Tannakian Categories

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The proof reader at work.

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Chapter I

Symmetric Monoidal Categories

A monoidal category is one in which every finite ordered set of objects has a well-defined tensor product. A symmetric monoidal category is one in which every (unordered) finite set of objects has a well-defined tensor product. A tensor category over a field k is a k-linear abelian symmetric monoidal category in which all objects have duals and $k \simeq \mathrm{End}(\mathbb{1})$. A tannakian category over k is a tensor category over k that admits a fibre functor with values in a nonzero k-algebra.

1 Monoidal categories

A monoid can be defined to be a set together with an associative multiplication such that there exists an element 1 with the following property: $11 = 1$ and the maps $x \mapsto 1x$ and $x \mapsto x1$ are injective. Then $1x = 11x$, so $x = 1x$, and $x1 = x11$, so $x1 = x$. Therefore, this definition agrees with the usual definition of a monoid. Moreover, the element 1 is unique and the maps $x \mapsto 1x$ and $x \mapsto x1$ are bijective. These observations suggest the definition of a monoidal category in 1.2 below.

Let C be a category and let

$$\otimes\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}, \quad (X,Y) \rightsquigarrow X\otimes Y$$

be a functor.

An ***associativity constraint*** for $(\mathsf{C},\otimes)$ is a natural isomorphism

$$\alpha_{X,Y,Z}\colon X\otimes(Y\otimes Z) \to (X\otimes Y)\otimes Z$$

such that, for all objects X, Y, Z, T, the following diagram commutes,

$$\begin{tikzcd}
& W\otimes(X\otimes(Y\otimes Z)) \arrow[dl, "W\otimes\alpha_{X,Y,Z}"'] \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
W\otimes((X\otimes Y)\otimes Z) \arrow[d, "\alpha_{W,X\otimes Y,Z}"'] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d, "\alpha_{W\otimes X,Y,Z}"] \\
(W\otimes(X\otimes Y))\otimes Z \arrow[rr, "\alpha_{W,X,Y}\otimes Z"] & & ((W\otimes X)\otimes Y)\otimes Z
\end{tikzcd} \tag{1}$$

This is the ***pentagon axiom*** (Saavedra 1972, I, 1.1.1.1; Mac Lane 1998, p. 162).[1]

DEFINITION 1.1 A pair (U, u) consisting of an object U of C and an isomorphism $u\colon U\otimes U \to U$ is a ***unit*** of $(\mathsf{C}, \otimes)$ if the functors

$$X \rightsquigarrow U \otimes X\colon \mathsf{C} \to \mathsf{C}$$
$$X \rightsquigarrow X \otimes U\colon \mathsf{C} \to \mathsf{C}$$

are fully faithful.

DEFINITION 1.2 A triple $(\mathsf{C}, \otimes, \alpha)$ consisting of a category C, a functor $\otimes\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}$, and an associativity constraint α satisfying the pentagon axiom is a ***monoidal category*** if there exists a unit (U, u).

Note that the set of isomomorphism classes of objects in an essentially small monoidal category becomes a monoid with the multiplication defined by $\otimes$.

Let (U, u) be a unit in a monoidal category. We wish to define morphisms (left and right unit constraints)

$$\lambda_X\colon U\otimes X \to X,$$
$$\rho_X\colon X\otimes U \to X.$$

Because of the fully faithfullness of the functors in 1.1, it suffice to define $U\otimes\lambda_X$ and $\rho_X\otimes U$. These we take to be the composites

$$U\otimes(U\otimes X) \xrightarrow{\alpha_{U,U,X}} (U\otimes U)\otimes X \xrightarrow{u\otimes X} U\otimes X$$
$$(X\otimes U)\otimes U \xrightarrow{\alpha^{-1}_{X,U,U}} X\otimes(U\otimes U) \xrightarrow{X\otimes u} X\otimes U.$$

Note that λ and ρ are natural isomorphisms of the functors $U\otimes -$ and $-\otimes U$ with the identity functor (because they become isomorphisms when tensored with U).

PROPOSITION 1.3 *The following triangles commute,*

$$\begin{tikzcd}
U\otimes(X\otimes Y) \arrow[rr, "\alpha_{U,X,Y}"] \arrow[dr, "\lambda_{X\otimes Y}"'] & & (U\otimes X)\otimes Y \arrow[dl, "\lambda_X\otimes Y"] \\
& X\otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes(Y\otimes U) \arrow[rr, "\alpha_{X,Y,U}"] \arrow[dr, "X\otimes\rho_Y"'] & & (X\otimes Y)\otimes U \arrow[dl, "\rho_{X\otimes Y}"] \\
& X\otimes Y &
\end{tikzcd} \tag{2}$$

(left and right unity diagrams).

PROOF To show that the first diagram,

$$\begin{tikzcd}
U\otimes(X\otimes Y) \arrow[r, "\lambda_{X\otimes Y}"] \arrow[d, "\alpha_{U,X,Y}"] & X\otimes Y \arrow[d, equal] \\
(U\otimes X)\otimes Y \arrow[r, "\lambda_X\otimes Y"] & X\otimes Y,
\end{tikzcd}$$

[1]In some sources, the arrow α has the opposite direction. A similar remark applies to other arrows in this chapter.

commutes, it suffices to show that

$$\begin{tikzcd}
U \otimes (U \otimes (X \otimes Y)) \arrow[r, "\alpha_{U,U,X\otimes Y}"] \arrow[d, "U\otimes\alpha_{U,X,Y}"] & (U \otimes U) \otimes (X \otimes Y) \arrow[r, "u\otimes(X\otimes Y)"] \arrow[d, "\alpha_{U\otimes U,X,Y}"] & U \otimes (X \otimes Y) \arrow[d, "\alpha_{U,X,Y}"] \\
U \otimes ((U \otimes X) \otimes Y) \arrow[r] & ((U \otimes U) \otimes X) \otimes Y \arrow[r, "(u\otimes X)\otimes Y"] & (U \otimes X) \otimes Y
\end{tikzcd}$$

commutes. The left-hand square commutes because of the pentagon axiom (the unmarked arrow involves two applications of α) and the right-hand square commutes because of the naturality of α. This proves the statement for λ_X, and the proof for ρ_X is similar. □

PROPOSITION 1.4 *For any object X,*

$$\lambda_{U\otimes X} = U \otimes \lambda_X$$
$$\rho_{X\otimes U} = \rho_X \otimes U.$$

PROOF The diagram

$$\begin{tikzcd}
U \otimes (U \otimes X) \arrow[r, "U\otimes\lambda_X"] \arrow[d, "\lambda_{U\otimes X}"] & U \otimes X \arrow[d, "\lambda_X"] \\
U \otimes X \arrow[r, "\lambda_X"] & X
\end{tikzcd}$$

commutes because of the naturality of λ. As λ_X is an isomorphism, this proves the first equality. Similarly, the second equality follows from the naturality of ρ. □

PROPOSITION 1.5 *The following diagram commutes for all X, Y,*

$$\begin{tikzcd}
X \otimes (U \otimes Y) \arrow[rr, "\alpha_{X,U,Y}"] \arrow[dr, "X\otimes\lambda_Y"'] & & (X \otimes U) \otimes Y \arrow[dl, "\rho_X\otimes Y"] \\
& X \otimes Y &
\end{tikzcd} \tag{3}$$

(middle unity diagram).

PROOF Consider the diagram,

$$\begin{tikzcd}
((X \otimes U) \otimes U) \otimes Y \arrow[dr, "(\rho_X\otimes U)\otimes Y"'] & & (X \otimes (U \otimes U)) \otimes Y \arrow[ll, "\alpha_{X,U,U}\otimes Y"'] \arrow[dl, "(X\otimes u)\otimes Y"] \\
& (X \otimes U) \otimes Y & \\
& X \otimes (U \otimes Y) \arrow[u, "\alpha_{X,U,Y}"] & \\
(X \otimes U) \otimes (U \otimes Y) \arrow[uuu, "\alpha_{X\otimes U,U,Y}"] \arrow[ur, "\rho_X\otimes(U\otimes Y)"] & & X \otimes ((U \otimes U) \otimes Y) \arrow[uuu, "\alpha_{X,U\otimes U,Y}"'] \arrow[ul, "X\otimes(u\otimes Y)"'] \\
& X \otimes (U \otimes (U \otimes Y)). \arrow[ul, "\alpha_{X,U,U\otimes Y}"] \arrow[ur, "X\otimes\alpha_{U,U,Y}"'] \arrow[uu, "X\otimes\lambda_{U\otimes Y}"] &
\end{tikzcd}$$

The triangle at lower left is (3), except with Y replaced by $U \otimes Y$. Because $Y \rightsquigarrow U \otimes Y$ is fully faithful, it suffices to show that this triangle commutes. The outside pentagon is that in the pentagon axiom, and so it suffices to show that each of the remaining subdiagrams commutes. For the two rectangles, this follows from the functoriality of α; for the top triangle, this follows from the definition of ρ_X; and for the lower-right triangle, it follows from the definition of $U \otimes \lambda_Y$ and the equality $\lambda_{U\otimes Y} = U \otimes \lambda_Y$. □

PROPOSITION 1.6 *If (U,u) is a unit, then*

$$\lambda_U = u = \rho_U.$$

PROOF From the left unity diagram (1.3),

$$(\lambda_U \otimes U) \circ \alpha_{U,U,U} = \lambda_{U\otimes U} \overset{1.4}{=} U \otimes \lambda_U.$$

From the middle unity diagram (1.5),

$$(\rho_U \otimes U)\circ\alpha_{U,U,U} = U \otimes \lambda_U.$$

From the definition of λ_U,

$$(u \otimes U) \circ \alpha_{U,U,U} = U \otimes \lambda_U.$$

As $\alpha_{U,U,U}$ is an isomorphism, this implies that

$$\lambda_U \otimes U = u \otimes U = \rho_U \otimes U,$$

which, in turn, implies the required statement. □

PROPOSITION 1.7 *If (U,u) is a unit then*

$$U \otimes u = (u \otimes U)\circ\alpha_{U,U,U} \tag{4}$$

and the functors $X \rightsquigarrow U \otimes X$ and $X \rightsquigarrow X \otimes U$ are equivalences of categories.

PROOF The equality (4) is the special case of (3) with $X = Y = U$. For the second part of the statement, use that λ and ρ are natural isomorphisms of the functors $U \otimes -$ and $- \otimes U$ with the identity functor. □

PROPOSITION 1.8 *For any two units (U,u) and (U',u') of a monoidal category, there is a unique isomorphism $a\colon U \to U'$ making the following diagram commute,*

$$\begin{tikzcd}
U \otimes U \arrow[r, "u"] \arrow[d, "a\otimes a"] & U \arrow[d, "a"] \\
U' \otimes U' \arrow[r, "u'"] & U'.
\end{tikzcd}$$

PROOF Define a to be the isomorphism

$$U \xleftarrow{\rho_U} U \otimes U' \xrightarrow{\lambda_{U'}} U'.$$

Using the left and right unity diagrams, it is easy to show that a makes the above diagram commute. To show that a is unique, it suffices to show that if $b\colon U \to U$ is an isomorphism such that

$$\begin{tikzcd} U \otimes U \arrow[r, "u"] \arrow[d, "b\otimes b"] & U \arrow[d, "b"] \\ U \otimes U \arrow[r, "u"] & U \end{tikzcd}$$

commutes, then $b = \mathrm{id}_U$. For any morphism $c\colon U \to U$, the diagram

$$\begin{tikzcd} U \otimes U \arrow[r, "u"] \arrow[d, "c\otimes U"] & U \arrow[d, "c"] \\ U \otimes U \arrow[r, "u"] & U \end{tikzcd}$$

commutes because $u = \rho_U$. Therefore, $b \otimes b = b \otimes U$, and hence $b = \mathrm{id}_U$. □

EXAMPLE 1.9 The category Cat of small categories and functors becomes a monoidal category when we take the tensor product of two categories to be their product. Any category with only one object and one arrow is a unit object.

Notes

1.10 In Kock 2008, a unit in our sense is called a Saavedra unit.

1.11 Saavedra 1972, I, 1.3.2, defines a “unité reduit” to be a pair (U, u) consisting of an object U and an isomorphism $u\colon U \otimes U \to U$ such that the functors $X \rightsquigarrow U \otimes X$ and $X \rightsquigarrow X \otimes U$ are equivalences. According to 1.7, this agrees with our notion of a unit.

1.12 Define an LR-unit to be a triple (U, λ, ρ) such that the middle unity diagram (3) commutes. It follows from Proposition 1.3 that, to give the structure of an LR-unit on an object is the same as giving the structure of a Saavedra unit.

1.13 A monoidal category is classically defined to be a triple $(C, \otimes, \alpha)$ together with an LR-unit (Mac Lane 1998, p. 162). According to 1.12, this is the same as giving a triple together with a Saavedra unit. In our definition of a monoidal category, instead of specifying a unit object we only required it to exist. According to 1.8, this makes little difference.

2 Symmetric monoidal categories

Let C be a category and let

$$\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}, \quad (X, Y) \rightsquigarrow X \otimes Y$$

be a functor.

A ***commutativity constraint*** for $(\mathsf{C}, \otimes)$ is a natural isomorphism

$$\gamma_{X,Y}\colon X \otimes Y \to Y \otimes X$$

such that, for all objects X, Y,

$$\gamma_{Y,X} \circ \gamma_{X,Y}\colon X \otimes Y \to X \otimes Y$$

is the identity morphism on $X \otimes Y$ (Saavedra 1972, I, 1.2.1).

An associativity constraint α and a commutativity constraint γ are ***compatible*** if, for all objects X, Y, Z, the following diagram commutes,[2]

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[dl, "X \otimes \gamma"'] & (X \otimes Y) \otimes Z \arrow[dr, "\gamma"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\alpha"] & & & Z \otimes (X \otimes Y) \arrow[dl, "\alpha"] \\
 & (X \otimes Z) \otimes Y \arrow[r, "\gamma \otimes Y"] & (Z \otimes X) \otimes Y &
\end{tikzcd} \tag{5}$$

This is the ***hexagon axiom*** (Saavedra 1972, I, 2.1.1.1; Mac Lane 1998, p. 184).

DEFINITION 2.1 A ***symmetric monoidal category*** is a system $(\mathsf{C}, \otimes, \alpha, \gamma)$, where $(\mathsf{C}, \otimes, \alpha)$ is a monoidal category and γ is a compatible commutativity constraint.

EXERCISE 2.2 Show that in a symmetric monoidal category, the following diagram commutes

$$\begin{tikzcd}
X \otimes U \arrow[rr, "\gamma_{X,U}"] \arrow[dr, "\rho_X"'] & & U \otimes X \arrow[dl, "\lambda_X"] \\
 & X &
\end{tikzcd}$$

EXERCISE 2.3 Show that in a symmetric monoidal category, the following diagram commutes,

$$\begin{tikzcd}
X \otimes (U \otimes Y) \arrow[r, "\alpha_{X,U,Y}"] \arrow[d, "X \otimes \lambda_Y"] & (X \otimes U) \otimes Y \arrow[d, "\gamma \otimes Y"] \\
X \otimes Y & (U \otimes X) \otimes Y \arrow[l, "\lambda_X \otimes Y"']
\end{tikzcd}$$

We sometimes denote a unit of a symmetric monoidal category by $(\mathbb{1}, e)$ and call $\mathbb{1}$ an ***identity object***.

EXAMPLE 2.4 Let R be a commutative ring. The category $\mathsf{Mod}(R)$ becomes a symmetric monoidal category with the usual tensor product and the obvious constraints. (If one perversely takes α to the negative of the obvious isomorphism, then the pentagon (1) fails to commute by a sign.) A pair (U, e) consisting of a free R-module U of rank 1 and a basis element e determines a unit (U, u) of Mod_R – take u to be the unique isomorphism

[2] When we use the associativity constraint to omit parentheses, this becomes the triangle

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\gamma_{X \otimes Y, Z}"] \arrow[dr, "X \otimes \gamma_{Y,Z}"'] & & Z \otimes X \otimes Y. \\
 & X \otimes Z \otimes Y \arrow[ur, "\gamma_{X,Z} \otimes Y"'] &
\end{tikzcd}$$

$U \otimes U \to U$ sending $e \otimes e$ to e. Every unit is of this form. In this case, there is a canonical unit, namely, $(R, R \otimes R \xrightarrow{\text{mult.}} R)$.

EXAMPLE 2.5 The category of complex Hilbert spaces and bounded linear maps becomes a symmetric monoidal category with the completed tensor product $\hat{\otimes}$ as tensor product. The pair $(\mathbb{C}, \mathbb{C} \otimes \mathbb{C} \xrightarrow{\text{mult.}} \mathbb{C})$ is a unit.

For other examples, see §8 below.

Extending $\otimes$

Let $(\mathbb{C}, \otimes, \alpha)$ be a monoidal category. Any functor $\mathsf{C}^n \to \mathsf{C}$ defined by repeated application of $\otimes$ is called an ***iterate*** of $\otimes$. If $F, F' : \mathsf{C}^n \to \mathsf{C}$ are iterates of $\otimes$, then it is possible to construct an isomorphism of functors $F \to F'$ using only α and α^{-1}. The significance of the pentagon axiom is that it implies that the isomorphism is unique: any two iterates of $\otimes$ to C^n are isomorphic by a unique isomorphism constructed out of α and α^{-1} (Mac Lane 1963, 1998, VII, 2). This means that there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes^n : \mathsf{C}^n \to \mathsf{C}$ for all $n \geq 0$. Similarly, when $(\mathsf{C}, \otimes, \alpha, \gamma)$ is a symmetric monoidal category, there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes_{i \in I} : \mathsf{C}^I \to \mathsf{C}$, where I is any (unordered) finite indexing set. In other words, the tensor product of any finite family of objects of C is well-defined up to a unique isomorphism. We can make this statement more precise.

PROPOSITION 2.6 *The tensor structure on a symmetric monoidal category* $(\mathsf{C}, \otimes)$ *admits an extension as follows: for each finite set I, there is a functor*

$$\bigotimes_{i \in I} : \mathsf{C}^I \to \mathsf{C},$$

and, for each map $a : I \to J$ *of finite sets, there is a natural isomorphism*

$$\chi(a) : \bigotimes_{i \in I} X_i \to \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_i \right)$$

satisfying the following conditions,

(a) *if I is a singleton, then* $\bigotimes_{i \in I}$ *is the identity functor* $X \rightsquigarrow X$*; if a is a map between singletons, then* $\chi(a)$ *is the identity automorphism of the identity functor;*

(b) *the isomorphisms defined by maps* $I \xrightarrow{a} J \xrightarrow{b} K$ *give rise to a commutative diagram*

$$\begin{tikzcd}
\bigotimes_{i \in I} X_i \arrow[rr, "\chi(a)"] \arrow[d, "\chi(ba)"] & & \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_i \right) \arrow[d, "\chi(b)"] \\
\bigotimes_{k \in K} \left(\bigotimes_{i \mapsto k} X_i \right) \arrow[rr, "\otimes(\chi(a|I_k))"] & & \bigotimes_{k \in K} \left(\bigotimes_{j \mapsto k} \left(\bigotimes_{i \mapsto j} X_i \right) \right),
\end{tikzcd}$$

where $I_k = (ba)^{-1}(k)$.

PROOF Apply the coherence theorems of Mac Lane 1963, 1998. See also Elmendorf 2025. □

By $(\bigotimes_{i\in I},\chi)$ being an extension of the tensor structure on C, we mean that $\bigotimes_{i\in I} X_i = X_1 \otimes X_2$ when $I=\{1,2\}$ and that the isomorphisms

$$X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$
$$X\otimes Y\to Y\otimes X$$

induced by χ are equal to α and γ respectively. It is automatic that $(\bigotimes_{\emptyset} X_i, \chi(\emptyset\to\{1,2\})$ is a unit and that $\chi(\{2\}\hookrightarrow\{1,2\})$ is $\lambda_X\colon X\to \mathbb{1}\otimes X$. If $(\bigotimes'_{i\in I},\chi')$ is a second such extension, then there is a unique system of natural isomorphisms $\bigotimes_{i\in I} X_i\to\bigotimes'_{i\in I} X_i$ compatible with χ and χ' and such that, when $I=\{i\}$, the isomorphism is id_{X_i}.

Whenever a symmetric monoidal category $(\mathsf{C},\otimes)$ is given, we shall always assume that an extension as in Proposition 2.6 has been made. Note that the constraints imposed are the minimum necessary to force the proposition to hold.

Invertible objects

Let $(\mathsf{C},\otimes)$ be a symmetric monoidal category. An object L of C is ***invertible*** if

$$X\rightsquigarrow L\otimes X\colon \mathsf{C}\to\mathsf{C}$$

is an equivalence of categories. For example, an object L of $\mathsf{Modf}(R)$ is invertible if and only if it is projective of rank 1.

If L is invertible, then there exists an L' such that $L\otimes L'$ is a unit object. The converse assertion is also true: if $L\otimes L'=\mathbb{1}$, then $L\otimes -$ and $-\otimes L'$ are quasi-inverse functors.

An ***inverse*** of L is any pair (L^{-1},δ) with L^{-1} and object and δ a morphism,

$$\delta\colon \bigotimes_{i\in\{\pm\}} X_i\xrightarrow{\simeq}\mathbb{1},\quad X_+=L,\quad X_-=L^{-1}.$$

Note that this definition is symmetric: (L,δ) is an inverse of L^{-1}. If (L_1,δ_1) and (L_2,δ_2) are both inverses of L, then there is a unique isomorphism $a\colon L_1\to L_2$ such that the composite

$$\delta_2\circ(1\otimes a)\colon L\otimes L_1\to L\otimes L_2\to\mathbb{1}$$

is δ_1 (because the functors $-\otimes L_1$ and $-\otimes L_2$ are both quasi-inverse to $L\otimes -$).

Notes

2.7 In Deligne and Milne 1982, symmetric monoidal categories are called tensor categories.

2.8 Our notion of a symmetric monoidal category essentially agrees with that in Mac Lane 1998, p. 184 (see 1.13).

2.9 Our notion of a symmetric monoidal category is the same as that of a "$\otimes$-catégorie AC unifère" in Saavedra 1972 and, because of 1.8, it is essentially the same as the notion of a "$\otimes$-catégorie ACU" in ibid., I, 2.4.1 (cf. ibid., I, 2.4.3).

2.10 Proposition 2.6 suggests the notion of an "unbiased symmetric monoidal category" in which no preference is given to the functor $\otimes\colon\mathsf{C}^2\to\mathsf{C}$ and the constraints are canonical. See Leinster 2004.

2.11 There is a large literature on coherence in monoidal and symmetric monoidal categories, beginning with Mac Lane 1963. For a recent review, see Mimram 2025.

2.12 There is a large literature on monoidal categories satisfying a commutativity constraint weaker than that we have imposed. For example, without the condition $\gamma_{Y,X} = \gamma_{X,Y}^{-1}$, the category is said to be braided.

3 Symmetric monoidal functors

DEFINITION 3.1 Let $(\mathsf{C}, \otimes)$ and $(\mathsf{D}, \otimes)$ be monoidal categories. A ***monoidal functor*** $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a pair (F, c) consisting of a functor $F\colon \mathsf{C} \to \mathsf{D}$ and a natural isomorphism $c_{X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y)$ such that

(a) for all $X, Y, Z \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "FX\otimes c"] \arrow[d, "\alpha_{FX,FY,FZ}"] & FX \otimes F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\alpha_{X,Y,Z})"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "c\otimes FZ"] & F(X \otimes Y) \otimes FZ \arrow[r, "c"] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

commutes, and

(b) if $U \otimes U \xrightarrow{u} U$ is a unit in C, then

$$F(U) \otimes F(U) \xrightarrow{c_{U,U}} F(U \otimes U) \xrightarrow{F(u)} F(U)$$

is a unit in D.

When $(\mathsf{C}, \otimes)$ and $(\mathsf{D}, \otimes)$ are symmetric, (F, c) is said to be ***symmetric*** if in addition

(c) for all $X, Y \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "c"] \arrow[d, "\gamma_{FX,FY}"] & F(X \otimes Y) \arrow[d, "F(\gamma_{X,Y})"] \\
FY \otimes FX \arrow[r, "c"] & F(Y \otimes X)
\end{tikzcd}$$

commutes;

3.2 Let (F, c) be a symmetric monoidal functor $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$. For any finite family $(X_i)_{i \in I}$ of objects in C, c gives rise to a well-defined isomorphism

$$c\colon \bigotimes_{i\in I} F(X_i) \to F(\bigotimes_{i \in I} X_i).$$

Moreover, for any map $a\colon I \to J$, the following diagram commutes,

$$\begin{tikzcd}
\bigotimes_{i\in I} F(X_i) \arrow[rr, "c"] \arrow[d, "\chi(a)"] & & F\left(\bigotimes_{i\in I} X_i\right) \arrow[d, "F(\chi(a))"] \\
\bigotimes_{j\in J}(\bigotimes_{i\mapsto j} F(X_i)) \arrow[r, "c"] & \bigotimes_{j\in J}(F(\bigotimes_{i\mapsto j} X_i)) \arrow[r, "c"] & F(\bigotimes_{j\in J}(\bigotimes_{i\mapsto j} X_i)).
\end{tikzcd}$$

For example, if (U, u) and (U', u') are units in C and D, then there exists a unique isomorphism $U' \to FU$ compatible with u' and Fu.

3.3 Let $(F, c^F)\colon \mathsf{C} \to \mathsf{D}$ and $(G, c^G)\colon \mathsf{D} \to \mathsf{E}$ be monoidal functors. When we define

$$c^{G\circ F}_{X,Y} : (G\circ F)(X) \otimes (G\circ F)(Y) \to (G\circ F)(X \otimes Y)$$

to be the composite of

$$(G\circ F)(X) \otimes (G\circ F)(Y) \xrightarrow{c^G_{FX,FY}} G(FX \otimes FY) \xrightarrow{G(c^F_{X,Y})} GF(X \otimes Y),$$

then

$$(G\circ F, c^{G\circ F})\colon \mathsf{C} \to \mathsf{E}$$

is a monoidal functor. If (F, c^F) and (G, c^G) are symmetric mononoidal categories, then so also is $(G\circ F, c^{G\circ F})$.

DEFINITION 3.4 Let (F, c^F) and (G, c^G) be monoidal functors $\mathsf{C} \to \mathsf{D}$. A natural transformation $\alpha\colon F \to G$ is ***monoidal*** if the following diagrams commute for all X, Y and units U, U',

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "c^F_{X,Y}"] \arrow[d, "\alpha_X \otimes \alpha_Y"] & F(X \otimes Y) \arrow[d, "\alpha_{X\otimes Y})"] \\
G(X) \otimes G(Y) \arrow[r, "c^G_{X,Y}"] & G(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[r, "\simeq"] \arrow[d, equal] & F(U) \arrow[d, "\alpha_U"] \\
U' \arrow[r, "\simeq"] & G(U).
\end{tikzcd}$$

If α_X is an isomorphism for all X, then we call α a ***monoidal natural isomorphism***. The definition is the same for symmetric monoidal functors, i.e, there is no extra condition to be a ***symmetric monoidal natural transformation.***

3.5 If $\alpha\colon G \to F$ a monoidal natural transformation, then for all finite families $(X_i)_{i\in I}$ of objects in C, the diagram

$$\begin{tikzcd}
\bigotimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\bigotimes_{i\in I} \lambda_{X_i}"] & F(\bigotimes_{i\in I} X_i) \arrow[d, "\lambda_{\bigotimes_{i\in I} X_i}"] \\
\bigotimes_{i\in I} G(X_i) \arrow[r, "d"] & G(\bigotimes_{i\in I} X_i)
\end{tikzcd} \tag{6}$$

commutes. The diagrams in 3.4 correspond to the cases $I = \{1, 2\}$ and $I = \emptyset$.It suffices to check that the diagram (6) commutes when I is $\{1, 2\}$ or the empty set. For the empty set, (6) becomes

(7)

DEFINITION 3.6 A symmetric monoidal functor $(F, c)\colon (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a ***symmetric monoidal equivalence*** if $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories.

This definition is justified by the following remark, which shows that a symmetric monoidal equivalence admits a symmetric monoidal functor as a quasi-inverse.

3.7 Let $(F, c)\colon (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ be a symmetric monoidal equivalence. To say that $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories means that there exists a functor $G\colon \mathsf{D} \to \mathsf{C}$ and natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathsf{C}} \xrightarrow{\simeq} GF, \quad \epsilon\colon \mathrm{id}_{\mathsf{D}} \xrightarrow{\simeq} FG.$$

After possibly replacing ϵ with a different natural isomorphism $\mathrm{id}_{\mathsf{D}} \xrightarrow{\simeq} FG$, we obtain a system (F,G,η,ϵ) satisfying the triangle identities (see A.5). There then exists a d such that (G,d) is symmetric monoidal functor and η and ϵ are symmetric monoidal natural isomorphisms (Saavedra 1972, I, 4.4).

EXAMPLE 3.8 Let $(\mathsf{C},\otimes,\alpha)$ be a monoidal category. The triple $(\mathsf{C},\otimes\circ S,\alpha^{-1})$, where S is the transposition functor

$$(X,Y) \rightsquigarrow (Y,X) : \mathsf{C}\times\mathsf{C} \to \mathsf{C}\times\mathsf{C},$$

is a monoidal category, called the ***opposite*** of $(\mathsf{C},\otimes,\alpha)$. A compatible commutativity constraint defines an equivalence of $(\mathsf{C},\otimes,\alpha)$ with its opposite.

We let $\mathrm{Hom}^{\otimes}(F,G)$ denote the collection of symmetric monoidal natural transformations $(F,c) \to (G,d)$. For a field k and k-algebra R, we let

$$\phi_R : \mathsf{Vecf}(k) \to \mathsf{Mod}(R),$$

denote the functor $V \rightsquigarrow V\otimes_k R$. When (F,c) and (G,d) are symmetric monoidal functors $\mathsf{C} \to \mathsf{Vecf}(k)$, we define $\mathcal{H}om^{\otimes}(F,G)$ to be the functor of k-algebras such that

$$\mathcal{H}om^{\otimes}(F,G)(R) = \mathrm{Hom}^{\otimes}(\phi_R\circ F, \phi_R\circ G). \tag{8}$$

NOTATION 3.9 We sometimes abbreviate “symmetric monoidal widget” to “$\otimes$-widget”. For example a $\otimes$-category is a symmetric monoidal category, and a $\otimes$-functor is a symmetric monoidal functor.

NOTES

3.10 Our notion of a monoidal functor agrees with that in most of the literature. There are also weak monoidal functors, in which c need not be an isomorphism, and strong monoidal functors, in which c is equality. For example, for a ring R, the forgetful functor $(\mathsf{Mod}(R),\otimes_R) \to (\mathsf{Ab},\otimes)$ is only a weak monoidal functor.

3.11 In Saavedra 1972, I, 4.2.3, a symmetric monoidal functor is called a “$\otimes$-foncteur AC unifère”.

4 Internal Homs and duals in symmetric monoidal categories

In this section, $(\mathsf{C},\otimes)$ is a symmetric monoidal category.

Internal Homs

DEFINITION 4.1 Let $X,Y \in \mathrm{ob}\,\mathsf{C}$. When the functor

$$T \rightsquigarrow \mathrm{Hom}(T\otimes X,Y) : \mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$$

is representable, we let $\mathcal{H}om(X,Y)$ denote the representing object and

$$\mathrm{ev}_{X,Y} : \mathcal{H}om(X,Y)\otimes X \to Y$$

the morphism corresponding to $\mathrm{id}_{\mathcal{H}om(X,Y)}$. Thus, to every morphism $g\colon T\otimes X\to Y$ there corresponds a unique morphism $f\colon T\to\mathcal{H}om(X,Y)$ such that $\mathrm{ev}_{X,Y}\circ(f\otimes X)=g$,

$$\begin{tikzcd} T \arrow[d, dashed, "f"'] \\ \mathcal{H}om(X,Y) \end{tikzcd} \qquad \begin{tikzcd} T\otimes X \arrow[d, dashed, "f\otimes X"'] \arrow[dr, "g"] & \\ \mathcal{H}om(X,Y)\otimes X \arrow[r, "\mathrm{ev}_{X,Y}"'] & Y. \end{tikzcd} \tag{9}$$

More succinctly,

$$\mathrm{Hom}(T\otimes X,Y)\simeq\mathrm{Hom}(T,\mathcal{H}om(X,Y)). \tag{10}$$

If, for a given X, $\mathcal{H}om(X,Y)$ exists for all Y, then the functor $\mathcal{H}om(X,-)$ is right adjoint to $-\otimes X$.[3]

DEFINITION 4.2 A symmetric monoidal category is ***closed*** if internal Homs exist, i.e., if for all objects X, the functor $-\otimes X$ has a right adjoint $\mathcal{H}om(X,-)$.

EXAMPLE 4.3 Let R be a commutative ring. For R-modules M,N, $\mathrm{Hom}_R(M,N)$ is again an R-module, and

$$\mathrm{Hom}_R(T\otimes_R M,N)\simeq\mathrm{Hom}_R(T,\mathrm{Hom}_R(M,N)),\quad \theta\leftrightarrow(t\mapsto(m\mapsto\theta(t\otimes m))$$

(Bourbaki A, II, 4.1). Thus, $\mathcal{H}om(M,N)$ exists for all modules M,N, and equals $\mathrm{Hom}_R(M,N)$ (viewed as an R-module). In this case, $\mathrm{ev}_{M,N}$ is

$$f\otimes x\mapsto f(x)\colon\ \mathrm{Hom}_R(M,N)\otimes M\to N,$$

which explains its name.

Closed symmetric monoidal categories

In this subsection, $(\mathsf{C},\otimes)$ is a symmetric monoidal category in which internal Homs exist.

There is a composition morphism

$$\mathcal{H}om(Y,Z)\otimes\mathcal{H}om(X,Y)\to\mathcal{H}om(X,Z), \tag{11}$$

corresponding to

$$\mathcal{H}om(Y,Z)\otimes\mathcal{H}om(X,Y)\otimes X\xrightarrow{\mathrm{id}\otimes\mathrm{ev}}\mathcal{H}om(Y,Z)\otimes Y\xrightarrow{\mathrm{ev}}Z.$$

From the canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}(T,\mathcal{H}om(Z,\mathcal{H}om(X,Y))) &\overset{(10)}{\simeq}\mathrm{Hom}(T\otimes Z,\mathcal{H}om(X,Y))\\ &\overset{(10)}{\simeq}\mathrm{Hom}(T\otimes Z\otimes X,Y)\\ &\overset{(10)}{\simeq}\mathrm{Hom}(T,\mathcal{H}om(Z\otimes X,Y)),\end{aligned}$$

[3] Strictly speaking, this is the left internal Hom. The right internal Hom is right adjoint to $X\otimes-$, so $\mathrm{Hom}(X\otimes T,Y)\simeq\mathrm{Hom}(T,\mathcal{H}om(X,Y))$. Because of the commutativity constraint, the two notions essentially coincide.

and the Yoneda lemma, we get a canonical isomorphism

$$\mathcal{H}om(Z, \mathcal{H}om(X, Y)) \simeq \mathcal{H}om(Z \otimes X, Y). \tag{12}$$

Note that

$$\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X, Y)) \overset{(10)}{\simeq} \mathrm{Hom}(\mathbb{1} \otimes X, Y) \simeq \mathrm{Hom}(X, Y). \tag{13}$$

The ***weak dual*** $X^\vee$ of an object X is defined to be $\mathcal{H}om(X, \mathbb{1})$. The morphism

$$\mathrm{ev}_{X,\mathbb{1}} : X^\vee \otimes X \to \mathbb{1}$$

induces a bijection

$$f \mapsto \mathrm{ev}_{X,\mathbb{1}} \circ (f \otimes X) : \ \mathrm{Hom}(T, X^\vee) \to \mathrm{Hom}(T \otimes X, \mathbb{1}), \tag{14}$$

natural in T, and this property characterizes $(X^\vee, \mathrm{ev}_{X,\mathbb{1}})$. The map $X \mapsto X^\vee$ can be made into a contravariant functor by sending $f : X \to Y$ to the unique morphism ${}^t f : Y^\vee \to X^\vee$ rendering commutative the diagram

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{{}^t f \otimes X} & X^\vee \otimes X \\ \Big\downarrow{\scriptstyle Y^\vee \otimes f} & & \Big\downarrow{\scriptstyle \mathrm{ev}_{X,\mathbb{1}}} \\ Y^\vee \otimes Y & \xrightarrow{\mathrm{ev}_{Y,\mathbb{1}}} & \mathbb{1}. \end{array} \tag{15}$$

In other words, ${}^t f$ is the morphism corresponding to $\mathrm{ev}_{Y,\mathbb{1}} \circ (Y^\vee \otimes f)$ under the isomorphism (14)

$$g \mapsto \mathrm{ev}_{X,\mathbb{1}} \circ (g \otimes X) : \ \mathrm{Hom}(Y^\vee, X^\vee) \to \mathrm{Hom}(Y^\vee \otimes X, \mathbb{1}).$$

When f is an isomorphism, we let $f^\vee = ({}^t f)^{-1} : X^\vee \to Y^\vee$, so that

$$\mathrm{ev}_{Y,\mathbb{1}} \circ (f^\vee \otimes f) = \mathrm{ev}_{X,\mathbb{1}} : X^\vee \otimes X \to \mathbb{1}. \tag{16}$$

EXAMPLE 4.4 In $\mathsf{Mod}(R)$, $M^\vee = \mathrm{Hom}_R(M, R)$ and ${}^t f$ is determined by the equation

$$\langle {}^t f(y), x \rangle_M = \langle y, f(x) \rangle_N, \quad y \in N^\vee \quad x \in M,$$

where we have written $\langle \ , \ \rangle_M$ and $\langle \ , \ \rangle_N$ for ev_M and ev_N. We have

$$\langle f^\vee(x'), f(x) \rangle_N = \langle x', x \rangle_M, \quad x' \in M^\vee, x \in M.$$

Let $i_X : X \to X^{\vee\vee}$ be the morphism corresponding in (14) to $\mathrm{ev}_X \circ \gamma : X \otimes X^\vee \to \mathbb{1}$. When i_X is an isomorphism, X is said to be ***reflexive***. If X has an inverse

$$(X^{-1}, \delta : X^{-1} \otimes X \xrightarrow{\simeq} \mathbb{1}),$$

then X is reflexive and the morphism $X^{-1} \to X^\vee$ corresponding to δ in (10) is an isomorphism.

For finite families of objects $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$, there is a morphism

$$\bigotimes_{i \in I} \mathcal{H}om(X_i, Y_i) \to \mathcal{H}om(\bigotimes_{i \in I} X_i, \bigotimes_{i \in I} Y_i) \tag{17}$$

corresponding in (10) to

$$(\bigotimes_{i\in I}\mathcal{H}om(X_i,Y_i))\otimes(\bigotimes_{i\in I}X_i)\xrightarrow{\simeq}\bigotimes_{i\in I}(\mathcal{H}om(X_i,Y_i)\otimes X_i)\xrightarrow{\otimes\,\mathrm{ev}}\bigotimes_{i\in I}Y_i.$$

In particular, there are morphisms

$$\bigotimes_{i\in I}X_i^\vee\to(\bigotimes_{i\in I}X_i)^\vee \tag{18}$$

and

$$X^\vee\otimes Y\to\mathcal{H}om(X,Y) \tag{19}$$

obtained respectively by taking $Y_i=\mathbb{1}$ all i, and $X_1=X, X_2=\mathbb{1}=Y_1, Y_2=Y$.

Duals

In this subsection, $(\mathsf{C},\otimes)$ is a symmetric monoidal category.

DEFINITION 4.5 Let X be an object in C. A pair $(X', X'\otimes X\xrightarrow{\varepsilon}\mathbb{1})$ is a ***dual***[4] of X if there exists a morphism $\delta\colon\ \mathbb{1}\to X\otimes X'$ such that the following equalities hold,

$$\begin{aligned}&(X\simeq\mathbb{1}\otimes X\xrightarrow{\delta\otimes X}X\otimes X'\otimes X\xrightarrow{X\otimes\varepsilon}X\otimes\mathbb{1}\simeq X)=\mathrm{id}_X\\&(X'\simeq X'\otimes\mathbb{1}\xrightarrow{X'\otimes\delta}X'\otimes X\otimes X'\xrightarrow{\varepsilon\otimes X'}\mathbb{1}\otimes X'\simeq X')=\mathrm{id}_{X'}.\end{aligned} \tag{20}$$

Here ε and δ are called the ***evaluation*** and ***coevaluation*** morphisms, and are often denoted ev_X and coev_X. If it exists, coev_X is unique (see 4.9 below).

4.6 Let (X',ev) be a dual of X with coevaluation map coev. For any $\otimes$-functor F, the pair $(F(X'),F(\mathrm{ev}))$ is a dual of $F(X)$ with coevaluation map $F(\mathrm{coev})$. To prove this, apply F to the diagrams (20). In particular, if (X',ev) is the dual of X in C, then it is the dual of X in any symmetric monoidal category containing C as a symmetric monoidal subcategory.

EXAMPLE 4.7 Let M be a free R-module of finite rank, let $N=\mathcal{H}om(M,R)$, and let ε be the evaluation map $f\otimes m\mapsto f(m)\colon\ N\otimes M\to R$. Let (e_i) and (e_i') be dual bases for M and N, and let $\delta\colon\ R\to M\otimes N$ be the map sending 1 to $\sum e_i\otimes e_i'$. Then δ is independent of the choice the basis (e_i), and the following equalities hold,

$$\begin{aligned}&(M\simeq R\otimes M\xrightarrow{\delta\otimes M}M\otimes N\otimes M\xrightarrow{M\otimes\varepsilon}M\otimes R\simeq M)=\mathrm{id}_M\\&(N\simeq N\otimes R\xrightarrow{N\otimes\delta}N\otimes M\otimes N\xrightarrow{\varepsilon\otimes M}R\otimes N\simeq N)=\mathrm{id}_N.\end{aligned} \tag{21}$$

Thus, (N,ε) is the dual of M.

PROPOSITION 4.8 *Let $\varepsilon\colon\ X'\otimes X\to\mathbb{1}$ be a morphism in* C. *The pair (X',ε) is a dual of X if and only if the map*

$$\Psi_{S,T}\colon\ \mathrm{Hom}(S,T\otimes X')\to\mathrm{Hom}(S\otimes X,T)$$

sending $f\colon\ S\to T\otimes X'$ to the composite $S\otimes X\xrightarrow{f\otimes X}T\otimes X'\otimes X\xrightarrow{T\otimes\varepsilon}T\otimes\mathbb{1}\simeq T$ is a bijection for all $S,T\in\mathrm{ob}\,\mathsf{C}$.

[4] Strictly speaking, this is a left dual – the right dual is a morphism $X\otimes X'\to\mathbb{1}$ such that there exists a morphism $\mathbb{1}\to X'\otimes X$ making the similar diagrams commute.

PROOF We have functors

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{C} \qquad \begin{aligned} FS &= S \otimes X \\ GT &= T \otimes X', \end{aligned}$$

and a natural transformation

$$\epsilon\colon FG \to \mathrm{id}_{\mathsf{C}}, \quad T \otimes X' \otimes X \xrightarrow{T\otimes\varepsilon} T \otimes \mathbb{1} \simeq T.$$

According to A.4, $\psi_{S,T}$ is bijective for all S, T, i.e., (F, G, ψ^{-1}) is an adjunction, if and only if there exists a natural transformation $\eta\colon\ \mathrm{id}_{\mathsf{C}} \to GF$ such that the triangle identities

$$(S \otimes X \longrightarrow S \otimes X \otimes X' \otimes X \longrightarrow S \otimes X) = \mathrm{id}_{S\otimes X}$$
$$(T \otimes X' \to T \otimes X' \otimes X \otimes X' \to T \otimes X') = \mathrm{id}_{T\otimes X}$$

hold for all $S, T \in \operatorname{ob}\mathsf{C}$, i.e., (X', ε) is a dual of X. □

COROLLARY 4.9 *Let $(X^\vee, \mathrm{ev})$ be the dual of X.*

(a) *The functor $-\otimes X^\vee$ is right adjoint to $-\otimes X$.*

(b) *The coevaluation map* $\mathrm{coev}\colon\ \mathbb{1} \to X \otimes X^\vee$ *is uniquely determined by* $(X^\vee, \mathrm{ev})$.

PROOF (a) This restates the fact that Ψ is an isomorphism.

(b) Under the adjunction isomorphism

$$\mathrm{Hom}(\mathbb{1}, X \otimes X^\vee) \simeq \mathrm{Hom}(X, X),$$

coev corresponds to id_X. □

Assume that $\mathcal{H}om(X, T)$ exists for all T, and consider

$$\mathrm{ev}_{X,\mathbb{1}}\colon\ \mathcal{H}om(X, \mathbb{1}) \otimes X \to \mathbb{1}.$$

As in 4.8, this defines a morphism $\psi_{S,T}$, and from

$$\mathrm{Hom}(S, T \otimes \mathcal{H}om(X, \mathbb{1})) \xrightarrow{\psi_{S,T}} \mathrm{Hom}(S \otimes X, T) \simeq \mathrm{Hom}(S, \mathcal{H}om(X, T)), \tag{22}$$

we get (by the Yoneda lemma) a canonical morphism, natural in T,

$$\phi_{X,T}\colon\ T \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T). \tag{23}$$

The morphism $\phi_{X,T}$ can also be obtained from the composition morphism (11)

$$\mathcal{H}om(\mathbb{1}, T) \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T)$$

by replacing $\mathcal{H}om(\mathbb{1}, T)$ with T.

PROPOSITION 4.10 *An object X of C admits a dual if and only if, for all $T \in \operatorname{ob}\mathsf{C}$, $\mathcal{H}om(X, T)$ exists and $\phi_{X,T}$ is an isomorphism.*

PROOF Assume that $\mathcal{H}om(X, T)$ exists and (23) is an isomorphism for all T. Then the composite of the morphisms in (22) is an isomorphism for all S and T, so $\psi_{S,T}$ is an isomorphism. According to 4.8, this implies that $(\mathcal{H}om(X, T), \mathrm{ev}_{X,\mathbb{1}})$ is a dual of X.

Conversely, if X has a dual $(X^\vee, \varepsilon)$, then

$$\psi_{S,T}\colon\ \mathrm{Hom}(S, T \otimes X^\vee) \to \mathrm{Hom}(S \otimes X, T),$$

is an isomorphism for all S (see 4.8). Therefore, $\mathcal{H}om(X, T)$ exists and equals $T \otimes X^\vee$. □

COROLLARY 4.11 *If an object X of C admits a dual $(X^\vee, \mathrm{ev}_X)$, then*

$$(X^\vee, \mathrm{ev}_X) \simeq (\mathcal{H}om(X, \mathbb{1}), \mathrm{ev}_{X,\mathbb{1}})$$
$$T \otimes X^\vee \simeq \mathcal{H}om(X, T).$$

In particular, if $(X^\vee, \mathrm{ev}_X)$ exists, then it is unique up to a unique isomorphism, and the morphism (9)

$$X^\vee \otimes T \to \mathcal{H}om(X, T)$$

is an isomorphism.

PROOF This was shown in the above proof. □

For an $M \in \mathrm{ob}\,\mathsf{Mod}(R)$, the morphism (23) becomes

$$T \otimes \mathrm{Hom}(M, R) \to \mathrm{Hom}(M, T), \quad t \otimes f \mapsto (m \mapsto f(m)t). \tag{24}$$

PROPOSITION 4.12 *The following conditions on an R-module M are equivalent:*

(a) *M admits a dual;*

(b) *the map (24) is an isomorphism for all T;*

(c) *M is finitely generated and projective.*

PROOF (a) $\Rightarrow$ (b): Special case of 4.10 (it can also be proved directly).

(b) $\Rightarrow$ (c): In particular, $M \otimes \mathrm{Hom}(M, R) \simeq \mathrm{End}_R(M)$. If $\sum_{i\in I} m_i \otimes f_i$ corresponds to id_M under this isomorphism, so that $\sum_{i\in I} f_i(m)m_i = m$ for all $m \in M$, then

$$M \xrightarrow{m\mapsto(f_i(m))} R^I \xrightarrow{(a_i)\mapsto\sum a_i m_i} M$$

is a factorization of id_M. Therefore M is a direct summand of a free module of finite rank, and so it is finitely generated and projective.

(c) $\Rightarrow$ (a): When M is free of finite rank, we saw in 4.7 that there exists a dual. In the general case, there exists a finite family $(f_i)_{i\in I}$ of elements of R generating R as an ideal and such that, for each i, the R_{f_i}-module M_{f_i} is free. Thus, there exists a δ_i for each i, and the uniqueness assertion in Corollary 4.9 implies that they patch together to give a δ for M. □

For example, a module over a Dedekind domain admits a dual if and only if it is finitely generated and torsion free, and a vector space over a field admits a dual if and only if has finite dimension (for an infinite-dimensional vector space V, there is no coevaluation map $k \to V \otimes V^\vee$).

SUMMARY 4.13 Let $(\mathsf{C}, \otimes)$ be a $\otimes$-category in which internal Homs exist. For any object X in C, the following statements are equivalent:

(a) $\phi_{X,T} : T \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T)$ is an isomorphism for all T;

(b) $\phi_{X,X} : X \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, X)$ is an isomorphism;

(c) X admits a dual, in which case the dual is $(\mathcal{H}om(X, \mathbb{1}), \mathrm{ev}_{X,\mathbb{1}})$.

NOTES This section follows Dold and Puppe 1980.

5 Rigid symmetric monoidal categories

Definition

DEFINITION 5.1 A symmetric monoidal category $(\mathsf{C}, \otimes)$ is ***rigid*** if, for all objects X and Y,

(a) $\mathcal{H}om(X,Y)$ exists, and

(b) the canonical morphism (23)

$$Y \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X,Y)$$

is an isomorphism.

Equvalent condition (4.10): every object of C admits a dual (in the sense of 4.5).

Let $(\mathsf{C}, \otimes)$ be a rigid symmetric monoidal category. The opposite category $\mathsf{C}^{\mathrm{opp}}$ has a tensor structure for which $\otimes X_i^{\mathrm{opp}} = (\otimes X_i)^{\mathrm{opp}}$. The $\otimes$-functor

$$\{X, f\} \rightsquigarrow \{X^\vee, {}^t f\} \colon \mathsf{C}^{\mathrm{opp}} \to \mathsf{C}$$

is a contravariant equivalence of categories because its composite with itself is isomorphic to the identity functor. It is therefore a contravariant equivalence of symmetric monoidal categories (3.7). In particular,

$$f \mapsto {}^t f \colon \operatorname{Hom}(X,Y) \to \operatorname{Hom}(Y^\vee, X^\vee) \tag{25}$$

is an isomorphism. There is also a canonical isomorphism

$$\mathcal{H}om(X,Y) \to \mathcal{H}om(Y^\vee, X^\vee), \tag{26}$$

namely, the composite of the isomorphisms

$$\mathcal{H}om(X,Y) \xleftarrow{(4.11)} X^\vee \otimes Y \xrightarrow{X^\vee \otimes i_Y} X^\vee \otimes Y^{\vee\vee} \xrightarrow{\gamma} Y^{\vee\vee} \otimes X^\vee \xrightarrow{(4.11)} \mathcal{H}om(Y^\vee, X^\vee).$$

PROPOSITION 5.2 *Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. If C is rigid, then $X \rightsquigarrow D(X) \overset{\text{def}}{=} X^\vee$ is a functor equipped with a natural isomorphism*

$$\psi_{X,Y,Z} \colon \operatorname{Hom}(X \otimes Y, Z) \simeq \operatorname{Hom}(X, Z \otimes D(Y)).$$

Conversely, if there exists a functor $D \colon \mathsf{C} \to \mathsf{C}$ and a natural isomorphism ψ, then C is rigid; moreover, $(D(Y), \varepsilon)$, where ε corresponds to the identity map under $\psi_{D(Y),Y,\mathbb{1}}$, is the dual of Y.

PROOF Restatement of 4.8. □

NOTES Saavedra 1972, I, 5.1.1 defines a symmetric monoidal category to be rigid if $\mathcal{H}om(X,Y)$ exists for all X, Y and the canonical morphism (17)

$$\mathcal{H}om(X_1, Y_1) \otimes \mathcal{H}om(X_2, Y_2) \to \mathcal{H}om(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

is an isomorphism for all X_1, X_2, Y_1, Y_2. This is equivalent to our definition.

Traces

Let $(\mathsf{C},\otimes)$ be a rigid symmetric monoidal category.

For any object X of C, there are morphisms

$$\mathcal{H}om(X,X) \xrightarrow[(19)]{\simeq} X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}. \tag{27}$$

On applying the functor $\mathrm{Hom}(\mathbb{1},-)$ to this, we obtain by (13) a morphism

$$\mathrm{Tr}_X\colon\ \mathrm{End}(X) \to \mathrm{End}(\mathbb{1}) \tag{28}$$

called the ***trace morphism***. More directly, the trace of $f: X \to X$ is the composite of the morphisms

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f\otimes X^\vee} X \otimes X^\vee \xrightarrow{\gamma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}.$$

We sometimes write $\mathrm{Tr}(f|X)$ for $\mathrm{Tr}_X(f)$. The ***(categorical) dimension*** (or ***rank***) of X is defined to be $\mathrm{Tr}_X(\mathrm{id}_X)$, so $\dim X$ is the composite of the morphisms

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{\gamma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}.$$

Note that $\dim(X)$ is an element of the ring $\mathrm{End}(\mathbb{1})$. In particular, it need not be an integer, much less a positive integer (see 9.9 and 9.11 for example). In the tensor category of finite-dimensional vector spaces over a field k of characteristic p, a vector space of dimension p has categorical dimension 0.

PROPOSITION 5.3 *There are the following equalities.*

(a) *If* $X \oplus Y$ *exists, then* $\mathrm{Tr}_{X\oplus Y}(f \oplus g) = \mathrm{Tr}_X(f) + \mathrm{Tr}_Y(g)$*, where* $f: X \to X$, $g: Y \to Y$.

(b) $\mathrm{Tr}_{X\otimes Y}(f \otimes g) = \mathrm{Tr}_X(f)\cdot \mathrm{Tr}_Y(g)$*, where* $f: X \to X$, $g: Y \to Y$.

(c) $\mathrm{Tr}_{X^\vee}(f^\vee) = \mathrm{Tr}_X(f)$*, where* $f: X \to X$.

(d) $\mathrm{Tr}_{\mathbb{1}}(f) = f$

(e) $\mathrm{Tr}_X(g\circ f) = \mathrm{Tr}_{Y\otimes X}(\gamma_{X,Y}\circ(g \otimes f)) = \mathrm{Tr}_Y(f\circ g)$*, where* $f: X \longrightarrow Y$, $g: Y \to X$.

PROOF Only the proof of (e) presents problems. For a morphism $f: X \to Y$, let $\delta(f)\colon\ \mathbb{1} \to Y \otimes X^\vee$ denote the composite

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f\otimes X^\vee} Y \otimes X^\vee.$$

Thus, when $Y = X$, $\mathrm{Tr}_X(f) = \mathrm{ev}_X \circ \gamma_{X,X^\vee} \circ \delta(f)$.

For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the morphism $\delta(g\circ f)$ is the composite

$$\mathbb{1} \simeq \mathbb{1} \otimes \mathbb{1} \xrightarrow{\delta(g)\otimes\delta(f)} Z \otimes Y^\vee \otimes Y \otimes X^\vee \xrightarrow{Z\otimes \mathrm{ev}_Y \otimes X^\vee} Z \otimes \mathbb{1} \otimes X^\vee \simeq Z \otimes X^\vee. \tag{29}$$

Thus, when $Z = X$,

$$\mathrm{Tr}_X(g\circ f) \overset{\mathrm{def}}{=} \mathrm{ev}_X \circ \gamma_{X,X^\vee} \circ \delta(g\circ f) = \mathrm{Tr}(\gamma_{X,Y}\circ(g \otimes f)).$$

Similarly,

$$\mathrm{Tr}_Y(f\circ g) = \mathrm{Tr}(\gamma_{Y,X}\circ(f \otimes g)) = \mathrm{Tr}(\gamma_{X,Y}\circ(g \otimes f)).$$

□

COROLLARY 5.4 *For any objects X, Y of C,*

$$\begin{aligned}\dim(X \oplus Y) &= \dim X + \dim Y \\ \dim(X \otimes Y) &= \dim X \cdot \dim Y \\ \dim(X^\vee) &= \dim(X) \\ \dim(\mathbb{1}) &= \mathrm{id}_{\mathbb{1}}\,.\end{aligned}$$

PROOF Apply the proposition to the identity morphisms. □

REMARK 5.5 (DELIGNE 1990, 7.2) Consider, as in (e) of Proposition 5.3, morphisms $f: X \to Y$ and $g: Y \to X$. The tensor product of the unordered set $\{X, Y\}$ is well-defined up to a canonical isomorphism, as is the morphism $\otimes\{f, g\}: \otimes\{X, Y\} \to \otimes\{X, Y\}$. The proposition says that the trace of this morphism is $\mathrm{Tr}(g \circ f)$, hence, by symmetry, also $\mathrm{Tr}(f \circ g)$.

More generally, consider morphisms

$$X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} X_n \xrightarrow{u_n} X_1.$$

These give rise to a morphism,

$$\bigotimes_{i \in \mathbb{Z}/(n)} u_i : \bigotimes_{i \in \mathbb{Z}/(n)} X_i \longrightarrow \bigotimes_{i \in \mathbb{Z}/(n)} X_i,$$

which we denote by $\otimes u_i$. On iterating (29), we find that $\delta(u_n \circ \cdots \circ u_1)$ is the composite

$$\mathbb{1} \xrightarrow{\delta(u_n)\otimes\cdots\otimes\delta(u_1)} X_1 \otimes X_n^\vee \otimes X_n \otimes \cdots \otimes X_2 \otimes X_1^\vee \xrightarrow{X_1 \otimes \mathrm{ev}_{X_n} \otimes \cdots \otimes \mathrm{ev}_{X_2} \otimes X_1^\vee} X_1 \otimes X_1^\vee,$$

and it follows that

$$\mathrm{Tr}(u_n \circ \cdots \circ u_1) = \otimes_i (\mathrm{ev}_{X_i} \circ \gamma_{X_i, X_i^\vee}) \circ \otimes_i \delta(u_i) = \mathrm{Tr}(\otimes_i u_i). \tag{30}$$

⊗-functors of rigid ⊗-categories

It is clear from Definition 4.5 that a ⊗-functor $F: \mathsf{C} \to \mathsf{D}$ of rigid ⊗-categories preserves duals:

$$F(X^\vee) \simeq F(X)^\vee.$$

Moreover, it induces a homomorphism $F: \mathrm{End}(\mathbb{1}_{\mathsf{C}}) \to \mathrm{End}(F(\mathbb{1}_{\mathsf{C}})) \simeq \mathrm{End}(\mathbb{1}_{\mathsf{D}})$. This preserves traces: for any $f: X \to X$,

$$\mathrm{Tr}_{F(X)} F(f) = F(\mathrm{Tr}_X(f)).$$

In particular, $\dim(F(X)) = F(\dim(X))$.

PROPOSITION 5.6 *Let $(F, c): (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ be a ⊗-functor of rigid symmetric monoidal categories. For any $X, Y \in \mathrm{ob}(\mathsf{C})$,*

$$F(\mathcal{H}om(X, Y)) \simeq \mathcal{H}om(FX, FY). \tag{31}$$

In particular,

$$F(X^\vee) \simeq F(X)^\vee.$$

PROOF The morphism (31) comes by adjunction (10) from

$$F(\mathrm{ev}_{X,Y})\colon F(\mathcal{H}om(X,Y))\otimes FX \to FY.$$

The commutative diagram

$$\begin{tikzcd}
F(\mathcal{H}om(X,Y)) \arrow[r] & \mathcal{H}om(FX,FY) \\
F(X^\vee\otimes Y) \arrow[r, "\simeq"] \arrow[u, "\simeq", "4.11"'] & (FX)^\vee \otimes FY. \arrow[u, "\simeq", "4.11"']
\end{tikzcd}$$

shows that it is an isomorphism. □

PROPOSITION 5.7 *Let (F,c) and (G,d) be $\otimes$-functors $\mathsf{C}\to\mathsf{D}$ of $\otimes$-categories. If C is rigid, then every monoidal natural transformation $u\colon F\to G$ is an isomorphism.*

PROOF The morphism $v\colon G\to F$ making the diagrams

$$\begin{tikzcd}
F(X^\vee) \arrow[r, "v_{X^\vee}"] \arrow[d, "\simeq"] & G(X^\vee) \arrow[d, "\simeq"] \\
F(X)^\vee \arrow[r, "{}^t(u_X)"] & G(X)^\vee
\end{tikzcd}$$

commute for all $X\in\mathrm{ob}(\mathsf{C})$, is an inverse for u. Note that v_X is the composite

$$G(X)\simeq G(X^\vee)^\vee \xrightarrow{{}^tu_{X^\vee}} F(X^\vee)^\vee \simeq F(X).$$

To see that $u_X\circ v_X=\mathrm{id}_{GX}$, chase around the outside of the following diagram in two ways, starting from the GX at lower left. The outer diagram commutes because each subdiagram does.

$$\begin{tikzcd}
FX\otimes F(X^\vee)\otimes GX \arrow[r, "\mathrm{id}\otimes u_{X^\vee}\otimes\mathrm{id}"] \arrow[dr, "u_X\otimes u_{X^\vee}\otimes\mathrm{id}"] & FX\otimes G(X^\vee)\otimes GX \arrow[r, "\mathrm{id}\otimes\varepsilon_{GX}"] \arrow[d, "u_X\otimes\mathrm{id}\otimes\mathrm{id}"] & FX \arrow[d, "u_X"] \\
GX \arrow[u, "\delta_{FX}\otimes\mathrm{id}"] \arrow[r, "\delta_{GX}\otimes\mathrm{id}"] & GX\otimes G(X^\vee)\otimes GX \arrow[r, "\mathrm{id}\otimes\varepsilon_{GX}"] & GX
\end{tikzcd}$$

The proof that $v_X\circ u_X=\mathrm{id}_{GX}$ is similar. □

For a k-algebra R, let ϕ_R denote the functor $-\otimes_k R\colon \mathsf{Vecf}(k)\to\mathsf{Mod}(R)$. Let $\omega\colon\mathsf{C}\to\mathsf{Vecf}(k)$ be a $\otimes$-functor, and define functors $\mathcal{E}nd^\otimes(\omega)$ and $\mathcal{A}ut^\otimes(\omega)$ by by

$$\mathcal{E}nd^\otimes(\omega)(R)\overset{\mathrm{def}}{=}\mathrm{End}(\phi_R\circ\omega)$$
$$\mathcal{A}ut^\otimes(\omega)(R)\overset{\mathrm{def}}{=}\mathrm{Aut}(\phi_R\circ\omega).$$

(cf. (8), p. 13). If C is rigid, then $\mathrm{End}(\phi_R\circ\omega)\simeq\mathrm{Aut}(\phi_R\circ\omega)$ (5.7), and so

$$\mathcal{E}nd^\otimes(\omega)\simeq\mathrm{Aut}(\omega). \qquad (32)$$

PROPOSITION 5.8 *Let $F\colon\mathsf{C}\to\mathsf{D}$ be a $\otimes$-functor between rigid symmetric monoidal categories. The following conditions on F are equivalent:*

(a) *every object of D is a subobject of $F(X)$ for some object X of C;*

(b) *every object of* D *is a quotient of* $F(X)$ *for some object* X *of* C.

PROOF Assume (a), and let Y be an object of D. There exists an object X of C and a monomorphism $i\colon Y^\vee \to F(X)$. In a rigid symmetric monoidal category, the functor $^\vee$ is a contravariant equivalence with itself as a quasi-inverse, and $\otimes$-functors preserve duals (5.6). Hence

$$F(X^\vee) \simeq F(X)^\vee \xrightarrow{i^\vee} Y^{\vee\vee} \simeq Y$$

is an epimorphism. We have shown that (a) implies (b), and the converse is proved similarly. □

DEFINITION 5.9 A $\otimes$-functor of rigid tensor categories is ***dominant*** if it satisfies the equivalent conditions of 5.8.

ASIDE 5.10 Let $F\colon \mathsf{C} \to \mathsf{D}$ be a k-linear exact $\otimes$-functor of essentially small tensor categories over k. By the special adjoint functor theorem, F has a right adjoint $F_*\colon \operatorname{Ind}\mathsf{D} \to \operatorname{Ind}\mathsf{C}$. When C and D are locally finite, the functor F is dominant if and only if F_* is faithful. See Coulembier 2026, 1.3.4.

DEFINITION 5.11 Let C' be a strictly full subcategory of a monoidal category C. We say that C' is a ***monoidal subcategory*** of C if it is stable under the formation of finite tensor products (it suffices to check that it contains a unit and that it contains $X \otimes Y$ whenever it contains X and Y). It becomes a monoidal category under the induced tensor structure. If C is symmetric, then C' is symmetric, and if C is symmetric and rigid, then C' is rigid if it contains contains $X^\vee$ whenever it contains X.

NOTES For a proof that the diagram in Proposition 5.6 commutes, see sx4932465. The proof of Proposition 5.7 was extracted from ncatlab.org.

6 Abelian rigid symmetric monoidal categories

Our convention, that functors between additive categories are additive, forces the following definition.

DEFINITION 6.1 A symmetric monoidal category $(\mathsf{C}, \otimes)$ is ***additive*** (resp. ***abelian***) if C is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

If $(\mathsf{C}, \otimes)$ is an additive symmetric monoidal category and $(\mathbb{1}, e)$ is a unit, then $R \overset{\text{def}}{=} \operatorname{End}(\mathbb{1})$ is a ring that acts through the isomorphism $\lambda_X\colon \mathbb{1} \otimes X \xrightarrow{\simeq} X$ on each object X of C. The action of R on X commutes with endomorphisms of X and so, in particular, R is commutative. If $(\mathbb{1}', e')$ is a second unit, then the unique isomorphism $a\colon (\mathbb{1}, e) \to (\mathbb{1}', e')$ (see 1.8) defines an isomorphism $R \simeq \operatorname{End}(\mathbb{1}')$. The category C is R-linear in the sense that each Hom-set is equipped with an R-module structure and $\circ$ and $\otimes$ are R-bilinear.

In more detail, we have a bi-additive map

$$(a, f) \mapsto a \cdot f\colon \operatorname{End}(\mathbb{1}) \times \operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, Y)$$

given by

$$\begin{tikzcd} \mathbb{1} \otimes X \arrow[r, "a \otimes f"] \arrow[d, "\lambda_X", "\simeq"'] & \mathbb{1} \otimes Y \arrow[d, "\lambda_Y", "\simeq"'] \\ X \arrow[r, "a \cdot f"] & Y. \end{tikzcd}$$

That this defines an action of the ring R on $\mathrm{Hom}(X,Y)$ follows from the middle unity diagram 1.5. That the action is R-bilinear follows from the definition of the action and the functoriality of $\otimes$. That $\otimes$ is R-bilinear follows from the middle unity diagram 1.5, the left unity diagram 1.3, and 2.2. See sx5080743.

When C is rigid and $R = \mathrm{End}(\mathbb{1})$, the trace morphism is an R-linear map

$$\mathrm{Tr}\colon\ \mathrm{End}(X) \to R.$$

PROPOSITION 6.2 *Let* $(\mathsf{C},\otimes)$ *be a symmetric monoidal category. If* C *is closed, then* $\otimes$ *commutes with inductive limits in each variable. If* C *is rigid, then* $\otimes$ *also commutes with projective limits in each variable.*

PROOF If C is closed, then the functor $\mathcal{H}om(X,-)$ is right adjoint to $-\otimes X$ (see (10)), and so $-\otimes X$ commutes with inductive limits. If C is rigid, then $X^\vee \otimes -$ is both a left and a right adjoint of $-\otimes X$. □

COROLLARY 6.3 *In an abelian rigid symmetric monoidal category,* $\otimes$ *is exact in each variable.*

Note that $\otimes$ is not usually exact in $\mathsf{Mod}(R)$.

PROPOSITION 6.4 *Let* $(\mathsf{C},\otimes)$ *be an abelian rigid symmetric monoidal category. If* U *is a subobject of* $\mathbb{1}$*, then* $\mathbb{1} \simeq U \oplus U^\perp$*, where* $U^\perp = \mathrm{Ker}(\mathbb{1} \to U^\vee)$*. Consequently,* $\mathbb{1}$ *is a simple object if* $\mathrm{End}(\mathbb{1})$ *is a field.*

PROOF Let $V = \mathrm{Coker}(U \to \mathbb{1})$, so

$$0 \longrightarrow U \longrightarrow \mathbb{1} \longrightarrow V \longrightarrow 0$$

is exact. On applying $\otimes$ to the commutative diagram

$$\begin{tikzcd}
U\times\mathbb{1} \arrow[r] & \mathbb{1}\times\mathbb{1} \arrow[r] & V\times\mathbb{1} \\
U\times U \arrow[r] \arrow[u] & \mathbb{1}\times U \arrow[r] \arrow[u] & V\times U \arrow[u]
\end{tikzcd}$$

in $\mathsf{C}\times\mathsf{C}$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & \mathbb{1} \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & U\otimes U \arrow[r] \arrow[u, hook] & U \arrow[r] \arrow[u, hook] \arrow[ur, dashed, "0"] & V\otimes U \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

in C. The bottom row is (top row) $\otimes U$, so it also is exact. It follows that $V\otimes U = 0$ and that $U\otimes U = U$ as a subobject of $\mathbb{1}\otimes\mathbb{1} = \mathbb{1}$.

For any object T, the morphism $T\otimes U \to T$, obtained from $U \hookrightarrow \mathbb{1}$ by tensoring with T, is a monomorphism. This proves the first equivalence in

$$T\otimes U = 0 \iff \text{the morphism } T\otimes U \to T \text{ is zero} \iff \text{the morphism } T \to U^\vee\otimes T \text{ is zero},$$

and the second follows from the canonical isomorphisms

$$\mathrm{Hom}(T\otimes U, T) \overset{1.6.5}{\simeq} \mathrm{Hom}(T\otimes U\otimes T^\vee, \mathbf{1}) \overset{1.6.5}{\simeq} \mathrm{Hom}(T, U^\vee\otimes T).$$

For an object X of C and subobject T, we deduce from the diagram

$$\begin{tikzcd} X \arrow[r] & U^\vee \otimes X \\ T \arrow[r] \arrow[u, hook] & U^\vee \otimes T \arrow[u, hook] \end{tikzcd}$$

that

$$\operatorname{Ker}(X \to U^\vee \otimes X)$$

is the largest subobject K of X such that $K \otimes U = 0$. On tensoring the exact sequence

$$0 \to U^\perp \to \mathbb{1} \to U^\vee \to 0$$

with X, we see that $K = U^\perp \otimes X$.

We apply this statement to the object V. As $V \otimes U = 0$, we deduce that $U^\perp \otimes V = V$.

We apply the statement to the object U. Let T be a subobject of U. The argument in the first paragraph shows that $T \otimes T = T$, and so

$$T \otimes U = 0 \implies T = 0.$$

We deduce that $U^\perp \otimes U = 0$.

From the exact sequence

$$0 \to U^\perp \otimes U \to U^\perp \otimes \mathbb{1} \to U^\perp \otimes V \to 0$$

we now deduce that the composite $U^\perp \to \mathbb{1} \to V$ is an isomorphism, so $\mathbb{1} \simeq U^\perp \oplus U$. □

PROPOSITION 6.5 *Let* C *be an abelian symmetric monoidal category with internal Homs. If* $\mathbb{1}$ *is simple, then the product of any two nonzero objects of* C *is nonzero provided one admits a strong dual.*

PROOF Let X and Y be nonzero objects of C with X admitting a dual. As X is nonzero, the equality (4.5)

$$(\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X$$

shows that the evaluation morphism $\mathrm{ev}_X \colon X^\vee \otimes X \to \mathbb{1}$ is nonzero, and hence an epimorphism because $\mathbb{1}$ is simple. Now $X^\vee \otimes X \otimes Y \to Y$ is an epimorphism (6.2), which implies that $X \otimes Y$ is nonzero. □

REMARK 6.6 It follows from the proposition shows that, in an abelian rigid symmetric monoidal category, there is a one-to-one correspondence between subobjects of $\mathbb{1}$ and idempotents in $\operatorname{End}(\mathbb{1})$. Such an idempotent e determines a decomposition of symmetric monoidal categories $\mathsf{C} = \mathsf{C}' \times \mathsf{C}''$ in which the objects of C' (resp. C'') are those on which e (resp. $1 - e$) acts as the identity morphism.

PROPOSITION 6.7 *Let* C *and* C' *be abelian symmetric monoidal categories with identity objects* $\mathbb{1}$ *and* $\mathbb{1}'$. *If* C *is rigid,* $\mathbb{1}$ *is simple and* $\mathbb{1}' \neq 0$, *then every right exact* $\otimes$*-functor* $F \colon \mathsf{C} \to \mathsf{C}'$ *is faithful.*

PROOF As F is additive and exact, it suffices to show that $F(X) \neq 0$ when $X \neq 0$. If $X \neq 0$, then $\mathrm{ev} \colon X^\vee \otimes X \to \mathbb{1}$ is an epimorphism because $\mathbb{1}$ is simple, and so

$$F(X^\vee) \otimes F(X) \simeq F(X^\vee \otimes X) \to F(\mathbb{1}) \simeq \mathbb{1}'$$

is an epimorphism. As $\mathbb{1}' \neq 0$, this implies that $F(X) \neq 0$. □

Note that $\mathbb{1}$ is simple if $\operatorname{End}(\mathbb{1})$ is a field, and $\mathbb{1}' \neq 0$ if $\mathsf{C}' \neq 0$.

Traces in abelian rigid symmetric monoidal categories

In this subsection, the underlying category is an abelian rigid symmetric monoidal category. We show that traces are additive on short exact sequences.

PROPOSITION 6.8 (DELIGNE) *For any exact commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[r] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0, \end{tikzcd}$$

$$\mathrm{Tr}_X(f) = \mathrm{Tr}_{X'}(f') + \mathrm{Tr}_{X''}(f'').$$

In particular,

$$\dim(X) = \dim(X') + \dim(X'').$$

PROOF For an object X of such a category, let t_X denote the morphism (27)

$$\mathcal{H}om(X,X) \simeq X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}.$$

Then $\mathrm{Hom}(\mathbb{1}, t_X)$ is the trace map $\mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$.

For a short exact sequence

$$\Sigma\colon\ 0 \to X' \to X \to X'' \to 0,$$

let

$$\mathcal{H}om(\Sigma,\Sigma) = \mathrm{Ker}\left(\mathcal{H}om(X,X) \to \mathcal{H}om(X',X'')\right).$$

Thus,

$$\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(\Sigma,\Sigma)) \simeq \mathrm{End}(\Sigma) \overset{\text{def}}{=} \{f \in \mathrm{End}(X) \mid f \text{ respects } \Sigma\}.$$

It suffices to show that the diagram

$$\begin{tikzcd} \mathcal{H}om(\Sigma,\Sigma) \arrow[r] \arrow[d] & \mathcal{H}om(X',X') \oplus \mathcal{H}om(X'',X'') \arrow[d, "t_{X'}+t_{X''}"] \\ \mathcal{H}om(X,X) \arrow[r, "t_X"] & \mathbb{1} \end{tikzcd} \tag{33}$$

commutes. On tensoring Σ with its dual, we get a diagram with exact rows and columns (6.2),

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & X''^\vee \otimes X' \arrow[r] \arrow[d] & X''^\vee \otimes X \arrow[r] \arrow[d] & X''^\vee \otimes X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^\vee \otimes X' \arrow[r] \arrow[d] & X^\vee \otimes X \arrow[r] \arrow[d] & X^\vee \otimes X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X'^\vee \otimes X' \arrow[r] \arrow[d] & X'^\vee \otimes X \arrow[r] \arrow[d] & X'^\vee \otimes X'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

From a diagram chase, we see that $(X''^{\vee} \otimes X) \oplus (X^{\vee} \otimes X')$ maps onto the kernel of $X^{\vee} \otimes X \to X'^{\vee} \otimes X''$. Hence $\mathcal{H}om(X'', X) \oplus \mathcal{H}om(X, X')$ maps onto $\mathcal{H}om(\Sigma, \Sigma)$, and so it suffices to show that (33) commutes with $\mathcal{H}om(\Sigma, \Sigma)$ replaced by each of $\mathcal{H}om(X'', X)$ and $\mathcal{H}om(X, X')$. On $\mathcal{H}om(X'', X)$, the map to $\mathcal{H}om(X', X')$ vanishes, and so we have to show that

$$\begin{tikzcd}
\mathcal{H}om(X'',X) \arrow[r] \arrow[d] & \mathcal{H}om(X'',X'') \arrow[d, "t_{X''}"] \\
\mathcal{H}om(X,X) \arrow[r, "t_X"] & \mathbb{1}
\end{tikzcd}$$

commutes, and dually for $\mathcal{H}om(X, X')$.

From the identity $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$, we see that the diagram

$$\begin{tikzcd}
\mathcal{H}om(X,X'') \otimes \mathcal{H}om(X'',X) \arrow[r] \arrow[d] & \mathcal{H}om(X'',X'') \arrow[d, "t_{X''}"] \\
\mathcal{H}om(X,X) \arrow[r, "t_X"] & \mathbb{1}
\end{tikzcd}$$

commutes. On composing with the morphism $\mathbb{1} \to \mathcal{H}om(X, X'')$ that corresponds to the given morphism $X \to X''$ under the isomorphism

$$\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X, X'')) \simeq \mathrm{Hom}(X, X''),$$

we obtain the required diagram for $\mathcal{H}om(X'', X)$. □

COROLLARY 6.9 *The trace of any nilpotent endomorphism is zero.*

PROOF If the endomorphism f of X is nilpotent, then there exists a finite decreasing filtration F of X such that $f(F^i) \subset F^{i+1}$, for example, the filtration with $F^i = f^i(X)$. Then $f = 0$ on F^i/F^{i+1}, and so

$$\mathrm{Tr}(f) \overset{6.8}{=} \sum \mathrm{Tr}(f|F^i/F^{i+1}) = 0.$$ □

The corollary fails for categories that are not abelian.

NOTES This subsection follows a letter of Deligne, dated July 11, 2003.

Traces and fibre functors

Let C be a k-linear abelian rigid $\otimes$-category. By a fibre functor on C with values in a k-algebra R, we mean a k-linear right-exact $\otimes$-functor $\mathsf{C} \to \mathsf{Mod}(R)$.

6.10 Let $\omega : \mathsf{C} \to \mathsf{Mod}(R)$ be a fibre functor on C, and let (U, u) be a unit in C. There is a unique isomorphism $a_\omega : R \to \omega(U)$ such that the top square in the following diagram commutes,

$$\begin{tikzcd}
R \otimes R \arrow[r] \arrow[d, "a_\omega \otimes a_\omega"] & R \arrow[d, "a_\omega"] \\
\omega(U) \otimes \omega(U) \arrow[r, "\omega(u)"] \arrow[d, "\eta_U \otimes \eta_U"] & \omega(U) \arrow[d, "\eta_U"] \\
\omega'(U) \otimes \omega'(U) \arrow[r, "\omega'(u)"] & \omega'(U)
\end{tikzcd}$$

(see 1.8). If ω' is a second R-valued fibre functor on C and $\eta\colon\ \omega \to \omega'$ is an isomorphism of $\otimes$-functors, then

$$a_{\omega'} = \eta_U \circ a_\omega. \tag{34}$$

This follows from the above diagram.

6.11 For an object X of C and an R-valued fibre functor ω, we define

$$\mathrm{Tr}_\omega\colon\ \mathrm{End}(X) \to R$$

by

$$\mathrm{Tr}_\omega(\alpha) = \mathrm{Tr}(\omega(\alpha) \mid \omega(X)), \quad \alpha \in \mathrm{End}(X).$$

Alternatively, Tr_ω is the composite of

$$\mathrm{End}(X) \xrightarrow{\mathrm{Tr}_X} \mathrm{End}(U) \xrightarrow{\omega} \mathrm{End}(\omega(U)) \simeq R,$$

where the canonical isomorphism is defined by a_ω. From this second description and (34), we see that $\mathrm{Tr}_\omega = \mathrm{Tr}_{\omega'}$ if ω and ω' are isomorphic, or even if they become isomorphic after a faithfully flat homomorphism $R \to R'$.

6.12 Assume now that the fibre functors on C are locally isomorphic for the fpqc topology, i.e., if ω and ω' are R-valued fibre functors on C, then there exists a faithfully flat homomorphism $R \to R'$ such that $\omega \otimes_R R' \approx \omega' \otimes_R R'$.

Then, for any R-valued fibre functor ω,

$$\mathrm{Tr}_\omega(\alpha) \in k, \quad \text{all } X \in \mathrm{ob}\,\mathsf{C} \text{ and } \alpha \in \mathrm{End}(X).$$

To see this, let ω be an R-valued fibre functor on C and consider the exact sequence

$$0 \longrightarrow k \longrightarrow R \underset{i_2}{\overset{i_1}{\rightrightarrows}} R \otimes_k R.$$

Then $\omega_1 \overset{\text{def}}{=} \omega \otimes_{R,i_1} R \otimes_k R$ and $\omega_2 \overset{\text{def}}{=} \omega \otimes_{R,i_2} R \otimes_k R$ are $R \otimes_k R$-valued fibre functors on C. Let α be an endomorphism of an object X of C. Under the maps i_1 and i_2, $\mathrm{Tr}_\omega(\alpha)$ maps to $\mathrm{Tr}_{\omega_1}(\alpha)$ and $\mathrm{Tr}_{\omega_2}(\alpha)$. As ω_1 and ω_2 are locally isomorphic, $\mathrm{Tr}_{\omega_1}(\alpha)$=$\mathrm{Tr}_{\omega_2}(\alpha)$ (by 6.11), and so $\mathrm{Tr}_\omega(\alpha) \in k$.

6.13 If C admits a nonzero fibre functor and any two fibre functors are locally isomorphic, then

$$\mathrm{End}(\mathbb{1}) = k.$$

Apply 6.12 to an endomorphism of $\mathbb{1}$.

7 Tensor categories

In this section, k is a field.

DEFINITION 7.1 A ***tensor category over*** k is a k-linear abelian rigid symmetric monoidal category such that the structure homomorphism $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism.[5]

[5]Following Deligne 1990, 2.1. The condition $k \simeq \mathrm{End}(\mathbb{1})$ is necessary for the fibre functors to form a gerbe (6.13). Some authors do not require tensor categories over k to be symmetric and some require them to be essentially small or locally finite.

In other words, a tensor category over k is an abelian rigid symmetric monoidal category equipped with an isomorphism $k \simeq \mathrm{End}(\mathbb{1})$.

DEFINITION 7.2 A ***tensor subcategory*** of a tensor category over k is a strictly full additive subcategory stable under direct sums, tensor products, duals, and subquotients.

In particular, a tensor subcategory is an abelian subcategory (abelian category such that the inclusion map is exact), contains an identity object, and is a tensor category over k.

DEFINITION 7.3 A ***tensor functor*** of tensor categories over k is a right-exact k-linear symmetric monoidal functor.

