CHAPTER **16**

Descent Theory

Consider fields $k \subset \Omega$. A variety V over k defines a variety V_Ω over Ω by extension of the base field. Descent theory attempts to answer the following question: what additional structure do you need to place on a variety over Ω, or regular map of varieties over Ω, to ensure that it comes from k?

In this chapter, we shall make free use of the axiom of choice (usually in the form of Zorn's lemma).

a Models

Let $\Omega \supset k$ be fields, and let V be a variety over Ω. Recall that a model of V over k (or a ***k-structure*** on V) is a variety V_0 over k together with an isomorphism $\varphi\colon V \to V_{0\Omega}$ (Chapter 11). Recall also that a variety over Ω need not have a model over k, and when it does it typically will have many nonisomorphic models.

Consider an affine variety. An embedding $V \hookrightarrow \mathbb{A}^n_\Omega$ defines a model of V over k if $I(V)$ is generated by polynomials in $k[X_1,\ldots,X_n]$, because then $I_0 \stackrel{\text{def}}{=} I(V) \cap k[X_1,\ldots,X_n]$ is a radical ideal, $k[X_1,\ldots,X_n]/I_0$ is an affine k-algebra, and $V(I_0) \subset \mathbb{A}^n_k$ is a model of V. Moreover, every model (V_0,φ) arises in this way, because every model of an affine variety is affine. However, different embeddings in affine space will usually give rise to different models. Similar remarks apply to projective varieties.

Note that the condition that $I(V)$ be generated by polynomials in $k[X_1,\ldots,X_n]$ is stronger than asking that V be the zero set of some polynomials in $k[X_1,\ldots,X_n]$. For example, let $V = V(X+Y+\alpha)$ where α is an element of Ω such that $\alpha^p \in k$ but $\alpha \notin k$. Then V is the zero set of the polynomial $X^p + Y^p + \alpha^p$, which has coefficients in k, but $I(V) = (X+Y+\alpha)$ is not generated by polynomials in $k[X,Y]$.

This is Chapter 16 of Algebraic Geometry by J.S. Milne. July 9, 2015

b Fixed fields

Let $\Omega \supset k$ be fields, and let Γ be the group $\mathrm{Aut}(\Omega/k)$ of automomorphisms of Ω (as an abstract field) fixing the elements of k. Define the ***fixed field*** Ω^{Γ} of Γ to be

$$\{a \in \Omega \mid \sigma a = a \text{ for all } \sigma \in \Gamma\}.$$

PROPOSITION 16.1. *The fixed field of Γ equals k in each of the following two cases:*

(a) *Ω is a Galois extension of k (possibly infinite);*

(b) *Ω is a separably closed field and k is perfect.*

PROOF. (a) See FT 7.9.
(b) See FT 9.29. □

REMARK 16.2. (a) The proof of Proposition 16.1 definitely requires the axiom of choice. For example, it is known that every measurable homomorphism of Lie groups is continuous, and so any measurable automorphism of $\mathbb{C}$ is equal to the identity map or to complex conjugation. Therefore, without the axiom of choice, $\Gamma \stackrel{\text{def}}{=} \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ has only two elements, and $\mathbb{C}^{\Gamma} = \mathbb{R}$.

(b) Suppose that Ω is algebraically closed and k is not perfect. Then k has characteristic $p \neq 0$ and Ω contains an element α such that $\alpha \notin k$ but $\alpha^p = a \in k$. As α is the unique root of $X^p - a$, every automorphism of Ω fixing k also fixes α, and so $\Omega^{\Gamma} \neq k$.

The ***perfect closure*** of k in Ω is the subfield

$$k^{p^{-\infty}} = \{\alpha \in \Omega \mid \alpha^{p^n} \in k \text{ for some } n\}$$

of Ω. Then $k^{p^{-\infty}}$ is purely inseparable over k, and when Ω is algebraically closed, it is the smallest perfect subfield of Ω containing k.

COROLLARY 16.3. *If Ω is separably closed, then Ω^{Γ} is a purely inseparable algebraic extension of k.*

PROOF. When k has characteristic zero, $\Omega^{\Gamma} = k$, and there is nothing to prove. Thus, we may suppose that k has characteristic $p \neq 0$. Choose an algebraic closure Ω^{al} of Ω, and let $k^{p^{-\infty}}$ be the perfect closure of k in Ω^{al}. As Ω^{al} is purely inseparable over Ω, every element σ of Γ extends uniquely to an automorphism $\tilde{\sigma}$ of Ω^{al}: let $\alpha \in \Omega^{\mathrm{al}}$ and let $\alpha^{p^n} \in \Omega$; then $\tilde{\sigma}(\alpha)$ is the unique root of $X^{p^n} - \sigma(\alpha^{p^n})$ in Ω. The action of Γ on Ω^{al} identifies it with $\mathrm{Aut}(\Omega^{\mathrm{al}}/k^{p^{-\infty}})$. According to the proposition, $(\Omega^{\mathrm{al}})^{\Gamma} = k^{p^{-\infty}}$, and so

$$k^{p^{-\infty}} \supset \Omega^{\Gamma} \supset k.$$

□

c Descending subspaces of vector spaces

In this subsection, $\Omega \supset k$ are fields such that k is the fixed field of $\Gamma = \mathrm{Aut}(\Omega/k)$.

Let V be a k-subspace of an Ω-vector space $V(\Omega)$ such that the map

$$c \otimes v \mapsto cv \colon \Omega \otimes_k V \to V(\Omega)$$

is an isomorphism. Equivalent conditions: V is the k-span of an Ω-basis for $V(\Omega)$; every k-basis for V is an Ω-basis for $V(\Omega)$. The group Γ acts on $\Omega \otimes_k V$ through its action on Ω:

$$\sigma(\sum c_i \otimes v_i) = \sum \sigma c_i \otimes v_i, \quad \sigma \in \Gamma, \quad c_i \in \Omega, \quad v_i \in V. \tag{1}$$

Correspondingly, there is a unique action of Γ on $V(\Omega)$ fixing the elements of V and such that each $\sigma \in \Gamma$ acts σ-linearly:

$$\sigma(cv) = \sigma(c)\sigma(v) \text{ all } \sigma \in \Gamma, c \in \Omega, v \in V(\Omega). \tag{2}$$

LEMMA 16.4. *The following conditions on a subspace W of $V(\Omega)$ are equivalent:*

(a) *$W \cap V$ spans W;*

(b) *$W \cap V$ contains an Ω-basis for W;*

(c) *the map $\Omega \otimes_k (W \cap V) \to W$, $c \otimes v \mapsto cv$, is an isomorphism.*

PROOF. (a) $\Longrightarrow$ (b,c) A k-linearly independent subset of V is Ω-linearly independent in $V(\Omega)$. Therefore, if $W \cap V$ spans W, then any k-basis $(e_i)_{i \in I}$ for $W \cap V$ will be an Ω-basis for W. Moreover, $(1 \otimes e_i)_{i \in I}$ will be an Ω-basis for $\Omega \otimes_k (W \cap V)$, and since the map $\Omega \otimes_k (W \cap V) \to W$ sends $1 \otimes e_i$ to e_i, it is an isomorphism.

(c) $\Longrightarrow$ (a), (b) $\Longrightarrow$ (a). Obvious. □

LEMMA 16.5. *For any k-vector space V, $V = V(\Omega)^{\Gamma}$.*

PROOF. Let $(e_i)_{i \in I}$ be a k-basis for V. Then $(1 \otimes e_i)_{i \in I}$ is an Ω-basis for $\Omega \otimes_k V$, and $\sigma \in \Gamma$ acts on $v = \sum c_i \otimes e_i$ according to the rule (1). Thus, v is fixed by Γ if and only if each c_i is fixed by Γ and so lies in k. □

LEMMA 16.6. *Let V be a k-vector space, and let W be a subspace of $V(\Omega)$ stable under the action of Γ. If $W^{\Gamma} = 0$, then $W = 0$.*

PROOF. Suppose $W \neq 0$. As V contains an Ω-basis for $V(\Omega)$, every nonzero element w of W can be expressed in the form

$$w = c_1 e_1 + \cdots + c_n e_n, \quad c_i \in \Omega \smallsetminus \{0\}, \quad e_i \in V, \quad n \geq 1.$$

Choose w to be a nonzero element for which n takes its smallest value. After scaling, we may suppose that $c_1 = 1$. For $\sigma \in \Gamma$, the element

$$\sigma w - w = (\sigma c_2 - c_2)e_2 + \cdots + (\sigma c_n - c_n)e_n$$

lies in W and has at most $n-1$ nonzero coefficients, and so is zero. Thus, $w \in W^{\Gamma} = \{0\}$, which is a contradiction. □

PROPOSITION 16.7. *A subspace W of $V(\Omega)$ is of the form $W = \Omega W_0$ for some k-subspace W_0 of V if and only if it is stable under the action of Γ.*

PROOF. Certainly, if $W = \Omega W_0$, then it is stable under Γ (and $W = \Omega(W \cap V)$). Conversely, assume that W is stable under Γ, and let W' be a complement to $W \cap V$ in V, so that

$$V = (W \cap V) \oplus W'.$$

Then

$$(W \cap \Omega W')^{\Gamma} = W^{\Gamma} \cap \left(\Omega W'\right)^{\Gamma} = (W \cap V) \cap W' = 0,$$

and so, by (16.6),

$$W \cap \Omega W' = 0. \tag{3}$$

As $W \supset \Omega(W \cap V)$ and

$$V(\Omega) = \Omega(W \cap V) \oplus \Omega W',$$

this implies that $W = \Omega(W \cap V)$. □

d Descending subvarieties and morphisms

In this subsection, $\Omega \supset k$ are fields such that k is the fixed field of $\Gamma = \mathrm{Aut}(\Omega/k)$ and Ω is separably closed. Recall that for any variety V over Ω, $V(\Omega)$ is Zariski dense in V (Chapter 11). In particular, two regular maps $V \to V'$ coincide if they agree on $V(\Omega)$.

