and $(\Lambda', \psi')$ correspond to the same triple $(A', i', \varphi)$ with $A' \overset{\alpha}{\cong} A$, then $(\Lambda', \psi') = \alpha' \circ \alpha^{-1}(\Lambda, \psi)$. 


Write $X = X^p \times X_p$ with

$$X^p = \{ (\Lambda^p, \tilde{\psi}) \mid (\Lambda, \tilde{\psi}) \in X \}$$
$$X_p = \{ \Lambda_p \mid (\Lambda, \tilde{\psi}) \in X \}.$$

We may regard $T_p^p A$ as a free module of rank $4d$ over $\mathbb{Z}_f$, $V_f^p A$ as $T_p^p A \otimes \mathbb{Q}$, and any $\Lambda^p$ as a $\mathbb{Z}_f$-lattice in $V_f^p A$ in the usual sense. The following lemma is obvious.

**Lemma 3.** The map

$$G(\mathbb{A}_f^p) \to X^p, \quad g \mapsto (g(T_f A), \varphi_A \circ g^{-1})$$

induces a bijection

$$G(\mathbb{A}_f^p)/K^p \to X^p.$$

We have $\Lambda_p = \lim_{\to -n} \Lambda_{(p^n)} \subset V_p(A) = \lim_{\to -n} A(p)(p^n)$. For $n$ sufficiently large, $p^n T_p A \subset \Lambda_p$ and then we can identify $\Lambda_{(p^n)}$ with $\Lambda_p / p^n T_p A$. Thus

$$\text{Ker}(\Lambda_{(p^n+1)} \to \Lambda_{(p^n)}) = p^n T_p A / p^{n+1} T_p A \cong \Lambda_p$$

and

$$A(p)/\Lambda_{(p^n+1)} \xrightarrow{p} A(p)/\Lambda_{(p^n)}$$

is an isomorphism. Moreover, $A(p)/\Lambda_{(p^n)}$ determines $\Lambda_p$ (because $\Lambda_{(p^{n+r})} = \text{Ker}(A(p) \xrightarrow{p^r} A(p)/\Lambda_{(p^n)})$ for $r \geq 0$) and the Dieudonné module of the $p$-divisible group $A(p)/\Lambda_{(p^n)}$ determines it. We have therefore:

**Lemma 4.** The map $\Lambda \mapsto \frac{1}{p^n} D(A(p))/\Lambda_{(p^n)} \subset D'A, n \gg 0$, identifies $X_p$ with the set of all submodules $M$ of $D'A$ such that:

(a) $M$ is free of rank $4d$ over $W$;

(b) $M$ is stable under $F$ and $V$;

(c) $M$ is stable under the action of $\mathcal{O}_B$;

(d) $M/F M$ satisfies the condition (*).

In summary:

**Theorem 5.** There exists a bijection

$$Y \approx \frac{H(\mathbb{Q}) \setminus G(\mathbb{A}_f^p) \times X_p}{K^p}$$

with $H(\mathbb{Q}) = E'^\times$ in the case (NS) and $H(\mathbb{Q}) = B'^\times$ in the case (S). Moreover, $\Phi$ acts as $1$ on $G(\mathbb{A}_f^p)$ and by $M \mapsto F M$ on $X_p$; the Hecke operator corresponding to $g \in G(\mathbb{A}_f^p)$ acts by multiplication on the right by $G(\mathbb{A}_f^p)$.

It remains to describe $X_p$ more explicitly.
LEMMA 6. There exists a bijection

\[ X_p \to \prod_{p|p'} X_p \]

where \( X_p \) is the set of all submodules \( M \) of \( D'A(p) \) which are free of rank \( 4d_p \) over \( W \) and which satisfy the conditions (b), (c), (d) of Lemma 4 (with \( O_E \otimes F_p \) replaced by \( O_{E_p} \otimes \overline{F}_p \) in (d)).

PROOF. We have

\[ O_E \otimes \mathbb{Z}_p \cong \prod_{p|p'} O_{E_p}. \]

Let \( e_p \) be the corresponding idempotents in \( O_E \otimes \mathbb{Z}_p \), so that \( O_{E_p} = e_p(O_E \otimes \mathbb{Z}_p) \). Note that \( e_p M \) has rank \( 4d_p \) over \( W \) because the trace of an element \( \alpha \in O_E \) acting on \( M \) (or \( A' \)) is four times its trace in the extension \( E \supset \mathbb{Q} \) [4, 7.6.1]. We therefore obtain a bijection

\[ M \to (\ldots, e_p M, \ldots). \]

\[ \square \]

Note that \( B_p =_{df} B \otimes E_p \cong M_2(E_p) \) acts on \( D'A(p) \). Let \( e_{11}, e_{21}, \ldots \in O_B \otimes O_{E_p} \) be the elements corresponding to the elements

\[ (\frac{0}{0}, \frac{1}{1}), (\frac{0}{0}, 1) \in M_2(O_{E_p}), \]

and write \( D'_p = e_{11}D'A(p) \); it is a module of dimension \( 2d_p \) over \( W' =_{df} W \otimes \mathbb{Q} \). If \( M \subset D'A(p) \) is in \( X_p \), then

\[ M = e_{11} M \oplus e_{22} M \]

and the map \( e_{11} x \mapsto e_{21} e_{11} x \) is an isomorphism \( e_{11} M \to e_{22} M \) with inverse \( e_{22} x \mapsto e_{12} e_{22} x \). Thus, \( e_{11} M \) determines \( M \), and we have