PROPOSITION 7.4 *Tensor functors between tensor categories over k are exact and faithful.*

PROOF This follows from 6.7 and the next more precise statement. □

PROPOSITION 7.5 (DELIGNE 1990, 2.10) *Let $F\colon \mathsf{C} \to \mathsf{D}$ be a right exact $\otimes$-functor between abelian $\otimes$-categories. If C is rigid and the tensor product on D is right exact, then F is exact.*

PROOF Let $0 \to X \to Y \to Z$ be an exact sequence in C. On applying F to the dual sequence, we get an exact sequence

$$F(Z)^\vee \to F(Y)^\vee \to F(X)^\vee \to 0 \tag{35}$$

in D. Let $U \to V \to W \to 0$ be an exact sequence in D. If $\mathcal{H}om(U,\mathbb{1})$ and $\mathcal{H}om(V,\mathbb{1})$ exist in D, then $\mathcal{H}om(W,\mathbb{1})$ exists in D and the sequence

$$0 \to \mathcal{H}om(W,\mathbb{1}) \to \mathcal{H}om(V,\mathbb{1}) \to \mathcal{H}om(U,\mathbb{1}) \tag{36}$$

is exact. As $F(X)$ is dual to $F(X)^\vee$, it equals $\mathcal{H}om(F(X)^\vee,\mathbb{1})$ (by 4.11) and similarly for Y and Z. For (35), the sequence (36) becomes

$$0 \to F(X) \to F(Y) \to F(Z),$$

which is therefore exact. □

Let $P \in \mathbb{N}[x,y]$. When X, Y are objects of a symmetric monoidal category, we can define $P(X,Y)$ by interpreting addition as $\oplus$ and multiplication as $\otimes$, so

$$P(x,y) = \sum\nolimits_{i,j} m_{i,j} x^i y^j \implies P(X,Y) = \bigoplus\nolimits_{i,j} m_{i,j} X^{\otimes i} \otimes Y^{\otimes j}.$$

DEFINITION 7.6 For an object X of an abelian rigid symmetric monoidal category C, we let $\langle X\rangle^\otimes$ denote the strictly full subcategory of C whose objects are subquotients of $P(X,X^\vee)$ for some $P \in \mathbb{N}[t,s]$. It is again an abelian rigid symmetric monoidal category. We call X a ***tensor generator*** of C if $\mathsf{C} = \langle X\rangle^\otimes$.

In other words, $\langle X\rangle^\otimes$ is the strictly full subcategory of C whose objects are those that can be constructed from X by iterated applications of the operations of direct sum, tensor product, dual, passage to a subobject or a quotient object. When C is a tensor category over k, $\langle X\rangle^\otimes$ is the smallest tensor subcategory containing X.

DEFINITION 7.7 A tensor category over k is ***monogenic***[6] if it admits a tensor generator.

In the literature, a monogenic tensor category over k is said to be finitely generated. Note that if a tensor category is generated by objects $X_1, \ldots, X_n$, i.e., is the smallest tensor subcategory over k containing the X_i, then it equals $\langle X \rangle^{\otimes}$, where $X = X_1 \oplus \cdots \oplus X_n$.

DEFINITION 7.8 A k-linear abelian category is ***locally finite*** if every object has finite length and every k-vector space $\mathrm{Hom}(X, Y)$ has finite dimension.

An object of an abelian category is said to have finite length if it admits a (finite) composition series, in which case all composition series have the same lengkth. Both the Jordan–Hölder and Krull–Schmidt theorems hold in locally finite k-linear abelian categories.

PROPOSITION 7.9 *A tensor category over k is locally finite if every object has finite length.*

PROOF Certainly, $\mathrm{Hom}(\mathbb{1}, \mathbb{1}) = k$ is finite dimensional. For all objects X, Y,

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X, Y))$$

((13), p. 15). As $\mathbb{1}$ is simple (6.4), the morphism $\mathbb{1}^n \to \mathcal{H}om(X, Y)$ defined by linearly independent morphisms $f_1, \ldots, f_n : X \to Y$ is a monomorphism. Therefore,

$$\dim_k \mathrm{Hom}(X, Y) \leq \mathrm{length}(\mathcal{H}om(X, Y)).$$

□

Extension of scalars

In this subsection, k is a field. Let C be a k-linear abelian category.

7.10 For a finite-dimensional vector space V and an object X in C, we define $V \otimes X$ to represent the functor $T \rightsquigarrow V \otimes \mathrm{Hom}(T, X)$. The choice of a basis $(e_i)_{i \in I}$ of V realizes $V \otimes X$ as a direct sum of copies of X indexed by I.

In more detail, there is a functor $V, X \rightsquigarrow V \otimes X : \mathsf{Vecf}(k) \times \mathsf{C} \to \mathsf{C}$ such that

$$\begin{aligned} \mathrm{Hom}(T, V \otimes X) &\simeq V \otimes_k \mathrm{Hom}(T, X) \\ \mathrm{Hom}(V \otimes X, T) &\simeq V \otimes_k \mathrm{Hom}(X, T) \end{aligned}$$

for all X, T in C and V in $\mathsf{Vecf}(k)$, and such that for any k-linear functor $F : \mathsf{C} \to \mathsf{Vecf}(k)$, we have $F(V \otimes X) \simeq V \otimes F(X)$. First define the functor on the skeleton of $\mathsf{Vecf}(k)$ consisting of the vector spaces of the form k^n, and then choose a quasi-inverse to the inclusion of the skeleton into $\mathsf{Vecf}(k)$.

7.11 Let k' be a finite extension of k. We define $\mathsf{C}_{(k')}$ to be the category whose objects are the pairs (X, α), where X is an object of C and α is a k'-module structure on X, i.e., a k-algebra homomorphism $k' \to \mathrm{End}(X)$.[7] A morphism $(X, \alpha) \to (Y, \beta)$ is a morphism $f : X \to Y$ such that $f \circ \alpha(c) = \beta(c) \circ f$ for all $c \in k'$. The category $\mathsf{C}_{(k')}$ is a k'-linear abelian category, said to have been obtained from C by ***extension of scalars***. For X in C,

[6]OED: Mathematics. Of a group, algebra, etc.: generated by a single element.

[7]Equivalently: let $k' \otimes X$ denote the tensor product in the sense of 7.10; a k'-module structure on X can be defined to be a morphism $\mu : k' \otimes X \to X$ satisfying the usual conditions (10.4).

there is a natural k'-module structure on $k' \otimes X$ (tensor product in the sense of 7.10), and in this way we get a functor

$$i_{k'/k}\colon \mathsf{C} \to \mathsf{C}_{(k')}, \quad X \rightsquigarrow k' \otimes X, \quad \text{(extension of scalars)}.$$

The forgetful functor is a functor

$$r_{k'/k}\colon \mathsf{C}_{(k')} \to \mathsf{C}, \quad (X,\alpha) \rightsquigarrow X, \quad \text{(restriction of scalars)},$$

and

$$\mathrm{Hom}_{\mathsf{C}}(X, r_{k'/k}Y) \simeq \mathrm{Hom}_{\mathsf{C}_{(k')}}(i_{k'/k}X, Y), \quad X \in \mathsf{C}, Y \in \mathsf{C}_{(k')}.$$

In particular,

$$\mathrm{Hom}_{\mathsf{C}_{(k')}}(i_{k'/k}X, i_{k'/k}Y) \simeq \mathrm{Hom}_{\mathsf{C}}(X, k' \otimes Y) \simeq k' \otimes_k \mathrm{Hom}_{\mathsf{C}}(X,Y), \quad X, Y \in \mathrm{ob}\,\mathsf{C}.$$

As $r_{k'/k}$ and $i_{k'/k}$ are right and left adjoints, the first is left exact and the second is right exact. In fact, both functors are exact. For example, to see that $r_{k'/k}$ is right exact, let $(X,\alpha) \to (Y,\beta)$ be an epimorphism in $\mathsf{C}_{(k')}$, and let C be the cokernel of $X \to Y$ in C; then C acquires a k'-module structure from those on X and Y, and so it is 0. See Deligne 2014, 5.1.

EXAMPLE 7.12 Let A be a finite-dimensional k-algebra and k' a finite extension of k. The k-linear abelian category $\mathsf{Modf}(A)$ is locally finite, and $\mathsf{Modf}(A)_{(k')}$ can be identified with $\mathsf{Modf}(k' \otimes_k A)$. In this case, $i_{k'/k}$ and $r_{k'/k}$ are the usual extension and restriction functors.

PROPOSITION 7.13 *Let C be a locally finite k-linear abelian category.*

(a) *Let X be a simple object of C.*

  i) $D \overset{\text{def}}{=} \mathrm{End}(X)$ *is a division algebra, finite-dimensional over k, and its centre F is a finite extension of k.*

  ii) $i_{k'/k}(X)$ *is semisimple in $\mathsf{C}_{(k')}$ if and only if $k' \otimes_k F$ is a product of fields.*

(b) *If $Y = (X,\alpha)$ is a semisimple object of $\mathsf{C}_{(k')}$, then X is a semisimple object of C.*

(c) *Let X be an object of C. If $i_{k'/k}(X)$ is simple (resp. semisimple), then so also is X.*

PROOF (a) (i). This is obvious.

(ii). The functor $\mathrm{Hom}(X,-)$ is an equivalence to Vecf_D from the strictly full subcategory of C whose objects are the finite direct sums of copies of X. This remains an equivalence when we extend scalars to k', and $i_{k'/k}(X)$ corresponds to the right $k' \otimes_k D$-module $k' \otimes_k D$, which is semisimple if and only if $F \otimes_k k'$ is a product of fields (Bourbaki A, VIII, §7).

(b) We may suppose that Y is simple. The sum of the simple subobjects of X is a nonzero k'-submodule of X, hence equals X (by the simplicity of Y).

(c) If $i_{k'/k}(X)$ is semisimple, then X is semisimple because it is a subobject of $r(i(X))$, which is semisimple by (b). If $i_{k'/k}(X)$ is simple, then X is obviously simple. □

PROPOSITION 7.14 *Let C be a locally finite k-linear abelian category, and let X be an object of C. If X is a sum of simple subobjects, say, $X = \sum_{i\in I} S_i$ (the sum need not be direct), then for every subobject Y of X, there is a subset J of I such that*

$$X = Y \oplus \bigoplus\nolimits_{i\in J} S_i.$$

In particular, X is a finite direct *sum of simple subobjects, and Y is a direct summand of X.*

PROOF Let J be maximal among the subsets of I such that the sum $S_J = \sum_{j\in J} S_j$ is direct and $Y \cap S_J = 0$. We claim that $Y + S_J = X$ (hence X is the direct sum of Y and the S_j with $j \in J$). For this, it suffices to show that each S_i is contained in $Y + S_J$. Because S_i is simple, $S_i \cap (Y + S_J)$ equals S_i or 0. In the first case, $S_i \subset Y + S_J$, and in the second $S_J \cap S_i = 0$ and $Y \cap (S_J + S_i) = 0$, contradicting the maximality of J. □

DEFINITION 7.15 A locally finite k-linear abelian category C is ***semisimple*** if every object is a sum of simple objects (hence a finite direct sum).

PROPOSITION 7.16 *Let C be a locally finite k-linear abelian category, and let k' be a finite extension of k. If $\mathsf{C}_{(k')}$ is semisimple, then so also is C, and the converse is true if k' is separable over k.*

PROOF The necessity follows directly from 7.13c, so suppose that k' is separable over k. If X in C is semisimple, then $i_{k'/k}(X)$ is semisimple (7.13a), and every object Y of $\mathsf{C}_{(k')}$ is a direct factor of such an object. More precisely, the adjunction morphism $(i\circ r)(Y) \to Y$ splits. The object $(i\circ r)(Y) = k' \otimes_k Y$ has two k'-module structures, that provided by k' and that provided by Y, and hence an action of $k' \otimes_k k'$. The adjunction morphism is the natural morphism $k' \otimes Y \to (k' \otimes Y) \otimes_{k'\otimes_k k'} k'$. This is split because $k' \otimes_k k' \to k'$ is projection onto a direct factor. □

Let $(\mathsf{C}, \otimes)$ be a tensor category over k and k' a finite extension of k. For objects X and Y of $\mathsf{C}_{(k')}$, let

$$X \otimes_{k'} Y = \operatorname{Coker}(X \otimes k' \otimes Y \rightrightarrows X \otimes Y). \tag{37}$$

The two morphisms are defined by the actions of k' on X and Y respectively.

PROPOSITION 7.17 *Let $(\mathsf{C}, \otimes)$ be a tensor category over k and k' a finite extension of k. With the tensor product (37), $\mathsf{C}_{(k')}$ becomes a k'-linear abelian $\otimes$-category, and $i_{k'/k} : \mathsf{C} \to \mathsf{C}_{(k')}$ is a k-linear right-exact $\otimes$-functor. Moreover, $i_{k'/k}$ sends unit objects to unit objects and duals to duals. Internal Homs exist in $\mathsf{C}_{(k')}$.*

PROOF The proof is straightforward, If $X' = (X, \mu_X)$ and $Y' = (Y, \mu_Y)$ are objects of $\mathsf{C}_{(k')}$, then $\mathcal{H}om(X', Y')$ is the intersection of the kernels of the morphisms

$$f \mapsto \lambda f - f\lambda \colon \mathcal{H}om(X, Y) \to \mathcal{H}om(X, Y),$$

where λ runs over a basis for k' as a k-vector space. □

PROPOSITION 7.18 *Let $(\mathsf{C}, \otimes)$ be a tensor category over k and k' a finite extension of k. If k'/k is separable, then $\mathsf{C}_{(k')}$ is a tensor category over k' and $i_{k'/k}$ is exact.*

PROOF If $\mathbb{1}$ is an identity object in C, then $i_{k'/k}(\mathbb{1})$ is an identity object in $\mathsf{C}_{(k')}$, and clearly $\operatorname{End}(i_{k'/k}(\mathbb{1})) \simeq k'$. To show that $\mathsf{C}_{(k')}$ is a tensor category, it remains to show that duals exist. If X in C has dual $X^\vee$, then $i_{k'/k}(X)$ has dual $i_{k'/k}(X^\vee)$, and every object Y of $\mathsf{C}_{(k')}$ is a direct factor of such an object. More precisely, the adjunction morphism $(i\circ r)(Y) \to Y$ splits (see the proof of 7.16). Finally, $i_{k'/k}$ is exact by 6.3 and 7.5, □

NOTES This subsection is based on Deligne 2014, 5.3.

Descent of tensor categories

7.19 Let k'/k be a finite Galois extension with Galois group Γ, and let C' be a tensor category over k'. A ***descent datum*** on C' relative to k'/k is

(a) a family $(\beta_\gamma)_{\gamma\in\Gamma}$ of equivalences of symmetric monoidal categories $\beta_\gamma : \mathsf{C}' \to \mathsf{C}'$, β_γ being semi-linear relative to γ, together with

(b) a family $(\mu_{\gamma',\gamma})$ of isomorphisms of tensor functors $\mu_{\gamma',\gamma} : \beta_{\gamma'\gamma} \xrightarrow{\approx} \beta_{\gamma'}\circ\beta_\gamma$ such that

$$\begin{tikzcd}
\beta_{\gamma''\gamma'\gamma}(X) \arrow[r, "\mu_{\gamma'',\gamma'\gamma}(X)"] \arrow[d, "\mu_{\gamma''\gamma',\gamma}(X)"] & \beta_{\gamma''}(\beta_{\gamma'\gamma}(X)) \arrow[d, "\beta_{\gamma''}(\mu_{\gamma',\gamma}(X))"] \\
\beta_{\gamma''\gamma'}(\beta_\gamma(X)) \arrow[r, "\mu_{\gamma'',\gamma'}(\beta_\gamma(X))"] & \beta_{\gamma''}(\beta_{\gamma'}(\beta_\gamma(X)))
\end{tikzcd}$$

commutes for all $X \in \mathrm{ob}(\mathsf{C})$.

7.20 A tensor category C over k gives rise to a tensor category $\mathsf{C}' = \mathsf{C}_{(k')}$ over k' together with a descent datum for which

$$\beta_\gamma(X, \alpha_X) = (X, \alpha_X\circ\gamma^{-1}).$$

Conversely, a tensor category C' over k' together with a descent datum relative to k'/k gives rise to a tensor category C over k whose objects are pairs $(X, (a_\gamma))$, where $X \in \mathrm{ob}(\mathsf{C}')$ and $(a_\gamma : X \to \beta_\gamma(X))_{\gamma\in\Gamma}$ is such that $(\mu_{\gamma',\gamma})_X\circ a_{\gamma'\gamma} = \beta_{\gamma'}(a_\gamma)\circ a_{\gamma'}$, and whose morphisms are morphisms in C' commuting with the a_γ. These two operations are quasi-inverse, so that to give a tensor category over k (up to a tensor equivalence, unique up to a unique isomorphism) is the same as giving a tensor category over k' together with a descent datum relative to k'/k. The proofs of these statements are elementary for k-linear abelian categories, and extend without difficulty to tensor categories over k. See Saavedra 1972, III, 1.2.

8 Tannakian categories

In this section, k is a field.

DEFINITION 8.1 Let $(\mathsf{C}, \otimes)$ be a tensor category over k and R a k-algebra.

(a) A ***fibre functor on*** C ***with values in*** R (or an R-***valued fibre functor on*** C) is a k-linear right exact $\otimes$-functor $\omega : \mathsf{C} \to \mathsf{Mod}(R)$.

(b) A ***morphism of fibre functors*** is a morphism of $\otimes$-functors (3.4); it is automatically an isomorphism (5.7).

More generally, a ***fibre functor on*** C ***over a*** k-***scheme*** S is an exact k-linear $\otimes$-functor from C to the category of quasi-coherent sheaves on S.

DEFINITION 8.2 A ***tannakian category over*** k is a tensor category over k that admits an R-valued fibre functor for some nonzero k-algebra R. If there exists a fibre functor with values in k itself, then the category is said to be ***neutral***.

In other words, a tannakian category over k is a k-linear abelian rigid symmetric monoidal category C such that the structure map $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism and such that there exists a fibre functor with values in some nonzero k-algebra R.

DEFINITION 8.3 A ***tannakian subcategory*** of a tannakian category over k is a tensor subcategory (7.2). In other words, it is a strictly full abelian subcategory closed under the formation of direct sums, tensor products, duals, and subquotients. It is again a tannakian category over k – any fibre functor restricts to a fibre functor.

There exist strictly full abelian subcategories of tannakian categories closed under the formation of tensor products and duals but not subquotients. See II, 5.5, below.

Let C be a tensor category over k and R a k-algebra.

PROPOSITION 8.4 *Every R-valued fibre functor ω on C is exact; moreover, ω is faithful if $R \neq 0$.*

PROOF That ω is exact follows from 7.5. Now assume that $R \neq 0$. As ω is exact, it suffices to show that $\omega(X) \neq 0$ if $X \neq 0$. For any nonzero X, $\mathrm{ev}: X^{\vee} \otimes X \to \mathbb{1}$ is an epimorphism, so $\omega(X)^{\vee} \otimes \omega(X)^{\vee} \to R$ is surjectivc, which implies that $\omega(X) \neq 0$. □

PROPOSITION 8.5 *Let ω be an R-valued fibre functor on C and X a nonzero object of C. Then $\omega(X)$ is finitely generated and projective, hence flat. It is faithfully flat if $R \neq 0$.*

PROOF As X admits a dual, so also does $\omega(X)$, and hence it is finitely generated and projective (4.12). In particular, it is flat. Assume that $R \neq 0$. To show that $\omega(X)$ is faithfully flat, we have to show that, for all maximal ideals $\mathfrak{m}$ of R, $\omega(X) \otimes_R R/\mathfrak{m} \neq 0$. The functor $X \rightsquigarrow \omega(X) \otimes_R R/\mathfrak{m}$ is again a fibre functor, so this follows from 8.4. □

Thus ω takes values in $\mathsf{Proj}(R)$, but is only exact as a functor to $\mathsf{Mod}(R)$.

PROPOSITION 8.6 *Let ω be an R-valued fibre functor on C. For any R-algebra R', the functor*

$$X \rightsquigarrow \omega_{R'}(X) \stackrel{\text{def}}{=} \omega(X) \otimes_R R'$$

is an R'-valued fibre functor on C.

PROOF Certainly, $\omega_{R'}$ is a k-linear right exact $\otimes$-functor $\mathsf{C} \to \mathsf{Mod}(R')$, hence a fibre functor. Here is a direct proof that ω exact implies $\omega_{R'}$ exact. If

$$0 \to X' \to X \to X'' \to 0$$

is a short exact sequence in C, then

$$0 \to \omega(X') \to \omega(X) \to \omega(X'') \to 0$$

is exact in $\mathsf{Mod}(R)$. As $\omega(X'')$ is projective, the sequence splits, and so it remains exact when tensored with any R-algebra. Thus $\omega_{R'}$ is exact. □

COROLLARY 8.7 *Every tannakian category C admits a fibre functor with values in a field.*

PROOF Let ω be an R-valued fibre functor on C with $R \neq 0$. For any maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is field containing k and $X \rightsquigarrow \omega(X) \otimes_R R/\mathfrak{m}$ is a fibre functor with values in $R/\mathfrak{m}$. □

PROPOSITION 8.8 *Every exact $\otimes$-functor from a tannakian category to a nonzero abelian $\otimes$-category with $\mathbb{1} \neq 0$ is faithful. In particular, every R-valued fibre functor, $R \neq 0$, is faithful.*

PROOF This is a special case of 6.7. □

8.9 An object of a tannakian category C is said to be ***trivial*** if it is isomorphic to a finite sum of copies of $\mathbb{1}$. As $\mathbb{1}$ is simple (6.4), every such object is a finite *direct* sum of copies of $\mathbb{1}$, and the full subcategory of trivial objects is stable under the formation of subquotients in C. The functors

$$\mathsf{Vecf}(k) \underset{\mathrm{Hom}(\mathbb{1},X) \leftarrow\!\!\shortmid X}{\overset{V \rightsquigarrow V\otimes\mathbb{1}}{\rightleftarrows}} \mathsf{C}$$

are adjoint,

$$\mathrm{Hom}(V \otimes \mathbb{1}, X) \simeq \mathrm{Hom}(V, \mathrm{Hom}(\mathbb{1}, X)).$$

Here $V \otimes \mathbb{1}$ is as in 7.10. The unit of the adjunction

$$V \to \mathrm{Hom}(\mathbb{1}, V \otimes \mathbb{1})$$

(see A.2) is an isomorphism, whereas the counit

$$\mathrm{Hom}(\mathbb{1}, X) \otimes \mathbb{1} \to X$$

is a monomorphism, and it is an isomorphism if and only if X is trivial. Therefore, the functors define an equivalence of $\mathsf{Vecf}(k)$ with the full subcategory of trivial objects in C.

PROPOSITION 8.10 *Let C be a tannakian category over k, and let ω be a fibre functor with values in a nonzero k-algebra R. For all X, Y in C, the canonical map*

$$\mathrm{Hom}(X, Y) \otimes_k R \to \mathrm{Hom}_R(\omega(X), \omega(Y))$$

is injective.

PROOF Recall (13), p. 15, that $\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X, Y)) \simeq \mathrm{Hom}(X, Y)$. In particular, each element of $\mathrm{Hom}(X, Y)$ defines a morphism $\mathbb{1} \to \mathcal{H}om(X, Y)$, and linearly independent elements define linearly independent morphisms. Therefore, for each finite-dimensional subspace V of $\mathrm{Hom}(X, Y)$ we have a canonical monomorphism

$$V \otimes \mathbb{1} \hookrightarrow \mathcal{H}om(X, Y).$$

On applying ω to this monomorphism, we obtain an injective map

$$V \otimes_k R \hookrightarrow \omega(\mathcal{H}om(X, Y)) \simeq \mathrm{Hom}_R(\omega(X), \omega(Y)).$$

On passing to the inductive limit over the subspaces V, we obtain the required injective map. □

PROPOSITION 8.11 *Every tannakian category is locally finite.*

PROOF Let $k = \mathrm{End}(\mathbb{1})$. According to Proposition 8.7, the category has a fibre functor ω with values in a field K containing k. Then $\omega\colon \mathsf{C} \to \mathsf{Vecf}(K)$ is exact and faithful, and according to Proposition 8.10, the canonical map

$$\mathrm{Hom}(X, Y) \otimes_k K \to \mathrm{Hom}(\omega(X), \omega(Y))$$

is injective, and so $\mathrm{Hom}(X, Y)$ is finite-dimensional. Subobjects of X are determined by their images in $\omega(X)$, and so X has finite length. □

DEFINITION 8.12 A ***pre-tannakian category*** over k is a locally finite tensor category over k.

Proposition 8.11 may fail for tensor categories over k. See Example 9.11 below.

DEFINITION 8.13 A tannakian category is ***algebraic*** or ***monogenic*** if it admits a tensor generator (in the sense of I, 7.6).

This definition of algebraic tannakian category agrees with that in Saavedra 1972, III, 3.3.1 (ibid., 3.3.1.1).

EXAMPLE 8.14 There exists a tannakian category over $\mathbb{R}$ containing an object X of dimension 2 with endomorphism algebra the quaternions (see 9.5). As the quaternions cannot act on a real vector space of dimension 2, there does not exist an $\mathbb{R}$-valued fibre functor, so the category is not neutral.

Extension of scalars

Let $(\mathsf{C}, \otimes)$ be a tensor category over k, and let k' be a finite extension of k. Recall (7.11, 7.17) that the category $\mathsf{C}_{(k')}$ of k'-modules in C is a k'-linear abelian $\otimes$-category with internal Homs. Moreover, the extension functor $i_{k'/k}\colon \mathsf{C} \to \mathsf{C}_{(k')}$ is an exact k-linear $\otimes$-functor.

THEOREM 8.15 (GROTHENDIECK) *Let k' be a finite extension of k. If $(\mathsf{C}, \otimes)$ is a tannakian category over k, then $(\mathsf{C}_{(k')}, \otimes_{k'})$ is a tannakian category over k'.*

The proof requires several lemmas.

LEMMA 8.16 *If $(\mathsf{C}_{(k')}, \otimes_{k'})$ is rigid, then it is tannakian over k'.*

PROOF We have to show that $\mathsf{C}_{(k')}$ admits a fibre functor. Let ω be a fibre functor on C with values in a nonzero k-algebra R. For any object (X, μ) in $\mathsf{C}_{(k')}$, $\omega(X)$ has the structure of an $R \otimes_k k'$-module. The functor

$$(X, \mu) \rightsquigarrow \omega(X)\colon \mathsf{C}_{(k')} \to \mathsf{Mod}(R \otimes_k k')$$

is k'-linear and exact (because $(X, \mu) \rightsquigarrow X$ is exact), and

$$\omega(X \otimes_{k'} Y) \simeq \omega(X) \otimes_{R\otimes_k k'} \omega(Y),$$

from which it follows that it is monoidal and symmetric. It is therefore a fibre functor. □

If $k'' \supset k' \supset k$ are extensions of k, then $\mathsf{C}_{(k'')}$ can be identified with $(\mathsf{C}_{(k')})_{(k'')}$, and so it suffices to prove that $(\mathsf{C}_{(k')}, \otimes_{k'})$ is a tensor category over k' in the two cases, k'/k separable and k'/k purely inseparable of degree p. The first case was treated in 7.18, and so it remains to prove the purely inseparable case. As $(\mathsf{C}_{(k')}, \otimes_{k'})$ is a k'-linear abelian symmetric monoidal category and clearly $\mathrm{End}(\mathbb{1}') \simeq k'$, it remains to prove that it is rigid, i.e., that duals exist.

In the next two lemmas, $(\mathsf{C}, \otimes)$ is a tannakian category over k and $k' = k(a^{1/p})$, $a \in k$.

LEMMA 8.17 *Let ω be a fibre functor on C with values in an extension K of k, and let X be an object of C. The $k' \otimes_k K$-module structure on $\omega(X)$ defined by any k'-module structure on X makes it into a free $k' \otimes_k K$-module.*

PROOF Let $K' = k' \otimes_k K$. If K' is a field, then the statement is obvious. Otherwise, a is a pth power in K, say, $a = \alpha^p$, and $K' = K[\varepsilon]/(\varepsilon^p)$, where $\varepsilon = \alpha \otimes 1 - 1 \otimes \alpha$. Let d be the dimension of the K-vector space $\omega(X)/\varepsilon\omega(X)$. The K'-module $\omega(X)$ is isomorphic to a direct sum of d modules of the form $K[\varepsilon]/(\varepsilon^j)$, $1 \le j \le p$. From this, we see that it is free if and only if $\bigwedge_{K'}^d \omega(X)$ is free (of rank one). Let $\bigwedge_{k'}^d X$ denote the image of the antisymmetrization map

$$\bigotimes^d X \to \bigotimes^d X.$$

Because ω is exact, it maps $\bigwedge_{k'}^d X$ to $\bigwedge_{K'}^d \omega(X)$. After replacing X with $\bigwedge_{k'}^d X$, we may suppose that $\omega(X)$ is a nonzero monogenic K'-module.

The k'-module structure on X defines a morphism

$$k' \otimes \mathbb{1} \to \mathcal{H}om(X,X). \tag{38}$$

As $\mathbb{1}$ is simple (6.4), the kernel of (38) is of the form $A \otimes \mathbb{1}$, where A is a vector subspace of k'. As A is also an ideal in k', it is 0, and so (38) is a monomorphism. On applying ω, we deduce that $\omega(X)$ is a faithful K'-module, and we conclude by noting that every faithful monogenic K'-module is free. □

LEMMA 8.18 *The $\otimes$-category $(\mathsf{C}_{(k')}, \otimes_{k'})$ is rigid.*

PROOF Let $\mathbb{1}'$ be the unit object $k' \otimes \mathbb{1}$ of $\mathsf{C}_{(k')}$, and let $X \in \operatorname{ob} \mathsf{C}_{(k')}$. We show that the weak dual $\mathcal{H}om_{k'}(X, \mathbb{1}')$ of X is a dual. For this, it suffices to show that for all objects T of $\mathsf{C}_{(k')}$, the canonical map

$$T \otimes_{k'} \mathcal{H}om_{k'}(X, \mathbb{1}') \to \mathcal{H}om_{k'}(X,T) \tag{39}$$

is an isomorphism (see 4.10). Let ω be a K-valued fibre functor on C, and let ω' be the functor $(X,\mu) \rightsquigarrow \omega(X)$ taking values in $k' \otimes_k K$-modules. Then ω' transforms (39) into

$$\omega(T) \otimes_{k'} \operatorname{Hom}_{k' \otimes_k K}(\omega(X), k' \otimes_k K) \to \operatorname{Hom}_{k' \otimes_k K}(\omega(X), \omega(T)). \tag{40}$$

Lemma 8.17 shows that (40) is an isomorphism. The functor ω' is exact and such that

$$\omega'(Z) = 0 \implies Z = 0$$

because ω has these properties. Therefore ω' is conservative and so (39) is an isomorphism. □

8.19 (DELIGNE 2014, 5.10) Let C be tannakian category over k, and let k^a be an algebraic extension of k, for example, an algebraic closure of k. As k' runs over the finite extensions of k in k^a, the categories $\mathsf{C}_{(k')}$ form a 2-inductive system of abelian categories (not an inductive system, because we only have canonical isomorphisms $i_{k'''/k''} \circ i_{k''/k'} \simeq i_{k'''/k'}$ satisfying a cocycle condition). We define $\mathsf{C}_{(k^a)}$ to be the 2-inductive limit of this system. Specifically, an object of $\mathsf{C}_{(k^a)}$ is an object X of $\mathsf{C}_{(k')}$ for some $k' \subset k^a$ finite over k. If X' in $\mathsf{C}_{(k'')}$ is a second such object, then

$$\operatorname{Hom}_{\mathsf{C}_{(k^a)}}(X, X') = \varinjlim \operatorname{Hom}_{\mathsf{C}_{(k''')}}(i_{k'''/k'}X, i_{k'''/k''}X'),$$

where k''' runs over the finite extensions of k in k^a containing k' and k''.

The category $\mathsf{C}_{(k^a)}$ is a tensor category over k^a and $i_{k^a/k} : \mathsf{C} \to \mathsf{C}_{(k^a)}$ is an exact k-linear $\otimes$-functor. An R-valued fibre functor ω on C defines a $k' \otimes_k R$-valued fibre functor on $\mathsf{C}_{(k')}$ for each $k' \subset k^a$ (cf. the proof of 8.16). On passing to the limit, we obtain a $k^a \otimes_k K$-valued fibre functor on $\mathsf{C}_{(k^a)}$. Therefore, $\mathsf{C}_{(k^a)}$ is a tannakian category over k^a.

We list some consquences of Theorem 8.15 (Deligne 2014, 5.7, 5.9, 5.11).

COROLLARY 8.20 *If an object X of a tannakian category C over k admits a k'-module structure, where k' is a extension of k, then $[k' : k]$ is finite and divides* $\dim X$.

PROOF Let X be an object of C admitting a k'-module structure μ. Then (X, μ) is an object of $\mathsf{C}_{(k')}$. Let ω be a K-valued fibre functor on C. Then $(X, \mu) \rightsquigarrow \omega(X)$ is a $k' \otimes K$-valued fibre functor ω' on $\mathsf{C}_{(k')}$. As $\omega'(X, \mu)$ is locally free of rank equal to the (categorical) dimension of (X, μ), we have

$$\dim(X) = \dim_K \omega(X) = [k' \otimes K : K] \cdot \mathrm{rank}_{K'} \omega(X) = [k' : k] \cdot \dim(X, \mu).$$

Therefore, $[k' : k]$ divides $\dim(X)$, with quotient of the dimension of X as an object of $\mathsf{C}_{(k')}$. □

COROLLARY 8.21 *Let X be a simple object of a tannakian category C over a field k of characteristic $p \neq 0$. If $\dim X < p$, then the centre of* $\mathrm{End}(X)$ *is separable extension of k.*

PROOF Indeed, according to 8.20, it is an extension of degree $< p$. □

COROLLARY 8.22 *L et C be a tannakian category over a field k of characteristic p and k^a an algebraic extension of k. Let X be an object of C.*

(a) *If $i_{k^a/k}(X)$ is semisimple in $\mathsf{C}_{(k^a)}$, then X is semisimple in C.*

(b) *If X is semisimple in C and* $\dim X < p$, *then $i_{k^a/k}(X)$ is semisimple in $\mathsf{C}_{(k^a)}$.*

PROOF See Deligne 2014, 5.11. □

NOTES Theorem 8.15 is due to Grothendieck. The exposition follows Deligne 2014, 5.4.

9 Examples

EXAMPLE 9.1 The category $\mathsf{Vecf}(k)$ of finite-dimensional vector spaces over a field k is a tannakian category over k with the forgetful functor as a k-valued fibre functor. All the above definitions take on a familiar meaning when applied to $\mathsf{Vecf}(k)$. For example, $\mathrm{Tr}\colon \mathrm{End}(X) \to k$ is the usual trace map.

EXAMPLE 9.2 The category $\mathsf{Mod}(R)$ of modules over a commutative ring R is an abelian symmetric monoidal category such that $\mathrm{End}(\mathbb{1}) = R$. In general it is not rigid because not all R-modules will admit duals (in the sense of 4.5). The category $\mathsf{Modf}(R)$ of finitely generated R-modules is an R-linear abelian symmetric monoidal category if R is coherent, for example, noetherian.

EXAMPLE 9.3 The category $\mathsf{Proj}(R)$ of finitely generated projective modules over a commutative ring R is an additive symmetric monoidal category such that $\mathrm{End}(\mathbb{1}) = R$. It is rigid (4.12) but not in general abelian..

EXAMPLE 9.4 Let G be an affine group scheme over a field k. The category $\mathsf{Repf}(G)$ of linear representations of G on finite-dimensional k-vector spaces is a tannakian category over k with the forgetful functor as a k-valued fibre functor.

EXAMPLE 9.5 (A NONNEUTRAL TANNAKIAN CATEGORY) Let T be the category of pairs (V,F) consisting of a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space $V = V^0 \oplus V^1$ and an ι-semilinear endomorphism F of V such that $F^2 = (-1)^i$ on V^i. With the obvious tensor structure, T becomes a nonneutral tannakian category.

Let $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ denote the usual quaternion algebra over $\mathbb{R}$. Let $V = \mathbb{H}$ regarded as a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space of degree 1, and let F act on V as

$$a + bj \mapsto j(a + bj) = -\bar{b} + \bar{a}j.$$

Then (V,F) is an object in T of (categorical) dimension 2, and the action of $\mathbb{H}$ on itself by right multiplication defines an isomorphism $\mathbb{H}^{\mathrm{opp}} \to \mathrm{End}(V,F)$. The forgetful functor is a $\mathbb{C}$-valued fibre functor on T. Any $\mathbb{R}$-valued fibre functor on T would send (V,F) to a 2-dimensional $\mathbb{R}$-vector space on which $\mathbb{H}^{\mathrm{opp}}$ acts, which is impossible: there is no homomorphism of $\mathbb{R}$-algebras $\mathbb{H}^{\mathrm{opp}} \to M_2(\mathbb{R})$ because $\mathbb{H}$ is a division algebra.

EXAMPLE 9.6 (MORE NONNEUTRAL TANNAKIAN CATEGORIES) For a central simple algebra A of dimension n^2 over k, we construct a tannakian category T over k containing an object with categorical dimension n and endomorphism algebra A^{opp}. If $A \neq M_n(k)$, then T is not neutral.

As a first attempt, one might try defining T to be the category of finite-dimensional vector spaces over k equipped with a left action of A, but this is not rigid. Instead, we proceed as follows.

Suppose first that A contains a maximal subfield L Galois over k. Let E denote the group of invertible elements a of A such that $aLa^{-1} = L$. Each $a \in E$ defines an element $x \mapsto axa^{-1}$ of $\mathrm{Gal}(L/k)$, and, according to the Noether–Skolem theorem, all elements of $\mathrm{Gal}(L/k)$ arise this way. Because L is a maximal subfield of A, it is its own centralizer in A, and so the sequence

$$1 \to L^\times \to E \to \mathrm{Gal}(L/k) \to 1$$

is exact.

Define T to be the category of representations of E on finite-dimensional L-vector spaces. Such a representation is a finite-dimensional $\mathbb{Z}$-graded L-vector space $V = \bigoplus_i V^i$ with $a \in L^\times$ acting as a^i on V^i together with an extension of the action of $L^\times$ on V to E.

Let V denote A regarded as a graded L-vector space of degree 1. The action of A on V by left multiplication induces an action r of E on V. The left and right actions of A on V define an isomorphism of k-algebras

$$A \otimes_k A^{\mathrm{opp}} \to \mathrm{End}_k(V).$$

This maps the centralizer A^{opp} of A in $A \otimes_k A^{\mathrm{opp}}$ isomorphically onto $\mathrm{End}(V,r)$.

Consider, for example, the case of a cyclic k-algebra A. Such an algebra equals $L[\gamma]$, where γ is an element such that $a \mapsto \gamma a \gamma^{-1} \colon L \to L$ is a generator σ of $\mathrm{Gal}(L/k)$ and $\gamma^n = c \in k$. The objects of T are the pairs (V,F) consisting of a finite-dimensional $\mathbb{Z}$-graded L-vector space V and a σ-semilinear endomorphism $F \colon V \to V$ such that F^n acts as c^i on V^i. For example, let V denote A regarded as a graded L-vector space of degree 1, and let F act on V as left multiplication by γ. Then (V,F) is an object of T and $\mathrm{End}(V,F) \simeq A^{\mathrm{opp}}$.

When no maximal subfield of A is Galois over k, we replace A with a suitable similar algebra. The tannakian category will then contain an object with endomorphism algebra $M_m(A)^{\mathrm{opp}}$, and a suitable direct factor of the object will have endomorphism algebra A^{opp}.

EXAMPLE 9.7 Let V' and V'' be vector spaces over k of the same dimension, each equipped with a nondegenerate quadratic form. Then the categories $\mathsf{Repf}(\mathrm{O}(V'))$ and $\mathsf{Repf}(\mathrm{O}(V''))$ are canonically equivalent as k-linear symmetric monoidal categories. To see this, note that $\mathrm{Isom}(V', V'')$ is an $\mathrm{O}(V')$ torsor with an action of $\mathrm{O}(V'')$. Twisting a representation of $\mathrm{O}(V')$ by the torsor gives the equivalence.

EXAMPLE 9.8 The finite groups D_4 and Q_8 have isomorphic representation rings (over $\mathbb{C}$, say) because they have the same character tables, but the categories $\mathsf{Repf}(D_4)$ and $\mathsf{Repf}(Q_8)$ are not equivalent as $\mathbb{C}$-linear tensor categories because a direct calculation shows that they have different associativity constraints. Alternatively, suppose that there is a tensor equivalence, which we may assume commutes with the forgetful fibre functors. Such an equivalence sends the trivial representation $\mathbb{1}_D$ of D_4 to the trivial representation $\mathbb{1}_Q$ of Q_8 (they are identity objects) and it sends the simple two-dimensional representation V_D of D_4 to the simple two-dimensional representation V_Q of Q_8 (they are unique). In particular, we would get a $\mathbb{C}$-linear isomorphism $g\colon V_D \to V_Q$. Now

$$g \otimes g\colon V_D \otimes V_D \to V_Q \otimes V_Q$$

sends the unique identity object $\mathbb{1}_D$ in $V_D \otimes V_D$ to the unique identity object $\mathbb{1}_Q$ in $V_Q \otimes V_Q$. But no such g can exist because the map $v \otimes w \mapsto w \otimes v$ acts as 1 on $\mathbb{1}_D$ and -1 on $\mathbb{1}_Q$.

It may be an open problem to classify the tannakian categories with the same representation ring as Q_8. See mo282292.

EXAMPLE 9.9 (SUPER VECTOR SPACES) Let $\mathsf{sVec}(k)$ denote the category whose objects are pairs (V^0, V^1) of finite-dimensional vector spaces over k, i.e., $\mathbb{Z}/(2)$-graded k-vector spaces. We give $\mathsf{sVec}(k)$ the tensor structure whose commutativity constraint is determined by the Koszul sign rule, i.e., that defined by the isomorphisms

$$v \otimes w \mapsto (-1)^{ij} w \otimes v\colon V^i \otimes W^j \to W^j \otimes V^i.$$

Then $\mathsf{sVec}(k)$ is a tensor category over k, but it is does not admit a nonzero fibre functor because

$$\dim(V^0, V^1) = \dim(V^0) - \dim(V^1),$$

which need not be positive.

EXAMPLE 9.10 Let C be a rigid additive symmetric monoidal category and T an object of C. We say that C is the ***rigid additive symmetric monoidal category freely generated by*** T if $\mathrm{End}(\mathbb{1})$ is the polynomial ring $\mathbb{Z}[t]$ and

$$F \rightsquigarrow F(T)\colon \mathsf{Hom}^{\otimes}(\mathsf{C}, \mathsf{C}') \to \mathsf{C}'$$

is an equivalence of categories for all rigid additive symmetric monoidal categories C' (t will turn out to be the categorical dimension of T). We show how to construct such a pair (C, T) – clearly it is unique up to a unique equivalence of symmetric monoidal categories preserving T.

Let V be a free module of finite rank over a commutative ring k and let $T^{a,b}(V)$ be the space of tensors $V^{\otimes a} \otimes V^{\vee \otimes b}$. A morphism $f\colon T^{a,b}(V) \to T^{c,d}(V)$ can be identified with a tensor "f" in $T^{b+c,a+d}(V)$. When $a + d = b + c$, $T^{b+c,a+d}(V)$ contains a special element, namely, the $(a+d)$th tensor power of "id"$\in T^{1,1}(V)$, and other elements can be obtained by allowing an element of the symmetric group S_{a+d} to permute the contravariant components of this special element. We therefore have a map

$$\epsilon\colon S_{a+d} \to \mathrm{Hom}(T^{a,b}, T^{c,d}) \qquad (\text{when } a+d = b+c).$$

The induced map $k[S_{a+d}] \to \mathrm{Hom}(T^{a,b}, T^{c,d})$ on the group algebra is injective provided $\mathrm{rank}(V) \geq a+d$. One checks that the composite of two such maps $\epsilon(\sigma)\colon T^{a,b}(V) \to T^{c,d}(V)$ and $\epsilon(\tau)\colon T^{c,d}(V) \to T^{e,f}(V)$ is given by a universal formula

$$\epsilon(\tau)\cdot\epsilon(\sigma) = (\mathrm{rank}\, V)^N \cdot \epsilon(\rho) \tag{41}$$

with ρ and N depending only on a, b, c, d, e, f, σ, and τ.

We define C' to be the category having as objects symbols $T^{a,b}$ ($a, b \in \mathbb{N}$), and for which $\mathrm{Hom}(T^{a,b}, T^{c,d})$ is the free $\mathbb{Z}[t]$-module with basis S_{a+d} if $a+d = b+c$ and is zero otherwise. Composition of morphisms is defined to be $\mathbb{Z}[t]$-bilinear and to agree on basis elements with the universal formula (41) but with $\mathrm{rank}(V)$ replaced by the indeterminate t. The associativity law holds for this composition because it does whenever t is replaced by a large enough positive integer (it becomes the associativity law in a category of modules). Tensor products are defined by

$$T^{a,b} \otimes T^{c,d} = T^{a+c,b+d}$$

and by an obvious rule for morphisms. We define T to be $T^{1,0}$.

The category C is deduced from C' by formally adjoining direct sums of objects. Its universality follows from the fact that the formula (41) holds in any rigid additive category.

EXAMPLE 9.11 (GL_t) Let (C, T) be the rigid additive symmetric monoidal category freely generated by T in Example 9.10. Let n be an integer, and use $t \mapsto n\colon \mathbb{Z}[t] \to \mathbb{C}$ to extend the scalars in C from $\mathbb{Z}[t]$ to $\mathbb{C}$. If V is an n-dimensional complex vector space and if $a+d \leq n$, then the map

$$\mathrm{Hom}(T^{a,b}, T^{c,d}) \otimes_{\mathbb{Z}[t]} \mathbb{C} \to \mathrm{Hom}_{\mathrm{GL}_V}(T^{a,b}(V), T^{c,d}(V))$$

is an isomorphism. For any sum T' of objects $T^{a,b}$ and sufficiently large integer n, $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{C}$ is therefore a product of matrix algebras. This implies that $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t)$ is a semisimple algebra.

After extending the scalars in C to $\mathbb{Q}(t)$, i.e., replacing $\mathrm{Hom}(T', T'')$ with

$$\mathrm{Hom}(T', T'') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t),$$

and passing to the pseudo-abelian envelope (formally adjoining images of idempotents), we obtain a pseudo-abelian rigid symmetric monoidal category GL_t with $\mathrm{End}(\mathbb{1}) = \mathbb{Q}(t)$. As its endomorphism algebras are semisimple, GL_t is semisimple abelian (VI, **??** below), hence a semisimple tensor category over $\mathbb{Q}(t)$. The dimension of $T \in \mathrm{ob}(\mathsf{GL}_t)$ is $t \notin \mathbb{N}$ and so GL_t is not tannakian.

For any object X of dimension t of a tensor category T over a field k containing t, there exists an exact $\otimes$-functor from GL_t to T sending the universal object X_t to X. Two such $\otimes$-functors are isomorphic, and the tensor automorphisms of such a $\otimes$-functor are those of X. In particular, we have a symmetric monoidal functor

$$X_t \rightsquigarrow \mathbb{1} \oplus X_{t-1}\colon \mathsf{GL}_t \to \mathsf{GL}_{t-1}.$$

Iterate this construction, and let T be the inductive limit of the categories GL_{t-n}, $n \geq 0$. This tensor category over a field k is freely generated by an object X_t of dimension t equipped with a decomposition

$$X_t = \mathbb{1} \oplus X_{t-1}, \; X_{t-1} = \mathbb{1} \oplus X_{t-2}, \; \dots$$

In GL_{t-n}, $X_t = \mathbb{1}^n \oplus X_{i-n}$ has endomorphism ring $M_n(k) \times k$. Passing to the limit, we find that the ring of endomorphisms of X_t in T is the ring of matrices of the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & \lambda I \end{array}\right)$$

The object X_t of T is not of finite length, and $\mathrm{Hom}(X_t, X_t)$ is not of finite dimension over k.

For more on these categories, see Deligne 2007 and the many articles citing it.

10 Algebraic geometry in a tensor category

Throughout this section, T is a tensor category over a field k. We explain, following Deligne 1989, how to do algebraic geometry in T.

10.1 For objects S, T, X, Y in T, the canonical morphism (17), p. 15,

$$\mathcal{H}om(S, X) \otimes \mathcal{H}om(T, Y) \to \mathcal{H}om(S \otimes T, X \otimes Y)$$

is an isomorphism – it is essentially the isomorphism

$$(S^\vee \otimes X) \otimes (T^\vee \otimes Y) \overset{\gamma}{\simeq} T^\vee \otimes S^\vee \otimes X \otimes Y \simeq (S \otimes T)^\vee \otimes X \otimes Y.$$

On applying $\mathrm{Hom}(\mathbb{1}, -)$, we obtain an isomorphism of k-vector spaces,

$$\mathrm{Hom}(S, X) \otimes \mathrm{Hom}(T, Y) \simeq \mathrm{Hom}(S \otimes T, X \otimes Y).$$

(The first $\otimes$ is in the category of k-vector spaces.) To give a morphism $X \otimes Y \to Z$ from $X \otimes Y$ to another object Z is the same as giving a homomorphism of k-vector spaces,

$$\mathrm{Hom}(S \otimes T, X \otimes Y) \to \mathrm{Hom}(S \otimes T, Z),$$

natural in S and T (Yoneda lemma). On combining the last two statements, we see that to give a morphism $X \otimes Y \to Z$ is the same as giving a homomorphism of k-vector spaces

$$\mathrm{Hom}(S, X) \otimes \mathrm{Hom}(T, Y) \to \mathrm{Hom}(S \otimes T, Z)$$

10.2 The category $\mathrm{Ind}\,\mathsf{T}$ of ind-objects of T is again a k-linear abelian category (Appendix B). The tensor product and the associativity and commutativity constraints on T extend to $\mathrm{Ind}\,\mathsf{T}$, making it a tensor category over k. Moreover, the tensor product on $\mathrm{Ind}\,\mathsf{T}$ is exact in each variable (6.2, B.7). The objects of $\mathrm{Ind}\,\mathsf{T}$ can be identified with the small filtered inductive limits of representable functors $\mathsf{T}^{\mathrm{opp}} \to \mathsf{Vec}(k)$,

$$X = \text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim h_{X_\alpha} = h_X, \quad h_{X_\alpha} \overset{\text{def}}{=} \mathrm{Hom}(-, X_\alpha).$$

To give a morphism $X \otimes Y \to Z$ in $\mathrm{Ind}\,\mathsf{T}$ is the same as giving a k-bilinear map

$$h_X(S) \times h_Y(T) \to h_Z(S \otimes T)$$

natural in S and T.

10.3 A ***ring*** A (associative with 1) in $\operatorname{Ind}\mathsf{T}$ is an object A equipped with morphisms $m\colon A\otimes A\to A$ and $1\colon \mathbb{1}\to A$ such that the two composed morphisms

$$A\otimes A\otimes A \underset{A\otimes m}{\overset{m\otimes A}{\rightrightarrows}} A\otimes A \xrightarrow{\;m\;} A$$

are equal and the two morphisms

$$A\simeq \mathbb{1}\otimes A \xrightarrow{1\otimes \mathrm{id}_A} A\otimes A \xrightarrow{m} A$$
$$A\simeq A\otimes \mathbb{1} \xrightarrow{A\otimes 1} A\otimes A \xrightarrow{m} A$$

equal id_A. For example, when $\mathsf{T}=\mathsf{Vec}(k)$, a ring in $\operatorname{Ind}\mathsf{T}$ is a k-algebra (associative with 1) in the usual sense. A ***homomorphism*** of rings in $\operatorname{Ind}\mathsf{T}$ is a morphism of objects compatible with the multiplication and the morphisms 1.

The multiplication morphism $m\colon A\otimes A\to A$ corresponds to a k-bilinear map

$$x,y\mapsto xy\colon\ h_A(S)\times h_A(T)\to h_A(S\otimes T),$$

natural in S and T, and the associativity of m becomes the equality

$$(xy)z=x(yz)\quad \text{for } x\in h_A(S),\ y\in h_A(T),\ z\in h_A(U).$$

Here $xy\in h_A(S\otimes T)$, $yz\in h_A(T\otimes U)$, and $(xy)z,\ x(yz)\in h_A(S\otimes T\otimes U)$. That $1\colon \mathbb{1}\to A$ is an identity becomes the equalities

$$1x=x=x1 \text{ for } x\in h_A(S).$$

Here $1\in h_A(\mathbb{1})$, so $1x\in h_A(\mathbb{1}\otimes S)\simeq h_A(S)$ and $x1\in h_A(S\otimes\mathbb{1})\simeq h_A(S)$.

10.4 A left A-module is an object M of $\operatorname{Ind}\mathsf{T}$ together with a morphism $\mu\colon A\otimes M\to M$ such that the two morphisms

$$A\otimes A\otimes M \underset{A\otimes \mu}{\overset{m\otimes M}{\rightrightarrows}} A\otimes M \xrightarrow{\;\mu\;} M$$

are equal, and such that

$$M\simeq \mathbb{1}\otimes M \xrightarrow{1\otimes M} A\otimes M \xrightarrow{\mu} M$$

equals id_M. Right A-modules are defined similarly. The left A-modules in $\operatorname{Ind}\mathsf{T}$ form an abelian category ${}_A\mathsf{Mod}$ – to form the kernel or cokernel of a morphism in ${}_A\mathsf{Mod}$, first form it in $\operatorname{Ind}\mathsf{T}$, and then equip it with the induced A-module structure.

When M is a right A-module and N a left A-module, we define $M\otimes_A N$ to be the coequalizer of the pair of morphisms

$$M\otimes A\otimes N \underset{M\otimes\mu_N}{\overset{\mu_M\otimes N}{\rightrightarrows}} M\otimes N$$

10.5 When A is commutative, $\otimes_A$ makes $\mathsf{Mod}(A)\overset{\text{def}}{=}\mathsf{Mod}_A$ into an A-linear symmetric monoidal category. The unit is (A,m). There is a canonical functor

$$X\rightsquigarrow X\otimes A\colon\ \mathsf{T}\to\mathsf{Mod}(A).$$

This sends $X^\vee$ to the dual of $X \otimes A$ in $\mathsf{Mod}(A)$,

$$X^\vee \otimes A = (X \otimes A)^\vee.$$

The categorical dimension of X in T becomes the categorical dimension of $X \otimes A$ in $\mathsf{Mod}(A)$,

$$\dim(X) = \dim_A(X \otimes A), \tag{42}$$

once we identify $k = \mathrm{End}(\mathbb{1})$ with a subring of $\mathrm{End}_A(A) \simeq \mathrm{Hom}(\mathbb{1}, A)$ using the morphism $e\colon \mathbb{1} \to A$.

DEFINITION 10.6 A homomorphism $f\colon A \to B$ of commutative rings in $\mathrm{Ind}\,\mathsf{T}$ is ***flat*** (resp. ***faithfully flat***) if the functor

$$M \rightsquigarrow M \otimes_A B\colon \mathsf{Mod}(A) \to \mathsf{Mod}(B)$$

is exact (resp. exact and faithful).

To check that a flat homomorphism $A \to B$ is faithfully flat, it suffices to show that

$$M \neq 0 \implies M \otimes_A B \neq 0.$$

LEMMA 10.7 *Let $f\colon A \to B$ be faithfully flat. A sequence*

$$(N)\,:\, N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$$

of A-modules is exact if (and only if)

$$(N) \otimes B\,:\, N' \otimes_A B \xrightarrow{\alpha\otimes 1} N \otimes_A B \xrightarrow{\beta\otimes 1} N'' \otimes_A B$$

is exact.

PROOF Let C be the cokernel of $N' \xrightarrow{\alpha} \mathrm{Ker}(\beta)$, so that

$$N' \xrightarrow{\alpha} \mathrm{Ker}(\beta) \longrightarrow C \longrightarrow 0$$

is exact. As f is flat, the sequence

$$N' \otimes_A B \xrightarrow{\alpha\otimes 1} \mathrm{Ker}(\beta) \otimes_A B \longrightarrow C \otimes_A B \longrightarrow 0$$

is exact, and as $(N) \otimes_A B$ is exact, $C \otimes_A B = 0$. This implies that $C = 0$ because f is faithful, and so (N) is exact. □

LEMMA 10.8 *For any nonzero commutative ring $(A, m, 1)$ in T, the morphism $1\colon \mathbb{1} \to A$ is faithfully flat.*

PROOF The morphism 1 is a monomorphism because it is nonzero and $\mathbb{1}$ is simple (6.4). As $\otimes$ is exact in each variable (6.2), for any object M, the morphism $M \otimes 1\colon M \otimes \mathbb{1} \to M \otimes A$ is a monomorphism. It follows that the functor $-\otimes A$ is exact and that $M \otimes A \neq 0$ whenever $M \neq 0$. The functor is therefore also faithful. □

Faithfully flat descent

PROPOSITION 10.9 *Let $f: A \to B$ be a faithfully flat homomorphism of commutative rings in* Ind T. *For any A-module M, the sequence*

$$M \xrightarrow{d_0} B \otimes_A M \underset{e_1 \otimes \mathrm{id}_M}{\overset{e_0 \otimes \mathrm{id}_M}{\rightrightarrows}} B \otimes_A B \otimes_A M \tag{43}$$

is exact, i.e., d_0 is the equalizer of the parallel pair of arrows. Here $e_0, e_1: B \to B \otimes_A B$ are the morphisms

$$B \simeq A \otimes_A B \xrightarrow{f \otimes \mathrm{id}_B} B \otimes_A B, \quad b \mapsto 1_A \otimes b,$$
$$B \simeq B \otimes_A A \xrightarrow{\mathrm{id}_B \otimes f} B \otimes_A B, \quad b \mapsto b \otimes 1_A,$$

and d_0 is the morphism

$$M \simeq A \otimes_A M \xrightarrow{f \otimes \mathrm{id}_M} B \otimes_A M.$$

PROOF Let $d_1 = e_0 \otimes \mathrm{id}_M - e_1 \otimes \mathrm{id}_M$. Clearly $d_1 \circ d_0 = 0$. Assume first that there exists an A-linear section to f, i.e., an A-linear map $g: B \to A$ such that $g \circ f = \mathrm{id}_A$. Define k_0 and k_1 to be the A-linear morphisms

$$\begin{cases} B \otimes_A M \xrightarrow{g \otimes M} A \otimes_A M \simeq M \\ B \otimes_A B \otimes_A M \xrightarrow{g \otimes B \otimes M} A \otimes_A B \otimes_A M \simeq B \otimes_A M. \end{cases}$$

Then $k_0 \circ d_0 = \mathrm{id}_M$, which shows that d_0 is a monomorphism. Moreover,

$$k_1 \circ d_1 + d_0 \circ k_0 = \mathrm{id}_{B \otimes_A M},$$

which shows that d_0 maps onto the kernel of d_1.

We now consider the general case. Because $A \to B$ is faithfully flat, it suffices to prove that the sequence (43) becomes exact after tensoring with B. But the sequence obtained by tensoring (43) with B is isomorphic to the sequence (43) for the faithfully flat homomorphism $B \simeq A \otimes_A B \xrightarrow{f \otimes B} B \otimes_A B$ and the B-module $B \otimes_A M$ because, for example,

$$B \otimes_A (B \otimes_A M) \simeq (B \otimes_A B) \otimes_B (B \otimes_A M).$$

Now $B \to B \otimes_A B$ has a B-linear section, namely, that defined by multiplication $B \otimes B \to B$, and so we can apply the first case. □

10.10 Let $f: A \to B$ be a faithfully flat homomorphism of commutative rings in Ind T, and let M be an A-module. Set $M' = f_* M \stackrel{\text{def}}{=} B \otimes_A M$. The modules $e_{0*} M'$ and $e_{1*} M'$ can be identified with $B \otimes_A M$ and $M \otimes_A B$ respectively, with the natural action of $B \otimes_A B$. There is a canonical isomorphism $\phi: e_{1*} M' \to e_{0*} M'$, namely,

$$e_{1*} M' \simeq (e_1 f)_* M = (e_0 f)_* M \simeq e_{0*} M'.$$

Moreover, M can be recovered from the pair (M', ϕ) as the equalizer of

$$M' \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B \otimes_A M', \qquad \begin{cases} \alpha: M' \simeq A \otimes_A M' \xrightarrow{f \otimes M'} B \otimes_A M' \\ \beta: M' \simeq M' \otimes_A A \xrightarrow{M' \otimes f} M' \otimes_A B \xrightarrow{\phi} B \otimes_A M'. \end{cases}$$

Conversely, every pair (M', ϕ), with M' a B-module and ϕ a $B \otimes_A B$-linear morphism $\phi\colon M' \otimes_A B \to B \otimes_A M'$ satisfying a certain natural condition arises in this way from an A-module. Given ϕ, we construct morphisms

$$\begin{aligned}
&\phi_1\colon B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M', && \phi_1 = B \otimes \phi \\
&\phi_2\colon M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M', && \phi_2 = (\gamma_{B,B} \otimes M') \circ \phi \circ (\gamma_{M',B} \otimes B) \\
&\phi_3\colon M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B, && \phi_3 = \phi \otimes B.
\end{aligned}$$

On points, ϕ_1, ϕ_2, and ϕ_3 are obtained by tensoring ϕ with id_B in the first, second, and third positions respectively. A pair (M', ϕ) arises from an A-module M as above if and only if $\phi_2 = \phi_1 \circ \phi_3$. The necessity is easy to check. For the sufficiency, define M to be the equalizer of $\alpha, \beta\colon M' \rightrightarrows B \otimes_A M'$ with α and β as above. There is a canonical morphism $B \otimes_A M \to M'$, and it suffices to show that this is an isomorphism and that the map arising from M is ϕ. The diagram

$$\begin{tikzcd}
M' \otimes_A B \arrow[r, shift left, "\alpha\otimes B"] \arrow[r, shift right, "\beta\otimes B"'] \arrow[d, "\phi", "\simeq"'] & B \otimes_A M' \otimes_A B \arrow[d, "\phi_1", "\simeq"'] \\
B \otimes_A M' \arrow[r, shift left, "e_0\otimes M'"] \arrow[r, shift right, "e_1\otimes M'"'] & B \otimes_A B \otimes_A M'
\end{tikzcd}$$

commutes with either the upper or the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1 \circ \phi_3$), and so ϕ induces an isomorphism on the equalizers. But, by definition of M, the equalizer of the pair $(\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id})$ is $M \otimes_A B$, and, according to Proposition 10.9, the equalizer of the pair $(e_0 \otimes \mathrm{id}, e_1 \otimes \mathrm{id})$ is M'. This completes the proof.

10.11 More precisely, $M \rightsquigarrow (B \otimes_A M, \phi)$ is an equivalence from $\mathsf{Mod}(A)$ to the category of pairs (M', ϕ) satisfying $\phi_2 = \phi_1 \circ \phi_3$. This statement holds also when M and M' are equipped with A-algebra structures preserved by ϕ.

Remark 10.12 When T is the category of finite-dimensional vector spaces over k, so $\mathrm{Ind}\,\mathsf{T}$ is the category of all vector spaces over k, then a ring in $\mathrm{Ind}\,\mathsf{T}$ is a k-algebra (associative with 1) in the usual sense, and the results become familiar, for example, 10.10 is faithfully flat descent in the usual sense (e.g., Waterhouse 1979, 17.1, 17.2).

Affine schemes

10.13 Following Deligne 1989, in order to have a geometric language at our disposal, we define the ***category of affine schemes in*** $\mathrm{Ind}\,\mathsf{T}$ to be the opposite of that of commutative rings (associative with 1) in $\mathrm{Ind}\,\mathsf{T}$. We call an object of the category an ***affine scheme in*** $\mathrm{Ind}\,T$ or an ***affine*** T-***scheme***, and we write $\mathrm{Sp}(A)$ for the affine T-scheme defined by A. Fibre products exist, and correspond to tensor products. An A-module M is called a ***module*** over $\mathrm{Sp}(A)$, and, when $\mathrm{Sp}(B)$ is an affine scheme over $\mathrm{Sp}(A)$, $B \otimes_A M$ is called the ***inverse image*** of M over $\mathrm{Sp}(B)$.

For example, $\mathrm{Sp}(0)$ is the empty scheme and $\mathrm{Sp}(\mathbb{1})$ is the point (pt) – they are the initial and final objects in the category. We say that $S = \mathrm{Sp}(A)$ is nonempty if $A \neq 0$. Every S is either empty or faithfully flat over (pt). For X and S affine schemes in T, the set $X(S)$ of S-***points*** of X is $\mathrm{Hom}(S, X)$.

An ***affine group*** T-***scheme*** is a group object in the category of affine T-schemes. Let H be an affine group scheme in T. An H-***torsor*** is a nonempty affine T-scheme P equipped with a right action $\rho\colon P\times H\to P$ such that, for all S, $P(S)$ is either empty or a torsor under $H(S)$. The condition "empty or a torsor" means that, for all S, $(\mathrm{pr}_1,\rho)\colon P(S)\times H(S)\to P(S)\times P(S)$ is bijective, i.e., that $(\mathrm{pr}_1,\rho)\colon P\times H\to P\times P$ is an isomorphism.

EXAMPLE 10.14 (VECTORIAL SCHEMES IN T) For M in $\operatorname{Ind}\mathsf{T}$, put $\Gamma(M)=\operatorname{Hom}(\mathbb{1},M)$, the global sections of M over S. When M is a module over $S=\operatorname{Sp}(A)$, we have

$$\Gamma(M)=\operatorname{Hom}(\mathbb{1},M)\simeq\operatorname{Hom}_A(A,M).$$

Note that the functor Γ need not be exact. For example, when $\mathsf{T}=\mathsf{Repf}(G)$, it is the functor of G-invariants.

An object X of T defines for each $S=\operatorname{Sp}(A)$ a module $X_S=A\otimes X$, the inverse image of X by $S\to(\mathrm{pt})$. The functor $S\rightsquigarrow\Gamma(X_S)$ is representable,

$$\operatorname{Hom}(\mathbb{1},A\otimes X)=\operatorname{Hom}(X^\vee,A)=\operatorname{Hom}_{\mathrm{rings}}(\operatorname{Sym}(X^\vee),A).$$

We sometimes denote by X the T-scheme $\operatorname{Sp}(\operatorname{Sym}(X^\vee))$ representing this functor. This is similar, when V is a finite-dimensional k-vector space, to using V to denote the scheme $\operatorname{Spec}(\operatorname{Sym}^*(V^\vee))$, which has V for its k-points.

The functor $S\rightsquigarrow\Gamma(X_S)$ is a functor to groups. The T-scheme X is therefore a group scheme in T. The group structure corresponds to the usual Hopf algebra structure on $\operatorname{Sym}^*(X^\vee)$.

EXAMPLE 10.15 (AN AFFINE k-SCHEME IS AN AFFINE T-SCHEME) Since $\operatorname{End}(\mathbb{1})=k$, the subcategory of T of sums of copies of $\mathbb{1}$ is naturally equivalent to that of vector spaces of finite dimension over k, by a functor $V\rightsquigarrow V\otimes\mathbb{1}$. The choice of a basis $e_1,\ldots,e_n$ of V identifies $V\otimes\mathbb{1}$ with $\mathbb{1}^n$. See 8.9.

Passing to the ind-objects, we obtain a functor from the category of (all) vector spaces over k to $\operatorname{Ind}\mathsf{T}$. Under this functor, an affine scheme over k defines a scheme in T. Similarly, for affine group schemes, torsors, and so on. The point $\operatorname{Spec}(k)$ defines the T-scheme (pt).

10.16 Let G be an affine group T-scheme and X an object of T. To give an ***action*** of G on X is to give, for every S, an action of $G(S)$ on the S-module X_S, compatible with base changes S'/S. Such an action is determined by the action of the universal element $\mathrm{id}_G\in G(G)$ on X_G. For $G=\operatorname{Sp}(A)$, this is an A-linear morphism $A\otimes X\to A\otimes X$, which is determined by its restriction to $X\to A\otimes X$. The morphism $X\to A\otimes X$ makes X a comodule over the Hopf algebra A (commutative with 1) in $\operatorname{Ind}\mathsf{T}$.

10.17 (THE CASE OF $\mathsf{Repf}(G)$) Let G be an affine group scheme over k and let $\mathsf{T}=\mathsf{Repf}(G)$. The ind-objects of T are the linear representations – not necessarily of finite dimension – of G (B.13). The affine T-schemes are the affine schemes over k equipped with an action of G, an affine group T-scheme H is an affine group scheme over k equipped with an action of G, an H-torsor is an G-equivariant H-torsor (in the usual sense), a vectorial T-scheme is the equivariant affine scheme of a finite-dimensional representation of G, and the inclusion of affine k-schemes into affine T-schemes is "equip with the trivial action of G".

When T is a neutral tannakian category, the choice of a fibre functor identifies T with a category $\mathsf{Repf}(G)$ (see Chapter II), so this interpretation allows us to routinely reduce questions on T-schemes to questions in usual algebraic geometry.

On reversing the arrows in 10.11, we obtain faithfully flat descent for affine schemes.

THEOREM 10.18 *Let $a\colon V \to U$ be a faithfully flat map of affine schemes in* $\mathrm{Ind}\,\mathsf{T}$. *To give an affine scheme W over U is the same as giving an affine scheme W' over V together with an isomorphism $\phi\colon\ \mathrm{pr}_1^* W' \to \mathrm{pr}_2^* W'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi).$$

Here p_{ji} denotes the projection $V \times V \times V \to V \times V$ such that $p_{ji}(w_1, w_2, w_3) = (w_j, w_i)$.

ASIDE 10.19 Let T be a pre-tannakian category over a perfect field. In this case, Deligne (1990, 8.13) constructs a T-group $\pi(\mathsf{T})$ called the ***fundamental group*** of T. In particular, $\pi(\mathsf{T})$ acts on the objects of T. For example, if $\mathsf{T} = \mathsf{Repf}(G)$, $\pi(\mathsf{T})$ is (the Hopf algebra of) G with G acting by conjugation.

When T is tannakian, the construction of the fundamental group is easier – see **??**, **??**. For any fibre functor ω of T over a scheme S, $\omega(\pi(\mathsf{T}))$ is an affine group scheme over S, and the action of $\pi(\mathsf{T})$ on an object X defines an action of $\omega(\pi(\mathsf{T}))$ on $\omega(X)$, natural in X and compatible with tensor products, which for varying X gives an isomorphism

$$\omega(\pi(\mathsf{T})) \xrightarrow{\simeq} \mathcal{A}ut_S^{\otimes}(\omega).$$

NOTES This section largely follows Deligne 1989, §5,6.

11 An intrinsic characterization of tannakian categories

We show that a tensor category over a field of characteristic zero is tannakian, i.e., admits a fibre functor, if its objects have categorical dimension an integer ≥ 0.

THEOREM 11.1 (DELIGNE 1990, 7.1) *Let* T *be an essentially small tensor category over a field k of characteristic zero. The following conditions are equivalent:*

(a) T *is tannakian, i.e., there exists a fibre functor with values in a nonzero ring;*

(b) *for all X in* T, $\dim X$ *is an integer* ≥ 0*;*

(c) *for all X in* T, *there exists an integer $n \geq 0$ such that* $\bigwedge^n X = 0$.

The proof will occupy the rest of this section. Throughout, T is a tensor category over a field k of characteristic 0.

The decomposition of $X^{\otimes n}$ under the action of the symmetric group

Let $X \in \mathrm{ob}\,\mathsf{T}$. Because T is symmetric, there is a natural action of the symmetric group S_n on $X^{\otimes n}$. The nth ***exterior power*** $\bigwedge^n X$ of X is defined to be the image of the antisymmetrization map

$$a \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma\colon X^{\otimes n} \to X^{\otimes n}.$$

As $\mathrm{char}\, k = 0$, it is also the image of the projector $a/n!$, and so

$$\dim(\textstyle\bigwedge^n X) = \mathrm{Tr}(a/n!) = \mathrm{Tr}(a)/n!. \tag{44}$$

PROPOSITION 11.2 *We have*

$$\dim(\textstyle\bigwedge^n X) = \binom{\dim X}{n} \stackrel{\text{def}}{=} \frac{(\dim X)(\dim X - 1)\cdots \dim(X - n + 1)}{n \cdot n - 1 \cdot \cdots \cdot 1}.$$

PROOF Let σ be a cyclic permutation of order n. It follows from (30), p. 21, applied to $X_i = X$ and $u_i = \mathrm{id}_X$, that

$$\mathrm{Tr}(\sigma \mid X^{\otimes n}) = \dim(X).$$

If σ has $r(\sigma)$ cycles (including cycles of length 1), then

$$\mathrm{Tr}(\sigma \mid X^{\otimes n}) = \dim(X)^{r(\sigma)}.$$

It follows from (44) that there exists a universal polynomial $P \in \mathbb{Q}[T]$ such that $\dim \bigwedge^n X = P(\dim X)$. Taking T to be $\mathsf{Vecf}(k)$, we find that, for all $d \in \mathbb{N}$,

$$P(d) = \binom{d}{n} \stackrel{\text{def}}{=} \frac{d!}{(d-n)!n!}.$$

We deduce that $P(T) = \binom{T}{n}$, and the statement follows. □

TODO 1 Fix t124a (11.3)

11.3 For each partition λ of n, let t_λ be the canonical Young tableau of shape λ (so the boxes are numbered $1, 2, \ldots, n$, starting at the left of first row, filling the first row, and then continuing to the next row ...). Let

$$c_\lambda = \sum_{\sigma,\tau} \mathrm{sign}(\tau)\sigma\tau \in \mathbb{Q}S_n,$$

where σ and τ respectively run over the row and column permutations, be the corresponding Young symmetrizer. Then $S_\lambda \stackrel{\text{def}}{=} \mathbb{Q}[S_n]c_\lambda$ is an absolutely simple representation of S_n over $\mathbb{Q}$, and the S_λ as λ runs over the partitions of n form a complete system of simple representations of S_n.

The dimension of S_λ is the number of standard Young tableau of shape λ. The hook length formula (Fulton and Harris, 2004, Exercise 6.4) says that

$$\dim S_\lambda = \prod\nolimits_\kappa \frac{n + c(\kappa)}{h(\kappa)}, \tag{45}$$

where the product is taken over all boxes κ of the Young diagram corresponding to t_λ and $c(\kappa)$ denotes the content of κ (the number of boxes to the left of κ minus the number above κ) and $h(\kappa)$ denotes the hook length of κ (the number of boxes directly below or directly to the right of the box, with the box itself counted once). For example, if $\lambda = (n, 0, \ldots, 0)$, then $S_\lambda = \mathrm{Sym}^n(V)$, and

$$\dim S_\lambda = \binom{2n}{n}.$$

Let $e_\lambda = \frac{\dim S_\lambda}{n!} c_\lambda$. Then the e_λ form a complete set of orthogonal primitive idempotents for $\mathbb{Q}[S_n]$.

11.4 Let $X \in \mathrm{ob}\,\mathsf{T}$. The action of S_n on $X^{\otimes n}$ extends to an action of the group algebra $k[S_n]$, to which we can apply the results of the last paragraph. Each idempotent of $k[S_n]$ defines a direct summand of $X^{\otimes n}$, and we have

$$X^{\otimes n} = \bigoplus_\lambda X_\lambda, \quad X_\lambda \stackrel{\text{def}}{=} e_\lambda X^{\otimes n}, \quad \lambda \text{ a partition of } n.$$

For example, the idempotent attached to the partition (n) is $\sum_{\sigma \in S_n} \sigma/n!$, and the corresponding direct summand is $\mathrm{Sym}^n(X)$. The idempotent attached to the partition $(1, 1, \ldots, 1)$ is $\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma/n!$, and the corresponding direct summand is $\bigwedge^n X$.

11.5 The argument in the proof of Proposition 11.2 shows that there exists a universal polynomial P_λ in $\mathbb{Z}[T]$ such that $\dim X_\lambda = P_\lambda(\dim X)$. Now (45) shows that, for T an integer > 0, hence always, we have

$$P_\lambda(T) = \prod\nolimits_\varkappa \frac{T + c(\varkappa)}{h(\varkappa)}.$$

Thus,

$$\dim X_\lambda = \prod\nolimits_\varkappa \frac{\dim X + c(\varkappa)}{h(\varkappa)}, \tag{46}$$

For example,

$$\dim \operatorname{Sym}^n X = \binom{\dim X + n - 1}{n}.$$

It follows that, if $\dim X$ is an integer, and is ≥ 0 (resp. ≤ 0), then $\dim X_\lambda = 0$ whenever the number of rows (resp. the number of columns) in the Young diagram exceeds $|\dim X|$.

VARIANT 11.6 Equivalently, we could define

$$X_\lambda = (S_\lambda \otimes X^{\otimes n})^{S_n}$$

(it makes sense to tensor an object of a k-linear category with a k (or $\mathbb{Q}$) vector space). The image of the idempotent $\frac{1}{n!}\sum \sigma$ is invariant, and the argument in the proof of Proposition 11.2 gives

$$\dim X_\lambda = \frac{1}{n!}\sum_{S_n} \chi_\lambda(\sigma)(\dim \chi_\lambda)^{r(\sigma)},$$

where χ_λ is the character of S_λ and $r(\sigma)$ is the number of cycles in σ.

Linear algebra in a tensor category

PROPOSITION 11.7 (DELIGNE 1990, 7.14) *Let*

$$0 \to N \xrightarrow{i} E \longrightarrow M \to 0$$

be an exact sequence in T. *There exists a ring P in* $\operatorname{Ind}\mathsf{T}$ *such that*

$$0 \to N \otimes P \to E \otimes P \to M \otimes P \to 0$$

is split as a sequence of P-modules.

PROOF Suppose first that $N = \mathbb{1}$. Then i is a monomorphism $\mathbb{1} \to E$. Let $S^n = \operatorname{Sym}^n(E)$, and let

$$S^n \xrightarrow{i_n} E^{\otimes n} \xrightarrow{\rho_n} S^n$$

be the canonical factorization of id_{S^n}. Let $v_n \colon S^n \to S^{n+1}$ be the composite of the morphisms

$$S^n \xrightarrow{i_n} E^{\otimes n} \simeq E^{\otimes n} \otimes I \xrightarrow{\mathrm{id} \otimes i} E^{\otimes n+1} \xrightarrow{\rho_{n+1}} S^{n+1},$$

and let $P = \varinjlim_n (S^n, v_n)$. Then v_n is a monomorphism with cokernel $\operatorname{Sym}^{n+1}(M)$. In particular, $P \neq 0$.

The next diagram defines a ring structure on P,

$$\begin{tikzcd}
& E^{\otimes m} \otimes E^{\otimes n} \arrow[dr, "\rho_{m+n}"] \arrow[dd, "(\mathrm{id} \otimes i)\otimes(\mathrm{id} \otimes i)"] & \\
S^m \otimes S^n \arrow[ur, "i_m \otimes i_n"] \arrow[rr, dashed] \arrow[dd, "v_m \otimes v_n"'] & & S^{m+n} \arrow[dd, "v_{m+n+1} \circ v_{m+n}"] \\
& E^{\otimes m+1} \otimes E^{\otimes n+1} \arrow[dr, "\rho_{m+1} \otimes \rho_{n+1}"] & \\
S^{m+1} \otimes S^{n+1} \arrow[ur, "i_{m+1} \otimes i_{n+1}"] \arrow[rr, dashed] & & S^{m+n+2}.
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
E^{\otimes n} \otimes I \arrow[r, "\mathrm{id}_{E^{\otimes n}} \otimes i"] & E^{\otimes m} \otimes E \arrow[r, equal] & E^{\otimes n+1} \arrow[d, "\rho_{n+1}"] \\
S^n \arrow[u, "i_n"] \arrow[r, "\mathrm{id}_{S^n} \otimes i"] & S^n \otimes E \arrow[u, "i_n \otimes \mathrm{id}"] \arrow[r, dashed, "u_n"] & S^{n+1},
\end{tikzcd}$$

where u_n is defined to make the right-hand square commute. From the diagram, we see that

$$u_n \circ (\mathrm{id}_{S^n} \otimes i) = v_n.$$

On passing to the limit, we obtain a morphism $u\colon P \otimes E \to E$ such that

$$u \circ (\mathrm{id}_P \otimes v) = \mathrm{id}_P\,.$$

Moreover, u is a P-module homomorphism because the diagram

$$\begin{tikzcd}
S^m \otimes S^n \otimes E \arrow[r, "u_n"] \arrow[d] & S^m \otimes S^{n+1} \arrow[d] \\
S^{m+n} \otimes E \arrow[r, "u_{m+n}"] & S^{m+n+1}
\end{tikzcd}$$

is obviously commutative as it does not involve i. This completes the proof of the lemma in the case $N = \mathbb{1}$.

Now consider an arbitrary N. In the next diagram, the bottom row is the pushout of the top row by $\mathrm{ev} \circ \gamma_{N,N^\vee}$,

$$\begin{tikzcd}
0 \arrow[r] & N \otimes N^\vee \arrow[r, "i \otimes \mathrm{id}_{N^\vee}"] \arrow[d, "\mathrm{ev} \circ \gamma_{N,N^\vee}"] & E \otimes N^\vee \arrow[r] \arrow[d, "h"] & M \otimes N^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{1} \arrow[r, "i'"] & E' \arrow[r] & M' \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact because internal Homs are exact in T. The first part of the proof gives us a morphism $u'\colon E' \to \mathbb{1}$ such that $u' \circ h \circ (i \otimes \mathrm{id}_{N^\vee}) = \mathrm{ev}_N \circ \gamma_{N,N^\vee}$. Consider the morphism

$$u\colon E \to N, \quad u = (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_E \otimes \delta_N).$$

Then

$$\begin{aligned}
u \circ i &= (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_E \otimes \delta_N) \circ i \\
&= (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(i \otimes \mathrm{id}_{N \otimes N^\vee})(\mathrm{id}_N \otimes \delta_N) \\
&= (\mathrm{id}_N \otimes u' \circ h)(\mathrm{id}_N \otimes i \otimes \mathrm{id}_{N^\vee})(\gamma_{N,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_N \otimes \gamma_N) \\
&= (\mathrm{id}_N \otimes \mathrm{ev}_N \gamma_{N,N^\vee})(\gamma_{N,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_N \otimes \delta_N) \\
&= (\mathrm{id}_N \otimes \mathrm{ev}_N)(\mathrm{id}_N \otimes \delta_N) \\
&= \mathrm{id}_N,
\end{aligned}$$

where in the third and fifth equations, we used the naturality of γ. This completes the proof. □

COROLLARY 11.8 *Let $F : \mathsf{C} \to \mathsf{D}$ be a k-linear $\otimes$-functor of tensor categories over a field k of characteristic* 0. *If F is faithful, then it is exact.*

PROOF Extend F to a functor $\mathrm{Ind}\,\mathsf{C} \to \mathrm{Ind}\,\mathsf{D}$, again denoted F – it is again a faithful k-linear $\otimes$-functor. Let $\xi : 0 \to N \to E \to M \to 0$ be an exact sequence in C, and let P be as in 11.7. Then $\xi \otimes P$ is split-exact, and so $F(\xi) \otimes P \simeq F(\xi) \otimes F(P)$ is exact because $\mathrm{Ind}\,F$ is additive. Therefore $F(\xi)$ is exact (10.7, 10.8). □

COROLLARY 11.9 *Let T be a tensor category over a field k of characteristic* 0, *let R be a k-algebra, and let $F : \mathsf{T} \to \mathsf{Modf}(R)$ be a k-linear $\otimes$-functor. If F is faithful, then it is exact.*

PROOF Extend F to a functor $\mathrm{Ind}\,\mathsf{T} \to \mathsf{Mod}(R)$, again denoted F – it is again a faithful k-linear $\otimes$-functor. Let $\xi : 0 \to N \to E \to M \to 0$ be an exact sequence in T, and let P be as in 11.7. Then $F(E)$ is a faithfully flat R-module (8.5), so $F(P) \overset{\text{def}}{=} \varinjlim \mathrm{Sym}^n(F(E))$ is a faithfully flat R-algebra. As $\xi \otimes P$ is split-exact and F is additive, the sequence $F(\xi \otimes P)$ is exact. But $F(\xi \otimes P) \simeq F(\xi) \otimes F(P)$, and so $F(\xi)$ is exact. □

Proof of Theorem 11.1

It is obvious that (a) of Theorem 11.1 implies (b). We next show that (c) implies (b). If $\bigwedge^n X = 0$, then $\dim \bigwedge^n X = 0$, and 11.2 shows that $\dim X = 0, 1, \ldots$ or $n-1$; in particular, it is an integer ≥ 0. It remains to show that (b) $\Rightarrow$ (a), (c).