For any variety V over k, Γ acts on $V(\Omega)$. For example, if V is embedded in $\mathbb{A}^n$ or $\mathbb{P}^n$ over k, then Γ acts on the coordinates of a point. If $V = \mathrm{Spm} A$, then

$$V(\Omega) = \mathrm{Hom}_{k\text{-algebra}}(A, \Omega),$$

and Γ acts through its action on Ω.

PROPOSITION 16.8. *Let V be a variety over k, and let W be a closed subvariety of V_{Ω} such that $W(\Omega)$ is stable under the action of Γ on $V(\Omega)$. Then there is a closed subvariety W_0 of V such that $W = W_{0\Omega}$.*

PROOF. Suppose first that V is affine, and let $I(W) \subset \Omega[V_{\Omega}]$ be the ideal of regular functions zero on W. Recall that $\Omega[V_{\Omega}] = \Omega \otimes_k k[V]$ (Chapter 11). Because $W(\Omega)$ is stable under Γ, so also is $I(W)$, and Proposition 16.7 shows that $I(W)$ is spanned by $I_0 = I(W) \cap k[V]$. Therefore, the zero set of I_0 is a closed subvariety W_0 of V with the property that $W = W_{0\Omega}$.

To deduce the general case, cover V with open affines $V = \bigcup V_i$. Then $W_i \overset{\text{def}}{=} V_{i\Omega} \cap W$ is stable under Γ, and so it arises from a closed subvariety W_{i0} of V_i; a similar statement holds for $W_{ij} \overset{\text{def}}{=} W_i \cap W_j$. Define W_0 to be the variety obtained by patching the varieties W_{i0} along the open subvarieties W_{ij0}. □

PROPOSITION 16.9. *Let V and W be varieties over k, and let $f\colon V_{\Omega} \to W_{\Omega}$ be a regular map. If f commutes with the actions of Γ on $V(\Omega)$ and $W(\Omega)$, then f arises from a (unique) regular map $V \to W$ over k.*

PROOF. Apply Proposition 16.8 to the graph of f, $\Gamma_f \subset (V \times W)_{\Omega}$. □

COROLLARY 16.10. *A variety V over k is uniquely determined (up to a unique isomorphism) by the variety V_{Ω} together with action of Γ on $V(\Omega)$.*

PROOF. More precisely, we have shown that the functor

$$V \rightsquigarrow (V_{\Omega},\ \text{action of } \Gamma \text{ on } V(\Omega)) \tag{4}$$

is fully faithful. □

REMARK 16.11. In Theorems 16.42 and 16.43 below, we obtain sufficient conditions for a pair to lie in the essential image of the functor (4).

e Galois descent of vector spaces

Let Γ be a group acting on a field Ω, and let k be a subfield of Ω^{Γ}. By an ***action*** of Γ on an Ω-vector space V we mean a homomorphism $\Gamma \to \mathrm{Aut}_k(V)$ satisfying (2), i.e., such that each $\sigma \in \Gamma$ acts σ-linearly.

LEMMA 16.12. *Let S be the standard $M_n(k)$-module (i.e., $S = k^n$ with $M_n(k)$ acting by left multiplication). The functor $V \mapsto S \otimes_k V$ from k-vector spaces to left $M_n(k)$-modules is an equivalence of categories.*

PROOF. Let V and W be k-vector spaces. The choice of bases $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ for V and W identifies $\mathrm{Hom}_k(V,W)$ with the set of matrices $(a_{ji})_{(j,i)\in J\times I}$, $a_{ji} \in k$, such that, for a fixed i, all but finitely many a_{ji} are zero. Because S is a simple $M_n(k)$-module and $\mathrm{End}_{M_n(k)}(S) = k$, the set $\mathrm{Hom}_{M_n(k)}(S \otimes_k V, S \otimes_k W)$ has the same description, and so the functor $V \mapsto S \otimes_k V$ is fully faithful.

The functor $V \mapsto S \otimes_k V$ sends a vector space V with basis $(e_i)_{i\in I}$ to a direct sum of copies of S indexed by I. Therefore, to show that the functor is essentially surjective, we have to prove that every left $M_n(k)$-module is a direct sum of copies of S.

We first prove this for $M_n(k)$ regarded as a left $M_n(k)$-module. For $1 \le i \le n$, let $L(i)$ be the set of matrices in $M_n(k)$ whose entries are zero except for those in the ith column. Then $L(i)$ is a left ideal in $M_n(k)$, and $L(i)$ is isomorphic to S as an $M_n(k)$-module. Hence,

$$M_n(k) = \bigoplus_i L(i) \simeq S^n \qquad \text{(as a left } M_n(k)\text{-module).}$$

We now prove it for an arbitrary left $M_n(k)$-module M, which we may suppose to be nonzero. The choice of a set of generators for M realizes it as a quotient of a sum of copies of $M_n(k)$, and so M is a sum of copies of S. It remains to show that the sum can be made direct. Let I be the set of submodules of M isomorphic to S, and let Ξ be the set of subsets J of I such that the sum $N(J) \stackrel{\text{def}}{=} \sum_{N\in J} N$ is direct, i.e., such that for any $N_0 \in J$ and finite subset J_0 of J not containing N_0, $N_0 \cap \sum_{N\in J_0} N = 0$. If $J_1 \subset J_2 \subset \ldots$ is a chain of sets in Ξ, then $\bigcup J_i \in \Xi$, and so Zorn's lemma implies that Ξ has maximal elements. For any maximal J, $M = N(J)$ because otherwise, there exists an element S' of I not contained in $N(J)$; because S' is simple, $S' \cap N(J) = 0$, and it follows that $J \cup \{S'\} \in \Xi$, contradicting the maximality of J. □

ASIDE 16.13. Let A and B be rings (not necessarily commutative), and let S be A-B-bimodule (this means that A acts on S on the left, B acts on S on the right, and the actions commute). When the functor $M \mapsto S \otimes_B M\colon \mathsf{Mod}_B \to \mathsf{Mod}_A$ is an equivalence of categories, A and B are said to be ***Morita equivalent through*** S. In this terminology, the lemma says that $M_n(k)$ and k are Morita equivalent through S.

PROPOSITION 16.14. *Let Ω be a finite Galois extension of k with Galois group Γ. The functor $V \rightsquigarrow \Omega \otimes_k V$ from k-vector spaces to Ω-vector spaces endowed with an action of Γ is an equivalence of categories.*

PROOF. Let $\Omega[\Gamma]$ be the Ω-vector space with basis $\{\sigma \in \Gamma\}$, and make $\Omega[\Gamma]$ into a k-algebra by setting

$$\left(\textstyle\sum_{\sigma\in\Gamma} a_\sigma \sigma\right)\left(\textstyle\sum_{\tau\in\Gamma} b_\tau \tau\right) = \textstyle\sum_{\sigma,\tau} (a_\sigma \cdot \sigma b_\tau)\sigma\tau.$$

Then $\Omega[\Gamma]$ acts k-linearly on Ω by the rule

$$(\textstyle\sum_{\sigma\in\Gamma} a_\sigma \sigma)c = \sum_{\sigma\in\Gamma} a_\sigma(\sigma c),$$

and Dedekind's theorem on the independence of characters (FT 5.14) implies that the homomorphism

$$\Omega[\Gamma] \to \mathrm{End}_k(\Omega)$$

defined by this action is injective. By counting dimensions over k, one sees that it is an isomorphism. Therefore, Lemma 16.12 shows that $\Omega[\Gamma]$ and k are Morita equivalent through Ω, i.e., the functor $V \mapsto \Omega \otimes_k V$ from k-vector spaces to left $\Omega[\Gamma]$-modules is an equivalence of categories. This is precisely the statement of the lemma. □

When Ω is an infinite Galois extension of k, we endow Γ with the Krull topology, and we say that an action of Γ on an Ω-vector space V is ***continuous*** if every element of V is fixed by an open subgroup of Γ, i.e., if

$$V = \textstyle\bigcup_\Delta V^\Delta \qquad \text{(union over the open subgroups } \Delta \text{ of } \Gamma\text{)}.$$

For example, the action of Γ on Ω is obviously continuous, and it follows that, for any k-vector space V, the action of Γ on $\Omega \otimes_k V$ is continuous.

