

FLAT HOMOLOGY

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In this note we define "homology groups" relative to the flat site, and list some of their properties, in the case that the base scheme is algebraic over a field.

X_{fl} denotes the big f.p.p.f. site over a scheme X and $S(X_{fl})$ the corresponding category of sheaves. $S = \text{spec } k$, where k is a field of characteristic p . $A(al)$ denotes the category of commutative algebraic group schemes over S and $A(u, f) \supset A(u) \supset A(uf) \supset A(f)$ the subcategories consisting of those affine groups which are respectively unipotent or finite, unipotent, unipotent and finite, finite. The letter A always stands for one of these categories and $\text{Pro-}A$ for the corresponding pro-category. The notations for derived categories are as in [6].

1. THEOREM (Universal Coefficient Theorem). *For any morphism $\pi: X \rightarrow S$ of finite type and any A , there exists a complex $L_*(X/S, A)$ in $K^-(\text{Pro-}A)$ such that:*

(a) $L_s(X/S, A)$ is a projective object, all s ;

(b) $\text{Hom}_{\text{Pro-}A}(L_*(X/S, A), N) \xrightarrow{\cong} \mathbf{R}(\pi_*)N_X$ in $D^+(S(S_{fl}))$ for all N in A . $\left(\prod_X\right)$

Moreover, $L_*(X/S, A)$ is unique, up to isomorphism, in $K^-(\text{Pro-}A)$.

PROOF. Choose a conservative family of points for X_{fl} , and let $C^*(F)$ be the corresponding Godement resolution of a sheaf F [1, XVII 4.2]. Choose L_s to pro-represent the functor $N \mapsto \Gamma(X, C^s(N_X)): A \rightarrow Ab$.

2. COROLLARY. Write $H_s(X/S, A)$ for $H_s(L_*(X/S, A))$. There is a spectral sequence

$$\text{Ext}_{\text{Pro-}A}^r(H_s(X/S, A), N) \Rightarrow H^{r+s}(X_{fl}, N_X) \quad \text{for all } N \text{ in } A.$$

3. DEFINITION. $L_*(X/S, A)$ is the flat homology complex of X/S relative to A , and $H_s(X/S, A)$ is the s th flat homology group.

4. REMARKS. (a) Theorem 1 is basically as conjectured by Grothendieck [5, p. 316].

(b) $L_*(X/S, A)$ and $H_s(X/S, A)$ are covariant functors in X/S .

(c) If $\omega_0: A(al) \rightarrow A(f)$ is the functor taking a group scheme to its maximal finite quotient, then $\omega_0(L_*(X/S, A(al))) = L_*(X/S, A(f))$. Thus there

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is a third-quadrant spectral sequence $\omega_r(H_s(X/S, A(al))) \Rightarrow H_{r+s}(X/S, A(f))$ where $\omega_r = L^r \omega_0$.

5. THEOREM. Assume k to be algebraically closed and let M be the functor taking a group scheme to its Dieudonné module (in the sense of [4, III]). Then

$$M(L_*(X/S, A(u, f))) = H^*(X_{Zar}, \underline{W}) \oplus (H^*(X_{fl}, \mu_{p^\infty}) \otimes_{\mathbb{Z}} W(k)),$$

where W_n is the group scheme of Witt vectors of length n and $\underline{W} = \varinjlim W_n(O_X)$ and $W(k) = \varprojlim W_n(k)$.

PROOF. Immediate from the definitions of L_* and M .

6. COROLLARY. $M(H_s(X/S, A(u, f))) = H^s(X_{Zar}, \underline{W}) \oplus H^s(X_{fl}, \mu_{p^\infty}) \otimes_{\mathbb{Z}} W(k)$.

PROOF. " $\varinjlim$ " W_n and " $\varinjlim$ " μ_{p^n} behave as injectives in A .

7. REMARK. $M(H_1)$ is equal to the group $I(X)$ studied in [7, §4].

8. THEOREM. Assume k to be algebraically closed and X/S to be proper. Then $L_*(X/S, A(u))$ is isomorphic (in $K^-(\text{Pro-}A(u))$) to $L_*(X/S, A(uf))$.

PROOF. $H_s(X/S, A(u)) \in \text{Pro-}A(uf)$ for otherwise $H^s(X, O_X)$ would have infinite dimension over k .

9. THEOREM. Write $N^\sim$ for the formal group associated to an affine group scheme N by Cartier duality (see [4, II.4]), and write $H_s^\sim$ for $H_s(L_*)^\sim = H^s(L_*^\sim)$ where $L_* = L_*(X/S, A(u))$. Then $H_s^\sim$ is a connected formal group of finite-type (see [4, p. 35]) and represents the functor of finite S -schemes.

$$T \mapsto \text{Ker}(\Gamma(T, R^s \pi_* \mathbf{G}_m) \rightarrow \Gamma(T_{red}, R^s \pi_* \mathbf{G}_m)).$$

PROOF. Regard $U = \text{Ker}(\mathbf{G}_{m,T} \rightarrow \mathbf{G}_{m,T_{red}})$ as a sheaf on T_{red} , and use (8).

10. COROLLARY. Write $\Phi^s(T) = \text{Ker}(H^s(X_T, \mathbf{G}_m) \rightarrow H^s(X_{T_{red}}, \mathbf{G}_m))$. If Φ^{s-1} is a formally smooth functor then Φ^s is represented by a formal group.

PROOF. Immediate from the theorem.

11. REMARKS. (a) Intuitively (9) says that $L_*^\sim$ represents $\mathbf{R}^* \pi_* \mathbf{G}_m$ infinitesimally.

(b) Generalizations of (10), but not (9), may be found in [2].

12. THEOREM. Assume that k is algebraically closed, X is projective and smooth over k , and $p > \dim(X)$. Then

$$\text{Hom}_W(K/W, M(H_s(X/S, A(f)))) \otimes_W K \xrightarrow{\cong} (H^s(X/W, O_{X/W}) \otimes_W K)_{[0,1]}$$

as F -isocrystals, where $W = W(k)$, $K =$ field of fractions of W , and the right-hand term is the part of crystalline cohomology with slopes between 0 and 1 (inclusive).

PROOF. Follows from [3] and (6).

13. REMARKS. (a) The last theorem states that (modulo torsion) the knowledge of the flat cohomology of finite constant group schemes on X is equivalent to the knowledge of the part of crystalline cohomology with slopes between 0 and 1.

(b) (12) differs from the "hope" expressed by Grothendieck [5, p. 316].

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