For the remainder of this section we assume that T *satisfies (b) of Theorem* 11.1, *i.e., for all X in* T, $\dim X$ *is an integer* ≥ 0.

LEMMA 11.10 *Let $X \in \mathrm{ob}\,\mathsf{T}$. If* $\dim X = 0$, *then* $X = 0$.

PROOF If $X \neq 0$, then $\mathrm{id}_X \neq 0$ and the map $\delta_X : \mathbb{1} \to X \otimes X^\vee$ is not the zero map. Because $\mathbb{1}$ is simple (see 6.4), it is a monomorphism. Now

$$0 \leq \dim(\mathrm{Coker}(\delta_X)) \overset{6.8}{=} \dim(X \otimes X^\vee) - 1 \overset{5.3}{=} (\dim X)(\dim X^\vee) - 1,$$

and so $\dim X \neq 0 \neq \dim X^\vee$. □

11.11 From Lemma 11.10, we see that $X_\lambda = 0$ for all λ of length $> \dim X$. Let X be an object of T of dimension $n > 0$. Let $\mathrm{GL}_n(k)$ act on $V = k^n$ according to the standard representation. Every simple $\mathrm{GL}_n(k)$-module has the form $V_\lambda \otimes \left(V^\vee_{(1,\ldots,1)}\right)^{\otimes m}$ for a partition λ of length at most n and an $m \in \mathbb{N}$. Then

$$V_\lambda \otimes \left(V_{(1,\ldots,1)}\right)^{\vee \otimes m} \rightsquigarrow X_\lambda \otimes \left(X^\vee_{(1,\ldots,1)}\right)^{\otimes m} : \mathsf{Repf}(\mathrm{GL}_n) \to \mathsf{T}$$

is an exact $\otimes$-functor sending the standard representation to X.

LEMMA 11.12 *Let A be a commutative ring in* $\operatorname{Ind}\mathsf{T}$ *and let M be an A-module that is a direct summand (as an A-module) of $A \otimes X$ for some $X \in \operatorname{ob}\mathsf{T}$.*

(a) *If $\dim_A M = 0$, then $M = 0$.*

(b) *If $\dim_A M > 0$, then there exists an algebra P, faithfully flat over A, such that $M \otimes_A P = P \oplus N$ as P-modules.*

(c) *If $\dim_A M = d > 0$, then there exists a faithfully flat extension B of A such that $M \otimes_A B \approx B^{\oplus d}$.*

PROOF (a) Let $n = \dim X$ and $A \otimes X = M \oplus N$. Then

$$\bigwedge\nolimits_A^{n+1}(A \otimes X) = A \otimes \bigwedge^{n+1} X = 0.$$

On the other hand, as

$$\bigwedge^m (U \oplus V) \simeq \bigoplus_{p+q=m} \bigwedge^p U \otimes \bigwedge^q U,$$

we see that $M \otimes_A \bigwedge_A^n N$ is a direct summand of $\bigwedge_A^{n+1}(A \otimes X)$; hence is zero. As

$$\dim_A(N) = \dim_A(A \otimes X) = \dim X = d,$$

we have $\dim_A(\bigwedge_A^n N) = 1$. Let $Q = \bigwedge_A^n N$. Then A is a direct summand of $Q \otimes_A Q^\vee$ because the dimension of Q over A is 1. Consequently,

$$M \otimes_A Q \otimes_A Q^\vee = 0 \implies M = 0.$$

(b) We can now argue as in 11.11 to obtain an exact faithful functor from $\mathsf{Repf}(\mathrm{GL}_n)$ to $\mathsf{Mod}(A)$ sending V to $A \otimes X$. Let $e_1, \dots, e_n$ be a basis for V and $f_1, \dots f_n$ the dual basis for $V^\vee$. The group GL_n acts on the polynomial algebra $k[e_i, f_j]$, which can therefore be viewed as an algebra in $\operatorname{Ind}(\mathsf{Repf}(\mathrm{GL}_n))$. The relation $\sum e_i f_i = 1$ is invariant under the action of GL_n, and hence the quotient algebra

$$C = \frac{k[e_i, f_i]}{(\sum e_i f_i - 1)}$$

is also an algebra in $\mathsf{Repf}(\mathrm{GL}_n)$.

The image of P of C under the above functor is the required algebra in $\mathsf{Mod}(A)$. Indeed, as M is a direct summand of $A \otimes X$, which is flat over A, M is flat over A. Therefore P is also flat over A, and it is faithfully flat because it contains A as a direct summand (as C contains k as a direct summand). Finally, there exists a decomposition $V \otimes C = C \oplus Q$ in $\mathsf{Repf}(\mathrm{GL}_d(k))$: the projection $V \otimes C \to C$ is $v \otimes p \mapsto v \cdot p$ and the embedding $C \to V \otimes C$ is $p \mapsto \sum e_i \otimes f_i(p)$, $p \in C$.

(c) Applying (b) d times, we get a faithfully flat A-algebra B such that $M \otimes_A B \approx B^{\oplus d} \oplus N$ with $\dim_B N = 0$. □

LEMMA 11.13 *Let A be a commutative ring in* $\operatorname{Ind}\mathsf{T}$ *and M an A-module such that $\dim_A M = d > 0$. If M is a direct summand (as A-modules) of $A \otimes X$ for some $X \in \operatorname{ob}\mathsf{T}$, then there exists a faithful extension B of A such that $M \otimes_A B \simeq B^{\otimes d}$.*

PROOF Applying (b) d times, we get a faithfully flat A-algebra B such that $M \otimes_A B \approx B^{\oplus d} \oplus N$ with $\dim_B N = 0$. □

We now prove the theorem. For each object X of C, we have constructed a commutative ring A_X such that $A_X \otimes X \approx A_X^{\oplus \dim X}$ as A_X-modules (11.10, 11.13). Define a partial order on the isomorphism classes of objects of C by $[X] \leq [Y]$ if and only if $\langle X \rangle \subset \langle Y \rangle$, and apply transfinite induction to obtain an algebra A in $\operatorname{Ind}\mathsf{T}$ such that

(a) for all X in T, $A \otimes X \approx A^{\oplus \dim X}$;

(b) for all exact sequences $0 \to M \to N \to P \to 0$ in T, the sequence of A-modules $0 \to M \otimes A \to N \otimes A \to P \otimes A \to 0$ is split exact.

Now

$$X \rightsquigarrow \operatorname{Hom}_A(A, X \otimes A) \colon \mathsf{T} \to \mathsf{Mod}(R), \quad R \stackrel{\text{def}}{=} \operatorname{End}_A(A),$$

is a fibre functor.

TODO 2 Details to be added. Note that if $A \otimes X \approx A^{\oplus \dim X}$, then $A \otimes Y \approx A^{\oplus \dim Y}$ for all Y in $\langle X \rangle$. See Simental.

QUESTION 11.14 Let T be a tensor category over a field k of characteristic zero. Assume that, for all X in T, $\dim X$ is an integer ≥ 0. We then showed that there exists a fibre functor on T. If T is algebraic, i.e., $\mathsf{T} = \langle X \rangle^{\otimes}$ for some X in T, is it possible to show that there exists a fibre functor on T with values in a finite extension of k?

NOTES

11.15 Corollaries 11.8 and 11.9 also hold for tensor categories over fields k of nonzero characteristic. See 13.36 below.

11.16 When k has nonzero characteristic, condition (c) of Theorem 11.1 no longer implies that the category is tannakian – one needs an additional hypothesis (Coulembier 2020, Theorem B).

11.17 For an analogue of Theorem 11.1 in super mathematics (i.e., $\mathbb{Z}/2\mathbb{Z}$-graded mathematics), see Deligne 2002.

NOTES The exposition in this section follows Deligne 1990, §7, and Hái 2002.

12 Tensor products of tensor categories

In this section, we explain a construction that will be needed in the next chapter.

Tensor products of abelian categories: definition and preliminaries

In this subsection, k is a commutative ring. Unadorned tensor products are over k.

DEFINITION 12.1 (DELIGNE 1990, 5.1) Let A and B be k-linear abelian categories. A k-linear abelian category $\mathsf{A} \boxtimes \mathsf{B}$ together with a k-bilinear functor

$$\boxtimes \colon \mathsf{A} \times \mathsf{B} \to \mathsf{A} \boxtimes \mathsf{B},$$

right exact in each variable, is the ***tensor product*** of A and B if, for all small k-linear abelian categories D, the functor

$$F \rightsquigarrow F \circ \boxtimes \colon \mathsf{Rex}_k(\mathsf{A} \boxtimes \mathsf{B}, \mathsf{D}) \to \mathsf{Rex}_k(\mathsf{A} \times \mathsf{B}, \mathsf{D}) \tag{47}$$

is an equivalence of categories. On the left, Rex_k is the category of k-linear right exact functors, and on the right, it is the category of k-bilinear functors $\mathsf{A} \times \mathsf{B} \to \mathsf{D}$ right exact in each variable.

TODO 3 Coulembier: Did you intend to let A,B be essentially small in definition 12.1 (and impose that $A \boxtimes B$ needs to be essentially small)? I think this might be used in the proof of Lemma 12.2.

LEMMA 12.2 *If it exists,* $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ *is unique up to an equivalence, which itself is unique up to isomorphism.*

PROOF If $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ and $(\mathsf{A} \boxtimes' \mathsf{B}, \boxtimes')$ are two tensor products of A and B, then there exists an equivalence $\alpha\colon \mathsf{A} \boxtimes \mathsf{B} \to \mathsf{A} \boxtimes' \mathsf{B}$ such that $\alpha \circ \boxtimes \approx \boxtimes'$,

$$\begin{tikzcd} \mathsf{A} \times \mathsf{B} \arrow[r, "\boxtimes"] \arrow[dr, "\boxtimes'"'] & \mathsf{A} \boxtimes \mathsf{B} \arrow[d, dashed, "\alpha"] \\ & \mathsf{A} \boxtimes' \mathsf{B}. \end{tikzcd}$$

and any two such α are isomomorphic. □

Tensor products of abelian categories in the sense of 12.1 have become known in the literature as ***Deligne tensor products*** (see, for example, ncatlab.org).

We shall prove the existence for locally finite k-linear abelian categories. Without some finiteness condition, the tensor product need not exist, even for $k = \mathbb{Q}$ (see 12.16 below).

Let A and R be rings (not necessarily commutative), and let M be an (A,R)-module. The functor

$$F_M\colon \mathsf{Mod}_A \to \mathsf{Mod}_R, \quad X \rightsquigarrow X \otimes_A M$$

is right exact and commutes with all direct sums. It has been known since about 1960 (Eilenberg, Gabriel, Watts) that every such functor arises in this way. We shall prove a more precise statement. Note that F_M is natural in M: a homomorphism $f\colon M \to N$ of (A,R)-bimodules defines a natural transformation $F_M \to F_N$ whose value on X is

$$X \otimes f\colon X \otimes_A M \to X \otimes_A N.$$

PROPOSITION 12.3 *The functor*

$$M \rightsquigarrow F_M\colon {}_A\mathsf{Mod}_R \to \mathsf{Rex}(\mathsf{Modf}_A, \mathsf{Mod}_R)$$

is an equivalence with quasi-inverse $F \rightsquigarrow F(A_A)$.

PROOF From a natural transformation $u\colon F_M \to F_N$, we get a morphism

$$M \simeq A_A \otimes_A M \xrightarrow{u_A} A_A \otimes_A N \simeq N.$$

When applied to $u = F_f$, where $f\colon M \to N$, this gives back f. Thus, the functor $M \rightsquigarrow F_M$ is faithful.

Let $u\colon F_M \to F_N$ be a natural transformation. We shall show that, for all X in Modf_A,

$$u_X = X \otimes_A u_A \quad \text{(as maps } X \otimes_A M \to X \otimes_A N\text{)}.$$

Let $\mathcal{C}$ be the collection of X for which this is true. Certainly $\mathcal{C}$ contains A_A, and it is closed under finite direct sums. If $X, Y \in \mathcal{C}$ and $X \to Y \to Z \to 0$ is exact, then the exact commutative diagram

$$\begin{tikzcd} X \otimes_A M \arrow[r] \arrow[d, "u_X"', "X\otimes u_A"] & Y \otimes_A M \arrow[r] \arrow[d, "u_Y"', "Y\otimes u_A"] & Z \otimes_M \arrow[r] \arrow[d] & 0 \\ X \otimes_A N \arrow[r] & Y \otimes_A N \arrow[r] & Z \otimes_A N \arrow[r] & 0 \end{tikzcd}$$

shows that $Z \in \mathcal{C}$. By definition, every X in Modf_A arises as a cokernel

$$A^{\oplus m} \to A^{\oplus n} \to X \to 0,$$

and so it lies in $\mathcal{C}$, completing the proof.

From $u_X = X \otimes u_A$ we deduce that $u = F(u_A)$. In particular, the functor $M \rightsquigarrow F_M$ is full.

Let $F\colon \mathsf{Modf}_A \to \mathsf{Mod}_R$ be right exact, and let $M = F(A_A)$. Then M is a right R-module by definition, and it becomes a left A-module through the map

$$A = \mathrm{End}_A(A_A) \xrightarrow{F} \mathrm{End}_R(F(A_A)) = \mathrm{End}_R(M).$$

Note that, for $a \in A$, $r \in R$, $m \in M$,

$$(am)r \stackrel{\text{def}}{=} (F(a)(m))r = F(a)(mr) = a(mr),$$

and so $M \in {}_A\mathsf{Mod}_R$. We shall show that F is isomorphic to F_M.

For X in Modf_A, we regard $Hom_R(M, FX)$ as a right A-module by

$$(f \cdot a)(m) = f(am), \quad f\colon M \to FX,\ a \in A,\ m \in M.$$

Then

$$f_X \stackrel{\text{def}}{=} \left(X \simeq \mathrm{Hom}_A(A, X) \xrightarrow{F} \mathrm{Hom}_R(M, FX)\right)$$

is A-linear: for $a \in A$, $m \in M$, $t\colon A \to X$,

$$F(ta)(m) = F(t \circ a)(m) = F(t)(F(a)(m)) = F(t)(am) = (F(t)a)(m).$$

Let $g_X\colon X \otimes M \to FX$ be the R-linear map corresponding to f_X under the canonical isomorphism

$$\mathrm{Hom}_A(X, \mathrm{Hom}_R(M, FX)) \simeq \mathrm{Hom}_R(X \otimes M, FX),$$

and let $\mathcal{C}$ be the collection of X such that g_X is an isomorphism. Then $\mathcal{C}$ contains A_A, is closed under finite direct sums, and contains the cokernel of $X \to Y$ if it contains X and Y. As before, this implies that $\mathcal{C} = \mathrm{ob}\,\mathsf{Mod}_A$, and so g is an isomorphism $F_M \simeq F$. □

EXAMPLE 12.4 Let k be a field and A a finite-dimensional k-algebra. Every left A-module M defines a right exact k-linear functor

$$F_M\colon \mathsf{Modf}_A \to \mathsf{Vec}(k), \quad X \rightsquigarrow X \otimes_A M.$$

The functor

$$M \rightsquigarrow F_M\colon {}_A\mathsf{Modf} \to \mathsf{Rex}_k(\mathsf{Modf}_A, \mathsf{Vecf}(k))$$

is an equivalence of categories, with quasi-inverse $F \rightsquigarrow F(A_A)$.

Tensor products of abelian categories: construction

In this subsection k is a field. Let A and R be k-algebras (not necessarily commutative) with A finite-dimensional, and let ${}_A\mathsf{Mod}_R$ denote the category of (A, R)-bimodules on which the two actions of k coincide.

For M in ${}_A\mathsf{Mod}_R$, F_M is k-linear, and so we have the following k-linear variant of Proposition 12.3.

PROPOSITION 12.5 *The functor*

$$M \rightsquigarrow F_M \colon {}_A\mathsf{Mod}_R \to \mathsf{Rex}_k(\mathsf{Modf}_A, \mathsf{Mod}_R)$$

is an equivalence of categories with quasi-inverse $F \rightsquigarrow F(A_A)$

PROPOSITION 12.6 *Let A, B be finite-dimensional k-algebras and R a k-algebra. The functor*

$$F \rightsquigarrow F_{(A,B)} \colon \mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{Mod}_R) \to {}_{A\otimes B}\mathsf{Mod}_R$$

is an equivalence of categories.

PROOF The proof of Proposition 12.3 generalizes. □

PROPOSITION 12.7 *Let A and B be finite-dimensional k-algebras. Then*

$$\otimes \colon \mathsf{Modf}_A \times \mathsf{Modf}_B \to \mathsf{Modf}_{A\otimes_k B}, \quad (M,N) \rightsquigarrow M \otimes_k N,$$

is the Deligne tensor product of the categories Modf_A and Modf_B.

PROOF We have to show that, for any small k-linear abelian category D, the functor

$$F \rightsquigarrow F \circ \otimes \colon \mathsf{Rex}_k(\mathsf{Modf}_{A\otimes B}, \mathsf{D}) \to \mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{D}) \tag{48}$$

is an equivalence of categories. If $\mathsf{D} \to \mathsf{D}'$ is fully faithful and exact, and the statement is true for D', then it is true for D. Now a k-linear variant of the full embedding theorem (Mitchell 1965, VI, 7.2) allows us to replace D with Mod_R for some k-algebra R. In this case, (48) has quasi-inverse

$$\mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{Mod}_R) \xrightarrow{12.6} {}_{A\otimes B}\mathsf{Mod}_R \xrightarrow{12.5} \mathsf{Rex}_k(\mathsf{Modf}_{A\otimes B}, \mathsf{Mod}_R).$$ □

THEOREM 12.8 (DELIGNE 1990, 5.13) *Let A and B be essentially small k-linear abelian categories (k a field). If A and B are locally finite (7.8), then their tensor product exists and is locally finite.*

PROOF Write each of A and B as filtered unions of subcategories $\mathsf{A} = \bigcup_\alpha \mathsf{A}_\alpha$ and $\mathsf{B} = \bigcup_\beta \mathsf{B}_\beta$ as in (II, 3.18). Now $\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta$ exists for each α, β (by 12.7), and the transition map

$$\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta \to \mathsf{A}_{\alpha'} \boxtimes \mathsf{B}_{\beta'}, \quad \alpha \leq \alpha, \quad \beta \leq \beta',$$

is fully faithful and exact because it can be identified with the map

$$\mathsf{Mod}_{A_\alpha \otimes B_\alpha} \to \mathsf{Mod}_{\bar{A}_\alpha \otimes \bar{B}_\alpha}$$

defined by a surjective map of k-algebras $A_\alpha \otimes B_\beta \to \bar{A}_\alpha \otimes \bar{B}_\beta$ (see II, 3.6). Define $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ to be the inductive 2-limit

$$\varinjlim_{\alpha,\beta} (\boxtimes \colon \mathsf{A}_\alpha \times \mathsf{B}_\beta \to \mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta).$$

Specifically, an object of $\mathsf{A} \boxtimes \mathsf{B}$ is an object X of $\mathsf{A}_{\alpha_0} \boxtimes \mathsf{B}_{\beta_0}$ for some (α_0, β_0). Such an X defines an object $X_{(\alpha,\beta)}$ of $\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta$ for all $(\alpha, \beta) \geq (\alpha_0, \beta_0)$. If Y is a second object of $\mathsf{A} \boxtimes \mathsf{B}$, then

$$\mathrm{Hom}_{\mathsf{A}\boxtimes\mathsf{B}}(X, Y) = \varinjlim \mathrm{Hom}_{\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta}(X_{(\alpha,\beta)}, Y_{(\alpha,\beta)}),$$

where the limit is over the pairs (α, β) such that $X_{(\alpha,\beta)}$ and $Y_{(\alpha,\beta)}$ are defined. It is straightforward to verify that $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ has the required properties. □

PROPOSITION 12.9 *Let* A, B, *and* C *be locally finite* k*-linear abelian categories. There are canonical equivalences of categories*

$$\mathsf{A}\boxtimes(\mathsf{B}\boxtimes\mathsf{C})\xleftrightarrow{\approx}(\mathsf{A}\boxtimes\mathsf{B})\boxtimes\mathsf{C}$$
$$\mathsf{A}\boxtimes\mathsf{B}\xleftrightarrow{\approx}\mathsf{B}\boxtimes\mathsf{A}.$$

PROOF Obvious from the definitions. □

PROPOSITION 12.10 *Let* A *and* B *be locally finite* k*-linear categories, and let* $(\mathsf{A}\boxtimes\mathsf{B},\boxtimes)$ *be their tensor product.*

(a) *The functor* $\boxtimes\colon\ \mathsf{A}\times\mathsf{B}\to\mathsf{A}\boxtimes\mathsf{B}$ *is exact in each variable.*

(b) *The functor* $\boxtimes$ *induces isomorphisms*

$$\mathrm{Hom}_{\mathsf{A}}(X_1,X_2)\otimes\mathrm{Hom}_{\mathsf{B}}(Y_1,Y_2)\simeq\mathrm{Hom}(X_1\boxtimes Y_1,X_2\boxtimes Y_2),$$

all $X_1,X_2\in\mathrm{ob}\,\mathsf{A}$, $Y_1,Y_2\in\mathrm{ob}\,\mathsf{B}$.

(c) *The functor* $\boxtimes$ *makes* $(\mathsf{A}\boxtimes\mathsf{B})^{\mathrm{opp}}$ *the tensor product of* $\mathsf{A}^{\mathrm{opp}}$ *and* $\mathsf{B}^{\mathrm{opp}}$.

(d) *For all small* k*-linear abelian categories* D, *the functor* $\boxtimes$ *induces an equivalence of categories*

$$\mathsf{Lex}(\mathsf{A}\boxtimes\mathsf{B},\mathsf{D})\xrightarrow{\approx}\mathsf{Lex}(\mathsf{A}\times\mathsf{B},\mathsf{D}),$$

where Lex *denotes the category of* k*-linear functors left exact (in each variable).*

PROOF It suffices to prove each statement for $\mathsf{A}=\mathsf{Modf}_A$ and $\mathsf{B}=\mathsf{Modf}_B$, where A and B are finite-dimensional k-algebras (see the proof of 12.8).

(a) As $\boxtimes=\otimes_k$, this is obvious.

(b) For finitely generated A and B modules, we have a bijection

$$\mathrm{Hom}_k(M_1,M_2)\otimes\mathrm{Hom}_k(N_1,N_2)\simeq\mathrm{Hom}_k(M_1\otimes N_1,M_2\otimes N_2)$$
$$f\otimes g\mapsto(m\otimes n\mapsto f(m)\otimes g(n)).$$

Under the bijection, maps that are A and B linear correspond to maps that are $A\otimes B$-linear.

(c) For a finite-dimensional k-algebra A, the functor

$$M\rightsquigarrow M^\vee\overset{\mathrm{def}}{=}\mathrm{Hom}_k(M,k)\colon\ (\mathsf{Modf}_A)^{\mathrm{opp}}\to\mathsf{Modf}_{A^{\mathrm{opp}}}$$

is an equivalence of categories because $M^{\vee\vee}\simeq M$ and $(A\otimes B)^{\mathrm{opp}}\simeq A^{\mathrm{opp}}\otimes B^{\mathrm{opp}}$. Therefore,

$$\mathsf{Modf}_A^{\mathrm{opp}}\boxtimes\mathsf{Modf}_B^{\mathrm{opp}}\sim\mathsf{Modf}_{A^{\mathrm{opp}}\otimes B^{\mathrm{opp}}}\approx\mathsf{Modf}_{(A\otimes B)^{\mathrm{opp}}}\approx(\mathsf{Modf}_A\boxtimes\mathsf{Modf}_B)^{\mathrm{opp}}.$$

This proves the statement.

(d) Under $\mathsf{A}\mapsto\mathsf{A}^{\mathrm{opp}}$, right exact functors go to left-exact functors. Therefore, this follows from (c) . □

PROPOSITION 12.11 *Suppose that the functors* $F\colon\ \mathsf{A}\times\mathsf{B}\to\mathsf{D}$ *and* $F'\colon\ \mathsf{A}\boxtimes\mathsf{B}\to\mathsf{D}$ *correspond under (47), p. 54. If* F *is exact in both variables and* k *is perfect, then* F' *is exact.*

PROOF Again, it suffices to prove this in the key case 12.13 below. □

12.12 Let A and B be finite-dimensional k-algebras and S a simple $A\otimes_k B$-module. There exist a simple A-module M and a simple B-module N, both finite-dimensional over k, such that S is a quotient of $M\otimes_k N$ (Bourbaki A, 7.7, Pptn 8). The centres k_M and k_N of $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$ are finite field extensions of k. If k is perfect, they are separable over k, and hence $k_M\otimes k_N$ is a semisimple k-algebra, which implies that $M\otimes_k N$ is a semisimple $A\otimes_k B$-algebra (Bourbaki A, 7.4, Thm 2). In this case, S is a direct summand of $M\otimes_k N$.

LEMMA 12.13 *Let A and B be finite-dimensional k-algebras, and suppose that the functors $F\colon \mathsf{Modf}_A\times\mathsf{Modf}_B\to\mathsf{Mod}_R$ and $F'\colon\mathsf{Modf}_{A\otimes B}\to\mathsf{Mod}_R$ correspond under (48), p. 57. If F is exact in both variables and k is perfect, then F' is exact.*

PROOF Let $M=F(A_A,B_B)=F'((A\otimes B)_{A\otimes B})$. Then

$$F(X,Y)=(X\otimes Y)\otimes_{A\otimes B}M$$
$$F'(Z)=Z\otimes_{A\otimes B}M.$$

We choose projective resolutions $X^\bullet\to X$ and $Y^\bullet\to Y$ for X and Y, and form the complex $(X^\bullet\otimes Y^\bullet)\otimes_{A\otimes B}M$. Let $\mathrm{Tor}_n^{A,B}((X,Y),M)$ denote the nth homology group of this complex. After our assumption, $\mathrm{Tor}_n^{A,B}((X,Y),M)=0$ for $n>0$. For $Z\in\mathrm{ob}\,\mathsf{Modf}_{A\otimes B}$, we define $\mathrm{Tor}_n^{A\otimes B}(Z,M)$ similarly. We have to show that $\mathrm{Tor}_n^{A\otimes B}(Z,M)=0$ for $n>0$. If $Z=X\otimes Y$ as above, then $X^\bullet\otimes Y^\bullet\to Z$ is a projective resolution, and therefore

$$\mathrm{Tor}_n^{A\otimes B}(X\otimes Y,M)=\mathrm{Tor}_n^{A,B}((X,Y),M)=0.$$

An exact sequence

$$0\to Z'\to Z\to Z''\to 0$$

in $\mathsf{Modf}_{A\otimes B}$ gives a long exact sequence

$$\cdots\to\mathrm{Tor}_n^{A\otimes B}(Z',M)\to\mathrm{Tor}_n^{A\otimes B}(Z,M)\to\mathrm{Tor}_n^{A\otimes B}(Z'',M).$$

As every finitely generated $A\otimes B$-module has finite length, it suffices to show that $\mathrm{Tor}_n^{A\otimes B}(Z,M)$ for all $n>0$ and all simple $A\otimes B$-modules Z. Because

$$\mathrm{Tor}_n^{A\otimes B}(Z\otimes Z',M)\simeq\mathrm{Tor}_n^{A\otimes B}(Z,M)\otimes\mathrm{Tor}_n^{A\otimes B}(Z',M)$$

(Mac Lane 1963, V, §§7,8) this follows from 12.12. □

TODO 4 Need to check the last proof. See aax complaint.

NOTES The exposition in this subsection follows the original in Deligne 1990, §5, except that we have been more careful in the passage to the inductive limits. See also Lattermann 1989, 3.5.

Tensor products of abelian categories: an alternative approach

Let k be a commutative ring. Recall that a category is finitely cocomplete if it has finite inductive limits. For example, abelian categories are finitely cocomplete.

12.14 (KELLY 1982) Let A and B be small k-linear finitely cocomplete categories. There exists a k-linear finitely cocomplete category $\mathsf{A}\bullet\mathsf{B}$ and a k-bilinear functor

$$\bullet\colon\ \mathsf{A}\times\mathsf{B}\to\mathsf{A}\bullet\mathsf{B},$$

right exact in each variable, with the following universal property: for all k-linear finitely complete categories D, the functor

$$F \rightsquigarrow F\circ\bullet\colon \operatorname{Rex}(\mathsf{A}\bullet\mathsf{B},\mathsf{D}) \to \operatorname{Rex}(\mathsf{A}\times\mathsf{B},\mathsf{D})$$

is an equivalence of categories.

12.15 (LÓPEZ FRANCO 2013, THEOREM 3) If k is a field, and A and B are locally finite k-linear abelian categories, then $\mathsf{A}\bullet\mathsf{B}$ is abelian and $(\mathsf{A}\bullet\mathsf{B},\bullet)$ is the tensor product of A and B in the sense of 12.1. This gives a second proof of Theorem 12.8. In fact, Lopez Franco (ibid., Theorem 1) shows that two small abelian categories have a tensor product in the sense of 12.1 if and only if their tensor product in the sense of 12.14 is abelian, in which case the two tensor products coincide. This can be used to show that the Deligne tensor product may not exist.

12.16 A k-algebra R is said to be ***left coherent*** if the category of finitely presented left R-modules is abelian (for this it suffices to check that the left ideals in R are finitely presented). If R and S are left coherent k-algebras such that $R\otimes_k S$ is not left coherent, then the tensor product of ${}_R\mathsf{Modf}$ and ${}_S\mathsf{Modf}$ exists in the sense of 12.14, but it is not abelian, and so the tensor product in the sense of 12.1 does not exist. For example, $\mathbb{Q}^{\mathbb{N}}[[T,U]]$ and $\mathbb{Q}[X]$ are coherent $\mathbb{Q}$-algebras, but $\mathbb{Q}^{\mathbb{N}}[[T,U]]\otimes_{\mathbb{Q}}\mathbb{Q}[X]\simeq\mathbb{Q}^{\mathbb{N}}[[T,U]][X]$ is not coherent.

Tensor products of tensor categories

Let $(\mathsf{C},\otimes)$ be a locally finite k-linear abelian rigid symmetric monoidal category. Then $\otimes$ is k-bilinear and exact in each variable (I, 6.2), so it factors through $\mathsf{C}\boxtimes\mathsf{C}$,

$$\begin{tikzcd} \mathsf{C}\times\mathsf{C} \arrow[r, "\boxtimes"] \arrow[dr, "\otimes"'] & \mathsf{C}\boxtimes\mathsf{C} \arrow[d, "T"] \\ & \mathsf{C} \end{tikzcd}$$

When k is perfect, T is exact (12.11). After 12.10(a), $\boxtimes$ is k-linear and exact in each variable.

Let C_1 and C_2 be k-linear abelian rigid categories.

PROPOSITION 12.17 *Let k be a perfect field. If C_1 and C_2 are tensor categories over k, then so also is $\mathsf{C}_1\boxtimes\mathsf{C}_2$.*

PROOF Let $\mathsf{C}=\mathsf{C}_1\boxtimes\mathsf{C}_2$. For $i=1,2$, we have a tensor $T_i\colon \mathsf{C}_i\boxtimes\mathsf{C}_i\to\mathsf{C}_i$. On taking their exterior tensor product, we get a natural transformation T

$$\begin{tikzcd} \mathsf{C}\boxtimes\mathsf{C} \arrow[r, equal] \arrow[rrr, bend left, "T"] & (\mathsf{C}_1\boxtimes\mathsf{C}_2)\boxtimes(\mathsf{C}_1\boxtimes\mathsf{C}_2) \simeq (\mathsf{C}_1\boxtimes\mathsf{C}_1)\boxtimes(\mathsf{C}_2\boxtimes\mathsf{C}_2) \arrow[r, "T_1\boxtimes T_2"'] & \mathsf{C}, \end{tikzcd}$$

which is k-linear and exact (12.11). The middle isomorphism comes from the canonical isomorphisms in Proposition 12.9. On composing T with $\boxtimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}\boxtimes\mathsf{C}$, we get a functor

$$\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}.$$

As unit object, we choose $\mathbb{1}\boxtimes\mathbb{1}$. It follows from (12.10(b)) that $\operatorname{End}(\mathbb{1}\otimes\mathbb{1})\simeq k$.

It remains to prove that C is rigid. The equivalences (I, §5)

$$X_i \rightsquigarrow X_i^\vee : \mathsf{C}_i^{\mathrm{opp}} \to \mathsf{C}_i$$

induce an equivalence

$$\mathsf{C}^{\mathrm{opp}} \simeq \mathsf{C}_1^{\mathrm{opp}} \boxtimes \mathsf{C}_2^{\mathrm{opp}} \to \mathsf{C},$$

denoted $X \rightsquigarrow X^\vee$, which is characterized by

$$(X_1 \boxtimes X_2)^\vee = X_1^\vee \boxtimes X_2^\vee, \quad X_i \in \mathsf{C}_i.$$

We shall show that internal Homs exist by constructing a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(Y, X^\vee \otimes Z), \quad X, Y, Z \in \mathrm{ob}\,\mathsf{C}.$$

Let $F_1, F_2 : \mathsf{C}^{\mathrm{opp}} \times \mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \mathsf{Vecf}(k)$, denote the two functors. Both are left exact in each variable, so this is equivalent to constructing an isomorphism between the functors

$$\widetilde{F}_1, \widetilde{F}_2 : \mathsf{C}_1^{\mathrm{opp}} \times \mathsf{C}_2^{\mathrm{opp}} \times \mathsf{C}_1^{\mathrm{opp}} \times \mathsf{C}_2^{\mathrm{opp}} \times \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{Vecf}(k),$$

where

$$\begin{aligned}\widetilde{F}_1(X_1, X_2, Y_1, Y_2, Z_1, Z_2) &= \mathrm{Hom}((X_1 \boxtimes X_2) \boxtimes (Y_1 \boxtimes Y_2), Z_1 \boxtimes Z_2) \\ &\simeq \mathrm{Hom}((X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2), Z_1 \boxtimes Z_2) \\ &\simeq \mathrm{Hom}(X_1 \otimes Y_1, Z_1) \boxtimes \mathrm{Hom}(X_2 \otimes Y_2, Z_2)\end{aligned}$$

and

$$\begin{aligned}\widetilde{F}_2(X_1, X_2, Y_1, Y_2, Z_1, Z_2) &= \mathrm{Hom}((Y_1 \boxtimes Y_2), (X_1^\vee \boxtimes X_2^\vee) \otimes (Z_1 \boxtimes Z_2)) \\ &\simeq \mathrm{Hom}(Y_1, X_1^\vee \otimes Z_1) \boxtimes \mathrm{Hom}(Y_2, X_2^\vee \otimes Z_2).\end{aligned}$$

For this, combine the isomorphisms (I, 4.8)

$$\begin{aligned}&\mathrm{Hom}(X_1 \otimes Y_1, Z_1) \simeq \mathrm{Hom}(Y_1, X_1^\vee \otimes Z_1), \quad X_1, Y_1, Z_1 \in \mathrm{ob}\,\mathsf{C}_1 \\ &\mathrm{Hom}(X_2 \otimes Y_2, Z_2) \simeq \mathrm{Hom}(Y_2, X_2^\vee \otimes Z_2), \quad X_2, Y_2, Z_3 \in \mathrm{ob}\,\mathsf{C}_{2.}\end{aligned}$$

It remains to show that the canonical morphism (23), p. 17,

$$\phi : \mathcal{H}om(X, \mathbb{1}) \otimes Y \to \mathcal{H}om(X, Y)$$

is an isomorphism for all $X, Y \in \mathrm{ob}(\mathsf{C})$. We regard ϕ as a morphism of k-bilinear left exact functors $\mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \mathsf{C}$. Then it suffices to show that $\phi_{X,Y}$ is an isomorphism for $X = X_1 \boxtimes X_2$ and $Y = Y_1 \boxtimes Y_2$ with $X_i, Y_i \in \mathrm{ob}\,\mathsf{C}_i$. When X is of the form $X_1 \boxtimes X_2$, it has a dual, namely, $(X_1^\vee \boxtimes X_2^\vee, \mathrm{ev}_{X_1} \boxtimes \mathrm{ev}_{X_2})$ and $\delta_{X_1} \boxtimes \delta_{X_2}$. From this the proposition follows. □

COROLLARY 12.18 *Let k be a perfect field. If C_1 and C_2 are tannakian categories over k, then so also is $\mathsf{C}_1 \boxtimes \mathsf{C}_2$.*

PROOF There exist fibre functors $\omega_i : \mathsf{C}_i \to \mathsf{Mod}(B_i)$, where B_i is a nonzero commutative k-algebra. The functor

$$(X_1, X_2) \rightsquigarrow \omega_1(X_1) \otimes \omega_2(X_2) : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{Mod}(B_1 \otimes B_2)$$

is k-linear and left exact in both variables, and so it induces a k-linear left exact functor

$$X_1 \boxtimes X_2 \rightsquigarrow \omega_1(X_1) \otimes \omega_2(X_2) \colon \mathsf{C}_1 \boxtimes .\mathsf{C}_2 \to \mathsf{Mod}(B_1 \otimes B_2).$$

We wish to define a natural isomorphism between $\omega(X \otimes Y)$ and $\omega X \otimes_{B_1 \otimes B_2} \omega Y$. It suffices to do this with $X = X_1 \boxtimes X_2$ and $Y = Y_1 \boxtimes Y_2$. In this case, we have

$$\begin{aligned}
\omega((X_1 \boxtimes X_2) \otimes (Y_1 \boxtimes Y_2)) &\simeq \omega((X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2)) \\
&\simeq \omega_1(X_1 \otimes Y_1) \boxtimes \omega_2 (X_2 \otimes Y_2) \\
&\simeq \left(\omega_1 X_1 \otimes_{B_1} \omega_1 Y_1\right) \boxtimes \left(\omega_2 X_2 \otimes_{B_2} \omega_2 Y_2\right) \\
&\simeq (\omega_1 X_1 \otimes \omega_2 X_2) \otimes_{B_1 \otimes B_2} (\omega_1 Y_1 \otimes \omega_2 Y_2) \\
&\simeq \omega\,(X_1 \boxtimes X_2) \otimes_{B_1 \otimes B_2} \omega\,(Y_1 \boxtimes Y_2)\,.
\end{aligned}$$

We have shown that $\mathsf{C}_1 \boxtimes \mathsf{C}_2$ has a fibre functor with values in the nonzero commutative k-algebra $B_1 \otimes B_2$. □

ASIDE 12.19 Over nonperfect fields, it seems not to be known whether tensor products of tensor categories always exist. For tannakian categories, this is is proved in Deligne 1990, 5.18. It is also an immediate consequence of **??**, Theorem 1.1, whose proof, however, uses Corollary 12.18; see **??**, §5.

EXAMPLE 12.20 Let T be a tensor category over k, and let k' be a finite extension of k. For a finite-dimensional k'-vector space V and object X of T, we define $V \otimes X$ to represent the functor

$$\mathsf{T} \rightsquigarrow V \otimes \mathrm{Hom}(T, X)$$

(cf. 7.10). Then $V \otimes X$ has a natural action of k', and so lies in $\mathsf{T}_{(k')}$. The functor

$$\mathsf{Vecf}(k') \times \mathsf{T} \to \mathsf{T}_{(k')}$$

induces a functor

$$\mathsf{Vecf}(k') \boxtimes \mathsf{T} \to \mathsf{T}_{(k')},$$

which is an equivalence of tensor categories over k'.

13 Complements

In this section, we extend some earlier results. Throughout, k is a field.

Abelian $\otimes$-categories with internal Homs

This subsection collects some results, mostly standard, about closed abelian $\otimes$-categories.

Let $(\mathsf{C}, \otimes)$ be an abelian $\otimes$-category with internal Homs (see §4). Recall (4.1) that this means that, for all objects X, the functor $- \otimes X$ has a right adjoint $\mathcal{H}om(X, -)$, and so $- \otimes X$ is right exact and $\mathcal{H}om(X, -)$ is left exact. Let $X^\vee = \mathcal{H}om(X, \mathbb{1})$ (weak dual).

PROPOSITION 13.1 *If $X \to Y \to Z \to 0$ is exact, then $0 \to Z^\vee \to Y^\vee \to X^\vee$ is exact.*

PROOF For all objects T,

$$T \otimes X \to T \otimes Y \to T \otimes Z \to 0$$

is exact, so

$$0 \to \mathrm{Hom}(T \otimes Z, \mathbb{1}) \to \mathrm{Hom}(T \otimes Y, \mathbb{1}) \to \mathrm{Hom}(T \otimes X, \mathbb{1})$$

is exact. By adjointness (10), it follows that

$$0 \to \mathrm{Hom}(T, Z^\vee) \to \mathrm{Hom}(T, Y^\vee) \to \mathrm{Hom}(T, X^\vee)$$

is exact (all T), and so

$$0 \to Z^\vee \to Y^\vee \to X^\vee.$$

□

DEFINITION 13.2 An object X of C is ***flat*** if the functor $X \otimes -$ is left exact (hence exact).

DEFINITION 13.3 An object X of C is ***rigid*** if $(X^\vee, \mathrm{ev}_{X,\mathbb{1}})$ is dual to X, i.e., if there exists a morphism $\mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee$ satisfying (20).

If X is rigid, then $X^\vee$ is its (strong) dual, which is also rigid (the condition is essentially symmetric),[8] and $X \simeq X^{\vee\vee}$.

EXAMPLE 13.4 In the category $\mathsf{Rep}(G)$ of representations of an affine group scheme G on k-vector spaces, the rigid objects are those in $\mathsf{Repf}(G)$. Similarly, if T is a tensor category over k, then the rigid objects in $\mathrm{Ind}\,\mathsf{T}$ are those in T.

PROPOSITION 13.5 *If X and Y are nonzero objects of C and X is rigid, then $X \otimes Y$ is nonzero.*

PROOF As X is rigid, the identities (20), p. 16, imply that the composite

$$Y \to X^\vee \otimes X \otimes Y \to Y$$

is the identity. Therefore, Y is a direct summand of $X^\vee \otimes X \otimes Y$. If $X \otimes Y = 0$, then $X^\vee \otimes X \otimes Y = 0$, contradicting $Y \neq 0$. □

TODO 5 Check this proof, or assume that $\mathbb{1}$ is simple (6.4), this follows from 6.5.

PROPOSITION 13.6 *Every rigid object X of C is flat.*

PROOF Let $0 \to A \to B \to C$ be exact. As X is rigid, the functors $- \otimes X$ and $- \otimes X^\vee$ are adjoint (4.8), and so, for all objects T, the sequence

$$0 \to \mathrm{Hom}(T, A \otimes X) \to \mathrm{Hom}(T, B \otimes X) \to \mathrm{Hom}(T, C \otimes X)$$

is isomorphic to the sequence

$$0 \to \mathrm{Hom}(T \otimes X^\vee, A) \to \mathrm{Hom}(T \otimes X^\vee, B) \to \mathrm{Hom}(T \otimes X^\vee, C),$$

which is obviously exact. Therefore $0 \to A \otimes X \to B \otimes X \to C \otimes X$ is exact. □

PROPOSITION 13.7 *The tensor product of a projective object with a rigid object is again projective. In particular, if the unit object is projective, then all rigid objects are projective.*

[8] Without the symmetry, if one is the left dual, the other is the right dual.

PROOF Let X be rigid and P projective. As X is rigid, $\mathrm{Hom}(P \otimes X, T) \simeq \mathrm{Hom}(P, T \otimes X^\vee)$. Let $B \to C$ be an epimorphism, and consider a morphism $P \otimes X \to C$,

$$\begin{tikzcd} & P \otimes X \arrow[d] \\ B \arrow[r, "\text{epi}"] & C. \end{tikzcd}$$

On using the adjointness, we get a diagram

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed] \\ B \otimes X^\vee \arrow[r, "\text{epi}"] & C \otimes X^\vee. \end{tikzcd}$$

As P is projective, there is a morphism $P \to B \otimes X^\vee$ making the diagram commute. By adjointness again, this corresponds to a morphism $X \to B$ making the top diagram commute. □

PROPOSITION 13.8 *The tensor product of an injective object with a rigid object is again injective.*

PROOF The proof is similar to that of Proposition 13.7, using that, for a rigid object X, $\mathrm{Hom}(T, I \otimes X) \simeq \mathrm{Hom}(T \otimes X^\vee, I)$. □

EXAMPLE 13.9 The unit object is projective if every epimorphism $X \to \mathbb{1}$ splits. In $\mathsf{Mod}(R)$, the unit object is R, which is projective. For $\mathsf{Repf}(G)$, $\mathbb{1}$ projective means that the functor $V \rightsquigarrow V^G$ is exact. In general, it means that the functor $A \rightsquigarrow \mathrm{Hom}(\mathbb{1}, A)$ is exact.

PROPOSITION 13.10 *Assume that every object of* C *is a quotient of a flat object, and let* Z *be a flat object of* C. *If*

$$0 \to X \to Y \to Z \to 0,$$

is exact, then

$$0 \to X \otimes T \to Y \otimes T \to Z \otimes T \to 0$$

is exact for all objects T.

PROOF Let T be an object of C. By assumption, there exists an exact sequence

$$0 \to R \to P \to T \to 0$$

with P flat. There then exists an exact commutative diagram

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & X \otimes R \arrow[r] \arrow[d] & Y \otimes R \arrow[r] \arrow[d] & Z \otimes R \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \otimes P \arrow[r] \arrow[d] & Y \otimes P \arrow[r] \arrow[d] & Z \otimes P \arrow[d] & \\ & X \otimes T \arrow[r] \arrow[d] & Y \otimes T \arrow[r] \arrow[d] & Z \otimes T \arrow[d] & \\ & 0 & 0 & 0 & \end{tikzcd}$$

The zeros at right and bottom come from the right exactness of the tensor product, and the zeros at left and top come from the flatness of P and Z. Now the snake lemma shows that the bottom row is exact with a zero at left. □

REMARK 13.11 In an abelian category with enough projectives, a sequence $B \to C \to 0$ is exact if and only if $\mathrm{Hom}(P,B) \to \mathrm{Hom}(P,C) \to 0$ is exact for all projectives P.

Certainly, if P is projective, then

$$B \to C \to 0 \quad \text{exact} \Rightarrow \mathrm{Hom}(P,B) \to \mathrm{Hom}(P,C) \to 0 \quad \text{exact.}$$

For the converse, the sequence $B \to C \to C/B \to 0$ is exact, so

$$\mathrm{Hom}(P,B) \to \mathrm{Hom}(P,C) \to \mathrm{Hom}(P,C/B) \to 0$$

is exact for all projectives P. If the first map is surjective, then $\mathrm{Hom}(P,C/B) = 0$, but, by assumption, there exists an epimorphism $P \to C/B$ with P projective, which is zero only if C/B is zero.

PROPOSITION 13.12 *Assume that every object is a quotient of a flat object. Let* $0 \to A \to B \to C \to 0$ *be exact with C flat. For all objects I, the sequence*

$$0 \to \mathcal{H}om(C,I) \to \mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$$

is left exact. If I is injective and there are enough projectives, then the sequence is also right exact.

PROOF The hypothese imply that the sequence

$$0 \to T \otimes A \to T \otimes B \to T \otimes C \to 0$$

is exact for all T. Therefore

$$0 \to \mathrm{Hom}(T \otimes C, I) \to \mathrm{Hom}(T \otimes B, I) \to \mathrm{Hom}(T \otimes A, I) \to 0$$

is left exact, and right exact if I is injective. This sequence is isomorphic to the sequence

$$0 \to \mathrm{Hom}(T, \mathcal{H}om(C,I)) \to \mathrm{Hom}(T, \mathcal{H}om(B,I)) \to \mathrm{Hom}(T, \mathcal{H}om(A,I)) \to 0.$$

The first assertion follows. Now assume that I is injective. Then

$$\mathrm{Hom}(P, \mathcal{H}om(B,I)) \to \mathrm{Hom}(P, \mathcal{H}om(A,I)) \to 0$$

is exact for all projective P, which, by the remark, implies that $\mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$ is exact. □

PROPOSITION 13.13 *Assume that every object is a quotient of a flat object, and that there are enough projectives. If* $0 \to A \to B \to C \to 0$ *is exact and C is rigid, then*

$$0 \to \mathcal{H}om(C,I) \to \mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$$

is exact for all objects I.

PROOF Under the hypotheses, projective objects are flat because they are direct summands of flat objects. It suffices to prove that

$$0 \to \mathrm{Hom}(P, \mathcal{H}om(C, I)) \to \mathrm{Hom}(P, \mathcal{H}om(B, I)) \to \mathrm{Hom}(P, \mathcal{H}om(A, I)) \to 0$$

is exact with P projective (by the remark). By adjunction (10), this is isomorphic to the sequence

$$0 \to \mathrm{Hom}(P \otimes C, I) \to \mathrm{Hom}(P \otimes B, I) \to \mathrm{Hom}(P \otimes A, I) \to 0.$$

As P is flat,

$$0 \to P \otimes A \to P \otimes B \to P \otimes C \to 0$$

is exact. From this, we get an exact sequence

$$0 \to \mathrm{Hom}(P \otimes C, I) \to \mathrm{Hom}(P \otimes B, I) \to \mathrm{Hom}(P \otimes A, I) \to \mathrm{Ext}^1(P \otimes C, I).$$

Note that $\mathrm{Hom}(P \otimes C, -) \simeq \mathrm{Hom}(P, - \otimes C^\vee)$. As $C^\vee$ is flat, the second functor is exact, so $P \otimes C$ is projective. Hence $\mathrm{Ext}^1(P \otimes C, -) = 0$. □

PROPOSITION 13.14 *Assume that every object is a quotient of a flat object. Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence whose (weak) dual is exact. If A and C are rigid, then B is rigid.

PROOF Recall (4.10) that an object X is rigid if and only if the canonical morphism

$$T \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T)$$

is bijective for all T. If A is rigid, so also is $A^\vee$, hence it is flat (13.6), so, for all T, the top row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & T \otimes C^\vee \arrow[r] \arrow[d, "\simeq"] & T \otimes B^\vee \arrow[r] \arrow[d] & T \otimes A^\vee \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathcal{H}om(C, T) \arrow[r] & \mathcal{H}om(B.T) \arrow[r] & \mathcal{H}om(A, T) &
\end{tikzcd}$$

is exact (13.10). Now the five-lemma shows that the middle vertical arrow is bijective. □

REMARK 13.15 (a) In the situation of Proposition 13.14, the dual sequence is exact if the category has enough projectives (13.13).

(b) Not all duals of exact sequences are exact. For example, the dual of the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ in $\mathsf{Mod}(\mathbb{Z})$ is $0 \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$ (but in this example, C is not rigid).

(c) Let $0 \to A \to B \to C \to 0$ be exact. If A and C are rigid and C is projective, then $B \approx A \oplus C$, so is rigid.

PROPOSITION 13.16 *Assume that C has enough injectives and that $\mathcal{H}om(-, I)$ is right exact when I is injective. Then every projective object P of C is flat.*

PROOF Let $0 \to A \to B \to C$ be exact. By assumption, for any injective object I, $\mathcal{H}om(C,I) \to \mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$ is exact. As P is projective,

$$\mathrm{Hom}(P, \mathcal{H}om(C,I)) \to \mathrm{Hom}(P, \mathcal{H}om(B,I)) \to \mathrm{Hom}(P, \mathcal{H}om(A,I)) \to 0$$

is exact. By adjunction (10), this implies that

$$\mathrm{Hom}(P \otimes C, I) \to \mathrm{Hom}(P \otimes B, I) \to \mathrm{Hom}(P \otimes A, I) \to 0$$

is exact. Hence, the kernel K of $P \otimes A \to P \otimes B$ has the property that $\mathrm{Hom}(K,I) = 0$ for all injectives. As C has enough injectives, this implies that $K = 0$. □

NOTES I thank Kevin Coulembier for providing me with Proposition 13.13.

The category of ind-objects of a tensor category

Let T be an essentially small tensor category over the field k.

13.17 The category $\mathrm{Ind}\,\mathsf{T}$ is a k-linear abelian category and the canonical functor $\iota_{\mathsf{T}} \colon \mathsf{T} \to \mathrm{Ind}\,\mathsf{T}$ is fully faithful and exact. The finitely presentable objects of $\mathrm{Ind}\,\mathsf{T}$ are exactly the objects in the essential image of ι_T.

Any set of representatives for the isomorphism clases of T generates $\mathrm{Ind}\,\mathsf{T}$, and so T is a Grothendieck category. In particular, $\mathrm{Ind}\,\mathsf{T}$ admits small inductive limits, which are exact when filtered, and it has enough injectives.

The objects of T are noetherian, and so $\mathrm{Ind}\,\mathsf{T}$ is locally noetherian. An object of $\mathrm{Ind}\,\mathsf{T}$ is noetherian if and only if it is finitely presented.

This summarizes part of Appendix B.

13.18 For $X = \varinjlim_i X_i$ and $Y = \varinjlim_j Y_j$, we set

$$X \otimes Y = \varinjlim_{i,j} X_i \otimes Y_j.$$

With this tensor functor, $\mathrm{Ind}\,\mathsf{T}$ becomes a k-linear $\otimes$-category, and ι_{T} becomes a $\otimes$-functor. The functor $\otimes$ commutes with inductive limits in each variable, and so internal Homs exist in $\mathrm{Ind}\,\mathsf{T}$.

13.19 The functor $\otimes \colon \mathrm{Ind}\,\mathsf{T} \otimes \mathrm{Ind}\,\mathsf{T} \to \mathrm{Ind}\,\mathsf{T}$ is exact in each variable. To see this, let X be an object of $\mathrm{Ind}\,\mathsf{T}$ and ξ a short exact sequence in $\mathrm{Ind}\,\mathsf{T}$. Write X as a filtered inductive limit $X = \varinjlim X_i$ of objects of T. As X_i is rigid in T, it is rigid in $\mathrm{Ind}\,\mathsf{T}$, and hence flat (13.6). Therefore, each sequence $X_i \otimes \xi$ is exact, and so $X \otimes \xi \simeq \varinjlim(X_i \otimes \xi)$ is exact.

13.20 Let I be an injective object of $\mathrm{Ind}\,\mathsf{T}$ and X an object of T. For all T,

$$\mathrm{Hom}(T, I \otimes \mathcal{H}om(X, \mathbb{1})) \overset{(4.10)}{\simeq} \mathrm{Hom}(T, \mathcal{H}om(X, I)) \overset{(10)}{\simeq} \mathrm{Hom}(T \otimes X, I).$$

This last functor is exact in T, and so $I \otimes \mathcal{H}om(X, \mathbb{1})$ is injective. On applying this remark to the dual of an object X of T, we see that for all injective objects I of $\mathrm{Ind}\,\mathsf{T}$, $I \otimes X$ is injective.

Extension of scalars for tensor categories

In this section, we prove Theorem 8.15 for an arbitrary field extension of k.

13.21 Let k be a commutative ring and C a k-linear category. For a k-algebra R, we define $\mathsf{Mod}(R,\mathsf{C})$ to be the category of R-modules in C, i.e., the category of pairs (X,μ) with X an object of C and μ a homomorphism $R \to \mathrm{End}(X)$ of k-algebras. Then $\mathsf{Mod}(R,\mathsf{C})$ is an R-linear category, and the forgetful functor $\mathsf{Mod}(R,\mathsf{C}) \to \mathsf{C}$ is k-linear and faithful. If C is abelian, then $\mathsf{Mod}(R,\mathsf{C})$ is abelian, and the forgetful functor is exact. If C admits small inductive limits, then, for all X in $\mathsf{Mod}(R,\mathsf{C})$ the functor

$$Y \rightsquigarrow \mathrm{Hom}_{\mathsf{C}}(X,Y)\colon \mathsf{C} \to \mathsf{Mod}(R)$$

has a left adjoint

$$M \rightsquigarrow M \otimes_R X\colon \mathsf{Mod}(R) \to \mathsf{C},$$

so

$$\mathrm{Hom}_{\mathsf{C}}(M \otimes_R X, Y) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathsf{C}}(X,Y)) \qquad M \in \mathsf{Mod}(R), \quad X \in \mathsf{Mod}(R,\mathsf{C}), \quad Y \in \mathsf{C}.$$

See Kashiwara and Schapira 2006, 8.5.1, 8.5.2, 8.5.3.

Now let k be a field and C an essentially small tensor category over k. Then, as we saw in the last subsection, $\mathsf{D} \stackrel{\text{def}}{=} \mathrm{Ind}\,\mathsf{C}$ is a k-linear abelian $\otimes$-category with internal Homs. The canonical functor $\mathsf{C} \to \mathsf{D}$ is fully faithful with essential image the subcategory of finitely presentable objects (equivalently, noetherian objects).

For X in D and V a k-vector space, the functor $Y \rightsquigarrow \mathrm{Hom}(V, \mathrm{Hom}(X,Y))$ is representable, and we let $V \otimes_k X$ denote the object of D representing it, so

$$\mathrm{Hom}_{\mathsf{D}}(V \otimes_k X, Y) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_{\mathsf{D}}(X,Y)). \tag{*}$$

If $(e_i)_{i\in I}$ is a basis for V (possibly infinite), then we can take $V \otimes_k X$ to be a direct sum of copies of X indexed by I.

Let k' be a field extension of k, and let $\mathsf{D}' = \mathsf{Mod}(k', \mathsf{D})$ be the category of k'-modules in D. Then D' is a k'-linear abelian $\otimes$-category, and the forgetful functor $\mathsf{D}' \to \mathsf{D}$ is k-linear, faithful, and exact.

PROPOSITION 13.22 *The forgetful functor* $\mathsf{D}' \to \mathsf{D}$ *has a left adjoint* $X \rightsquigarrow k' \otimes X$, *so*

$$\mathrm{Hom}_{\mathsf{D}'}(k' \otimes X, Y) \simeq \mathrm{Hom}_{\mathsf{D}}(X,Y).$$

PROOF The vector space $k' \otimes_k X$ defined above has a natural structure of a k'-module, and so lies in D'. The isomorphism (*) induces the required isomorphism. □

Let C' be the full subcategory of finitely presentable objects of D'. Then C' is stable under finite sums and subquotients in D'.

PROPOSITION 13.23 *The functor* "$\varinjlim X_\alpha$" $\rightsquigarrow \varinjlim X_\alpha\colon$ $\mathrm{Ind}\,\mathsf{C}' \to \mathsf{D}'$ *is an equivalence of categories with quasi-inverse the functor sending an object of* D' *to the inductive system of its noetherian subobjects.*

PROOF All objects of C are noetherian, small filtered inductive limits exist in D and are exact, and every object of D' is a small filtered inductive limit of objects in C'. Therefore the statement follows from Proposition B.10 of the Appendix. □

Let $\mathsf{C}_{(k')}$ be the full subcategory of D' whose objects X are those generated as a k'-module by a subobject in C, i.e., such that there exist a $Y \subset X$, $Y \in \operatorname{ob}\mathsf{T}$, for which the composite

$$k' \otimes_k Y \to k' \otimes_k X \to X$$

is an epimorphism.

PROPOSITION 13.24 *The forgetful functor* $\mathsf{D}' \to \mathsf{D}$ *induces an equivalence of categories* $\mathsf{C}' \to \mathsf{C}_{(k')}$.

PROOF If $X \in \operatorname{ob}\mathsf{D}'$ is generated as a k'-module by a D-subobject $Y \subset X$ with $Y \in \operatorname{ob}\mathsf{C}$, then it is a quotient of $k' \otimes_k Y$ and is therefore in C'. Conversely, if $X \in \operatorname{ob}\mathsf{C}'$, we write it in D as a filtered union of its subobjects X_α in C. Since X is noetherian in C', the system of submodule images of the $k' \otimes_k X_\alpha$ in X is stationary. Therefore X is a quotient of $k' \otimes_k X_\alpha$ for α sufficiently large, and so lies in $\mathsf{C}_{(k')}$. □

ASIDE 13.25 The pairing

$$(V, X) \rightsquigarrow V \otimes_k X : \mathsf{Vecf}(k') \times \mathsf{C} \to \mathsf{C}'$$

is universal, and induces an isomorphism from the Kelly tensor product (12.14) to C',

$$V \boxtimes \mathsf{C} \to \mathsf{C}'.$$

Now let T be a tensor category over k. Then $\operatorname{Ind}\mathsf{T}$ is a k-linear abelian $\otimes$-category with internal Homs.

PROPOSITION 13.26 *Every object of* $\mathsf{T}_{(k')}$ *is a quotient of a rigid object.*

PROOF For any object Y of T, $k' \otimes Y$ is rigid as an object of $\mathsf{T}_{(k')}$. □

PROPOSITION 13.27 *Every simple object X of* $\mathsf{T}_{(k')}$ *is rigid.*

PROOF Let Y generate X. As X is simple, any simple subobject of Y also generates X. But, if Y is simple in T, then $k' \otimes Y$ is simple in $\mathsf{T}_{(k')}$, and so the epimorphism $k' \otimes Y \to X$ is an isomorphism. □

THEOREM 13.28 *Let* T *be a tannakian category over a field k. For any field k' containing k,* $\mathsf{T}_{(k')}$ *is a tannakian category over k'.*

PROOF Recall (8.7) that T has a fibre functor ω with values in a field K. In proving the theorem, we may suppose that k' is finitely generated over k, and then that k' is either a finite extension of k or equal to $k(t)$. We proved the first case in 8.15. For the second case, note that $\mathsf{T}_{(k')}$ has a fibre functor with values in $k(t) \otimes_k K$ and that

$$k(t) \otimes K = \bigcup_f k[t]_f \otimes K \simeq \bigcup_f K[t]_f,$$

where f runs over the nonzero polynomials in $k[t]$. Therefore, the theorem follows Lemma 8.20 below applied to $\mathsf{T}_{(k')}$. □

LEMMA 13.29 *Let R be an integral domain and T a nonzero torsion module. Then* $\bigwedge^{n+1}(R^{\oplus n} \oplus T)$ *is a nonzero torsion module.*

PROOF See Schäppi 2018, 3.2.3. □

LEMMA 13.30 (SCHÄPPI 2018, 3.2.4) *Let* T *be a* k*-linear abelian* $\otimes$*-category such that* $k \simeq \mathrm{End}(\mathbb{1})$*. Assume that every object of* T *is a quotient of a rigid object. If there exists an exact faithful* $\otimes$*-functor* $\omega\colon \mathsf{T} \to \mathsf{Mod}(R)$*, where* R *is a filtered union of principal ideal domains, then* T *is tannakian.*

PROOF We have to proof that T is rigid. Because ω is exact and faithful, it reflects isomorphisms. Therefore it suffices to show that, for all X, $\omega(X)$ is a finitely generated projective R-module (cf. the proof of 8.18).

As X is the quotient of rigid object, $\omega(X)$ is a finitely presented R-module. The assumption on R therefore implies that it is a finite direct sum of cyclic modules. Thus, $\omega(X) = F_n \oplus T$, where F_n is free of rank n and T is torsion. It remains to show that $T = 0$.

If not

$$\omega(\textstyle\bigwedge^{n+1} X) = \bigwedge^{n+1}(F_n \oplus T),$$

is a nonzero torsion module (Lemma 13.29). We may replace X with $\bigwedge^{n+1} X$, and so suppose that $\omega(X)$ is a nonzero torsion module.

By assumption, there exists an epimorphism $p\colon V \to X$ with V rigid. Let V' denote its kernel in $\mathrm{Ind}\,\mathsf{T}$. From the assumption on the ring R, we see that $\omega(p)\colon \omega(V) \to \omega(X)$ arises from a homomorphism over a principal ideal subdomain of R. As R is flat over the subdomain, the kernel of $\omega(p)$ also arises from the subdomain, and hence is finitely generated and free. As V' is the kernel of the map $(\mathrm{Ind}\,\omega)(p)$, it follows that V' is rigid, hence finitely presented, and so lies in T. On passing to the field of fractions of R, we see that $\omega(V)$ and $\omega(V')$ are free of the same rank.

It remains to show that $\varphi\colon V' \to V$ is an isomorphism, and for this it suffices to show that $\omega(\varphi)$ is an isomorphism. As it is a map between free R-modules of the same rank, n say, it suffices to check that the determinant

$$E^n(\omega(\varphi)) \simeq \textstyle\bigwedge^n(\omega(\varphi)) \simeq \omega(\bigwedge^n(\varphi))$$

is invertible (E^n denotes the exterior power). As $\omega(V')$ and $\omega(V)$ are flat, $\varphi^{\otimes n}\colon (V')^{\otimes n} \to V^{\otimes n}$ is a monomorphism. Thus $\bigwedge^n \varphi$ is a monomorphism between two ind-objects whose images in $\mathsf{Mod}(R)$ are free of rank one. But any such ind-object is invertible (in the sense of §2) in $\mathrm{Ind}\,\mathsf{T}$ (so lies in T). Moreover, as the unit object of T is simple, all invertible objects of T are simple. Thus $\bigwedge^n \varphi$ is indeed an isomorphism. This shows that $\omega(X) = 0$, giving a contradiction. □

COROLLARY 13.31 *Let* T *be a tannakian category over* k*, and let* K *be a field containing* k*. If* T *has a* K*-valued fibre functor, then* $\mathsf{T}_{(K)}$ *is a neutral tannakian category over* K*.*

PROOF Let ω be a K-valued fibre functor on T. Then $(X,\mu) \rightsquigarrow \omega(X)$ is a $K \otimes_k K$-valued fibre functor ω' on $\mathsf{T}_{(K)}$ (cf. the proof of 8.16). The multiplication map $K \times K \to K$ defines a k-algebra homomorphism $K \otimes_k K \to K$. Now $\omega' \otimes_{K\otimes_k K} K$ is a K-valued fibre functor on $\mathsf{T}_{(K)}$. □

NOTES

13.32 Corollary 13.31, when combined with the results of the next chapter, shows that any two fibre functors on a tannakian category are locally isomorphic for the fpqc topology (II, 8.8).

13.33 Deligne (1990) proved Theorem 13.28 as a consequence of his description of tannakian categories in terms of affine groupoid schemes (see IV, 9.1). The elementary proof given above is from Schäppi 2018.

13.34 Here is a third proof of Theorem 13.28.

(a) Case k' is algebraic over k. This case follows from Theorem 8.15 (cf. 8.19).

(b) Case k is algebraically closed. In this case, it is known that, if T is a tensor category over k, then $\mathsf{T}_{(k')}$ is a tensor category (see the next note).

(c) General case. Let $\bar{k}$ be an algebraic closure of k and K a field containing both k' and $\bar{k}$. Then $\mathsf{T}_{(K)} \sim \left(\mathsf{T}_{(\bar{k})}\right)_{(K)}$ so it is tannakian by (a) and (b), and it follows that $\mathsf{T}_{(k')}$ is tannakian (because $\left(\mathsf{T}_{(k')}\right)_{(K)} \approx \mathsf{T}_{(K)}$ is tannakian).

13.35 One can ask whether Theorem 13.28 holds for tensor categories: does a tensor category T over k remain a tensor category under extension of the base field?

When k is algebraically closed, Etingof and Ostrik (2021, 2.2)[9] suggest the following argument: as simple objects in $\mathsf{T}_{(k')}$ are rigid and the objects of $\mathsf{T}_{(k')}$ are noetherian, it suffices to prove that an extension of rigid objects in $\mathsf{T}_{(k')}$ is again rigid. The author has not been able to find a proof of this last statement.

In a future paper, Coulembier and Sherman plan to include a proof that a tensor category over an algebraically closed field remains a tensor category under extension of the base field.

Faithful $\otimes$-functors of tensor categories are exact

We prove that Corollaries 11.8 and 11.9 hold over all fields.

PROPOSITION 13.36 *Let $F : \mathsf{C} \to \mathsf{D}$ be a k-linear $\otimes$-functor of tensor categories over k. If F is faithful, then it is exact.*

PROPOSITION 13.37 *Let T be a tensor category over k and R a nonzero k-algebra. Every faithful k-linear $\otimes$-functor $\omega : \mathsf{T} \to \mathsf{Mod}(R)$ is exact (hence a fibre functor).*

Therefore, in both cases, F is faithful if and only if it is exact (6.7).

FIRST PROOF OF PROPOSITION 13.36

Let

$$\xi : 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

be an exact sequence in C, and let I be a nonzero injective object of $\operatorname{Ind}\mathsf{T}$, which exists because $\operatorname{Ind}\mathsf{T}$ is a Grothendieck category (13.17). Then

$$\xi \otimes I : 0 \to A \otimes I \to B \otimes I \to C \otimes I \to 0$$

is exact (13.19), and even split-exact because $A \otimes I$ is injective (13.8). Therefore $F(\xi \otimes I)$ is exact, and hence also the isomorphic sequence

$$F(\xi) \otimes F(I) : 0 \to F(A) \otimes F(I) \to F(B) \otimes F(I) \to F(C) \otimes F(I) \to 0.$$

[9] See also Coulembier et al. 2023, 5.1.3(2).

Let H be some homology object of the complex

$$F(\xi)\colon 0 \to F(A) \to F(B) \to F(C) \to 0.$$

Because $-\otimes F(I)$ is exact (13.19), $H \otimes F(I)$ is the corresponding homology object of the complex $F(\xi)\otimes F(I)$, which is zero. As H is rigid, this implies that $H = 0$ (by 6.5).

PROOF OF PROPOSITIONS 13.36 AND 13.37

The proofs of the propositions are based on the following lemma.

LEMMA 13.38 *Let* T *be a tensor category over* k*. A sequence*

$$\xi\colon 0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$$

is exact in T *if and only if there exists a nonzero morphism* $u\colon \mathbb{1} \to U$ *such that the map of chain complexes*

$$\begin{tikzcd}
\xi \arrow[d, "u\otimes\xi"] & 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
U\otimes\xi & 0 \arrow[r] & U\otimes X \arrow[r] & U\otimes Y \arrow[r] & U\otimes Z \arrow[r] & 0
\end{tikzcd}$$

is null homotopic.

PROOF For the sufficiency, let $u\colon \mathbb{1} \to U$ be a nonzero morphism such that $u\otimes\xi$ is null homotopic. Define the ring A_u in $\operatorname{Ind}\mathsf{T}$ to be the ind-object

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "u"] & U \arrow[r, shift left, "u\otimes U"] \arrow[r, shift right, "U\otimes u"'] & U\otimes U \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U\otimes U\otimes U \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots
\end{tikzcd}$$

(indexed by a category whose objects are the natural numbers). Let I be a nonzero injective object in $\operatorname{Ind}\mathsf{T}$ (exists by 13.17). Then $I\otimes u$ is a monomorphism of injective objects (13.18), so there exists a morphism $h\colon I\otimes U \to I$ such that $h\circ(I\otimes u) = \mathrm{id}_I$. The morphism h can be extended to a morphism $f\colon I\otimes A_u \to I$,

$$\begin{tikzcd}
I \arrow[r, "I\otimes u"] \arrow[d, equal] & I\otimes U \arrow[r, shift left, "I\otimes u\otimes U"] \arrow[r, shift right, "I\otimes U\otimes u"'] \arrow[dl, "h"] & I\otimes U\otimes U \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[dll, bend left, "h\circ(h\otimes U)"] & I\otimes U\otimes U\otimes U \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[dlll, bend left, "h\circ(h\otimes U)\circ(h\otimes U\otimes U)"] & \cdots, \\
I & & & &
\end{tikzcd}$$

such that $f\circ(I\otimes u) = \mathrm{id}_I$. Therefore $A_u \neq 0$

Let $v\colon \mathbb{1} \to A_u$ be the composite $I \xrightarrow{u} U \to A_u$. As $u\otimes\xi$ is null homotopic, the map of chain complexes $v\otimes\xi\colon \xi \to A_u\otimes\xi$ is also null homotopic, which implies that the identity map $A_u\otimes\xi \to A_u\otimes\xi$ is null homotopic. This last statement means that $A_u\otimes\xi$ is split-exact in $\mathsf{Mod}(A_u)$. As the morphism $\mathbb{1} \to A_u$ is faithfully exact (10.8), this implies that ξ is exact.

Now assume that ξ is exact. Form the pushout diagram,

$$\begin{tikzcd}
X\otimes X^\vee \arrow[r, "a\otimes X^\vee"] \arrow[d, "\mathrm{ev}_{X^\vee}"] & Y\otimes X^\vee \arrow[d] \\
\mathbb{1} \arrow[r, "e"] & E.
\end{tikzcd}$$

On applying adjunction, we see that the left hand triangle in the diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "e\otimes X"'] & Y \arrow[dl] \arrow[d, "e\otimes Y"] \\ E\otimes X \arrow[r, "E\otimes a"] & E\otimes Y \end{tikzcd}$$

commutes. Hence the difference between the two morphisms $Y \to E \otimes Y$ restricts to zero when composed with a. As the sequence ξ is exact, this difference factors through $b\colon Y \to Z$. This gives the required homotopy. □

PROOF (13.36) Let ξ be a short exact sequence in C. By Lemma 13.38, there exists a nonzero morphism $u\colon \mathbb{1} \to U$ such that $u \otimes \xi$ is null homotopic, As F is faithful

$$\mathbb{1}_{\mathsf{D}} \simeq F(\mathbb{1}_{\mathsf{C}}) \xrightarrow{F(U)} F(U)$$

is nonzero, and it induces a null homotopic morphism $F(\xi) \to F(U) \otimes \xi$ because F preserves homotopies. By Lemma 13.38 again, $F(\xi)$ is exact. □

PROOF (13.37) Let ξ be a short exact sequence in T and let $u\colon \mathbb{1} \to U$ be a nonzero morphism such that $u \otimes \xi$ is null homotopic. Let A_u be as in the proof of Lemma 13.38. Then, as in the proof of the lemma, $A_u \neq 0$ and $A_u \otimes \xi$ is split-exact in $\mathsf{Mod}(A_u)$, so $F(A_u \otimes \xi)$ is exact (because F is additive). The R-module $F(U)$ faithfully flat (8.5). Therefore $F(A_u)$, being an inductive limit of faithfully flat R-modules, is also faithfully flat. As $F(A_u \otimes \xi) \simeq F(A_u) \otimes F(\xi)$, we see that $F(\xi)$ is exact. □

NOTES In characteristic zero, Proposition 13.36 is an immediate consequence of statements in Deligne 1990 (see 11.8). It is proved in general in Coulembier 2020. The above proofs follow Coulembier et al. 2023, 2.4.1, 2.4.2.

Abelian envelopes

In this section, all pseudo-tensor and tensor categories over k are essentially small.

13.39 When we relax the condition of being abelian to being pseudo-abelian, we arrive at the notion of a pseudo-tensor category. Thus, a k-linear $\otimes$-category D is a ***pseudo-tensor category over*** k if

(a) $k \simeq \mathrm{End}(\mathbb{1})$,

(b) D is rigid (all objects have duals),

(c) D is pseudo-abelian (additive and idempotent complete).

From a k-linear $\otimes$-category satisfying (a) and (b), we can obtain a pseudo-tensor category over k by passing to the pseudo-abelian envelope. In this subsection, we examine the possibility of obtaining a tensor category over k by passing to an abelian envelope.

Let D be a pseudo-tensor category over k. Roughly speaking, we say that a fully faithful $\otimes$-functor from D to a tensor category T over k is an abelian envelope of D if every faithful $\otimes$-functor from D to a $\otimes$-category T′ extends essentially uniquely to an exact $\otimes$-functor from T,

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[dr] & \mathsf{T} \arrow[d, dashed] \\ & \mathsf{T}' \end{tikzcd}$$

DEFINITION 13.40 (COULEMBIER ET AL. 2023, 2.1.1, 2.5.1) Let D be a pseudo-tensor category over k. A fully faithful k-linear $\otimes$-functor $F\colon \mathsf{D} \to \mathsf{T}$ from D to a tensor category T over k is an ***abelian envelope*** of D if, for every tensor category T' over k, composition with F defines an equivalence from

- the category of k-linear exact $\otimes$-functors $\mathsf{T} \to \mathsf{T}'$ to
- the category of k-linear faithful $\otimes$-functors $\mathsf{D} \to \mathsf{T}'$.

This definition makes sense because k-linear exact $\otimes$-functors of tensor categories are faithful (7.4).

By a ***monoidal category over*** k, we mean a k-linear monoidal category such that $k \simeq \mathrm{End}(\mathbb{1})$.

PROPOSITION 13.41 (COULEMBIER 2022, 4.4.4) *Consider a k-linear faithful monoidal functor $u\colon \mathsf{A} \to \mathsf{T}$, where A is a rigid monoidal category over k and T is an abelian rigid monoidal category over k. Assume that*

(a) *every object in T is a quotient of $u(X)$ for some object X in A, and*

(b) *for every morphism $a\colon u(X) \to u(Y)$ in T, there exists a morphism $q\colon X' \to X$ in A such that $u(q)$ is an epimorphism and $a \circ u(q)$ is in the image of u.*

Then, for every abelian rigid monoidal T' over k, u defines an equivalence from the category of k-linear exact tensor functors $\mathsf{T} \to \mathsf{T}'$ to the category of k-linear faithful $\otimes$-functors $\mathsf{A} \to \mathsf{T}'$.

COROLLARY 13.42 *Let D be a pseudo-tensor category over k and T a tensor category over k. A fully faithful $\otimes$-functor $F\colon \mathsf{D} \to \mathsf{T}$ is an **abelian envelope** of D if every object of T is a quotient of $F(X)$ for some X in D.*

PROOF Immediate consequence of the proposition. □

PROPOSITION 13.43 (COULEMBIER ET AL. 2023, THEOREM A) *A pseudo-abelian tensor category D over k admits an abelian envelope if and only if the following two conditions hold:*

(a) *every nonzero morphism $U \to \mathbb{1}$ in D, is the equalizer of the pair of arrows*

$$U \otimes U \rightrightarrows U.$$

(b) *for every arrow $X \to Y$ in D, there exists a nonzero M in $\mathrm{Ind}\,\mathsf{D}$ such that the morphism $M \otimes X \to M \otimes Y$ is split in $\mathrm{Ind}\,\mathsf{D}$.*

A morphism in an additive category is ***split*** if it is a direct sum of an isomorphism and a zero morphism. In a pseudo-abelian category, a morphism $f\colon X \to Y$ is split if and only if there exists a $g\colon Y \to X$ such that $f = f \circ g \circ f$. For example, an endomorphism f of X is split if and only if it is von Neumann regular in $\mathrm{End}(X)$.

EXAMPLE 13.44 Let T be a tensor category over k, and let K be a field containing k. Let $K \otimes \mathsf{T}$ be the category with the same objects as T but with Hom replaced by $K \otimes_k \mathrm{Hom}$. Its pseudo-abelian envelope T_K is a K-linear pseudo-abelian rigid $\otimes$-category with $K \simeq \mathrm{End}(\mathbb{1})$.

Now let T be a tannakian category over k, and let K be a field containing k. Then $\mathsf{T}_{(K)}$ is the abelian envelope of T_K. Moreover, $\mathrm{Ind}(\mathsf{T}_{(K)})$ is equivalent to $\mathrm{Ind}(\mathsf{T})_K$.

EXAMPLE 13.45 (COULEMBIER ET AL. 2023, 5.1.1, 5.1.2, 5.1.4) Let K be a field containing k and D a pseudo-tensor category over k. The naive extension of scalars $K \otimes \mathsf{D}$ is the category with the same objects as D but with Hom replace by $K \otimes_k$ Hom. Its pseudo-abelian envelope is a pseudo-tensor category, denoted $K\dot{\otimes}\mathsf{C}$. If the K-linear $\otimes$-category T_K is a tensor category over K, then it is the abelian envelope of $K\dot{\otimes}\mathsf{T}$; moreover, $\mathrm{Ind}(\mathsf{T}_{(K)}) \sim (\mathrm{Ind}\,\mathsf{T})_K$.

QUESTION 13.46 The category of motives $M_H(k)$ over k relative to a Weil cohomology theory H is a pseudo-abelian tensor category over k. If it admits an abelian envelope, then the Lefschetz standard conjecture holds over k. Is the converse true?

14 Some 2-categories

We show that monoidal categories form a (strict) 2-category $\mathscr{M}on\mathscr{C}at$. This allows us to deduce that other categories, for example, tannakian categories over a fixed field k also form a 2-category. Our convention concerning units in monoidal categories so simplifies the proofs that we can leave those concerning units to the reader.

The 2-category $\mathscr{M}on\mathscr{C}at$

- The objects of $\mathscr{M}on\mathscr{C}at$ are the small monoidal categories.
- The 1-morphisms of $\mathscr{M}on\mathscr{C}at$ are the monoidal functors.
- The 2-morphisms of $\mathscr{M}on\mathscr{C}at$ are the monoidal natural transformations.

Thus, a 1-morphism $\mathsf{C} \to \mathsf{D}$ is a functor $F\colon \mathsf{C} \to \mathsf{D}$ together with a natural isomorphism $c^F_{X,Y}\colon FX \otimes FY \to F(X \otimes Y)$ such that certain diagrams commute (3.1) and F maps unit objects to unit objects. A 2-morphism

$$\alpha\colon F \Rightarrow G$$

is a natural transformation $F \Rightarrow G$ such that certain diagrams commute (3.4).

Recall that $\mathscr{C}at$ is a 2-category (Appendix, A.9). The forgetful functor $\mathscr{M}on\mathscr{C}at \to \mathscr{C}at$ sends a monoidal category to its underlying category, a monoidal functor to its underlying functor, and a monoidal natural transformation to the natural transformation. The associativity and interchange laws follow from those in $\mathscr{C}at$ once we show that the various constructions preserve monoidal functors and monoidal natural transformations.

VERTICAL COMPOSITION

Given monoidal categories C, D, the Hom category

$$\mathsf{Hom}(\mathsf{C}, \mathsf{D})$$

is the category whose objects are the monoidal functors $F\colon \mathsf{C} \to \mathsf{D}$ and whose morphisms are the monoidal natural transformations. We check that this is indeed a category,

Given a monoidal functor F, the identity natural transformation $F \Rightarrow F$ is clearly monoidal. Given monoidal natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, we see from the commutative diagram

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "c^F"] \arrow[d, "\alpha_X\otimes\alpha_Y"] \arrow[dd, bend right=60, "(\beta\circ\alpha)_X\otimes(\beta\circ\alpha)_Y"'] & F(X\otimes Y) \arrow[d, "\alpha_{X\otimes Y}"] \arrow[dd, bend left=60, "(\beta\circ\alpha)_{X\otimes Y}"] \\
G(X)\otimes G(Y) \arrow[r, "c^G"] \arrow[d, "\beta_X\otimes\beta_Y"] & G(X\otimes Y) \arrow[d, "\beta_{X\otimes Y}"] \\
H(X)\otimes H(Y) \arrow[r, "c^H"] & H(X\otimes Y)
\end{tikzcd}$$

that $\beta\circ\alpha$ is monoidal. Therefore vertical composition is well-defined, and it is associative and unital because natural transformations have these properties.