PROPOSITION 16.15. *Let Ω be a Galois extension of k (possibly infinite) with Galois group Γ. For any Ω-vector space V equipped with a continuous action of Γ, the map*

$$\textstyle\sum c_i \otimes v_i \mapsto \sum c_i v_i \colon \Omega \otimes_k V^\Gamma \to V$$

is an isomorphism.

PROOF. Suppose first that Γ is finite. Proposition 16.14 allows us to assume that $V = \Omega \otimes_k W$ for some k-subspace W of V. Then $V^\Gamma = (\Omega \otimes_k W)^\Gamma = W$, and so the statement is true.

When Γ is infinite, the finite case shows that $\Omega \otimes_k (V^\Delta)^{\Gamma/\Delta} \simeq V^\Delta$ for every open normal subgroup Δ of Γ. Now pass to the direct limit over Δ, recalling that tensor products commute with direct limits (CA 10.2). □

f Descent data

For a homomorphism of fields $\sigma\colon F \to L$, we sometimes write σV for V_L (the variety over L obtained by base change). For example, if V is embedded in affine or projective space, then σV is the affine or projective variety obtained by applying σ to the coefficients of the equations defining V.

A regular map $\varphi\colon V \to W$ defines a regular map $\varphi_L\colon V_L \to W_L$ which we also denote $\sigma\varphi\colon \sigma V \to \sigma W$. Note that $(\sigma\varphi)(\sigma Z) = \sigma(\varphi(Z))$ for any subvariety Z of V. The map $\sigma\varphi$ is obtained from φ by applying σ to the coefficients of the polynomials defining φ.

Let $\Omega \supset k$ be fields, and let $\Gamma = \mathrm{Aut}(\Omega/k)$. An Ω/k-***descent system*** on a variety V over Ω is a family $(\varphi_\sigma)_{\sigma\in\Gamma}$ of isomorphisms $\varphi_\sigma\colon \sigma V \to V$ satisfying the following cocycle condition:

$$\begin{tikzcd} \sigma\tau V \arrow[r, "\sigma\varphi_\tau"'] \arrow[rr, bend left, "\varphi_{\sigma\tau}"] & \sigma V \arrow[r, "\varphi_\sigma"'] & V \end{tikzcd}$$

$$\varphi_\sigma \circ (\sigma\varphi_\tau) = \varphi_{\sigma\tau} \text{ for all } \sigma,\tau \in \Gamma.$$

A model (V_0,φ) of V over a subfield K of Ω containing k ***splits*** $(\varphi_\sigma)_{\sigma\in\Gamma}$ if $\varphi_\sigma = \varphi^{-1}\circ\sigma\varphi$ for all σ fixing K:

$$\begin{tikzcd} \sigma V \arrow[r, "\sigma\varphi"'] \arrow[rr, bend left, "\varphi_\sigma"] & \sigma(V_{0\Omega}) = V_{0\Omega} & V. \arrow[l, "\varphi"] \end{tikzcd}$$

A descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ is said to be ***continuous*** if it is split by some model over a subfield K of Ω that is *finitely generated* over k. A ***descent datum*** is a continuous descent system. A descent datum is ***effective*** if it is split by some model over k. In a given situation, we say that ***descent is effective*** or that ***it is possible to descend the base field*** if every descent datum is effective.

Let V_0 be a variety over k, and let $V = V_{0\Omega}$. Then $V = \sigma V$ because the two varieties are obtained from V_0 by extension of scalars with respect to the maps $k\to L$ and $k \to L \xrightarrow{\sigma} L$, which are equal. Write φ_σ for the identity map $\sigma V \to V$; then $(\varphi_\sigma)_{\sigma\in\Gamma}$ is a descent datum on V.

Let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be an Ω/k descent system on a variety V, and let $\Gamma' = \mathrm{Aut}(\Omega^{\mathrm{sep}}/k)$. Every k-automorphism of Ω extends to a k-automorphism of Ω^{sep}, and $(\varphi_\sigma)_{\sigma\in\Gamma}$ extends to the Ω^{sep}/k descent system $(\varphi'_\sigma)_{\sigma\in\Gamma'}$ on $V_{\Omega^{\mathrm{sep}}}$ with $\varphi'_\sigma = \left(\varphi_{\sigma|\Omega}\right)_{\Omega^{\mathrm{sep}}}$. A model of V over a subfield K of Ω splits $(\varphi_\sigma)_{\sigma\in\Gamma}$ if and only if it splits $(\varphi'_\sigma)_{\sigma\in\Gamma'}$. This observation sometimes allows us to assume that Ω is separably closed.

PROPOSITION 16.16. *Assume that k is the fixed field of $\Gamma = \mathrm{Aut}(\Omega/k)$, and that (V_0,φ) and (V_0',φ') split descent data $(\varphi_\sigma)_{\sigma\in\Gamma}$ and $(\varphi'_\sigma)_{\sigma\in\Gamma}$ on varieties V and V' over Ω. To give a regular map $\psi_0\colon V_0\to V_0'$ amounts to giving a regular map $\psi\colon V\to V'$ such that $\psi\circ\varphi_\sigma = \varphi'_\sigma\circ\sigma\psi$ for all $\sigma\in\Gamma$, i.e., such that*

$$\begin{tikzcd} \sigma V \arrow[r, "\varphi_\sigma"] \arrow[d, "\sigma\psi"] & V \arrow[d, "\psi"] \\ \sigma V' \arrow[r, "\varphi'_\sigma"] & V' \end{tikzcd} \tag{5}$$

commutes for all $\sigma\in\Gamma$.

PROOF. Given ψ_0, define ψ to make the right hand square in

$$\begin{tikzcd} \sigma V \arrow[r, "\sigma\varphi"] \arrow[d, "\sigma\psi"] & V_{0\Omega} \arrow[d, "\psi_{0\Omega}"] & V \arrow[l, "\varphi"'] \arrow[d, "\psi"] \\ \sigma V' \arrow[r, "\sigma\varphi'"] & V'_{0\Omega} \arrow[r, "\varphi'"] & V' \end{tikzcd}$$

commute. The left hand square is obtained from the right hand square by applying σ, and so it also commutes. The outer square is (5).

In proving the converse, we may assume that Ω is separably closed. Given ψ, use φ and φ' to transfer ψ to a regular map $\psi'\colon V_{0\Omega}\to V'_{0\Omega}$. Then the hypothesis implies that ψ' commutes with the actions of Γ on $V_0(\Omega)$ and $V_0'(\Omega)$, and so is defined over k (16.9). □

COROLLARY 16.17. *Assume that k is the fixed field of $\Gamma = \mathrm{Aut}(\Omega/k)$, and that (V_0,φ) splits the descent datum $(\varphi_\sigma)_{\sigma\in\Gamma}$. Let W be a variety over k. To give a regular map $W\to V_0$ (resp. $V_0\to W$) amounts to giving a regular map $\psi\colon W_\Omega\to V$ (resp. $\psi\colon V\to W_\Omega$) compatible with the descent datum, i.e., such that $\varphi_\sigma\circ\sigma\psi=\psi$ (resp. $\psi\circ\varphi_\sigma = \sigma\psi$).*

PROOF. Special case of the proposition in which W_Ω is endowed with its natural descent datum. □

REMARK 16.18. Proposition 16.16 implies that the functor taking a variety V over k to V_Ω over Ω endowed with its natural descent datum is fully faithful.

Let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be an Ω/k-descent system on V. For a subvariety W of V, we set ${}^\sigma W = \varphi_\sigma(\sigma W)$. Then the following diagram commutes:

$$\begin{tikzcd} \sigma V \arrow[r, "\varphi_\sigma", "\simeq"'] & V \\ \sigma W \arrow[u, hook] \arrow[r, "\varphi_\sigma|\sigma W", "\simeq"'] & {}^\sigma W \arrow[u, hook] \end{tikzcd}$$

LEMMA 16.19. *The following hold.*

(a) *For all $\sigma,\tau\in\Gamma$ and $W\subset V$, ${}^\sigma({}^\tau W) = {}^{\sigma\tau}W$.*

(b) *Suppose that $(\varphi_\sigma)_{\sigma\in\Gamma}$ is split by a model (V_0,φ) of V over k_0, and let W be a subvariety of V. If $W=\varphi^{-1}(W_{0\Omega})$ for some subvariety W_0 of V_0, then ${}^\sigma W = W$ for all $\sigma\in\Gamma$; the converse is true if $\Omega^\Gamma = k$.*

PROOF. (a) By definition

$$^\sigma({}^\tau W) = \varphi_\sigma(\sigma(\varphi_\tau(\tau W)) = (\varphi_\sigma\circ\sigma\varphi_\tau)(\sigma\tau W) = \varphi_{\sigma\tau}(\sigma\tau W) = {}^{\sigma\tau}W.$$

In the second equality, we used that $(\sigma\varphi)(\sigma W)=\sigma(\varphi W)$.