LEMMA 7. The set \( X_p \) can be identified with the set of all submodules \( M \) of \( D'_p \) such that:

(a) \( M \) is free of rank \( 2d_p \) over \( W \);

(b) \( M \) is stable under \( F \) and \( V \);

(c) \( M \) is stable under \( O_{E_p} \);

(d) \( M/FM \) is a free \( O_{E_p} \otimes \overline{F}_p \)-module of rank 1.

LEMMA 8. Let \( e_1, \ldots, e_{d_p} \) be the idempotents in \( O_{E_p} \otimes W \) corresponding to the decomposition \( O_{E_p} \otimes W \cong W \times \cdots \times W \). Then \( N_j = e_j D'_p \) has dimension 2 over \( W' \) and \( D'_p = N_1 \oplus \cdots \oplus N_{d_p} \). If \( F_{jl} : N_l \to N_j \) is the map induced by \( F : D'_p \to D'_p \), then \( F_{jl} = 0 \) for \( l \neq j - 1 \mod d_p \), and it is an isomorphism otherwise. It is possible to choose a basis \( \{ \varepsilon, \varepsilon' \} \) for \( N_1 \) such that \( F_{d_p} : N_1 \to N_1 \) corresponds to a matrix

\[
\delta = \begin{pmatrix} p^{m_0} & 0 \\ 0 & p^{n_0'} \end{pmatrix} \quad \text{if } p \text{ splits in } E' \text{ (case NS)}
\]

\[
= \begin{pmatrix} p^{d_p/2} & 0 \\ 0 & p^{d_p/2} \end{pmatrix} \quad \text{if } d \text{ is even (case NS or S)}
\]

\[
= p^{(d_p - 1)/2} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \quad \text{otherwise.}
\]
PROOF. The same argument as in the proof of Lemma 6 shows that $N_j$ has dimension two over $W'$. Let $\sigma$ be the Frobenius automorphism of $W'$. When $E_p$ is identified with a subfield of $W'$, the mapping

$$E_p \to E_p \otimes_{Q_p} W' \cong W' \times \cdots \times W'$$

becomes

$$a \mapsto (a, \sigma a, \ldots, \sigma^{d_p-1} a).$$

Thus, for $\beta = (\beta_1, \ldots, \beta_{d_p}) \in D'_p = N_1 \times \cdots \times N_{d_p}$ and $a \in E_p$, we have

$$a \beta = (a \beta_1, \ldots, \sigma^{j-1}(a) \beta_j, \ldots).$$

Since $aF = Fa$ on $D'_p$, we have

$$\sigma^{j-1}(a) \sum_i F_{ji} \beta_i = \sum_i F_{ji} \sigma^{l-1}(a) \beta_i = \sigma^l(a) \sum_i F_{ji} \beta_i.$$

Therefore, $F_{ji} = 0$ if $l \not\equiv j - 1 \mod d_p$. It is clear that $F_{ji}$ is an isomorphism for $l \equiv j - 1 \mod d_p$ because $F: D'_p \to D'_p$ is.

In case (NS), if $p$ splits in $E'$ and $m'_p \neq m''_p$, then $N_1$ is a $W'[F^{d_p}]$-module of rank 2 over $W'$ whose slopes are $m'_p$ and $m''_p$ (relative to $F^{d_p}$). Therefore, it is clear that there exists a basis $\{\epsilon, \epsilon'\}$ such that $F^{d_p} \epsilon = p^{m'_p} \epsilon$ and $F^{d_p} \epsilon' = p^{m''_p} \epsilon'$.

In the contrary case, all the slopes of $D'_p$ equal $1/2$. Therefore, $D'_p$ is a direct sum of $W'[F]$-modules of rank 2 over $W'$ on which $F$ acts by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d_p = \begin{pmatrix} p^{d_p/2} & 0 \\ 0 & p^{d_p/2} \end{pmatrix}$ when $d_p$ is even and $p^{(d_p-1)/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ otherwise, $D'_p$ is evidently an isotypic semisimple $W'[F^{d_p}]$-module, which completes the proof.

\[\square\]