Composition of 1-morphisms

Let $(F, c^F)\colon \mathsf{C} \to \mathsf{D}$ and $(G, c^G)\colon \mathsf{D} \to \mathsf{E}$ be monoidal functors. Their composite is the monoidal functor (3.3)

$$(G\circ F, c^{G\circ F})\colon \mathsf{C} \to \mathsf{E}.$$

Strict associativity of 1-morphism composition

Let

$$F\colon \mathsf{A} \to \mathsf{B}, \quad G\colon \mathsf{B} \to \mathsf{C}, \quad H\colon \mathsf{C} \to \mathsf{D}$$

be monoidal functors. Then

$$(H\circ G)\circ F = H\circ(G\circ F)$$

as functors, and $c^{(HG)F}_{X,Y}$ and $c^{H(GF)}_{X,Y}$ both equal

$$HG(c^F_{X,Y})\circ H(c^G_{F(X),F(Y)})\circ c^H_{GF(X),GF(Y)}.$$

Horizontal composition

Let F, G be monoidal functors $\mathsf{C} \to \mathsf{D}$ and H, I functors $\mathsf{D} \to \mathsf{E}$. The horizontal composite of monoidal natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon H \Rightarrow I$,

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E},
\end{tikzcd}$$

is the natural transformation $\beta * \alpha\colon H\circ F \to I\circ G$ such that $(\beta * \alpha)_X$ is the diagonal map in the commutative square

$$\begin{tikzcd}
(H\circ F)(X) \arrow[r, "\beta_{F(X)}"] \arrow[d, "H(\alpha_X)"] & (I\circ F)(X) \arrow[d, "I(\alpha_X)"] \\
(H\circ G)(X) \arrow[r, "\beta_{G(X)}"] & (I\circ G)(X).
\end{tikzcd}$$

Thus

$$I(\alpha_X)\circ\beta_{F(X)} \overset{\text{def}}{=} (\beta * \alpha)_X \overset{\text{def}}{=} \beta_{G(X)}\circ H(\alpha_X).$$

It remains to see that $\beta * \alpha$ is monoidal. The required compatibility diagram follows by pasting the monoidality squares for α and β together with the naturality square for the tensor constraint of I. Explicitly, one checks that

$$(\beta * \alpha)_{X\otimes Y}\circ c^{HF}_{X,Y} = c^{IG}_{X,Y}\circ((\beta * \alpha)_X \otimes (\beta * \alpha)_Y).$$

We omit the verification.

THE INTERCHANGE LAW

Since horizontal and vertical composition of monoidal natural transformations This follows directly from the interchange law in $\mathscr{C}\mathit{at}$.

Now that we have shown $\mathscr{M}\mathit{on}\mathscr{C}\mathit{at}$ to be a 2-category, it follows easily that the following are also 2-categories.

The 2-category of symmetric monoidal categories

- The objects are the small symmetric monoidal categories.
- The 1-morphisms are the symmetric monoidal functors.
- The 2-morphisms are the monoidal natural transformations.

If monoidal functors F and G are symmetric, so also is $G\circ F$.

Now fix a field k.

The 2-category of tensor categories over k.

- The objects are the small tensor categories over k.
- The 1-morphisms are the exact k-linear symmetric monoidal functors.
- The 2-morphisms are the k-linear monoidal natural transformations.

If symmetric monoidal functors F and G are k-linear and exact, so also is $G\circ F$.

The 2-category of tannakian categories over k, $\mathscr{T}\mathit{ann}_k$.

- The objects are the small tannakian categories over k.
- The 1-morphisms are the exact k-linear symmetric monoidal functors.
- The 2-morphisms are the k-linear monoidal natural transformations.

NOTES That monoidal categories form a (strict) 2-category is well-known to the experts, but I was unable to find a published proof.

TODO 6 In the absence of an available proof that the monoidal categories form a 2-category, I should probably include all details. Since none of the constructions seem to involve α or γ, if the monoidal categories form a bicategory, they surely form a 2-category, and if they don't form a bicategory, there's probably something wrong with the definition.

Chapter II

Neutral tannakian categories

In this chapter, we describe neutral tannakian categories and their fibre functors in terms of affine group schemes and their torsors.

Throughout, k is a field except when indicated otherwise. Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$.

1 Affine group schemes and their representations

We review the basic theory of affine group schemes and their representations. For more details, see, for example, Milne 2017 or Waterhouse 1979. After 1.10, all bialgebras and Hopf algebras are commutative.

Affine monoid schemes and bialgebras

In this subsection, we allow k to be a commutative ring.

1.1 Let $G = \operatorname{Spec} A$ be an affine scheme over k, and $m\colon G \times G \to G$ and $e\colon \operatorname{Spec} k \to G$ morphisms. The triple (G, m, e) is an ***affine monoid scheme*** over k if $(G(R), m(R), e(R))$ is a monoid for all k-algebras R. This condition can be expressed in terms of diagrams: the associativity condition requires that the two composed morphisms

$$G \times G \times G \overset{m\times \mathrm{id}_G}{\underset{\mathrm{id}_G \times m}{\rightrightarrows}} G \times G \xrightarrow{m} G \tag{49}$$

are equal and the condition that $e(R)$ is a neutral element requires that the two morphisms

$$\begin{aligned} G \simeq G \times \operatorname{Spec} k &\xrightarrow{\mathrm{id}_G \times e} G \times G \xrightarrow{m} G \\ G \simeq \operatorname{Spec} k \times G &\xrightarrow{e \times \mathrm{id}_G} G \times G \xrightarrow{m} G \end{aligned} \tag{50}$$

equal id_G.

1.2 An ***algebra*** over k (associative with 1) is a k-module V together with k-linear maps $m\colon A \otimes A \to A$ and $e\colon k \to A$ such that the two composed maps

$$A \otimes A \otimes A \overset{m\otimes \mathrm{id}_A}{\underset{\mathrm{id}_A \otimes m}{\rightrightarrows}} A \otimes A \xrightarrow{m} A \tag{51}$$

are equal and the two maps

$$\begin{aligned} A \simeq k \otimes A &\xrightarrow{e \otimes \mathrm{id}} A \otimes A \xrightarrow{m} A \\ A \simeq A \otimes k &\xrightarrow{\mathrm{id} \otimes e} A \otimes A \xrightarrow{m} A \end{aligned} \tag{52}$$

equal id_A.

1.3 A ***coalgebra*** over k (co-associative with co-identity) is a k-module C together with k-linear maps $\Delta\colon C \to C \otimes C$ and $\epsilon\colon C \to k$ such that the two composed maps

$$C \xrightarrow{\Delta} C \otimes C \underset{\mathrm{id} \otimes \Delta}{\overset{\Delta \otimes \mathrm{id}}{\rightrightarrows}} C \otimes C \otimes C$$

are equal and the two maps

$$\begin{aligned} C \xrightarrow{\Delta} C \otimes C &\xrightarrow{\epsilon \otimes \mathrm{id}} k \otimes C \simeq C \\ C \xrightarrow{\Delta} C \otimes C &\xrightarrow{\mathrm{id} \otimes \epsilon} C \otimes k \simeq C \end{aligned}$$

equal id_C.

Thus “algebra” and “coalgebra” are opposite (dual) notions.

1.4 A ***bialgebra*** over k is a quintuple $(A, m, e, \Delta, \epsilon)$ such that

(a) (A, m, e) is an algebra over k,

(b) (A, Δ, ϵ) is a coalgebra over k,

(c) Δ and ϵ are algebra homomorphisms,

(d) m and e are coalgebra homomorphisms.

In the presence of (a) and (b), the conditions (c) and (d) are equivalent (Milne 2017, 9.40). A bialgebra $(A, m, e, \Delta, \epsilon)$ is said to be ***commutative*** if the underlying algebra (A, m, e) is commutative.

1.5 Let $G = \operatorname{Spec} A$ be an affine scheme over k. Then A is a commutative k-algebra, and the monoid structures (m, e) on G correspond exactly to the coalgebra structures (Δ, ϵ) on A given by algebra homomorphisms, i.e., to the bialgebra structures on A. The functor $A \rightsquigarrow \operatorname{Spec} A$ defines a contravariant equivalence between the category of commutative bialgebras over k and the category of affine monoid schemes over k.

1.6 Let (C, Δ, ϵ) and (C', Δ', ϵ') be coalgebras over k. Then $C \otimes C'$ becomes a coalgebra over k with the comultiplication

$$C \otimes C' \xrightarrow{\Delta \otimes \Delta'} (C \otimes C) \otimes (C' \otimes C') \xrightarrow{C \otimes \gamma_{C,C'} \otimes C'} (C \otimes C') \otimes (C \otimes C'),$$

where $\gamma_{C,C'}(c \otimes c') = c' \otimes c$, and co-identity

$$\epsilon \otimes \epsilon'\colon C \otimes C' \to k \otimes k \simeq k.$$

If G and G' are affine monoid schemes over k with associated coalgebras C and C', then $G \times G'$ is an affine monoid scheme over k with associated coalgebra $C \otimes C'$.

Affine group schemes and Hopf algebras

In this subsection, we allow k to be a commutative ring.

1.7 Let G be an affine scheme over k and $m\colon\ G\times G\to G$ a k-morphism. The pair (G,m) is an ***affine group scheme*** over k if $(G(R),m(R))$ is a group for all k-algebras R. In terms of diagrams, this condition says that there exist morphisms $e\colon\ \operatorname{Spec} k\to G$ and $\mathrm{inv}\colon\ G\to G$ (necessarily unique) such that (G,m,e) is a monoid scheme over k and

$$\begin{tikzcd} G \arrow[r, "(\mathrm{inv},\mathrm{id})"] \arrow[d] & G\times G \arrow[d, dashed, "m"] & G \arrow[l, "(\mathrm{id},\mathrm{inv})"'] \arrow[d] \\ \operatorname{Spec} k \arrow[r, "e"] & G & \operatorname{Spec} k \arrow[l, "e"'] \end{tikzcd}$$

commutes.

1.8 Let A be a k-algebra (not necessarily commutative) and $\Delta\colon\ A\to A\otimes A$ a homomorphism of k-algebras. The pair (A,Δ) is a ***Hopf algebra*** over k if there exist k-algebra homomorphisms $\epsilon\colon\ A\to k$ and $S\colon\ A\to A$ (necessarily unique) such that (A,Δ,ϵ) is a bialgebra over k, and

$$(S,\mathrm{id}_A)\circ\Delta=\epsilon=(\mathrm{id}_A,S)\circ\Delta.$$

Such an S is called an ***antipode***. Thus a bialgebra over k is a Hopf algebra if and only if there exists an antipode. A Hopf algebra is said to be ***commutative*** if the algebra A is commutative.

1.9 Let $G=\operatorname{Spec} A$ be an affine scheme over k. To give the structure of a group scheme on G is the same as giving the structure of a commutative Hopf algebra on A. The functor $A\rightsquigarrow\operatorname{Spec} A$ defines a contravariant equivalence between the category of commutative Hopf algebras over k and the category of affine group schemes over k.

1.10 An affine group scheme $G=\operatorname{Spec} A$ over k is of finite type over k if A is finitely generated as a k-algebra. By an ***algebraic group over*** k we mean an affine group scheme of finite type over k.

Henceforth, all bialgebras and Hopf algebras are commutative.

1.11 Let $A\subset B$ be Hopf algebras over a field. Then B is faithfully flat over A. See Waterhouse 1979, 14.1.

Representations

In this subsection, k is a field.

1.12 Let G be an affine group (or monoid) scheme over k. A ***representation*** of G on a k-vector space V is a homomorphism

$$G(R)\to\mathrm{Aut}_{R\text{-linear}}(V(R))$$

natural in R. In other words, it is a family of homomorphisms $G(R)\to\mathrm{GL}(V(R))$, indexed by the k-algebras R, compatible with extension of scalars. When V is finite-dimensional, this is the same as a homomorphism $G\to\mathrm{GL}_V$ of affine group (or monoid) schemes over k. We let $\mathsf{Repf}(G)$ denote the category of representations of G on finite-dimensional k-vector spaces.

1.13 A ***(right) comodule*** over a k-coalgebra C is a vector space V over k together with a k-linear map $\rho\colon V \to V \otimes_k C$ such that the two composed maps

$$V \xrightarrow{\rho} V \otimes C \underset{\rho \otimes \mathrm{id}_C}{\overset{\mathrm{id}_V \otimes \Delta}{\rightrightarrows}} V \otimes C \otimes C$$

are equal and

$$V \xrightarrow{\rho} V \otimes C \xrightarrow{\mathrm{id}_V \otimes \epsilon} V \otimes k \simeq V$$

equals id_V. For example, Δ defines an C-comodule structure on C.

PROPOSITION 1.14 *Let $G = \operatorname{Spec} A$ be an affine group (or monoid) scheme and V a k-vector space. To give an A-comodule structure on V is the same as giving a linear representation of G on V.*

PROOF A representation r of G on V is determined by its action on the "universal" element

$$\mathrm{id}_G \in \operatorname{Hom}(G, G) = G(A).$$

Now $r(\mathrm{id}_G)$ is an A-isomorphism $V \otimes A \to V \otimes A$ whose restriction to $V = V \otimes k \subset V \otimes A$ determines it and is an A-comodule structure ρ on V. Conversely, a comodule structure ρ on V determines a representation of G on V such that, for every k-algebra R and $g \in G(R) \overset{\text{def}}{=} \operatorname{Hom}_k(A, R)$, the restriction of $g_V\colon V \otimes R \to V \otimes R$ to $V \otimes k \subset V \otimes R$ is

$$(\mathrm{id}_V \otimes g) \circ \rho\colon V \to V \otimes A \to V \otimes R.$$

For more details, see Milne 2017, 4.1. □

Let $G = \operatorname{Spec} A$. The representation of G on A defined by the A-comodule structure $\Delta\colon A \to A \otimes A$ is called the ***regular representation*** of G.

PROPOSITION 1.15 *Let C be a k-coalgebra and (V, ρ) a comodule over C. Every finite subset S of V is contained in a sub-comodule of V having finite dimension over k.*

PROOF Let $\{c_i\}$ be a basis for C over k (possibly infinite). For each v in S, write $\rho(v) = \sum v_i \otimes c_i$ (finite sum). The k-subspace spanned by the v and the v_i occurring in such a sum is a sub-comodule over C containing S (see Milne 2017, 4.7). □

COROLLARY 1.16 *Every C-comodule is a filtered union of finite-dimensional sub-comodules.*

PROOF Let C be a k-coalgebra, and V a C-comodule. The set of all sub-comodules of V finite-dimensional over k is partially ordered by inclusion, filtered (any two are contained in a third), and has union V (by the proposition). □

COROLLARY 1.17 *Every representation of an affine group scheme (or monoid scheme) on a vector space is a filtered union of finite-dimensional subrepresentations.*

PROOF According to Proposition 1.14, this is a restatment of Corollary 1.16. □

A representation $r\colon G \to \mathrm{GL}_V$ is ***faithful*** if $r(R)\colon G(L) \to \operatorname{Aut}_{R\text{-linear}}(V(R))$ is injective for all k-algebras R. When V is finite dimensional, this is equivalent to r being a closed immersion (Milne 2017, 3.35).

PROPOSITION 1.18 *An affine group scheme G is algebraic if and only if it has a faithful finite-dimensional representation over k.*

PROOF If G admits a faithful representation on a finite-dimensional vector space, then it admits a closed immersion into scheme of finite type over k, and so it is of finite type over k.

For the converse, let $G = \operatorname{Spec} A$ with A a finitely generated k-algebra, and let (Δ, ϵ) be the corresponding k-coalgebra structure on A. A sub-comodule V of A provides a faithful representation of G if it contains a set of generators for the k-algebra A (Milne 2017, proof of 4.9). According to 1.15, we can choose V to be finite-dimensional over k. □

PROPOSITION 1.19 *Let C be a coalgebra over k. Every finite subset of C is contained in a finite-dimensional k-subcoalgebra.*

PROOF According to 1.15, the finite subset is contained in a finite-dimensional k-subspace V of A such that $\Delta(V) \subset V \otimes_k C$. Let $\{v_j\}$ be a basis for V, and let $\Delta(v_j) = \sum v_i \otimes a_{ij}$ (finite sum). Then $\Delta(a_{ij}) = \sum a_{il} a_{lj}$, and so the k-subspace V' spanned by the v_i and a_{ij} satisfies $\Delta(V') \subset V' \otimes_k V'$. Now V' is the required k-coalgebra. □

PROPOSITION 1.20 *Let A be a Hopf algebra over k. Every finite subset of A is contained in a Hopf subalgebra B that is finitely generated as a k-algebra.*

PROOF According to 1.19, the finite subset is contained in a finite-dimensional k-subspace C of A such that $\Delta(C) \subset C \otimes_k C$. If $\Delta a = \sum b_i \otimes c_i$, then $\Delta(Sa) = \sum Sb_i \otimes Sc_i$, and so the k-subspace V spanned by C and SC satisfies $\Delta(V) \subset V \otimes_k V$ and $S(V) \subset V$. Let B be the k-algebra generated by V. Then

$$1 \in B, \quad \Delta(B) \subset B \otimes B, \quad \epsilon(B) \subset k, \quad S(B) \subset B.$$

□

COROLLARY 1.21 *Let G be an affine group scheme over k. Then G is a filtered projective limit $G = \varprojlim G_i$ of (affine) algebraic groups over k in which the transition maps $G_j \to G_i$, $i \leq j$, are faithfully flat.*

PROOF Let A be a the Hopf algebra over G. According to the proposition, A is a filtered union $A = \bigcup A_i$ of Hopf subalgebras A_i that are finitely generated as k-algebras. The functor Spec transforms the inductive limit $A = \varinjlim A_i$ into an projective limit $G = \varprojlim G_i$. As Hopf algebras are always faithfully flat over Hopf subalgebras (1.11), A_j is faithfully flat over A_i, and so the transition map $G_j \to G_i$ is faithfully flat. □

More precisely, if G is an affine group scheme over k, then $G = \varprojlim G/N$, where N runs over the set of normal affine subgroup schemes of G such that G/N is algebraic. Projective limits of affine group schemes are again affine group schemes. See Demazure and Gabriel 1970, III, §3, n° 7.

PROPOSITION 1.22 *Let G be an algebraic group over k and (V, r) a faithful finite-dimensional representation of G. Every finite-dimensional representation of G can be constructed from V by iterated applications of the operations of direct sum, tensor product, dual, passage to a subobject or quotient object.*

PROOF See Milne 2017, 4.14. □

In other words, (V, r) is a tensor generator (I, 7.6) for the rigid symmetric monoidal category $\mathsf{Repf}(G)$.

THEOREM 1.23 (CHEVALLEY) *Let G be an algebraic group over k. Every algebraic subgroup of H of G arises as the stabilizer of a one-dimensional subspace in a finite-dimensional representation of G.*

PROOF See Milne 2017, 4.27. □

2 Recovering a group scheme from its representations

Let G be an affine group scheme over k. Let R be a k-algebra, and let $g \in G(R)$. For any finite-dimensional representation (V, r_V) of G over k, we have an R-linear isomorphism

$$\lambda_V \stackrel{\text{def}}{=} r_V(g)\colon V_R \to V_R.$$

These maps satisfy the following conditions:

(a) for all representations V and W, $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$;

(b) $\lambda_{\mathbb{1}}$ is the identity map (here $\mathbb{1} = k$ with the trivial action);

(c) for all G-equivariant maps $u\colon V \to W$, $\lambda_W \circ u_R = u_R \circ \lambda_V$.

THEOREM 2.1 *Let G be an affine group scheme over k, and let R be a k-algebra. Suppose that, for every finite-dimensional representation (V, r_V) of G over k, we are given an R-linear map $\lambda_V\colon V_R \to V_R$. If the family $(\lambda_V)_V$ satisfies the conditions (a, b, c), then there exists a unique $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all V. In particular, the maps λ_V are isomorphisms.*

PROOF Let $A(G,R)$ denote the set of families $\lambda = (\lambda_V)_V$ satisfying the conditions (a,b,c). Suppose first that G is algebraic, and let (W, r) be a faithful representation of G. It follows from Proposition 1.22 that an element $(\lambda_V)_V$ of $A(G,R)$ is determined by λ_W, and so we have inclusions

$$G(R) \hookrightarrow A(G,R) \hookrightarrow \mathrm{GL}_W(R), \quad g \mapsto (r_V(g))_V \mapsto r_W(g).$$

According to Chevalley's theorem (1.23), there exists a representation (W, r_W) of G, which we may choose to be faithful, and a one-dimensional subspace L in W such that G is the stabilizer of L in GL_W. Let $u\colon L \to W$ be the inclusion map, and let $\lambda \in A(G,R)$. As $\lambda_W \circ u_R = u_R \circ \lambda_L$, we see that λ_W stabilizes L, and so it lies in $G(R)$, as required.

In the general case, let $V \in \mathsf{Repf}(G)$, and let $\langle V\rangle^{\otimes}$ be the full subcategory of $\mathsf{Repf}(G)$ of objects isomorphic to a subquotient of $P(V, V^{\vee})$ for some $P \in \mathbb{N}[t,s]$. Let G_V be the image of G in GL_V. It is an algebraic quotient of G acting faithfully on V, and so

$$G_V(R) \simeq A(G_V, R) \hookrightarrow \mathrm{GL}_V(R).$$

Define a partial ordering on the set of isomorphism classes of objects of $\mathsf{Repf}(G)$ by the rule

$$[V] \leq [V'] \iff \langle V\rangle^{\otimes} \subset \langle V'\rangle^{\otimes}.$$

Note that $[V], [V'] \leq [V \oplus V']$, and so the set is filtered. If $[V] \leq [V']$, then restriction gives a commutative diagram

$$\begin{tikzcd} G_{V'}(R) \arrow[r, "\simeq"] \arrow[d] & A(G_{V'}, R) \arrow[d] \\ G_V(R) \arrow[r, "\simeq"] & A(G_V, R) \end{tikzcd}$$

Now $G = \varprojlim G_V$ so, on passing to the projective limit, we get bijections

$$G(R) \simeq \varprojlim G_V(R) \simeq \varprojlim A(G_V, R) \simeq A(G,R). \qquad \square$$

COROLLARY 2.2 *Let G be an affine group scheme over k and let ω be the forgetful functor on* $\mathsf{Repf}(G)$*. Then the canonical morphism*

$$G \to \mathcal{A}ut^{\otimes}(\omega)$$

is an isomorphism of functors (of k-algebras).

PROOF For any k-algebra R,

$$\mathcal{A}ut^{\otimes}(\omega)(R) \overset{\text{def}}{=} \operatorname{End}(\phi_R \circ \omega) = A(G,R).$$

□

Let $f : G \to H$ be a homomorphism of affine group schemes over k. Using f, we can regard an H-module as a G-module. In this way, we get a $\otimes$-functor

$$\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G) \tag{53}$$

such that $\omega^G_{\text{forget}} \circ \omega^f = \omega^H_{\text{forget}}$. Our next result shows that all such functors arise in this fashion.

COROLLARY 2.3 *Let G and H be affine k-group schemes, and let $F : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ be a $\otimes$-functor such that $\omega^G_{\text{forget}} \circ \omega^f = \omega^H_{\text{forget}}$. There exists a unique homomorphism $f : G \to H$ such that $F = \omega^f$.*

PROOF Such an F defines a homomorphism (functorial in the k-algebra R)

$$F^* : \mathcal{A}ut^{\otimes}(\omega^G_{\text{forget}})(R) \to \mathcal{A}ut^{\otimes}(\omega^H_{\text{forget}})(R), \quad F^*(\lambda)_V = \lambda_{F(V)}.$$

Corollary 2.2 allows us to regard F^* as a functorial homomorphism $G(R) \to H(R)$. According to the Yoneda lemma this arises from a unique homomorphism $f : G \to H$. Clearly, the maps $F \mapsto f$ and $f \mapsto \omega^f$ are inverse. □

REMARK 2.4 (a) Theorem 2.1 holds also for affine monoid schemes, but with a different proof as we cannot use Chevalley's theorem (see Milne 2017, 9.2).

(b) Proposition 2.2 shows that G is determined by the triple $(\mathsf{Repf}(G), \otimes, \omega^G)$. In fact, the coalgebra of G is already determined by $(\mathsf{Repf}(G), \omega^G)$ (see the proof of Theorem 3.1 below).

EXERCISE 2.5 Let G be an algebraic group over k, and let ω be the forgetful functor $\mathsf{Repf}(G) \to \mathsf{Vecf}(k)$. We have seen that $G(k)$ can be identified with set of natural automorphisms $(\lambda_V)_V$ of ω such that $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$ and $\lambda_{\mathbb{1}} = \text{id}$. Show that the Lie algebra of G can be identified with the set of natural endomorphisms $(\lambda_V)_V$ of ω such that $\lambda_{V\otimes W} = \lambda_V \otimes \text{id} + \text{id} \otimes \lambda_W$ and $\lambda_{\mathbb{1}} = 0$.

3 The main theorem

Recall that a neutral tannakian category over k is a k-linear rigid abelian symmetric monoidal category such that there exists a fibre functor ω to $\mathsf{Vecf}(k)$. These conditions imply that $k \simeq \operatorname{End}(\mathbb{1})$ and that fibre functors are faithful.

Statements

THEOREM 3.1 *Let* $(\mathsf{C}, \otimes)$ *be an essentially small neutral tannakian category over* k *and* ω *a* k*-valued fibre functor.*

(a) *The functor* $\mathcal{A}ut^{\otimes}(\omega)$ *of* k*-algebras is represented by an affine group scheme* G*.*

(b) *The functor* $\mathsf{C} \to \mathsf{Repf}(G)$ *defined by* ω *is an equivalence of* $\otimes$*-categories.*

Thus every neutral tannakian category is equivalent (in possibly many different ways) to the category of finite-dimensional representations of an affine group scheme.

The proof will occupy the rest of this section. After a series of preliminaries on abelian categories, we construct the coalgebra A of G without using the tensor structure on C (Theorem 3.20). The tensor structure then allows us to define an algebra structure on A, and the rigidity of C implies that A is a Hopf algebra (so that G is a group scheme rather than a monoid scheme).

COROLLARY 3.2 *A neutral tannakian category* $(\mathsf{C}, \otimes)$ *is monogenic if and only if, for one (hence every)* k*-valued fibre functor* ω*, the affine group scheme* $\mathcal{A}ut^{\otimes}(\omega)$ *is algebraic.*

PROOF If $(\mathsf{C}, \otimes)$ is monogenic, with tensor generator X, then $\omega(X)$ is a faithful finite-dimensional representation of $\mathcal{A}ut^{\otimes}(\omega)$ for any fibre functor ω, which is therefore algebraic (1.18).

Conversely, suppose that for some k-valued fibre functor ω on C, the affine group scheme $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}(\omega)$ is algebraic. Then $\mathsf{Repf}(G)$ has a tensor generator (1.22), which corresponds to a tensor generator of $(\mathsf{C}, \otimes)$ under the tensor equivalence $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$ defined by ω. □

Abelian categories as module categories

Let A be an abelian category. An object P of A is a ***generator*** if the functor $h^P \overset{\text{def}}{=} \mathrm{Hom}(P, -)$ is faithful and it is ***projective*** if h^P is exact. Thus P is a ***projective generator*** if and only if h^P is exact and faithful.

LEMMA 3.3 *Let* A *be an abelian category and* P *a generator* A*. For any noetherian object* X *of* A*, there exists an epimorphism* $P^{\oplus n} \to X$*, some* n*.*

PROOF If X is nonzero, then there exists a nonzero morphism $\phi_1 : P \to X$ (otherwise P would not separate the identity morphism $X \to X$ from the zero morphism). Similarly, if ϕ_1 is not an epimorphism, then there exists a morphism $\phi_2 : P \to X$ such that the image of $(\phi_1, \phi_2) : P^2 \to X$ is strictly larger than that of ϕ_1 (otherwise P would not separate the projection $X \to X/\mathrm{Im}(\phi_1)$ from zero). Continuing in this fashion, and using that X is noetherian, we obtain an epimorphism $(\phi_1, \dots, \phi_n) : P^{\oplus n} \to X$. □

PROPOSITION 3.4 *Let* A *be an abelian category whose objects are noetherian. Assume that* A *has a projective generator* P*, and let* $A = \mathrm{End}(P)$*. Then* A *is a right noetherian ring, and* h^P *is an equivalence from* A *to the category of finitely generated right* A*-modules.*

PROOF We first show that A is right noetherian. Let $\mathfrak{a}$ be a right ideal in A, and let $\mathfrak{a}P$ be the subobject of P generated by the images of the maps $\alpha : P \to P$ with $\alpha \in \mathfrak{a}$. It suffices to show that $\mathfrak{a} = \mathrm{Hom}(P, \mathfrak{a}P)$ because then the ascending chain condition on the submodules of P implies it for the right ideals of A. Certainly $\mathfrak{a} \subset \mathrm{Hom}(P, \mathfrak{a}P)$.

Let $\alpha \in \mathrm{Hom}(P, \mathfrak{a}P)$. There exist $\phi_1, \ldots, \phi_n \in \mathfrak{a}$ such that $(\phi_1, \ldots, \phi_n)\colon P^n \to X$ is an epimorphism. Now the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "\alpha"] & \\ P^n \arrow[r, "(\phi_1,\ldots,\phi_n)"'] & X \arrow[r] & 0 \end{tikzcd}$$

shows that $\alpha = \sum \phi_i \circ h \in \mathfrak{a}$.

The left action of A on P defines a right action of A on $h^P(X)$, natural in X, and so h^P is a functor $\mathsf{A} \to \mathsf{Mod}_A$. Because P is a projective generator, h^P is exact and faithful.

Let X an object of A. For some m, n, we have an exact sequence

$$P^{\oplus m} \to P^{\oplus n} \to X \to 0.$$

On applying h^P to this sequence, we get an exact sequence

$$A^{\oplus m} \to A^{\oplus n} \to h^P(X) \to 0,$$

which shows that $h^P(X)$ has finite presentation. Conversely, because A is right noetherian, every finitely generated right A-module M is finitely presented, say,

$$A^{\oplus m} \xrightarrow{u} A^{\oplus n} \to M \to 0,$$

where u is an $m \times n$ matrix with coefficients in A. This matrix defines a morphism $P^{\oplus m} \to P^{\oplus n}$ whose cokernel X has the property that $h^P(X) \simeq M$. This shows that h^P is essentially surjective, and it remains to show that it is full.

Let X, Y be objects of A, and choose an exact sequence $P^{\oplus m} \to P^{\oplus n} \to X \to 0$. Then

$$\mathrm{Hom}(P^{\oplus m}, Y) \simeq h^P(Y)^{\oplus m} \simeq \mathrm{Hom}(A^{\oplus m}, h^P(Y)) \simeq \mathrm{Hom}(h^P(P^{\oplus m}), h^P(Y)),$$

and the composite of these maps is that defined by h^P. From the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}(P^n,Y) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(P^m,Y) \arrow[d, "\simeq"] \\ 0 \arrow[r] & \mathrm{Hom}(h^P(X),h^P(Y)) \arrow[r] & \mathrm{Hom}(h^P(P^n),h^P(Y)) \arrow[r] & \mathrm{Hom}(h^P(P^m),h^P(Y)), \end{tikzcd}$$

we see that $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(h^P(X), h^P(Y))$ is an isomorphism, and so h^P is full. □

EXAMPLE 3.5 Let A be a locally finite k-linear abelian category. Assume that A has a projective generator P, and let $A = \mathrm{End}(P)$. Then A is a finite-dimensional k-algebra, and h^P is an equivalence from A to the category of finitely generated right A-modules.

EXAMPLE 3.6 Let A be abelian with noetherian objects, and let B be a full subcategory of A stable under finite direct sums and subquotients. For X in A, let $q(X) = X/\bigcap \mathrm{Ker}(\alpha)$, where α runs over the epimorphisms $X \to Y$ with $Y \in \mathrm{ob}\,\mathsf{B}$. As X is noetherian,

$$q(X) = X/\bigcap_{\alpha\in F} \mathrm{Ker}(\alpha)$$

for some finite set F of α. Note that $q(X)$ is a subobject of $\bigoplus_{\alpha\in F} X/\mathrm{Ker}(\alpha)$, and so it lies in B. Clearly, $q(X)$ is the largest quotient of X in B – any other epimorphism

$X \to Y$ factors uniquely through $q(X)$ – and so q is a left adjoint for the inclusion functor $i\colon \mathsf{B} \to \mathsf{A}$,

$$\mathrm{Hom}_{\mathsf{B}}(q(X), Y) \simeq \mathrm{Hom}_{\mathsf{A}}(X, i(Y)), \quad X \in \mathrm{ob}\,\mathsf{A}, \quad Y \in \mathrm{ob}\,\mathsf{B}.$$

Now assume that A has a projective generator P, and let $A = \mathrm{End}(P)$. Then $Q \stackrel{\text{def}}{=} q(P)$ is a projective generator for B, and $B \stackrel{\text{def}}{=} \mathrm{End}(Q)$ is a quotient $A/\mathfrak{a}$ of A.

According to Proposition 3.4, the functor $\mathrm{Hom}(P, -)$ identifies A with the category Modf_A. In this model, P is A_A, Q is B_B, and B consists of the right A-modules killed by $\mathfrak{a}$. In summary:

$$\begin{tikzcd} \mathsf{B} \arrow[r, "i"] \arrow[d, "h^Q"', "\sim"] & \mathsf{A} \arrow[d, "h^P"', "\sim"] \\ \mathsf{Mod}_B \arrow[r, "j"] & \mathsf{Mod}_A \end{tikzcd} \qquad \begin{cases} A = \mathrm{End}(P), \quad Q = q(P), \\ B = \mathrm{End}(Q), \\ j \text{ defined by } A \twoheadrightarrow B. \end{cases}$$

NOTES Proposition 3.4 and its infinite version (without the noetherian hypothesis) have been variously credited to Freyd, Gabriel, Lam, and Mitchell.

Abelian categories with a "fibre" functor

EXAMPLE 3.7 Let $\mathsf{A} = \mathsf{Modf}_A$, where A is a finite-dimensional k-algebra. Let Q be a projective generator of A, and let $\omega = h^Q$. Then ω is an exact faithful k-linear functor $\mathsf{A} \to \mathsf{Vecf}(k)$. If $\mathrm{End}(\omega) = B$, then $\mathrm{End}(Q) = B^{\mathrm{opp}}$, and, according to 3.4, ω is an equivalence

$$\mathsf{Modf}_A \xrightarrow{\omega} \mathsf{Modf}_{B^{\mathrm{opp}}} = {}_B\mathsf{Modf}.$$

PROPOSITION 3.8 *Let A be a k-linear abelian category, and let $\omega\colon \mathsf{A} \to \mathsf{Vecf}(k)$ be an exact faithful k-linear functor. Assume that A has a projective generator. Then $B \stackrel{\text{def}}{=} \mathrm{End}(\omega)$ is a finite-dimensional k-algebra, and ω is an equivalence $\mathsf{A} \to {}_B\mathsf{Modf}$.*

PROOF Note first that the existence of ω implies that the k-linear category A is locally finite: certainly $\mathrm{Hom}(X, Y) \subset \mathrm{Hom}(\omega(X), \omega(Y))$ is finite dimensional, and a subobject Z of X is determined by the subspace $\omega(Z)$ of $\omega(X)$, and so the set of subobjects of X can be identified with a subset of the lattice of subspaces of $\omega(X)$.

The left action of B on ω defines a left action of B on $\omega(X)$, natural in X, and so ω is a functor $\mathsf{A} \to {}_B\mathsf{Modf}$. By definition, it is exact and faithful, and it remains to show that it is full and essentially surjective.

Let P be a projective generator for A, and let $A = \mathrm{End}(P)$, so h^P is an equivalence $\mathsf{A} \to \mathsf{Modf}_A$. Then A acts on $\omega(P)$ on the left, and the map

$$\alpha \otimes p \mapsto \omega(\alpha)(p)\colon\ \mathrm{Hom}(P, X) \otimes_A \omega(P) \to \omega(X)$$

is natural in $X \in \mathrm{ob}\,\mathsf{A}$. This map is obviously an isomorphism when $X = P$. As both functors commute with finite direct sums and are right exact, it follows (as in the proof of Proposition 3.4) that it is an isomorphism for all X in A, and so

$$h^P(-) \otimes \omega(P) \simeq \omega(-).$$

In other words, up to a natural isomorphism, we have a factorization

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\sim", "h^P"'] \arrow[rr, bend left, "\omega"] & \mathsf{Modf}_A \arrow[r, "-\otimes\omega(P)"'] & {}_B\mathsf{Modf}. \end{tikzcd}$$

As ω is exact, it follows from the diagram, that the A-module $\omega(P)$ is flat, and hence projective (because it is finitely presented). Let $Q = \mathrm{Hom}_A(\omega(P), {}_AA)$. Then $h^Q(-) \simeq - \otimes \omega(P)$, and so h^Q is exact and faithful. Thus, Q is a projective generator for Mod_A, and so

$$h^Q : \mathsf{Mod}_A \xrightarrow{\approx} \mathsf{Modf}_{B^{\mathrm{opp}}} = {}_B\mathsf{Modf},$$

where $B^{\mathrm{opp}} = \mathrm{End}(Q) \simeq \mathrm{End}(h^Q)^{\mathrm{opp}} \simeq \mathrm{End}(\omega)^{\mathrm{opp}}$. □

REMARK 3.9 Let $\mathsf{A} = \mathsf{Modf}_A$, where A is a finitely generated k-algebra. Then every exact faithful k-linear functor $\omega : \mathsf{A} \to \mathsf{Vecf}(k)$ is of the form h^Q for some projective generator Q. Indeed, on applying the proof of 3.8 with $\mathsf{A} = \mathsf{Modf}_A$ and $P = A_A$, we find that $- \otimes_A \omega(A_A) = \omega(-)$, and so $\omega = h^Q$ with Q the A-linear dual of $\omega(A_A)$.

Existence of a projective generator

Let A be an abelian category. For an object X of A, we let $\langle X \rangle$ denote the strictly full subcategory of A whose objects are subquotients of a finite direct sum of copies of X. It is the smallest strictly full abelian subcategory of A containing X.

An object of A is ***simple*** if it is nonzero and contains no proper nonzero subobject. A ***composition series*** for an object X of A is a finite decreasing filtration

$$X = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

with simple successive quotients F^i/F^{i+1}. An object is said to have ***finite length*** if it admits a composition series, in which case any two composition series have the same length and multiset of quotients (taken up to isomorphism). We let $\lg(X)$ denote the common length of the composition series for X, and, for a simple object S, we let $\lg_S(X)$ denote the number of $i$such that F^i/F^{i+1} is isomorphic to S. In the Grothendieck group of A,

$$[X] = \sum \lg_S(X) \cdot [S],$$

where $[S]$ runs over the isomorphism classes of simple objects in A.

LEMMA 3.10 *Let A be an abelian category whose objects have finite length. If*

(a) *there are only finitely many isomorphism classes of simple objects, and*

(b) *every simple object is a quotient of a projective object,*

then there exists a projective generator for A.

PROOF Let $S_1, \ldots, S_m$ be a set of representatives for the simple objects, and, for each i, let $P_i \to S_i$ be an epimorphism with P_i projective. We claim that $P \overset{\text{def}}{=} P_1 \oplus \cdots \oplus P_m$ is a projective generator. Certainly, it is projective, and to show that h^P is faithful, it suffices to show that $M \neq 0 \implies h^P(M) \neq 0$. Every nonzero M has a simple quotient, so there exists an epimorphism $M \to S_i$ for some i. The epimorphism $P_i \to S_i$ extends (by zero) to an epimorphism $P \to S_i$, which lifts to a nonzero morphism $P \to M$. □

3.11 An ***essential extension*** of an object Y is an epimorphism $\alpha : E \to Y$ such that no subobject $E' \subset E$, distinct from E, maps onto Y. When Y is simple, this just says that the kernel of α contains every $E' \subset E$ distinct from E.

LEMMA 3.12 *Assume* $\mathsf{A} = \langle X \rangle$, *and let* $E \twoheadrightarrow S$ *be an essential extension of a simple object* S *in* A. *For all* $Y \in \operatorname{ob} \mathsf{A}$,

$$\dim_k \operatorname{Hom}(E, Y) \leq \lg_S(Y) \cdot \dim_k \operatorname{End}(S). \tag{54}$$

If equality holds for X, *then it holds for all objects of* A, *and* E *is projective.*

PROOF When Y is simple, every nonzero morphism $E \to Y$ factors through $E \twoheadrightarrow S$ (because its kernel is contained in $\operatorname{Ker}(E \twoheadrightarrow S)$) and induces an isomorpism $S \to Y$. Thus, $\operatorname{Hom}(E, Y) \simeq \operatorname{Hom}(S, Y)$, and so

$$\begin{cases} \operatorname{Hom}(E, Y) \approx \operatorname{End}(S) \text{ if } Y \approx S \\ \operatorname{Hom}(E, Y) = 0 \text{ otherwise.} \end{cases}$$

In both cases, (54) holds with equality.

An exact sequence

$$0 \to Y' \to Y \to Y'' \to 0 \tag{55}$$

gives an exact sequence

$$0 \to \operatorname{Hom}(E, Y') \to \operatorname{Hom}(E, Y) \to \operatorname{Hom}(E, Y''), \tag{56}$$

from which it follows that

$$\dim_k \operatorname{Hom}(E, Y) \leq \dim_k \operatorname{Hom}(E, Y') + \dim_k \operatorname{Hom}(E, Y'').$$

As the right-hand side of (54) is additive on short exact sequences, the inequality (54) now follows by induction on the length of Y. If equality holds for Y, then equalities hold throughout the following diagram

$$\begin{array}{ccccc} \operatorname{RHS}(Y) & = & \operatorname{RHS}(Y') & + & \operatorname{RHS}(Y'') \\ \parallel & & \vee\!| & & \vee\!| \\ \operatorname{LHS}(Y) & \leq & \operatorname{LHS}(Y') & + & \operatorname{LHS}(Y''), \end{array}$$

so equality holds for each of Y' and Y'' and the sequence (56) is exact with a 0 on the right.

If equality holds in (54) for $Y = X$, then it holds for $X^{\oplus m}$, and so also for all subquotients of $X^{\oplus m}$. Moreover, the sequence (56) is exact with a 0 on the right, which says that E is projective. □

PROPOSITION 3.13 (GABBER) *Let* A *be a locally finite k-linear abelian category. For any* X *in* A, *the subcategory* $\langle X \rangle$ *of* A *has a projective generator.*

PROOF The quotients of any composition series for X represent the isomorphism classes of simple objects in $\langle X \rangle$, and so $\langle X \rangle$ satisfies the condition (a) of 3.10. We shall complete the proof by showing that every simple object S of $\langle X \rangle$ admits an essential extension $P(S) \twoheadrightarrow S$ with $P(S)$ projective.

Let

$$X = F^0 \supset \cdots \supset F^i \supset F^{i+1} \supset \cdots \supset F^r = 0$$

be a composition series for X. We construct, by induction on i, an essential extension $P_i \twoheadrightarrow S$ of S such that

$$\dim_k \operatorname{Hom}(P_i, X/F^i) = \lg_S(X/F^i) \cdot \dim_k \operatorname{End}(S). \tag{57}$$

We can take $P_1 = S$. Supposing P_i to have been constructed, we now construct P_{i+1}. Let $f_1, \dots, f_s$ span $\mathrm{Hom}(P_i, X/F^i)$. Define Q_j by the fibre product diagram

$$\begin{tikzcd} Q_j \arrow[r] \arrow[d] & X/F^{i+1} \arrow[d] \\ P_i \arrow[r, "f_j"] & X/F^i, \end{tikzcd}$$

and let Q be a subobject of $Q_1 \times_{P_i} \cdots \times_{P_i} Q_s$ minimal among those mapping onto P_i. Then Q is an essential extension of P_i, hence also of S. As Q maps onto P_i, we have an inclusion

$$\mathrm{Hom}(P_i, X/F^i) \hookrightarrow \mathrm{Hom}(Q, X/F^i),$$

but

$$\dim_k \mathrm{Hom}(Q, X/F^i) \overset{(54)}{\leq} \mathrm{lg}_S(X/F^i) \cdot \dim_k \mathrm{End}(S) \overset{(57)}{=} \dim_k \mathrm{Hom}(P_i, X/F^i),$$

and so

$$\mathrm{Hom}(P_i, X/F^i) \simeq \mathrm{Hom}(Q, X/F^i). \tag{58}$$

From the diagram

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[dr, "\text{onto}"'] & Q_1 \times_{P_i} \cdots \times_{P_i} Q_s \arrow[r] \arrow[d] & Q_1 \times \cdots \times Q_s \arrow[r, "\text{project}"] \arrow[d] & Q_j \arrow[r] \arrow[d] & X/F^{i+1} \arrow[d] \\ & P_i \arrow[r, "\text{diagonal}"] \arrow[rrr, bend right, "f_j"'] & P_i \times \cdots \times P_i \arrow[r, "\text{project}"] & P_i \arrow[r, "f_j"] & X/F^i \end{tikzcd}$$

we see that f_j, hence every element of $\mathrm{Hom}(P_i, X/F^i)$, lifts to an element of $\mathrm{Hom}(Q, X/F^{i+1})$, and so the map

$$\mathrm{Hom}(Q, X/F^{i+1}) \to \mathrm{Hom}(P_i, X/F^i) \overset{(58)}{\simeq} \mathrm{Hom}(Q, X/F^i)$$

is surjective. Therefore, the sequence

$$0 \to \mathrm{Hom}(Q, F^{i+1}/F^i) \to \mathrm{Hom}(Q, X/F^{i+1}) \to \mathrm{Hom}(Q, X/F^i) \to 0$$

is exact. The dimensions of the end terms are respectively $\mathrm{lg}_S(F^i/F^{i+1}) \cdot \dim_k \mathrm{End}(S)$ (because F^{i+1}/F^i is simple) and $\mathrm{lg}_S(X/F^i) \cdot \dim_k \mathrm{End}(S)$ (use (58) and the induction hypothesis). Therefore,

$$\dim_k \mathrm{Hom}(Q, X/F^{i+1}) = \mathrm{lg}_S(X/F^{i+1}) \cdot \dim_k \mathrm{End}(S),$$

and we can take $P_{i+1} = Q$.

The induction ends with an essential extension $P(S) \overset{\text{def}}{=} P_r$ of S such that (54) is an equality for $E = P(S)$ and $Y = X$. Lemma 3.12 now shows that $P(S)$ is projective, as required. □

COROLLARY 3.14 *A locally finite k-linear abelian category* A *has a projective generator if and only if* $\mathsf{A} = \langle X \rangle$ *for some object* X.

PROOF The sufficiency follows from the proposition. For the necessity, if A admits a projective generator P, then $\mathsf{A} \approxeq \mathsf{Modf}_A$ with $A = \mathrm{End}(P)$ (see 3.5), and $\mathsf{Modf}_A = \langle A_A \rangle$. □

COROLLARY 3.15 *Let* A *be a* k*-linear abelian category, and let* $\omega\colon \mathsf{A} \to \mathsf{Vecf}(k)$ *be an exact faithful* k*-linear functor. For any* X *in* A, $B \stackrel{\text{def}}{=} \mathrm{End}(\omega|\langle X\rangle)$ *is a finite-dimensional* k*-algebra, and* ω *is an equivalence* $\langle X\rangle \xrightarrow{\approxeq} {}_B\mathsf{Modf}$.

PROOF The hypotheses imply that A is locally finite, and hence $\langle X\rangle$ has a projective generator. We can now apply Proposition 3.8. □

REMARK 3.16 For a more direct proof of 3.15, not using Gabber's result 3.13, see Deligne and Milne 1982, 2.12, 2.13.

REMARK 3.17 Let A be a locally finite k-linear abelian category, and let S be a simple object of A. If S is not projective, then there exists a nonsplit extension

$$0 \to S' \to E_1 \to S \to 0$$

with S' simple. If E_1 is not projective, then we can repeat the construction with E_1 for S. Continuing in this way we obtain a sequence

$$E_i \twoheadrightarrow E_{i-1} \twoheadrightarrow \cdots \twoheadrightarrow E_1 \twoheadrightarrow S$$

with each E_j an essential extension of S. If the process terminates with a projective object, then we have shown that S is a quotient of a projective object, as required. It is obvious that the process terminates when A has a generator, but we only know something weaker, hence the more elaborate argument in the above proof.

NOTES Proposition 3.13 is Deligne 1990, 2.14, where Deligne writes “Le résultat suivant m'a été signalé par O. Gabber”. There are many similar results in the literature, but this seems to have been new in 1990.

Locally finite k-linear abelian categories as unions of module categories

PROPOSITION 3.18 *Every locally finite* k*-linear abelian category* A *is a filtered union of strictly full subcategories* A_α *such that*

- *each* A_α *is stable under finite direct sums and subquotients,*
- *each* A_α *is equivalent to* Modf_{A_α} *for some finite-dimensional* k*-algebra* A_α *(not necessarily commutative).*

PROOF The category A is a union of the subcategories of the form $\langle X\rangle$. The union is filtered because $\langle X\rangle, \langle Y\rangle \subset \langle X \oplus Y\rangle)$. Each category $\langle X\rangle$ satisfies the first condition by definition, and it satisfies the second by 3.5 and 3.13. □

Locally finite k-linear abelian categories as comodule categories

The next proposition allows us to express the results of the last subsection in terms of coalgebras, which are more convenient for passage to the limit.

Recall that $V^\vee$ denotes the linear dual $\mathrm{Hom}_{k\text{-linear}}(V,k)$ of a k-vector space V. Note that $V^\vee \otimes V^\vee \subset (V \otimes V)^\vee$, with equality if (and only if) V is finite-dimensional.

PROPOSITION 3.19 (a) *If* (C,Δ,ϵ) *is a coalgebra over* k*, then* $(C^\vee, \Delta^\vee|_{C^\vee\otimes C^\vee}, \epsilon^\vee)$ *is an algebra over* k.

(b) *If (A, m, e) is a finite-dimensional algebra over k, then $(A^\vee, m^\vee, e^\vee)$ is a coalgebra over k.*

(c) *Let C be a coalgebra over k. Every right C-comodule is a left $C^\vee$-module.*

(d) *Let A be an algebra over k. If A is finite dimensional, then every left A-module is a right $A^\vee$-comodule.*

PROOF For (a) and (b), compare the definitions 1.2 and 1.3.

For (c), let (M, ρ) be a right C-comodule. For $m \in M$, write $\rho(m) = \sum_i m_i \otimes c_i$, and for $f \in C^\vee$, define

$$f \cdot m = \sum\nolimits_i f(c_i) \cdot m_i.$$

This makes M into a left $C^\vee$-module.

For (d), let M be a left A-module. For $m \in M$, let $\{m_1, \ldots, m_n\}$ be a basis of $A \cdot m$. Then there exist $f_i \in A^\vee$ such that $a \cdot m = \sum f_i(a) m_i$ for all $a \in A$. Now

$$\rho(m) = \sum m_i \otimes f_i$$

defines a co-action of $A^\vee$ on M. □

The operations in (c) and (d) are inverse, so, when C is finite-dimensional, to give a right co-action of C on a k-vector space V is the same as giving a left action of $C^\vee$ on V.

Let C be an essentially small k-linear abelian category and $\omega \colon \mathsf{C} \to \mathsf{Vecf}(k)$ an exact faithful k-linear functor. Recall that the existence of ω implies that C is locally finite. For an object X of C, let $A_X = \mathrm{End}(\omega|\langle X \rangle)$, and let $C_X = A_X^\vee$. For any Y in $\langle X \rangle$, $\omega(Y)$ is a left A_X-module, hence also a right C_X-comodule. The functor $Y \rightsquigarrow \omega(Y)$ is an equivalence of categories (3.8, 3.13, 3.19)

$$\langle X \rangle \xrightarrow{\approx} \mathsf{coModf}(C_X).$$

Define a partial ordering on the set of isomorphism classes of objects in C by the rule

$$[X] \leq [Y] \text{ if } \langle X \rangle \subset \langle Y \rangle.$$

Note that $[X], [Y] \leq [X \oplus Y]$, so the set is filtered, and that if $[X] \leq [Y]$, then restriction defines a homomorphism $A_Y \to A_X$. On passing to the inductive limit over the isomorphism classes, we obtain the following statement.

THEOREM 3.20 *Let C be an essentially small k-linear abelian category and $\omega \colon \mathsf{C} \to \mathsf{Vecf}_k$ an exact faithful k-linear functor. Let $C(\omega)$ denote the k-coalgebra $\varinjlim_{[X]} \mathrm{End}(\omega|\langle X \rangle)^\vee$. Then ω is an equivalence $\mathsf{C} \xrightarrow{\approx} \mathsf{coModf}(C(\omega))$.*

In more detail, the natural actions of $C(\omega)$ on the objects $\omega(Y)$, $Y \in \mathsf{C}$, allow us to regard ω as a functor

$$\mathsf{C} \to \mathsf{coModf}(C(\omega)).$$

This functor is an equivalence of categories carrying ω into the forgetful functor.

EXAMPLE 3.21 Let A be a finite-dimensional k-algebra (not necessarily commutative) and ω the forgetful functor ${}_A\mathsf{Mod} \to \mathsf{Vecf}(k)$. For R a commutative k-algebra, let ϕ_R denote the functor $R \otimes - \colon \mathsf{Vecf}(k) \to \mathsf{Mod}(R)$. The action of $R \otimes A$ on $R \otimes \omega(M)$ defines a map

$$\alpha \colon R \otimes A \to \mathrm{End}(\phi_R \circ \omega),$$

which we shall show to be an isomorphism by describing an inverse β. For $\lambda \in \mathrm{End}(\phi_R \circ \omega)$, set $\beta(\lambda) = \lambda_A(1 \otimes 1)$. Clearly $\beta \circ \alpha = \mathrm{id}$, and so we need only show that $\alpha \circ \beta = \mathrm{id}$. For $M \in \mathrm{ob}(\mathsf{Mod}_A)$, let $M_0 = \omega(M)$. The A-module $A \otimes M_0$ is a direct sum of copies of A, and the additivity of λ shows that $\lambda_{A \otimes M_0} = \lambda_A \otimes \mathrm{id}_{M_0}$. The map $a \otimes m \mapsto am\colon A \otimes M_0 \to M$ is A-linear, and hence

$$\begin{tikzcd} R \otimes A \otimes M_0 \arrow[r] \arrow[d, "\lambda_A \otimes \mathrm{id}_{M_0}"] & R \otimes M \arrow[d, "\lambda_M"] \\ R \otimes A \otimes M_0 \arrow[r] & R \otimes M \end{tikzcd}$$

is commutative. Therefore

$$\lambda_M(1 \otimes m) = \lambda_A(1) \otimes m = (\alpha \circ \beta(\lambda))_M(1 \otimes m) \text{ for } m \in M,$$

i.e., $\alpha \circ \beta = \mathrm{id}$.

We have shown that $A \simeq \mathrm{End}(\omega)$, and it follows that, if in (3.20) we take $\mathsf{C} = {}_A\mathsf{Mod}$, so that $\mathsf{C} = \langle {}_A A \rangle$, then the equivalence of categories obtained sends a left A-module to the associated right $A^\vee$-comodule (3.19).

NOTES

3.22 It is necessary in Theorem 3.20 that C be essentially small, because otherwise the underlying "set" of $C(\omega)$ may be a proper class. For example, let S be a proper class, and let C be the category of finite-dimensional k-vector spaces graded by S. In this case $C(\omega)$ contains an idempotent for each element of S, and so cannot be a set.

In terms of Grothendieck universes, we can say that C is always equivalent to a category of comodules, but possibly only in a larger universe.

3.23 It is possible to realize the category in Theorem 3.20 as a category of modules over k-algebra, but for this it is necessary to introduce topologies. The category of coalgebras over k is the ind-category of the category of finite coalgebras over k. Hence, its opposite is the pro-category of the category of finite k-algebras, i.e., the category of profinite k-algebras. The category of right comodules over a coalgebra is equivalent, as a k-linear category, to the category of finite continuous modules over the corresponding profinite k-algebra. See Serre 1993, 2.2.

3.24 Theorem 3.24 can be expressed in terms of coends. Let C be a small category and $F\colon \mathsf{C} \times \mathsf{C}^{\mathrm{opp}} \to \mathsf{D}$ a functor. A ***coend*** for F is an object D of D together a family of morphisms $\zeta_X\colon F(X,X) \to D$ indexed by the objects X of C such that

(a) for all morphisms $X \to Y$ in C,

$$\begin{tikzcd} & F(X,X) \arrow[dr, "\zeta_X"] & \\ F(X,Y) \arrow[ur, "F(X, X\to Y)"] \arrow[dr, "F(X\to Y, Y)"'] & & D \\ & F(Y,Y) \arrow[ur, "\zeta_Y"'] & \end{tikzcd}$$

commutes;

(b) the pair (D,ζ) is universal with respect to (a).

When it exists, the coend is uniquely determined up to a unique isomorphism. The definition can be extended to essentially small categories by choosing a skeleton for the category.

Now consider a category C and functor $\omega\colon \mathsf{C}\to\mathsf{Vecf}(k)$, as in 3.20. Let

$$C(\omega)=\varinjlim_{[X]}\mathrm{End}(\omega|\langle X\rangle)^{\vee}.$$

For each X in C, we have an isomorphism

$$\omega(X)\otimes\omega(X)^{\vee}\to\mathrm{End}(\omega(X))^{\vee},\quad (a\otimes f)(\alpha)=f(\alpha(a)),$$

and hence morphism $\zeta_X\colon \omega(X)\otimes\omega(X)^{\vee}\to C(\omega)$. The pair $(C(\omega),\zeta)$ is the coend for the functor $\omega\otimes\omega^{\vee}\colon \mathsf{C}\times\mathsf{C}^{\mathrm{opp}}\to\mathsf{Vecf}(k)$.

NOTES Theorem 3.20 is proved Takeuchi 1977, but was known earlier to Bourbaki (Serre 1993, §2, Thm 3).

Categories of comodules

Let (C,Δ,ϵ) be a coalgebra over the field k (co-associative with co-identity), and let ω be the forgetful functor $\mathsf{coModf}(C)\to\mathsf{Vecf}(k)$.

By definition, an object of $\mathsf{coModf}(C)$ is a pair (M,ρ_M) with M a finite-dimensional k-vector space and $\rho_M\colon M\to M\otimes C$ a k-linear map satisfying certain conditions. On varying M, we obtain a natural transformation $\rho\colon \omega\to\omega\otimes C$. On combining ρ with a k-linear map $a\colon C\to V$, where V is a k-vector space (not necessarily finite-dimensional), we get a natural transformation $\alpha\colon \omega\to\omega\otimes V$.

PROPOSITION 3.25 *The map of k-vector spaces*

$$a\mapsto\alpha\colon\ \mathrm{Hom}_{k\text{-linear}}(C,V)\to\mathrm{Nat}(\omega,\omega\otimes V)\tag{59}$$

is an isomorphism.

PROOF We construct an inverse. Let α be natural transformation $\omega\to\omega\otimes V$. Let $c\in C$, and let M be a finite-dimensional subcomodule containing c (which exists by 1.15). Then $\alpha_M(c)\in M\otimes V$, and its image under $\epsilon\otimes\mathrm{id}_V$ lies in V. The map $\alpha\mapsto(c\mapsto(\epsilon\otimes\mathrm{id}_V)(\alpha_M(c))$ is well-defined, and is the required inverse. □

The map (59) is natural in V, and so we have a natural isomorphism of functors

$$\mathrm{Hom}_{k\text{-linear}}(C,-)\simeq\mathrm{Nat}(\omega,\omega\otimes -).\tag{60}$$

REMARK 3.26 The comultiplication map $\Delta\colon C\to C\otimes C$ corresponds under (59) to a natural transformation

$$\omega\to\omega\otimes(C\otimes C).$$

This can be shown to be the composite

$$\omega\xrightarrow{\rho}\omega\otimes C\xrightarrow{\rho\otimes C}(\omega\otimes C)\otimes C\simeq\omega\otimes(C\otimes C).$$

The coidentity map $\epsilon\colon C\to k$ corresponds under (59) to the natural isomorphism $\omega\to\omega\otimes k$. As the underlying vector space of C represents the functor $\mathrm{Nat}(\omega,\omega\otimes -)$, we see that (C,Δ,ϵ) can be recovered from the pair $(\mathsf{coModf}(C),\omega)$ (uniquely, up to a unique isomorphism).

A homomorphism $m\colon C\otimes C\to C$ of k-coalgebras defines a C-coaction $\rho^m_{M,N}$

$$M\otimes N\xrightarrow{\rho_M\otimes\rho_N} M\otimes C\otimes N\otimes C\xrightarrow{M\otimes\gamma_{C,N}\otimes N} M\otimes N\otimes C\otimes C\xrightarrow{M\otimes N\otimes m} M\otimes N\otimes C$$

on $M\otimes N$ for any C-comodules M,N. On varying M and N in $\mathsf{coModf}(C)$, we get a natural transformation

$$\rho^m\colon \omega\otimes\omega\to\omega\otimes\omega\otimes C.$$

PROPOSITION 3.27 *The map*

$$m\mapsto\rho^m\colon\ \mathrm{Hom}_{k\text{-coalgebra}}(C\otimes C,C)\to\mathrm{Nat}(\omega\otimes\omega,\omega\otimes\omega\otimes C)$$

is an isomorphism of k-vector spaces.

PROOF Construct an inverse, as in 3.25. □

TODO 7 Add proof, aax**3.

Define

$$\phi^m\colon \mathsf{coModf}_C\times\mathsf{coModf}_C\to\mathsf{coModf}_C$$

to be the functor sending a pair of C-comodules (M,ρ_M), (N,ρ_N) to $(M\otimes N,\rho^m_{M,N})$.

PROPOSITION 3.28 *The map $m\mapsto\phi^m$ defines a one-to-one correspondence between the set of k-coalgebra homomorphisms $m\colon C\otimes_k C\to C$ and the set of k-bilinear functors*

$$\phi\colon \mathsf{coModf}_C\times\mathsf{coModf}_C\to\mathsf{coModf}_C$$

with the property that $\phi(M,N)=M\otimes N$ as k-vector spaces.

(a) *The homomorphism m is associative (51) if and only if, for all M, N, P in $\mathsf{coModf}(C)$, the canonical isomorphism of k-vector spaces*

$$M\otimes(N\otimes P)\simeq(M\otimes N)\otimes P$$

is an isomorphism of C-comodules.

(b) *There exists a k-coalgebra homomorphism $e\colon k\to C$ satisfying (52), p. 80, if and only if there exists a C-comodule U with underlying k-vector space of dimension 1 such that the canonical isomorphism of k-vector spaces*

$$U\otimes U\simeq U$$

is an isomorphism of C-comodules.

(c) *The homomorphism m is commutative (i.e., $m(a\otimes b)=m(b\otimes a)$ for all $a,b\in C$) if and only if, for all M, N in $\mathsf{coModf}(C)$, the canonical isomorphism of k-vector spaces*

$$M\otimes N\simeq N\otimes M$$

is an isomorphism of C-comodules.

PROOF The first assertion is a restatement 3.27.

(a) Similar to (c).

(b) The map

$$M \simeq M \otimes k \xrightarrow{\rho_M \otimes e} M \otimes C \otimes C \xrightarrow{M \otimes m} M \otimes C, \quad M \in \mathsf{coModf}(C),$$

corresponds under (60) to $c \mapsto m(c \otimes e) : C \to C$. Therefore the right identity property (52) holds if and only if the above map is ρ_M (see **??**). This means precisely that the canonical isomorphism $M \otimes k \simeq M$ of k-vector spaces is a C-comodule map.

(c) Note that m is commutative if and only if $m = m \circ \gamma$, where $\gamma : C \otimes C \to C \otimes C$ is $c \otimes d \mapsto d \otimes c$. This holds if and only if

$$(\rho^m : M \otimes N \to M \otimes N \otimes C) = (\rho^{m \circ \gamma} : M \otimes N \to M \otimes N \otimes C).$$

From the definition of ρ^m, we see that this is the case if and only if the canonical isomorphism of k-vector spaces $M \otimes N \simeq N \otimes M$ is an isomorphism of C-comodules. □

NOTES Proposition 3.28 is from Saavedra (1972, II, 2.6.3). For another exposition, see Szamuely 2009, 6.2.

TODO 8 TBA Need to rewrite the proof of 3.28. See IV, **??**, **??**. See aax.

Summary

3.29 Let C be an essentially small k-linear abelian category, and let $\omega : \mathsf{C} \to \mathsf{Vecf}_k$ be an exact faithful k-linear functor.

- The k-coalgebra
$$C(\omega) \stackrel{\text{def}}{=} \varinjlim_{[X]} \operatorname{End}(\omega|\langle X \rangle)^\vee$$
acts on each object $\omega(Y)$, $Y \in \operatorname{ob}\mathsf{C}$, and so ω defines a functor $\mathsf{C} \to \mathsf{coModf}(C(\omega))$. This is an equivalence of categories.
- If C is a monoidal category and ω is a monoidal functor, then $C(\omega)$ is a bialgebra. Specifically, the monoidal structure on C provides $C(\omega)$ with a multiplication.
- If, in addition, C is symmetric, then $C(\omega)$ is a commutative bialgebra.
- If, in addition, C is rigid, then $C(\omega)$ is a Hopf algebra.
- The k-coalgebra $C(\omega)$ is equal to the coend of the the functor
$$\omega \otimes \omega^\vee : \mathsf{C} \times \mathsf{C}^{\mathrm{opp}} \to \mathsf{Vecf}(k).$$

Completion of the proof of the main theorem 3.1

THEOREM 3.30 *Let $(\mathsf{C}, \otimes)$ be an essentially small k-linear abelian $\otimes$-category and $\omega : \mathsf{C} \to \mathsf{Vecf}(k)$ an exact faithful k-linear $\otimes$-functor.*

(a) *The functor $\mathcal{E}nd^{\otimes}(\omega)$ of k-algebras is represented by an affine monoid scheme G over k.*

(b) *The functor $\mathsf{C} \to \mathsf{Repf}(G)$ defined by ω is an equivalence of $\otimes$-categories.*

PROOF Because of the correspondence between monoid schemes and bialgebras (and their representations) (see §1), this is simply a restatement of part of 3.29. □

If $(\mathsf{C},\otimes)$ is rigid, ω is automatically faithful (I, 6.7). Moreover, $\mathcal{E}nd^{\otimes}(\omega)=\mathcal{A}ut^{\otimes}(\omega)$ (I, 5.7), and so $G(R)$ is a group for all k-algebras R, which implies that G is a group scheme. This completes the proof of Theorem 3.1.

REMARK 3.31 (a) Let (C,ω) be $(\mathsf{Repf}(G),\omega^G)$. On following through the proof of Theorem 3.1 in this case one recovers Corollary 2.2: $\mathcal{A}ut^{\otimes}(\omega^G)$ is represented by G.

(b) In the proof of Theorem 3.30, it is possible to replace Proposition 3.28 with V, **??**.

ASIDE 3.32 If in Theorem 3.30, $(\mathsf{C},\otimes)$ is a monoidal rather than a symmetric monoidal category, then the Hopf algebra C may be neither commutative nor cocommutative, and this is an interesting way of constructing such Hopf algebras. For more on this theme, see Breen 1994, 1.5.

4 Nori's criterion to be a neutral tannakian category

The following is a sometimes useful criterion for a category with a tensor product and forgetful functor to be tannakian.

PROPOSITION 4.1 *Let* C *be an essentially small* k*-linear abelian category and* $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ *a functor,* k*-linear in each variable. Suppose that there are given an exact faithful* k*-linear functor* $F\colon \mathsf{C}\to\mathsf{Vecf}(k)$ *and natural isomorphisms*

$$\alpha_{X,Y,Z}\colon X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$
$$\gamma_{X,Y}\colon X\otimes Y\to Y\otimes X$$

with the following properties,

(a) $F\circ\otimes=\otimes\circ(F\times F)$;

(b) $F(\alpha_{X,Y,Z})$ *is the usual associativity constraint in* $\mathsf{Vecf}(k)$*;*

(c) $F(\gamma_{X,Y})$ *is the usual commutativity constraint in* $\mathsf{Vecf}(k)$*;*

(d) *there exists a unit object* (U,u) *in* $(\mathsf{C},\otimes)$ *such that* (FU,Fu) *is a unit object in* $\mathsf{Vecf}(k)$*;*

(e) *if* $F(L)$ *has dimension* 1*, then there exists an object* L^{-1} *in* C *and an isomorphism* $L\otimes L^{-1}\to U$.

Then $(C,\otimes,\alpha,\gamma)$ *is a tannakian category over* k*, and* F *is a* k*-valued fibre functor.*

PROOF Certainly $(\mathsf{C},\otimes,\alpha,\gamma)$ is a k-linear abelian $\otimes$-category, and Theorem 3.30 shows that F defines an equivalence of $\otimes$-categories $\mathsf{C}\xrightarrow{\approx}\mathsf{Repf}(G)$, where G is the affine monoid scheme over k representing $\mathcal{E}nd^{\otimes}(F)$. Thus, we may assume $\mathsf{C}=\mathsf{Repf}(G)$ and that F is the forgetful functor. Let (U,u) be as in (d). Because it is a unit object in $\mathsf{Repf}(G)$, U can be identified with k (trivial action of G). Let $\lambda\in G(R)$. If L in $\mathsf{Repf}(G)$ has dimension 1, then $\lambda_L\colon L\to L$ is invertible, as follows from the existence of a G-isomorphism $L\otimes L^{-1}\to U$. It follows that λ_X is invertible for all X in $\mathsf{Repf}(G)$, because

$$\det(\lambda_X)\overset{\text{def}}{=}\bigwedge\nolimits^{d}\lambda_X=\lambda_{\bigwedge^d X},\quad d=\dim X,$$

is invertible. Thus, G is an affine group scheme. □

NOTES Nori (1976, §1) adopts the statement of 4.1 as a definition of tannakian category.

TODO 9 Check this once again. aax 2.8

5 The functor defined by a homomorphism of group schemes

Let $f: G \to H$ be a homomorphism of affine group schemes over k. Using f, we can regard an H-module as a G-module. In this way, we get a k-linear exact $\otimes$-functor

$$\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$$

compatible with the forgetful functors.

THEOREM 5.1 *Let $f: G \to H$ be a homomorphism of affine group schemes over k.*

(a) *The homomorphism f is faithfully flat if and only if the functor ω^f is fully faithful and its essential image is closed under subobjects.*[1]

(b) *The homomorphism f is a closed immersion if and only if every object of $\mathsf{Repf}(G)$ is a subquotient of an object in the image of ω^f.*

Sometimes it is possible to omit the condition on subobjects in (a).

COROLLARY 5.2 *A homomorphism $f: G \to H$ of affine group schemes is faithfully flat if ω^f is fully faithful and at least one of the following conditions holds:*

(a) $\mathsf{Repf}(G)$ *is semisimple (for example, G reductive and $\mathrm{char}(k) = 0$);*

(b) *G or H is profinite.*

PROOF We have to show that the essential image is closed under subobjects.

(a) When B is semisimple, every subobject of $F(X)$ is of the form $e(F(X))$ with e an idempotent in $\mathrm{End}(F(X))$, and $e(F(X)) \simeq F(e'X)$ with e' the idempotent in $\mathrm{End}(X)$ mapping to e.

(b) See Biswas et al. 2021, Lemma 2.1. □

Proof of (a) of Theorem 5.1

For a k-algebra A (not necessarily commutative), we let ${}_A\mathsf{Modf}_k$ denote the category of left A-modules finite-dimensional over k.

Let $f: A \to B$ be a homomorphism of k-algebras. Using f, we can regard a B-module as an A-module and ${}_B\mathsf{Modf}_k$ as a subcategory of ${}_A\mathsf{Modf}_k$.

LEMMA 5.3 *Assume that B is finite-dimensional over k. The homomorphism $f: A \to B$ is surjective if and only if ${}_B\mathsf{Modf}_k$ is a full subcategory of ${}_A\mathsf{Modf}_k$ stable under forming submodules.*

PROOF If f is surjective, then the subcategory ${}_B\mathsf{Modf}_k$ certainly has the claimed properties. For the converse, let $\bar{A}$ denote the image of A in B. Then $\bar{A}$ is an A-submodule of B, and hence also a B-submodule. As it contains the identity element 1 of B, it equals B. □

Let $f: C \to D$ be a homomorphism of k-coalgebras. Using f, we can regard a C-comodule as a D-comodule and $\mathsf{coModf}(C)$ as a subcategory of $\mathsf{coModf}(D)$.

LEMMA 5.4 *The homomorphism $f: C \to D$ is injective if and only if $\mathsf{coModf}(C)$ is a full subcategory of $\mathsf{coModf}(D)$ closed under subobjects.*

[1] Let $F: \mathsf{A} \to \mathsf{B}$ be an exact fully faithful functor of abelian categories. We say that its essential image is ***closed under subobjects*** if, for all objects X of A, every subobject of $F(X)$ is isomorphic to $F(Y)$ for some subobject Y of X.

PROOF If C is finite-dimensional over k, this follows from 5.3 applied to $f^\vee : D^\vee \to C^\vee$ (see 3.19). In the general case, we can write C as a union $C = \bigcup C_i$ of finite-dimensional k-subcoalgebras (1.19), and correspondingly $\mathsf{coModf}(C) = \bigcup_i \mathsf{coModf}(C_i)$. Now the statement for C follows from the statement for the C_i. □

We now prove (a) of Proposition 5.1.

If $f : G \to H$ is faithfully flat, and therefore an epimorphism, then $\mathsf{Repf}(H)$ can be identified with the subcategory of $\mathsf{Repf}(G)$ of representations of G factoring through H. It is therefore obvious that ω^f has the stated properties. Conversely, if ω^f has the stated properties, then the homomorphism $\mathcal{O}(H) \to \mathcal{O}(G)$ of Hopf algebras over k is injective (5.4), and so faithfully flat (1.11).

Proof of (b) *of Proposition 5.1*

Let $f : G \to H$ be a homomorphism of affine group schemes over k. Let C be the strictly full subcategory of $\mathsf{Repf}(G)$ whose objects are subobjects of objects of the form of $\omega^f(Y)$, $Y \in \mathrm{ob}(\mathsf{Repf}(H))$. The functors

$$\mathsf{Repf}(H) \to \mathsf{C} \to \mathsf{Repf}(G)$$

correspond (see 3.20) to homomorphisms of k-coalgebras

$$\mathcal{O}(H) \to B \to \mathcal{O}(G).$$

As C is closed under taking subobjects in $\mathsf{Repf}(G)$, we see that $B \to \mathcal{O}(G)$ is injective (Lemma 5.4). Moreover, for $Y \in \mathrm{ob}(\mathsf{Repf}(H))$,

$$\mathrm{End}(\omega_G | \langle \omega^f(Y) \rangle) \to \mathrm{End}(\omega_H | \langle Y \rangle)$$

is injective, where ω_G and ω_H are the forgetful functors, and so $\mathcal{O}(H) \to B$ is surjective.

We now prove (b) of Proposition 5.1.

If f is a closed immersion, then $\mathcal{O}(H) \to \mathcal{O}(G)$ is surjective, and it follows that $B \simeq \mathcal{O}(G)$ and $\mathsf{C} = \mathsf{Repf}(G)$.

Conversely, if $\mathsf{C} = \mathsf{Repf}(G)$, then $B = \mathcal{O}(G)$, and $\mathcal{O}(H) \to \mathcal{O}(G)$ is surjective, i.e., f is a closed immersion.

Complements

5.5 Let $f : G \to H$ be a homomorphism of affine algebraic groups over k. If f is an epimorphism in the category of affine group schemes, then $\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ is fully faithful, and so $\mathsf{Repf}(H)$ is a strictly full abelian subcategory of $\mathsf{Repf}(G)$ closed under the formation of direct sums, tensor products, and duals, but not subquotients unless f is faithfully flat. For example, the inclusion $G \hookrightarrow H$ of a proper parabolic subgroup G into a smooth connected algebraic group H is an epimorphism but is not faithfully flat. See Milne 2017, 5.44, and Brion 2017, Theorem 1.

5.6 Let G be an affine group scheme over k and N a normal subgroup scheme of G. Then $\mathsf{Repf}(G/N)$ is the full subcategory of $\mathsf{Repf}(G)$ whose objects are the representations of G on which N acts trivially. In this way, we get a one-to-one correspondence between the normal subgroup schemes of G and the tannakian subcategories of $\mathsf{Repf}(G)$ closed under subobjects.

5.7 Statement (a) of Theorem 5.1 generalizes. A homomorphism $f : G \to H$ of flat affine group schemes over a noetherian ring R is faithfully flat if and only if the functor $\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ on the categories of representations on finite R-modules is fully faithful and its essential image is closed under subobjects. See Hai et al. 2025, Theorem 7.2.

ToDo 10 Fix this.

5.8 Let

$$1 \to N \xrightarrow{f} G \xrightarrow{g} Q \to 1$$

be a sequence of homomorphisms of affine group schemes over k. Assume that f is a closed immersion and that g is faithfully flat. The sequence is exact if and only if the following three statements hold.

(a) Let $V \in \mathsf{Repf}(G)$. Then $\omega^f(V)$ is trivial if and only if $V \approx \omega^g(W)$ for some $W \in \mathsf{Repf}(Q)$.

(b) Let $V \in \mathsf{Repf}(G)$. There exists a subobject $V_0 \subset V$ such that $\omega^f(V_0)$ is the largest trivial subobject of $\omega^f(V)$.

(c) Every object of $\mathsf{Repf}(N)$ is a subobject of $\omega^f(V)$ for some V in $\mathsf{Repf}(G)$ (i.e., ω^f is dominant, 5.8).

See Esnault et al. 2008, Appendix A . For a different criterion, see dos Santos 2015, §4.2.

6 Properties of G reflected in Repf(G)

In view of the previous theorems, it is natural to ask how properties of G are reflected in $\mathsf{Repf}(G)$.

Finiteness and connectedness

Proposition 6.1 *An affine group scheme G over k is finite if and only if* $\mathsf{Repf}(G) = \langle X\rangle$ *for some representation X, i.e., the objects of* $\mathsf{Repf}(G)$ *are subquotients of a direct sum of copies of X.*

Proof If G is finite, then the regular representation X of G is finite-dimensional and has the required property. Conversely if $\mathsf{Repf}(G) = \langle X\rangle$, then $G = \operatorname{Spec} B$, where B is the linear dual of the finite k-algebra A_X in the proof of 3.20. □

Proposition 6.2 *An algebraic group G has no nontrivial finite quotients if and only if, for every representation X on which G acts nontrivially, the subcategory $\langle X\rangle$ is not stable under $\otimes$.*

Proof According to 5.1(a) and Proposition 6.1, the finite quotients of G correspond to the subcategories of $\mathsf{Repf}(G)$ of the form $\langle X\rangle$ stable under tensor products (hence tannakian). □

Corollary 6.3 *In characteristic zero, an algebraic group G is connected if* $\mathsf{Repf}(G)$ *has no symmetric monoidal subcategory with only finitely many simple objects up to isomorphism.*

Proof In characteristic zero, G is disconnected $\Rightarrow$ G has a nontrivial finite quotient $\Rightarrow$ $\mathsf{Repf}(G)$ has a tensor subcategory of the form $\langle X\rangle$. Such a category has only finitely many simple objects (the quotients of any composition series for X represent the isomorphism classes of simple objects in $\langle X\rangle$). □

The converse to the corollary is false: in characteristic zero, every unipotent algebraic group G is connected, but $\mathsf{Repf}(G)$ has a single simple object (up to isomorphism).

Algebraicity

PROPOSITION 6.4 *An affine group scheme G over k is algebraic if and only if $\mathsf{Repf}(G)$ admits a tensor generator, i.e., $\mathsf{Repf}(G) = \langle X\rangle^{\otimes}$ for some object X.*

PROOF Restatement of 3.2. □

This helps explain the definition 8.13: a tannakian category is algebraic if it is monogenic.

Smoothness; reducedness

6.5 Let G be an affine group scheme over a field k. Is there a criterion on $\mathsf{Repf}(G)$ for G to be smooth or reduced? In characteristic zero, every algebraic group is smooth, and over a perfect field of characteristic p, an algebraic group is smooth if and only if it is reduced (equivalently, geometrically reduced) (Milne 2017, 1.26). Note that $\mathsf{Repf}(\mu_p)$ is semisimple even though μ_p is not reduced (hence not smooth).

6.6 Let G be an affine group scheme over a perfect field k of characteristic p. Then G is reduced if and only if the relative Frobenius $\mathrm{Frob}_{G/k}$ is faithfully flat. Thus Proposition 5.1(a) applies.

6.7 Let T be a neutral tannakian category over a perfect field k of characteristic p. Define $\mathsf{T}^{(p)}$ to be category with the same objects as T but with

$$\mathrm{Hom}_{\mathsf{T}^{(p)}}(X,Y) = \mathrm{Hom}_{\mathsf{T}}(X,Y) \otimes_{k,x\mapsto x^p} k.$$

There is a k-linear exact $\otimes$-functor $\mathrm{Fr}_+ : \mathsf{T}^{(p)} \to \mathsf{T}$, and the affine group scheme attached to a fibre functor on T is reduced if and only if Fr_+ is fully faithful and its essential image is closed under subobjects. See Deninger and Wibmer 2023, Corollary 4.2.

Unipotent groups

6.8 An algebraic group G over k is ***unipotent*** if its only simple representations are the one-dimensional representations with G acting trivially. Thus, if G is unipotent, then every nonzero representation has a nonzero fixed vector, and an easy induction argument shows that, for any (V,r) in $\mathsf{Repf}(G)$, there exists a basis of V for which

$$r(G) \subset \mathbb{U}_n \stackrel{\text{def}}{=} \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & * \\ & & \ddots & \ddots & \\ & 0 & & 1 & * \\ & & & & 1 \end{pmatrix}.$$

i.e., there exists a flag such that G acts trivially on the quotients. Conversely, every algebraic subgroup of $\mathbb{U}_n$ is unipotent (see, for example, Milne 2017, 14.5). An algebraic group over k is unipotent if and only if every nontrivial algebraic subgroup of it admits a nontrivial homomorphism to $\mathbb{G}_a$ (ibid., 14.22). In characteristic zero, unipotent algebraic groups are connected.

Trigonalizable groups

6.9 An algebraic group G over k is ***trigonalizable*** if its only simple representations are those of dimension 1. An easy induction argument shows that, for any (V, r) in $\mathsf{Repf}(G)$, there exists a basis of V for which

$$r(G) \subset \mathbb{T}_n \stackrel{\text{def}}{=} \begin{pmatrix} * & * & * & \cdots & * \\ & * & * & & * \\ & & \ddots & \ddots & \\ & 0 & & * & * \\ & & & & * \end{pmatrix}.$$

Conversely, every algebraic subgroup of $\mathbb{T}_n$ is trigonalizable (see, for example, Milne 2017, 16.2). A smooth connected algebraic group over an algebraically closed field is solvable if and only if it is trigonalizable (Lie–Kolchin theorem, ibid., 16.30).

Reductive groups

6.10 A smooth connected algebraic group G over k is ***reductive*** if it has no nontrivial smooth connected normal unipotent algebraic subgroup and this condition continues to hold under extension of the base field k. When k is perfect, the condition has to be checked only over k, and a smooth connected algebraic group is reductive if and only if it has a faithful semisimple representation (see Milne 2017, 6.46, 19.17).

6.11 Let G be an affine group scheme over a field of characteristic zero. Then G is a reductive if and only if

(a) $\mathsf{Repf}(G)$ has a tensor generator (so G is algebraic; 6.4),

(b) $\mathsf{Repf}(G)$ contains no nontrivial object X such that $\langle X\rangle$ is stable under $\otimes$ (so G is connected; 6.2), and

(c) $\mathsf{Repf}(G)$ is semisimple (so G is reductive; 6.15 below).