(b) Let $W=\varphi^{-1}(W_{0\Omega})$. By hypothesis $\varphi_\sigma=\varphi^{-1}\circ\sigma\varphi$, and so

$$^\sigma W = (\varphi^{-1}\circ\sigma\varphi)(\sigma W) = \varphi^{-1}(\sigma(\varphi W)) = \varphi^{-1}(\sigma W_{0\Omega}) = \varphi^{-1}(W_{0\Omega}) = W.$$

Conversely, suppose ${}^\sigma W = W$ for all $\sigma\in\Gamma$. Then

$$\varphi(W) = \varphi({}^\sigma W) = (\sigma\varphi)(\sigma W) = \sigma(\varphi(W)).$$

Therefore, $\varphi(W)$ is stable under the action of Γ on $V_{0\Omega}$, and so is defined over k (see 16.8). □

For a descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on V and a regular function f on an open subset U of V, we define ${}^\sigma f$ to be the function $(\sigma f)\circ\varphi_\sigma^{-1}$ on ${}^\sigma U$, so that ${}^\sigma f({}^\sigma P) = \sigma(f(P))$ for all $P\in U$. Then ${}^\sigma({}^\tau f) = {}^{\sigma\tau}f$, and so this defines an action of Γ on the regular functions.

The ***Krull topology*** on Γ is that for which the subgroups of Γ fixing a subfield of Ω finitely generated over k form a basis of open neighbourhoods of 1 (see FT Chapter 7). An action of Γ on an Ω-vector space V is ***continuous*** if

$$V = \bigcup_\Delta V^\Delta \qquad \text{(union over the open subgroups } \Delta \text{ of } \Gamma).$$

For a subfield L of Ω containing k, let $\Delta_L = \mathrm{Aut}(\Omega/L)$.

PROPOSITION 16.20. *Assume that Ω is separably closed. A descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on an affine variety V is continuous if and only if the action of Γ on $\Omega[V]$ is continuous.*

PROOF. If $(\varphi_\sigma)_{\sigma\in\Gamma}$ is continuous, it is split by a model of V over a subfield K of Ω finitely generated over k. By definition, Δ_K is open, and $\Omega[V]^{\Delta_K}$ contains a set $\{f_1,\ldots,f_n\}$ of generators for $\Omega[V]$ as an Ω-algebra. Now $\Omega[V]=\bigcup L[f_1,\ldots,f_n]$ where L runs over the subfields of Ω containing K and finitely generated over k. As $L[f_1,\ldots,f_n]=\Omega[V]^{\Delta_L}$, this shows that $\Omega[V]=\bigcup\Omega[V]^{\Delta_L}$.

Conversely, if the action of Γ on $\Omega[V]$ is continuous, then for some subfield L of Ω finitely generated over k, $\Omega[V]^{\Delta_L}$ will contain a set of generators $f_1,\ldots,f_n$ for $\Omega[V]$ as an Ω-algebra. According to (16.3), Ω^{Δ_L} is a purely inseparable algebraic extension of L, and so, after replacing L with a finite extension, the embedding $V\hookrightarrow\mathbb{A}^n$ defined by the f_i will determine a model of V over L. This model splits $(\varphi_\sigma)_{\sigma\in\Gamma}$, which is therefore continuous. □

PROPOSITION 16.21. *A descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on a variety V over Ω is continuous if there exists a finite set S of points in $V(\Omega)$ such that*

(a) *any automorphism of V fixing all $P\in S$ is the identity map, and*

(b) *there exists a subfield K of Ω finitely generated over k such that ${}^\sigma P=P$ for all $\sigma\in\Gamma$ fixing K.*

PROOF. Let (V_0,φ) be a model of V over a subfield K of Ω finitely generated over k. After possibly replacing K by a larger finitely generated field, we may suppose (i) that ${}^\sigma P=P$ for all $\sigma\in\Gamma$ fixing K and all $P\in S$ (because of (b)) and (ii) that $\varphi(P)\in V_0(K)$ for all $P\in S$ (because S is finite). Then, for $P\in S$ and every σ fixing K,

$$\varphi_\sigma(\sigma P)\overset{\text{def}}{=}{}^\sigma P\overset{\text{(i)}}{=}P$$

$$(\sigma\varphi)(\sigma P)=\sigma(\varphi P)\overset{\text{(ii)}}{=}\varphi P,$$

and so φ_σ and $\varphi^{-1}\circ\sigma\varphi$ are isomorphisms $\sigma V\to V$ sending σP to P. Therefore, φ_σ and $\varphi^{-1}\circ\sigma\varphi$ differ by an automorphism of V fixing the $P\in S$, which implies that they are equal. This says that (V_0,φ) splits $(\varphi_\sigma)_{\sigma\in\Gamma}$. □

PROPOSITION 16.22. *Let V be a variety over Ω whose only automorphism is the identity map. A descent datum on V is effective if V has a model over k.*

PROOF. Let (V,φ) be a model of V over k. For $\sigma\in\Gamma$, the maps φ_σ and $\varphi^{-1}\circ\sigma\varphi$ are both isomorphisms $\sigma V\to V$, and so differ by an automorphism of V. Therefore they are equal, which says that (V,φ) splits $(\varphi_\sigma)_{\sigma\in\Gamma}$. □

Of course, in Proposition 16.21, S doesn't have to be a finite set of points. The proposition will hold with S any additional structure on V that rigidifies V (i.e., is such that $\mathrm{Aut}(V,S)=1$) and is such that (V,S) has a model over a finitely generated extension of k.

g Galois descent of varieties

In this subsection, Ω is a Galois extension of k with Galois group Γ.

THEOREM 16.23. *A descent datum $(\varphi_\sigma)_{\sigma\in\Gamma}$ on a variety V is effective if V is covered by open affines U with the property that ${}^\sigma U=U$ for all $\sigma\in\Gamma$.*

PROOF. Assume first that V is affine, and let $A = k[V]$. A descent datum $(\varphi_\sigma)_{\sigma\in\Gamma}$ defines a continuous action of Γ on A (see 16.20). From (16.15), we know that

$$c \otimes a \mapsto ca\colon \Omega \otimes_k A^\Gamma \to A \tag{6}$$

is an isomorphism. Let $V_0 = \operatorname{Spm} A^\Gamma$, and let φ be the isomorphism $V \to V_{0\Omega}$ defined by (6). Then (V_0, φ) splits the descent datum.

In the general case, write V as a finite union of open affines U_i such that ${}^\sigma U_i = U_i$ for all $\sigma \in \Gamma$. Then V is the variety over Ω obtained by patching the U_i by means of the maps

$$U_i \hookleftarrow U_i \cap U_j \hookrightarrow U_j. \tag{7}$$

Each intersection $U_i \cap U_j$ is again affine (AG, 5.29), and so the system (7) descends to k. The variety over k obtained by patching the descended system is a model of V over k splitting the descent datum. □

COROLLARY 16.24. *If each finite set of points of $V(\Omega^{\mathrm{sep}})$ is contained in an open affine subvariety of $V_{\Omega^{\mathrm{sep}}}$, then every descent datum on V is effective.*

PROOF. As we noted before, an Ω/k-descent datum for V extends in a natural way to an Ω^{sep}/k-descent datum for $V_{\Omega^{\mathrm{sep}}}$, and if a model (V_0, φ) over k splits the second descent datum, then it also splits the first. Thus, we may suppose that Ω is separably closed.

Let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be a descent datum on V, and let U be a subvariety of V. By definition, (φ_σ) is split by a model (V_1, φ) of V over some finite extension k_1 of k. After possibly replacing k_1 with a larger finite extension, there will exist a subvariety U_1 of V_1 such that $\varphi(U) = U_{1\Omega}$. Now (16.19b) shows that ${}^\sigma U$ depends only on the coset $\sigma\Delta$ where $\Delta = \mathrm{Gal}(\Omega/k_1)$. In particular, $\{{}^\sigma U \mid \sigma \in \Gamma\}$ is finite. The subvariety $\bigcap_{\sigma\in\Gamma} {}^\sigma U$ is stable under Γ, and so (see 16.8, 16.19) ${}^\tau(\bigcap_{\sigma\in\Gamma} {}^\sigma U) = (\bigcap_{\sigma\in\Gamma} {}^\sigma U)$ for all $\tau \in \Gamma$.