REMARK 9. Let $G_p(Z_p) = \text{End}_{O_B}(A(p))^\times$ and $G_p(Q_p) = (\text{End}_{O_B}(A(p)) \otimes_{Z_p} Q_p)^\times$. Then $G_p(Q_p)$ is the multiplicative group of the commutant of $B_p$ in $\text{End}_{W'[F]}(D' A(p))$ or, after Lemma 7, the multiplicative group of the commutant of $E_p$ in $\text{End}_{W'[F]}(D'_p)$. But if, for $\alpha \in \text{End}_{W'[F^{d_p}]}(N_1)$ and $\beta = (\beta_1, \ldots, \beta_{d_p}) \in D'_p$, we put $\alpha(\beta) = (\alpha \beta_1, \ldots, \alpha \beta_{d_p})$, then $\text{End}_{W'[F^{d_p}]}(N_1)$ is identified with this last commutant. Thus

$$G_p(Q_p) = \{ (a b) \mid a, b \in E_p, ab \neq 0 \}$$

in case (NS) when $m'_p \neq m''_p$,

$$= \text{GL}_2(E_p) \quad \text{when } m'_p \neq m''_p \text{ and } d_p \text{ is even},$$

$$= \mathbb{H} \times \text{GL}_2(E_p) \quad \text{when } m'_p \neq m''_p \text{ and } d_p \text{ is odd} \quad (\mathbb{H} \text{ is the quaternion algebra over } E_p).$$

LEMMA 10. The set $X_p$ can be identified with the set of sequences of lattices $(L_j)_{j \in \mathbb{Z}}$ in $W' \times W'$ such that

(a) $L_j \supsetneq L_{j-1} \supsetneq pL_j$

(b) $\sigma^{d_p} \delta L_{j+d_p} = L_j$ with $\delta$ as in Lemma 8.
PROOF. For \( M \in X_p \), we have \( M = M_1 \oplus \cdots \oplus M_{d_p} \) with \( M_j = e_j M \). Since
\[
FM = FM_{d_p} \oplus FM_1 \oplus \cdots \oplus FM_{d_p-1}
\]
with \( FM_j \subset N_{j+1} \), the conditions (b) and (d) of Lemma 7 imply that \( FM_{d_p} \subset M_1 \), \( FM_1 \subset M_2 \), ... and that \( M_1/FM_{d_p}, M_2/FM_1, \ldots \) have dimension 1 over \( \overline{\mathbb{F}}_p \).

Choose a basis \( \{ \varepsilon, \varepsilon' \} \) for \( N_1 \) as in Lemma 8 and let \( \varphi_j : N_j \overset{\cong}{\to} W' \times W' \) be the mapping
\[
a(F^j \varepsilon) + b(F^j \varepsilon') \mapsto (\sigma^{1-j}(a), \sigma^{1-j}(b)).
\]
Note that \( \varphi_{j+1} F(x) = \varphi_j(x) \) and \( \varphi_j(F^{\sigma_b} x) = \sigma^{\delta} \varphi_j(x) \). Put \( L_j = \varphi_j(M_j) \) for \( 1 \leq j \leq d_p \) and \( L_{j-d_p} = \varphi_j(F^{\sigma_b} M_j) = \sigma^{\delta} \delta L_j \).

\[ \square \]

REMARK 11. \( \Phi(L_j)_{j \in \mathbb{Z}} = (L'_j)_{j \in \mathbb{Z}} \) with \( L'_j = L_{j-1} \). The group \( (E_p')^\infty \) (respectively \( (B'_p)^\infty = (B' \otimes_E E_p)^\infty \)) acts on \( X_p \) via the embedding \( E_p' \hookrightarrow \tilde{G}_p(\mathbb{Q}_p) \) (respectively \( B'_p \hookrightarrow \tilde{G}_p(\mathbb{Q}_p) \)). The group \( H(\mathbb{Q}) \) acts on \( X_p \) via the obvious embedding \( H(\mathbb{Q}) \hookrightarrow \tilde{G}(\mathbb{Q}_p) \).

In summary:

THEOREM 12. \( X_p \cong \prod_{p \mid P} X_p \) where \( X_p \) is the set of sequences of lattices satisfying the conditions of Lemma 10, and \( \Phi \) and \( H(\mathbb{Q}) \) act as described in Remark 11.

REMARK 13. Let \( \Omega \) be the maximal unramified extension of \( \mathbb{Q}_p \), and let \( O_\Omega \) the ring of integers in \( \Omega \). Then \( W' \) is the completion of \( \Omega \). One can write \( D'A = \tilde{D}'A \otimes_\Omega W' \) with \( \tilde{D}'A \) a module over \( \Omega[F] \) (see [2, p85]). If \( M \) is as in Lemma 4, then \( M \) is the image of \( D_{\alpha} : DA' \to DA \) for a certain isogeny \( \alpha : A' \to A \). Since \( \alpha \) is defined over a finite subfield \( k \) of \( \overline{\mathbb{F}}_p \), and \( W'_k \subset \Omega \), we have \( M = \tilde{M} \otimes_\Omega W' \) for a submodule \( \tilde{M} \subset \tilde{D}'A \). Therefore, \( X_p \) can be identified with the set of submodules of \( \tilde{D}'A \), and \( X_p \) with the set of sequences of lattices \( (L_j)_{j \in \mathbb{Z}}, L_j \subset \Omega \times \Omega \), satisfying the conditions of Lemma 10.

Bibliography

[1] 2L. Breen: Exposé IV of the same seminar.

\[ \text{jsm 08.12.02} \]

\[ ^2 \text{This is only a summary of [3].} \]