Semisimple groups

6.12 A smooth connected algebraic group G over k is ***semisimple*** if it has no nontrivial smooth connected normal solvable algebraic subgroup and this condition continues to hold under extension of the base field k. When k is perfect, the condition has to be checked only over k. See Milne 2017, 6.44.

6.13 A reductive group G is semisimple if and only if its centre is finite. The centre of G is reflected in the gradations on $\mathsf{Repf}(G)$ (see 9.2 below). For example, let D be a diagonalizable algebraic group with character group M. To give a homomorphism $D \to Z(G)$ is the same as giving an M-gradation on $\mathsf{Repf}(G)$.

6.14 Let G be a reductive group. Then G/G^{der} is a torus, which is trivial if and only if G is semisimple. It follows that G is semisimple if and only if there do not exist nontrivial representations V and W of G such that $V \otimes W$ is trivial.

Semisimple tannakian categories

In this subsection, k is a field of characteristic 0 and $\bar{k}$ is an algebraic closure of k.

THEOREM 6.15 *Let G be a connected affine group scheme over k. The category $\mathsf{Repf}(G)$ is semisimple if and only if G is pro-reductive (i.e., a projective limit of reductive groups).*

The theorem fails (both implications are false) if k has nonzero characteristic. The theorem will be proved as a consequence of a series of propositions.

PROPOSITION 6.16 *The category $\mathsf{Repf}(G)$ is semisimple if and only if $\mathsf{Repf}_{\bar{k}}(G_{\bar{k}})$ is semisimple.*

PROOF Let G be an affine group scheme over k, and let k' be a finite extension of k. A representation of $G_{k'}$ on a finite-dimensional k'-vector space induces a representation of G on the underlying k-vector space together with a k-linear action of k' on the representation. In this way, we get an equivalence of categories

$$\mathsf{Repf}_{k'}(G_{k'}) \to \mathsf{Repf}_k(G)_{(k')}$$

(see B.19b). Now I, 7.16, shows that $\mathsf{Repf}_k(G)$ is semisimple if and only if $\mathsf{Repf}_{k'}(G_{k'})$ is semisimple (because k' is separable over k). The proposition follows. □

PROPOSITION 6.17 *Let (V, r) be a representation of a connected algebraic group G over k. A subspace $W \subset V$ is stable under G if and only if it is stable under* $\mathrm{Lie}(G)$.

PROOF We have

$$W \text{ is stable under } G \iff \mathrm{Stab}_G(W) = G.$$

As $\mathrm{Stab}_G(W)$ is smooth and G is connected,

$$\mathrm{Stab}_G(W) = G \iff \mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Lie}(G)$$

(Milne 2017, 10.15). On the other hand,

$$W \text{ is stable under } \mathrm{Lie}(G) \iff \mathrm{Stab}_{\mathrm{Lie}(G)}(W) = \mathrm{Lie}(G).$$

As $\mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Stab}_{\mathrm{Lie}(G)}(W)$ (ibid., 10.31), the statement follows. □

PROPOSITION 6.18 (WEYL) *Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Every finite-dimensional representation of $\mathfrak{g}$ is semisimple.*

PROOF For an algebraic proof, see Humphreys 1972, 6.3. We sketch Weyl's original proof. We can assume that $k = \mathbb{C}$. Let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$, and let G_0 be a connected simply-connected real Lie group with Lie algebra $\mathfrak{g}_0$. Then G_0 is compact, so every finite-dimensional representation (V, r) of it carries a $\mathfrak{g}_0$-invariant positive-definite form, namely, $\langle x, y\rangle_0 = \int_{G_0} \langle x, y\rangle dg$, where $\langle\, ,\, \rangle$ is any positive-definite form on V. It follows that every finite-dimensional (real or complex) representation of G_0 is semisimple, but, for any complex vector space V, the restriction map is an isomorphism

$$\mathrm{Hom}(G, \mathrm{GL}_V) \simeq \mathrm{Hom}(G_0, \mathrm{GL}_V),$$

and so every complex representation of G is semisimple. □

COROLLARY 6.19 *Let G be an algebraic group over k. If G is semisimple, then $\mathsf{Repf}(G)$ is semisimple.*

PROOF After 6.16, we may suppose that k is algebraically closed. The Lie algebra of G is semisimple (Milne 2017, 23.68), and so this follows from 6.17 and 6.18. □

PROPOSITION 6.20 *Let G be an algebraic group over an algebraically closed field k, and let N be a normal algebraic subgroup of G. If (V,r) is a semisimple representation of G, then $(V,r|N)$ is a semisimple representation of N.*

PROOF We may suppose that (V,r) is simple. Let W be a nonzero simple N-submodule of V. For any $g \in G(k)$, gW is an N-module and it is simple because $g \mapsto g^{-1}S$ maps N-submodules of gW to N-submodules of W. The sum $\sum gW$, $g \in G(k)$, is G-stable and nonzero, and therefore equals V. Thus V, being a sum of simple N-submodules, is semisimple (I, 7.14). □

We now prove Theorem 6.15. After 6.16, we may suppose that k is algebraically closed. As every finite-dimensional representation $G \to \mathrm{GL}_V$ of G factors through an algebraic quotient of G, and (by definition) G is pro-reductive if and only its algebraic quotients are reductive, we may suppose that G is algebraic.

If G is reductive, then $G = Z \cdot G'$, where Z is the centre of G and G' is the derived subgroup of G (Milne 2017, 19.25). Let $r\colon G \to \mathrm{GL}_V$ be a finite-dimensional representation of G. The algebraic group Z is a diagonalizable, so $r|Z$ is diagonalizable: $V = \bigoplus_i V_i$, where Z acts on the V_i through distinct characters χ_i (ibid., 12.12). Each V_i is stable under G'. The algebraic group G' is semisimple, so each V_i is a sum of simple G'-modules (6.19). Each of these simple submodules is stable under $G = Z \cdot G'$, and so V is a sum of simple G-modules.

Conversely, suppose that Repf(G) is semisimple and choose a faithful representation V of G. Let N be a normal unipotent algebraic subgroup of G. We have to show that $N = 1$. According to 6.20, V is semisimple as a representation of N: it is a sum of its simple subrepresentations. As N is unipotent, it acts trivially on each simple subrepresentation, hence on V, but V was chosen to be faithful, so this implies that $N = 1$, as required.

REMARK 6.21 The proposition can be strengthened as follows: the identity component G° of an affine group scheme G over k is pro-reductive if and only if Repf(G) is semisimple.

To prove this, we have to show that the category Repf(G) is semisimple if and only if Repf(G°) is semisimple. We may suppose that G is algebraic. As G° is a normal algebraic subgroup of G, the necessity follows from 6.20. For the sufficiency, let V be a representation of G. Replace G with its image in GL_V. Let W be a G-stable subspace of V. By assumption, there is a G°-equivariant map $p\colon V \to W$ such that $p|W = \mathrm{id}$. Define

$$q\colon \bar{k} \otimes V \to \bar{k} \otimes W, \quad q = \frac{1}{n}\sum_g g_W p g_V^{-1},$$

where $n = (G(\bar{k})\colon G^\circ(\bar{k}))$ and g runs over a set of coset representatives for $G^\circ(\bar{k})$ in $G(\bar{k})$. One checks easily that q has the following properties:

(a) it is independent of the choice of the coset representatives;

(b) for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, $\sigma(q) = q$;

(c) for all $y \in \bar{k} \otimes W$, $q(y) = q$;

(d) for all $g \in G(\bar{k})$, $g_W \cdot q = q \cdot g_V$.

Thus q is defined over k, restricts to the identity map on W, and is G-equivariant.

REMARK 6.22 An algebraic group G is said to be ***linearly reductive*** if $\mathsf{Repf}(G)$ semisimple. Thus, in characteristic zero, G is linearly reductive if and only if G° is reductive. An algebraic group G over a field of characteristic $p \neq 0$ is linearly reductive if and only if G° is of multiplicative type and p does not divide the index $(G : G^\circ)$. This was proved by Nagata in 1961 for smooth algebraic groups, and is often referred to as Nagata's theorem. See Demazure and Gabriel 1970, IV, §3, 3.6.

Tannakian categories with the Chevalley property

A tensor category over k is said to have the ***Chevalley property*** if the tensor product of any two semisimple objects is semisimple, or, in other words, if the subcategory of semisimple objects is a tensor subcategory. Chevalley's theorem says that the category of finite-dimensional representations of an abstract group over a field of characteristic zero has the Chevalley property (Milne 2017, 22.45). We can extend this to nonneutral tannakian categories.

THEOREM 6.23 *Let T be a tannakian category over a field k of characteristic zero. Then T has the Chevalley property.*

PROOF We may suppose that T is algebraic. Then there exists a finite extension k' of k such that $\mathsf{T}_{(k')}$ is a neutral tannakian category (IV, 10.1). Let V and W be semisimple objects in T, and let V' and W' be their images in $\mathsf{T}_{(k')}$. Then V' and W' are semisimple (I, 7.13a), and so $V' \otimes W'$ is semisimple. As $(V \otimes W)' \simeq V' \otimes W'$, it follows that $V \otimes W$ is semisimple (I, 7.13c). □

In nonzero characteristic, there is only the following theorem of Deligne and Serre.

THEOREM 6.24 *Let T be a tannakian category over a field of characteristic $p \neq 0$, and let $V_1, \dots, V_m$ be objects of T. If the V_i are semisimple and $\sum_{i=1}^{m} (\dim V_i - 1) < p$, then $V_1 \otimes \cdots \otimes V_m$ is semisimple.*

PROOF As for the preceding theorem, it suffices to prove this for $\mathsf{Repf}(G)$. In that case, it is proved in Serre 1994 when G is smooth and in Deligne 2014 in general. □

The bound $\sum (\dim V_i - 1) < p$ in the theorem is optimal, as the following example shows. Let $G = \mathrm{SL}_2$, and for $d \in \mathbb{N}$, let $V(d)$ denote the k-vector space of homogeneous polynomials of degree d in two symbols, so $\dim_k V(d) = d + 1$. There is a canonical action of G on $V(d)$ for which $V(d)$ is simple if $d < p$ and nonsemisimple if $d = p$. Let $d_1, \dots, d_m$ be integers such that $1 \leq d_i \leq p - 1$ and $\sum d_i = p$. Then

$$\sum (\dim V(d_i) - 1) = p,$$

and the homomorphism

$$(f_1, \dots, f_m) \mapsto f_1 \cdots f_m : V(d_1) \otimes \cdots \otimes V(d_m) \to V(p)$$

is surjective. As $V(p)$ is not semisimple, neither is $V(d_1) \otimes \cdots \otimes V(d_m)$.

EXERCISE 6.25 Let k be a field of characteristic p and G a finite group. Show that $\mathsf{Repf}(G)$ has the Chevalley property if and only if G has only one Sylow p-subgroup.

ASIDE 6.26 Let k be an algebraically closed field. A k-linear abelian rigid monoidal category is a ***fusion category*** if it is semisimple, has only finitely many isomorphism classes of simple objects, and $k \simeq \mathrm{End}(\mathbb{1})$. Note that these conditions imply that the category equals $\langle X \rangle$ for some object X. When G is an affine group scheme over k, $\mathsf{Repf}(G)$ is a fusion category if and only if G is a finite discrete group whose order is prime to the characteristic exponent of k (6.22, 6.1). Every symmetric fusion category such that $\dim(X) \in \mathbb{N}$ for all objects X is of this form (11.1).

7 Torsors

Basic definitions

In this subsection, we allow k to be a commutative ring. Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$.

7.1 Let G be an affine group scheme flat[2] over k, and let X be an affine scheme over k. An ***action*** of G on X is a morphism $X \times G \to X$ of k-schemes such that, for every k-algebra R, $X(R) \times G(R) \to X(R)$ is an action of the group $G(R)$ on the set $X(R)$. In other words, the following two diagrams commute,

$$\begin{tikzcd} X \times G \times G \arrow[r, "\mu \times G"] \arrow[d, "X \times m"] & X \times G \arrow[d, "\mu"] \\ X \times G \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} X \times \operatorname{Spec}(k) \arrow[r, "X \times e"] \arrow[dr, "\simeq"'] & X \times G \arrow[d, "\mu"] \\ & X. \end{tikzcd}$$

The action is said to be ***simply transitive*** if, for every R and pair of points (x_1, x_2) in $X(R)$, there is a unique g in $G(R)$ such that $gx_1 = x_2$. In other words, for all R, the map

$$(t, g) \mapsto (t, tg)\colon\ X(R) \times G(R) \to X(R) \times X(R)$$

is a bijection. This is equivalent to the morphism

$$X \times G \to X \times X$$

being an isomorphism.

7.2 Let $\mu\colon\ X \times G \to X$ be a simply transitive action of G on an affine k-scheme X. We say that (X, μ) is a ***torsor under*** G ***over*** k (for the fpqc topology) if $X(R) \neq \emptyset$ for some faithfully flat k-algebra R. For example, G acting on itself by right translation is a torsor under G over k (this is the ***trivial torsor***). There is an obvious notion of a morphism of torsors under G over k.

7.3 Let (X, μ) be a torsor under G over k. By assumption, there exists a $P \in X(R)$ for some R faithfully flat over k. For all R-algebras R',

$$g \mapsto Pg\colon\ G(R') \to X(R')$$

is a bijection compatible with the actions of $G(R')$, and so $X_R \simeq G_R$ as G_R-torsors. In other words, (X, μ) is locally trivial for the fpqc topology on $\operatorname{Spec} k$. Note that X is faithfully flat over k because it becomes faithfully flat over the faithfully flat k-algebra R.

We sometimes write "G-torsor over k" instead of a "torsor under G over k".

7.4 A ***sheaf torsor under*** G ***over*** k is a sheaf F for the fpqc topology on $\operatorname{Spec}(k)$ together with an action of G such that, locally, F is G-equivariantly isomorphic to G. When regarded as a sheaf, a torsor under G over k becomes a sheaf torsor under G over k. In this way, we get an equivalence from the category of torsors under G over k to the category of sheaf torsors under G over k. That the functor is essentially surjective follows from descent theory: let F be a sheaf torsor on $\operatorname{Spec}(k)$; by assumption, there exists a faithfully flat k-algebra R such that the restriction of F to $\operatorname{Spec} R$ is representable, and this implies that F is representable over $\operatorname{Spec} k$ (V, **??**).

[2] Hence faithfully flat, because of the existence of the identity section.

7.5 Let G be an affine group scheme over k, and suppose that G acts on a sheaf X in such a way that $G(R)$ acts simply transitively on $X(R)$ for all k-algebras X. Then X is torsor under G over k if it satisfies any one of the following equivalent conditions:

(a) for some faithfully flat k-algebra R, $X(R) \neq \emptyset$;

(b) X is represented by an affine scheme faithfully flat over k.

7.6 Let $G \to H$ be a homomorphism of affine group schemes flat over k, and let X be a torsor under G over k. The quotient of the sheaf $X \times H$ by the diagonal action $(x,h)g = (xg, g^{-1}h)$ of G is a sheaf torsor, hence a torsor, under H over k. The torsor is denoted by $X \wedge^G H$, and called the ***contracted product***. If $G \to H$ is faithfully flat, then so also is $X \to X \wedge^G H$.

ASIDE 7.7 The correspondence between affine group schemes over a ring and commutative Hopf algebras over the ring extends to a correspondence between torsors and Hopf-Galois extensions. For noncommutative Hopf algebras this gives a noncommutative notion of torsor. See ncatlab. We leave it as an exercise to the reader to interpret the construction in 7.6 in terms of Hopf algebras.

Projective limits

In this subsection, k is a field. We show that torsors for affine group schemes over algebraically closed fields are trivial.

THEOREM 7.8 *Let X be a torsor under an affine group scheme G over k. Let $G = \varprojlim G_i$, as in 1.21, and let $X_i = X \wedge^G G_i$. For all i, the map $X(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ is surjective; in particular, $X(k^{\mathrm{al}}) \neq \emptyset$.*

We first need a lemma from topology. Let $(X_i)_{i\in I}$, $(\phi_{i,j})_{i\leq j}$, be a filtered projective system of topological spaces and continuous maps. If the X_i are non-empty and compact (i.e., quasi-compact and T_2), then $\varprojlim X_i$ is nonempty according to a standard theorem. The next lemma shows that we may weaken T_2 to T_1 in this statement provided that we require the transition maps to be closed.

LEMMA 7.9 *Let $(X_i)_{i\in I}$, $(\phi_{i,j})_{i\leq j}$ be a filtered projective system of topological spaces and continuous maps. If*

(a) the X_i are non-empty, quasi-compact, and T_1, and

(b) the $\phi_{i,j}$ are closed maps,

then $\varprojlim X_i$ is nonempty. Furthermore, if, for some fixed i, the maps $\phi_{i,j} \colon X_j \to X_i$ are surjective for all $j \geq i$, then the map $\varprojlim X_j \to X_i$ is surjective.

PROOF Let $\mathcal{S}$ be the set of families $(A_i)_{i\in I}$ such that A_i is a nonempty closed subset of X_i and $\phi_{i,j}(A_j) \subset A_i$ for all $i, j \in I$ with $j \geq i$. Define a partial ordering on $\mathcal{S}$ by setting $(A_i) \leq (B_i)$ if $A_i \subset B_i$ for all $i \in I$. By quasicompactness $(\mathcal{S}, \geq)$ satisfies the hypotheses of Zorn's lemma, and so there exists an element $(A_i)_{i\in I}$ of $\mathcal{S}$ that is minimal with respect to $\leq$. By (b), $B_i \overset{\text{def}}{=} \bigcap_{j\geq i} \phi_{i,j}(A_j)$ is closed in A_i, and it is easy to see that $(B_i)_{i\in I} \in \mathcal{S}$. By minimality, $B_i = A_i$ for all i. We have shown that, for a minimal family (A_i), $\phi_{i,j}$ maps A_j onto A_i for all $j \geq i$.

For some fixed i, let $x_i \in A_i$, and define

$$C_j = \begin{cases} \phi_{ij}^{-1}(x_i) \cap A_j & \text{if } j \geq i \\ A_j & \text{otherwise.} \end{cases}$$

The condition T_1 implies that C_j is closed, hence $(C_j)_{j\in I} \in \mathcal{S}$. By minimality, $A_i = C_i = \{x_i\}$. As this is true for all i, we see that $(x_i)_{i\in I} \in \varprojlim X_i$. This proves the first statement, that $\varprojlim X_i \neq \emptyset$, and the second statement follows from the first applied to the projective system with $Y_j = \phi_{ij}^{-1}(x_i)$ for $j \geq i$, where x_i is any element of X_i. □

PROOF (OF THEOREM 7.8) We first remark, that if X is an affine scheme of finite type over k, say, $X = \operatorname{Spec} A$ with A a nonzero finitely generated k-algebra, then $X(k^{\mathrm{al}}) \neq \emptyset$. Indeed, for any maximal ideal $\mathfrak{m}$ of A, $k' \overset{\text{def}}{=} A/\mathfrak{m}$ is a finite extension of k (Zariski's lemma), and $X(k') \neq \emptyset$. Similarly, if $Y \to X$ is a faithfully flat map of affine schemes of finite type over k, then $Y(k^{\mathrm{al}}) \to X(k^{\mathrm{al}})$ is surjective.

In the situation of the theorem, if I contains a countable cofinal subset, then we may suppose that $I = \mathbb{N}$. The maps

$$\cdots \to X_j(k^{\mathrm{al}}) \to \cdots \to X_{i+1}(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$$

are surjective, and so the statement is obvious in this case.

For more general I, we want to apply Lemma 7.9. However, the transition maps $X_j(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ are not closed for the Zariski topology, and so we need to define a new "orbit topology".

Let Y be a torsor under an algebraic group H over k, and consider the collection $\mathcal{C}$ of subsets of $Y(k^{\mathrm{al}})$ that are finite unions of orbits $yH'(k^{\mathrm{al}})$, where y ranges over $Y(k^{\mathrm{al}})$ and H' ranges over the algebraic subgroups of H. These sets are closed for the Zariski topology on $Y(k^{\mathrm{al}})$, which is noetherian, and so any infinite intersection of such subsets is actually a finite intersection. As $y_1H_1(k^{\mathrm{al}}) \cap y_2H_2(k^{\mathrm{al}})$ is either empty or equal to $z(H_1 \cap H_2)(k^{\mathrm{al}})$ for any element z of the intersection, we see that every finite intersection of sets in $\mathcal{C}$, hence every intersection, lies in $\mathcal{C}$. It follows that the elements of $\mathcal{C}$ are the closed sets of a topology on $X(k^{\mathrm{al}})$ – this is the ***orbit topology***. As every $y \in H(k^{\mathrm{al}})$ is an orbit of the trivial group, the topology is T_1. It is quasi-compact because of the property we proved for infinite intersections.

We now prove Theorem 7.8. The transition maps $\phi_{ij}: X_j(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ are surjective. When we endow each set $X_i(k)$ with its orbit topology, they are continuous because, if H is an algebraic subgroup of G_i and $x \in X_i(k^{\mathrm{al}})$, then $\phi_{ij}^{-1}(xH(k^{\mathrm{al}})) = x'H'(k^{\mathrm{al}})$, where x' is any preimage of x in $X_j(k^{\mathrm{al}})$ and H' is the preimage of H in G_j. They are also closed because, if H is an algebraic subgroup of G_j and $x \in X_j(k^{\mathrm{al}})$, then $\phi_{ji}(xH(k^{\mathrm{al}})) = \phi_{ji}(x)H'(k^{\mathrm{al}})$, where H' is the image of H in G_i (an algebraic subgroup). Thus, the theorem follows from Lemma 7.9. □

COROLLARY 7.10 *Let G be an affine k-group scheme, and write G as a projective limit $\varprojlim G_i$, as in 1.21. For all i, the map $G(k^{\mathrm{al}}) \to G_i(k^{\mathrm{al}})$ is surjective.*

PROOF Apply Theorem 7.8 to $X = G$. □

REMARK 7.11 For any faithfully flat homomorphism $f: G \to H$ of affine group schemes over k, the map $f(k^{\mathrm{al}}): G(k^{\mathrm{al}}) \to H(k^{\mathrm{al}})$ is surjective (Demazure and Gabriel 1970, III, §3, 7.6).

REMARK 7.12 Let $G \to H$ be a faithfully flat homomorphism of algebraic groups over a field k and $X \to Y$ an equivariant morphism of homogeneous spaces. When $X(k^{\mathrm{al}})$ and $Y(k^{\mathrm{al}})$ are endowed with the orbit topology, the map $X(k^{\mathrm{al}}) \to Y(k^{\mathrm{al}})$ is closed and continuous. Using this, it is possible to prove Theorem 7.8 for homogeneous spaces (Stewart 1975, §2).

NOTES The orbit topology and Lemma 7.9 are used to prove Corollary 7.10 in Hochschild and Mostow 1957. In Stewart 1975 they are used to study projective limits of coset varieties.

8 Classification of the fibre functors

Let C be a neutral tannakian category over k. By definition, there exists a fibre functor ω with values in k and we proved (3.1) that the functor $\mathcal{A}ut^{\otimes}(\omega)$ of k-algebras is represented by an affine group scheme G over k. Moreover, in a natural way, ω takes values in $\mathsf{Repf}(G)$ and defines an equivalence $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$. For any fibre functor η with values in a k-algebra R, composition defines a pairing

$$\mathcal{H}om^{\otimes}(\omega_R,\eta) \times \mathcal{A}ut^{\otimes}(\omega_R) \to \mathcal{H}om^{\otimes}(\omega_R,\eta)$$

of functors of R-algebras. According to I, Proposition 5.7, $\mathcal{H}om^{\otimes}(\omega_R,\eta) = \mathcal{I}som^{\otimes}(\omega_R,\eta)$, so $\mathcal{A}ut^{\otimes}(\omega)(R')$ acts simply transitively on $\mathcal{H}om^{\otimes}(\omega,\eta)(R')$ for all R-algebras R'. To prove that $\mathcal{H}om^{\otimes}(\omega,\eta)$ is a G-torsor, it remains to show that it is representable by an affine scheme over k (7.5).

THEOREM 8.1 *Let C be a neutral tannakian category over k. let ω be a k-valued fibre functor, and let $G = \mathcal{A}ut^{\otimes}(\omega)$.*

(a) *For any fibre functor η on C with values in R, $\mathcal{H}om^{\otimes}(\omega_R,\eta)$ is representable by an affine scheme faithfully flat over* Spec R*; it is therefore a G-torsor.*

(b) *The functor $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega_R,\eta)$ determines an equivalence between the category of fibre functors on C with values in R and the category of G-torsors over R.*

PROOF Recall (I, 7.10), that we have defined $V \otimes X$ when V and X are objects of Vecf_k and C respectively. We let $\mathcal{H}om(V,X) = V^{\vee} \otimes X$. If $W \subset V$ and $Y \subset X$, then the ***transporter*** of W to Y is

$$(Y : W) \stackrel{\text{def}}{=} \operatorname{Ker}(\mathcal{H}om(V,X) \to \mathcal{H}om(W,X/Y).$$

Let $X \in \operatorname{ob}(\mathsf{C})$, and define

$$\begin{cases} A_X \subset \operatorname{End}(\omega X), & A_X = \bigcap_Y (\omega Y : \omega Y), \quad Y \subset X^n, \quad n \geq 1 \\ P_X \subset \operatorname{End}(\omega X, X), & P_X = \bigcap_Y (Y : \omega Y), \quad Y \subset X^n, \quad n \geq 1. \end{cases}$$

Then $\omega(P_X) = A_X$ and $P_X \in \operatorname{ob}(\langle X \rangle)$. For any R-algebra R', $\mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R')$ is the subspace of $\operatorname{Hom}(\omega(P_X) \otimes_k R', \eta(P_X) \otimes_R R')$ of maps respecting all $Y \subset X^n$, some n. It therefore equals $\eta(P_X) \otimes R'$. Thus

$$\mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \xrightarrow{\simeq} \operatorname{Hom}_{R\text{-linear}}(\eta(P_X^{\vee}), R').$$

Let Q be the ind-object $(P_X^{\vee})_X$, and let $B = \varinjlim A_X^{\vee}$. We saw in 3.28, that the tensor structure on C defines an algebra structure on B. Because ω is an equivalence $\mathsf{C} \to \mathsf{Repf}(G)$, it also defines a ring structure on Q, i.e., a map $Q \otimes Q \to Q$ in $\operatorname{Ind}(\mathsf{C})$ making $\omega(Q) \to B$ into an isomorphism of k-algebras. We have

$$\begin{aligned} \mathcal{H}om(\omega,\eta)(R') &= \varprojlim \mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \\ &= \varprojlim \operatorname{Hom}_{R\text{-linear}}(\eta(P_X^{\vee}), R') \\ &= \operatorname{Hom}_{R\text{-linear}}(\eta(Q), R), \end{aligned}$$

where $\eta(Q) \overset{\text{def}}{=} \varinjlim \eta(P_X^\vee)$. Under this correspondence,

$$\mathcal{H}om^{\otimes}(\omega,\eta)(R') = \mathrm{Hom}_{R\text{-algebra}}(\eta(Q), R'),$$

and so $\mathcal{H}om^{\otimes}(\omega,\eta)$ is represnted by $\eta(Q)$. We know (I, 8.5) that $\eta(P_X^\vee)$ is a projective R-module, in particular flat, and so $\eta(Q) = \varinjlim \eta(P_X^\vee)$ is flat over R. For each X, there is an epimorphism $P_X \twoheadrightarrow \mathbb{1}$, and the exact sequence

$$0 \to \mathbb{1} \to P_X^\vee \to P_X^\vee/\mathbb{1} \to 0$$

gives rise to an exact sequence

$$0 \to \eta(\mathbb{1}) \to \eta(P_X^\vee) \to \eta(P_X^\vee/\mathbb{1}) \to 0.$$

As $\eta(\mathbb{1}) = R$ and $\eta(P_X^\vee/\mathbb{1})$ is flat, this shows that $\eta(P_X^\vee)$ is a faithfully flat R-module. Hence $\eta(Q)$ is faithfully flat over R, which completes the proof that $\mathcal{H}om^{\otimes}(\omega,\eta)$ is a G-torsor.

To show that $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega,\eta)$ is an equivalence, we construct a quasi-inverse. Let T be a G-torsor over R. For a fixed X, define $R' \rightsquigarrow \eta_T(X)(R')$ to be the sheaf associated with

$$R' \rightsquigarrow T(R') \wedge^{G(R')} \left(\omega(X) \otimes R'\right).$$

Then $X \rightsquigarrow \eta_T(X)$ is a fibre functor on C with values in R. □

TODO 11 Rewrite this using Gabber's result. Indeed, we have

$$\omega = h^P \otimes \omega(P), \quad \eta = h^P \otimes \eta(P).$$

See aax 2.3.

TODO 12 Given a neutral tannakian category and a k-valued fibre functor, identify the affine scheme X in the category such that $\eta(X)$ represents $\mathcal{I}som^{\otimes}(\omega,\eta)$.

EXAMPLE 8.2 Let G be an affine group scheme, and let ω be the forgetful functor. Then $\mathcal{O}_G = \varinjlim$ TBA.

COROLLARY 8.3 *Any two fibre functors on a neutral tannakian category over k are locally isomorphic for the fpqc topology.*

PROOF Let η be an R-valued fibre functor and ω a k-valued fibre functor. Then $\mathcal{H}om^{\otimes}(\omega_R,\eta)$ is a torsor over $\operatorname{Spec} R$ for the fpqc topology, and so becomes trivial over some faithfully flat R-algebra R'. This means that ω and η become isomorphic over R'. If η_1 and η_2 are fibre functors with values in R_1 and R_2 respectively, then they become isomorphic to ω over some common ring faithfully flat over R_1 and R_2. □

COROLLARY 8.4 *Let C be a neutral tannakian category over k. Any two k-valued fibre functors ω and η of C become isomorphic over k^{al} (and over a finite extension of k if C is algebraic).*

PROOF Let ω and η be k-valued fibre functors on C. If C is algebraic, then $\mathcal{H}om(\omega,\eta)$ is represented by a scheme $X = \operatorname{Spec} A$, where A is a nonzero finitely generated k-algebra. For any maximal ideal of A, $A/\mathfrak{m}$ is a finite extension of k (Zariski's lemma), and $X(A/\mathfrak{m}) \neq \emptyset$.

In the general case, $\mathcal{H}om^{\otimes}(\omega,\eta)$ is represented by a torsor under the affine group scheme $\mathcal{A}ut^{\otimes}(\omega)$ over k, and so has a k^{al}-point by Theorem 7.8. □

A ***nonassociative algebra***[3] in a symmetric monoidal category is a pair (X,t) consisting of an object X and a morphism $t\colon X\otimes X\to X$ (no conditions). Let $A=(V,t)$ be a nonassociative algebra in $\mathsf{Vecf}(k)$. The functor of commutative k-algebras

$$R \rightsquigarrow \mathrm{Aut}(A\otimes R) \quad \text{(automorphisms of } R\text{-algebras)}$$

is represented by an algebraic subgroup of GL_V, denoted $\mathcal{A}ut(A)$ or $\mathcal{A}ut(V,t)$.

COROLLARY 8.5 *Let* $(\mathsf{C},\otimes)$ *be a neutral algebraic tannakian category over* k. *There exists a nonassociative algebra* (X,t) *in* C *such that, for any fibre functor over an extension* k' *of* k,

$$\mathcal{A}ut^{\otimes}(\omega)_{k'}=\mathcal{A}ut(\omega(X),\omega(t)).$$

PROOF As C is neutral, there exists a k-valued fibre functor ω_0, and ω_0 defines an equivalence of symmetric monoidal categories $\mathsf{C}\xrightarrow{\approxeq}\mathsf{Repf}(G)$, where $G=\mathcal{A}ut^{\otimes}(\omega_0)$. According to Milne 2020a, Theorem 1, $G=\mathcal{A}ut(A)$ for some nonassociative algebra $A=(V,t_V)$ in $\mathsf{Repf}(G)$. There exists a nonassociative algebra (X,t) in C and an isomorphism $\omega_0(X,t)\simeq(V,t_V)$ (unique up to a unique isomorphism). For any fibre functor ω with values in an extension k' of k,

$$\mathcal{A}ut^{\otimes}(\omega)_{k'}\subset\mathcal{A}ut(\omega(X),\omega(t)),$$

but ω becomes isomorphic to ω_0 over some extension of k', and so the inclusion is an equality. □

REMARK 8.6 Let (X,t) be as in 8.5, and let $A=\omega(X,t)$ for some k-valued fibre functor ω. For any k-algebra R, the functor $\eta\rightsquigarrow\eta(X,t)$ is an equivalence from the category of fibre functors over R to the category of forms of A over R (i.e., nonassociative algebras over R that become isomorphic to A over some faithfully flat R-algebra).

EXAMPLE 8.7 Let V and V' be vector spaces of the same dimension, each equipped with a nondegenerate quadratic form. There is a canonical equivalence between $\mathsf{Repf}(O(V))$ and $\mathsf{Repf}(O(V'))$ given by the $O(V)$-torsor of isomorphisms $V\to V'$.

By using Theorem 13.28 of Chapter I, we can extend Corollary 8.3 to nonneutral tannakian categories.

COROLLARY 8.8 *Any two fibre functors on a tannakian category are locally isomorphic for the fpqc topology.*

PROOF Let T be a tannakian category over k, and let ω_1 and ω_2 be R_1- and R_2- valued fibre functors. Then T has a fibre functor ω over a field K (I, 8.7), and $\mathsf{T}_{(K)}$ is a neutral tannakian category (I, 13.31) with fibre functor ω',

$$(X,\mu)\rightsquigarrow\omega(X)\otimes_{K\otimes_k K}K.$$

The functor ω_1 defines an $R_1\otimes_k K$-valued fibre functor ω_1' on $\mathsf{T}_{(K)}$,

$$(X,\mu)\rightsquigarrow\omega_1(X),$$

and similarly for ω_2. According to 8.3, there exists a K-algebra R, faithfully flat over both $R_1\otimes_k K$ and $R_2\otimes_k K$ such that $\omega_1'\otimes_{R_1\otimes K}R\approx\omega_2'\otimes_{RL2\otimes K}R$. Such an isomorphism defines an isomorphism $\omega_1\otimes_{R_1}R\to\omega_2\otimes_{R_2}R$. □

EXERCISE 8.9 Rewrite the proof of Theorem 8.1 in terms of Hopf–Galois extensions.

[3] This is shorthand for “possibly nonassociative algebra”.

NOTES

8.10 Let G be an affine group scheme over k. Let $\mathsf{C} = \mathsf{Repf}(G)$, and let ω be a fibre functor on C. If ω is the forgetful functor, then $G \simeq \mathcal{A}ut^{\otimes}(\omega)$, but otherwise $\mathcal{A}ut^{\otimes}(\omega)$ is the inner twist of G by the G-torsor $\mathcal{H}om(\omega_{\text{forget}}, \omega)$. Thus, except when k is algebraically closed, C only determines G up to an inner twist (i.e., it determines the band of G).

8.11 We noted in 8.10 that, over an algebraically closed field, an affine group scheme G is determined up to isomorphism by the pair $(\mathsf{Repf}(G), \otimes)$. Without $\otimes$, we have only the following results.

Let G be a connected reductive group over an algebraically closed field of characteristic zero. Then G is determined up to isomorphism by the set of isomorphism classes of its finite-dimensional semisimple representations endowed with the obvious sum and product (i.e., by the Grothendieck semiring of $\mathsf{Repf}(G)$). See, for example, Kazhdan et al. 2014.

A connected split reductive group G over a field of characteristic 0 is determined up to isomorphism by specifying a maximal torus T of G, the set of isomorphism classes of simple representations of G, and the character homomorphism from the Grothendieck ring of G to that of T. More precisely, it is possible to determine all isomorphisms compatible with the specified data.

9 Examples

Graded vector spaces

9.1 Let C be the category whose objects are the families $(V^n)_{n\in\mathbb{Z}}$ of vector spaces over k with finite-dimensional direct sum $V = \bigoplus V^n$. There is an obvious rigid symmetric monoidal structure on C for which $\mathrm{End}(\mathbb{1}) = k$ and $\omega\colon (V^n) \rightsquigarrow \bigoplus V^n$ is a fibre functor. Thus, according to Theorem 3.1, ω defines an equivalence of symmetric monoidal categories $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$ for some affine k-group scheme G. This equivalence is easy to describe: let $G = \mathbb{G}_m$ and send (V^n) to the representation of $\mathbb{G}_m$ on $V = \bigoplus V^n$ for which $\mathbb{G}_m$ acts on V^n through the character $\lambda \mapsto \lambda^n$.

Gradations on tannakian categories

9.2 Let M be a set. An ***M*-gradation** on an object X of an additive category is a decomposition $X = \bigoplus_{m\in M} X^m$. An ***M*-gradation** on an additive functor $u\colon \mathsf{C} \to \mathsf{C}'$ is an M-gradation on each $u(X)$, $X \in \mathrm{ob}(\mathsf{C})$, depending functorially on X.

Suppose now that M is an abelian group, and let D be the diagonalizable group over k whose character group is M. For example, if $M = \mathbb{Z}$, then $D = \mathbb{G}_m$, and if $M = \mathbb{Z}/n\mathbb{Z}$, then $D = \mu_n$.

An ***M*-gradation** on a tannakian category C over k can be variously described as follows:

(a) an M-gradation, $X = \bigoplus X^m$, on each object X of C that depends functorially on X and is compatible with tensor products in the sense that

$$(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes Y^s;$$

(b) an M-gradation on the identity functor id_{C} of C that is compatible with tensor products;

(c) a homomorphism $D \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$;

(d) a central homomorphism $D \to G$, where $G = \mathcal{A}ut^{\otimes}(\omega)$, for one (or every) fibre functor ω.

Definition (a) is simply a restatement of (b). By a central homomorphism in (d), we mean a homomorphism from D into the centre of G defined over k. Although G need not be defined over k, its centre is, and equals $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$, from which the equivalence of (c) and (d) follows. Finally, a homomorphism $w\colon D \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ defines a gradation $X = \bigoplus X^m$ on every $X \in \mathrm{ob}\,\mathsf{C}$: let X^m be the subobject on which $w(d)$ acts as $m(d) \in k$.

Note that, when regarded as a map into the centre of $\mathcal{A}ut^{\otimes}(\omega)$, the homomorphism in (d) is independent of ω (in fact, it is equal to the homomorphism in (c)).

NOTES The results in this subsection are from Saavedra 1972, IV, 1.1.

Representations of groups of multiplicative type

9.3 Let $\bar{k}$ be a separable algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. An algebraic group G over k is of ***multiplicative type*** if every representation of G becomes diagonalizable over $\bar{k}$. In characteristic zero, this is equivalent to G being commutative and its identity component being a torus. The character group $X^*(G) \stackrel{\text{def}}{=} \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_m)$ of a group G of multiplicative type is a finitely generated abelian group on which Γ acts continuously. Let $M = X^*(G)$, and let $k' \subset \bar{k}$ be a Galois extension of k over which all elements of M are defined. For any finite-dimensional representation V of G, we have a decomposition

$$V \otimes_k k' = \bigoplus_{m \in M} V_m, \quad V_m \stackrel{\text{def}}{=} \{v \in V \otimes_k k' \mid gv = m(g)v \text{ all } g \in G(k)\}.$$

A finite-dimensional vector space V over k together with a decomposition

$$k' \otimes V = \bigoplus_{m \in M} V_m$$

arises from a representation of G if and only if $V^{\sigma(m)} = \sigma V^m$ for all $m \in M$ and $\sigma \in \Gamma$. Thus an object of $\mathsf{Repf}(G)$ can be identified with a finite-dimensional vector space V over k together with an M-gradation on $V \otimes k'$ compatible with the action of Γ. See III, 5.26 below.

Filtrations of $\mathsf{Repf}(G)$

Let V be a vector space. A homomorphism $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_V$ defines a filtration

$$\cdots \supset F^nV \supset F^{n+1}V \supset \cdots, \quad F^nV = \bigoplus\nolimits_{i \geq n} V_i$$

of V, where $V = \bigoplus_i V_i$ is the gradation defined by λ.

Let G be an algebraic group over a field k. A homomorphism $\lambda\colon \mathbb{G}_m \to G$ defines a filtration $F^\bullet$ on V for every representation (V, r) of G, namely, that corresponding to $r \circ \lambda$. These filtrations are compatible with the formation of tensor products and duals, and they are exact in the sense that the functor $V \rightsquigarrow \mathrm{Gr}^\bullet(V)$ is exact. A functor $(V, r) \rightsquigarrow (V, F^\bullet)$ from representations of G to filtered vector spaces satisfying these conditions is called a ***filtration*** $F^\bullet$ of $\mathsf{Repf}(G)$, and a homomorphism $\lambda\colon \mathbb{G}_m \to G$ defining $F^\bullet$ is said to ***split*** $F^\bullet$. We write $\mathrm{Filt}(\lambda)$ for the filtration defined by λ.

Define F^0G to be the algebraic subgroup of G respecting the filtration on each representation of G, and, for $n \geq 1$, define F^nG to be the algebraic subgroup of F^0G acting trivially on the graded module $\bigoplus_i F^iV/F^{i+n}V$ attached to each representation V of G. Clearly, F^nG is unipotent for $n \geq 1$.

THEOREM 9.4 *Let G be a reductive group over a field k, and let $F^\bullet$ be a filtration of* $\mathsf{Repf}(G)$. *From the adjoint action of G on $\mathfrak{g}$, we acquire a filtration of $\mathfrak{g}$.*

(a) *There exists a cocharacter λ of G splitting the filtration $F^\bullet$.*

(b) *F^0G is a parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$.*

(c) *F^1G is the unipotent radical of F^0G, and* $\mathrm{Lie}(F^1G) = F^1\mathfrak{g}$.

(d) *The centralizer $Z(\lambda)$ of any cocharacter λ splitting $F^\bullet$ is a connected algebraic subgroup of F^0G such that the quotient map $q\colon F^0G \to F^0G/F^1G$ induces an isomorphism $Z(\lambda) \to F^0G/F^1G$, so*

$$F^0G = F^1G \rtimes Z(\lambda),$$

and the composite $q\circ\lambda$ of λ with q is central.

(e) *Two cocharacters λ and λ' of G define the same filtration of G if and only if they define the same group F^0G and $q\circ\lambda = q\circ\lambda'$; the cocharacters λ and λ' are then conjugate under F^1G.*

PROOF Choose a faithful representation V of G, and let P be the algebraic subgroup of G preserving the filtration on V. Then P is obviously parabolic, and so $P = P(\lambda)$ for some cocharacter λ of G, i.e., P is the unique smooth algebraic subgroup of G such that

$$P(k^{\mathrm{al}}) = \left\{g \in G(k^{\mathrm{al}}) \;\middle|\; \lim_{t\to 0} \lambda(t)\cdot g \cdot \lambda(t)^{-1} \text{ exists in } G(k^{\mathrm{al}})\right\}$$

see Milne 2017, 25.1. Now λ splits the filtration, and so (a), (b), and (c) follow from ibid., 13.3, 25.6. □

REMARK 9.5 It sometimes more convenient to work with ascending filtrations. To turn a descending filtration $F^\bullet$ into an ascending filtration $W_\bullet$, set $W_i = F^{-i}$; if μ splits $F^\bullet$, then $z \mapsto \mu(z)^{-1}$ splits $W_\bullet$. With this terminology, we have $W_0G = W_{-1}G \rtimes Z(\mu)$.

NOTES See Saavedra 1972, especially IV, 2.2.5.

TODO 13 Should add more details. See mo50566 for a confused discussion and a better discussion in mo467613. Should finally write out a detailed proof of this. aax 2.4

Filtered fibre functors

Let T be a tannakian category (not necessarily neutral) over a field k, and let R be a commutative k-algebra.

A ***graded R-module*** is an R-module M together with a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M^n$. With the obvious tensor structure, the graded R-modules become an R-linear abelian symmetric monoidal category. A graded R-module admits a dual if and only if it is finitely generated and projective. To give a gradation on an R-module M is the same as giving a representation of $\mathbb{G}_m$ on M.

An R-valued ***graded fibre functor*** on T is an exact k-linear $\otimes$-functor from T to the category of graded R-modules. It takes values in the finitely generated projective graded R-modules.

A ***filtered R-module*** is an R-module M together with a family $(F^nM)_{n\in\mathbb{Z}}$ of submodules

$$\cdots \supset F^nM \supset F^{n+1}M \supset \cdots$$

such that

$$\bigcup_{n\in\mathbb{Z}} F^nM = M, \qquad \bigcap_{n\in\mathbb{Z}} F^nM = 0.$$

A ***morphism*** $(M,(F^nM)) \to (N,(F^nN))$ of filtered R-modules is an R-linear map $f\colon M \to N$ such that $f(F^nM) \subset F^nN$ for all n. With the obvious tensor structure, the filtered R-modules become an R-linear symmetric monoidal category. A filtered R-module $(M,(F^nM))$ admits a dual if and only if M is finitely generated and projective and the submodules F^nM are direct summands of M locally for the Zariski topology on $\operatorname{Spec} R$. There is a $\otimes$-functor

$$M \rightsquigarrow \operatorname{Gr}^\bullet(M) \overset{\text{def}}{=} \bigoplus\nolimits_n F^nM/F^{n+1}M$$

from filtered R-modules to graded R-modules.

An R-valued ***filtered fibre functor*** on T is a k-linear $\otimes$-functor ω from T to the category of filtered R-modules such that the functor $X \rightsquigarrow \operatorname{Gr}^\bullet(\omega(X))$ is exact. For example, if ω is an R-valued graded fibre functor on T, then

$$X \rightsquigarrow (M,(F^nM)), \quad M \overset{\text{def}}{=} \bigoplus\nolimits_{i\in\mathbb{Z}} \omega(X)^i, \quad F^nM \overset{\text{def}}{=} \bigoplus\nolimits_{i\geq n} \omega(X)^i,$$

is a filtered fibre functor on T. Filtered fibre functors of this form are said to be ***splittable***.

In the last subsection, we defined a filtration on $\mathsf{Repf}(G)$ to be a filtration on its forgetful functor, and we saw that such a filtration is splittable if G is reductive.

THEOREM 9.6 *Let ω be a filtered fibre functor on T with values in a k-algebra R.*

(a) *There exists a faithfully flat R-algebra R' such that $\omega \otimes_R R'$ is splittable.*

(b) *Let ω' be the composite of ω with the forgetful functor to* $\mathsf{Mod}(R)$. *If the affine group scheme over R representing $\mathcal{A}ut^{\otimes}(\omega')$ is pro-smooth (i.e., a projective limit of smooth algebraic groups), then ω is splittable.*

PROOF See Ziegler 2015, Theorems 1.2 and 1.3. □

NOTES Saavedra (1972, IV, 2.2.1) states 9.6(a) as an open problem. In ibid., IV, 2.4, he gives proofs (due to Deligne) of the theorem under various additional hypotheses, for example, if T is neutral and k has characteristic zero.

Real envelopes; compact groups

9.7 Let K be a topological group. The category $\mathsf{Repf}_{\mathbb{R}}(K)$ of continuous representations of K on finite-dimensional $\mathbb{R}$-vector spaces is, in a natural way, a neutral tannakian category with the forgetful functor as an $\mathbb{R}$-valued fibre functor. There is therefore an affine group scheme $\widetilde{K}$ over $\mathbb{R}$, called the ***real envelope*** of K, and an equivalence of categories $\mathsf{Repf}_{\mathbb{R}}(\widetilde{K}) \xrightarrow{\approx} \mathsf{Repf}_{\mathbb{R}}(K)$ compatible with the forgetful fibre functors. This equivalence arises from a homomorphism $K \to \widetilde{K}(\mathbb{R})$. When K is compact, $K = \widetilde{K}(\mathbb{R})$ and $\widetilde{K}$ is of finite type if and only if K is a Lie group. See Serre 1993, §5.

An algebraic group G over $\mathbb{R}$ is said to be ***compact*** if $G(\mathbb{R})$ is compact and the canonical functor $\mathsf{Repf}_{\mathbb{R}}(G) \longrightarrow \mathsf{Repf}_{\mathbb{R}}(G(\mathbb{R}))$ is an equivalence. The second condition is equivalent to each connected component of $G(\mathbb{C})$ containing a real point (or to $G(\mathbb{R})$ being Zariski dense in G).

ASIDE 9.8 For the original Tannaka duality, see Tannaka 1938. For a contemporary review (by van der Waerden), see zbMATH 0020.00904. In modern language:

> Let G be a topological group and let $\Pi(G)$ be the symmetric monoidal category of representations of G on finite-dimensional $\mathbb{C}$-vector spaces. Let $F\colon \Pi(G) \to \mathsf{Vecf}(\mathbb{C})$ be the forgetful functor. Endow the collection of natural transformations $\alpha\colon F \to F$ with the coarsest topology such that the projections $\alpha \mapsto \alpha_V\colon \mathrm{End}(F) \to \mathrm{End}(V)$ are continuous. We say that a natural transformation α is tensor-preserving if it is the identity map on the trivial representation and the equality $\alpha_{V\otimes W} = \alpha_V \otimes \alpha_W$ holds, and that it is self-conjugate if it equals its complex conjugate. The set $\mathcal{T}(G)$ of all tensor-preserving self-conjugate natural transformations of F is a compact subgroup of $\mathrm{End}(F)$. If G has sufficiently many almost-periodic functions, then every element of G defines an element of $\mathcal{T}(G)$ and Tannaka shows that the embedding $G \to \mathcal{T}(G)$ has dense image. When G is compact, this map is an isomorphism of topological groups. The main tool in the proof is an examination of the prime ideals in the ring of almost-periodic functions on G.

For a historical study of Tannaka's 1938 paper, see Kümmerle 2024.

Representations of Lie algebras

9.9 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. The category $\mathsf{Repf}(\mathfrak{g})$ of representations of $\mathfrak{g}$ on finite-dimensional k-vector spaces is a tannakian category over k with the forgetful functor ω as a k-valued fibre functor (apply 4.1). We examine this category in two cases.

(a) Let $\mathfrak{g}$ be one-dimensional and assume that k is algebraically closed. The affine group scheme $\mathcal{A}ut_k^{\otimes}(\omega)$ is $D(M) \times \mathbb{G}_a$, where $D(M)$ is the diagonalizable group scheme with character group $M = (k, +)$, i.e., k viewed as an abelian group under addition. This can be deduced from Iwahori 1954.

(b) Let $\mathfrak{g}$ be semisimple, and assume that k has characteristic zero. Then $\mathcal{A}ut_k^{\otimes}(\omega)$ is the simply connected semisimple algebraic group over k with Lie algebra $\mathfrak{g}$ (Cartier 1956; Milne 2007).

9.10 Recall that a diagram is a root system (V, R) together with a lattice X such that $Q(R) \subset X \subset P(R)$. It is possible to attach a tannakian category to a diagram (V, R, X) without mentioning algebraic groups. Indeed, let $\mathfrak{g}$ be a split Lie algebra over a field k of characteristic zero with root system (V, R), and let $\mathsf{Repf}(\mathfrak{g})^X$ be the full subcategory of $\mathsf{Repf}(\mathfrak{g})$ of representations whose simple components have highest weight in X. This is a tannakian category over k, and the group attached to the forgetful fibre functor is a split semisimple algebraic group over k with diagram (V, R, X). See Milne 2007, 8.3.

Nori's (true) fundamental group

TODO 14 Look at this again. aax

9.11 Let S be a scheme over a field k. By a ***vector sheaf*** on S, we mean a locally free sheaf of finite rank (equivalently, a vector bundle). A vector sheaf E on S is ***finite*** if there exist polynomials $g, h \in \mathbb{N}[t]$, $g \neq h$, such that $g(E) \approx h(E)$. For example, an invertible L sheaf on S is finite if and only of $L^{\otimes m} \approx \mathcal{O}_S$ for some m. With the obvious structures, the finite vector sheaves form k-linear rigid symmetric monoidal category with $\mathcal{O}_S$ as the unit object, but it is not necessarily abelian.

Define a vector sheaf E on a curve to be ***semi-stable*** if for every vector subsheaf $E' \subset E$,

$$\frac{\deg(E')}{\operatorname{rank}(E')} \leq \frac{\deg(E)}{\operatorname{rank}(E)} \stackrel{\text{def}}{=} \mu(E).$$

Let S be a complete connected reduced scheme over a field k. Following Nori 1976, we say that a vector sheaf on S is ***semi-stable*** if for every nonconstant morphism $f: C \to S$ with C a complete connected normal curve, f^*E is semi-stable with slope $\mu(f^*E) = 0$. Let $\mathsf{C}(S/k)$ denote the category of semi-stable vector sheaves on S that are subquotients of finite vector sheaves. If S has a k-rational point s, then $\mathsf{C}(S/k)$ is a tannakian category over k (in particular, abelian) with a k-valued fibre functor $\omega_s : E \rightsquigarrow E_s$. The affine group scheme attached to (C, ω_s) is called the ***fundamental group scheme*** $\pi_1^N(S, s)$ of S. It is a projective limit of finite group schemes over k. Note that $\mathsf{C}(S/k)$ also has a tautologous fibre functor η over S such that $\eta_s = \omega_s$.

Fix an $s \in S(k)$, and consider the triples (G, P, p), where G is a finite group scheme over k, P is a torsor under G over S, and $p \in P(k)$ maps to $s \in S(k)$. They form a category $N(S/k, s)$ in an obvious way. From a homomorphism $\pi_1^N(S, s) \to G$ from $\pi_1^N(S, s)$ to a finite group scheme G over k, we get an exact $\otimes$-functor $\phi : \mathsf{Repf}(G) \to \mathsf{C}(S/k)$, and hence a fibre functor $\eta = \eta \circ \phi$ over S, a fibre functor $\omega = \omega_s \circ \phi$ over k, and an isomorphism $\eta_s \to \omega$. Now $\mathcal{H}om(\eta, \omega_S)$ is a torsor under G over S together with a k-rational point lying over s. In this way, we obtain an equivalence of categories

$$\operatorname{Hom}(\pi_1^N(S, s), -) \to N(S/k, s), \tag{61}$$

where $\operatorname{Hom}(\pi_1^N(S, s), -)$ is the category whose objects are finite group schemes over k equipped with a homomorphism from $\pi_1^N(S, s)$ and whose morphisms are the homomorphisms of k-group schemes compatible with the morphism from $\pi_1^N(S, s)$.

Let $\pi_1^{\text{ét}}(S, s)$ denote the largest pro-étale quotient of $\pi_1^N(S, s)$. When k is algebraically closed, $\pi_1^{\text{ét}}(S, s)(k)$ is the usual étale fundamental group $\pi_1(S, s)$ of S since they classify the same objects. Let k be a perfect field, and let $\bar{S} = S \times \operatorname{Spec}(\bar{k})$, where $\bar{k}$ is an algebraic closure of k. The point $s \in X(k)$ defines a point $\bar{s} \in X(\bar{k})$. Because $\bar{s}$ comes from a k-rational point, $\operatorname{Gal}(\bar{k}/k)$ acts continuously on the étale fundamental group $\pi_1(\bar{S}, \bar{s})$. The associated pro-étale group scheme over k is $\pi_1^{\text{ét}}(\bar{S}, \bar{s})$. For example, both groups are trivial if $S = \operatorname{Spec}(k)$.

See Nori 1976 for the original account and Szamuely 2009, 6.7, for a more recent account. There is, by now, a large literature on the Nori fundamental group scheme and various extensions of it.

The Galois theory of linear differential equations.

A ***differential field*** is a field K, which we shall always assume to have characteristic zero, equipped with derivation, i.e., an additive map $\partial : K \to K$ such that $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$. We sometimes write a' for ∂a. For example, $(\mathbb{C}(T), \frac{d}{dT})$, is a differential field.

Let (K, ∂) be a differential field. A ***differential module*** (V, ∇) is a finite-dimensional K-vector space V together with an additive map ∇ such that $\nabla(fm) = f'm + f\nabla m$ for all $f \in K$, $m \in V$. The choice of a basis for V gives rise to a ***matrix differential equation***

$$y' = Ay, \quad A \in M_n(K), \quad y \in K^n.$$

Roughly speaking, the Picard–Vessiot field for a differential module is the differential field extension generated by the solutions of the matrix differential equation. The ***differential Galois group*** is then the group of differential k-algebra automorphisms of the Picard–Vessiot field. It has a natural structure of an algebraic group. The classical theory provides a Galois correspondence between the algebraic subgroups of the differential Galois group and the differential subfields of the Picard–Vessiot ring. The basic problem in the theory is how to compute the differential Galois group of a given differential module.

There is a tannakian interpretation of the above theory, which provides new insights, and which we now briefly describe.

Let (K,∂) be a differential field. The kernel of ∂ is a subfield k of K, called the ***constant field***. Let (V,∇) be a differential module over (K,∂). The subset $V^{\nabla} \overset{\text{def}}{=} \operatorname{Ker}(\nabla)$ is k-subspace, whose elements are called the ***horizontal vectors.*** Given an extension $(L,\partial) \subset (K,\partial)$ of differential fields, define (V_L,∇_L) to be the differential module over (L,∂) with $V_L = V \otimes_K L$ and such that, relative to a basis for V, ∇_L gives rise to a differential equation with the same matrix A as ∇.

The differential modules over (K,∂) form a tensor category over k. For example, the tensor product of (V_1,∇_1) and (V_2,∇_2) is $(V_1 \otimes V_2, \nabla_1 \otimes \nabla_2)$, where

$$(\nabla_1 \otimes \nabla_2)(v_1 \otimes v_2) = \nabla_1(v_1) \otimes v_2 + v_1 \otimes \nabla_2(v_2),$$

and the dual of (V,∇) is $(V^{\vee},\nabla^{\vee})$, where

$$\nabla^{\vee}(\phi)(v) = \partial(\phi(v)) - \phi(\nabla(v)).$$

The forgetful functor $(V,\nabla) \rightsquigarrow V$ is a K-valued fibre functor, and so the category is even tannakian.

Definition 9.12 Let (V,∇) be a differential module over a differential field (K,∂). A differential field extension $(L,\partial) \supset (K,\partial)$ is a ***Picard–Vessiot extension*** for (V,∇) if it has the following properties:

(a) (L,∂) has the same constant field k as (K,∂);
(b) the k-subspace $V_L^{\nabla_L}$ of horizontal vectors in V_L spans V_L as an L-vector space;
(c) the coordinates of the horizontal vectors of V_L, relative to an L-basis of V_L coming from a K-basis of V, generate the field L as an extension of K.

Note that if (L,∂) is a Picard–Vessiot extension for (V,∇), then the condition (b) holds for all differential modules in the tannakian subcategory $\langle(V,\nabla)\rangle^{\otimes}$ generated by (V,∇).

Theorem 9.13 *Let (V,∇) be a differential module over a differential field (K,∂).*

(a) *Let (L,∂) be a Picard–Vessiot extension for (V,∇). The functor*

$$\omega_L : \langle(V,\nabla)\rangle^{\otimes} \to \mathsf{Vecf}(k), \quad (W,\nabla) \rightsquigarrow (W_L)^{\nabla_L},$$

is a fibre functor on $\langle(V,\nabla)\rangle^{\otimes}$, and $\mathcal{A}ut^{\otimes}(\omega_L)$ is canonically isomorphic to the differential Galois group of (V,∇).

(b) *Every k-valued fibre functor on $\langle(V,\nabla)\rangle^{\otimes}$ arises, as in (a), from a Picard–Vessiot extension.*

In particular, when k is algebraically closed, (IV, 10.1) implies that there exists a Picard–Vessiot extension for (V,∇) (unique, up to a nonunique isomorphism).

Notes For more, see Deligne 1990, §9, van der Put and Singer 2003, and Szamuely 2009, 6.6.

Real Hodge structures

9.14 A ***real Hodge structure*** is a finite-dimensional vector space V over $\mathbb{R}$ together with a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus\nolimits_{p,q} V^{p,q}$$

such that $V^{p,q}$ and $V^{q,p}$ are conjugate complex subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$. There is an obvious rigid symmetric monoidal structure on the category $\mathsf{Hod}_{\mathbb{R}}$ of real Hodge structures, and

$$\omega\colon (V, (V^{p,q})) \rightsquigarrow V$$

is a fibre functor. The group corresponding to $\mathsf{Hod}_{\mathbb{R}}$ and ω is the real algebraic group $\mathbb{S}$ obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$, that is, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ (the Deligne torus). The real Hodge structure $(V, (V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V such that an element $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ acts on $V^{p,q}$ as $\lambda^{-p}\bar{\lambda}^{-q}$. There is a ***weight gradation*** $V = \bigoplus V_m$, where $V_m \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q}$. The functor $(V, (V^{p,q})) \rightsquigarrow (V_m)$ from $\mathsf{Hod}_{\mathbb{R}}$ to the category of graded real vector spaces corresponds to the homomorphism $\mathbb{G}_m \to \mathbb{S}$ that, on real points, is

$$t \mapsto t^{-1}\colon \mathbb{R}^{\times} \to \mathbb{C}^{\times}.$$

Rational Hodge structures

9.15 A ***rational Hodge structure*** is a $\mathbb{Q}$-vector space V together with a real Hodge structure on $V \otimes \mathbb{R}$ such that the weight gradation is defined over $\mathbb{Q}$. Thus, to give a rational Hodge structure on V is the same as giving a gradation $V = \bigoplus_m V_m$ on V together with a real Hodge structure of weight m on $V_m \otimes \mathbb{R}$ for each m. The ***Tate Hodge structure*** $\mathbb{Q}(m)$ is defined to be the $\mathbb{Q}$-subspace $(2\pi i)^m\mathbb{Q}$ of $\mathbb{C}$ with $h(z)$ acting as $(z\bar{z})^m$. It has weight $-2m$ and type $(-m, -m)$.

For a real Hodge structure (V, h), the $\mathbb{R}$-linear map $C = h(i)$ is called the ***Weil operator***. It acts as i^{q-p} on $V^{p,q}$, and $C^2 = h(-1)$ acts as $(-1)^m$ on V_m.

A ***polarization*** of a real Hodge structure (V, h) of weight m is a morphism of Hodge structures

$$\psi\colon V \otimes V \to \mathbb{R}(-m), \quad m \in \mathbb{Z},$$

such that

$$(x, y) \mapsto (2\pi i)^m \psi(x, Cy)\colon V \times V \to \mathbb{R}$$

is symmetric and positive-definite. A ***polarization*** of a rational Hodge structure V is a morphism of rational Hodge structures $\psi\colon V \otimes V \to \mathbb{Q}(-m)$ such that $\psi \otimes \mathbb{R}$ is a polarization of $V \otimes \mathbb{R}$. A rational Hodge structure is polarizable, i.e., admits a polarization, if and only if $V \otimes \mathbb{R}$ is polarizable. See Deligne 1979, 1.1.

The polarizable rational Hodge structures form a tannakian category $\mathsf{Hod}_{\mathbb{Q}}$ with the forgetful functor as a fibre functor. Let G denote the associated affine group scheme over $\mathbb{Q}$. Then,

(a) all algebraic quotients of G are reductive (in particular, connected);

(b) the quotient of G by its derived group is the (well-known) Serre protorus;

(c) the derived group of G is simply connected (hence a product of simply connected almost-simple algebraic groups over $\mathbb{Q}$);

(d) the simple factors of the adjoint group of G are the groups of the form $\mathrm{Res}_{F/\mathbb{Q}}(H)$, where F is a totally real number field and H is a geometrically simple algebraic group over F such that, for all embeddings σ of F in $\mathbb{R}$, the real algebraic group σH contains a compact maximal torus.

See, for example, Milne 2020b.

Hodge–Tate modules

9.16 Let K be a field of characteristic zero, complete with respect to a discrete valuation, whose residue field is perfect of characteristic $p \neq 0$. Let C be the completion of an algebraic closure K^{al} of K, and let $G = \mathrm{Gal}(K^{\mathrm{al}}/K)$. The cyclotomic character $\chi\colon G \to \mathbb{Z}_p^{\times}$ is defined by the action G on the p^nth roots of 1 in K^{al},

$$\sigma\zeta = \zeta^{\chi(\sigma)}, \quad \sigma \in G, \quad \zeta^n = 1 \text{ some } n.$$

Let V be a finite-dimensional C-vector space equipped with a semilinear action of G,

$$\sigma(cv) = \sigma(c)\sigma(v), \quad \sigma \in G, \quad c \in C, \quad v \in V.$$

For $i \in \mathbb{Z}$, let V^i denote the subset of V on which G acts through χ^i,

$$V^i = \{v \in V \mid \sigma v = \chi(\sigma)^i v, \text{ all } \sigma \in G\}.$$

It is a K-subspace of V, and we set $V(i) = V^i \otimes_K C$. The inclusion $V^i \hookrightarrow V$ extends to a C-linear map $V(i) \to V$, and Tate showed that the map

$$\bigoplus_i V(i) \to V$$

is injective. If it is also surjective, then V is said to be Hodge–Tate.

Now assume that the residue field is algebraically closed. An action of G on a finite-dimensional K-vector space V extends to a semilinear action of G on $V \otimes C$, and we say that V is Hodge–Tate if $V \otimes C$ is Hodge–Tate in the above sense. The Hodge-Tate modules for K form a tannakian category (Serre 1979).

10 Neutral tannakian categories over a base ring

In this section, we explain how to extend some of the results in this chapter to more general base rings. Throughout, R is a commutative ring. By a ***finite flat*** R-module, we mean an R-module that is finitely presented and flat (equivalently, finitely generated and projective, or locally free of finite rank). All Hopf algebras are commutative.

Affine group schemes and their representations

The correspondence between affine group schemes and Hopf algebras described in §1 holds over any commutative ring R. Let G be the affine group scheme corresponding to the Hopf algebra A, and let V be an R-module. To give a representation of G on V is the same as giving an A-comodule structure on V. In this way, we obtain an isomorphism(!) of categories

$$\mathsf{Rep}(G) \approxeq \mathsf{coMod}(A)$$

When G is flat over R, it is possible to endow the kernel and cokernel of a homomorphism of A-comodules with an A-comodule structure in such a way that $\mathsf{coMod}(A)$ becomes an abelian category (Serre 1968, 1.3). If, in addition, R is noetherian, then the category $\mathsf{coModf}(A)$ of right A-modules finitely generated over R is also an abelian category.

PROPOSITION 10.1 *When R is noetherian, every comodule over a flat R-coalgebra is a filtered union of its finitely generated subcomodules.*

PROOF See Serre 1993, 1.4. □

Correspondingly, when G is a affine group scheme flat over a noetherian ring R, every G-module is a filtered union of finite generated G-submodules.

PROPOSITION 10.2 *Let G be an affine group scheme flat and of finite type over a Dedekind domain R. There exists a finite flat R-submodule V of $\mathcal{O}(G)$, stable under G, such that the homomorphism $G \to \mathrm{GL}_V$ is a closed immersion.*

PROOF There exists a finitely generated R-submodule V of $\mathcal{O}(G)$, stable under G, containing a set of generators for $\mathcal{O}(G)$ (by 10.1). Now G flat over $R \implies \mathcal{O}(G)$ is torsion-free (as an R-module) $\implies V$ is torsion-free $\implies V$ is flat (because R is a Dedekind domain). It remains to show that the homomorphism $\alpha\colon \mathcal{O}(\mathrm{GL}_V) \to \mathcal{O}(G)$ defined by the action of G on V is surjective.

We may suppose that V is free. Let $\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ be the comultiplication map and $\epsilon\colon \mathcal{O}(G) \to R$ the co-identity map. Let $(e_i)_{1 \le i \le n}$ be a basis for V, and write $\Delta(e_j) = \sum_i e_i \otimes a_{ij}$, $a_{ij} \in \mathcal{O}(G)$. The image of α contains the a_{ij} (the choice of the basis (e_i), determines an isomorphism $\mathcal{O}(\mathrm{GL}_V) \simeq R[T_{ij}]$, and α maps T_{ij} to a_{ij}). As $\epsilon\colon \mathcal{O}(G) \to R$ is the co-identity,

$$e_j = (\epsilon \otimes \mathrm{id}_A)\Delta(e_j) = \sum\nolimits_i \epsilon(e_i) a_{ij},$$

and so the image of α contains V, which we chose to generate $\mathcal{O}(G)$. □

Thus, if R is a Dedekind domain and G is flat, then there is an embedding of G into GL_V for some finite flat R-module V. Such a V is a direct summand $F = V \oplus W$ of a free R-module F of finite rank. Extend the action of G on V to F by letting it act trivially on W and choose a basis for F. Now G is a closed algebraic subgroup of GL_n, $n = \operatorname{rank} F$.

Let S be a noetherian scheme and $\mathcal{E}$ a locally free $\mathcal{O}_S$-module of finite rank. The functor sending an S-scheme $f\colon T \to S$ to $\mathrm{Aut}_{\mathcal{O}_T}(f^*\mathcal{E})$ is representable by an affine group scheme $\mathrm{GL}_{\mathcal{E}}$ over S.

DEFINITION 10.3 A flat group scheme G over a noetherian scheme S is said to be ***linear*** if there exists a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of finite rank and an S-morphism $G \to \mathrm{GL}_{\mathcal{E}}$ that is both a closed immersion and a homomorphism.

Grothendieck showed that an algebraic group scheme of multiplicative type over a connected noetherian scheme S is linear if and only it is isotrivial, i.e., split by a finite étale covering $S' \to S$ of S. Gille extended this to reductive groups as follows. A group scheme G smooth and affine over a scheme S is said to be ***reductive*** if its geometric fibres are reductive. For such a group scheme, $\mathrm{rad}(G)$ is defined to be the largest central torus of G.

PROPOSITION 10.4 (GILLE 2022, 1.1) *A reductive group over a connected noetherian scheme S is linear if and only if* $\mathrm{rad}(G)$ *is isotrivial.*

Let S be affine connected and noetherian, and let $\bar{s}$ be a geometric point of S. Let T be a torus over S, and let T^f be the quotient of T corresponding to the submodule of $X^*(T_{\bar{s}}) \stackrel{\text{def}}{=} \mathrm{Hom}_{\bar{s}}(T_{\bar{s}}, \mathbb{G}_m)$ consisting of the elements with finite $\pi_1(S,\bar{s})$-orbits. Then T^f is the universal isotrivial quotient of T.

COROLLARY 10.5 *Let G be a reductive group over S (affine, connected, and noetherian), and let G^f be the quotient of G by the kernel of* $\mathrm{rad}(G) \to \mathrm{rad}(G)^f$. *Then G^f is linear.*

PROOF Indeed, $\mathrm{rad}(G^f) = \mathrm{rad}(G)^f$, which is isotrivial. □

Let V be a finite flat R-module. By a ***line*** in V, we mean a finite flat submodule L of V of rank 1 that, locally on $\mathrm{Spec}\, R$, is a direct summand of V. Given a line L in a finite flat R-module V, we let $\mathcal{A}ut(V,L)$ denote the functor of R-algebras

$$R' \rightsquigarrow \{\alpha \in \mathrm{Aut}_{R'}(V \otimes_R R') \mid \alpha(L_{R'}) = LR'\}.$$

PROPOSITION 10.6 *Let G be a group scheme flat of finite type over a Dedekind domain R. There exists a finite flat R-submodule V and a line L in V such that*

$$G = \mathcal{A}ut(V,L).$$

PROOF The usual proof over fields extends to Dedekind domains (Broshi 2013, 1.1). □

The statement (1.11), that a Hopf algebra over a field is faithfully flat over any Hopf subalgebra, has an extension to base rings.

THEOREM 10.7 *Let $A \subset B$ be Hopf algebras flat over a noetherian ring R. The algebra B is faithfully flat over A if and only if B/A is a flat R-module.*

PROOF From the criterion,

> a homomorphism $A \to B$ of commutative rings is faithfully flat if and only if it is injective, flat, and B/A is a flat A-module,

we see that it suffices to show that $A \to B$ is faithfully flat if B/A is a flat R-module. Note that this last condition implies that[4]

$$0 \to A \otimes_R \kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p}) \to (B/A) \otimes_R \kappa(\mathfrak{p}) \to 0$$

is exact for all prime ideals $\mathfrak{p}$ of R. Now the original theorem 1.11 shows that $A \otimes_R \kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p})$ is faithfully flat for all $\mathfrak{p}$, and the next criterion shows that $A \to B$ is faithfully flat.

> Let R be a noetherian ring and $f\colon A \to B$ a homomorphism of flat R-algebras. Then f is faithfully flat if $f \otimes_R \kappa(\mathfrak{p})$ is faithfully flat for all prime ideals $\mathfrak{p}$ of R (Hai et al. 2025, 3.3). □

When R is a Dedekind domain, 10.7 is proved in Duong and Hai 2018, 4.1.1. For a noetherian ring, it is Hai et al. 2025, 7.1.

[4] $\kappa(\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$.

Recovering G from $\mathsf{Rep}(G)$

PROPOSITION 10.8 *Let G be an affine group scheme flat over a Dedekind domain R, and let ω be the forgetful functor $\mathsf{Repf}(G) \to \mathsf{Mod}(R)$. Then ω defines an isomorphism $G \to \mathcal{A}ut^{\otimes}(\omega)$.*

PROOF Once 10.6 has been acquired, the proof of Theorem 2.1 applies. □

TODO 15 Explain this.

PROPOSITION 10.9 *Let G be a reductive group over scheme S, assumed to be affine noetherian and connected, and let G^f be the quotient of G by the kernel of $\mathrm{rad}(G) \to \mathrm{rad}(G)^f$. Then the homomorphism $G \to \mathcal{A}ut^{\otimes}(\omega)$ defined by the forgetful functor ω induces an isomorphism*

$$G^f \simeq \mathcal{A}ut^{\otimes}(\omega).$$

PROOF See Zhao 2022, 3.2.3. □

For example, the map $G \to \mathcal{A}ut^{\otimes}(\omega)$ is an isomorphism when G is a reductive group over a Dedekind domain and it need not be an isomorphism when G is a torus of dimension 2 over a curve with a node.

Tannakian duality over a Dedekind domain

When no longer working over a field, we have to take account of the fact that not all finitely generated modules will be projective. For a Dedekind domain, there is the following result.

PROPOSITION 10.10 *Let L be a flat coalgebra over a Dedekind domain, and let M be a finitely generated L-comodule. Then M is a quotient of a finite flat L-comodule,*

PROOF See Serre 1968, 2.2. □

Hence, for any finitely generated L-comodule, there exists an exact sequence of L-comodules

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0 and P_1 projective and finitely generated.

PROPOSITION 10.11 *Let R be a Dedekind domain. Let A be an R-linear abelian category and $\omega\colon \mathsf{A} \to \mathsf{Modf}(R)$ an exact R-linear faithful R-linear functor. Assume that there exists a full subcategory A° of A such that*

(a) *every object of A is a quotient of an object of A° and*

(b) *$\omega(X)$ is a finite flat R-module for all X in A°.*

Then the coalgebra $L(\omega) \overset{\text{def}}{=} \mathrm{Coend}(\omega \otimes \omega^{\vee})$ is flat over R, acts naturally on each object of A, and defines an equivalence $\mathsf{A} \to \mathsf{coModf}(L)$.

PROOF Saavedra 1972, II, 2.3.5, 2.6.1, or Duong and Hai 2018. □

TODO 16 Show that if $\mathsf{A} \approxeq \mathsf{coMod}(L)$, then $L = \mathrm{Coend}(\omega \otimes \omega^{\vee})$. Add a proof of this proposition.

Recall (13.3) that an object of a symmetric monoidal category is rigid if it admits a dual.

DEFINITION 10.12 Let R be a Dedekind domain. A ***neutral tannakian category*** over R is an R-linear abelian $\otimes$-category T such that

(a) every object is a quotient of a rigid object, and

(b) there exists an R-valued ***fibre functor***, i.e., an exact faithful R-linear $\otimes$-functor $\omega\colon \mathsf{T} \to \mathsf{Mod}(R)$.

EXAMPLE 10.13 Let L be a flat Hopf algebra over a Dedekind domain R. Then $\mathsf{coModf}(L)$ is a neutral tannakian category over R. Proposition 10.10 shows that it satisfies condition (a), and the forgetful functor satisfies (b).

THEOREM 10.14 (SAAVEDRA 1972, II, 4.1.1) *Let T be an R-linear abelian category and $\omega\colon \mathsf{C} \to \mathsf{Modf}(R)$ an exact faithful R-linear $\otimes$-functor. Let C_0 be the full subcategory of X such that $\omega(X)$ is a finitely generated projective R-module.There exists an affine monoid scheme G flat over R and a $\otimes$-equivalence*

$$\mathsf{T} \to \mathsf{Repf}(G)$$

whose composite with the forgetful functor is ω if and if every object of C is a quotient of an object of C_0, in which case

$$G \simeq \mathcal{E}nd^{\otimes}(\omega) \simeq \mathcal{E}nd^{\otimes}(\omega|\mathsf{C}_0).$$

PROOF The pair (T, ω) satisfies the hypotheses of 10.11 with T° the subcategory of rigid objects. Therefore ω defines an equivalence $\mathsf{T} \to \mathsf{coModf}(L(\omega))$, where $L(\omega)$ is the coend of $\omega \otimes \omega^{\vee}$. Now, as in the proof of Theorem 3.1, the tensor structure on T defines a Hopf algebra structure on $L(\omega)$, so $G = \mathrm{Spec}(L(\omega))$ is an affine group scheme flat over R with the required properties. □

The functor defined by a homomorphism of group schemes

Let L be a coalgebra over a noetherian ring R. A right L-comodule is an R-module V together with an R-linear map $\rho\colon V \to V \otimes L$ satisfying the conditions in 1.13. We write $\mathsf{coMod}(L)$ for the category of such comodules, $\mathsf{coModf}(L)$ for the full subcategory of those such that V is finitely generated as an R-module, and $\mathsf{coModf}^{\circ}(L)$ for the full subcategory of those such that V is a finite flat R-module.

A submodule M of an L-module N is said to be ***special*** if N/M is flat over R. A ***special subquotient*** M of N is a special submodule of a quotient of N, or, equivalently, a quotient of a special submodule.

Let $f\colon L' \to L$ be a homomorphhism of coalgebras over R. From an L-module (V, ρ) we get an L'-module by composing $\rho\colon V \to V \otimes L'$ with $\mathrm{id}_V \otimes f\colon V \otimes L' \to V \otimes L$. In this way, we get a functor

$$\omega^{\circ}\colon \mathsf{coModf}^{\circ}(L) \to \mathsf{coModf}^{\circ}(L').$$

PROPOSITION 10.15 *Let $f\colon L' \to L$ be a homomorphism of flat coalgebras over a Dedekind domain R. The homomorphism f is surjective if and only if every object of $\mathsf{coModf}^{\circ}(L')$ is a special subquotient of an object in the image of ω°.*

PROOF See Duong and Hai 2018, 3.2.5. □

Let G be an affine group scheme over a noetherian ring R. We write $\mathsf{Repf}^{\circ}(G)$ for the category of representations G on finite flat R-modules. Let $f\colon G \to H$ be a homomorphism of affine group schemes over R. Using f, we can regard an H-module as a G-module. In this way, we get a R-linear exact $\otimes$-functor

$$\omega^{\circ}\colon \mathsf{Repf}^{\circ}(H) \to \mathsf{Repf}^{\circ}(G)$$

compatible with the forgetful functors.

THEOREM 10.16 (DUONG AND HAI 2018, 4.1.2) *Let $f\colon G \to H$ be a homomorphism of affine group schemes flat over a Dedekind domain R. The homomorphism f is a closed immersion if and only if every object of $\mathsf{Repf}^{\circ}(G)$ is a special subquotient of an object in the image of ω^{f}.*

PROOF Restatement of 10.15. □

PROPOSITION 10.17 *Let $f\colon L' \to L$ be a homomorphism of flat coalgebras over a noetherian ring R. The following are equivalent:*

- *the homomorphism f is injective and L/L' is a flat R-module;*
- *ω° is fully faithful and its essential image is closed under subobjects.*

PROOF See Duong and Hai 2018, 3.2.1, when R is a Dedekind domain, Hai et al. 2025, 7.3, in the general case. □

THEOREM 10.18 *Let $f\colon G \to H$ be a homomorphism of affine group schemes flat over a noetherian ring R. The homomorphism f is faithfully flat if and only if the functor ω° is fully faithful and its essential image is closed under subobjects.*

PROOF Combine 10.7 and 10.17. □

Theorem 10.18 is Duong and Hai 2018, 4.1.2(i), when R is a Dedekind domain, Hai et al. 2025, 7.2, in the general case.

Classification of the fibre functors

THEOREM 10.19 *Let T be a neutral tannakian category over a Dedekind domain R, let ω be an R-valued fibre functor, and let $G = \mathcal{A}ut^{\otimes}(\omega)$.*

(a) *For any fibre functor η on T over an R-algebra R', $\mathcal{H}om^{\otimes}(\omega_{R'},\eta)$ is representable by an affine scheme faithfully flat over* Spec R'*; it is therefore a torsor under $G_{R'}$ over R'.*

(b) *The functor $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega_{R'},\eta)$ determines an equivalence between the category of fibre functors on T with values in R' and the category of torsors under $G_{R'}$ over R'.*

PROOF After Theorem 10.14, it suffices to prove this in the case $\mathsf{T} = \mathsf{Repf}(G)$ with G a flat affine group. For this, see Broshi 2013. See also **?**, II, 3.2.3. □

Tannakian lattices over Dedekind domains

There are situations where one would like to use Saavedra's theorem 10.14 to construct an affine group schemes but the tensor category does not have all the properties of Saavedra's definition. One therefore wants to reconstruct an affine group scheme from a smaller part of its representation category. There are at least three approaches to this problem, one in Wedhorn 2004, one in Bruguiéres 2004 following an idea of M. Nori, which is used in dos Santos 2009, and one in Duong and Hai 2018. Here we follow the last approach. First, a little category theory.

10.20 A category is ***regular*** if

(a) it is finitely complete (i.e., has finite products and equalizers),

(b) every kernel pair admits a coequalizer,

(c) regular epimorphisms are stable under pullbacks.

A morphism is a regular epimorphism if it is the coequalizer of some parallel pair of morphisms. In an additive category, regular epimorphisms are just cokernels. See Borceux 1994b, 2.1.1.

10.21 In a regular category, every morphism factors into a regular epimorphism followed by a monomorphism, and this factorization is unique up to a unique isomorphism.

In more detail, every morphism $f\colon A\to B$ in a regular category factors into a composite

$$A\xrightarrow{q} I\xrightarrow{m} B \tag{62}$$

with q a regular epimorphism and m a monomorphism. If $f=m'\circ q'$ with m' a monomorphism, then there exists a unique morphism j such that $j\circ q=q'$ and $m=m'\circ j$,

$$\begin{tikzcd} & I \arrow[dd, "j"] \arrow[dr, "m"] & \\ X \arrow[ur, "q"] \arrow[dr, "q'"'] & & Y. \\ & I' \arrow[ur, "m'"'] & \end{tikzcd}$$

When q' is a regular epimorphism, j is an isomorphism. See Borceux 1994b, 2.1.3.

10.22 In an additive regular category, every cokernel $p\colon X\to Y$ has a kernel, namely, its pullback along $0\to Y$.

Let $f\colon X\to Y$ be morphism in an additive regular category, and let $f=m\circ p$ be its cokernel-monomorphism factorization. Then $\operatorname{Ker} p=\operatorname{Ker} f$, so f admits a kernel. Moreover, $p=\operatorname{Coker}(\operatorname{Ker} f)$ so all kernels have cokernels. See Henrard et al. 2023, 4.4.