Let $P \in V$. Because $\{{}^\sigma P \mid \sigma \in \Gamma\}$ is finite, it is contained in an open affine U of V. Now $U' = \bigcap_{\sigma\in\Gamma} {}^\sigma U$ is an open affine in V containing P and such that ${}^\sigma U' = U'$ for all $\sigma \in \Gamma$. It follows that the variety V satisfies the hypothesis of Theorem 16.23. □

COROLLARY 16.25. *Descent is effective in each of the following two cases:*

(a) *V is quasiprojective, or*

(b) *an affine algebraic group G acts transitively on V.*

PROOF. (a) Apply (AG, 6.28) (whose proof applies unchanged over any infinite base field).

(b) We may assume Ω to be separably closed. Let S be a finite set of points of $V(\Omega)$, and let U be an open affine in V. For each $P \in S$, there is a nonempty open subvariety G_P of G such that $G_P \cdot P \subset U$. Because Ω is separably closed, there exists a $g \in (\bigcap_{P\in S} G_P \cdot P)(\Omega)$ (see Chapter 11; the separable points are dense in a variety). Now $g^{-1}U$ is an open affine containing S. □

h Weil restriction

Let K/k be a finite extension of fields, and let V be a variety over K. A pair (V_*, φ) consisting of a variety V_* over k and a regular map $\varphi\colon V_{*K} \to V$ is called the K/k-***Weil restriction*** of V if it has the following universal property: for any variety T over k and regular map $\varphi'\colon T_K \to V$, there exists a unique regular map $\psi\colon T \to V$ (of k-varieties) such that $\varphi \circ \psi_K = \varphi'$, i.e., given

$$\begin{tikzcd} T_K \arrow[dr, "\varphi'"] & \\ V_{*K} \arrow[r, "\varphi"] & V \end{tikzcd} \quad \text{there exists a unique} \quad \begin{tikzcd} T \arrow[d, "\psi"'] \\ V_* \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} T_K \arrow[d, "\psi_K"'] \arrow[dr, "\varphi'"] & \\ V_{*K} \arrow[r, "\varphi"] & V \end{tikzcd} \quad \text{commutes.}$$

In other words, (V_*, φ) is the K/k-Weil restriction of V if φ defines an isomorphism

$$\mathrm{Mor}_k(T, V_*) \to \mathrm{Mor}_K(T_K, V)$$

(natural in the k-variety T); in particular,

$$V_*(A) \simeq V(K \otimes_k A)$$

(natural in the affine k-algebra A). If it exists, the K/k-Weil restriction of V is uniquely determined by its universal property (up to a unique isomorphism).

When (V_*, φ) is the K/k-Weil restriction of V, the variety V_* is said to have been obtained from V ***by (Weil) restriction of scalars*** or ***by restriction of the base field***.

PROPOSITION 16.26. *If V satisfies the hypothesis of (16.24) (for example, if V is quasiprojective) and K/k is separable, then the K/k-Weil restriction exists.*

PROOF. Let Ω be a Galois extension of k large enough to contain all conjugates of K, i.e., such that $\Omega \otimes_k K \simeq \prod_{\tau: K \to \Omega} \tau K$. Let $V' = \prod \tau V$ — this is a variety over Ω. For $\sigma \in \mathrm{Gal}(\Omega/k)$, define $\varphi_\sigma\colon \sigma V' \to V'$ to be the regular map that acts on the factor $\sigma(\tau V)$ as the canonical isomorphism $\sigma(\tau V) \simeq (\sigma\tau)V$. Then $(\varphi_\sigma)_{\sigma \in \mathrm{Gal}(\Omega/k)}$ is a descent datum, and so defines a model (V_*, φ_*) of V' over k.

Choose a $\tau_0\colon K \to \Omega$. The projection map $V' \to \tau_0 V$ is invariant under the action of $\mathrm{Gal}(\Omega/\tau_0 K)$, and so defines a regular map $(V_*)_{\tau_0 K} \to \tau_0 V$ (16.9), and hence a regular map $\varphi\colon V_{*K} \to V$. It is easy to check that this has the correct universal property. □

i Generic fibres and specialization

In this subsection, k is an algebraically closed field.

Let $\varphi\colon V \to U$ be a dominant map with U irreducible, and let $K = k(U)$. Then there is a regular map $\varphi_K\colon V_K \to \mathrm{Spm} K$, called the ***generic fibre*** of φ. For example, if V and U are affine, so that φ corresponds to an injective homomorphism of rings $f\colon A \to B$, then φ_K corresponds to $A \otimes_k K \to B \otimes_k K$. In the general case, we replace U with any open affine and write V as a finite union of affines $V = \bigcup_i V_i$; then $V_K = \bigcup_i V_{iK}$.

Let K be a field finitely generated over k, and let V be a variety over K. For any irreducible k-variety U with $k(U) = K$, there will exist a dominant map $\varphi\colon V \to U$ with generic fibre V. For example, we can take $U = \mathrm{Spm}(A)$ where A is any finitely generated k-subalgebra of K containing a set of generators for K and containing the coefficients of some set of polynomials defining V. Let P be a point in the image of φ. Then the fibre of V over P is a variety $V(P)$ over k, called the ***specialization*** of V at P.

Similar statements are true for morphisms of varieties.

j Rigid descent

LEMMA 16.27. *Let V and W be varieties over an algebraically closed field k. If V and W become isomorphic over some field containing k, then they are already isomorphic over k.*

PROOF. The hypothesis implies that, for some field K finitely generated over k, there exists an isomorphism $\varphi\colon V_K \to W_K$. Let U be an affine k-variety such that $k(U) = K$. After possibly replacing U with an open subset, we can extend φ to an isomorphism $\varphi_U\colon U \times V \to U \times W$. The fibre of φ_U at any point of U is an isomorphism $V \to W$. □

Consider fields $\Omega \supset K_1, K_2 \supset k$. Recall (Chapter 11) that K_1 and K_2 are said to be linearly disjoint over k if the homomorphism

$$\sum a_i \otimes b_i \mapsto \sum a_i b_i\colon K_1 \otimes_k K_2 \to K_1 \cdot K_2$$

is injective.

LEMMA 16.28. *Let $\Omega \supset k$ be algebraically closed fields, and let V be a variety over Ω. If there exist models of V over subfields K_1, K_2 of Ω finitely generated over k and linearly disjoint over k, then there exists a model of V over k.*

PROOF. The model of V over K_1 extends to a model over an irreducible affine variety U_1 with $k(U_1) = K_1$, i.e., there exists a surjective map $V_1 \to U_1$ of k-varieties whose generic fibre is a model of V over K_1. A similar statement applies to the model over K_2. Because K_1 and K_2 are linearly disjoint, $K_1 \otimes_k K_2$ is an integral domain with field of fractions $k(U_1 \times U_2)$. From the map $V_1 \to U_1$, we get a map $V_1 \times U_2 \to U_1 \times U_2$, and similarly for V_2.

Assume initially that $V_1 \times U_2$ and $U_1 \times V_2$ are isomorphic over $U_1 \times U_2$, so that we have a commutative diagram:

$$\begin{tikzcd}
V_1 \arrow[d] & V_1 \times U_2 \arrow[l] \arrow[rr, "\approx"] \arrow[dr] & & U_1 \times V_2 \arrow[r] \arrow[dl] & V_2 \arrow[d] \\
U_1 & & U_1 \times U_2 \arrow[ll] \arrow[rr] & & U_2
\end{tikzcd}$$

Let P be a point of U_1. When we pull back the triangle to the subvariety $P \times U_2$ of $U_1 \times U_2$, we get the diagram at left below. Note that $P \times U_2 \simeq U_2$ and that $P \simeq \operatorname{Spm} k$ (because k is algebraically closed).

$$\begin{tikzcd}
V(P) \times U_2 \arrow[rr, "\approx"] \arrow[dr] & & P \times V_2 \arrow[dl] \\
& P \times U_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
V(P)_{K_2} \arrow[rr, "\approx"] \arrow[dr] & & V_{2K_2} \arrow[dl] \\
& \operatorname{Spm}(K_2) &
\end{tikzcd}$$

The generic fibre of this diagram is the diagram at right. Here $V_1(P)_{K_2}$ is the variety over K_2 obtained from $V_1(P)$ by extension of scalars $k \to K_2$. As V_{2K_2} is a model V over K_2, it follows that $V_1(P)$ is a model of V over k.

We now prove the general case. The varieties $(V_1 \times U_2)_{k(U_1\times U_2)}$ and $(U_1 \times V_2)_{k(U_1\times U_2)}$ become isomorphic over some finite field extension L of $k(U_1 \times U_2)$. Let $\bar{U}$ be the normalization[1] of $U_1 \times U_2$ in L, and let U be a dense open subset of $\bar{U}$ such that some isomorphism of $(V_1 \times U_2)_L$ with $(U_1 \times V_2)_L$ extends to an isomorphism over U. The going-up theorem (AG, 1.44) shows that $\bar{U} \to U_1 \times U_2$ is surjective, and so the image U' of U in $U_1 \times U_2$ contains a nonempty (hence dense) open subset of $U_1 \times U_2$ (see AG, 9.1). In particular, U' contains a subset $P \times U_2'$ with U_2' a nonempty open subset of U_2. Now the previous argument gives us varieties $V_1(P)_{K_2}$ and V_{2K_2} over K_2 that become isomorphic over $k(U'')$ where U'' is the inverse image of $P \times U_2'$ in $\bar{U}$. As $k(U'')$ is a finite extension of K_2, this again shows that $V_1(P)$ is a model of V over k. □

EXAMPLE 16.29. Let E be an elliptic curve over Ω with j-invariant $j(E)$. There exists a model of E over a subfield K of Ω if and only if $j(E) \in K$. If $j(E)$ is transcendental, then any two such fields contain $k(j(E))$, and so can't be linearly disjoint. Therefore, the hypothesis in the proposition implies $j(E) \in k$, and so E has a model over k.

LEMMA 16.30. *Let Ω be algebraically closed of infinite transcendence degree over k, and assume that k is algebraically closed in Ω. For any $K \subset \Omega$ finitely generated over k, there exists a $\sigma \in \mathrm{Aut}(\Omega/k)$ such that K and σK are linearly disjoint over k.*

PROOF. Let $a_1,\ldots,a_n$ be a transcendence basis for K/k, and extend it to a transcendence basis $a_1,\ldots,a_n,b_1,\ldots,b_n,\ldots$ of Ω/k. Let σ be any permutation of the transcendence basis such that $\sigma(a_i) = b_i$ for all i. Then σ defines a k-automorphism of $k(a_1,\ldots a_n,b_1,\ldots,b_n,\ldots)$, which we extend to an automorphism of Ω.

Let $K_1 = k(a_1,\ldots,a_n)$. Then $\sigma K_1 = k(b_1,\ldots,b_n)$, and certainly K_1 and σK_1 are linearly disjoint. In particular, $K_1 \otimes_k \sigma K_1$ is an integral domain. Because k is algebraically closed in K, $K \otimes_k \sigma K$ is an integral domain (cf. AG, 5.17). This implies that K and σK are linearly disjoint. □

LEMMA 16.31. *Let $\Omega \supset k$ be algebraically closed fields such that Ω is of infinite transcendence degree over k, and let V be a variety over Ω. If V is isomorphic to σV for every $\sigma \in \mathrm{Aut}(\Omega/k)$, then V has a model over k.*

PROOF. There will exist a model V_0 of V over a subfield K of Ω finitely generated over k. According to Lemma 16.30, there exists a $\sigma \in \mathrm{Aut}(\Omega/k)$ such that K and σK are linearly disjoint. Because $V \approx \sigma V$, σV_0 is a model of V over σK, and we can apply Lemma 16.28. □

In the next two theorems, $\Omega \supset k$ are fields such that the fixed field of $\Gamma = \mathrm{Aut}(\Omega/k)$ is k and Ω is algebraically closed

THEOREM 16.32. *Let V be a quasiprojective variety over Ω, and let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be a descent system for V. If the only automorphism of V is the identity map, then V has a model over k splitting (φ_σ).*

PROOF. According to Lemma 16.31, V has a model (V_0,φ) over the algebraic closure k^{al} of k in Ω, which (see the proof of 16.22) splits $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k^{\mathrm{al}})}$.

Now $\varphi'_\sigma \overset{\text{def}}{=} \varphi^{-1} \circ \varphi_\sigma \circ \sigma\varphi$ is stable under $\mathrm{Aut}(\Omega/k^{\mathrm{al}})$, and hence is defined over k^{al} (16.9). Moreover, φ'_σ depends only on the restriction of σ to k^{al}, and $(\varphi'_\sigma)_{\sigma\in\mathrm{Gal}(k^{\mathrm{al}}/k)}$ is a

[1] Let $U_1 \times U_2 = \mathrm{Spm}\, C$; then $\bar{U} = \mathrm{Spm}\, \bar{C}$, where $\bar{C}$ is the integral closure of C in L.

descent system for V_0. It is continuous by (16.21), and so V_0 has a model (V_{00}, φ') over k splitting $(\varphi'_\sigma)_{\sigma \in \mathrm{Gal}(k^{\mathrm{al}}/k)}$. Now $(V_{00}, \varphi \circ \varphi'_\Omega)$ splits $(\varphi_\sigma)_{\sigma \in \mathrm{Aut}(\Omega/k)}$. □

We now consider pairs (V, S) where V is a variety over Ω and S is a family of points $S = (P_i)_{1 \le i \le n}$ of V indexed by $[1, n]$. A morphism $(V, (P_i)_{1 \le i \le n}) \to (W, (Q_i)_{1 \le i \le n})$ is a regular map $\varphi \colon V \to W$ such that $\varphi(P_i) = Q_i$ for all i.

THEOREM 16.33. *Let V be a quasiprojective variety over Ω, and let $(\varphi_\sigma)_{\sigma \in \mathrm{Aut}(\Omega/k)}$ be a descent system for V. Let $S = (P_i)_{1 \le i \le n}$ be a finite set of points of V such that*

(a) *the only automorphism of V fixing each P_i is the identity map, and*

(b) *there exists a subfield K of Ω finitely generated over k such that ${}^\sigma P = P$ for all $\sigma \in \Gamma$ fixing K.*

Then V has a model over k splitting (φ_σ).

PROOF. Lemmas 16.27–16.31 all hold for pairs (V, S) (with the same proofs), and so the proof of Theorem 16.32 applies. □

EXAMPLE 16.34. Theorem 16.33 can be used to prove that certain abelian varieties attached to algebraic varieties in characteristic zero, for example, the generalized Jacobian varieties, are defined over the same field as the variety.[2] We illustrate this with the usual Jacobian variety J of a complete nonsingular curve C. For such a curve C over $\mathbb{C}$, there is a principally polarized abelian variety $J(C)$ such that, as a complex manifold,

$$J(C)(\mathbb{C}) = \Gamma(C, \Omega^1)^\vee / H_1(C, \mathbb{Z}).$$

The association $C \mapsto J(C)$ is a functorial, and so a descent datum $(\varphi_\sigma)_{\sigma \in \mathrm{Aut}(\Omega/k)}$ on C defines a descent system on $J(C)$. It is known that if we take S to be the set of points of order 3 on $J(C)$, then condition (a) of the theorem is satisfied (see, for example, Milne 1986[3], 17.5), and condition (b) can be seen to be satisfied by regarding $J(C)$ as the Picard variety of C.

k Weil's descent theorems

THEOREM 16.35. *Let k be a finite separable extension of a field k_0, and let I be the set of k-homomorphisms $k \to k_0^{\mathrm{al}}$. Let V be a quasiprojective variety over k; for each pair (σ, τ) of elements of I, let $\varphi_{\tau,\sigma}$ be an isomorphism $\sigma V \to \tau V$ (of varieties over k_0^{al}). Then there exists a variety V_0 over k_0 and an isomorphism $\varphi \colon V_{0k} \to V$ such that $\varphi_{\tau,\sigma} = \tau\varphi \circ (\sigma\varphi)^{-1}$ for all $\sigma, \tau \in I$ if and only if the $\varphi_{\tau,\sigma}$ are defined over k_0^{sep} and satisfy the following conditions:*

(a) $\varphi_{\tau,\rho} = \varphi_{\tau,\sigma} \circ \varphi_{\sigma,\rho}$ *for all $\rho, \sigma, \tau \in I$;*

(b) $\varphi_{\tau\omega,\sigma\omega} = \omega\varphi_{\tau,\sigma}$ *for all $\sigma, \tau \in I$ and all k_0-automorphisms ω of k_0^{al} over k_0.*

Moreover, when this is so, the pair (V_0, φ) is unique up to isomorphism over k_0, and V_0 is quasiprojective or quasi-affine if V is.

PROOF. This is Theorem 3 of Weil 1956,[4] p515. It is essentially a restatement of (a) of Corollary 16.25 (and (V_0, φ) is unique up to a *unique* isomorphism over k_0). □

[2] This was pointed out to me by Niranjan Ramachandran.

[3] Milne, J.S., Abelian varieties, in Arithmetic Geometry, Springer, 1986.

[4] Weil, André, The field of definition of a variety. Amer. J. Math. 78 (1956), 509–524.

An extension K of a field k is said to be ***regular*** if it is finitely generated, admits a separating transcendence basis, and k is algebraically closed in K. These are precisely the fields that arise as the field of rational functions on geometrically irreducible algebraic variety over k.