10.23 Every additive regular category C admits an full embedding into an abelian category A, characterized by the following properties

(a) C is closed under subobjects in A,

(b) every object of A is a quotient of an object of C.

See Henrard et al. 2023, 1.3.

EXAMPLE 10.24 Let G be an affine group scheme flat over a noetherian ring R. Then $\mathsf{Repf}(G)$ is an abelian R-linear category. If R is a Dedekind domain, then the subcategory $\mathsf{Repf}^{\circ}(G)$ of $\mathsf{Repf}(G)$ is closed under subobjects, and hence regular.

Let R be a Dedekind domain with field of fractions K.

DEFINITION 10.25 A ***tensor lattice*** over R is an R-linear regular rigid $\otimes$-category T such that $R \simeq \mathrm{End}(\mathbb{1})$ and T_K is abelian. An R'-valued ***fibre functor*** on T is a faithful R-linear $\otimes$-functor $\omega\colon \mathsf{T} \to \mathsf{Mod}(R')$ preserving kernels and cokernel-monomorphism factorizations and such that $R' \simeq \mathrm{End}(\omega(\mathbb{1}))$.

DEFINITION 10.26 A ***neutral tannakian lattice*** over R is a tensor lattice over R that admits an R-valued fibre functor.

THEOREM 10.27 *Let T be neutral tannakian lattice over R and ω an R-valued fibre functor.*

(a) *The functor $\mathcal{A}ut^{\otimes}(\omega)$ is represented by an affine group scheme flat over R, and ω induces an equivalence $\mathsf{T} \to \mathsf{Repf}^{\circ}(G)$.*

(b) *For any fibre functor η on T with values in an R-algebra R', $\mathcal{I}som^{\otimes}(\omega_{R'}, \eta)$ is a torsor under G over R'.*

PROOF See Duong and Hai 2018, 2.3.2, 2.4.8. □

Chapter III

Tannakian categories and Galois groupoids

In this chapter, we describe tannakian categories over k with a $\bar{k}$-valued fibre functor in terms of Galois groupoids (following Langlands and Rapoport 1987).

Throughout, k is a field and $\bar{k}$ is a fixed separable algebraic closure of k. We let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Finite extensions of k are required to be subfields of $\bar{k}$. All algebraic groups over k are affine and smooth, and all affine group schemes are pro-smooth. Recall that if k has characteristic zero, this last condition is automatic (and $\bar{k}$ is then an algebraic closure of k).

1 Galois groupoids

We assume that the reader is familiar with Galois descent, as described in §5 below.

DEFINITION 1.1 Let G be an algebraic group over $\bar{k}$. A $\bar{k}/k$-***Galois groupoid with kernel*** G is an exact sequence of groups

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1, \tag{63}$$

together with a section $\sigma \mapsto s_\sigma\colon\ \mathrm{Gal}(\bar{k}/K) \to \mathcal{G}$ for a suitable finite extension K of k, such that the automorphisms

$$\gamma_\sigma\colon\ G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma \stackrel{\text{def}}{=} (g \mapsto s_\sigma \cdot g \cdot s_\sigma^{-1}), \quad \sigma \in \mathrm{Gal}(\bar{k}/K),$$

define a K-structure on G. It is to be understood that K can be replaced by a larger finite extension, so that s is actually a germ of a section for the Krull topology on $\mathrm{Gal}(\bar{k}/k)$.

In more detail, consider an extension $\mathcal{G}$ of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. Let $K \subset \bar{k}$ be a finite extension of k and s a homomorphism such that

$$\begin{tikzcd} & \mathrm{Gal}(\bar{k}/K) \arrow[dl, "s"'] \arrow[d, hook] \\ \mathcal{G} \arrow[r] & \mathrm{Gal}(\bar{k}/k) \end{tikzcd}$$

commutes. To say that the automorphisms $\gamma_\sigma\colon\ G(\bar{k}) \to G(\bar{k})$, $\sigma \in \mathrm{Gal}(\bar{k}/K)$, define a K-structure on G means that there exists an algebraic group G_0 over K and an isomorphism

$\varphi\colon G_{0\bar{k}} \to G$ such that

$$\varphi(\sigma g) = \gamma_\sigma(\varphi(g)), \quad \text{all } \sigma \in \mathrm{Gal}(\bar{k}/K), \quad g \in G_0(\bar{k}).$$

A necessary and sufficient condition for this is that the homomorphisms γ_σ be σ-linear and that the action of $\mathrm{Gal}(\bar{k}/K)$ on $G(\bar{k})$ they define be continuous (Appendix, 6.4).

Two sections $s\colon\ \mathrm{Gal}(\bar{k}/K) \to \mathcal{G}$ and $s'\colon\ \mathrm{Gal}(\bar{k}/K') \to \mathcal{G}$ are said to be ***equivalent*** if they agree on $\mathrm{Gal}(\bar{k}/K'')$ for some field K'' containing K and K' and finite over k. A ***germ*** of sections is an equivalence class. Now a $\bar{k}/k$-Galois groupoid with kernel G can be defined to be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$ together with a germ of sections satisfying the above conditions.

1.2 A ***homomorphism*** of Galois groupoids $\mathcal{G} \to \mathcal{H}$ is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, "\phi"', "\text{homomorphism}"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] \arrow[l, bend right, dashed, "s"'] & 1 \\
1 \arrow[r] & H(\bar{k}) \arrow[r] & \mathcal{H} \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[l, bend left, dashed, "s'"] & 1
\end{tikzcd}$$

such that ϕ preserves the germs of sections and such that the restriction of ϕ to $G(\bar{k})$ is regular (i.e., defined by a homomorphism of algebraic groups).

From a different point-of-view, a homomorphism of Galois groupoids is a homomorphism of algebraic groups $\varphi\colon\ G \to H$ together with an extension of $\varphi(\bar{k})$ to a homomorphism $\mathcal{G} \to \mathcal{H}$ compatible with the germs of sections and inducing the identity map on the Galois groups.[1]

For example, conjugation by an element g of $G(\bar{k})$ defines an automorphism $\mathrm{ad}(g)$ of $\mathcal{G}$,

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d, "\mathrm{ad}\, g"] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{ad}\, g"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] \arrow[l, bend right, dashed, "s"'] & 1 \\
1 \arrow[r] & G(\bar{k}) \arrow[r] & \mathcal{G} \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[l, bend left, dashed, "\mathrm{ad}(g)\circ s"] & 1.
\end{tikzcd}$$

1.3 Let $\mathcal{G}$ and $\mathcal{H}$ be Galois groupoids with kernels G and H, and let ϕ and ϕ' be homomorphisms $\mathcal{G} \to \mathcal{H}$. A ***morphism*** $\phi \to \phi'$ is an element g of $H(\bar{k})$ such that $\mathrm{ad}(g)\circ\phi = \phi'$, i.e., such that

$$g \cdot \phi(x) \cdot g^{-1} = \phi'(x), \quad \text{all } x \in \mathcal{G}.$$

With this structure, the Galois groupoids form a 2-category.

EXAMPLE 1.4 Let G be an algebraic group over k. The extension

$$1 \to G(\bar{k}) \to G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

[1] Recall that if V is a geometrically reduced scheme of finite type over a separably closed field $\bar{k}$, then $V(\bar{k})$ is schematically dense in V. As G is smooth over $\bar{k}$, it is geometrically reduced, and so the homomorphism $\varphi\colon\ G \to H$ is determined by $\varphi(\bar{k})$

together with the section $\sigma \mapsto (1,\sigma)$ defines Galois gerbe $\mathcal{G}_G$ with kernel G. A $\bar{k}/k$-Galois groupoid $\mathcal{G}$ is ***split*** if it is isomorphic to $\mathcal{G}_G$ for some G. Equivalently, if there exists a section s to the homomorphism $\pi\colon \mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$, compatible with the germ of sections, such that the automorphisms

$$\gamma_\sigma\colon G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

define a k-structure on G.

1.5 Let

$$1 \to G(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

be a $\bar{k}/k$-Galois groupoid with kernel G. The choice of a section s in the germ determines a model (G_0,φ) of G over some finite extension K of k and a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G_0(\bar{k}) \arrow[r] \arrow[d, "\varphi(\bar{k})", "\simeq"'] & G_0(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/K) \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\bar{k}/K) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G(\bar{k}) \arrow[r] & \mathcal{G} \arrow[r, "\pi"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

We call such a diagram a ***splitting*** of $\mathcal{G}$ over K.

1.6 Let E be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. When k' is a subfield of $\bar{k}$ containing k, we can form an extension of $\mathrm{Gal}(\bar{k}/k')$ by $G(\bar{k})$ by pullback. It is uniquely determined, up to a unique isomorphism, by the diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\bar{k}/k') \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G(\bar{k}) \arrow[r] & E \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

When E is a $\bar{k}/k$-Galois groupoid, E' becomes a $\bar{k}/k'$-Galois groupoid with the obvious germ of sections.

1.7 Let E be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. When $G \to H$ is a homomorphism of algebraic groups over $\bar{k}$, we can form an extension E' of $\mathrm{Gal}(\bar{k}/k)$ by $H(\bar{k})$ by pushout. It is uniquely determined, up to a unique isomorphism, by the diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H(\bar{k}) \arrow[r] & E' \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

When E is a $\bar{k}/k$-Galois groupoid with kernel G, E' becomes a $\bar{k}/k$-Galois groupoid with kernel H and the obvious germ of sections.

1.8 Let

$$1 \to G(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

be a $\bar{k}/k$-Galois groupoid with G is algebraic, and let $s\colon\ \mathrm{Gal}(\bar{k}/k) \to \mathcal{G}$ be a set-theoretic section to π such that the restriction of s to $\mathrm{Gal}(\bar{k}/K)$ lies in the given germ for some $K \subset \bar{k}$ finite over k.[2] Then $\mathcal{G}$ is determined by the following data:

[2]For example, choose a group-theoretic section $\mathrm{Gal}(\bar{k}/K) \to \mathcal{G}$ in the given germ, and extend it to a set-theoretic section on $\mathrm{Gal}(\bar{k}/k)$.

(a) the family $(\gamma_\sigma)_\sigma$ of automorphisms of G (as an algebraic group over $\bar{k}$) given by

$$\gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k), \quad g \in G(\bar{k});$$

(b) the family $(a_{\sigma,\tau})_{\sigma,\tau}$ of elements of $G(\bar{k})$ given by

$$s_\sigma s_\tau = a_{\sigma,\tau} s_{\sigma\tau}, \quad \sigma,\tau \in \mathrm{Gal}(\bar{k}/k).$$

Indeed, every element of $\mathcal{G}$ can be written uniquely

$$g \cdot s_\sigma, \quad g \in G(\bar{k}), \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

and

$$(g \cdot s_\sigma)(h \cdot s_\tau) = (g\ \gamma_\sigma(h)\ a_{\sigma,\tau}) \cdot s_{\sigma\tau}.$$

PROPOSITION 1.9 *A homomorphism $\phi\colon \mathcal{G} \to \mathcal{G}'$ of $\bar{k}/k$-Galois groupoids is an isomorphism if and only if it is an isomorphism on the kernels.*

PROOF If ϕ has an inverse ϕ', then $\phi'|G'$ is an inverse for $\phi|G$. For the converse, suppose that $\phi|G$ is an isomorphism, and choose a section s as in 1.8. Then $\sigma \mapsto s'_\sigma \overset{\text{def}}{=} \phi \circ s_\sigma$ is a section for $\mathcal{G}'$ with similar properties, and clearly $g \cdot s_\sigma \mapsto \phi(g \cdot s_\sigma) = \phi(g) \cdot s'_\sigma$ is a bijection. Alternatively, this follows from a diagram chase. □

The band of a Galois groupoid

1.10 Let $\mathcal{G}$ be a Galois groupoid with kernel an algebraic group G. Let s be a set-theoretic section to π, as in 1.8. The automorphisms

$$\gamma_\sigma\colon G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

form a descent datum modulo inner automorphisms on G, and so define the structure of a band on G (Appendix C, §5). This structure is independent of the choice of the section s. The band $B(\mathcal{G})$ of $\mathcal{G}$ is well-defined up to a unique isomorphism.

When the kernel is commutative, the family of automorphisms $(\gamma_\sigma)_\sigma$ is a descent datum on G, and so G acquires a model over k, well-defined up to a unique isomorphism.

The cohomology class of a Galois groupoid

1.11 We fix a band B over k, and consider $\bar{k}/k$-Galois groupoids equipped with an isomorphism $B \simeq B(\mathcal{G})$. The cohomology set $H^2(k,B)$ is the set of isomorphism classes of such systems. This agrees with the definition in Appendix C, §6.

1.12 When B is commutative, we can identify it with a commutative algebraic group G over k. When G is of algebraic over k, $H^2(k,G)$ becomes the familiar Galois cohomology group $H^2(\mathrm{Gal}(\bar{k}/k), G(\bar{k}))$.

Galois groupoids admitting a special section

In this subsection, we study an important class of Galois groupoids, which includes all those with commutative kernels.

Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with algebraic kernel G, and let s be a set-theoretic section to π, as in 1.8. We say that $(s_\sigma)_\sigma$ is ***special*** if the family $(\gamma_\sigma)_\sigma$ satisfies the cocycle condition (6.3),

$$\gamma_{\sigma\tau} = \gamma_\sigma \circ (\sigma\gamma_\tau), \quad \sigma,\tau \in \mathrm{Gal}(\bar{k}/k).$$

Then $(\gamma_\sigma)_\sigma$ is a continuous cocycle and so defines a model G_0 of G over k. Moreover,

$$a_{\sigma,\tau} \stackrel{\text{def}}{=} \gamma_\sigma \cdot \sigma\gamma_\tau \cdot \gamma_{\sigma\tau}^{-1}$$

is a continuous 2-cocycle on $\mathrm{Gal}(\bar{k}/k)$ with values in $Z_G(\bar{k})$, where Z_G is the centre of G.

Consider the category whose objects are the pairs (G,a), where G is an algebraic group over k and $a=(a_{\sigma,\tau})_{\sigma,\tau\in\Gamma}$ is a continuous 2-cocycle on Γ with values in $Z_G(\bar{k})$. A morphism $(G',a')\to(G,a)$ is a pair (φ,f), where $\varphi\colon G'\to G$ is a homomorphism of algebraic groups over $\bar{k}$ and $f=(f_\sigma)_{\sigma\in\Gamma}$ is a continuous 1-cochain with values in $G(\bar{k})$ such that, for all $\sigma,\tau\in\mathrm{Gal}(\bar{k}/k)$,

$$\begin{cases} a_{\sigma,\tau}\cdot f_\sigma\sigma(f_\tau)f_{\sigma\tau}^{-1}=\varphi(a'_{\sigma,\tau}) \\ \mathrm{ad}(f_\sigma)\circ\sigma^*(\varphi)=\varphi. \end{cases}$$

Composition of morphisms is given by

$$(\varphi,f)\circ(\varphi',f')=(\varphi\circ\varphi',f''),$$

where $f''_\sigma=\varphi(f'_\sigma)f_\sigma$, $\sigma\in\Gamma$.

For an object (G,a), we let $\mathcal{G}$ be the extension

$$1\to G(\bar{k})\to\mathcal{G}\to\mathrm{Gal}(\bar{k}/k)\to 1$$

defined by the 2-cocycle a: the elements of $\mathcal{G}$ are the pairs $(g,\sigma)\in G(\bar{k})\times\mathrm{Gal}(\bar{k}/k)$, and

$$(g,\sigma)(h,\tau)=(g\cdot\sigma h\cdot a_{\sigma,\tau},\sigma\tau).$$

For example, if $a_{\sigma,\tau}=1$ for all σ,τ, then $\mathcal{G}$ is the split Galois groupoid $G(\bar{k})\rtimes\mathrm{Gal}(\bar{k}/k)$.

PROPOSITION 1.13 *The functor $(G,a)\rightsquigarrow\mathcal{G}$ from pairs (G,a) to $\bar{k}/k$-Galois groupoids is fully faithful; its essential image consists of the Galois groupoids admitting a special section.*

PROOF Routine verification using descent theory. □

REMARK 1.14 Let $\mathcal{P}$ be a commutative $\bar{k}/k$-groupoid with kernel P. Let $\varphi\colon P\to G$ be a homomorphism of algebraic groups over k, and let Z_φ be the centralizer of $\varphi(P)$ in G. Assume that φ extends to a homomorphism $\phi\colon\mathcal{P}\to\mathcal{G}_G$, and let $I_\phi=\mathcal{A}ut(\phi)$. Then I_ϕ is an inner form of Z_ϕ whose cohomology class can be described as follows. Choose special section s, and let $(d_{\rho,\tau})$ be the corresponding 2-cocycle. When we write $\phi(s(\rho))=(c_\rho,\rho)$, we obtain a 1-cochain (c_ρ) splitting the cocycle $(\varphi(d_{\rho,\tau}))$:

$$c_\rho\cdot\rho c_\tau=\varphi(d_{\rho,\tau})\cdot c_{\rho\tau}.$$

For $p\in P(\bar{k})$ we have

$$\rho\varphi(p)=\varphi(\rho p)=\varphi(s(\rho)\cdot p\cdot s(\rho)^{-1})=(c_\rho,\rho)\cdot\varphi(p)\cdot(c_\rho,\rho)^{-1}=c_\rho\cdot\rho\varphi(p)\cdot c_\rho^{-1},$$

and so $c_\rho\in Z_\varphi(\bar{k})$. The formula displayed above shows that the image of (c_ρ) in $Z_\varphi/\varphi(P)$ is a cocycle. Its class in $H^1(k,Z_\varphi/\varphi(P))$ depends only on the isomorphism class of φ, and it is the cohomology class of I_φ.

The category of Galois groupoids

TODO 17 Need to fix this subsection. However they are defined, the proposition should hold.

1.15 When G is an affine group scheme over k, we define a $\bar{k}/k$-***Galois groupoid with kernel*** G to be an extension of groups

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

such that, for all algebraic quotients $G \to G_\alpha$ of G, the pushout

$$1 \to G_\alpha(\bar{k}) \to \mathcal{G}_\alpha \to \mathrm{Gal}(\bar{k}/k) \to 1$$

is a $\bar{k}/k$-Galois groupoid with kernel G_α. In particular, this means that we are given a compatible system of germs of sections on the $\mathcal{G}_\alpha$.

PROPOSITION 1.16 *Let* D *be the category of* $\bar{k}/k$*-Galois groupoids and* C *the full subcategory of* $\bar{k}/k$*-Galois groupoids with algebraic kernel.*

(a) The objects of C *are artinian.*

(b) The functor

$$\text{“}\varprojlim\text{”} G_\alpha \rightsquigarrow \varprojlim G_\alpha \colon \ \mathrm{Pro}\, C \to \mathsf{D}$$

is an equivalence of categories, with quasi-inverse the functor sending an affine group scheme G to the projective system of its algebraic quotients.

PROOF Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G. The subobjects of $\mathcal{G}$ are in one-to-one correspondence with the subgroup schemes of G (see 1.9), and so $\mathcal{G}$ is artinian if G is algebraic. The rest of the proof is opposite to that of Appendix B, B.10. □

In other words, the category of $\bar{k}/k$-Galois groupoids is the category of pro-objects in the category of $\bar{k}/k$-Galois groupoids with algebraic kernel.

Galois groupoids and groupoid schemes

1.17 Let $S_0 = \mathrm{Spec}\, k$ and $S = \mathrm{Spec}\, \bar{k}$. Let $(G, (t, s), \circ)$ be a k-groupoid scheme acting transitively on S (see IV, 2.8). Then $(S(\bar{k}), G(\bar{k}), (t, s), \circ)$ is a transitive groupoid (in sets). In particular, $(t, s)\colon\ G(\bar{k}) \to S(\bar{k}) \times S(\bar{k})$ is surjective, so

$$G(\bar{k}) = \bigsqcup_{(b,a)\in S(\bar{k})\times S(\bar{k})} G(\bar{k})_{b,a} = \bigsqcup_{(b,a)\in S(\bar{k})\times S(\bar{k})} \mathrm{Hom}(a, b),$$

where $G(\bar{k})_{b,a}$ is the fibre of $G(\bar{k})$ over (b, a). Note that $\mathrm{Gal}(\bar{k}/k)$ acts simply transitively on $S(\bar{k}) = \mathrm{Hom}(\bar{k}, \bar{k})$. Let a denote the identity element in S, and let

$$E = \{f \in G \mid t(f) = a\} = \bigsqcup_{\sigma\in\mathrm{Gal}(\bar{k}/k)} G_{a,\sigma a} = \bigsqcup_{\sigma\in\mathrm{Gal}(\bar{k}/k)} \mathrm{Hom}(\sigma a, a).$$

Write π for the map $E \to \mathrm{Gal}(\bar{k}/k)$ sending $f \in G_{a,\sigma a}$ to σ. Define the product of $f_1\colon\ \sigma_1 a \to a$ and $f_2\colon\ \sigma_2 a \to a$ by

$$f_1 \cdot f_2 = f_1 \circ \sigma_1 f_2.$$

Then $f_1 \cdot f_2 \in G_{a,\sigma_1\sigma_2 a}$, and so

$$\pi(f_1 \cdot f_2) = \sigma_1\sigma_2 = \pi(f_1)\pi(f_2).$$

Moreover,

$$\begin{cases} (f_1 \cdot f_2) \cdot f_3 = f_1 \circ \sigma_1 f_2 \circ \sigma_1 \sigma_2 f_3 = f_1 \cdot (f_2 \cdot f_3) \\ f \cdot \mathrm{id}_a = f = \mathrm{id}_a \cdot f \\ f \cdot \sigma^{-1} f^{-1} = \mathrm{id}_a = \sigma^{-1} f^{-1} \cdot f, \end{cases}$$

so E is a group. We have constructed an exact sequence

$$1 \to G_{a,a} \to E \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

of abstract groups and group homomorphisms. This is a $\bar{k}/k$-Galois groupoid. This explains our choice of name. See IV, 2.23, for more details.

2 Representations of Galois groupoids

2.1 Let V be a vector space over $\bar{k}$ and let $\sigma \in \mathrm{Gal}(\bar{k}/k)$. An additive map $\alpha : V \to V$ is said to be σ-***linear*** if $\alpha(c \cdot v) = \sigma c \cdot \alpha(v)$ for all $c \in \bar{k}$ and $v \in V$. Note that if α_1 is σ_1-linear and α_2 is σ_2-linear, then $\alpha_2 \circ \alpha_1$ is $\sigma_2\sigma_1$-linear,

$$(\alpha_2 \circ \alpha_1)(c \cdot v) = \alpha_2(\sigma_1 c \cdot \alpha_1(v)) = \sigma_2 \sigma_1 c \cdot \alpha_2 \circ \alpha_1(v).$$

2.2 Let V be a finite-dimensional vector space over $\bar{k}$. Let $\mathcal{G}_V$ be the collection of all additive isomorphisms $g : V \to V$ that are σ-linear for some $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Then $\mathcal{G}_V$ becomes a group under composition, and there is an exact sequence

$$1 \to \mathrm{GL}(V) \longrightarrow \mathcal{G}_V \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

in which π sends a σ-linear map to σ. The choice of a basis $(v_i)_i$ for V over $\bar{k}$ determines a section $s : \ \mathrm{Gal}(\bar{k}/k) \to \mathcal{G}_V$ to π,

$$s(\sigma)(\sum c_i \cdot v_i) = \sum \sigma c_i \cdot v_i.$$

A different basis determines an equivalent section and so, in this way, we get a Galois groupoid $\mathcal{G}_V$ with kernel GL_V. Note that the choice of a $\bar{k}$-basis for V determines a $\mathbb{Q}$-structure on V and a splitting of $\mathcal{G}_V$ over k.

2.3 Let $\mathcal{G}$ be a Galois groupoid with kernel G. A ***representation*** of $\mathcal{G}$ on a finite dimensional $\bar{k}$-vector space V is a homomorphism of $\bar{k}/k$-Galois groupoids $\rho : \ \mathcal{G} \to \mathcal{G}_V$. In other words, a representation of $\mathcal{G}$ on V is a representation $\rho_0 : G \to \mathrm{GL}_V$ of G on V together with an extension ρ of $\rho_0(\bar{k})$ to a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d, "\rho_0(\bar{k})"] & \mathcal{G} \arrow[r] \arrow[d, "\rho"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathcal{G}_V \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1 \end{tikzcd}$$

compatible with the germs of sections.

2.4 A ***morphism*** of representations (V,ρ_V) and (W,ρ_W) of $\mathcal{G}$ is a $\bar{k}$-linear map $\varphi\colon V\to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_V(g)"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

commutes for all $g\in\mathcal{G}$ (equality of σ-linear maps).

2.5 Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid. The category $\mathsf{Repf}(\mathcal{G})$ of representations of $\mathcal{G}$ is on finite-dimensional $\bar{k}$-vector spaces is abelian and k-linear. It has a natural structure of a symmetric monoidal category. For example, the trivial representation

$$\begin{tikzcd} 1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\rho_{\text{trivial}}"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{GL}(\bar{k}) \arrow[r] & \mathcal{G}_{\bar{k}} \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1. \end{tikzcd}$$

in which $\rho_{\text{trivial}}(g)=\pi(g)=\sigma$ (viewed as a σ-linear isomorphism $\bar{k}\to\bar{k}$) is a unit object. Duals exist, and obviously $\mathrm{End}(\mathbb{1})=k$, so $\mathsf{Repf}(\mathcal{G})$ is a tensor category over k. The functor sending a representation to its underlying vector space is a $\bar{k}$-valued fibre functor, so $\mathsf{Repf}(\mathcal{G})$ is tannakian category over k.

2.6 Let G be an algebraic group scheme over k, and let $\mathcal{G}$ be a $\bar{k}/k$ Galois groupoid with kernel $G_{\bar{k}}$. We suppose given a section s to $\mathcal{G}\to\mathrm{Gal}(\bar{k}/k)$ in the germ such that $\mathbb{Q}$-structure on $G_{\bar{k}}$ defined by s is G (see 1.4). For example, $\mathcal{G}$ could be $\mathcal{G}_G\overset{\text{def}}{=}G(\bar{k})\rtimes\mathrm{Gal}(\bar{k}/k)$ with its canonical section.

A representation $\rho\colon G\to\mathrm{GL}_V$ of G on a finite-dimensional k-vector space defines a representation $\bar{\rho}\overset{\text{def}}{=}\rho(\bar{k})\colon G(\bar{k})\to\mathrm{GL}(V(\bar{k}))$ of $G(\bar{k})$ on $V(\bar{k})$ with the property that

$$\bar{\rho}(\sigma g)=\sigma\rho(g),\quad g\in G(\bar{k}),\quad \sigma\in\mathrm{Gal}(\bar{k}/k).$$

Every element of $\mathcal{G}$ can be written uniquely $g\cdot s(\sigma)$ with $g\in G(\bar{k})$ and $\sigma\in\mathrm{Gal}(\bar{k}/k)$, and we extend $\bar{\rho}$ to a representation $\widetilde{\rho}$ of $\mathcal{G}$ on $V(\bar{k})$ by setting

$$\widetilde{\rho}(g\cdot s(\sigma))=\bar{\rho}(g)\sigma.$$

Thus, a representation of G defines a representation of $\mathcal{G}$. Conversely, a representation $\mathcal{G}\to\mathcal{G}_V$ restricts to a representation $G(\bar{k})\to V$, which commutes with the action of $\mathrm{Gal}(\bar{k}/k)$ (for a suitable $\mathbb{Q}$-structure on V), and so arises from a representation of G on V. In this way, we get an equivalence of tensor categories over k,

$$\mathsf{Repf}(G)\to\mathsf{Repf}(\mathcal{G}).$$

In the next three paragraphs, we generalize this statement.

2.7 Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G. The choice of a section s,

$$\begin{tikzcd} & \mathrm{Gal}(\bar{k}/K) \arrow[dl, "s"'] \arrow[d, hook] \\ \mathcal{G} \arrow[r] & \mathrm{Gal}(\bar{k}/k), \end{tikzcd}$$

in the germ of $\mathcal{G}$ determines a model of G over K, which we again denote G, and a homomorphism

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d, "\simeq"] & G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/K) \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\bar{k}/K) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G(\bar{k}) \arrow[r] & \mathcal{G} \arrow[r, "\pi"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1,
\end{tikzcd}$$

The group-theoretic section s can be extended to a set-theoretic section s of $\mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$ with the property that

$$s(\sigma\tau) = s(\sigma)s(\tau), \quad \text{all } \sigma \in \mathrm{Gal}(\bar{k}/K), \quad \tau \in \mathrm{Gal}(\bar{k}/k).$$

See Lattermann 1989, 4.2.6.

2.8 In the situation of 2.7, we define a functor

$$\mathsf{Repf}_K(G) \to \mathsf{Repf}(\mathcal{G})_{(K)}.$$

A representation $\rho\colon G \to \mathrm{GL}_V$ of G on a finite-dimensional K-vector space V defines a representation $\bar{\rho} \overset{\text{def}}{=} \rho(\bar{k})$ of $G(\bar{k})$ on $\bar{V} \overset{\text{def}}{=} V \otimes \bar{k}$ with the property that

$$\bar{\rho}(\sigma g) = \sigma\left(\bar{\rho}(g)\right), \quad g \in G(\bar{k}) \quad \sigma \in \mathrm{Gal}(\bar{k}/K).$$

Because G is smooth, ρ can be recovered from $\bar{\rho}$ and the action of $\mathrm{Gal}(\bar{k}/K)$. Write gv for $\bar{\rho}(g)v$, and let

$$\mathrm{Ind}\,\bar{V} = \{f\colon\ \mathrm{Gal}(\bar{k}/k) \to \bar{V} \mid f(\sigma\tau) = \sigma f(\tau), \text{ all } \sigma \in \mathrm{Gal}(\bar{k}/K)\}.$$

Then $\mathrm{Ind}\,\bar{V}$ is a $\bar{k}$-vector space by

$$(\lambda f)(\tau) = (\tau\lambda)(f(\tau)), \quad \lambda \in \bar{k}, \quad f \in \mathrm{Ind}\,\bar{V}, \quad \tau \in \mathrm{Gal}(\bar{k}/k),$$

and a $\mathrm{Gal}(\bar{k}/k)$-module by

$$(\sigma f)(\tau) = f(\tau\sigma), \quad f \in \mathrm{Ind}\,\bar{V}, \quad \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

Every element x of $\mathcal{G}$ can be written uniquely $x = g \cdot s(\sigma)$ with $g \in G(\bar{k})$ and $\sigma \in \mathrm{Gal}(\bar{k}/k)$, and we define an action of $\mathcal{G}$ on $\bar{V}$ by setting

$$(xf)(\tau) = (s(\tau) \cdot g \cdot s(\sigma)s(\tau\sigma)^{-1})f(\tau\sigma).$$

Then $g \cdot s(\sigma)$ acts σ-linearly on $\mathrm{Ind}\,\bar{V}$, and this way we get a representation of $\mathcal{G}$ on $\mathrm{Ind}\,\bar{V}$. There is an action of K on $\mathrm{Ind}\,\bar{V}$,

$$(\lambda f)(\tau) = \lambda f(\tau), \quad \lambda \in K, \quad f \in \mathrm{Ind}\,\bar{V}, \quad \tau \in \mathrm{Gal}(\bar{k}/k),$$

and so we have defined an object of $\mathsf{Repf}(\mathcal{G})_{(K)}$. See Lattermann 1989, 4.2.7.

2.9 In the situation of 2.7, we define a functor

$$\mathsf{Repf}(\mathcal{G})_{(K)} \to \mathsf{Repf}_K(G).$$

Let V be a finite-dimensional $\bar{k}$-vector space on which $\mathcal{G}$ and K act compatibly, and let V_1 be the subspace of V on which the two actions of K coincide. Then V_1 is a finite-dimensional $\bar{k}$-vector space on which $G(\bar{k})$ acts compatibly with the actions of $\mathrm{Gal}(\bar{k}/K)$, and so this action arises from a representation of G on a finite-dimensional K-vector space. See Lattermann 1989, 4.2.8.

2.10 The functors

$$\mathsf{Repf}_K(G) \to \mathsf{Repf}(\mathcal{G})_{(K)}$$
$$\mathsf{Repf}(\mathcal{G})_{(K)} \to \mathsf{Repf}_K(G)$$

defined in 2.8 and 2.9 are quasi-inverse. See Lattermann 1989, 4.2.9.

3 Tannakian categories and Galois groupoids

Preliminaries

PROPOSITION 3.1 *Let T be a tannakian category over k, and let k' be a finite separable extension of k. If T has a k'-valued fibre functor, then $\mathsf{T}_{(k')}$ is neutral.*

PROOF Let ω be a k'-valued fibre functor on T. For any (X,μ) in $\mathsf{T}_{(k')}$, $\omega(X)$ has a natural $k'\otimes_k k'$-structure, and $(X,\mu) \rightsquigarrow \omega(X)$ is a $k'\otimes_k k'$-valued fibre functor (see the proof of I, 8.16). The multiplication map $k'\times k' \to k'$ defines a homomorphism of k-algebras $k'\otimes_k k' \to k'$. Now $(X,\mu) \rightsquigarrow \omega(X)\otimes_{k'\otimes_k k'} k'$ is a k'-valued fibre functor on $\mathsf{T}_{(k')}$ (I, 8.6). □

COROLLARY 3.2 *Let ω' be the fibre functor $(X,\mu) \rightsquigarrow \omega(X)\otimes_{k'\otimes_k k'} k'$ on $\mathsf{T}_{(k')}$. Then ω' defines an equivalence*

$$\mathsf{T}_{(k')} \to \mathsf{Repf}(\mathcal{A}ut^{\otimes}(\omega')),$$

of tensor categories over k'.

PROOF Combine the proposition with II, 3.1. □

PROPOSITION 3.3 *Let T be a tannakian category over k. If T has a $\bar{k}$-valued fibre functor ω, then $\mathsf{T}_{(\bar{k})}$ is neutral.*

PROOF As in the above proof, for any subfield k' of $\bar{k}$ finite over k, the functor

$$(X,\mu) \rightsquigarrow \omega(X)\otimes_{k'\otimes_k \bar{k}} \bar{k}\otimes_k \bar{k}$$

is a $\bar{k}\otimes_k \bar{k}$-valued functor on $\mathsf{T}_{(k')}$. Using the description of $\mathsf{T}_{(\bar{k})}$ in I, 8.19, we deduce that there exists a $\bar{k}\otimes_k\bar{k}$-valued fibre functor $\widetilde{\omega}$ on $\mathsf{T}_{(k)}$ agreeing with this fibre functor on each subcategory $\mathsf{T}_{(k')}$. Now choose a homomorphism $\bar{k}\otimes_k \bar{k}\to\bar{k}$, and set $\bar{\omega} = \widetilde{\omega}\otimes_{\bar{k}\otimes_k\bar{k}}\bar{k}$. □

PROPOSITION 3.4 *Let T be a tannakian category over k. Any two $\bar{k}$-valued fibre functors on T are isomorphic over $\bar{k}$.*

PROOF We saw in the last proof that every $\bar{k}$-valued fibre functor ω on T, extends to a $\bar{k}\otimes_k\bar{k}$-valued fibre functor $\widetilde{\omega}$ on $\mathsf{T}_{(\bar{k})}$. If ω' is a second $\bar{k}$-valued fibre functor, then, because $\mathsf{T}_{(\bar{k})}$ is neutral, $\widetilde{\omega}$ and $\widetilde{\omega}'$ are isomorphic, and this implies that ω and ω' are isomorphic. □

PROPOSITION 3.5 *Let T be a tannakian category over k, and let ω be a $\bar{k}$-valued fibre functor on T. For any algebraic tannakian subcategory T_0 of T, there exists a finite extension $K\subset\bar{k}$ of k such that ω restricts to a K-valued fibre functor on T_0.*

PROOF Let $\mathsf{T}_0 = \langle X\rangle^{\otimes}$, and let $V=\omega(X)$. There exists a family t of tensors for V such that $\mathcal{A}ut^{\otimes}(\omega) = \mathcal{A}ut(V,t)$. Now the pair (V,t) has a model over a finite extension $K\subset\bar{k}$ of k and $\omega|\langle X\rangle^{\otimes}$ is K-valued. □

Let T be a tannakian category over k, and let ω be a $\bar{k}$-valued fibre functor on T (assumed to exist). For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, define ${}^{\sigma}\omega$ to be the fibre functor $X \rightsquigarrow \omega(X) \otimes_{\bar{k},\sigma} \bar{k}$. Then

$$^{\sigma_2}({}^{\sigma_1}\omega) \simeq {}^{\sigma_2\sigma_1}\omega.$$

Define

$$\mathcal{G}(\omega) = \bigsqcup_{\sigma \in \Gamma} \mathrm{Isom}^{\otimes}({}^{\sigma}\omega, \omega),$$

and let $\pi \colon \mathcal{G}(\omega) \to \mathrm{Gal}(\bar{k}/k)$ be the map sending the elements of $\mathrm{Isom}^{\otimes}({}^{\sigma}\omega,\omega)$ to σ. From an isomorphism $f \colon {}^{\sigma}\omega \to \omega$ and an element ρ of $\mathrm{Gal}(\bar{k}/k)$, we obtain an isomorphism ${}^{\rho}f \colon {}^{\rho\sigma}\omega \to {}^{\rho}\omega$ by applying the functor $-\otimes_{\bar{k},\rho}\bar{k}$. We define the product of the elements $f_1 \colon {}^{\sigma_1}\omega \to \omega$ and $f_2 \colon {}^{\sigma_2}\omega_2 \to \omega$ of $\mathcal{G}$ by the rule

$$f_1 \cdot f_2 = f_1 \circ {}^{\sigma_1}f_2, \qquad \begin{tikzcd} {}^{\sigma_1\sigma_2}\omega \arrow[r, "{}^{\sigma_1}f_2"'] \arrow[rr, bend left, "f_1\cdot f_2"] & {}^{\sigma_1}\omega \arrow[r, "f_1"'] & \omega. \end{tikzcd}$$

Then

$$\pi(f_1 \cdot f_2) = \sigma_1\sigma_2 = \pi(f_1)\pi(f_2),$$

and

$$\begin{cases} (f_1 \cdot f_2)\cdot f_3 = f_1 \circ {}^{\sigma_1}f_2 \circ {}^{\sigma_1\sigma_2}f_3 = f_1 \cdot (f_2 \cdot f_3) \\ f \cdot \mathrm{id}_{\omega} = f = \mathrm{id}_{\omega} \cdot f \\ f \cdot {}^{\pi(f)^{-1}}(f^{-1}) = \mathrm{id}_{\omega} = {}^{\pi(f)^{-1}}(f^{-1}) \cdot f, \end{cases}$$

and so $\mathcal{G}$ has a group structure for which π is a homomorphism. We have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^{\otimes}(\omega) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \longrightarrow 1.$$

of abstract groups. The homomorphism π is surjective because any two fibre functors on T with values in $\bar{k}$ are isomorphic. Every algebraic tannakian subcategory admits a fibre functor with values in a finite extension K of k in $\bar{k}$, which can be used to define a section of π over $\mathrm{Gal}(\bar{k}/K)$. From C and ω we have constructed a $\bar{k}/k$-Galois groupoid $\mathcal{G} \overset{\text{def}}{=} \mathcal{A}ut_k^{\otimes}(\omega)$ with kernel $\mathcal{A}ut_{\bar{k}}^{\otimes}(\omega)$.

THEOREM 3.6 *Let T be an essentially small tannakian category over k and ω a $\bar{k}$-valued fibre functor on T.*

(a) *The extension $\mathcal{G}(\omega)$ is a $\bar{k}/k$-Galois groupoid with kernel $\mathcal{A}ut_{\bar{k}}^{\otimes}(\omega)$.*

(b) *The functor $\mathsf{T} \to \mathsf{Repf}(\mathcal{G}(\omega))$ defined by ω is an equivalence of tensor categories over $\bar{k}$.*

Conversely, if $\mathcal{G}$ is a $\bar{k}/k$-Galois groupoid, then $\mathsf{Repf}(\mathcal{G})$ is a tannakian category over k, the forgetful functor is a fibre functor, and $\mathcal{G} \simeq \mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

PROOF Statement (a) is proved above. In proving (b), we may suppose that T is algebraic. Then T has a fibre functor ω over a finite exension k', and

$$\mathsf{T} \xrightarrow{\omega} \mathsf{Repf}(\mathcal{G})$$

is an equivalence because it becomes an equivalence after we have extended scalars to k',

$$\mathsf{T}_{(k')} \underset{3.2}{\xrightarrow{\sim}} \mathsf{Repf}(\mathcal{A}ut^{\otimes}(\omega')) \underset{2.10}{\xrightarrow{\sim}} \mathsf{Repf}(\mathcal{G})_{(k')}.$$

The final statement is proved in 2.5. □

Notes

3.7 In 1.3 we defined a 2-category $\mathcal{GGrpd}$ of $\bar{k}/k$-Galois groupoids. Similarly (IV, 7.2) we can define a 2-category $\mathcal{Tann}^\bullet$ of essentially small tannakian categories over k equipped with a $\bar{k}$-valued fibre functor ω. There are 2-functors

$$\begin{aligned} \mathcal{Tann}^{\bullet\mathrm{opp}} &\to \mathcal{GGrpd}, \quad (\mathsf{T},\omega) \rightsquigarrow \mathcal{A}ut_k^{\otimes}(\omega) \\ \mathcal{GGrpd} &\to \mathcal{Tann}^{\bullet\mathrm{opp}}, \quad \mathcal{G} \rightsquigarrow \mathsf{Repf}(\mathcal{G}). \end{aligned}$$

These define biequivalence.

We have classified tannakian categories over k equipped with a $\bar{k}$-valued fibre functor. Later (IV, 10.1) we shall see that all essentially small tannakian categories over k with smooth band admit a $\bar{k}$-valued fibre functor.

3.8 Let k' be a finite extension of k. Essentially small tannakian categories over k equipped with a fibre functor over k' correspond to $\bar{k}/k$-Galois groupoids equipped with a splitting over k'.

3.9 To give a tannakian category T over k together with a $\bar{k}$-valued fibre functor ω is essentially the same as giving a tannakian category $\bar{\mathsf{T}}$ over $\bar{k}$ together with a $\bar{k}$-valued fibre functor $\bar{\omega}$ and a descent datum. If $\rho\colon \mathcal{A}ut_{\bar{k}}^{\otimes}(\omega) \to \mathrm{GL}_V$ is the representation attached to $(\bar{\mathsf{T}},\bar{\omega})$, then giving the descent datum amounts to extending ρ to a representation $\mathcal{A}ut_{\bar{k}}^{\otimes}(\omega) \to \mathcal{G}_V$ of the Galois gerbe attached to (T,ω).

4 Examples

Galois groupoids for $\mathbb{C}/\mathbb{R}$

4.1 Let

$$1 \to G(\mathbb{C}) \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

be a $\mathbb{C}/\mathbb{R}$-Galois groupoid with kernel G. Choose an $s \in \mathcal{G}$ such that $\pi(s) = \iota$ (complex conjugation). Let $\sigma\colon G(\mathbb{C}) \to G(\mathbb{C})$ be conjugation by s, and let $c = s^2$. Then σ is ι-linear, $c \in G(\mathbb{C})$, and

$$\sigma^2 = \mathrm{ad}(c), \quad \sigma(c) = c. \tag{*}$$

Every triple (G,σ,c) satisfying these conditions arises in this way from a $\mathbb{C}/\mathbb{R}$-Galois groupoid. When we replace the section s with sm, $m \in G(\mathbb{C})$, the pair (σ, c) is replaced by $(\sigma \circ \mathrm{ad}(m), \sigma(m)cm)$.

4.2 Let (G,σ,c) be a triple satisfying the conditions in 4.1, and let $\mathcal{G}$ be the corresponding $\mathbb{C}/\mathbb{R}$-Galois groupoid. Then $\mathsf{Repf}(\mathcal{G})$ is a tannakian category over $\mathbb{R}$ with a $\mathbb{C}$-valued (forgetful) fibre functor ω such that $G = \mathcal{A}ut^{\otimes}(\omega)$. Together with Theorem 3.6, this gives a description of tannakian categories over $\mathbb{R}$ that we exploit in Chapter V, §1.

5 Tannakian categories with band of multiplicative type

Let T be a tannakian category over a field k. If $\mathcal{A}ut^{\otimes}(\omega)$ is commutative for one fibre functor ω, then there exists an affine group scheme G over k such that $G_R \simeq \mathcal{A}ut^{\otimes}(\omega_R)$ for any fibre functor ω_R over a k-algebra R. We then say that T has commutative band G. In this section we study tannakian categories with band a group scheme of multiplicative type. They form an important class – for example, the category of motives over a finite field is conjecturally of this type.

Throughout, $\bar{k}$ is a separable algebraic closure of a field k, and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. We implicitly assume that tannakian categories are essentially small.

Review of the Brauer group

5.1 Two central simple algebras A and B over k are ***similar*** if $M_m(A) \approx M_n(B)$ for some m, n. This is an equivalence relation, and we let $[A]$ denote the similarity class of A. The ***Brauer group*** $\mathrm{Br}(k)$ of k is the set of similarity classes of central simple algebras over k with the multiplication defined by

$$[A] \cdot [B] = [A \otimes B].$$

It is a commutative group with identity element $[k]$. Let V denote A regarded as a vector space over k. Then $A \otimes_k A^{\mathrm{opp}}$ acts on V by $(a \otimes b)(v) = avb$, and $A \otimes_k A^{\mathrm{opp}} \simeq \mathrm{End}_{k\text{-linear}}(V)$. Therefore $[A]^{-1} = [A^{\mathrm{opp}}]$.

5.2 For a subfield L of $\bar{k}$, let $\mathrm{Br}(L/k)$ denote the kernel of

$$[A] \mapsto [A \otimes L] \colon \ \mathrm{Br}(k) \to \mathrm{Br}(L).$$

Then $\mathrm{Br}(k) = \bigcup \mathrm{Br}(L/k)$, where L runs over the subfields of $\bar{k}$ finite over k.

5.3 Let $L \subset \bar{k}$ be a finite Galois extension of k. Every class in $\mathrm{Br}(L/k)$ is represented by a central simple algebra A over k containing L as a maximal subfield (so $[A : k] = [L : k]^2$). Let E be the group of invertible elements $\alpha \in A$ such that $\alpha L \alpha^{-1} = L$. Then each $\alpha \in E$ defines an element $x \mapsto \alpha x \alpha^{-1}$ of $\mathrm{Gal}(L/k)$, and the Noether–Skolem theorem implies that every element of $\mathrm{Gal}(L/k)$ arises from an $\alpha \in E$. Because L is a maximal subfield of A, the centralizer of L in A is L itself, and so the sequence

$$1 \to L^{\times} \longrightarrow E \xrightarrow{\pi} \mathrm{Gal}(L/k) \to 1$$

is exact. Let $s \colon \ \mathrm{Gal}(L/k) \to E$ be a section to π, and write

$$s(\sigma)s(\tau) = a_{\sigma,\tau} s(\sigma\tau).$$

Then $a_{\sigma,\tau}$ is a 2-cocycle, whose class in $H^2(L/k, L^{\times})$ we denote by $\gamma(A)$.

THEOREM 5.4 *The map $A \mapsto \gamma(A)$ induces an isomorphism*

$$\mathrm{Br}(L/k) \to H^2(L/k, \mathbb{G}_m).$$

On passing to the limit over the finite Galois extensions of k in $\bar{k}$, we obtain an isomorphism

$$\mathrm{Br}(k) \simeq H^2(k, \mathbb{G}_m) \quad \textit{(Galois cohomology group).}$$

PROOF See Serre 2001. □

Numeric study of abelian categories

In this section, we define some numeric invariants of abelian and tannakian categories. Later we determine these invariants for tannakian categories with band of multiplicative type.

5.5 Let k be a field and A a locally finite k-linear abelian category. Recall that an object of an abelian category is simple if it is nonzero and contains no proper nonzero subobject. An object is ***isotypic*** if it is isomorphic to a direct sum of copies of a single simple object, whose isomorphism class is then called the ***type*** of the isotypic object. Two isotypic objects M and N have the same type if and only if $M^{\oplus m}$ is isomorphic to $N^{\oplus n}$ for some $m, n > 0$.

5.6 For a finite extension k' of k, we let $\mathsf{A}_{(k')}$ denote the category whose objects are the pairs consisting of an object of A together with an action of k'. There is an exact functor

$$i_{k'/k}\colon \mathsf{A} \to \mathsf{A}_{(k')}, \quad X \rightsquigarrow k' \otimes_k X,$$

and $\mathrm{Hom}_{\mathsf{A}_{(k')}}(k' \otimes X, k' \otimes Y) \simeq k' \otimes \mathrm{Hom}_{\mathsf{A}}(X, Y)$. The forgetful functor

$$r_{k'/k}\colon \mathsf{A}_{(k')} \to \mathsf{A}, \quad (X, \mu) \rightsquigarrow X,$$

is also exact, and is right adjoint to $i_{k'/k}$. See I, 7.11.

5.7 If M is a simple object of A, then the smallest abelian subcategory $\langle M\rangle$ of A containing M is equivalent to the category of finite-dimensional vector spaces over the division algebra $D = \mathrm{End}(M)$. The object M remains semisimple under an extension k'/k if and only if $D \otimes_k k'$ is semisimple, i.e., a product of matrix algebras over division algebras. This is true for all k' if it is true for one perfect field containing k, or, equivalently, if the centre of D is separable over k. One then says that M is ***absolutely semisimple***.

5.8 The Grothendieck group $K(\mathsf{A})$ of A is the free abelian group generated by the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects of A. If k' is an extension of k and M is a simple object of A such that $k' \otimes M$ is semisimple, then the decomposition of $k' \otimes M$ can be read off from the structure of $k' \otimes_k D = D'$: if

$$D' = \prod\nolimits_{1 \le i \le r} M_{n_i}(D_i), \quad D_i \text{ a division algebra}, \quad n_i \ge 1,$$

then $k' \otimes M$ decomposes into r isotypic components each of the form $S_i^{\oplus n_i}$ with S_i simple. If k' is Galois over k with Galois group Π, then the $[S_i]$, $1 \le i \le r$, are conjugate under the action of Π, and so there is a canonical bijection

$$\Sigma(\mathsf{A}) \simeq \Sigma(\mathsf{A}_{(k')})/\Pi.$$

5.9 Now assume that A is semisimple. Recall that this means that every object is a sum of its simple subobjects, and hence a finite direct sum of simple objects (I, 7.14). If e is simple, then every nonzero endomorphism of e is an isomorphism. Therefore, $\mathrm{End}(e)$ is a division algebra. It contains k in its centre, and is finite-dimensional over k. Let re denote the direct sum of r copies of e. Then $\mathrm{End}(re) \simeq M_r(\mathrm{End}(e))$. If e' is a second simple object, then either $e \approx e'$ or $\mathrm{Hom}(e, e') = 0$. Therefore, if $x = \sum r_i e_i$ ($r_i \ge 0$) and $y = \sum s_i e_i$ ($s_i \ge 0$) are two objects of A expressed as sums of copies of simple objects e_i such that $e_i \not\approx e_j$ if $i \ne j$, then

$$\mathrm{Hom}(x, y) \simeq \prod M_{s_i, r_i}(\mathrm{End}(e_i)).$$

Thus, the category A is determined up to equivalence by

(a) the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects in A;

(b) for each $\sigma \in \Sigma$, the isomorphism class $[D_\sigma]$ of the endomorphism algebra of a representative of σ.

We call $\Sigma(\mathsf{A})$ and $([D_\sigma])_{\sigma\in\Sigma(\mathsf{A})}$ the ***numeric characters*** of A.

Let k_σ denote the centre of D_σ. The isomorphism class of D_σ as a k-algebra is determined by the isomorphism class of k_σ as an extension of k and the class ζ_σ of D_σ in $\mathrm{Br}(k_\sigma)$.

5.10 Suppose in addition that A has a k-linear tensor structure. We call the ***numeric characters*** of A the following data:

(a) the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects of A;

(b) the multiplication in $K(\mathsf{A}) = \mathbb{Z}^{(\Sigma)}$ defined by the tensor product; this amounts to giving, for all $\sigma,\tau \in \Sigma$, the natural numbers $c^\rho_{\sigma,\tau}$ such that

$$\sigma\tau = \sum\nolimits_\rho c^\rho_{\sigma,\tau}\rho;$$

(c) the map

$$\sigma \mapsto (k_\sigma, \zeta_\sigma), \quad \zeta_\sigma \in \mathrm{Br}(k_\sigma),$$

considered in (5.9);

(d) the function

$$\dim\colon \mathsf{A} \to \mathbb{Z}$$

when A is assumed to be tannakian.

5.11 Note that to give (a) and (b) is equivalent to giving the commutative ring $K(\mathsf{A})$ together with its "effective subset"

$$\begin{aligned} K(\mathsf{A})^{\mathrm{eff}} &= \text{submonoid of } K(\mathsf{A}) \text{ generated by } [M], \quad M \in \mathrm{ob}\,A,\\ &= \mathbb{N}^{(\Sigma)}. \end{aligned}$$

Indeed, Σ can be recovered from $K(\mathsf{A})^{\mathrm{eff}} \subset K(\mathsf{A})$ as the set of minimal nonzero elements of $K(\mathsf{A})^{\mathrm{eff}}$. The datum (c) then allows us to recover, as described above, the functorial variance of the ring $K(\mathsf{A}_{(k')})$ with respect to k'. Finally, the datum in (d) allows us to recover in principle the characteristic polynomial (in particular, the trace and determinant) of a semisimple object M of A in terms of the reduced characteristic polynomial (resp. reduced trace, reduced norm) in the central simple algebra $\mathrm{End}(M_i)$ corresponding to the isotypic components M_i of M. Let M be isotypic of rank n, and suppose that $D = \mathrm{End}(M)$ has centre Z of rank r over k; then $n = n'r$ where n' is the rank of M over Z, and D is of rank d^2 over Z with $d|n'$. Moreover,

$$\begin{aligned} \det\nolimits_M f &= \mathrm{Nm}_{Z/k}(\mathrm{rNm}_{D/Z} f)^{n'/d}\\ \mathrm{Tr}_M f &= \mathrm{Tr}_{Z/k}(\mathrm{rTr}_{D/Z} f)\frac{n'}{d}\\ \mathrm{Pol}_M(f,t) &= \mathrm{Nm}_{Z/k}(\mathrm{rPol}_{D/Z}(f,t))^{n'/d}, \quad n'/d = n/dr. \end{aligned}$$

5.12 The knowledge of the numeric characters of a tannakian category A does not allow us to reconstruct it up to equivalence, even if k is an algebraically closed field of characteristic zero (in which case the datum (c) is vacuous) and A is semisimple, even in the particular case where A is the category of representations of a finite group.[3] As an exception to this remark, we shall show that a tannakian category with diagonalizable band is determined by its numeric characters.

Neutral tannakian categories with diagonalizable band

Let G be an affine group scheme over k.

5.13 A k-***character*** of G is a homomorphism $\chi\colon G\to\mathbb{G}_m$ (over k). The k-characters of G form an abelian group $X(G)$.

Let $(\mathcal{O}(G),\Delta)$ be the Hopf algebra of G. A ***group-like element*** of G is an element $m\in\mathcal{O}(G)^\times$ such that $\Delta(m)=m\otimes m$.

A k-character χ of G corresponds to a homomorphism of k-algebras $k[T,T^{-1}]\to\mathcal{O}(G)$ compatible with the comultiplications. Such a homomorphism sends T to a group-like element $a(\chi)$ of $\mathcal{O}(G)$ and is determined by it. Therefore $\chi\leftrightarrow a(\chi)$ is a one-to-one correspondence between the characters of G and the group-like elements of $\mathcal{O}(G)$.

5.14 Let $r\colon G\to\mathrm{GL}_V$ be a representation of G on a finite-dimensional k-vector space V and $\rho\colon V\to V\otimes\mathcal{O}(G)$ the corresponding co-action. For a k-character χ of G, we let

$$V_\chi=\{v\in V\mid \rho(v)=v\otimes a(\chi)\},$$

and call it the ***eigenspace for*** G ***with character*** χ.

The following conditions on G are equivalent:

(a) $\mathcal{O}(G)$ is spanned (as a k-vector space) by its group-like elements;

(b) for every finite-dimensional representation (V,r) of G,

$$V=\bigoplus_{\chi\in X(G)}V_\chi.$$

An affine group scheme satisfying these equivalent conditions is said to be ***diagonalizable.*** See Milne 2017, 12.12.

5.15 Let M be an abelian group written multiplicatively, and let $k[M]$ be the k-vector space with basis M. When equipped with the multiplication extending that of M and the comultiplication determined by

$$\Delta(m)=m\otimes m,\quad m\in M,$$

$k[M]$ becomes a Hopf algebra. We let $D(M)$ denote the corresponding affine group scheme over k. Then $D(M)$ is diagonalizable because the elements of M are group-like. Conversely, if G is diagonalizable, then $G\simeq D(M)$ with M the group of group-like elements in $\mathcal{O}(G)$. The functor D is an equivalence from the category of abelian groups to the category of affine group schemes of multiplicative type.

For example, $D(\mathbb{Z})=\mathbb{G}_m$ and $D(\mathbb{Z}/p\mathbb{Z})=\mu_p$, which is an infinitesimal group scheme if $\mathrm{char}(k)=p$ and a finite étale group scheme otherwise.

[3] The first order for which there are two nonisomorphic noncommutative groups is 8, and their character groups do not distinguish them. But see II, 8.11.

5.16 Let G be diagonalizable, and let $M = X(G)$, so $G \simeq D(M)$. Then $\mathsf{Rep}(G)$ is the category of finite-dimensional k-vector spaces V equipped with an M-gradation,

$$V = \bigoplus_{m \in M} V_m$$

(cf. II, 9.2).

THEOREM 5.17 *A neutral tannakian category has diagonalizable band if and only if*

(a) *it is semisimple,*

(b) *the tensor product of any two simple objects is simple.*

PROOF Let C be a neutral tannakian category, let ω be k-valued fibre functor, and let $G = \mathcal{A}ut^{\otimes}(\omega)$.

If G is diagonalizable, say, $G = D(M)$, then ω defines an equivalence of C with the category $\mathsf{Repf}(D(M))$, which has the properties (a) and (b).

For the converse, suppose that C has the properties (a) and (b). If S is a simple object of C, then $\mathrm{ev}_S : S^{\vee} \otimes S \to \mathbb{1}$ is a nonzero morphism between simple objects, hence an isomorphism. Therefore $\omega(S)^{\vee} \otimes \omega(S) \simeq k$, and so $\omega(S)$ has dimension 1. Every finite-dimensional representation of G is a sum of one-dimensional representations, which implies that G satisfies condition 5.14(b) to be diagonalizable. □

Tannakian categories with diagonalizable band

Let $G = D(M)$ be a diagonalizable algebraic group over k. The simple objects of $\mathsf{Repf}(G)$ are the one-dimensional k-vector spaces on which G acts through a character. The abelian group M can be recovered from $\mathsf{Repf}(G)$ as the set of isomorphism classes of simple objects with the addition defined by tensor product. Alternatively, it is the group of types of the isotypic objects.

PROPOSITION 5.18 *Let* T *be a tannakian category over* k *admitting a fibre functor over* $\bar{k}$. *Then* T *has diagonalizable band if and only if*

(a) *it is semisimple, and*

(b) *the tensor product of any two isotypic objects is isotypic.*

In this case, the set of types of isotypic objects M *forms a group under tensor product, and the band of* T *is* $D(M)$.

PROOF If T has diagonalizable band, then $\mathsf{T}_{(\bar{k})}$ is equivalent to $\mathsf{Repf}(D(M))$ with M set of types of isotypic classes in $\mathsf{T}_{(\bar{k})}$. Therefore $\mathsf{T}_{(\bar{k})}$ satisfies (a) and (b) (5.17), and it follows that T satisfies these conditions (see I, 7.16 and 5.7)

Conversely, suppose that (a) and (b) hold, and let M be the set of types of isotypic objects. As the tensor product of two isotypic objects is again isotypic, M has the structure of an abelian group, and the isotypic objects are the homogeneous objects for an M-gradation on T. The gradation defines a homomorphism

$$D(M) \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{T}}) = Z(\text{band of } \mathsf{T})$$

(II, 9.2) and it remains to prove that this induces an isomorphism of $D(M)$ onto the band of T. It suffices to do this after an extension of the base field, and so we may suppose that there is a k-valued fibre functor ω on T. Once we have shown that the simple objects of T have (categorical) dimension 1, the functor ω will define an equivalence from T to the tannakian category of M-graded finite-dimensional k-vector spaces, and hence to $\mathsf{Repf}(D(M))$, as required. If S is a simple object of dimension r, then $\dim(S \otimes S^\vee) = r^2$ by I, 5.4. As $S \otimes S^\vee$ is isotypic, the coevaluation morphism shows that it is trivial, and so $\mathrm{Hom}(\mathbb{1}, S \otimes S^\vee)$ has dimension r^2. Recall that $\mathrm{End}(S) = \mathrm{Hom}(\mathbb{1}, S \otimes S^\vee)$; see (13), p. 15. As ω induces an isomorphism

$$\mathrm{End}(S) \to \mathrm{End}(\omega(S)) \simeq M_r(k)$$

and $\mathrm{End}(S)$ is a division algebra, we see that $r = 1$. □

EXAMPLE 5.19 Let A be a central simple algebra over k. After possibly replacing A with a similar algebra, we may assume that A contains a maximal field L Galois over k. Consider the extension

$$1 \to L^\times \to E \to \mathrm{Gal}(L/k) \to 1 \tag{*}$$

defined in 5.3. For each $\sigma \in \mathrm{Gal}(L/k)$, let $s(\sigma)$ be an element of E mapping to σ. Then

$$s(\sigma) \cdot \sigma s(\tau) = a_{\sigma,\tau} s(\sigma\tau),$$

where $(a_{\sigma,\tau})_{\sigma,\tau}$ is a 2-cocycle representing the cohomology class of A in $\mathrm{Br}(k)$.

Let V denote A regarded a k-vector space, and let L act on it by left multiplication. The action of A on V defines an action r of E on V, and

$$\mathrm{End}(V, r) \simeq A^{\mathrm{opp}}$$

(see I, 9.6).

Let ${}^\sigma V$ equal V as a k-vector space, but with $c \in L$ acting as σc. Then left multiplication by $s(\sigma)$ is an L-linear map ${}^\sigma V \to V$, which we again denote by $s(\sigma)$. Now

$$s(\sigma) \circ \sigma s(\tau) = s(\sigma\tau)$$

as L-linear maps $\sigma\tau V \to V$.

Let T be the tannakian category of representations of E on finite-dimensional k-vector spaces, and let ω be the forgetful fibre functor. Note that E acts on V through the action of A, and we have $\omega(V) = V$ and ${}^\sigma\omega(V) = {}^\sigma V$. It follows that the cohomology class of T is the image of the class of the extension (*) under the canonical map

$$H^2(L/k, L^\times) \to H^2(\bar{k}/k, \bar{k}^\times) = H^2(k, \mathbb{G}_m).$$

EXERCISE 5.20 Show that a semisimple tannakian category has diagonalizable band if and only if, for all isotypic objects X,

$$\dim_k(\mathrm{End}(X)) = \dim(X)^2.$$

Let T be a tannakian category whose band is the diagonalizable group $D(M)$. An element m of M determines a homomorphism $\mathbb{Z} \to M$ and hence a homomorphism

$$\varphi_m : H^2(k, D(M)) \to H^2(k, D(\mathbb{Z})) \simeq \mathrm{Br}(k).$$

PROPOSITION 5.21 *For any isotypic object X of type m, the homomorphism φ_m sends the class of T in $H^2(k, D(M))$ to the class of $\mathrm{End}(X)^{\mathrm{opp}}$ in $\mathrm{Br}(k)$.*

PROOF We prove this using Galois groupoids, which requires us to assume M has no p-torsion if k has characteristic $p \neq 0$. For simplicity, we also assume that M is finitely generated. For a proof using gerbes, see Saavedra 1972, VI, 3.5.3.

On choosing a fibre functor over $\bar{k}$, we get a Galois groupoid

$$1 \to D(M)(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

whose cohomology class in $H^2(k, D(M))$ is the cohomology class of T.

Let X be an isotypic object of type m, and let $A = \mathrm{End}(X)$. After possibly replacing X with a different isotypic object of type m, we may suppose that A contains a maximal subfield $L \subset \bar{k}$ Galois over k. Let

$$1 \to L^\times \to E \to \mathrm{Gal}(L/k) \to 1$$

the extension attached to A as in 5.3. The category $\langle X \rangle^\otimes$ is equivalent to the category of the representations of this extension. Therefore the cohomology class of A^{opp} in $H^2(k, \mathbb{G}_m)$ is equal to the cohomology class of $\langle X \rangle^\otimes$ (5.19). Since φ_m sends the class of T in $H^2(k, D(M))$ to the class of $\langle X \rangle^\otimes$ in $H^2(k, \mathbb{G}_m)$, this completes the proof. □

From φ_m, we get a pairing

$$H^2(k, D(M)) \times M \to \mathrm{Br}(k), \quad x, m \mapsto \varphi_m(x).$$

PROPOSITION 5.22 *Assume that M is finitely generated. The homomorphism*

$$H^2(k, D(M)) \to \mathrm{Hom}(M, \mathrm{Br}(k)) \tag{64}$$

defined by the above pairing is an isomorphism.

PROOF When $M = \mathbb{Z}$, the homomorphism $\varphi_1 : H^2(k, \mathbb{G}_m) \to \mathrm{Br}(k)$ is the canonical isomorphism $H^2(k, \mathbb{G}_m) \simeq \mathrm{Br}(k)$. It follows that the proposition holds for $\mathbb{Z}$, hence for $\mathbb{Z}^{(I)}$, where I is any finite set, because both sides of (64) transform sums into products.

As M is finitely generated M, there is an exact sequence

$$0 \to \mathbb{Z}^{(I)} \to \mathbb{Z}^{(J)} \to M \to 0$$

with I and J finite, and hence an exact sequence

$$0 \to D(M) \to \mathbb{G}_m^J \to \mathbb{G}_m^I \to 0.$$

As for H^2, we have $H^1(k, \mathbb{G}_m^I) \simeq H^1(k, \mathbb{G}_m)^I$, which is zero, and so we have an exact sequence

$$0 \to H^2(k, D(M)) \to H^2(k, \mathbb{G}_m)^J \to H^2(k, \mathbb{G}_m)^I.$$

On comparing this with the exact sequence

$$0 \to \mathrm{Hom}(M, \mathrm{Br}(k)) \to \mathrm{Hom}(\mathbb{Z}^{(J)}, \mathrm{Br}(k)) \to \mathrm{Hom}(\mathbb{Z}^{(I)}, \mathrm{Br}(k)),$$

we obtain the statement for M. □

The statement is true also when M is not finitely generated provided one interprets $H^2(k, D(M))$ as an fpqc cohomology group (the fppf groups need not commute with infinite products).

COROLLARY 5.23 *If* $\mathrm{Br}(k) = 0$, *for example, if k is separably closed or finite, then every tannakian category over k with diagonalizable band is neutral.*

SUMMARY 5.24 Let T be tannakian category over k with band $D(M)$. Then T is determined up to tensor equivalence by a homomorphism

$$u\colon M \to \mathrm{Br}(k).$$

The category T is graded of type M. The homogeneous objects are the isotypic objects. For each $m \in M$, there is exactly one simple object (up to isomorphism) S_m homogeneous of degree m, and $\mathrm{End}(S_m)$ is a division algebra with centre k and invariant $u(m)$ in $\mathrm{Br}(k)$.

Let $d(m)$ be the order of $u(m)$ in $\mathrm{Br}(k)$. When k is a local or global field,

$$d(m) = \sqrt{\dim(\mathrm{End}(S_m))},$$

but not in general for other fields (period-index problem).

The Grothendieck ring of T is the subring $K(\mathsf{T}) \subset \mathbb{Z}[M]$ determined by

$$\sum n_m[S_m] \in K(\mathsf{T}) \iff n_m \equiv 0 \bmod d(m) \text{ for all } m \in M.$$

A canonical basis of $K(\mathsf{T})$ is formed by the elements $d(m)[S_m]$.

5.25 When $M = \mathbb{Z}^{(I)}$, i.e., $G = \mathbb{G}_m^I$, to give u amounts to giving an element $\xi_i \in \mathrm{Br}(k)$ for each $i \in I$. Choose, for each i, a nonzero isotypic object E_i of degree i, and let $A_i = \mathrm{End}(E_i)$ be a central simple k-algebra with invariant ξ_i. It is possible to reconstruct T (up to tensor equivalence) from the A_i.

Consider first the additive subcategory having as objects the elements $\gamma \in \mathbb{N}^{(I)}$, ($\gamma = (\gamma_i)_{i\in I}$ being interpreted as $\otimes E_i^{\gamma_i}$), with

$$\mathrm{Hom}(\gamma, \gamma') = \begin{cases} \otimes A_i^{\otimes \gamma_i} & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma'. \end{cases}$$

Composition of morphisms is obvious. Define in the obvious way the functor $\otimes$ (addition on $\mathbb{N}^{(I)}$ on the objects, tensor product on the homomorphisms) and the constraints so that one obtains a tensor subcategory. Its karoubian envelope is an abelian symmetric monoidal category, which can be identified with the full subcategory of T formed of objects whose nonzero homogeneous components have degree in $\mathbb{N}^{(I)}$. There is a fibre functor over k^{al}. On the other hand, for all i, if A_i has of rank d_i^2 (therefore E_i is of rank d_i), let $L_i = \det E_i \overset{\text{def}}{=} \bigwedge^{d_i} E_i$ (where E_i denotes the object of the category corresponding to the ith element of the canonical basis of $\mathbb{Z}^{(I)}$, say l_i ...); it is a pre-invertible object ($X \mapsto L_i \otimes X$ is fully faithful), and on inverting the L_i, one obtains (up a tensor equivalence) the category T. When M is not free, one finds an analogous construction in terms of suitable systems of generators $(\gamma_i)_{i\in I}$, but "putting equal to 1" certain elements. These constructions are well adapted for describing the $\otimes$-functors of T into a Tannakian category, and to describing base change. For this last reason notably, it would be maladroit in the general case to take the E_i to be simple.

Algebraic groups of multiplicative type and their representations

An algebraic group scheme G over k is of ***multiplicative type*** if it becomes diagonalizable over some extension of k, in which case it becomes diagonalizable over a finite separable extension of k. We let

$$X^*(G) = X(G_{\bar{k}}) \overset{\text{def}}{=} \operatorname{Hom}(G_{\bar{k}}, (\mathbb{G}_m)_{\bar{k}}).$$

The functor $G \rightsquigarrow X^*(G)$ is a contravariant equivalence from the category of algebraic groups of multiplicative type over k to the category of finitely generated $\mathbb{Z}$-modules equipped with a continuous action of $\Gamma \overset{\text{def}}{=} \operatorname{Gal}(\bar{k}/k)$.

PROPOSITION 5.26 *Let G be an algebraic group of multiplicative type over k. Then* $\mathsf{Repf}(G)$ *is a k-linear semisimple abelian category whose isomorphism classes are classified by the orbits of Γ acting on $X^*(G)$. Let (V,r) be the simple representation corresponding to an orbit Ξ, and let $\chi \in \Xi$; then* $\operatorname{End}(V,r) \simeq k(\chi)$, *where $k(\chi)$ is the subfield of $\bar{k}$ fixed by the subgroup of Γ fixing χ.*

PROOF The group G is split by a finite Galois extension K of k – let $\bar{\Gamma} = \operatorname{Gal}(K/k)$. Then $\bar{\Gamma}$ acts on $\mathcal{O}(G_K) \simeq K \otimes \mathcal{O}(G)$ through its action on K. Let (V,r) be a representation of G_K, and let ρ be the corresponding co-action. By a semilinear action of $\bar{\Gamma}$ on (V,r), we mean a semilinear action of $\bar{\Gamma}$ on V fixing ρ. It follows from descent theory (see §6) that the functor $V \rightsquigarrow V_K$ from $\mathsf{Rep}_k(G)$ to the category of objects of $\mathsf{Rep}_K(G_K)$ equipped with a semilinear action of $\bar{\Gamma}$ is an equivalence of categories. Let V be a representation of G over k. Then $K \otimes V$ is a representation of G_K equipped with a semilinear action of $\bar{\Gamma}$. As a representation of G_K, it decomposes into a direct sum

$$K \otimes V = \bigoplus\nolimits_{\chi \in X(G_K)} V_\chi.$$

An element γ of $\bar{\Gamma}$ acts on $K \otimes V$ by mapping V_χ isomorphically onto $V_{\gamma\chi}$. Thus, we see that the set of χ occurring in the sum is stable under the action of $\bar{\Gamma}$. Conversely, if Ξ is an orbit of $\bar{\Gamma}$ in $X(G_K)$, then $\bigoplus_{\chi \in \Xi} V_\chi$ has a natural semilinear action of G_K, and so arises from a (simple) representation of G over k. From this the statement follows easily. □

Tannakian categories with band of multiplicative type

Let T be an essentially small tannakian category over a field k of characteristic zero, and let ω be a fibre functor on T over $\bar{k}$. Assume that $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\bar{k}}(\omega)$ is an algebraic group of multiplicative type. Then

$$1 \to G(\bar{k}) \to \mathcal{G} \to \Gamma \to 1$$

with

$$\mathcal{G} = \mathcal{A}ut^{\otimes}_{k}\omega \overset{\text{def}}{=} \bigsqcup_{\sigma \in \Gamma} \operatorname{Isom}^{\otimes}({}^{\sigma}\omega, \omega)$$

is a $\bar{k}/k$-Galois groupoid and the functor

$$\mathsf{T} \to \mathsf{Repf}(\mathcal{G})$$

is an equivalence of tensor categories (3.6).

Let $\chi \in X^*(G)$, let $\Gamma(\chi)$ be the subgroup of Γ fixing χ, and let $k(\chi) = \bar{k}^{\Gamma(\chi)}$. Then χ is defined over $k(\chi)$, and from $\chi\colon G_{k(\chi)} \to \mathbb{G}_m$ (homomorphism of commutative algebraic groups over $k(\chi)$), we get a homomorphism

$$H^2(k(\chi), G_{k(\chi)}) \to H^2(k(\chi), \mathbb{G}_m).$$

On combining this with the restriction map $H^2(k, G) \to H^2(k(\chi), G_{k(\chi)})$ and the isomorphism $H^2(k(\chi), \mathbb{G}_m) \simeq \mathrm{Br}(k(\chi))$, we get a homomorphism

$$H^2(k, G) \to \mathrm{Br}(k(\chi)). \tag{*}$$

PROPOSITION 5.27 *Let* $V(\chi) = \bigoplus_{\chi' \in \Gamma\chi} V_{\chi'}$. *Then* $V(\chi)$ *is simple,* $\mathrm{End}(V(\chi))$ *is a division algebra with centre* $k(\chi)$*, and the homomorphism (*) sends the class of* T *in* $H^2(k, G)$ *to the class of* $\mathrm{End}(V(\chi))$ *in* $\mathrm{Br}(k(\chi))$.

PROOF This is a straightforward consequence of the above definitions (see the case of a torus). □

THEOREM 5.28 *The category* T *is a locally finite semisimple k-linear abelian category, and the fibre functor* ω *defines a bijection*

$$\Sigma(\mathsf{T}) \simeq \Gamma\backslash X^*(G),$$

where $G = \mathcal{A}ut_k^{\otimes}(\omega)$. *If* $V(\chi)$ *is the simple representation corresponding to the orbit* $\Gamma\chi$*, then* $\mathrm{End}(V(\chi))$ *is a division algebra with centre* $k(\chi)$*, and its class in* $\mathrm{Br}(k(\chi))$ *is the image of the class of* T *under the homomorphism* $*$.

PROOF This summarizes previous results. □

When do the numeric characters determine a tannakian category up to a numerical equivalence?

The question of deciding whether the numeric characters determine the Tannakian category up to equivalence comes down, for a fixed G, to determining whether an element $\xi \in H^2(k, G)$ is known when $u_\sigma(\xi)$ is known for all $u\colon G \to (\mathbb{G}_m)_{Z_\sigma/k}$ as before. This is true when G is diagonalizable. We look at some other examples.

5.29 Let G be a one-dimensional torus, hence either $\mathbb{G}_m$ or $\mathbb{G}_m$ twisted by a quadratic extension Z of k. In the second case, we have an exact sequence

$$0 \longrightarrow G \xrightarrow{u} (\mathbb{G}_m)_{Z/k} \xrightarrow{\mathrm{Nm}_{Z/k}} \mathbb{G}_m \longrightarrow 0$$

and we conclude from the exact cohomology sequence and Hilbert's Theorem 90, that $\mathrm{Ker}(u\colon H^2(k, G) \to H^2(k, (\mathbb{G}_m)_{Z/k})$ is zero. In this case, the numeric characters determine T up to equivalence.

5.30 Let P be the group of multiplicative type conjecturally attached to the category of motives over $\mathbb{F}$. An element of $H^2(\mathbb{Q}, P)$ is zero if its image in $H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\chi]/\mathbb{Q}})$ is zero for all characters χ of P. See the statement (*) and its proof following Lemma 3.15 in Milne 1994.

NOTES This section is partly based on Saavedra 1972, VI, 3.5, and an unpublished manuscript of Grothendieck.

6 Appendix: Galois descent

Let V be an algebraic scheme over a field k_1 and let $\sigma\colon k_1 \to k_2$ be a homomorphism of fields. We let σV denote the algebraic scheme over k_2 obtained by applying σ to the coefficients of the equations defining V, and for $P \in V(k_1)$, we let σP denote the point on σV obtained by applying σ to the coordinates of P.

6.1 Let V be an affine algebraic variety over $\bar{k}$. Let $A = \Gamma(V, \mathcal{O}_V)$, and let $\sigma \in \Gamma$. We say that a bijection $\lambda\colon V(\bar{k}) \to V(\bar{k})$ is ***σ-linear*** if there exists a σ-linear automorphism λ' of the $\bar{k}$-algebra A such that

$$(\lambda' f)(\lambda P) = \sigma(f(P)), \quad \text{all } f \in A,\ P \in V(\bar{k}) \,(= \mathrm{Hom}_{\bar{k}}(A, \bar{k})). \tag{65}$$

To give such a λ is the same as giving an isomorphism $\Lambda\colon \sigma V \to V$ of algebraic varieties over $\bar{k}$. Indeed, such an isomorphism defines an automorphism λ of $V(\bar{k})$,

$$V(\bar{k}) \xrightarrow{\mathrm{can}_\sigma} (\sigma V)(\bar{k}) \xrightarrow{\Lambda(\bar{k})} V(\bar{k}),$$

and a $\bar{k}$-algebra automorphism λ' of A,

$$\Gamma(V, \mathcal{O}_V) \xrightarrow{\mathrm{can}_\sigma} \Gamma(\sigma V, \mathcal{O}_{\sigma V}) \xrightarrow{\Gamma(V,\Lambda)^{-1}} \Gamma(V, \mathcal{O}_V),$$

which are related by (65). Moreover, Λ is uniquely determined by λ, and a λ arises from a Λ if and only if there exists a λ' satisfying (65).

An action $*\colon \Gamma \times V(\bar{k}) \to V(\bar{k})$ of Γ on $V(\bar{k})$ is said to be ***regular*** if, for all $\sigma \in \Gamma$, the map

$$\sigma P \mapsto \sigma{*}P\colon (\sigma V)(\bar{k}) \to V(\bar{k})$$

is regular, i.e., defined by a morphism $\sigma V \to V$ of algebraic varieties over $\bar{k}$. An action $*$ of Γ on $V(\bar{k})$ is regular if and only if the map

$$P \mapsto \sigma{*}P\colon V(\bar{k}) \to V(\bar{k})$$

is σ-linear for all $\sigma \in \Gamma$. There is a one-to-one correspondence between regular actions $*$ of Γ on $V(\bar{k})$ and actions $(\sigma, f) \mapsto {}^{\sigma}f\colon \Gamma \times A \to A$ of Γ on A by semilinear $\bar{k}$-algebra homomorphisms, given by

$$({}^{\sigma}f)(\sigma{*}P) = \sigma(f(P)), \quad \text{all } P \in V(\bar{k}).$$

For example, if $W = \mathrm{Spec}\, B$ is a variety over k, then the natural action of Γ on $W(\bar{k})$ is regular, and corresponds to the action of Γ on $B \otimes_k \bar{k}$ through its action on $\bar{k}$.

6.2 Let V be an algebraic variety over $\bar{k}$. An action $*$ of Γ on $V(\bar{k})$ is said to be ***continuous*** if there exists a subfield K of $\bar{k}$ finite over k and a model $(W, \varphi\colon V \xrightarrow{\approx} W_{\bar{k}})$ of V over K such that the action of $\mathrm{Gal}(\bar{k}/K)$ is that defined by W. This means that the bijection

$$\varphi(\bar{k})\colon V(\bar{k}) \to W(\bar{k})$$

is Γ-equivariant, that is,

$$\varphi(\bar{k})(\sigma * P) = \sigma\left(\varphi(\bar{k})(P)\right)$$

Two such models, over K and K' say, become isomorphic over some finite extension of $K \cdot K'$.

Suppose that V is affine, and let $A = \Gamma(V, \mathcal{O}_V)$. Call an action of Γ on A trivial if there exists a model A_0 of A over k such that the action is induced through the isomorphism $A_0 \otimes \bar{k} \simeq A$ from the action of Γ on $\bar{k}$. The action of Γ on V is continuous if and only if it becomes trivial on an open subgroup of Γ.

It is easy to write down actions that are not continuous, but, in practice, those "occurring in nature" are continuous.

6.3 Let V be an algebraic scheme over $\bar{k}$. A $\bar{k}/k$-***descent system*** on V is a family of isomorphisms $\varphi_\sigma : \sigma V \to V$, $\sigma \in \mathrm{Gal}(\bar{k}/k)$, satisfying the ***cocycle condition***

$$\varphi_{\sigma\tau} = \varphi_\sigma \circ (\sigma\varphi_\tau) \quad \text{for all } \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

A model (V_0, φ) of V over a subfield K of $\bar{k}$ ***splits*** the descent system if $\varphi_\sigma = \varphi^{-1} \circ \sigma\varphi$ for all σ fixing K. A descent system is ***continuous*** if it is split by some subfield K finite over k. A ***descent datum*** is a continuous descent system. A descent datum is ***effective*** if it is split by a model over k. When V is quasi-projective, every descent datum is effective (see, for example, Milne 2024, 7.3).

THEOREM 6.4 *The functor $V \rightsquigarrow (\bar{V}, *)$ sending an affine algebraic variety V over k to an affine algebraic variety over $\bar{V}$ over $\bar{k}$ equipped with an action of Γ on $\bar{V}(\bar{k})$ is fully faithful with essential image the pairs $(\bar{V}, *)$ such that $*$ is continuous and regular.*

PROOF For the fully faithfulness, see, for example, Milne 2024, 4.5. For the description of the essential image, let V be an affine algebraic variety over $\bar{k}$ equipped with a continuous regular action of Γ. For each $\sigma \in \Gamma$, there is an isomorphism of algebraic varieties $\varphi_\sigma : \sigma V \to V$ sending σP to $\sigma * P$. The family $(\varphi_\sigma)_\sigma$ obviously satisfies the cocycle condition, and it is continuous because the action is continuous. Therefore, it is a descent datum on V. As V is affine, this descent datum is effective. □

In particular, algebraic groups over k correspond to algebraic groups over $\bar{k}$ equipped with a continuous regular action of Γ on their $\bar{k}$-points. This gives a description of algebraic groups over k that is close to the classical (pre-scheme) description. In a step that some may consider retrograde, we extended this to a description of k-groupoids acting transitively on $\mathrm{Spec}\, \bar{k}$.