Let k be a field, and let $k(t)$, $t=(t_1,\ldots,t_n)$, be a regular extension of k (in Weil's terminology, t is a *generic point* of a variety over k). By $k(t')$ we shall mean a field isomorphic to $k(t)$ by $t\mapsto t'$, and we write $k(t,t')$ for the field of fractions of $k(t)\otimes_k k(t')$.[5] When V_t is a variety over $k(t)$, we shall write $V_{t'}$ for the variety over $k(t')$ obtained from V_t by base change with respect to $t\mapsto t'\colon k(t)\to k(t')$. Similarly, if f_t denotes a regular map of varieties over $k(t)$, then $f_{t'}$ denotes the regular map over $k(t')$ obtained by base change. Similarly, $k(t'')$ is a second field isomorphic to $k(t)$ by $t\mapsto t''$ and $k(t,t',t'')$ is the field of fractions of $k(t)\otimes_k k(t')\otimes_k k(t'')$.

THEOREM 16.36. *With the above notations, let V_t be a quasiprojective variety over $k(t)$; for each pair (t,t'), let $\varphi_{t',t}$ be an isomorphism $V_t\to V_{t'}$ defined over $k(t,t')$. Then there exists a variety V defined over k and an isomorphism $\varphi_t\colon V_{k(t)}\to V_t$ (of varieties over $k(t)$) such that $\varphi_{t',t}=\varphi_{t'}\circ\varphi_t^{-1}$ if and only if $\varphi_{t',t}$ satisfies the following condition:*

$$\varphi_{t'',t}=\varphi_{t'',t'}\circ\varphi_{t',t}\quad \text{(isomorphism of varieties over } k(t,t',t'')\text{)}.$$

Moreover, when this is so, the pair (V,φ_t) is unique up to an isomorphism over k, and V is quasiprojective or quasi-affine if V is.

PROOF. This is Theorem 6 and Theorem 7 of Weil 1956, p522. □

THEOREM 16.37. *Let Ω be an algebraically closed field of infinite transcendence degree over a perfect field k. Then descent is effective for quasiprojective varieties over Ω.*

PROOF. Let (φ_σ) be a descent datum on a variety V over Ω. Because (φ_σ) is continuous, it is split by a model of V over some subfield K of Ω finitely generated over k. Let k' be the algebraic closure of k in K; then k' is a finite extension of k and K is a regular extension of k. Write $K=k(t)$, and let (V_t,φ') be a model of V over $k(t)$ splitting (φ_σ). According to Lemma 16.30, there exists a $\sigma\in\mathrm{Aut}(\Omega/k)$ such that $\sigma k(t)=k(t')$ and $k(t)$ are linearly disjoint over k. The isomorphism

$$V_{t\Omega}\xrightarrow{\varphi'}V\xrightarrow{\varphi_\sigma^{-1}}\sigma V\xrightarrow{(\sigma\varphi')^{-1}}V_{t',\Omega}$$

is defined over $k(t,t')$ and satisfies the conditions of Theorem 16.36. Therefore, there exists a model (W,φ) of V over k' splitting $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k(t)}$.

For $\sigma,\tau\in\mathrm{Aut}(\Omega/k)$, let $\varphi_{\tau,\sigma}$ be the composite of the isomorphisms

$$\sigma W\xrightarrow{\sigma\varphi}\sigma V\xrightarrow{\varphi_\sigma}V\xrightarrow{\varphi_\tau^{-1}}\tau V\xrightarrow{\tau\varphi}\tau W.$$

Then $\varphi_{\tau,\sigma}$ is defined over the algebraic closure of k in Ω and satisfies the conditions of Theorem 16.35, which gives a model of W over k splitting $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$. □

[5] If $k(t)$ and $k(t')$ are linearly disjoint subfields of some large field Ω, then $k(t,t')$ is the subfield of Ω generated over k by t and t'.

l Restatement in terms of group actions

In this subsection, $\Omega \supset k$ are fields such that $k = \Omega^{\Gamma}$ and Ω is algebraically closed. Recall that for any variety V over k, there is a natural action of Γ on $V(\Omega)$. In this subsection, we describe the essential image of the functor

$$\{\text{quasiprojective varieties over } k\} \to \{\text{quasiprojective varieties over } \Omega + \text{action of } \Gamma\}.$$

In other words, we determine which pairs $(V, *)$, with V a quasiprojective variety over Ω and $*$ an action of Γ on $V(\Omega)$,

$$(\sigma, P) \mapsto \sigma * P \colon \Gamma \times V(\Omega) \to V(\Omega),$$

arise from a variety over k. There are two obvious necessary conditions for this.

REGULARITY CONDITION

Obviously, the action should recognize that $V(\Omega)$ is not just a set, but rather the set of points of an algebraic variety. For $\sigma \in \Gamma$, let σV be the variety obtained by applying σ to the coefficients of the equations defining V, and for $P \in V(\Omega)$ let σP be the point on σV obtained by applying σ to the coordinates of P.

DEFINITION 16.38. We say that the action $*$ is ***regular*** if the map

$$\sigma P \mapsto \sigma * P \colon (\sigma V)(\Omega) \to V(\Omega)$$

is regular isomorphism for all σ.

A priori, this is only a map of sets. The condition requires that it be induced by a regular map $\varphi_\sigma \colon \sigma V \to V$. If $V = V_{0\Omega}$ for some variety V_0 defined over k, then $\sigma V = V$, and φ_σ is the identity map, and so the condition is clearly necessary.

REMARK 16.39. The maps φ_σ satisfy the cocycle condition $\varphi_\sigma \circ \sigma\varphi_\tau = \varphi_{\sigma\tau}$. In particular, $\varphi_\sigma \circ \sigma\varphi_{\sigma^{-1}} = \mathrm{id}$, and so if $*$ is regular, then each φ_σ is an isomorphism, and the family $(\varphi_\sigma)_{\sigma\in\Gamma}$ is a descent system. Conversely, if $(\varphi_\sigma)_{\sigma\in\Gamma}$ is a descent system, then

$$\sigma * P = \varphi_\sigma(\sigma P)$$

defines a regular action of Γ on $V(\Omega)$. Note that if $* \leftrightarrow (\varphi_\sigma)$, then $\sigma * P = {}^{\sigma}P$.

CONTINUITY CONDITION

DEFINITION 16.40. We say that the action $*$ is ***continuous*** if there exists a subfield L of Ω finitely generated over k and a model V_0 of V over L such that the action of $\Gamma(\Omega/L)$ is that defined by V_0.

For an affine variety V, an action of Γ on V gives an action of Γ on $\Omega[V]$, and one action is continuous if and only if the other is.

Continuity is obviously necessary. It is easy to write down regular actions that fail it, and hence don't arise from varieties over k.

EXAMPLE 16.41. The following are examples of actions that fail the continuity condition ((b) and (c) are regular).

(a) Let $V = \mathbb{A}^1$ and let $*$ be the trivial action.

(b) Let $\Omega/k = \mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$, and let N be a normal subgroup of finite index in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ that is not open,[6] i.e., that fixes no extension of $\mathbb{Q}$ of finite degree. Let V be the zero-dimensional variety over $\mathbb{Q}^{\mathrm{al}}$ with $V(\mathbb{Q}^{\mathrm{al}}) = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})/N$ with its natural action.

(c) Let k be a finite extension of $\mathbb{Q}_p$, and let $V = \mathbb{A}^1$. The homomorphism $k^{\times} \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ can be used to twist the natural action of Γ on $V(\Omega)$.

RESTATEMENT OF THE MAIN THEOREMS

Let $\Omega \supset k$ be fields such that k is the fixed field of $\Gamma = \mathrm{Aut}(\Omega/k)$ and Ω is algebraically closed.

THEOREM 16.42. *Let V be a quasiprojective variety over Ω, and let $*$ be a regular action of Γ on $V(\Omega)$. Let $S = (P_i)_{1 \le i \le n}$ be a finite set of points of V such that*

(a) *the only automorphism of V fixing each P_i is the identity map, and*

(b) *there exists a subfield K of Ω finitely generated over k such that $\sigma * P = P$ for all $\sigma \in \Gamma$ fixing K.*

Then $*$ arises from a model of V over k.

PROOF. This a restatement of Theorem 16.33. □

THEOREM 16.43. *Let V be a quasiprojective variety over Ω with an action $*$ of Γ. If $*$ is regular and continuous, then $*$ arises from a model of V over k in each of the following cases:*

(a) *Ω is algebraic over k, or*

(b) *Ω is has infinite transcendence degree over k.*

PROOF. Restatements of (16.23, 16.25) and of (16.37). □

The condition "quasiprojective" is necessary, because otherwise the action may not stabilize enough open affine subsets to cover V.

m Faithfully flat descent

Recall that a homomorphism $f\colon A \to B$ of rings is flat if the functor "extension of scalars" $M \mapsto B \otimes_A M$ is exact. It is ***faithfully flat*** if a sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-modules is exact if and only if

$$0 \to B \otimes_A M' \to B \otimes_A M \to B \otimes_A M'' \to 0$$

[6] For a proof that such subgroups exist, see FT 7.26.

is exact. For a field k, a homomorphism $k \to A$ is always flat (because exact sequences of k-vector spaces are split-exact), and it is faithfully flat if $A \neq 0$.