NOTES Langlands and Rapoport (1987) introduced Galois groupoids (which they called Galois-gerbes) as a down-to-earth tool for studying tannakian categories over a field of characteristic zero equipped with a fibre functor over the algebraic closure of the field. This chapter is based on their article and on the exposition in Lattermann 1989.

Chapter IV

Tannakian categories and groupoids

In this chapter, we describe tannakian categories in terms of affine groupoid schemes (following Deligne 1990).

Throughout, k is the base commutative ring (usually a field). Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$.

In §1, we state Deligne's theorem relating tannakian categories with a fibre functor over a scheme to groupoids acting transitively on the scheme

$$(\mathsf{T}, \omega) \leftrightarrow (S, G).$$

In §2, we define groupoid schemes; in §3, we describe their representations; in §4, we interpret the representations as comodules over a coalgebroid; in §5, we prove the comonadicity theorem; in §6, we construct the coalgebroid of endomorphisms of a fibre functor and prove Deligne's theorem; in §7, we restate the results in terms of 2-categories.

1 Statement of the main theorem

In this section, k is a field.

A groupoid of sets is a small category in which all morphisms are isomorphisms. Thus giving a groupoid amounts to giving a set S (of objects), a set G (of arrows), maps $t, s\colon G \rightrightarrows S$ (sending an arrow to its target and source respectively), and a partial law of composition satisfying certain conditions. When S has only a single element, G is just a group, and so we can think of a groupoid as being a "group with many objects".

A groupoid scheme is a groupoid in the category Aff_k of affine schemes over k. Thus, it consists of an affine k-scheme S, an affine k-scheme G, morphisms $t, s\colon G \rightrightarrows S$, and a partial law of composition $\circ$ such that $(S(T), G(T), t, s, \circ)$ is a groupoid of sets for all affine k-schemes T. We usually refer to G as a k-groupoid acting on S. The morphism $(t, s)\colon G \to S\times S$ allows us to regard G as an $S\times S$-scheme. We say that G acts transitively on S if there exists a faithfully flat morphism $S' \to S \times S$ such that $\operatorname{Hom}_{S\times S}(S', G) \neq \emptyset$. Later we shall see that this is equivalent to G itself being faithfully flat over $S \times S$.

We present a detailed description of groupoids in §2. In §3, we define the category $\mathsf{Repf}(S{:}G)$ of representations of G on locally free sheaves of finite rank on S. This is a k-linear symmetric monoidal category.

When $(\mathsf{T},\otimes)$ is a tannakian category over k and ω is a fibre functor on T over a nonempty k-scheme S, we let $\mathcal{A}ut_k^{\otimes}(\omega)$ denote the functor sending an $S\times S$-scheme $(b,a)\colon T\to S\times S$ to the set of isomorphisms of $\otimes$-functors $a^*\omega\to b^*\omega$.

THEOREM 1.1 (DELIGNE 1990, 1.12) *Let* T *be an essentially small tannakian category over* k *and* ω *a fibre functor on* T *over a nonempty affine* k*-scheme* S*.*

(a) *The functor* $\mathcal{A}ut_k^{\otimes}(\omega)$ *of* $S\times S$*-schemes is represented by a* k*-groupoid* G *acting transitively on* S*.*

(b) *The functor* $\mathsf{T}\to\mathsf{Repf}(S{:}G)$ *defined by* ω *is a* $\otimes$*-equivalence.*

Conversely, if G *is a* k*-groupoid acting transitively on a nonempty* k*-scheme* S*, then* $\mathsf{Repf}(S{:}G)$ *is a tannakian category over* k *and* $G\simeq\mathcal{A}ut_k^{\otimes}(\omega_{\mathrm{forget}})$*.*

Thus the theorem provides a dictionary between tannakian categories over k equipped with a fibre functor over S and k-groupoids acting transitively on S.

EXAMPLE 1.2 When $S=\operatorname{Spec}k$, Theorem 1.1 becomes Theorem 3.1 of Chapter II.

NOTATION 1.3 Let T be a tannakian category over k. When ω_1 and ω_2 are fibre functors over S, we let $\mathcal{I}som_S^{\otimes}(\omega_1,\omega_2)$ denote the functor of S-schemes

$$(T\xrightarrow{u}S)\rightsquigarrow\mathrm{Isom}^{\otimes}(u^*\omega_1,u^*\omega_2).$$

When ω_1 and ω_2 are fibre functors over S_1 and S_2 respectively, we let

$$\mathcal{I}som_k^{\otimes}(\omega_2,\omega_1)=\mathcal{I}som_{S_1\times S_2}^{\otimes}(\mathrm{pr}_2^*\,\omega_2,\mathrm{pr}_1^*\,\omega_1),$$

so, as a functor of $S_1\times S_2$-schemes, $\mathcal{I}som_k^{\otimes}(\omega_2,\omega_1)$ is

$$(T\xrightarrow{(b,a)}S_1\times S_2)\rightsquigarrow\mathrm{Isom}^{\otimes}(a^*\omega_2,b^*\omega_1).$$

For a fibre functor ω over S, we put

$$\begin{aligned}\mathcal{A}ut_S^{\otimes}(\omega)&=\mathcal{I}som_S^{\otimes}(\omega,\omega)\\ \mathcal{A}ut_k^{\otimes}(\omega)&=\mathcal{I}som_k^{\otimes}(\omega,\omega)\overset{\text{def}}{=}\mathcal{I}som_{S\times S}^{\otimes}(\mathrm{pr}_2^*\,\omega,\mathrm{pr}_1^*\,\omega).\end{aligned}$$

Thus, $\mathcal{A}ut_S^{\otimes}(\omega)$ is the functor of S-schemes

$$(T\xrightarrow{u}S)\rightsquigarrow\mathrm{Aut}^{\otimes}(u^*\omega)$$

and $\mathcal{A}ut_k^{\otimes}(\omega)$ is the functor of $S\times S$-schemes

$$(T\xrightarrow{(b,a)}S\times S)\rightsquigarrow\mathrm{Isom}^{\otimes}(a^*\omega,b^*\omega).$$

As a functor of k-schemes, $\mathcal{A}ut_k^{\otimes}(\omega)$ sends a k-scheme T to

$$\{(b,a,\varphi)\mid b,a\colon T\to S,\quad\varphi\colon a^*\omega\xrightarrow{\approx}b^*\omega\}.$$

According to Theorem 1.1, $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by an $S\times S$-scheme $(t,s)\colon G\longrightarrow S\times S$ and the partial law of composition on $\mathcal{A}ut_k^{\otimes}(\omega)$ makes $(S,G,t,s,\circ)$ into a k-groupoid acting on S.

Before sketching the proof of the theorem, we prove an important corollary.

COROLLARY 1.4 *Let* T *be as in the theorem. Any two fibre functors on* T *over an affine k-scheme S become isomorphic over some faithfully flat covering of S.*

PROOF Let ω_1 and ω_2 be fibre functors on T over k-schemes S_1 and S_2 respectively. The functor sending a fibre functor over $S \stackrel{\text{def}}{=} S_1 \sqcup S_2$ to its restrictions to S_1 and S_2 is an equivalence of categories. Thus, ω_1 and ω_2 arise from a fibre functor ω over S, unique up to a unique isomorphism. On applying the theorem to (T, ω), we find that $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by a k-groupoid G acting transitively on S. The transitivity implies that there exists a faithfully flat morphism $S' \to S \times S$ such that $\mathrm{Hom}_{S\times S}(S', G) \neq \emptyset$, i.e., such that $\mathcal{A}ut_k^{\otimes}(\omega)(S') \neq \emptyset$. Note that

$$S \times S = (S_1 \times S_1) \sqcup (S_1 \times S_2) \sqcup (S_2 \times S_1) \sqcup (S_2 \times S_2).$$

We now take $S_1 = S_2 = S_0$, so ω_1 and ω_2 are fibre functors over S_0. The restriction of $\mathcal{A}ut_k^{\otimes}(\omega)$ to the diagonal S_0 in $S_2 \times S_1$ is $\mathcal{I}som_{S_0}^{\otimes}(\omega_1, \omega_2)$, and so $\mathcal{I}som_{S_0}^{\otimes}(\omega_1, \omega_2)(S'') \neq \emptyset$, where $S'' = S_0 \times_{S\times S} S'$. As S'' is faithfully flat over S_0, this completes the proof. □

We prove Theorem 1.1 in §6, after presenting various preliminaries in §2–§5. Before outlining the proof, we review the proof in the neutral case.

Let T be a tannakian category over k and ω a k-valued fibre functor on T. We want to realize (T, ω) as $(\mathsf{Repf}(G), \omega_{\text{forget}})$ for some affine group scheme G. Initially, we forget the tensor structure on T and simply regard it as an abelian category. A result of Gabber (II, 3.13) allows us to write T as a union of categories of the form Mod_A with A a finite-dimensional k-algebra. An elementary argument then allows us to replace Mod_A with coMod_C, where C is the dual coalgebra, and realize the whole of (the abelian category) T as the category of comodules over a k-coalgebra C. Now the tensor structure on T provides C with an algebra structure, and the existence of duals implies the existence of an antipode.

The proof in the general case is similar except that, at each stage, we must replace an object by its more complicated "-oid" generalization, and to realize T (as an abelian category) as a category of comodules, we appeal to the comonadic theorem in category theory.

In §2 we develop the basic theory of affine groupoid schemes, and in §3 we show that the category of representations $\mathsf{Repf}(S{:}G)$ is tannakian. In §4 we explain how to interpret representations of affine k-groupoids as comodules over coalgebroids in the same way that representations of affine k-groups can be interpreted as comodules over coalgebras.

In §5, we prove the comonadic theorem of Barr and Beck. This is a result in category theory that provides a solution to the following problem: given a faithful functor $F\colon \mathsf{C} \to \mathsf{B}$, use F to define a structure on B with the property that C can be recovered from B and the structure.

After these preparations, in §6 we prove Theorem 1.1. Let T be a tannakian category over k and ω a fibre functor on T over an affine k-scheme S. We first use the results on abelian categories proved in Chapter II and the comonadicity theorem to show that the abelian category T is equivalent to the category of comodules over the coalgebroid $L(\omega)$ of "endomorphisms of ω". Now the tensor structure on T allows us to provide $L(\omega)$ with an algebra structure, and the rigidity of T implies that Spec $L(\omega)$ is a k-groupoid G acting on S (rather than a k-monoidoid).

It remains to show that G is faithfully flat over $S\times S$. For this, we use the statement (I, 10.8) that for a ring (A,m,e) in a tensor category, the morphism $e\colon \mathbb{1}\to A$ is faithfully flat. In §10 of Chapter II, we constructed a tensor category $\mathsf{T}\boxtimes\mathsf{T}$ with the property that a fibre functor ω on T over $S=\operatorname{Spec}B$ defines a fibre functor $\omega\times\omega$ on $\mathsf{T}\boxtimes\mathsf{T}$ over $S\times S$. The $B\otimes B$-algebra $L(\omega)$ is faithfully flat because it is the image by $\omega\times\omega$ of an Ind-object containing $\mathbb{1}$ of $\mathsf{T}\otimes\mathsf{T}$.

The remaining sections §7–§10 add various complements.

REMARK 1.5 When one defines a groupoid scheme to be a groupoid in the category of all (not necessarily affine) schemes over k and allows fibre functors over nonaffine schemes, one arrives at the following statement.

> Let T be an essentially small tannakian category over k and ω a fibre functor on T over a nonempty k-scheme S.
>
> (a) The functor $\mathcal{A}ut_k^{\otimes}(\omega)$ of $S\times S$-schemes is represented by a k-groupoid G acting transitively on S.
>
> (b) The functor $\mathsf{T}\to\mathsf{Repf}(S{:}G)$ defined by ω is an equivalence of tensor categories.
>
> Conversely, if G is a k-groupoid acting transitively on a nonempty affine k-scheme S, then $\mathsf{Repf}(S{:}G)$ is a tannakian category over k and $G\simeq\mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

This is Deligne's original statement (Deligne 1990, 1.12). As he remarks (ibid., 1.13), it suffices to prove the statement with S affine. Moreover, $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by a groupoid G affine over $S\times S$ (see IV, **??** below). Thus, requiring everything to be affine changes little. The reader can consult the original works of Deligne for the more general statements.

2 Groupoid schemes

Throughout this section, k is a field.

Groupoids (in Set*)*

A ***groupoid*** (in Set) is a small category in which every morphism is an isomorphism. Thus giving a groupoid amounts to giving a set S (of objects), a set G (of arrows), two maps[1] $t,s\colon G\rightrightarrows S$ (sending an arrow to its target and source respectively), and a partial law of composition,

$$(g,h)\mapsto g\circ h\colon G\underset{s,S,t}{\times}G\to G,\quad \text{where } G\underset{s,S,t}{\times}G=\{(g,h)\in G\times G\mid s(g)=t(h)\},$$

satisfying the following conditions: composition of arrows is associative; each object has an identity arrow; each arrow has an inverse. We often refer to (S,G) as a groupoid or to G as a ***groupoid acting on*** S. The map $(t,s)\colon G\to S\times S$ allows us to regard G as a set over $S\times S$.

EXAMPLE 2.1 A group G defines a groupoid as follows: take S to be any singleton, t and s to be the unique maps to S, and $\circ$ to be multiplication on G. Conversely, if (S,G) is a groupoid with S a singleton, then $(G,\circ)$ is a group.

[1] In French, t and s become b and s (but and source), and in German z and q (Ziel and Quelle).

A groupoid (S,G) is said to be ***transitive*** if the map

$$(t,s)\colon G\to S\times S,$$

is surjective, i.e., if for every pair of objects (b,a) of S there exists an arrow $a\to b$.

Let G be a transitive groupoid. Write $G_{b,a}$ for the fibre of G over (b,a); thus

$$G_{b,a}=\{g\in G\mid s(g)=a,\ t(g)=b\}=\{g\mid g\colon a\to b\}=\mathrm{Hom}(a,b).$$

There is a law of composition

$$\begin{gathered}G_{c,b}\times G_{b,a}\to G_{c,a}\\ \mathrm{Hom}(b,c)\times\mathrm{Hom}(a,b)\to\mathrm{Hom}(a,c).\end{gathered}$$

This law makes $G_a\overset{\text{def}}{=}G_{a,a}$ into a group, called the ***vertex*** or ***isotropy group***, and $G_{b,a}$ into a right principal homogeneous space under G_a. The choice of an element $u_{b,a}\in G_{b,a}$ defines an isomorphism $\mathrm{ad}\,u_{b,a}\colon G_a\to G_b$, which is independent of $u_{b,a}$ up to an inner automorphism. The ***kernel*** G^{Δ} of G is the family $(G_a)_{a\in S}$. It can be viewed as a relative group over S.

If G is transitive and G_a is commutative for one (hence all) $a\in S$, then G is said to be ***commutative***. In this case the isomorphism $\mathrm{ad}\,u_{b,a}\colon G_a\to G_b$ is independent of the choice of $u_{b,a}$, and so there is a canonical isomorphism $G_o\times S\to G^{\Delta}$ for any $o\in S$. Therefore, G^{Δ} is a constant group over S.

EXAMPLE 2.2 Let S be a topological space. The ***fundamental groupoid*** Π of S is the groupoid acting on S for which $\Pi_{b,a}$ is the set of paths from a to b taken up to homotopy. The law of composition is the usual composition of paths. The group Π_a is the fundamental group $\pi_1(S,a)$. The fundamental groupoid Π acts transitively on S if S is path-connected.

This is the archetype that should be kept in mind when thinking of groupoids.

EXAMPLE 2.3 Let S be a set and k a field. Let $V=(V_a)_{a\in S}$ be a family of k-vector spaces indexed by S. For $a,b\in S$, let

$$G_{b,a}=\mathrm{Isom}(V_a,V_b).$$

Then $G(V)\overset{\text{def}}{=}\bigsqcup_{a,b\in S}G_{b,a}$ becomes a groupoid acting on S with $s(G_{b,a})=a$, $t(G_{b,a})=b$, and $G_{c,b}\times G_{b,a}\to G_{c,a}$ the composition of isomorphisms. If the V_a all have the same finite dimension, then $G(V)$ acts transitively on S.

A ***morphism of groupoids acting on*** S is a map $f\colon G\to H$ that, together with the identity map $S\to S$, forms a functor. A ***morphism*** $\alpha\colon f\to f'$ of morphisms $f,f'\colon G\rightrightarrows H$ of groupoids acting on S is simply a natural transformation. Thus it is a family of arrows $\alpha_a\colon a\to a$ in H, indexed by the elements of S, such that the diagrams

$$\begin{tikzcd} a \arrow[r, "\alpha_a"] \arrow[d, "f(g_{b,a})"'] & a \arrow[d, "f'(g_{b,a})"] \\ b \arrow[r, "\alpha_b"] & b \end{tikzcd}$$

commute for all $g_{b,a}\in G_{b,a}$. In this way, the groupoids acting on a fixed S form a 2-category.

DEFINITION 2.4 Let G be a groupoid acting on S, and let $V = (V_a)_{a\in S}$ be a family of k-vector spaces of the same finite dimension. A ***representation*** of G on V is a morphism $\rho\colon G \to G(V)$. Thus, for each $g \in G$, we have an isomorphism $\rho(g)\colon V_{s(g)} \to V_{t(g)}$, such that

(a) for the identity element e_a of G_a, $\rho(e_a)\colon V_a \to V_a$ is the identity map,

(b) $\rho(g\circ h) = \rho(g)\circ\rho(h)$ if $s(g) = t(h)$.

REMARK 2.5 Let (S,G) be a transitive groupoid. For any $a \in S$, the group G_a is a skeleton of the category (S,G). This example should discourage readers from considering equivalent categories, even with a given equivalence, as being "the same".

Groupoids internal to a category

A ***groupoid internal to a category*** is a diagram

$$\begin{tikzcd} G \underset{s,S,t}{\times} G \arrow[r, dashed, shift left=2, "\mathrm{pr}_2"] \arrow[r, "m"] \arrow[r, dashed, shift right=2, "\mathrm{pr}_1"'] & G \arrow[loop below, dotted, "\mathrm{inv}"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & S \arrow[l, "e"'] \end{tikzcd}$$

in the category such that the following equalities hold,

$$\begin{cases} s\circ m = s\circ \mathrm{pr}_2, \quad t\circ m = t\circ \mathrm{pr}_1, \\ s\circ e = \mathrm{id}_S = t\circ e, \\ m\circ(\mathrm{id}_G \times_S m) = m\circ(m \times_S \mathrm{id}_G), \\ m\circ(\mathrm{id}_G \times_S e)\circ(\mathrm{id}_G, s) = \mathrm{id}_G = m\circ(e \times_S \mathrm{id}_G)\circ(s, \mathrm{id}_G), \end{cases}$$

$$\begin{cases} s\circ \mathrm{inv} = t, \quad t\circ \mathrm{inv} = s, \\ m\circ(\mathrm{inv}, \mathrm{id}_G) = e\circ s, \\ m\circ(\mathrm{id}_G, \mathrm{inv}) = e\circ t \end{cases}$$

The first collection of equalities expresses that the diagram is an internal category (s and t send an arrow to its source and target respectively, e sends an object to its identity morphism, and m sends a pair of composable arrows to their composite), and the second collection expresses that inv sends an arrow to its inverse (its existence means that inverses exist).

EXAMPLE 2.6 The fundamental groupoid of a topological space S has a natural structure of a groupoid in the category of topological spaces if S is locally path connected and semilocally simply connected. It is Hausdorff if and only if S is Hausdorff.

EXAMPLE 2.7 A ***Lie groupoid*** is a groupoid $t,s\colon G \rightrightarrows S$ internal to the category of smooth manifolds, except that we require the maps t and s to be submersions so that $G \underset{s,S,t}{\times} G$ is a submanifold of $G\times G$.

Groupoid schemes

A groupoid scheme over k is a groupoid internal to the category Aff_k of affine schemes over k.

DEFINITION 2.8 A ***groupoid scheme over*** k consists of

- affine k-schemes S and G,
- a pair of morphisms $t, s\colon G \rightrightarrows S$ of k-schemes (making G into an $S \times S$-scheme),
- a morphism $\circ\colon G \underset{s,S,t}{\times} G \to G$ of $S \times S$-schemes

such that, for all affine k-schemes T, the system

$$S(T), \quad G(T), \quad t, s\colon G(T) \rightrightarrows S(T), \quad \circ\colon G(T) \underset{s,S(T),t}{\times} G(T) \to G(T)$$

is a groupoid (in Set). We usually call (S, G) a k-***groupoid acting on*** S or an S/k-***groupoid***.

2.9 The condition in the definition can be expressed in terms of diagrams.

(a) The associativity of composition says that the two morphisms

$$G \underset{s,S,t}{\times} G \underset{s,S,t}{\times} G \overset{\circ \times \mathrm{id}}{\underset{\mathrm{id} \times \circ}{\rightrightarrows}} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

are equal.

(b) The existence of identity maps says that there exists a morphism $e\colon S \to G$ of $S \times S$-schemes (regarding S as an $S \times S$-scheme by $\Delta\colon S \to S \times S$) such that both morphisms

$$G \simeq G \underset{s,S,\mathrm{id}}{\times} S \xrightarrow{\mathrm{id} \times e} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

$$G \simeq S \underset{\mathrm{id},S,t}{\times} G \xrightarrow{e \times \mathrm{id}} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

equal id_G.

(c) The existence of inverses says that there exists a morphism $\mathrm{inv}\colon G \to G$ of k-schemes such that

$$\begin{cases} s \circ \mathrm{inv} = t \\ t \circ \mathrm{inv} = s \end{cases}$$

and the diagrams

$$\begin{tikzcd} G \arrow[r, "(\mathrm{inv},\mathrm{id})"] \arrow[d, "s"] & G \underset{s,S,t}{\times} G \arrow[d, "\circ"] \\ S \arrow[r, "e"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id},\mathrm{inv})"] \arrow[d, "t"] & G \underset{s,S,t}{\times} G \arrow[d, "\circ"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

commute.

The morphisms e and inv, when they exist, are uniquely determined by $(S, G, t, s, \circ)$.

Remark 2.10 Without the condition (c), we get the notion of a ***monoidoid scheme over*** k, i.e., a small category internal to the category Aff_k.

Definition 2.11 Let S be an affine k-scheme, and let G and H be k-groupoids acting on S. A morphism $f\colon G \to H$ of $S \times S$-schemes is a ***morphism*** of k-groupoids acting on S if $f(T)\colon G(T) \to H(T)$ is a morphism of groupoids acting on $S(T)$ for every affine k-scheme T. This condition can also be expressed by saying that the diagrams

$$\begin{tikzcd} G \underset{s,S,t}{\times} G \arrow[r, "f\times f"] \arrow[d, "\circ_G"] & H \underset{s,S,t}{\times} H \arrow[d, "\circ_H"] \\ G \arrow[r, "f"] & H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "f"] & & H \\ & S \arrow[ul, "e_G"] \arrow[ur, "e_H"'] & \end{tikzcd} \tag{66}$$

commute.

DEFINITION 2.12 We say that an S/k-groupoid G is ***transitive***, or that G ***acts transitively on*** S, if the morphism $(t,s)\colon G \to S \times S$ is covering for the fpqc topology, i.e., there exists a commutative diagram

$$\begin{tikzcd} T \arrow[d] \arrow[dr, "\text{faithfully flat}"] & \\ G \arrow[r, "(t,s)"] & S \times S. \end{tikzcd}$$

Obviously, G is transitive if $(t,s)\colon G \to S \times S$ itself is faithfully flat. Below (2.18), we show that the converse is true.

EXAMPLE 2.13 When $S = \operatorname{Spec} k$, a k-groupoid acting on S is nothing but an affine group scheme over k. It is automatically transitive.

Let G be a k-groupoid acting on S. For a scheme $(b,a)\colon T \to S \times S$ over $S \times S$, we write $G_{b,a}$ for $(b,a)^*G$. Note that b and a are objects of the category $S(T)$, and $G_{b,a}$ can be thought of as the scheme of arrows $a \to b$,

$$G_{b,a} = \text{“}\mathrm{Hom}(a,b)\text{”}.$$

The law of composition provides morphisms (of schemes over T)

$$G_{c,b} \times_T G_{b,a} \to G_{c,a}.$$

This law makes $G_a \overset{\text{def}}{=} G_{a,a} \overset{\text{def}}{=} (a,a)^*G$ into an affine group scheme over T, which is flat if G is transitive.

EXAMPLE 2.14 Let S be an affine k-scheme and V a locally free $\mathcal{O}_S$-module of constant[2] finite rank. There exists a k-groupoid $G(V)$, acting transitively on S such that, for any scheme $(b,a)\colon T \to S \times S$ over $S \times S$,

$$G(V)(T) = \mathrm{Isom}_{\mathcal{O}_T}(a^*V, b^*V).$$

For $a = b = s \in S$, we have

$$G(V)_s = \mathrm{GL}_{V_s},$$

where V_s is the fibre of V over s (a finite-dimensional $\kappa(s)$-vector space).

[2]If the rank is not constant, $G(V)$ may fail to act transitively.

Groupoids regarded as fibred categories

2.15 Let $(G, S, (t, s), \circ)$ be a k-groupoid acting on S. By definition, for any affine k-scheme T, the quadruple

$$(S(T), G(T), (t, s), \circ)$$

is a groupoid (in sets). For varying T, these categories form a fibred category over Aff_k, which we denote by $\mathsf{G}^0(S{:}G)$, or just G^0.

Thus, for an affine k-scheme T,

$$\operatorname{ob} \mathsf{G}^0_T = S(T) \stackrel{\text{def}}{=} \operatorname{Hom}_k(T, S),$$

and, for $a, b \in \operatorname{ob} \mathsf{G}^0_T = S(T)$,

$$\operatorname{Hom}_{\mathsf{G}^0_T}(a, b) = \{h \in G(T) \mid s\circ h = a,\ t\circ h = b\}.$$

For a morphism $f\colon T' \to T$ of affine k-schemes, the inverse image functor f^* sends an object $a \in S(T)$ to $a\circ f \in S(T')$ and a morphism $h \in G(T)$ to $h\circ f$.

A morphism $f\colon G \to H$ of groupoids is a cartesian functor $\mathsf{G}^0(S{:}\ f)\colon \mathsf{G}^0(S{:}G) \to \mathsf{G}^0(S{:}\ H)$, i.e., a family of functors compatible with base change.

2.16 Let T be an affine k-scheme, and let a and b be objects of the fibre of G^0 over T, so $a, b \in S(T)$. Define $\mathcal{I}som_T(a, b)$ to be the functor

$$(T' \xrightarrow{f} T) \rightsquigarrow \operatorname{Isom}_{\mathsf{G}^0_{T'}}(f^*a, f^*b)\colon \mathsf{Aff}_T \to \mathsf{Grp}.$$

Note that

$$\begin{aligned}
\operatorname{Isom}_{\mathsf{G}^0_{T'}}(f^*a, f^*b) &\stackrel{\text{def}}{=} \{h\colon T' \to G \mid s\circ h = a\circ f,\ t\circ h = b\circ f\}\\
&= \operatorname{Isom}_{S\times S}(T', G)\\
&\simeq \operatorname{Isom}_T(T', G\times_{S\times S} T),
\end{aligned}$$

$$\begin{tikzcd}
 & G\times_{S\times S} T \arrow[r] \arrow[d] & G \arrow[d, "{(t,s)}"] \\
T' \arrow[r, "f"] \arrow[ur, dashed] \arrow[urr, bend left, "h"] & T \arrow[r, "{(b,a)}"] & S\times S,
\end{tikzcd}$$

so $\mathcal{I}som_T(a, b)$ is represented by the T-scheme $G\times_{S\times S} T$. In particular, $\operatorname{Isom}_T(a, b)$ is a sheaf on T for the fpqc topology.

For example, let Q be the tautological object of G^0 over S, namely, $Q = \mathrm{id}_S$. Then $\mathrm{pr}_1^* Q$ and $\mathrm{pr}_2^* Q$ are objects of the fibre of G^0 over $S\times S$, and the sheaf

$$\mathcal{I}som_{S\times S}(\mathrm{pr}_2^* Q, \mathrm{pr}_1^* Q)$$

on $S\times S$ is represented by the $S\times S$-scheme $(t, s)\colon G \to S\times S$.

Transitive groupoids

PROPOSITION 2.17 *Let G be a k-groupoid acting on S. The following are equivalent:*

(a) *G is transitive;*

(b) *the objects $\mathrm{pr}_1, \mathrm{pr}_2\colon S\times S \to S$ of $\mathsf{G}^0_{S\times S}$ are locally isomorphic;*

(c) *any two objects $a, b\colon T \to S$ of G^0_T, where T is an affine k-scheme, are locally isomorphic.*

PROOF To say that the objects pr_1 and pr_2 in $\mathsf{G}^0_{S\times S}$ are locally isomorphic means that there exists a faithfully flat map $f\colon T\to S\times S$ and an $h\colon T\to G$ such that

$$s\circ h = \mathrm{pr}_1\circ f, \quad t\circ h = \mathrm{pr}_2\circ f,$$

i.e., such that $h \in \mathrm{Hom}_{S\times S}(T, G)$. Thus the condition for pr_1 and pr_2 to be locally isomorphic is the definition of "transitive" (2.12).

Assume that pr_1 and pr_2 are locally isomorphic. We show that any two $a, b \in \mathrm{ob}\,\mathsf{G}^0_T = S(T)$ are locally isomorphic. From

$$\begin{tikzcd} T \arrow[r, "{(a,b)}"] \arrow[rr, bend left, "b"] \arrow[rr, bend right, "a"'] & S\times S \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\mathrm{pr}_1"'] & S, \end{tikzcd} \qquad \begin{cases} \mathrm{pr}_1\circ(a,b) = a \\ \mathrm{pr}_2\circ(a,b) = b \end{cases}$$

we see that $b = (a,b)^*\,\mathrm{pr}_2$ and $a = (a,b)^*\,\mathrm{pr}_1$. If pr_1 and pr_2 become isomorphic on a faithfully flat covering $U \to S\times S$ of $S\times S$, then a and b become isomorphic on the faithfully flat covering $T\times_{S\times S} U \to T$. □

PROPOSITION 2.18 *A k-groupoid G acting on S is transitive if and only if $(t,s)\colon G\to S\times S$ is faithfully flat.*

PROOF The necessity if obvious. For the sufficiency, let x be a point of $S\times S$, and let $T\to S\times S$ be a morphism with image x. On pulling back the diagram in 2.11 to T, we see that x is in the image of (t,s), and hence (t,s) is surjective. It remains to prove that it is flat.

Let $Q = \mathrm{id}_S \in S(S) = \mathrm{ob}\,\mathsf{G}^0_S$. We saw in 2.16 that that G, regarded as an $S\times S$-scheme, represents the sheaf $\mathcal{I}som_{S\times S}(\mathrm{pr}_2^*\,Q, \mathrm{pr}_1^*\,Q)$ on $S\times S$. On pulling back by the diagonal $\Delta\colon S\to S\times S$, we deduce that $G\times_{S\times S,\Delta} S$ represents the sheaf $\mathcal{A}ut_S(Q)$ on S. As a first step, we show that $\mathcal{A}ut_S(Q)$ (by which we mean the scheme representing it) is flat over S.

Let $a\colon P\to S$ be a morphism of affine k-schemes with P the spectrum of a field, and consider the diagram

$$\begin{tikzcd} S\times_{\mathrm{Spec}(k)} P \arrow[r, "\mathrm{pr}_P"] \arrow[d, "\mathrm{pr}_S"] & P \arrow[dl, "a"] \arrow[d] \\ S \arrow[r] & \mathrm{Spec}(k). \end{tikzcd}$$

As G is transitive, the two objects $\mathrm{pr}_S^*\,Q$ and $\mathrm{pr}_P^*(a^*Q)$ of $\mathsf{G}^0_{S\times P}$ become isomorphic on some affine U faithfully flat over $S\times P$. Therefore $\mathcal{A}ut_S(Q)$ and $\mathcal{A}ut_P(a^*Q)$ become isomorphic when restricted to U. As $\mathcal{A}ut_P(a^*Q)$ is obviously flat over P, this implies that the restriction of $\mathcal{A}ut_S(Q)$ to U is flat over U, which implies that $\mathcal{A}ut_S(Q)$ is flat over S because U is faithfully flat over S.

Recall that, as a scheme over $S \times S$, G represents $\mathcal{I}som_{S\times S}(\mathrm{pr}_2^* Q, \mathrm{pr}_1^* Q)$. As G is transitive, there is a faithfully flat morphism $f\colon U \to S \times S$ such that $\mathrm{pr}_2^* Q$ and $\mathrm{pr}_1^* Q$ become isomorphic over U. On choosing an isomorphism, we get an isomorphism of sheaves over U,

$$\mathcal{I}som_U(f^* \mathrm{pr}_2^* Q, f^* \mathrm{pr}_1^* Q) \approx \mathcal{A}ut_U(f^* \mathrm{pr}_1^* Q).$$

As the scheme representing the right-hand sheaf is flat over U, so is the scheme representing the left-hand sheaf. This is $G \times_{S\times S} U$, and so G is flat over $S \times S$. □

Pullbacks of groupoid schemes

Let G be a k-groupoid acting on S, and let $u\colon T \to S$ be a morphism of affine k-schemes. The ***pullback*** of G by $u \times u\colon T \times T \to S \times S$ is a k-groupoid acting on T, which we denote by G_T.

For example, let G be an affine group scheme over k, viewed as a k-groupoid acting on $\operatorname{Spec} k$. For any affine k-scheme S, we get a k-groupoid scheme

$$G_S \overset{\text{def}}{=} G \times (S \times S)$$

acting on S, which is called the ***neutral groupoid scheme*** defined by G. In the special case that G is the trivial group, $G_S = S \times S$ and is called the ***trivial*** S/k-groupoid.

ASIDE 2.19 There is a more abstract version of the above theory. Let E be a Grothendieck topos, i.e., the category of sheaves of sets on some small site, and let 1 be a terminal object of E. A groupoid $(S, G, (s,t), \circ)$ in E is a ***bouquet*** if

(a) the unique morphism $S \to 1$ is an epimorphism ((S,G) is said to be nonempty), and

(b) the morphism $(t,s)\colon G \to S \times S$ is an epimorphism, ((S,G) is said to be connected),

(Duskin 1982, 2013). A k-groupoid scheme acting transitively on an affine k-scheme S defines a bouquet in the topos of sheaves of sets on Aff_k for the fpqc topology.

Groupoids and Galois groupoids

2.20 Let $S_0 = \operatorname{Spec} k$ and $S = \operatorname{Spec} \bar{k}$. The left action of $\Gamma \overset{\text{def}}{=} \operatorname{Gal}(\bar{k}/k)$ on $\bar{k}$ defines a right action of Γ on S, and the map

$$S \times \Gamma \to S \times_{S_0} S, \quad (s,\sigma) \mapsto (s, s\cdot\sigma) \tag{67}$$

is an isomorphism of schemes. Here Γ is to be interpreted as a pro-finite scheme over k.

2.21 Let G be a k-groupoid scheme acting transitively on S, and assume that the kernel G^{Δ} is of finite type over $\bar{k}$. When we view (67) as a morphism of S-schemes (through projection on the first factor), it identifies $(S \times_{S_0} S)(S)$ with $\Gamma = \operatorname{Gal}(\bar{k}/k)$, and the morphism $(t,s)\colon G \to S \times_{S_0} S$ defines a map

$$\pi\colon G(\bar{k}) = G(S) \to (S \times_{S_0} S)(S) \simeq \operatorname{Gal}(\bar{k}/k) \overset{\text{def}}{=} \Gamma.$$

More precisely, $G(\bar{k})$ is the set of sections to the morphism t in the diagram

$$\begin{tikzcd} G \arrow[r, "{(t,s)}"] \arrow[dr, "t"'] & S \times_{S_0} S \arrow[d, "\mathrm{pr}_1"] & S \times \Gamma \arrow[l, "\simeq"'] \arrow[dl, "\mathrm{pr}_1"] \\ & S = \operatorname{Spec}(\bar{k}). & \end{tikzcd}$$

The map $G(\bar{k}) \to \Gamma$ is surjective because (t,s) is faithfully flat and $\bar{k}$ is algebraically closed.

2.22 Now $(G(\bar{k}), S(\bar{k}), (s,t), \circ)$ is a transitive groupoid (in Set) with a canonical point $a \in S(\bar{k})$, namely, the identity map. Therefore (see 1.17) there is a well-defined group structure on $G(\bar{k})$ for which the sequence

$$1 \to G^{\Delta}(\bar{k}) \to G(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

is exact.

2.23 In this way, a k-groupoid G acting transitively on S with kernel G^{Δ} defines a $\bar{k}/k$-Galois groupoid,

$$1 \to G^{\Delta} \to G(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

3 Representations of groupoid schemes

In this section, k is a field.

DEFINITION 3.1 Let G be a k-groupoid acting on an affine k-scheme S, and let V be a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. A ***representation*** of G on V is a morphism of S/k-groupoids $G \to G(V)$.

Explicitly, this means that for every affine k-scheme T and $g \in G(T)$, we have a morphism $\rho(g)\colon V_{g\circ s} \to V_{g\circ t}$ between the inverse images of V with respect to $g\circ s, g\circ t\colon T \rightrightarrows S$; these satisfy the following conditions (cf. 2.4),

(a) for the element e of $G(S)$, $\rho(e)$ is the identity map of $V = V_{s\circ e} = V_{t\circ e}$;

(b) $\rho(g\circ h) = \rho(g)\rho(h)$ if $s(g) = t(h)$;

(c) the formation of $\rho(g)$ commutes with base change $T' \to T$.

REMARK 3.2 We can use the explicit description to define a representation of G on any quasi-coherent sheaf V on S. If G is transitive and for some $s \in S$, the fibre V_s of V is a vector space of dimension n over the residue field at s, then V is locally free of rank n (see **??**, **??**).

3.3 Let $G = \mathrm{Spec}\, L$, let $S = \mathrm{Spec}\, B$, and let V be a finitely generated projective B-module (=locally free sheaf of finite rank on S). For $g \in G(R)$, let $s(g)$ denote the image of g under $s(R)\colon G(R) \to S(R) = \mathrm{Hom}(B,R)$, and similarly for $t(g)$. To give a representation of G on V is the same as giving, for every k-algebra R and $g \in G(R) = \mathrm{Hom}(L,R)$, a homomorphism of R-modules

$$\rho(g)\colon V \otimes_{B,s(g)} R \to V \otimes_{B,t(g)} R,$$

satisfying the following conditions,

(a) $\rho(e) = \mathrm{id}_V$,

(b) $\rho(g\circ h) = \rho(g)\circ\rho(h)$ whenever $g\circ h$ is defined,

(c) ρ is compatible with base change, i.e., for any homomorphism of k-algebras $u\colon R \to R'$, the following diagram commutes

$$\begin{tikzcd}
V \otimes_{B,s(g)} R \arrow[r, "\rho(g)"] \arrow[d, "V\otimes u"] & V \otimes_{B,t(g)} R \arrow[d, "V\otimes u"] \\
V \otimes_{B,s(g')} R' \arrow[r, "\rho(g')"] & V \otimes_{B,t(g')} R',
\end{tikzcd}$$

where $g' = G(u)(g) \in G(R')$.[3]

As G is a groupoid, all $\rho(g)$ are isomorphisms.

A ***morphism*** of representations (V, ρ_V) and (W, ρ_W) of G is a homomorphism of B-modules $\varphi\colon V \to W$ such that

$$\begin{tikzcd}
V \otimes_{B,s(g)} R \arrow[r, "\rho_V(g)"] \arrow[d, "\varphi\otimes R"] & V \otimes_{B,t(g)} R \arrow[d, "\varphi\otimes R"] \\
W \otimes_{B,s(g)} R \arrow[r, "\rho_W(g)"] & W \otimes_{B,t(g)} R
\end{tikzcd}$$

commutes for all k-algebras R and $g \in G(R)$.

Let $\mathsf{Repf}(S{:}G)$ denote the category of representations of G on finitely generated projective B-modules. We define the tensor product of two such representations by

$$(V, \rho_V) \otimes (W, \rho_W) = (V \otimes_B W, \rho_{V\otimes_B W}),$$

where $\rho_{V\otimes W}(g)$ is determined by the commutative diagram

$$\begin{tikzcd}
(V \otimes_{B,s(g)} R) \otimes_R (W \otimes_{B,s(g)} R) \arrow[r, "\rho_V(g)\otimes_R \rho_W(g)"] \arrow[d, "\simeq"] & (V \otimes_{B,t(g)} R) \otimes_R (W \otimes_{B,t(g)} R) \arrow[d, "\simeq"] \\
(V \otimes_B W) \otimes_{B,s(g)} R \arrow[r, "\rho_{V\otimes W}(g)"] & (V \otimes_B W) \otimes_{B,t(g)} R.
\end{tikzcd}$$

There are obvious associativity and commutativity constraints, and B, equipped with the trivial action, is an identity object. We define the dual of a representation (V, ρ) to be $(V^\vee, \rho^\vee)$, where $V^\vee$ is the B-linear dual of V and $\rho^\vee$ is determined by the commutative diagram

$$\begin{tikzcd}
V^\vee \otimes_{B,s(g)} R \arrow[r, "\rho^\vee(g)"] \arrow[d, "\simeq"] & V^\vee \otimes_{B,t(g)} R \arrow[d, "\simeq"] \\
(V \otimes_{B,s(g)} R)^\vee \arrow[r, "(\rho_V(g)^{-1})^\vee"] & (V \otimes_{B,t(g)} R)^\vee.
\end{tikzcd}$$

In this way, $\mathsf{Repf}(S{:}G)$ becomes a k-linear rigid symmetric monoidal category.

[3] When we use the same symbol for a map of affine schemes and the corresponding map of rings, we have

$$\begin{aligned}
s(g) &\overset{\text{def}}{=} g\circ s\colon\ B \to L \to R \\
g' &\overset{\text{def}}{=} u\circ g\colon\ B \to R \to R' \\
s(g') &\overset{\text{def}}{=} g'\circ s = u\circ s(g)\colon\ B \to R'.
\end{aligned}$$

PROPOSITION 3.4 *Let G be a k-groupoid acting transitively on S, and let $u: T \to S$ be a morphism of affine k-schemes with $T \neq \emptyset$. The pullback G_T of G to T is a k-groupoid acting transitively on T, and u induces an equivalence of categories*

$$\mathsf{Repf}(S{:}G) \xrightarrow{\approx} \mathsf{Repf}(T{:}G_T).$$

PROOF This is most conveniently proved in terms of gerbes. A groupoid G acting transitively on a scheme S defines a gerbe $\mathsf{G}(S:G)$, and there is a canonical equivalence of categories,

$$\mathsf{Repf}(\mathsf{G}(S:G)) \xrightarrow{\approx} \mathsf{Repf}(S:G).$$

As the morphism of gerbes $\mathsf{G}(S:G) \to \mathsf{G}(T:G_T)$ defined by $(S:G) \to (T:G_T)$ is an equivalence, the statement follows. For the details, see the first section of the next chapter (especially **??**), which is independent of the rest of this chapter – it could have been inserted here. □

For example, if there exists an $s \in S(k)$, then

$$\mathsf{Repf}(S{:}G) \xrightarrow{\approx} \mathsf{Repf}(G_s),$$

where G_s is the affine group scheme over k fixing s (the fibre of G over (s,s)).

For example, let G be an affine group scheme over k, and let $\bar{G}$ be the neutral k-groupoid scheme $G \times \mathrm{Spec}(\bar{k} \otimes \bar{k})$ acting on $\mathrm{Spec}(\bar{k})$. Then the restriction functor $\mathsf{Repf}(\bar{G}) \xrightarrow{\approx} \mathsf{Repf}(G)$ is an equivalence of categories.

PROPOSITION 3.5 *Let G be a k-groupoid acting transitively on S. Then $\mathsf{Repf}(S{:}G)$ is a tannakian category over k with the forgetful functor as a fibre functor over S.*

PROOF After 3.4, we may suppose that $S = \mathrm{Spec}\,B$, where B is a field. We know that $\mathsf{Repf}(S{:}G)$ is a k-linear rigid symmetric monoidal category.

We next show that it is abelian. Obviously, it is additive. Let

$$(V,\rho_V) \xrightarrow{f} (W,\rho_W)$$

be a morphism in $\mathsf{Repf}(S{:}G)$. There is an exact sequence

$$0 \to \mathrm{Ker}\,f \to V \to W \to \mathrm{Coker}\,f \to 0$$

of B-vector spaces, and, for each $g \in G(R)$, a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f)\otimes_{B,s(g)} R \arrow[r] \arrow[d, dashed] & V\otimes_{B,s(g)} R \arrow[r] \arrow[d, "\rho_V(g)", "\simeq"'] & W\otimes_{B,s(g)} R \arrow[r] \arrow[d, "\rho_W(g)", "\simeq"'] & \mathrm{Coker}(f)\otimes_{B,s(g)} R \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Ker}(f)\otimes_{B,t(g)} R \arrow[r] & V\otimes_{B,t(g)} R \arrow[r] & W\otimes_{B,t(g)} R \arrow[r] & \mathrm{Coker}(f)\otimes_{B,t(g)} R \arrow[r] & 0
\end{tikzcd}$$

(here we use that B is a field). The dashed arrows define an action of G on $\mathrm{Ker}\,f$ and $\mathrm{Coker}\,f$ making them the kernel and cokernel of f in $\mathsf{Repf}(S{:}G)$. Obviously, the canonical morphism $\mathrm{Coker}(\mathrm{Ker}(f)) \to \mathrm{Ker}(\mathrm{Coker}(f))$ is an isomorphism, and so $\mathsf{Repf}(S{:}G)$ is abelian. Moreover, the forgetful functor $\omega: \mathsf{Repf}(S{:}G) \to \mathsf{Vecf}_B$ is an exact faithful $\otimes$-functor, and so it remains to show that $\mathrm{End}(\mathbb{1}) = k$.

Let $G = \operatorname{Spec} L$, and denote the maps $B \rightrightarrows L$ defined by $s, t\colon G \to S$ by the same letter. An endomorphism of $\mathbb{1} = B$ is given by an element a of B, and $\operatorname{End}(\mathbb{1})$ is the equalizer of the pair of arrows $s, t\colon B \rightrightarrows L$. The diagram

$$\begin{tikzcd} B \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] \arrow[dr, shift left, "i_1"] \arrow[dr, shift right, "i_2"'] & & L \\ & B \otimes_k B \arrow[ur, "s\otimes t"'] & \end{tikzcd}$$

commutes with $i_1(a) = a \otimes 1$ and $i_2(a) = 1 \otimes a$. As $k \to B$ is faithfully flat, it is the equalizer of the parallel pair (i_1, i_2). The map $s \otimes t$ is faithfully flat, hence injective, and it follows that $\operatorname{End}(\mathbb{1}) = k$. □

REMARK 3.6 Let G be an affine k-groupoid acting transitively on an affine k-scheme S. Let C be the category $\mathsf{Repf}(S : G)$ of representations of G on locally free sheaves of finite rank on S, and D the category of representations of G on quasi-coherent sheaves on S. Then C and D satisfy the conditions (a), (b), and (c) of Proposition B.10. When S is the spectrum of a field, condition (c) follows from 6.6. The general case then follows from 3.4. Therefore, the functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim X_\alpha\colon \operatorname{Ind}\mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an object X of D to the inductive system of its subobjects in C.

If there is an $s \in S(k)$, then the functor “fibre at s” is an equivalence of $\mathsf{Repf}(S{:}G)$ with $\mathsf{Repf}(G_{s,s})$ (6.6), and we are back in the situation of Example B.13.

REMARK 3.7 Let G be an affine groupoid acting transitively on an affine scheme S over k, let $(S'{:}G')$ be the pair over k' deduced from $(S{:}G)$ by extension of scalars, and let D and D' be as in a 3.6. Then the system

$$\begin{tikzcd} \mathsf{Repf}(S{:}G) \arrow[r] \arrow[d, hook] & \mathsf{Repf}(S'{:}G') \arrow[d, hook] \\ \mathsf{D} \arrow[r, shift left, "\text{extension}"] & \mathsf{D}' \arrow[l, shift left, "\text{restriction}"] \end{tikzcd}$$

satisfies the conditions of B.18. Therefore the functor restriction of scalars induces an equivalence $\mathsf{Repf}(S'{:}G') \to \mathsf{Repf}(S{:}G)_{(k')}$. See **??** for a related result.

4 Representations of groupoids as comodules

In this section, k is a commutative ring. From 4.6, it is a field.

Just as representations of affine group schemes can be realized as comodules over coalgebras (II, §1), representations of groupoid schemes can be realized as comodules over coalgebroids.

TERMINOLOGY 4.1 Let R and S be rings (not necessarily commutative). An (R, S)-***bimodule*** is an abelian group M together with a left action of R and a commuting right action of S:

$$(r \cdot m) \cdot s = r \cdot (m \cdot s), \qquad r \in R,\ m \in M,\ s \in S.$$

Such a module is sometimes denoted ${}_RM_S$. If R, T, S are rings and ${}_RM_T$ and ${}_TM_S$ are bimodules, then

$$ {}_RM_T \underset{T}{\otimes} {}_TM_S $$

is a well-defined (R,S)-bimodule. An (R,R)-bimodule is also called an R-bimodule. When the rings are k-algebras (k a commutative ring), we require that the various actions of k coincide.

DEFINITION 4.2 Let B be a ring, not necessarily commutative.[4] A B-***coalgebroid*** (or ***coalgebroid acting on*** B) is a B-bimodule L equipped with two B-bimodule homomorphisms (comultiplication and coidentity)

$$ \begin{aligned} &c\colon L \to L \underset{B}{\otimes} L, \quad \text{i.e., } {}_BL_B \to {}_BL_B \underset{B}{\otimes} {}_BL_B \\ &\epsilon\colon L \to B \end{aligned} $$

such that the two composed maps

$$ L \xrightarrow{c} L \otimes_B L \overset{c\otimes L}{\underset{L\otimes c}{\rightrightarrows}} L \otimes_B L \otimes_B L $$

are equal and the two maps

$$ \begin{aligned} L \xrightarrow{c} L \otimes_B L \xrightarrow{\epsilon\otimes L} B \otimes_B L \simeq L \\ L \xrightarrow{c} L \otimes_B L \xrightarrow{L\otimes\epsilon} L \otimes_B B \simeq L \end{aligned} $$

equal the id_L.

A ***morphism of*** B-***coalgebroids*** $(L, c, \epsilon) \to (L', c', \epsilon')$ is a homomorphism $f\colon L \to L'$ of B-bimodules such that the following diagrams commute,

$$ \begin{tikzcd} L \otimes_B L \arrow[r, "f\otimes f"] & L' \otimes_B L' \\ L \arrow[u, "c"] \arrow[r, "f"] & L' \arrow[u, "c'"'] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "\epsilon"'] & & L' \arrow[dl, "\epsilon'"] \\ & B. & \end{tikzcd} $$

When B is an algebra over a commutative ring k, a B/k-***coalgebroid*** (or k-***coalgebroid acting on*** B) is a B-coalgebroid L such that the two k-module structures on L coincide.

DEFINITION 4.3 Let L be a B-coalgebroid. A ***representation*** of L is a right B-module M equipped with a ***coaction*** of L, i.e., a homomorphism $\rho\colon M \to M \otimes_B L$ of right B-modules such that the two composed maps

$$ M \xrightarrow{\rho} M \otimes_B L \overset{\rho\otimes L}{\underset{M\otimes c}{\rightrightarrows}} M \otimes_B L \otimes_B L \tag{68} $$

are equal and the map

$$ M \xrightarrow{\rho} M \otimes_B L \xrightarrow{M\otimes\epsilon} M \otimes_B B \simeq M \tag{69} $$

[4]We shall only need commutative B, but allowing noncommutative rings forces us to distinguish left from right correctly.

equals id_M. We call (M,ρ) an L-***comodule***.

A ***morphism of*** L-***comodules*** $(M,\rho_M)\to(N,\rho_N)$ is a homomorphism of B-modules $f\colon M\to N$ such that the following diagram commutes

$$\begin{array}{ccc} M\otimes_B L & \xrightarrow{f\otimes L} & N\otimes_B L \\ \uparrow{\scriptstyle \rho_M} & & \uparrow{\scriptstyle \rho_N} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

When L is a B-coalgebroid, we let $\mathsf{coModf}(L)$ denote the category of L-comodules that are finitely generated and projective as right B-modules.

EXAMPLE 4.4 If B is commutative and the two B-module structures on L agree, then L is a B-coalgebra (see II, §1), which helps explains the terminology. The notions of a B-comodule agree in the two cases.

EXAMPLE 4.5 Let L be a B-coalgebroid. If L is flat as a left B-module, then the category of L-comodules is abelian and the forgetful functor is exact. The proof of this is the same as the proof of the similar result for $\mathsf{Repf}(S\!:\!G)$ (see 3.5).

Now let k be a field.

EXAMPLE 4.6 Let G be a k-groupoid acting on S, and let $G=\operatorname{Spec}L$ and $S=\operatorname{Spec}B$. From the morphism $(t,s)\colon G\to S\times S$, we get the structure of a $B\otimes_k B$-module on L, i.e., a B-bimodule structure such that the two k-module structures coincide. We write the B-module structure defined by t (resp. s) on the left (resp. right). The composition law

$$G\underset{s,S,t}{\times}G\to G$$

corresponds to a map of B-bimodules

$$c\colon L\to L\underset{B}{\otimes}L$$

and the identity $e\colon S\to G$ corresponds to a map of B-bimodules

$$\epsilon\colon L\to B.$$

A comparison of the diagrams in 2.9 and 4.2 shows that these structures make L into a B/k-coalgebroid, which helps explains the terminology.

PROPOSITION 4.7 *Let $G=\operatorname{Spec}L$ be a k-groupoid acting transitively on $S=\operatorname{Spec}B$ (so L is a B/k-coalgebroid), and let M be a right B-module. There is a canonical one-to-one correspondence between the representations of G on M and the representations of L on M.*

PROOF A representation ρ of G on M is determined by its action on the "universal" element

$$u\overset{\text{def}}{=}\mathrm{id}_G\in\operatorname{Hom}(G,G)=G(L)$$

because, according to 3.3(c), the diagram

$$\begin{array}{ccc} M\otimes_{B,s}L & \xrightarrow{\rho(u)} & M\otimes_{B,t}L \\ \downarrow{\scriptstyle M\otimes g} & & \downarrow{\scriptstyle M\otimes g} \\ M\otimes_{B,s(g)}R & \xrightarrow{\rho(g)} & M\otimes_{B,t(g)}R \end{array}$$

commutes for all $g \in G(R) = \mathrm{Hom}(L, R)$. In turn, the L-linear map

$$\rho(u)\colon M \otimes_{B,s} L \to M \otimes_{B,t} L,$$

is determined by its restriction to a B-linear map

$$\rho_M\colon M \to M \otimes_{B,t} L.$$

In the other direction, given a coaction ρ_M of L on a B-module M and a $g \in G(R) = \mathrm{Hom}(L, R)$, we define $\rho(g)$ by the following diagram

$$\begin{tikzcd}
M \otimes_{B,s(g)} R \arrow[r, "\rho_M \otimes R"] \arrow[d, "\rho(g)"'] & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \arrow[d, "M\otimes g\otimes R"] \\
M \otimes_{B,t(g)} R & M \otimes_{B,t(g)} R \underset{s(g),B,s(g)}{\otimes} R. \arrow[l, "M\otimes\mathrm{mult.}"]
\end{tikzcd} \tag{70}$$

If ρ_M is obtained from a ρ as in the first paragraph, then the action of id_G given by (70) returns ρ_M. Conversely, let ρ be a representation of G on M. We get a coaction ρ_M as in the first paragraph, and from ρ_M a representation $\widetilde{\rho}$ of G. Now $\rho = \widetilde{\rho}$ because they agree on $u = \mathrm{id}_G$.

It remains to show that, under the correspondence $\rho \leftrightarrow \rho_M$, ρ satisfies the axioms for a representation if and only if ρ_M satisfies the axioms for a comodule.

If $g, h \in G(R)$ with $t(g) = s(h)$, then $\rho(h \circ g)$ is given by the diagram

$$\begin{tikzcd}
M \otimes_{B,s(g)} R \arrow[r, "\rho_M\otimes R"] \arrow[dr, "\rho(h\circ g)"'] & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \arrow[r, "M\otimes c\otimes R"] & M \otimes_{B,t} L \underset{s,B,t}{\otimes} L \underset{s,B,s(g)}{\otimes} R \arrow[d, "M\otimes h\otimes g\otimes R"] \\
& M \otimes_{B,t(h)} R & M \otimes_{B,t(h)} R \underset{s(h),B,t(g)}{\otimes} R \underset{s(h),B,s(g)}{\otimes} R. \arrow[l, "M\otimes R\otimes\mathrm{mult.}"']
\end{tikzcd}$$

and $\rho(h)\circ\rho(g)$ is given by

$$\begin{tikzcd}
M \otimes_{B,s(g)} R \arrow[r, "\rho_M\otimes R"] \arrow[ddd, "\rho(h)\circ\rho(g)"'] & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \arrow[r, "M\otimes g\otimes R"] & M \otimes_{B,t(g)} R \underset{s(g),B,s(g)}{\otimes} R \arrow[d, "M\otimes\mathrm{mult.}"] \\
& & M \otimes_{B,t(g)} R \arrow[d, "\rho_M\otimes R"] \\
& & \\
M \otimes_{B,t(h)} R & M \otimes_{B,t(h)} R \underset{s(h),B,s(h)}{\otimes} R \arrow[l, "M\otimes\mathrm{mult.}"] & M \otimes_{B,t} L \underset{s,B,s(h)}{\otimes} R. \arrow[l, "M\otimes h\otimes R"]
\end{tikzcd}$$

When we write $\rho_M(m) = \sum m_i \otimes \ell_i$, then these homomorphisms agree if and only if

$$(1 \otimes h \otimes g)\Big(\sum m_i \otimes c(\ell_i)\Big) = (1 \otimes h \otimes g)\Big(\sum \rho(m_i) \otimes \ell_i\Big)$$

for all k-algebra homomorphisms $h, g\colon L \to R$ with $t(g) = s(h)$. This is obviously equivalent to the maps in (68) agreeing.

Let ρ be a representation of G on M, and let ρ_M be the associated coaction. Then the action of $\rho(\epsilon)$ is given by the top row of the following commutative diagram

$$\begin{tikzcd}
M \otimes_B B \arrow[r] \arrow[d, "\simeq"] & M \otimes_{B,t} L \otimes_{s,B} B \arrow[r, "M\otimes\epsilon\otimes B"] \arrow[d, "\simeq"] & M \otimes_B B \otimes_B B \arrow[r, "\simeq"] & M \otimes_B B \\
M \arrow[r, "\rho_M"] & M \otimes_{B,t} L \arrow[urr, "\mathrm{id}\otimes\epsilon"'] & &
\end{tikzcd}$$

It follows that the map in (69) is the identity if and only if ϵ acts trivially on M. □

PROPOSITION 4.8 *Let $G = \operatorname{Spec} L$ be a groupoid acting transitively on $S = \operatorname{Spec} B$. The functor*

$$(M, \rho) \rightsquigarrow (M, \rho_M) \colon \mathsf{Repf}(S{:}G) \to \mathsf{coModf}(L)$$

is an equivalence of categories

PROOF After Proposition 4.7, it remains to show that a B-module homomorphism $f \colon M \to N$ is G-equivariant if and only if it is a morphism of L-comodules, but this is straighforward from the definitions. □

REMARK 4.9 If we define the tensor product of L-comodules $M, \rho_M)$ and (N, ρ_N) to be the L-comodule $(M \otimes N, \rho_{M\otimes N})$, where $\rho_{M\otimes N}$ is given by

$$\begin{tikzcd}
M \otimes_B N \arrow[r, "\rho_M\otimes\rho_N"] \arrow[d, "\rho_{M\otimes N}"] & (M \otimes_{B,t} L) \underset{B\otimes B}{\otimes} (N \otimes_{B,t} L) \arrow[d, "\simeq"] \\
(M \otimes_B N) \otimes_{B,t} L & (M \otimes_B N) \otimes_{B,t} (L \underset{B\otimes B}{\otimes} L), \arrow[l, "\mathrm{id}\otimes m"]
\end{tikzcd}$$

then the equivalence

$$\mathsf{Repf}(S{:}G) \xrightarrow{\approx} \mathsf{coModf}(L)$$

respects tensor products.

NOTES The exposition of the proof of 4.7 follows that in Lattermann 1989, 1.4.7.

5 The comonadic theorem and applications

A faithful functor $F \colon \mathsf{C} \to \mathsf{B}$ creates a "shadow" of C on B, and it is sometimes possible to recover C from B and the shadow. For example, in (II, Theorem 3.20), we were able to recover C from its shadow (a coalgebra) in Vecf_k. In the case we are interested in here, the shadow is a "comonad" on B, and a standard result in category theory (Theorem 5.12) describes C as the category of "G-modules" in B.

Comonads

DEFINITION 5.1 A ***comonad*** on a category B consists of

- a functor $G \colon \mathsf{B} \to \mathsf{B}$,
- a natural transformation $c \colon G \to G \circ G$ (the comultiplication),
- a natural transformation $\epsilon \colon G \to \mathrm{id}_{\mathsf{B}}$ (the counit)

such that the two natural transformations

$$\begin{tikzcd} G \arrow[r, "c"] & G \circ G \arrow[r, shift left, "cG"] \arrow[r, shift right, "Gc"'] & G \circ G \circ G \end{tikzcd}$$

are equal and the two natural transformations

$$\begin{tikzcd} G \arrow[r, "c"] & G \circ G \arrow[r, shift left, "\epsilon G"] \arrow[r, shift right, "G\epsilon"'] & G \end{tikzcd}$$

equal id_G. The counit, if it exists, is uniquely determined by (G, c).

EXAMPLE 5.2 Fix a set E, and let $G\colon \mathsf{Set} \to \mathsf{Set}$ be the functor $X \rightsquigarrow X \times E$. Then

$$(x, e) \mapsto (x, e, e)\colon X \times E \to X \times E \times E$$

is a natural transformation $c\colon G \to G \circ G$, and

$$(x, e) \mapsto e\colon X \times E \to X$$

is a natural transformation $G \to \mathrm{id}$. The triple (G, c, ϵ) is a comonad.

REMARK 5.3 Let C be a category. There is a monoidal category whose objects are the functors $\mathsf{C} \to \mathsf{C}$, whose morphisms are the natural transformations, and whose tensor product is $\circ$. A comonad is a comonoid in this category (monoid in the opposite category), which explains the similarity of the above diagrams to earlier diagrams.

Comonads from adjunctions

We refer the reader to A.2 for our notation concerning adjoint pairs.

PROPOSITION 5.4 *Let* $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ *be an adjoint pair with unit* $\eta\colon \mathrm{id}_{\mathsf{C}} \to U \circ F$ *and counit* $\epsilon\colon F \circ U \to \mathrm{id}_{\mathsf{B}}$. *Then* $F \circ U$ *is a comonad on* B *with comultiplication*

$$F\eta U\colon F \circ U \longrightarrow F \circ U \circ F \circ U$$

and counit ϵ.

PROOF The two composites in

$$\begin{tikzcd} FU \arrow[r, "F\eta U"] & FU \circ FU \arrow[r, shift left, "F\eta UFU"] \arrow[r, shift right, "FUF\eta U"'] & FU \circ FU \circ FU \end{tikzcd}$$

agree because they both equal the horizontal natural transformation in

$$\begin{tikzcd} & & \mathsf{B} \arrow[dr, "U"] & & \mathsf{B} \arrow[dr, "U"] & & \\ \mathsf{B} \arrow[r, "U"] & \mathsf{C} \arrow[ur, "F"] \arrow[rr, equal] & \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{C} \arrow[ur, "F"] \arrow[rr, equal] & \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{C} \arrow[r, "F"] & \mathsf{B}. \end{tikzcd}$$

The two natural transformations

$$\begin{tikzcd} F \circ U \arrow[r, "F\eta U"] & F \circ U \circ F \circ U \arrow[r, shift left, "\epsilon FU"] \arrow[r, shift right, "FU\epsilon"'] & F \circ U \end{tikzcd}$$

equal id_{FU} by the triangle identities for the adjunction. □

EXAMPLE 5.5 For the standard adjoint pair

$$\mathsf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Ab}, \qquad \begin{cases} F = \text{forget the group structure} \\ U = \text{form the free abelian group} \end{cases}$$

the endofunctor $F \circ U$ sends an abelian group A to the free abelian group on the underlying set of A.

Statement of the comonadicity theorem

DEFINITION 5.6 Let (G, c, ϵ) be a comonad. A ***coaction*** of G on an object X of B is a morphism $\rho_X : X \to GX$ such that (coassociativity) the two morphisms

$$X \xrightarrow{\rho_X} GX \underset{c_X}{\overset{G\rho_X}{\rightrightarrows}} G \circ GX$$

are equal and (counit) the morphism

$$X \xrightarrow{\rho_X} G_X \xrightarrow{\epsilon_X} X$$

is the identity. We call such a pair (X, ρ_X) a G-***comodule***.

A ***morphism*** of G-comodules $(B, \rho_B) \to (B', \rho_{B'})$ is a morphism $f : B \to B'$ such that the following diagram commutes,

$$\begin{tikzcd} GB \arrow[r, "Gf"] & GB' \\ B \arrow[u, "\rho_B"] \arrow[r, "f"] & B'. \arrow[u, "\rho_{B'}"'] \end{tikzcd}$$

The category B^G of G-modules in B is called the ***Eilenberg–Moore category*** of the comonad (G, c, ϵ).

DEFINITION 5.7 Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and (G, c, ϵ) the associated comonad. The (Eilenberg–Moore) ***comparison functor***

$$\Phi : \mathsf{C} \to \mathsf{B}^G$$

is defined to be

$$\begin{cases} \Phi C = (FC, F\eta c) \\ \Phi f = Ff. \end{cases}$$

It follows from the triangle identities and the naturality of η that ΦC is a G-comodule and Φf is a morphism of G-comodules.

DEFINITION 5.8 An adjoint pair $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ is ***comonadic*** if the comparison functor $\Phi : \mathsf{C} \to \mathsf{B}^G$ is an equivalence of categories.

Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and define functors

$$\mathsf{B}^G \underset{U^G}{\overset{F^G}{\rightleftarrows}} \mathsf{B} \qquad \begin{cases} F^G(B,\rho_B) = B \\ F^G f = f \end{cases} \qquad \begin{cases} U^G B = (GB, F\eta_{UB}) \\ U^G f = Gf. \end{cases}$$

Then the following diagrams commute,

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "F"] \arrow[dr, "\Phi"'] & & \mathsf{B} \\ & \mathsf{B}^G \arrow[ur, "F^G"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C} \arrow[dr, "\Phi"'] & & \mathsf{B} \arrow[ll, "U"'] \arrow[dl, "U^G"] \\ & \mathsf{B}^G. & \end{tikzcd}$$

LEMMA 5.9 *The functors* $\mathsf{B}^G \underset{U^G}{\overset{F^G}{\rightleftarrows}} \mathsf{B}$ *are an adjoint pair.*

PROOF Consider the morphisms $\eta\colon \mathrm{id}_{\mathsf{B}^G} \to U^G \circ F^G$, $\epsilon\colon F^G \circ U^G = G \to \mathrm{id}_{\mathsf{B}}$, and $\rho_B\colon (B,\rho_B) \to (GB, F\eta_{UB})$. The triangle identities for (η,ϵ) must be proven, which means that the compositions

$$(GB, F\eta_{UB}) \xrightarrow{F\eta_{UB}} (GGB, F\eta_{UGB}) \xrightarrow{G\epsilon_B} (GB, F\eta_{UB})$$

and

$$B \xrightarrow{\rho_B} GB \xrightarrow{\epsilon_B} B$$

are the identity for all $B \in \operatorname{ob}\mathsf{B}$, resp. $(B,\rho_B) \in \operatorname{ob}(\mathsf{B}^G)$. In the first case, this follows from the triangle identities (86), p. 198, and in the second from the co-identity axiom. □

DEFINITION 5.10 A ***split equalizer diagram*** consists of morphisms

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[l, bend left, "s"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[l, bend left, "t"] \end{tikzcd} \tag{71}$$

such that $f \circ h = g \circ h$, $s \circ h = \mathrm{id}_A$, $t \circ g = \mathrm{id}_B$, $t \circ f = h \circ s$.

Note that split equalizer diagrams are preserved by functors.

LEMMA 5.11 *In the diagram (71), h is the equalizer of f and g (hence the universal equalizer).*

PROOF The first condition says that h equalizes f and g. Suppose that $w\colon X \to B$ also equalizes f and g, so $f \circ w = g \circ w$. Let $j = s \circ w$. Then $h \circ j = h \circ s \circ w = t \circ f \circ w = t \circ g \circ w = w$, and so w factors through $A \to B$. The uniqueness of the factorization follows from the condition $s \circ h = \mathrm{id}_A$. □

THEOREM 5.12 (BECK'S COMONADICITY THEOREM) *An adjoint pair*

$$\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$$

is comonadic if and only if the following conditions hold:

(a) *F reflects isomorphisms, i.e., f is an isomorphism whenever Ff is.*

(b) *if the image under F of a parallel pair f,g of morphisms in C embeds in a split equalizer diagram in B, then the pair f,g has an equalizer in C that is preserved by F.*

In the terminology introduced below, (b) says that F-split equalizer pairs have equalizers in C, preserved by F.

We present the proof in the next subsection. For the proof of the opposite (dual) statement, see Borceux 1994b, Theorem 4.4.4, Mac Lane 1998, pp. 147–150, or Riehl 2016, 5.5.1.

COROLLARY 5.13 *Let $F\colon \mathsf{C}\to\mathsf{B}$ be an exact faithful functor of abelian categories. If F admits a right adjoint functor, then the comparison functor*

$$\Phi\colon \mathsf{C}\to\mathsf{B}^G,\quad \Phi(C)=(FC,F\eta c)$$

is an equivalence of categories.

PROOF Exact faithful functors of abelian categories reflect isomorphisms,[5] and *every* parallel pair in C has an equalizer in C, which is preserved by F. Therefore, F is comonadic. □

Proof of the comonadicity theorem

Before beginning the proof, we need some definitions.

DEFINITION 5.14 A morphism is a ***regular monomorphism*** if it is the equalizer of some parallel pair of morphisms.

As the name suggests, regular monomorphisms are monomorphisms.

DEFINITION 5.15 Let $F\colon \mathsf{C}\to\mathsf{B}$ be a functor. An F-***split equalizer pair*** is a parallel pair $f,g\colon A\rightrightarrows B$ in C together with an extension of $Ff,Fg\colon FA\rightrightarrows FB$ to a split equalizer diagram in B. If, in addition, there exists a $t\colon B\to A$ such that $t\circ f=t\circ g=\mathrm{id}_A$, then the pair $f,g\colon A\to B$ is said to be ***reflexive***.

Let $\mathsf{C}\underset{U}{\overset{F}{\rightleftarrows}}\mathsf{B}$ be an adjoint pair and $\Phi\colon \mathsf{C}\to\mathsf{B}^G$ the comparison functor.

LEMMA 5.16 *For all C,C' in C, Φ induces a bijection*

$$\mathrm{Hom}_{\mathsf{C}}(C,UFC')\simeq\mathrm{Hom}_{\mathsf{B}^G}(\Phi C,\Phi UFC').$$

PROOF We have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(C,UFC')&\simeq\mathrm{Hom}_{\mathsf{B}}(FC,FC')\\&\simeq\mathrm{Hom}_{\mathsf{B}}(F^G(FC,F\eta_C),FC')\\&\simeq\mathrm{Hom}_{\mathsf{B}^G}((FC,F\eta_C),U^GFC')\\&\simeq\mathrm{Hom}_{\mathsf{B}^G}((FC,F\eta_C),(GFC',F\eta_{UFC'}))\\&\simeq\mathrm{Hom}_{\mathsf{B}^G}(\Phi C,\Phi UFC').\end{aligned}$$

. □

[5] Let $f\colon A\to B$ be a morphism in C, and consider the exact sequence

$$0\to K\to A\xrightarrow{f}B\to C\to 0.$$

This sequence remains exact when we apply F. If Ff is an isomorphism, then $FK=0=FC$, so $K=0=C$, and f is an isomorphism.

LEMMA 5.17 *For all* (B, ρ_B) *in* B^G,

$$(GB, c_B) \underset{G\rho_B}{\overset{c_B}{\rightrightarrows}} (GGB, c_{GB})$$

is a reflexive F^G*-split equalizer pair.*

PROOF The naturality of c implies that $G\rho_B$ and c_B are morphisms of comodules. Their common left inverse is $G\epsilon_B$ (which is a comodule morphism for the same reason). We have

$$G\epsilon_B \circ c_B = FU\epsilon_B \circ F\eta_{UB} = F(U\epsilon_B \circ \eta_{UB}) = \mathrm{id}$$

because of the triangle identities (p. 198), and $G\epsilon_B \circ c_B = \mathrm{id}$ because of the co-identity axiom. It remains to show that

$$\begin{tikzcd} B \arrow[r, "\rho_B"] & B \arrow[l, bend left, "\epsilon_B"] \arrow[r, shift left, "c_B"] \arrow[r, shift right, "G\rho_B"'] & C \arrow[l, bend left=40, "\epsilon_{GB}"] \end{tikzcd} \tag{72}$$

is a split equalizer diagram in B. However, this follows directly from the triangle equalities and the comodule axioms. □

LEMMA 5.18 *In the situation of Lemma 5.17,*

$$(B, \rho_B) \xrightarrow{\rho_B} (GB, c_B) \underset{G\rho_B}{\overset{c_B}{\rightrightarrows}} (GGB, c_{GB})$$

is an equalizer in B^G.

PROOF Let $f : (B', \rho_{B'}) \to (GB, c_B)$ be a morphism of modules with $c_B \circ f = G\rho_B \circ f$. After 5.17 and 5.11, there exists a unique map $g : B' \to B$ such that $\rho_B \circ g = f$, and it remains to show that g is a morphism of comodules. For this, we consider

$$\begin{tikzcd} GB' \arrow[r, "Gg"] & GB \arrow[r, "G\rho_B"] & GGB \\ B' \arrow[u, "\rho_{B'}"] \arrow[r, "g"] & B \arrow[u, "\rho_B"] \arrow[r, "\rho_B"] & GB. \arrow[u, "c_B"] \end{tikzcd}$$

The outer rectangle commutes by assumption and the right-hand rectangle commutes because of coassociativity. It follows that

$$G\rho_B \circ Gg \circ \rho_{B'} = G\rho_B \circ \rho_B \circ g.$$

Since $G\rho_B$ has left inverse $G\epsilon_B$, the assertion follows. □

LEMMA 5.19 *For all* C *in* C,

$$UFC \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} UFUFC$$

is a reflexive F*-split equalizer pair.*

PROOF The triangle identities show that the two arrows have $U\xi_{FC}$ as a common left inverse. The assertion now follows from Lemma 5.17, applied to the comodule (FC, η_C). □

LEMMA 5.20 *If $\eta_C : C \to UFC$ is a regular monomorphism for all C in C, then*

$$C \xrightarrow{\eta_C} UFC \underset{\eta_{UFC}}{\overset{UF\eta_C}{\rightrightarrows}} UFUFC$$

is an equalizer for all C.

PROOF By assumption, η_C is the equalizer of a pair $A \underset{d^1}{\overset{d^0}{\rightrightarrows}} B$. We have to show that, for every morphism $w : X \longrightarrow UFC$,

$$UF\eta_C \circ w = \eta_{UFC} \circ w \iff d^0 \circ w = d^1 \circ w.$$

First of all $\eta_{UFC} \circ w = UF\eta_C \circ w$, and it follows

$$UF\eta_C \circ UFw = UFUF\eta_C \circ UFw.$$

We claim that

$$\begin{tikzcd} UFC \arrow[r, "UF\eta_C"] & UFUFC \arrow[l, bend left, "U\epsilon_{FC}"] \arrow[r, shift left, "UF\eta_{UFC}"] \arrow[r, shift right, "UFUF\eta_C"'] & UFUFUFC \arrow[l, bend left, "U\epsilon_{FUFC}"] \end{tikzcd}$$

is a split kernel pair in C. This follows from (72) applied to $(FC, F\eta_C)$ and (5.11). From the claim it follows that UFw factors through UFC. Let $h : X \to UFC$ be a morphism with $UFw = UF\eta_C \circ h$. It follows that

$$\begin{aligned} UFd^0 \circ UFw &= UFd^0 \circ UF\eta_C \circ h \\ &= UFd^1 \circ UF\eta_C \circ h \\ &= UFd^1 \circ UFw. \end{aligned}$$

Because of the naturality of η, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[d, "\eta_X"] & UFC \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] \arrow[d, "\eta_{UFC}"] & C' \arrow[d, "\eta_{C'}"] \\ UFX \arrow[r, "UFw"] & UFUFC \arrow[r, shift left, "UFd^0"] \arrow[r, shift right, "UFd^1"'] & UFC'. \end{tikzcd}$$

It follows that $\eta_{C'} \circ d^0 \circ w = \eta_{C'} \circ d^1 \circ w$, and from this the assertion follows because $\eta_{C'}$ is a monomorphism. Now the converse applies, $d^0 \circ w = d^1 \circ w$. Then $w = \eta_C \circ g$ for a $g : X \to C$. From this follows

$$\eta_{UFC} \circ w = \eta_{UFC} \circ \eta_C \circ g = UF\eta_C \circ \eta_C \circ g = UF\eta_C \circ w.$$ □

LEMMA 5.21 *If η_C is a regular monomorphism for all $C \in \operatorname{ob} \mathsf{C}$, then Φ is fully faithful.*

PROOF After Lemma 5.20,

$$C \xrightarrow{\eta_C} UFC \underset{\eta_{UFC}}{\overset{UF\eta_C}{\rightrightarrows}} UFUFC$$

is an equalizer for all $C \in \operatorname{ob} \mathsf{C}$, and after Lemma 5.18,

$$\Phi C \xrightarrow{\Phi\eta_C} \Phi UFC \underset{\Phi\eta_{UFC}}{\overset{\Phi UF\eta_C}{\rightrightarrows}} \Phi UFUFC$$

is an equalizer in B^G. So, for all $A \in \operatorname{ob} \mathsf{C}$, the rows in the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(A,C) \arrow[r] \arrow[d, "\Phi"] & \operatorname{Hom}(A,UFC) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\Phi"] & \operatorname{Hom}(A,UFUFC) \arrow[d, "\Phi"] \\
\operatorname{Hom}(\Phi A,\Phi C) \arrow[r] & \operatorname{Hom}(\Phi A,\Phi UFC) \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Hom}(\Phi A,\Phi UFUFC)
\end{tikzcd}$$

are equalizers. According to Lemma 5.16, the middle and right vertical arrows are bijective, and so the left is also. □

PROOF (OF THEOREM 5.12) Let $C \in \operatorname{ob} \mathsf{C}$. We first show that η_C is a regular homomorphism. According to Lemma 5.19,

$$UFC \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} UFUFC$$

is a reflexive F-split kernel pair in C. According to hypothesis (b) of the theorem, there exists an equalizer $K \xrightarrow{d} UFC$. Because of the naturalness of η, the morphism η_C equalizes η_{UFC} and $UF\eta_C$, and so it factors through K,

$$\begin{tikzcd}
K \arrow[r, "d"] & UFC \arrow[r, shift left, "\eta_{UFC}"] \arrow[r, shift right, "UF\eta_C"'] & UFUFC \\
C \arrow[u, "f"] \arrow[ur, "\eta_C"'] & &
\end{tikzcd}$$

According to (72) with $B = FC$, when we apply F to the bottom row we get an equalizer, and after hypothesis (b), when we apply F to the top row, we get an equalizer in B. Therefore is Ff an isomorphism, which implies that f is an isomorphism. Thus η_C is a regular monomorphism. According to Lemma 5.21, Φ is fully faithful. Now let $(B,\rho_B) \in \operatorname{ob} \mathsf{B}^G$. According to Lemma 5.17

$$(FUB, F\eta_{UB}) \underset{FU\rho_B}{\overset{F\eta_{UB}}{\rightrightarrows}} (FUFUB, F\eta_{UFUB})$$

is a reflexive F^G-split kernel pair in B^G. On the other hand, this pair is the image of

$$UB \underset{U\rho_B}{\overset{\eta_{UB}}{\rightrightarrows}} UFUB$$

under Φ. Therefore this is a reflexive F-split kernel pair in C, and the common left inverse of both arrows is $U\epsilon_B$. After hypothesis (b), there exists an equalizer $d\colon K \to UB$ and

$$FK \xrightarrow{Fd} FUB \underset{FU\rho_B}{\overset{F\eta_{UB}}{\rightrightarrows}} B$$

is an equalizer in B. After (72), $\rho_B\colon B \to FUB$ is the equalizer of the same pair, and so there exists a unique isomorphism $g\colon B \to FK$ such that $Fd \circ g = \rho_B$. It remains to show that g is a morphism of comodules. Consider

$$\begin{tikzcd} FUB \arrow[r, "FUg"] & FUFK \arrow[r, "FUFd"] & FUFUB \\ B \arrow[u, "\rho_B"] \arrow[r, "g"] & FK \arrow[u, "F\eta_K"] \arrow[r, "Fd"] & FUB. \arrow[u, "F\eta_{UB}"] \end{tikzcd}$$

The right-hand square commutes because of the naturalness of η, and the outer one because of the coassociativity. The map ρ_B is left invertible, and so Fd is also left invertible. It follows that the left square commutes. Thus $\Phi K \simeq (B, \rho_B)$, and we have shown that Φ is essentially surjective. □

NOTES As noted, this section is standard category theory, although usually expressed for monads rather than comonads. Our proof of the comonadicity theorem follows that in Lattermann 1989, 2.3.

TODO 18 Need to check and revise this proof. aax agrees.

Application to modules and comodules

5.22 In this subsection, A and B are rings (not necessarily commutative), and $M = {}_AM_B$ is an (A, B)-bimodule, finitely generated and projective as a right B-module, and faithfully flat as an A-module. The B-dual of M,

$$M' = {}_BM'_A \stackrel{\text{def}}{=} \mathrm{Hom}_B(M, B),$$

is a (B, A)-bimodule with the actions

$$\begin{cases} (b \cdot m')(m) = bm'(m) \\ (m' \cdot a)(m) = m'(am) \end{cases} \qquad a \in A, b \in B, m \in M, m' \in M'.$$

LEMMA 5.23 *For any right B-module Y, the homomorphism*

$$\left(y \otimes m'\right) \mapsto \left(m \mapsto ym'(m)\right) \colon Y_B \underset{B}{\otimes} {}_BM'_A \to \mathrm{Hom}_B(M, Y),$$

is an isomorphism.