The next theorem and its proof are quintessential Grothendieck.

THEOREM 16.44. *If $f: A \to B$ is faithfully flat, then the sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{d^0} B^{\otimes 2} \to \cdots \to B^{\otimes r} \xrightarrow{d^{r-1}} B^{\otimes r+1} \to \cdots$$

is exact, where

$$\begin{aligned} B^{\otimes r} &= B \otimes_A B \otimes_A \cdots \otimes_A B \qquad (r \text{ times}) \\ d^{r-1} &= \sum (-1)^i e_i \\ e_i(b_0 \otimes \cdots \otimes b_{r-1}) &= b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}. \end{aligned}$$

PROOF. It is easily checked that $d^r \circ d^{r-1} = 0$. We assume first that f admits a section, i.e., that there is a homomorphism $g: B \to A$ such that $g \circ f = 1$, and we construct a contracting homotopy $k_r: B^{\otimes r+2} \to B^{\otimes r+1}$. Define

$$k_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0) b_1 \otimes \cdots \otimes b_{r+1}, \qquad r \geq -1.$$

It is easily checked that

$$k_{r+1} \circ d^{r+1} + d^r \circ k_r = 1, \quad r \geq -1,$$

and this shows that the sequence is exact.

Now let A' be an A-algebra. Let $B' = A' \otimes_A B$ and let $f' = 1 \otimes f: A' \to B'$. The sequence corresponding to f' is obtained from the sequence for f by tensoring with A' (because $B^{\otimes r} \otimes A' \cong B'^{\otimes f}$ etc.). Thus, if A' is a faithfully flat A-algebra, it suffices to prove the theorem for f'. Take $A' = B$, and then $b \overset{f}{\mapsto} b \otimes 1: B \to B \otimes_A B$ has a section, namely, $g(b \otimes b') = bb'$, and so the sequence is exact. □

THEOREM 16.45. *If $f: A \to B$ is faithfully flat and M is an A-module, then the sequence*

$$0 \to M \xrightarrow{1 \otimes f} M \otimes_A B \xrightarrow{1 \otimes d^0} M \otimes_A B^{\otimes 2} \to \cdots \to M \otimes_B B^{\otimes r} \xrightarrow{1 \otimes d^{r-1}} B^{\otimes r+1} \to \cdots$$

is exact.

PROOF. As in the above proof, one may assume that f has a section, and use it to construct a contracting homotopy. □

REMARK 16.46. Let $f: A \to B$ be a faithfully flat homomorphism, and let M be an A-module. Write M' for the B-module $f_* M = B \otimes_A M$. The module $e_{0*} M' = (B \otimes_A B) \otimes_B M'$ may be identified with $B \otimes_A M'$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(b \otimes m) = b_1 b \otimes b_2 m$, and $e_{1*} M'$ may be identified with $M' \otimes_A B$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(m \otimes b) = b_1 m \otimes b_2 b$. There is a canonical isomorphism $\phi: e_{1*} M' \to e_{0*} M'$ arising from

$$e_{1*} M' = (e_1 f)_* M = (e_0 f)_* M = e_{0*} M';$$

explicitly, it is the map

$$(b \otimes m) \otimes b' \mapsto b \otimes (b' \otimes m): M' \otimes_A B \to B \otimes_A M.$$

Moreover, M can be recovered from the pair (M',ϕ) because

$$M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}.$$

Conversely, every pair (M',ϕ) satisfying certain obvious conditions does arise in this way from an A-module. Given $\phi\colon M' \otimes_A B \to B \otimes_A M'$, define

$$\begin{aligned}
\phi_1&\colon B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M'\\
\phi_2&\colon M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M',\\
\phi_3&\colon M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B
\end{aligned}$$

by tensoring ϕ with id_B in the first, second, and third positions respectively. Then a pair (M',ϕ) arises from an A-module M as above if and only if $\phi_2 = \phi_1 \circ \phi_3$. The necessity is easy to check. For the sufficiency, define

$$M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}.$$

There is a canonical map $b \otimes m \mapsto bm\colon B \otimes_A M \to M'$, and it suffices to show that this is an isomorphism (and that the map arising from M is ϕ). Consider the diagram

$$\begin{array}{ccc}
M' \otimes_A B & \overset{\alpha\otimes 1}{\underset{\beta\otimes 1}{\rightrightarrows}} & B \otimes_A M' \otimes_A B\\
\downarrow \phi & & \downarrow \phi_1\\
B \otimes_A M' & \overset{e_0\otimes 1}{\underset{e_1\otimes 1}{\rightrightarrows}} & B \otimes_A B \otimes_A M'
\end{array}$$

in which $\alpha(m) = 1 \otimes m$ and $\beta(m) = \phi(m) \otimes 1$. As the diagram commutes with either the upper of the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1 \circ \phi_3$), ϕ induces an isomorphism on the kernels. But, by defintion of M, the kernel of the pair $(\alpha \otimes 1, \beta \otimes 1)$ is $M \otimes_A B$, and, according to (16.45), the kernel of the pair $(e_0 \otimes 1, e_1 \otimes 1)$ is M'. This essentially completes the proof.

A regular map $\varphi\colon W \to V$ of algebraic spaces is ***faithfully flat*** if it is surjective on the underlying sets and $\mathcal{O}_{\varphi(P)} \to \mathcal{O}_P$ is flat for all $P \in W$, and it is ***affine*** if the inverse images of open affines in V are open affines in W.

THEOREM 16.47. *Let $\varphi\colon W \to V$ be a faithfully flat map of algebraic spaces. To give an algebraic space U affine over V is the same as to give an algebraic space U' affine over W together with an isomorphism $\phi\colon p_1^* U' \to p_2^* U'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi).$$

Here p_{ji} denotes the projection $W \times W \times W \to W \times W$ such that $p_{ji}(w_1, w_2, w_3) = (w_j, w_i)$.

PROOF. When W and V are affine, (16.46) gives a similar statement for modules, hence for algebras, and hence for algebraic spaces. □

EXAMPLE 16.48. Let Γ be a finite group, and regard it as an algebraic group of dimension 0. Let V be an algebraic space over k. An algebraic space ***Galois over*** V ***with Galois group*** Γ is a finite map $W \to V$ to algebraic space together with a regular map $W \times \Gamma \to W$ such that

(a) for all k-algebras R, $W(R) \times \Gamma(R) \to W(R)$ is an action of the group $\Gamma(R)$ on the set $W(R)$ in the usual sense, and the map $W(R) \to V(R)$ is compatible with the action of $\Gamma(R)$ on $W(R)$ and its trivial action on $V(R)$, and

(b) the map $(w,\sigma) \mapsto (w, w\sigma)\colon W \times \Gamma \to W \times_V W$ is an isomorphism.

Then there is a commutative diagram[7]

$$\begin{array}{ccccccc}
V & \leftarrow & W & \leftleftarrows & W \times \Gamma & \Lleftarrow & W \times \Gamma^2 \\
\| & & \| & & \downarrow \simeq & & \downarrow \simeq \\
V & \leftarrow & W & \leftleftarrows & W \times_V W & \Lleftarrow & W \times_V W \times_V W
\end{array}$$

The vertical isomorphisms are

$$\begin{aligned}
(w,\sigma) &\mapsto (w, w\sigma) \\
(w,\sigma_1,\sigma_2) &\mapsto (w, w\sigma_1, w\sigma_1\sigma_2).
\end{aligned}$$

Therefore, in this case, Theorem 16.47 says that to give an algebraic space affine over V is the same as to give an algebraic space affine over W together with an action of Γ on it compatible with that on W. When we take W and V to be the spectra of fields, then this becomes affine case of Theorem 16.23.

EXAMPLE 16.49. In Theorem 16.47, let φ be the map corresponding to a regular extension of fields $k \to k(t)$. This case of Theorem 16.47 coincides with the affine case of Theorem 16.36 except that the field $k(t,t')$ has been replaced by the ring $k(t) \otimes_k k(t')$.

MNOTE 1. The paper of Weil cited in subsection on Weil's descent theorems is the first important paper in descent theory. Its results haven't been superseded by the many results of Grothendieck on descent. In Milne 1999[8], Theorem 16.33 was deduced from Weil's theorems. The present more elementary proof was suggested by Wolfart's elementary proof of the 'obvious' part of Belyi's theorem (Wolfart 1997[9]; see also Derome 2003[10]).

[7] See Milne, J. S., Etale cohomology. Princeton, 1980, p100.

[8] Milne, J. S., Descent for Shimura varieties. Michigan Math. J. 46 (1999), no. 1, 203–208.

[9] Wolfart, Jürgen. The "obvious" part of Belyi's theorem and Riemann surfaces with many automorphisms. Geometric Galois actions, 1, 97–112, London Math. Soc. Lecture Note Ser., 242, Cambridge Univ. Press, Cambridge, 1997.

[10] Derome, G., Descente algébriquement close, J. Algebra, 266 (2003), 418–426.