PROOF After replacing B with B_f for f in a suitable finite set of elements of B, we may suppose that M is free as a B-module. For a fixed Y, both sides commute with finite direct sums, and so it suffices to check this for $M = B$, where it is obvious. □

When $Y = M$, the lemma says that

$$_AM_B \underset{B}{\otimes} {}_BM'_A \simeq \mathrm{End}_B(M), \qquad m \otimes m' \leftrightarrow (x \mapsto m \cdot m'(x)). \tag{73}$$

5.24 We have functors

$$\mathsf{Mod}_A \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Mod}_B \qquad \begin{cases} FX = X \otimes_A M \\ UY = Y \otimes_B M'. \end{cases} \tag{74}$$

For $X \in \mathrm{ob}(\mathsf{Mod}_A)$, $Y \in \mathrm{ob}(\mathsf{Mod}_B)$, there are canonical isomorphisms

$$\mathrm{Hom}_B(X \otimes_A M, Y) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y)) \simeq \mathrm{Hom}_A(X, Y \otimes_B M'),$$

natural in X and Y. The first is defined by the pairing

$$(f, x) \mapsto (m \mapsto f(x \otimes m)): \ \mathrm{Hom}_B(X \otimes_A M, Y) \times X \to \mathrm{Hom}_B(M, Y),$$

and the second is induced by the isomorphism in Lemma 5.23. Hence (74) is an adjunction. The corresponding unit and counit are

$$\begin{aligned} \eta &: X \to X \otimes_A M \otimes_B M', \quad && \eta(x) = x \otimes \delta \\ \epsilon &: Y \otimes_B M' \otimes_A M \to Y, \quad && \epsilon(y \otimes m' \otimes m) = ym'(m), \end{aligned}$$

where $\delta \in M \otimes_B M'$ corresponds to the identity map under the isomorphism (73).

5.25 We can now apply Proposition 5.4. Let L denote the B-bimodule

$$_BM'_A \underset{A}{\otimes} {}_AM_B,$$

The functor $G \overset{\text{def}}{=} F \circ U$ sends a right B-module Y to $Y \otimes_B L$, and $G \circ G$ sends Y to $Y \otimes_B {}_BL_B \otimes_B {}_BL_B$. We have B-bimodule homomorphisms

$$\begin{aligned} c &: L \to L \otimes_B L, \quad && m' \otimes m \mapsto m' \otimes \delta \otimes m \\ \epsilon &: L \to B, \quad && m' \otimes m \mapsto m'(m), \end{aligned}$$

which make L a B-coalgebroid (with $k = \mathbb{Z}$). The corresponding natural transformations

$$\begin{aligned} c &: G \to G \circ G \\ \epsilon &: G \to \mathrm{id} \end{aligned}$$

make G into a comonad. The functor F is faithful and exact because $_AM_B$ is faithfully flat over A.

In the next theorem, we construct a commutative diagram

$$\begin{tikzcd} \mathsf{Mod}_A \arrow[rr, "\omega"] \arrow[dr, "F"'] & & \mathsf{coMod}_L \arrow[dl, "\text{forget}"] \\ & \mathsf{Mod}_B. & \end{tikzcd}$$

THEOREM 5.26 (DELIGNE 1990, 4.4) *Let $M = {}_AM_B$ be an (A,B)-bimodule, finitely generated and projective as a B-module, and faithfully flat as an A-module; let $M^\vee$ be its B-dual and L the coalgebroid $M^\vee \otimes_A M$ defined above. The functor*

$$\omega: \mathsf{Mod}_A \longrightarrow \mathsf{coMod}_L, \quad X_A \rightsquigarrow (X_A \underset{A}{\otimes} {}_AM_B, \rho_X),$$

where

$$\rho_X(x \otimes m) = x \otimes \delta \otimes m \in (X \otimes_A M) \otimes (M^\vee \otimes_A M),$$

is an equivalence of categories. Under the equivalence, A-modules of finite type (resp. of finite presentation) correspond to comodules of finite type (resp. of finite presentation) as B-modules.

PROOF The first statement follows from Corollary 5.13 applied to the above discussion. It remains to prove the second statement.

If X is finitely generated over A, then there exists a surjection $A^n \to X$. On tensoring this with M, we get a surjection $M^n \to X \otimes_A M$. As M is finitely generated over B, so also are M^n and $X \otimes_A M$.

If $X \otimes_A M$ is finitely generated as a B-module, then it has a finite set of generators

$$x_i \otimes m_i, \quad x_i \in X, \quad m_i \in M, \quad i \in I.$$

The A-submodule X' of X generated by the x_i has the property that $X' \otimes_A M \simeq X \otimes_A M$. As M is faithfully flat over A, this implies that $X' = X$.

Let X be finitely generated over A. Then $X = A^n/R$ for some $n \geq 0$, and X is finitely presented if and only if R is finitely generated, and $\omega(X) \simeq {}_AM_B^n/\omega(R)$ is finitely presented if and only if $\omega(R)$ is finitely generated. Now use that R is finitely generated if and only if $\omega(R)$ is. □

REMARK 5.27 When A and B are k-algebras (k a commutative ring) and the two k-module structures on ${}_AM_B$ coincide, the coalgebroid $L \stackrel{\text{def}}{=} {}_BM_A^\vee \underset{A}{\otimes} {}_AM_B$ is a k-coalgebroid.

EXAMPLE 5.28 In the case $A = k$ is a field and B a k-algebra, we obtain from a finitely generated projective B-module M, a k-coalgebroid $L \stackrel{\text{def}}{=} M^\vee \otimes_k M$ whose coidentity is the evaluation map. This coalgebroid coacts on M (the image of k by $- \otimes_k M$) by

$$\rho_0 : M \to M \otimes_B L, \quad m \mapsto \delta \otimes m \in M \otimes_B M' \otimes_B M \simeq M \otimes_B L.$$

ASIDE 5.29 From the comonadicity theorem, it is possible to deduce a faithfully flat descent theorem for noncommutative rings, and from there the faithfully flat descent theorem for commutative rings. However, we do not need the first theorem and this approach to the second theorem is not easier than the direct approach (Waterhouse 1979, 17.2), even for those familiar with the comonadicty theorem. See Lattermann 1989, 2.4.10, 2.4.11.

6 Proof of the main theorem

After these preliminaries, we are ready to prove Theorem 1.1

The coalgebroid of endomorphisms of a fibre functor

In this subsection, we attach a coalgebroid $L(\omega)$ to a fibre functor ω on a tannakian category. Recall that when k is a commutative ring and B_1 and B_2 are k-algebras, we always require the two actions of k on a (B_1, B_2)-module to be equal.

6.1 We begin with a general definition. Let C be a small category and $F : \mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \mathsf{D}$ a functor. A ***cowedge*** $e : F \to w$ is an object w of D and a family of morphisms $e_c : F(c,c) \to w$, indexed by the objects c of C, such that, for all morphisms $f : c' \to c$, the following diagram commutes,

$$\begin{tikzcd} F(c,c') \arrow[r, "F(f{,}c')"] \arrow[d, "F(c{,}f)"] & F(c',c') \arrow[d, "e_{c'}"] \\ F(c,c) \arrow[r, "e_c"] & w. \end{tikzcd}$$

Given a cowedge $e\colon F \to w$ and a morphism $h\colon w \to v$, we obtain a cowedge $h\circ e\colon F \to v$ by composition. A cowedge $e\colon F \to w$ is a ***coend*** if it is universal, i.e., any other cowedge $e'\colon F \to w'$ factors uniquely through a morphism $h\colon w \to w'$. When C has direct sums, a wedge can be viewed as a morphism

$$e\colon \bigoplus_{c\in \mathrm{ob}\,\mathsf{C}} F(c,c) \to w,$$

and such a morphism is a coend if and only if it is the coequalizer of the pair of morphisms

$$\bigoplus_{(f\colon c'\to c)\in \mathrm{ar}\,\mathsf{C}} F(c,c') \underset{e_c\circ F(c,f)}{\overset{e_{c'}\circ F(f,c')}{\rightrightarrows}} \bigoplus_{c\in \mathrm{ob}\,\mathsf{C}} F(c,c).$$

6.2 Let B_1, B_2 be k-algebras (k a commutative ring) and ω_1, ω_2 functors from a small category C to the categories of finitely generated projective right modules over B_1, B_2,

$$\begin{cases} \omega_1\colon \mathsf{C} \to \mathrm{Proj}_{B_1} \\ \omega_2\colon \mathsf{C} \to \mathrm{Proj}_{B_2}. \end{cases}$$

Let $\omega_1(Y)^\vee$ denote the B_1-dual of $\omega_1(Y)$ – it is a left B_1-module (5.22). We define $L_k(\omega_1,\omega_2)$ to be the coend of the functor

$$(Y,X) \rightsquigarrow \omega_1(Y)^\vee \otimes_k \omega_2(X)\colon \mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \{(B_1,B_2)\text{-bimodules}\}.$$

Thus $L_k(\omega_1,\omega_2)$ is a (B_1,B_2)-bimodule equipped with a morphism of (B_1,B_2)-bimodules

$$\omega_1(X)^\vee \otimes_k \omega_2(X) \to L_k(\omega_1,\omega_2) \tag{75}$$

for each $X \in \mathrm{ob}\,\mathsf{C}$ such that certain diagrams commute and $L_k(\omega_1,\omega_2)$ is universal. More concretely, it is the coequalizer of the parallel pair of morphisms

$$\bigoplus_{(f\colon X\to Y)\in \mathrm{ar}\,\mathsf{C}} \omega_1(Y)^\vee \otimes \omega_2(X) \rightrightarrows \bigoplus_{X\in \mathrm{ob}\,\mathsf{C}} \omega_1(X)^\vee \otimes_k \omega_2(X).$$

When it causes no confusion, we drop the k from $L_k(\omega_1,\omega_2)$.

To give the morphisms (75) is equivalent to giving morphisms of B_2-modules

$$\lambda(X)\colon \omega_2(X) \to \omega_1(X) \otimes_{B_1} L(\omega_1,\omega_2), \qquad X \in \mathrm{ob}\,\mathsf{C}. \tag{76}$$

The commutativity of the diagrams means that $\lambda(X)$ is functorial in X, and the universal property of $L(\omega_1,\omega_2)$ says that, for all (B_1,B_2)-bimodules U, the map

$$f \mapsto (\mathrm{id}_{\omega_1(X)} \otimes f \circ \lambda(X))_X$$

sending a (B_1,B_2)-bimodule homomorphism $f\colon L(\omega_1,\omega_2) \to U$ into the system of morphisms $u(X)$, natural in X,

$$\begin{tikzcd} \omega_2(X) \arrow[r, "\lambda(X)"'] \arrow[rr, bend left, "u(X)"] & \omega_1(X) \otimes_{B_1} L(\omega_1,\omega_2) \arrow[r, "\mathrm{id}_{\omega_1(X)} \otimes f"'] & \omega_1(X) \otimes_{B_1} U \end{tikzcd}$$

is a bijection, so

$$\mathrm{Hom}_{(B_1,B_2)}(L(\omega_1,\omega_2), U) \simeq \mathrm{Nat}(\omega_2(-), \omega_1(-) \otimes_{B_1} U).$$

6.3 For three functors $\omega_1, \omega_2, \omega_3$, iterating (76), we obtain a morphism

$$\omega_3(X) \to \omega_2(X) \otimes_{B_2} L(\omega_2, \omega_3) \to \omega_1(X) \otimes_{B_1} L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3),$$

natural in X, and hence a morphism

$$L(\omega_1, \omega_3) \to L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3). \tag{77}$$

The coproduct (77) is coassociative: for four functors $\omega_1, \omega_2, \omega_3, \omega_4$, the diagram

$$\begin{tikzcd} L(\omega_1, \omega_4) \arrow[r] \arrow[d] & L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_4) \arrow[d] \\ L(\omega_1, \omega_3) \otimes_{B_3} L(\omega_3, \omega_4) \arrow[r] & L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3) \otimes_{B_3} L(\omega_3, \omega_4) \end{tikzcd}$$

is commutative. The evaluation maps $\omega_j(X)^\vee \otimes_{B_j} \omega_j(X) \to B_j$ define a counit $\epsilon\colon L(\omega_j, \omega_j) \to B_j$: the two maps

$$L(\omega_1, \omega_2) \to L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3) \xrightarrow{1\otimes\epsilon} L(\omega_1, \omega_2)$$
$$L(\omega_1, \omega_2) \to L(\omega_1, \omega_1) \otimes_{B_1} L(\omega_1, \omega_2) \xrightarrow{\epsilon\otimes 1} L(\omega_1, \omega_2)$$

equal the identity map.

6.4 The important case for us is $B_1 = B_2$ and $\omega_1 = \omega_2$. Let $B = B_1 = B_2$. Then the map (77) for $\omega_1 = \omega_2 = \omega_3 = \omega$ makes

$$L_k(\omega) \overset{\text{def}}{=} L_k(\omega, \omega)$$

into a k-coalgebroid acting on B. Note that $L_k(\omega)$ is the coequalizer of

$$\bigoplus_{(f\colon X \to Y) \in \operatorname{ar} \mathsf{C}} \omega(Y)^\vee \otimes \omega(X) \rightrightarrows \bigoplus_{X \in \operatorname{ob} \mathsf{C}} \omega(X)^\vee \otimes_k \omega(X).$$

The map (76)

$$\lambda(X)\colon\ \omega(X) \to \omega(X) \otimes_B L(\omega)$$

is a coaction of $L(\omega)$ on $\omega(X)$, functorial in X. The universal property of $L(\omega)$ says that, for any k-coalgebroid L' acting on B, to give an action of L' on $\omega(X)$, natural in X, is the same as giving a morphism of coalgebroids $L(\omega) \to L'$. We call $L(\omega) = L_k(\omega)$ the ***coalgebroid of** k**-endomorphisms*** of ω.

EXAMPLE 6.5 When C consists of a single object and its identity morphism, to give ω is the same as giving a finitely generated projective right B-module M. The coalgebroid $L(\omega)$ is $M^\vee \otimes_k M$ and its coaction on M is that in 5.28. The universal property of $L(\omega)$ says that, for any k-coalgebroid L' acting on B, the canonical isomorphism

$$\operatorname{Hom}_B(M, M \otimes_k L') \simeq \operatorname{Hom}_{(B,B)}(M^\vee \otimes_k M, L')$$

makes coactions of L' on M correspond to morphisms of coalgebroids $M^\vee \otimes_k M \to L'$.

PROPOSITION 6.6 *Let L be a coalgebroid acting on a division algebra B.*

(a) *Every representation V of L is a filtered union of subrepresentations of finite dimension over B.*

(b) *The coalgebroid L is a filtered union of subcoalgebras of finite type as $B \otimes B$-modules.*

PROOF (a) Let $\rho\colon V \to V \otimes_B L$ be a coaction. Each $a \in V \otimes_B L$ is contained in a subspace $V_1 \otimes_B L$ with V_1 finite-dimensional, and the smallest V_1 is the set of elements $\lambda(a)$, where λ runs over the morphisms $L \to B$ of left B-vector spaces. For $a = \rho(v)$, this V_1 contains v (take λ to be the coidentity) and is stable: if $\rho(v) = \sum v_i \otimes \ell_i$ with the ℓ_i linearly independent, then V_1 is generated by the v_i; the axiom for coactions gives

$$\sum \rho(v_i) \otimes \ell_i = \sum v_i \otimes c(\ell_i) \in V_1 \otimes L \otimes L,$$

and so $\rho(v_i) \in V_1 \otimes L$ for all i.

(b) The comultiplication $c\colon L \to L \otimes_B L$ is a coaction of L on L (the regular representation). After (a), L is a filtered union of the subrepresentations V_i of finite dimension over B. The coaction of L on V_i corresponds, after 6.5, to a morphism of coalgebroids $f_i\colon V_i^\vee \otimes V_i \to L$. The coidentity e of L induces a linear form on V_i, and $f_i(e|V_i \otimes x) = x$. The image of f_i therefore contains V_i. Because B is a division algebra, the image of f_i is a subcoalgebroid of L. The coalgebroid L is a filtered union of the images of the f_i, which completes the proof. □

FUNCTORIALITIES

6.7 (a) Consider homomorphisms $f\colon B_j \to B'_j$ and the corresponding extensions of scalars $\omega_j \mapsto \omega'_j$. We have

$$L(\omega_1, \omega_2) \otimes_{B_1 \otimes_k B_2} (B'_1 \otimes_k B'_2) \xrightarrow{\simeq} L(\omega'_1, \omega'_2). \tag{78}$$

(b) Consider a functor $T\colon \mathsf{D} \to \mathsf{C}$. The morphisms

$$\omega_2(T(D)) \to \omega_1(T(D)) \otimes L(\omega_1, \omega_2), \quad D \in \operatorname{ob} \mathsf{D},$$

define a morphism of coalgebroids

$$L(\omega_1 \circ T, \omega_2 \circ T) \to L(\omega_1, \omega_2). \tag{79}$$

(c) If C is a filtered inductive limit of categories C_i and T_i is the natural functor $\mathsf{C}_i \to \mathsf{C}$, then the morphisms (79) induce an isomorphism

$$\varinjlim L(\omega_1 \circ T_i, \omega_2 \circ T_i) \to L(\omega_1, \omega_2). \tag{80}$$

(d) Assume that B_1, B_2 are commutative. Let $(\mathsf{C}_i)_{i \in I}$ be a finite family of categories, and let

$$\begin{cases} \omega_1^i\colon \mathsf{C} \to \mathrm{Proj}_{B_1} \\ \omega_2^i\colon \mathsf{C} \to \mathrm{Proj}_{B_2} \end{cases} \quad i \in I$$

be functors. Define

$$\begin{cases} \bigotimes_i \omega_1^i\colon \mathsf{C} \to \mathrm{Proj}_{B_1}, & \left(\bigotimes_i \omega_1^i\right)(C) = \bigotimes_i \omega_1^i(C) \\ \bigotimes_i \omega_2^i\colon \mathsf{C} \to \mathrm{Proj}_{B_2}, & \left(\bigotimes_i \omega_2^i\right)(C) = \bigotimes_i \omega_2^i(C). \end{cases}$$

Then

$$L(\otimes \omega_1^i, \otimes \omega_2^i) \xrightarrow{\simeq} \bigotimes_{B_1 \otimes_k B_2} L(\omega_1^i, \omega_2^i).$$

As in 5.22 et seq., let ${}_AM_B$ be an (A,B)-module, finitely generated and projective as a right B-module, let ${}_BM_A^\vee$ be its B-dual, and let L be the B coalgebroid ${}_BM_A^\vee \otimes_A {}_AM_B$. We suppose that, for some commutative ring k, A and B are k-algebras and that the two k-module structures on ${}_AM_B$ coincide. For example, we could have $k=\mathbb{Z}$, in which case the hypothesis is automatic.

LEMMA 6.8 *Let C be a full subcategory of the category of right A-modules, containing A_A, and such, if $E \in \operatorname{ob}\mathsf{C}$, then $E \otimes_A {}_AM_B$ is finitely generated and projective over B. Let ω be the functor $E \rightsquigarrow E \otimes_A {}_AM_B$. By 5.26, L coacts on the $\omega(E)$, $E \in \operatorname{ob}\mathsf{C}$, and hence we have a morphism*

$$L(\omega) \to L. \tag{81}$$

This morphism is an isomorphism.

PROOF When C consists only of the A-module A, this is the definition of the tensor product over A. We now give the general proof.

Let D be the full subcategory of C having A_A as its only object. The functoriality 6.7(b) gives a map $L(\omega|\mathsf{D}) \to L(\omega)$. As we just noted, we have $L(\omega|\mathsf{D}) \xrightarrow{\simeq} L$, and the triangle

$$\begin{tikzcd} L(\omega|\mathsf{D}) \arrow[rr, "6.7(b)"] \arrow[dr, "(81)"'] & & L(\omega) \arrow[dl, "(81)"] \\ & L & \end{tikzcd}$$

commutes. The morphism (79) therefore admits a retraction, and it suffices to show that $L(\omega|\mathsf{D})$ maps onto $L(\omega)$, i.e., that for all C in C, the image in $L(\omega)$ of $\omega(C)^\vee \otimes_k \omega(C)$ is contained in $\omega(A)^\vee \otimes_k \omega(A)$. Every element of $\omega(C)^\vee \otimes \omega(C)$ is a finite sum $\sum \alpha_i \otimes x_i$ and each $x_i \in C \otimes_A {}_AM_B$ is a finite sum $\sum \alpha_{ij} \otimes m_j$, and therefore a sum of elements of the form $f(y)$ with $f : A \to C$ and $y \in \omega(\mathsf{A})$. It therefore suffices to show that an element of $\omega(C)^\vee \otimes_k \omega(C)$ of the form $\alpha \otimes f(y)$, $f : A \to C$, $y \in \omega(A)$, has image in $L(\omega)$ contained in that of $\omega(A)^\vee \otimes_k \omega(A)$. By definition of $L(\omega)$, the diagram

$$\begin{tikzcd} \omega(C)^\vee \otimes_k \omega(A) \arrow[r, "{}^t f \otimes 1"] \arrow[d, "1\otimes f"] & \omega(A)^\vee \otimes_k \omega(A) \arrow[d] \\ \omega(C)^\vee \otimes_k \omega(C) \arrow[r] & L \end{tikzcd}$$

commutes. Applying this to $\alpha \otimes y \in \omega(C)^\vee \otimes \omega(A)$, we find that the image of $\alpha \otimes f(y)$ in L is also the image of the element $f^t(\alpha) \otimes y$ of $\omega(A)^\vee \otimes_k \omega(A)$. □

Let B be a division k-algebra and L a k-coalgebroid acting on B. Let $\mathsf{coModf}(L)$ denote the category of L-comodules of finite dimension as B-vector spaces, and let ω be the forgetful functor. Since L coacts on each $\omega(X)$, the universal property of $L(\omega)$ furnishes a morphism u from $L(\omega)$ to L.

PROPOSITION 6.9 *The morphism $u : L(\omega) \to L$ is an isomorphism.*

PROOF By construction, a coaction of L on a finite-dimensional V has a natural lift to $L(\omega)$. Applying 6.6 and passing to the inductive limit, we see that the restriction "finite-dimensional" is unnecessary. Taking $V = L$ and the coaction $c : L \to L \otimes_B L$, we obtain $c_1 : L \to L \otimes_B L(\omega)$. Let $a = (\mathrm{counit}\otimes 1)\circ c_1 : L \to L(\omega)$. As $(1 \otimes u)\circ c_1 = c$, we have $ua = \mathrm{id}$.

Let V be equipped with a coaction ρ, which lifts to a coaction $\widetilde{\rho}$ of $L(\omega)$. Since $\rho\colon V \to V \otimes L$ is a morphism of vector spaces with coaction, the diagram

$$\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[d, "\bar{\rho}"] & V \otimes L \arrow[d, "1\otimes c_1"] & \\
V \otimes L\omega \arrow[r, "\rho\otimes 1"] & V \otimes L \otimes L\omega \arrow[r, "1\otimes\mathrm{counit}\otimes 1"] & V \otimes L\omega
\end{tikzcd}$$

commutes, and so $\widetilde{\rho} = (1 \otimes a)\rho$. The morphism deduced from $\rho\colon V^\vee \otimes V \to L\omega$ admits the factorization $V^\vee \otimes V \to L \xrightarrow{a} L\omega$. The definition of $L\omega$ shows that a is surjective, and therefore u is an isomorphism. □

TODO 19 Should rework and try to simplify the above proof. See the exposition in L 3.1.1–3.1.10.

Realizing (T, ω) *as a category of comodules*

The next theorem generalizes (II, 3.20).

THEOREM 6.10 (DELIGNE 1990, 6.2) *Let B be an algebra over a field k. Let A be an essentially small, locally finite, k-linear abelian category and $\omega\colon \mathsf{A} \to \mathrm{Proj}(B)$ an exact faithful k-linear functor. Let $L(\omega)$ be the k-coalgebroid of k-endomorphisms of ω (6.4). Then ω defines an equivalence of categories $\mathsf{A} \xrightarrow{\approx} \mathsf{coModf}(L(\omega))$ carrying ω into the forgetful functor.*

PROOF Recall (II, 3) that, for an object X of an abelian category, $\langle X\rangle$ denotes the strictly full subcategory whose objects are subquotients of a finite direct sum of copies of X. It is an abelian subcategory containing X.

For X in A, the category $\langle X\rangle$ admits a projective generator P (II, 3.13). Let $A = \mathrm{End}(P)$. Then the functor $Y \rightsquigarrow \mathrm{Hom}(P, Y)$ is an equivalence of $\langle X\rangle$ with the category Modf_A of right A-modules of finite type. Under this equivalence, P corresponds to A_A. Put ${}_AM_B = \omega(P)$. By (the proof of) 3.8, the right exact functor $\omega|\langle X\rangle$ can be identified with the functor $E \rightsquigarrow E \otimes_A {}_AM_B\colon \mathsf{Modf}_A \to \mathsf{Mod}_B$. Note that, as ω is k-linear, the two k-module structures on ${}_AM_B$ coincide.

After 6.8, $L(\omega|X)$ is the k-coalgebroid ${}_BM_A^\vee \otimes_A {}_AM_B$ of 5.26. By hypothesis, $\omega|\langle X\rangle$ is exact and faithful. The A-module ${}_AM_B$ is therefore faithfully flat over A. After 5.26 and 5.27, ω induces an equivalence of $\langle X\rangle$ with the category of right B-modules of finite type equipped with a coaction of $L(\omega|\langle X\rangle)$.

The category $\mathrm{Ind}\langle X\rangle$ of Ind-objects of $\langle X\rangle$ can be identified with that of all right A-modules Mod_A (see B.11). The extension of $L(\omega|\langle X\rangle)$ to Ind-objects, $\omega(\text{“}\varinjlim\text{”}X_i) = \varinjlim \omega(X_i)$ is again $E \rightsquigarrow E \otimes_A {}_AM_B$. By 5.26, this extension is an equivalence of $\mathrm{Ind}\langle X\rangle$ with the category of right B-modules equipped with a coaction of $L(\omega|\langle X\rangle)$. This, and the assumed properties of ω, show that any right B-module of finite type, equipped with a coaction of $L(\omega|\langle X\rangle)$, is finitely generated projective, and that any right B-module with a coaction is a inductive limit of finitely generated projective right B-modules, and, in particular, is flat.

The exactness of ω ensures that ${}_AM_B$ is A-flat. Moreover, for any right A-module N, the right B-module $N \otimes_A {}_AM_B$ is flat. The formula

$$L(\omega|\langle X\rangle) = {}_BM_A^\vee \otimes_A {}_AM_B$$

then shows that $L(\omega|\langle X\rangle)$ is flat for the two B-module structures.

If Y is in $\langle X\rangle$, i.e., $\langle Y\rangle \subset \langle X\rangle$, and $\mathfrak{a}$ is the 2-sided ideal of A such that $\langle Y\rangle$ corresponds to the A-modules killed by $\mathfrak{a}$ (see 3.6), we have

$$\begin{aligned} L(\omega|\langle Y\rangle) &= (A/\mathfrak{a} \otimes_A {}_A M_B)^\vee \otimes_{A/\mathfrak{a}} (A/\mathfrak{a} \otimes_A {}_A M_B) \\ &= (A/\mathfrak{a} \otimes_A {}_A M_B)^\vee \otimes_{A/\mathfrak{a}} M_B, \end{aligned}$$

and $(A/\mathfrak{a} \otimes_A {}_A M_B)^\vee$ is the kernel of the epimorphism ${}_B M_A^\vee \to (\mathfrak{a} \otimes_A {}_A M_B)^\vee$, and so there is an exact sequence

$$0 \to L(\omega|\langle Y\rangle) \to L(\omega|\langle X\rangle) \to (\mathfrak{a} \otimes_A {}_A M_B)^\vee \otimes_A {}_A M_B \to 0.$$

The morphism 6.7(b) of $L(\omega|\langle Y\rangle)$ into $L(\omega|\langle X\rangle)$ is therefore injective, with kernel flat as a left and as a right B-module.

A B-module of finite type with a coaction of $L(\omega|\langle X\rangle)$ corresponds to an object of $\langle Y\rangle$ if the coaction

$$N \to N \otimes L(\omega|\langle X\rangle)$$

factors through $N \otimes L(\omega|\langle Y\rangle)$.

The category A is the filtered union of the subcategories $\langle X\rangle$ and $L(\omega)$ is the inductive limit of the $L(\omega|\langle X\rangle)$ (6.7(c)). Passing to the limit, we obtain the theorem. □

EXAMPLE 6.11 Let A be a small locally finite k-linear abelian category, B an extension field of k, and $\omega\colon \mathsf{C} \to \mathsf{Vecf}_B$ an exact faithful k-linear functor. Then ω factors into

$$\mathsf{C} \xrightarrow{i} \mathsf{coModf}_{L(\omega)} \xrightarrow{\text{forget}} \mathsf{Vecf}_B.$$

The functor i is an equivalence of categories.

Proof of the main theorem, except for the faithful flatness

6.12 Let B be a commutative ring. We shall use the construction of $L(\omega_1, \omega_2)$ in 6.2 for $B_1 = B_2 = k = B$. Later, we shall need to consider two commutative k-algebras B_1, B_2 and we shall take B to be the commutative algebra $B_1 \otimes_k B_2$.

Suppose that we have three categories $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ and, for each category, two functors

$$\begin{cases} \omega_1^i\colon \mathsf{A}_i \to \mathsf{Proj}(B) \\ \omega_2^i\colon \mathsf{A}_2 \to \mathsf{Proj}(B). \end{cases}$$

Suppose also that we have a functor $\otimes\colon \mathsf{A}_1 \times \mathsf{A}_2 \to \mathsf{A}_3$ and isomorphisms of functors

$$\begin{cases} \omega_1^1(X_1) \otimes_B \omega_1^2(X_2) \to \omega_1^3(X_1 \otimes X_2) \\ \omega_2^1(X_1) \otimes_B \omega_j^2(X_2) \to \omega_j^3(X_1 \otimes X_2). \end{cases}$$

In 6.2, we defined B-modules $L_B(\omega_1^i, \omega_2^i)$, and 6.7b and 6.7d furnish a B-bilinear product

$$L_B(\omega_1^1, \omega_2^1) \otimes_B L_B(\omega_1^2, \omega_2^2) \to L_B(\omega_1^3, \omega_2^3). \tag{82}$$

For any X_1 and X_2 in A_1 and A_2, the morphism (76)

$$\omega_2^3(X_1 \otimes X_2) \to \omega_1^3(X_1 \otimes X_2) \otimes_B L_B(\omega_1^3, \omega_2^3)$$

can be deduced from the analogous morphisms for $\omega_1^j(X_j)$ and $\omega_2^j(X_j)$ by (82).

Let A be a symmetric monoidal category and ω_1 and ω_2 two $\otimes$-functors $\mathsf{A} \to \mathsf{Proj}(B)$ (see Chapter I). When we take $\mathsf{A}_i = \mathsf{A}$ and $\omega_j^i = \omega_j$ ($i = 1, 2, 3$), the product (82) becomes a product

$$L_B(\omega_1, \omega_2) \otimes_B L_B(\omega_1, \omega_2) \to L_B(\omega_1, \omega_2). \tag{83}$$

PROPOSITION 6.13 *The product (83) makes $L_B(\omega_1, \omega_2)$ into a commutative B-algebra.*

PROOF For $j = 1, 2$, we have essentially commutative diagrams,

$$\begin{tikzcd} \mathsf{A} \times \mathsf{A} \arrow[r, "(X,Y)\mapsto(Y,X)"] \arrow[d, "\omega_j \otimes \omega_j"] & \mathsf{A} \times \mathsf{A} \arrow[r, "\otimes"] \arrow[d, "\omega_j \otimes \omega_j"] & \mathsf{A} \arrow[d, "\omega_j"] \\ \mathsf{Mod}(B) \arrow[r, equal] & \mathsf{Mod}(B) \arrow[r, equal] & \mathsf{Mod}(B) \end{tikzcd}$$

The left-hand square is rendered commutative by the isomorphism of functors

$$\omega_j(X) \otimes_B \omega_j(Y) \to \omega_j(Y) \otimes_B \omega_j(X),$$

and the right-hand square by the isomorphism

$$\omega_j(X \otimes Y) \to \omega_j(X) \otimes_B \omega_j(Y).$$

By the definition of a $\otimes$-functor, the isomorphism of composed functors makes commutative the boundary of the diagram, and also the left-hand square once we identify $\omega(Y \otimes X)$ with $\omega(X \otimes Y)$ using the commutativity of $\otimes$ in A. Applying $L(\cdot, \cdot)$ to the diagram, we obtain the commutativity in (83). The associativity is obtained the same way. If $\{\mathbb{1}\}$ is the subcategory of A consisting only of the identity object and the identity arrow, then we obtain, by definition of $\otimes$-functor, that $\omega_j(1) \xrightarrow{\simeq} B$ and $L_B(\omega_1|\{\mathbb{1}\}, \omega_2|\{\mathbb{1}\}) = B$. The identity $B \to L_B(\omega_1, \omega_2)$ of $L_B(\omega_1, \omega_2)$ is defined by 6.7(b). □

6.14 Recall (1.3) that when A is a symmetric monoidal category and ω_1 and ω_2 are $\otimes$-functors from A to the quasi-coherent sheaves on an affine scheme S, we define $\mathcal{H}om_S^{\otimes}(\omega_1, \omega_2)$ (resp. $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$) to be the functor on Aff_S sending $u\colon T \to S$ to the set of morphisms (resp. isomorphisms) of tensor functors $u^*\omega_1 \to u^*\omega_2$. If ω_1 and ω_2 take values in the category of locally free modules of finite rank, for example, if A is rigid (I, 8.5), then these functors are representable by an affine scheme over S and

$$\mathcal{H}om_S^{\otimes}(\omega_1, \omega_2) \simeq \mathcal{I}som_S^{\otimes}(\omega_1, \omega_2).$$

6.15 As in 1.3, when ω_i has values in the category of quasi-coherent sheaves on S_i, we put

$$\mathcal{H}om_k^{\otimes}(\omega_2, \omega_1) = \mathcal{H}om_{S_1 \times S_s}^{\otimes}(\mathrm{pr}_2^*\,\omega_2, \mathrm{pr}_1^*\,\omega_1),$$

and similarly for $\mathcal{I}som$. For $\omega_1 = \omega_2$, we let

$$\begin{aligned} \mathcal{E}nd(\omega) &= \mathcal{H}om(\omega, \omega) \\ \mathcal{A}ut(\omega) &= \mathcal{I}som(\omega, \omega). \end{aligned}$$

PROPOSITION 6.16 *Let A be a symmetric monoidal category and ω_1 and ω_2 $\otimes$-functors $\mathsf{A} \to \mathsf{Proj}(B)$. Put $S = \operatorname{Spec} B$. The scheme $L_B(\omega_1, \omega_2)$ represents the functor $\mathcal{H}om_S^{\otimes}(\omega_2, \omega_1)$.*

PROOF Let $u\colon T \to S$, $T = \operatorname{Spec} C$, be an affine scheme over S. By definition (6.2), a morphism $f\colon L_B(\omega_1, \omega_2) \to C$ of B-modules can be identified with a functorial system of B-modules

$$f_X\colon \omega_2(X) \to \omega_1(X) \otimes_B C.$$

Giving the f_X is equivalent to giving C-linear morphisms

$$f'_X\colon \omega_2(X) \otimes_B C \to \omega_1(X) \otimes_B C,$$

functorial in X, i.e., a morphism f' of functors $u^*\omega_1$ to $u^*\omega_1$. It can be checked that f is a morphism of algebras if and only if f' is a morphism of $\otimes$-functors. □

6.17 Let k be a commutative ring, and let B_1 and B_2 be two commutative k-algebras. Let A be a k-linear symmetric monoidal category and ω_1 and ω_2 $\otimes$-functors from A to $\mathsf{Proj}(B_1)$ and $\mathsf{Proj}(B_2)$. By extension of scalars, ω_1 and ω_2 define $\otimes$-functors $\mathsf{A} \to \mathsf{Proj}(B_1 \otimes_k B_2)$, which we denote $\omega_1 \otimes 1$ and $1 \otimes \omega_2$ respectively. We have

$$L_k(\omega_1, \omega_2) = L_{B_1 \otimes_k B_2}(\omega_1 \otimes 1, 1 \otimes \omega_2).$$

After 6.16, $\operatorname{Spec} L_k(\omega_1, \omega_2)$ represents the functor $\mathcal{Hom}_k^{\otimes}(\omega_2, \omega_1)$, which is equal to $\mathcal{Isom}_k^{\otimes}(\omega_1, \omega_2)$ if A is rigid (see 6.14).

For three B_j and ω_j, composition of morphisms

$$\mathcal{Hom}_k^{\otimes}(\omega_3, \omega_2) \times_{S_2} \mathcal{Hom}_k^{\otimes}(\omega_2, \omega_1) \to \mathcal{Hom}_k^{\otimes}(\omega_3, \omega_1)$$

corresponds to a morphism of k-algebras

$$c\colon\ L_k(\omega_1, \omega_3) \to L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3).$$

By definition, the morphism of $L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3)$-modules deduced by extension of scalars (by c) from

$$\omega_3(X) \otimes_{B_3} L_k(\omega_1, \omega_3) \to \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_3)$$

is the composite of the morphisms deduced by extension of scalars from

$$\omega_3(X) \otimes_{B_3} L_k(\omega_2, \omega_3) \to \omega_2(X) \otimes_{B_2} L_k(\omega_2, \omega_3) \quad \text{and}$$
$$\omega_2(X) \otimes_{B2} L_k(\omega_1, \omega_2) \to \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2).$$

This comes down to the commutativity of

$$\begin{tikzcd}
\omega_3(X) \arrow[r] \arrow[d] & \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2) \arrow[d] \\
\omega_2(X) \otimes_{B_2} L_k(\omega_2, \omega_3) \arrow[r] & \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3)
\end{tikzcd}$$

and c is therefore (77).

6.18 Let k be a field. Let T be a tensor category over k and ω a fibre functor on T over $S = \operatorname{Spec} B$, B a nonzero k-algebra. We prove that ω induces an equivalence of T with $\mathsf{Repf}(S\colon G)$, where G is the groupoid $\mathcal{Aut}_k^{\otimes}(\omega)$.

In the above, we take $\mathsf{A} = \mathsf{T}$, $B_1 = B_2 = B$, $\omega_1 = \omega_2 = \omega$. According to (I, 8.11), T is locally finite, and so we can apply 6.10. After 6.17, $\mathcal{Aut}_k^{\otimes}(\omega)$ is the spectrum of $L(\omega) = L_k(\omega_1, \omega_2)$. The action of $\mathcal{Aut}_k^{\otimes}(\omega)$ on $\omega(X)$ is defined by the morphisms

$$\omega(X) \to \omega(X) \otimes_B L(\omega, \omega)$$

which defined L (76). By 6.17 again, the law of composition of the groupoid $\mathcal{Aut}_k^{\otimes}(\omega)$ is defined by the comultiplication of $L(\omega)$, and 6.10 is equivalent to the required statement.

Proof of the faithful flatness

PROPOSITION 6.19 *Let* C *be a tannakian category over* k *and let* ω *be a fibre functor with values in a (commutative) nonzero* k*-algebra. Let* $\mathrm{Ind}\,\omega\colon\ \mathrm{Ind}\,\mathsf{C}\to\mathsf{Mod}(B)$ *denote the extension of* ω *to* $\mathrm{Ind}\,\mathsf{C}$*. For all nonzero objects* T *in* $\mathrm{Ind}\,\mathsf{C}$*,* $\mathrm{Ind}(\omega)(T)$ *is faithfully flat over* B*.*

PROOF The category C is abelian and its objects are noetherian (I, 8.11). On applying Proposition B.9, we see that $T=\varinjlim X_i$ with $X_i\in\mathrm{ob}\,\mathsf{C}$, $X_i\subset T$. Fix an $X_j\neq 0$, and consider the exact sequence

$$0\to X_j\to T\to T/X_j\to 0$$

in $\mathrm{Ind}\,\mathsf{C}$. Because ω is right exact, $\mathrm{Ind}\,\omega$ commutes with arbitrary inductive limits (B.7), and so $(\mathrm{Ind}\,\omega)(T)=\varinjlim\omega(X_i)$. We deduce an exact sequence

$$0\to\omega X_j\to(\mathrm{Ind}\,\omega)(T)\to(\mathrm{Ind}\,\omega)(T/X_j)\to 0.$$

Now $\mathrm{Ind}(\omega)(T)$ is an inductive limit of finitely generated projective (hence flat) modules, and so is a flat B-module. Similarly, $\mathrm{Ind}(\omega)(T/X_j)$ is a flat B-module. Moreover, the B-module $\omega(X_j)$ is faithfully flat (8.5). If M is an arbitrary nonzero B-module, then the sequence

$$0\to\omega(X_j)\otimes_B M\to(\mathrm{Ind}\,\omega)(T)\otimes_B M\to(\mathrm{Ind}\,\omega)(T/X_j)\otimes_B M\to 0$$

is exact because $(\mathrm{Ind}\,\omega)(T/X_j)$ is flat.[6] As $\omega(X_j)\otimes_B M\neq 0$, we have $(\mathrm{Ind}(\omega)(T))\otimes_B M\neq 0$, and so $\mathrm{Ind}(\omega)(T)$ is a faithfully flat B-module. □

COROLLARY 6.20 *Let* C *be a tannakian category over a perfect field* k*, and let* ω *be a fibre functor with values in a (commutative)* k*-algebra* B*. Then* $L_k(\omega)$ *(see 6.4) is a faithfully flat* $B\otimes_k B$*-algebra.*

PROOF Define T to be the coequalizer of the parallel pair of morphisms

$$\bigoplus_{(f\colon X\to Y)\in\mathrm{ar}\,\mathsf{C}} Y^\vee\otimes X \rightrightarrows \bigoplus_{X\in\mathrm{ob}\,\mathsf{C}} X^\vee\otimes X$$

in $\mathrm{Ind}(\mathsf{C}\boxtimes\mathsf{C})$ (cf. 6.2). According to (12.18), we have a fibre functor $\omega\boxtimes\omega$ on $\mathsf{C}\boxtimes\mathsf{C}$, and clearly $\mathrm{Ind}(\omega\boxtimes\omega)(T)=L(\omega)$. If $B\neq 0$, then $L(\omega)\neq 0$, and so $T\neq 0$. Now Proposition 6.19 shows that $L(\omega)$ is faithfully flat. □

COROLLARY 6.21 *Let* C *be a tannakian category over a perfect field* k*, and let* ω *be a fibre functor with values in a* k*-algebra* $B\neq 0$*. Then there exists a faithfully flat homomorphism* $f\colon B\to B'$ *such that* $f^*\omega\approx f^*\omega'$*.*

PROOF See 1.4. □

[6] Let $0\to M'\to M\to M''\to 0$ be an exact sequence of right A-modules. If M'' is flat, then

$$0\to M'\otimes_A N\to M\otimes_A N\to M''\otimes_A N\to 0$$

is exact for all left A-modules N. This is most naturally proved using the Tor functor, but, for a proof without the Tor functor, see 13.10.

This completes the proof of Theorem 1.1 when k is perfect. For the case of a nonperfect base field, we refer the reader to Deligne 1990.

REMARK 6.22 We used the hypothesis that k is perfect to show that the functor $\mathsf{A}\boxtimes\mathsf{B}\to\mathsf{D}$ defined by a functor $\mathsf{A}\times\mathsf{B}\to\mathsf{D}$, exact in both variables, is exact (I, 12.11). This was used in the proof (I, 12.17) that a tensor product of tensor categories over a perfect field is tensor and hence in the proof of the similar statement (I, 12.18) for tannakian categories. For an explanation of how to remove the perfectness hypothesis in I, 12.18, hence in the proof of Theorem 1.1, see Deligne 1990, 5.18.

NOTES The exposition of the proof Theorem 1.1 and of its preliminaries largely follows the original (Deligne 1990, §§1–6). See also Lattermann 1989.

7 Restatement for 2-categories

Let S be a nonempty affine scheme over k. Theorem 1.1 can be interpreted as saying that the 2-category of k-groupoids acting transitively on S is biequivalent (not 2-equivalent) to the category of tannakian categories equipped with a fibre functor over S.

DEFINITION 7.1 The 2-category $\mathcal{G}\!rpd_S$ has

- objects the affine k-groupoids acting transitively on S;
- a 1-morphism from G to H is a morphism $f\colon G\to H$ of groupoids acting on S (2.11).
- a 2-morphism $f\to g$ is a natural transformation from f to g (viewing f and g as functors of affine $S\times S$-schemes).

DEFINITION 7.2 The 2-category $\mathcal{T}\!ann_S^{\bullet}$ of S-pointed tannakian categories over Aff_k has

- objects the pairs (T,ω), where T is an essentially small tannakian category over k and ω is a fibre functor on T over S;
- a 1-morphism from (T,ω) to (T',ω') is an exact k-linear $\otimes$-functor from T to T' carrying ω into ω';
- a 2-morphism is a morphism of $\otimes$-functors.

The functors we defined in §1 extend in an obvious way to 2-functors

$$\Phi\colon \mathcal{T}\!ann_S^{\bullet\mathrm{opp}}\to\mathcal{G}\!rpd_S,\quad (\mathsf{T},\omega)\rightsquigarrow \mathcal{A}ut^{\otimes}(\omega)$$
$$\Psi\colon \mathcal{G}\!rpd_S\to\mathcal{T}\!ann_S^{\bullet\mathrm{opp}},\quad G\rightsquigarrow \mathsf{Repf}(S{:}G),$$

and we also defined functors $\eta\colon\ \mathrm{id}\to\Psi\circ\Phi$ and $\epsilon\colon\ \Phi\circ\Psi\to\mathrm{id}$.

THEOREM 7.3 *The system*

$$\mathcal{T}\!ann_S^{\bullet\mathrm{opp}} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{G}\!rpd_S,\qquad \eta\colon\ \mathrm{id}\to\Psi\circ\Phi,\qquad \epsilon\colon\ \Phi\circ\Psi\to\mathrm{id},$$

is a biequivalence of $\mathcal{T}\!ann_S^{\bullet\mathrm{opp}}$ *and* $\mathcal{G}\!rpd_S$.

PROOF This is little more than a restatement of Theorem 1.1. □

COROLLARY 7.4 *Let* T_1 *and* T_2 *be tannakian categories over* k*, and let* ω_1 *and* ω_2 *be fibre functors on* T_1 *and* T_2 *over* S*. Then*

$$\mathsf{Hom}((\mathsf{T}_1,\omega_1),(\mathsf{T}_2,\omega_2)) \to \mathsf{Hom}(\mathcal{A}ut^{\otimes}(\omega_2),\mathcal{A}ut^{\otimes}(\omega_1))$$

is an equivalence of categories.

PROOF Immediate consequence of the theorem. □

TODO 20 TBA: add detailed proof of 7.3.

8 Properties of a tannakian category reflected in its band

As in (II, §5, §6), there is a dictionary relating the properties of tannakian categories and their morphisms to those of their bands. Here are some examples.

8.1 Let C be a tannakian category over k with band L.

(a) L is finite (i.e., locally represented by a finite group scheme) if and only if $\mathsf{C} = \langle X\rangle$ for some object X.

(b) When k has characteristic 0, L is pro-reductive if and only if C is semisimple.

8.2 Let $\omega : \mathsf{C}' \to \mathsf{C}$ be a morphism of tannakian categories over k, bound by a morphism of bands $u : L \to L'$.

(a) u is faithfully flat (i.e., an epimorphism of bands) if and only if ω is fully faithful and its essential image is closed under subobjects.

(b) u is injective (i.e., locally represented by a monomorphism of groups) if and only if every object of C is a subquotient of an object in the image of ω.

See Saavedra 1972, III, 3.3.3, p. 205.

TODO 21 Should add explanation. Saavedra 1972, III, 3.3.3, p. 205, states this without explanation.

9 Extension of scalars for tannakian categories

Let $(\mathsf{T},\otimes)$ be a tannakian category over k, and let ω be a fibre functor on T with values in a field K. Consider a diagram of fields

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k'. \end{array}$$

Let X be an object of $\mathrm{Ind}(\mathsf{T})$ equipped with a homomorphism $i : k' \to \mathrm{End}_k(X)$ of k-algebras. Recall (I, §13) that we say that $Y \subset X$ ***generates*** (X,i) ***as a*** k'***-module*** if it is not contained in any proper k'-module, and we define $\mathsf{T}_{(k')}$ to be the category whose objects are the k'-modules in $\mathrm{Ind}(\mathsf{T})$ that are generated as k'-modules by subobjects in T.

PROPOSITION 9.1 *The category* $\mathsf{T}_{(k')}$ *is a tannakian category over* k'*, the* K*-valued fibre functor* ω *extends to a* K'*-valued fibre functor* ω' *on* $\mathsf{T}_{(k')}$*, and the* K'/k'*-groupoid* $\mathcal{A}ut^{\otimes}_{k'}(\omega')$ *is the pullback of the* K/k*-groupoid* $\mathcal{A}ut^{\otimes}_{k}(\omega)$*.*

PROOF After Theorem 1.1, we may suppose that $\mathsf{T} = \mathsf{Repf}(S{:}G)$, where $S = \operatorname{Spec} K$, and that ω is the forgetful functor. Then the statement follows from 3.7. □

EXAMPLE 9.2 Take $k' = K' = K$, so T is a tannakian category over k and ω is a k'-valued fibre functor. The proposition then shows that $\mathsf{T}_{(k')}$ is a neutral tannakian category over k', and that ω extends to a k'-valued fibre functor ω' on $\mathsf{T}_{(k')}$. The affine group scheme attached to $(\mathsf{T}_{(k')}, \omega')$ is the kernel of the groupoid attached to (T, ω),

$$\mathcal{A}ut^{\otimes}_{k'}(\omega') \simeq \mathcal{A}ut^{\otimes}_{k}(\omega)^{\Delta}.$$

10 Existence of a fibre functor over k^{al}

In this section, we show that essentially small tannakian categories over algebraically closed fields are neutral.

THEOREM 10.1 (DELIGNE) *Let T be an essentially small tannakian category over k and k^{al} an algebraic closure of k. Then T has a fibre functor with values in k^{al}.*

We deduce this from the following more precise statements.

THEOREM 10.2 *Let T be an essentially small tannakian category over k. If T is algebraic, then it has a fibre functor with values in some finite extension of k.*

THEOREM 10.3 *Let T be an essentially small tensor category over k. If every monogenic tensor subcategory has a k^{al}-valued fibre functor, then T has an k^{al}-valued fibre functor. In particular, T is tannakian.*

Proof of Theorem 10.2.

We first state a result from commutative algebra.

LEMMA 10.4 *Let A_0 be a ring and let $A = \varinjlim_{i\in I} A_i$ be the limit of a filtered inductive system of A_0-algebras. Let B_0 be an A_0-algebra of finite presentation, and set $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$.*

(a) *If $\operatorname{spec} B \to \operatorname{spec} A$ is surjective, then there exists an i such that $\operatorname{spec} B_i \to \operatorname{spec} A_i$ is surjective.*

(b) *If B is flat over A, then there exists an i such that B_i is flat over A_i.*

PROOF (a) EGA IV, 8.10.5.
(b) EGA IV, 11.2.6.1. □

10.5 We now prove Theorem 10.2. By assumption, T has a fibre functor over a nonempty k-scheme S, which we may suppose to be the spectrum of a field, and then we may suppose that $\mathsf{T} = \mathsf{Repf}(S{:}G)$, where G is a groupoid of finite presentation and faithfully flat over $S \times S$ (Theorem 1.1). Let $S = \operatorname{Spec} B$, and write B as a union $B = \bigcup_{\alpha} B_{\alpha}$ of finite generated k-algebras B_{α}. For some α, G is the pullback of a groupoid of finite presentation over $S_{\alpha} \times S_{\alpha}$, where $S_{\alpha} = \operatorname{Spec} B_{\alpha}$. As G is faithfully flat over $S \times S$, G_{α} is faithfully flat over $S_{\alpha} \times S_{\alpha}$ for α sufficiently large (10.4). Now $\mathsf{T} = \mathsf{Repf}(S_{\alpha}{:}G_{\alpha})$ (see **??**). In particular, T has a fibre functor over S_{α}. As S_{α} has a K-point for some finite extension K of k (Zariski's lemma), it follows that T has a fibre functor with values in K.

10.6 The above proof of Theorem 10.2, using groupoid schemes, is from Deligne 1990, 6.20. Here is another proof, using gerbes, from Saavedra 1972, III, 3.3.1.1, p. 204. Let T be a tannakian category over a field k. The fibre functors on T form a fibred category over Aff_k, and even a stack for the fpqc topology. It is nonempty by definition, so to show that it a gerbe it remains to show that any two fibre functors are locally isomorphic for the fpqc topology. For a proof of this, independent of the results of this chapter, see II, 8.8. We now use that a fibred category over k is a gerbe for the fppf topology on Aff_k if it is a gerbe for the fpqc topology (C.22). Therefore T has a fibre functor with values in a finitely generated k-algebra R, hence in $R/\mathfrak{m}$ for $\mathfrak{m}$ a maximal ideal of R, and $R/\mathfrak{m}$ is a finite extension of k (Zariski's lemma).

Proof of Theorem 10.3

Let A_f denote the set of all subcategories T_α of T of the form $\langle X\rangle^{\otimes}$ for some $X \in \operatorname{ob}\mathsf{T}$, partially ordered by inclusion, and let $\operatorname{Fib}(T_\alpha)$ denote the set of all k^{al}-valued fibre functors of T_α. If $\mathsf{T}_\alpha \subset \mathsf{T}_\beta$, then the restriction map $\operatorname{Fib}(T_\beta) \to \operatorname{Fib}(T_\alpha)$ is surjective, and we have to show that $\varprojlim_\alpha \operatorname{Fib}(T_\alpha)$ is nonempty. If A_f contains a countable cofinal subset, then this follows from the axiom of dependent choice. For the general case, we need the full axiom of choice (in the form of Zorn's lemma). After I, 8.19, we may suppose that $k = k^{\mathrm{al}}$. To avoid set theoretical difficulties, we consider only fibre functors taking values in a suitable small category of vector spaces over k, for example, in the category of all vector spaces k^n, $n \in \mathbb{N}$. Also, we replace T with an equivalent small category.

Let A be the set of all tensor subcategories of T over k. For $\alpha \in A$, we denote by T_α the corresponding subcategory (so $\mathsf{T}_\alpha = \alpha$). The set A is partially ordered by inclusion. We already know the existence and uniqueness up to isomorphism of fibre functors for the T_α, $\alpha \in A_f$.

If $\mathsf{T}_\alpha \subset \mathsf{T}_\beta$, it makes sense to say that a fibre functor ω_β on T_β extends a fibre functor ω_α on T_α. If an extension up to isomorphism exists, then an actual extension exists too.

We consider the set $\mathcal{S}$ of pairs (α, ω_α) with $\alpha \in A$ and ω_α a fibre functor on T_α, partially ordered by

$$(\alpha, \omega_\alpha) \le (\beta, \omega_\beta) \iff \mathsf{T}_\alpha \subset \mathsf{T}_\beta \text{ and } \omega_\beta \text{ extends } \omega_\alpha.$$

The partially ordered set $\mathcal{S}$ is inductive: if I is a totally ordered subset, then

$$\mathsf{T} \stackrel{\text{def}}{=} \bigcup \{\mathsf{T}_\alpha \mid (\alpha, \omega_\alpha) \in I\}$$

lies in A and has a fibre functor ω characterized by $\omega|\mathsf{T}_\alpha = \omega_\alpha$. The pair (T, ω) is an upper bound for I. By Zorn's lemma $\mathcal{S}$ has a maximal element (T_1, ω_1). To show that $\mathsf{T}_1 = \mathsf{T}$, it suffices to prove the following statement.

10.7 *Let T' be in A and T'' in A_f. Then, every fibre functor ω on T' extends to $\langle \mathsf{T}', \mathsf{T}''\rangle^{\otimes}$, the smallest tensor subcategory of T containing T' and T''*

We first prove a lemma.

Lemma 10.8 *If T' is also in A_f, then restriction is an equivalence of categories*

$$\{\textit{fibre functors } \omega \textit{ on } \langle \mathsf{T}', \mathsf{T}''\rangle^{\otimes}\} \longrightarrow \left\{\begin{array}{l}\textit{triples } (\omega', \omega'', \tau) \textit{ with } \omega' \textit{ and } \omega'' \textit{ fibre func-}\\ \textit{tors on } \mathsf{T}' \textit{ and } \mathsf{T}'' \textit{ and } \tau \textit{ an isomorphism of}\\ \textit{the restrictions of } \omega' \textit{ and } \omega'' \textit{ to } \mathsf{T}' \cap \mathsf{T}''\end{array}\right\}.$$

Note that both $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$ and $\mathsf{T}' \cap \mathsf{T}''$ lie in A_f.

PROOF We use that the Noether isomomorphism theorems hold for affine group schemes over a field (Milne 2017, Chapter 5). After I, 3.1, and 10.4, we may suppose that $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$ is the category of representations of an algebraic group G over k. There exist normal algebraic subgroups A and B of G such that T' (resp. T'') is the subcategory of representations on which A (resp. B) acts trivially. To say that T' and T''generate $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$ means that $A \cap B = \{e\}$. The intersection $\mathsf{T}' \cap \mathsf{T}''$ is the category of representations on which AB acts trivially.

We claim that the triples $(\omega', \omega'', \tau)$ are all isomorphic. As all ω' (resp. all ω'') are isomorphic, it suffices to show that $(\omega', \omega'', \tau_1)$ and $(\omega', \omega'', \tau_2)$ are isomorphic, but τ_1 and τ_2 differ by an automorphism of $\omega' | \mathsf{T}' \cap \mathsf{T}''$, and such an automorphism lifts to an automorphism of ω': the homomorphism

$$(G/A)(k) \to (G/AB)(k)$$

is surjective.

Thus, the categories in the lemma each have just one isomorphism class of objects, and so it becomes a question of comparing their endomorphism rings (automorphism groups in this case). But

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/A \arrow[d] \\ G/B \arrow[r] & G/AB \end{tikzcd}$$

is a pullback diagram because $A \cap B = e$, and so the same is true when we take k-valued points. □

We now prove the statement 10.7. Let B be the set of T_α ($\alpha \in A_f$) contained in T'. We have

$$\mathsf{T}' = \bigcup_{\beta \in B} \mathsf{T}_\beta, \qquad \langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes} = \bigcup_{\beta \in B} \langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes},$$

and equivalences

$$\{\text{fibre functors on } \mathsf{T}'\} \xrightarrow{\approx} \left\{\begin{array}{l} \text{fibre functors } \omega_\beta \text{ on the } \mathsf{T}_\beta, \text{ plus a compatible system} \\ \text{of isomorphisms } \omega_\beta | \mathsf{T}_\gamma \xrightarrow{\sim} \omega_\gamma \text{ for } \mathsf{T}_\gamma \subset \mathsf{T}_\beta, \text{ plus a} \\ \text{compatibility condition for } \mathsf{T}_\delta \subset \mathsf{T}_\gamma \subset \mathsf{T}_\beta \end{array}\right\}$$

$$\{\text{fibre functors on } \langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}\} \xrightarrow{\approx} \left\{\begin{array}{l} \text{fibre functors on the } \langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes} \text{ plus similar} \\ \text{conditions.} \end{array}\right\}$$

There is a largest $\mathsf{T}_\beta \cap \mathsf{T}''$: if $\mathsf{T}'' = \mathsf{Repf}(G'')$, then each $\mathsf{T}_\beta \cap \mathsf{T}''$ corresponds to a normal algebraic subgroups N_β of G'', and among the N_β, there is a smallest one, namely, $\bigcap_\beta N_\beta$. If β_0 is such that $\mathsf{T}_{\beta_0} \cap \mathsf{T}''$ is the largest $\mathsf{T}_\beta \cap \mathsf{T}''$, then, for any $\beta > \beta_0$, extending $\omega_\beta \overset{\text{def}}{=} \omega | \mathsf{T}_\beta$ to $\langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes}$ amounts to extending $\omega_\beta | \mathsf{T}_\beta \cap \mathsf{T}''$ to T'' (Lemma 10.8), that is, to extending ω_{β_0} from $\mathsf{T}_{\beta_0} \cap \mathsf{T}''$ to T''. If we choose one such extension, then we get, up to *unique* isomorphisms, a system of extensions of the ω_β to $\langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes}$, and by gluing them an extension of ω to $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$.

This completes the proof of 10.7, which shows that T_1 contains $\langle X \rangle^{\otimes}$ for all X in T.

Notes

10.9 Only the proof of 10.5 requires Theorem 1.1 – the rest of the proof depends only on the first two chapters.

10.10 Without the essentially small hypotheses, the fibre functor may only exist after the ambient universe has been enlarged.

Notes Theorem 10.1 and its proof are from a letter of Deligne, dated November 30, 2011. Apparently, they were known to Deligne earlier (see the sentence following 0.6 in Deligne 2002).

Appendix A

Categories and 2-Categories

We review some basic definitions and statements. Throughout, 2-category means strict 2-category and bicategory means weak 2-category.

Equivalence of categories

A.1 Let C and D be categories. An ***equivalence*** of C and D is a system

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}, \quad \eta\colon \mathrm{id}_{\mathsf{C}} \to GF, \quad \epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}, \tag{84}$$

where F and G are functors and η and ϵ are invertible natural transformations. We call a functor $F\colon \mathsf{C} \to \mathsf{D}$ an ***equivalence*** if it can be extended to such a system. A functor F is an equivalence if and only if it is fully faithful and essentially surjective (i.e., every object of D is isomorphic to an object in the image of F).

In practice, these conditions may be too strong. For example, we shall need to consider functors $F\colon \mathsf{C} \to \mathsf{D}$ such that the objects of D are only "equivalent" to objects in the image. To be able talk about such functors, we need morphisms of morphisms. In other words, we need 2-categories.[1] But first, we review adjoint functors.

Review of adjoint functors

Let C and D be categories.

A.2 An ***adjunction*** is a triple (F, G, ϕ) consisting of a pair of functors

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \tag{85}$$

and a family of bijections

$$\phi_{X,Y}\colon \ \mathrm{Hom}_{\mathsf{D}}(FX, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, GY),$$

natural in $X \in \operatorname{ob}\mathsf{C}$ and $Y \in \operatorname{ob}\mathsf{D}$. We call (F, G) an ***adjoint pair***, with F the ***left adjoint*** of G and G the ***right adjoint*** of F . There are natural transformations

$$\begin{aligned} \eta\colon \ \mathrm{id}_{\mathsf{C}} \to GF, \quad & \eta_X = \phi(\mathrm{id}_{FX})\colon \ X \to GFX \\ \epsilon\colon \ FG \to \mathrm{id}_{\mathsf{D}}, \quad & \epsilon_Y = \phi^{-1}(\mathrm{id}_{GY})\colon \ FGY \to Y, \end{aligned}$$

[1]If a functor of 1-categories is an isomorphism of categories, then it is surjective on objects; if an equivalence, then it is surjective on isomorphism classes of objects; if part of a biequivalence of 2-categories, then it is surjective on equivalence classes of objects.

called the ***unit*** and ***counit*** of the adjunction, satisfying the ***triangle identities***,

$$\begin{aligned}(F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F) &= \mathrm{id}_F \\ (G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G) &= \mathrm{id}_G\,.\end{aligned} \tag{86}$$

See A.9 for the notation $F\eta$ and ϵF.

A.3 Let F and G be functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$, and let $\eta\colon \mathrm{id}_{\mathsf{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}$ be natural transformations satisfying the triangle identities. Then the map

$$\phi_{X,Y}\colon \mathrm{Hom}_{\mathsf{D}}(FX,Y) \to \mathrm{Hom}_{\mathsf{C}}(X,GY), \quad f \mapsto Gf \circ \eta_X,$$

is natural in X and Y, and has inverse

$$\psi_{X,Y}\colon \mathrm{Hom}_{\mathsf{C}}(X,GY) \to \mathrm{Hom}_{\mathsf{D}}(FX,Y), \quad f \mapsto \epsilon_X \circ Ff.$$

In particular, $\phi_{X,Y}$ is a bijection, and so the triple (F,G,ϕ) is an adjunction with unit η and counit ϵ.

A.4 Let F and G be functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$, and let $\epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}$ be a natural transformation. There exists a natural transformation $\eta\colon \mathrm{id}_{\mathsf{C}} \to GF$ such that the triangle identities hold if and only if the map

$$\psi_{X,Y}\colon \mathrm{Hom}_{\mathsf{C}}(X,GY) \to \mathrm{Hom}_{\mathsf{D}}(FX,Y), \quad f \mapsto \epsilon_X \circ Ff,$$

is bijective for all X in C and Y in D, in which case (F,G,ψ^{-1}) is an adjunction with η and ϵ as its unit and counit. The natural transformation η is unique if it exists.

A.5 An equivalence (F,G,η,ϵ) is an ***adjoint equivalence*** if it satisfies the triangle identities (A.2). Any equivalence can be made into an adjoint equivalence by replacing either one of η or ϵ. Indeed, the composite

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

is an isomorphism, but it need not be the identity. Modifying either ϵ or η to absorb the relevant inverse automorphism gives an adjoint equivalence (Riehl 2016, 4.4.5).

A.6 When the categories are abelian, every adjunction preserves the abelian group structure on the Hom-sets. Let C and D be abelian categories with small inductive limits, and assume that small filtered inductive limits in C are exact and that C admits a generator. Then an additive functor $F\colon \mathsf{C} \to \mathsf{D}$ has a right adjoint if and only if it preserves small inductive limits (special adjoint functor theorem). An additive functor between abelian categories preserves small inductive limits if and only if is right exact and preserves small direct sums.

2-categories

A 2-category is a category enriched over the category of small categories equipped with its cartesian monoidal structure (I, 1.9). When we forget the arrows in the Hom-categories, we get a category in the usual sense. Thus, a 2-category can be viewed as a category in the usual sense enriched with morphisms between morphisms.

DEFINITION A.7 A 2-***category*** has objects, 1-morphisms, and 2-morphisms. The objects and 1-morphisms form a category in the usual sense.

For each pair A, B of objects, there is a small category $\mathsf{Hom}(A, B)$ having the 1-morphisms $f\colon A \to B$ as objects and the 2-morphisms[2]

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \tag{87}$$

as morphisms (from f to g). Composition of morphisms in the category $\mathsf{Hom}(A, B)$ is called ***vertical composition***,

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & B. \end{tikzcd}$$

For each triple A, B, C of objects, there is a ***horizontal composition***

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "i\circ g"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & C. \end{tikzcd}$$

Finally, horizontal composition is functorial with respect to vertical composition. This means that the horizontal composite of identity 2-morphisms is an identity 2-morphism,

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_f"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_h"] & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "h\circ f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{h\circ f}"] & C, \end{tikzcd}$$

and the system satisfies the middle four interchange law: given a diagram,

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \arrow[r, bend left=60, "i"] \arrow[r, "j" description] \arrow[r, bend right=60, "k"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & C, \end{tikzcd}$$

the 2-morphism $i\circ f \Rightarrow k\circ h$ obtained by first composing vertically and then horizontally equals that obtained by first composing horizontally and then vertically,

$$(\delta\circ\gamma) * (\beta\circ\alpha) = (\delta * \beta)\circ(\gamma * \alpha).$$

See Riehl and Verity 2022, B.1.1.

[2]We sometimes use the symbol $\Rightarrow$ to denote 2-morphisms.

A.8 When A and B are objects of a 2-category $\mathscr{C}$, we sometimes write $\mathscr{C}(A,B)$ for the 1-category $\mathsf{Hom}(A,B)$. From horizontal composition we get

(a) a bifunctor

$$c_{ABC}\colon \mathscr{C}(A,B)\times\mathscr{C}(B,C)\to\mathscr{C}(A,C),$$

which is associative in the sense that

$$\begin{tikzcd}
\mathscr{C}(A,B)\times\mathscr{C}(B,C)\times\mathscr{C}(C,D) \arrow[r, "1\times c"] \arrow[d, "c\times 1"] & \mathscr{C}(A,B)\times\mathscr{C}(B,D) \arrow[d, "c"] \\
\mathscr{C}(A,C)\times\mathscr{C}(C,D) \arrow[r, "c"] & \mathscr{C}(A,D)
\end{tikzcd}$$

commutes for all objects A,B,C,D, and

(b) functors $u_A\colon \mathbb{1}\to\mathcal{C}(A,A)$, where $\mathbb{1}$ is the category with one object and one morphism, satisfying

$$c_{AAB}\circ(u_A\times 1) = 1 = c_{ABB}\circ(1\times u_B)$$

Conversely, to give a 2-category amounts to giving a class of objects, small categories $\mathscr{C}(A,B)$ for each pair of objects A and B, identity functors u_A for each object A, and composition functors satisfying (a) and (b) for each triple of objects A, B, and C (Borceux 1994a, 7.1.1).

When $\mathscr{C}$ is a 2-category, we write $\mathscr{C}_0$ for the collection of objects, $\mathscr{C}_1$ for the collection of 1-morphisms, and $\mathscr{C}_2$ for the collection of 2-morphisms. A 2-category $\mathscr{C}$ is ***small*** if $\mathscr{C}_0$ is a set.[3]

To distinguish them, we sometimes call categories in the usual sense 1-categories.

From a 2-category, we get a 1-category by forgetting the 2-morphisms. Conversely, a 1-category can be turned into a 2-category by regarding each Hom-set as a discrete category (i.e., the only morphisms are the identity morphisms).

EXAMPLE A.9 The 2-category $\mathscr{C}\!at$ has as objects the small categories, as 1-morphisms the functors between categories, and as 2-morphisms the natural transformations between functors.

Let F,G, and H be functors $\mathsf{A}\to\mathsf{B}$. The vertical composite of natural transformations $\alpha\colon F\Rightarrow G$ and $\beta\colon G\Rightarrow H$ is the natural transformation $\beta\circ\alpha\colon F\Rightarrow H$ such that

$$(\beta\circ\alpha)_a = \beta_a\circ\alpha_a\colon F(a)\to H(a)$$

for all objects a of A.

Let F,G be functors $\mathsf{A}\to\mathsf{B}$ and H,I functors $\mathsf{B}\to\mathsf{C}$. The horizontal composite of natural transformations $\alpha\colon F\Rightarrow G$ and $\beta\colon H\Rightarrow I$,

$$\begin{tikzcd}
\mathsf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C},
\end{tikzcd}$$

[3]Because we are requiring the Hom-categories to be small, this agrees with the usual definition, which requires that the objects form a set and the Hom-categories be small.

is the natural transformation $\beta * \alpha\colon H\circ F \to I\circ G$ such that $(\beta * \alpha)_a$ is the diagonal map in the commutative square

$$\begin{tikzcd}
(H\circ F)(a) \arrow[r, "\beta_{F(c)}"] \arrow[d, "H(\alpha_a)"] & (I\circ F)(a) \arrow[d, "I(\alpha_a)"] \\
(H\circ G)(a) \arrow[r, "\beta_{G(c)}"] & (I\circ G)(a)
\end{tikzcd}$$

for all objects a of A. For example, from

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "F"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C},
\end{tikzcd}$$

we get a natural transformation $\beta F\colon HF \to IF$, namely, $\beta * \mathrm{id}_F$, and from

$$\begin{tikzcd}
A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "H"] & C
\end{tikzcd}$$

we get a natural transformation $H\alpha\colon H\circ F \to H\circ G$. With this notation,

$$\beta * \alpha = (\beta G)\circ(H\alpha) = (I\alpha)\circ(\beta F).$$

See Riehl 2016, 1.7.

We can now make the following definition.

DEFINITION A.10 Let A and B be objects in a 2-category. An ***equivalence*** between A and B is a system

$$A \underset{g}{\overset{f}{\rightleftarrows}} B, \qquad \eta\colon \mathrm{id}_A \Rightarrow g\circ f, \qquad \epsilon\colon f\circ g \Rightarrow \mathrm{id}_B,$$

where f and g are 1-morphisms and η and ϵ are 2-isomorphisms (invertible 2-morphisms). We call a 1-morphism $f\colon A \to B$ an ***equivalence*** if it can be extended to such a system, and we say that A and B are ***equivalent***, denoted $A \approxeq B$, when there exists such an f.

A.11 An equivalence (f, g, η, ϵ) is an ***internal*** or ***adjoint equivalence*** if it satisfies the triangle identities (A.2). As in A.5, any equivalence can be made into an adjoint equivalence by changing at most one of η or ϵ (Johnson and Yau 2021, 6.2.4).

A.12 If $A \approxeq A'$ and $B \approxeq B'$, then $\mathsf{Hom}(A, B) \approxeq \mathsf{Hom}(A', B')$. More precisely, from equivalences between A and A' and between B and B', we get an equivalence between $\mathsf{Hom}(A, B)$ and $\mathsf{Hom}(A', B')$.

ASIDE A.13 As a rough rule of thumb, in a 1-category, objects can be considered to be the "same" if they are isomorphic, and parallel morphisms if they are equal. In a 2-category, objects can be considered to be the "same" if they are equivalent, parallel 1-morphisms if they are isomorphic, and parallel 2-morphisms if they are equal.

NOTES Because we require the Hom categories to be small, what we call a 2-category is called a locally small 2-category in Johnson and Yau 2021, 2.3.9. There are also "nonstrict" 2-categories, called bicategories (ibid., 2.1.3).

Bicategories

The definition of a ***bicategory*** is the same as that of a 2-category except that the conditions (a) and (b) (the associativity and unit axioms) hold only up to coherent invertible 2-morphisms. More precisely, the associativity and unit axioms hold up to natural isomorphisms, which are thus part of the data of a bicategory, and which themselves satisfy certain coherence conditions. See Borceux 1994a, 7.7.

Every bicategory is biequivalent to a 2-category (Johnson and Yau 2021, 8.4.1), but bicategories also arise naturally. In this work, we only use 2-categories, but we need to consider biequivalences in addition to 2-equivalences.

EXAMPLE A.14 Every monoidal category $(\mathsf{C}, \otimes)$ can be thought of as bicategory with one object $*$. The 1-morphisms are the objects of C and the 2-morphisms are the morphisms of C. Horizontal composition is given by tensor product,

$$\begin{tikzcd} * \arrow[r, bend left, "W"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & * \arrow[r, bend left, "Y"] \arrow[r, bend right, "Z"'] \arrow[r, phantom, "\Downarrow\beta"] & * \end{tikzcd} \rightsquigarrow \begin{tikzcd} * \arrow[r, bend left, "Y\otimes W"] \arrow[r, bend right, "Z\otimes X"'] \arrow[r, phantom, "\Downarrow\beta\otimes\alpha"] & * \end{tikzcd}.$$

Note that $X \otimes (Y \otimes Z)$ is not equal to $(X \otimes Y) \otimes Z$, but there is a natural isomorphism $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ satisfying certain coherence conditions.

2-functors and 2-equivalence

A 2-functor between 2-categories assigns objects to objects, 1-morphisms to 1-morphisms, and 2-morphisms to 2-morphisms, preserving the 2-category structure strictly.[4]

DEFINITION A.15 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-***functor*** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and a family of functors

$$F_{A,B} : \mathsf{Hom}_{\mathscr{C}}(A,B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B),$$

indexed by the pairs of objects A, B in $\mathscr{C}$, satisfying the following conditions,

(a) F is a functor between the underlying 1-categories of $\mathscr{C}$ and $\mathscr{D}$,

(b) F preserves horizontal compositions of 2-morphisms.

The conditions (a) and (b) require that the diagrams

$$\begin{tikzcd} \mathscr{C}(A,B)\times\mathscr{C}(B,C) \arrow[r, "c"] \arrow[d, "F_{A,B}\times F_{B,C}"] & \mathscr{C}(A,C) \arrow[d, "F_{A,C}"] \\ \mathscr{D}(FA,FB)\times\mathscr{D}(FB,FC) \arrow[r, "c"] & \mathscr{D}(FA,FC) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "u_A"] \arrow[dr, "u_{FA}"'] & \mathscr{C}(A,A) \arrow[d, "F_{A,A}"] \\ & \mathscr{D}(FA,FA) \end{tikzcd} \tag{88}$$

commute for all objects A, B, C in $\mathscr{C}$.

See Johnson and Yau 2021, 4.1.8

DEFINITION A.16 Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be a parallel pair of 2-functors of 2-categories. A 2-***(natural transformation)*** $\alpha : F \to G$ is a family of 1-morphisms

$$\alpha_A : FA \to GA$$

indexed by the objects A of $\mathscr{C}$ such that

[4] What we call 2-functor, 2-(natural transformation), etc. are often called strict 2-functor, strict 2-(natural transformation), etc.

(a) for each 1-morphism $f: A \to B$ in $\mathscr{C}$ the diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"] & GB \end{tikzcd}$$

commutes in $\mathscr{D}$, and

(b) for each 2-morphism $\theta: f \Rightarrow g$ in $\mathsf{Hom}_{\mathscr{C}}(A,B)$, the diagram

$$\begin{tikzcd} (Gf)\circ\alpha_A \arrow[r, equal] \arrow[d, "(G\theta)*\mathrm{id}_{\alpha_A}"] & \alpha_B\circ(Ff) \arrow[d, "\mathrm{id}_{\alpha_A}*(G\theta)"] \\ (Gg)\circ\alpha_A \arrow[r, equal] & \alpha_B\circ(Fg) \end{tikzcd}$$

commutes in $\mathsf{Hom}_{\mathscr{D}}(FA, GB)$.

Ibid., 4.2.11.

DEFINITION A.17 A 2-(natural transformation) $\alpha: F \to G$ is a 2-***(natural isomorphism)*** if there exists a 2-(natural transformation) $\beta: G \to F$ such that $\beta\alpha = \mathrm{id}_F$ and $\alpha\beta = \mathrm{id}_G$.

DEFINITION A.18 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-***equivalence*** of $\mathscr{C}$ and $\mathscr{D}$ is a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}, \quad \eta: \mathrm{id}_{\mathscr{C}} \to GF, \quad \epsilon: FG \to \mathrm{id}_{\mathscr{D}},$$

where F and G are 2-functors and η and ϵ are 2-(natural isomorphisms). We call a 2-functor $F: \mathscr{C} \to \mathscr{D}$ a 2-***equivalence*** if it can be extended to such a system.

A 2-equivalence between 2-categories is precisely the same as a strict $\mathscr{C}at$-enriched equivalence. Ibid., 1.3.11, 6.2.12.

THEOREM A.19 (WHITEHEAD THEOREM FOR 2-EQUIVALENCES) *Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-functor $F: \mathscr{C} \to \mathscr{D}$ is a 2-equivalence if and only if*

(a) *the underlying functor on the 1-categories is an equivalence, and*

(b) *F is fully faithful on 2-morphisms.*

PROOF See Johnson and Yau 2021, Theorem 7.5.8 and Definition 7.0.1. □

Condition (b) in the theorem means that, for all objects A, B in $\mathscr{C}$, the functor

$$F_{A,B}: \mathsf{Hom}_{\mathscr{C}}(A,B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

is fully faithful.

A.20 The conditions in the theorem are equivalent to

(a) F_0 is surjective on isomorphism classes of objects, and

(b) for all objects A, B in $\mathscr{C}$, the functor

$$F_{A,B}: \mathsf{Hom}_{\mathscr{C}}(A,B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

is an isomorphism of 1-categories.

As the condition (b) suggests, a 2-equivalence is the analogue for 2-categories of an isomorphism of 1-categories. As we explain in the next subsection, the analogue of the more useful notion of an equivalence of 1-categories is that of a biequivalence.

Pseudofunctors and biequivalences

We need to relax some conditions in the last subsection. The notion of a pseudo widget is obtained from that of a widget by allowing certain equalities to be isomorphisms.

DEFINITION A.21 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A ***pseudofunctor*** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and a family of functors

$$F_{A,B} : \mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

indexed by the pairs of objects A, B in $\mathscr{C}$ such that the diagrams (88) commute up to given invertible natural transformations. In detail, there are given invertible natural transformations

$$\begin{aligned} \gamma_{A,B,C} : c\circ(F_{A,B} \times F_{B,C}) &\Rightarrow F_{A,C}\circ c \\ \delta_A : u_{FA} &\Rightarrow F_{A,A}\circ u_A \end{aligned}$$

for all objects A, B, C of $\mathscr{C}$, satisfying the following conditions:

(a) (associativity) for every triple of 1-morphisms in $\mathscr{C}$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

the diagram

$$\begin{array}{ccc} Fh\circ Fg\circ Ff & \xrightarrow{\mathrm{id}_{Fh}*\gamma} & Fh\circ F(g\circ f) \\ \Big\downarrow{\scriptstyle \gamma*\mathrm{id}_{Ff}} & & \Big\downarrow{\scriptstyle \gamma*\mathrm{id}_{Ff}} \\ F(h\circ g)\circ Ff & \xrightarrow{\gamma} & F(h\circ g\circ f) \end{array}$$

commutes;

(b) (unity) for every 1-morphism $f : A \to B$ in $\mathscr{C}$,

$$\gamma_{\mathrm{id}_A,f}\circ(\mathrm{id}_{Ff} * \delta_A) = \mathrm{id}_{Ff} = \gamma_{f,\mathrm{id}_B}\circ(\delta_A * \mathrm{id}_{Ff}).$$

See Borceux 1994a, 7.5.1; Johnson and Yau 2021, 4.1.2).

A.22 Pseudo-functors preserve internal (adjoint) equivalences: if (f, g, η, ϵ) is an internal equivalence and F is a pseudofunctor, then Ff and Fg are adjoint members of an internal equivalence (Johnson and Yau 2021, 6.2.3).

DEFINITION A.23 Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be two pseudofunctors of 2-categories. A ***pseudo natural transformation*** $\alpha : F \Rightarrow G$ is a family of 1-morphisms

$$\alpha_A : FA \to GA$$

indexed by the objects A of $\mathscr{C}$ and a family of invertible 2-morphisms

$$\alpha_f : G(f)\circ\alpha_A \;\Rightarrow\; \alpha_B\circ F(f)$$

indexed by the 1-morphisms f of $\mathscr{C}$. The families are required to satisfy certain coherence conditions, ibid., 4.2.1.

Pseudo natural transformations are sometimes called strong transformations.

DEFINITION A.24 Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be pseudofunctors of 2-categories, and let $\alpha, \beta : F \Rightarrow G$ be pseudo natural transformations. A ***modification*** $\Gamma : \alpha \rightsquigarrow \beta$ is a family of 2-morphisms

$$\Gamma_A : \alpha_A \Rightarrow \beta_A,$$

indexed by the objects of $\mathscr{C}$, such that the diagram

$$\begin{array}{ccc} (Gf)\alpha_A & \xrightarrow{Gf*\Gamma_A} & (Gf)\beta_A \\ \big\downarrow{\scriptstyle \alpha_f} & & \big\downarrow{\scriptstyle \beta_f} \\ \alpha_B(Ff) & \xrightarrow{\Gamma_B*Ff} & \beta_B(Ff) \end{array}$$

commutes in $\mathscr{D}$ (ibid., 4.4.1, 4.4.3).

DEFINITION A.25 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories with $\mathscr{C}$ small. There exists a 2-category $\mathcal{P}s\mathcal{F}un(\mathscr{C}, \mathscr{D})$ with

- objects the pseudofunctors $\mathscr{C} \to \mathscr{D}$,
- 1-morphisms the pseudo natural transformations between such pseudofunctors,
- 2-morphisms the modifications between such pseudo natural transformations.

Ibid., 4.4.13, where the category is denoted $\mathsf{Bicat}^{\mathrm{ps}}(\mathscr{C}, \mathscr{D})$.

DEFINITION A.26 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A ***biequivalence*** of $\mathscr{C}$ and $\mathscr{D}$ is a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}. \qquad \mathrm{id}_{\mathscr{C}} \to GF, \quad FG \to \mathrm{id}_{\mathscr{D}},$$

where F and G are pseudofunctors and $\mathrm{id}_{\mathscr{C}} \to GF$ and $FG \to \mathrm{id}_{\mathscr{D}}$ are pseudo natural transformations that are equivalences as 1-morphisms in $\mathcal{P}s\mathcal{F}un(\mathscr{C}, \mathscr{C})$ and $\mathcal{P}s\mathcal{F}un(\mathscr{D}, \mathscr{D})$ respectively (A.10). We call a pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ a ***biequivalence*** if it can be extended to such a system (ibid., 6.2.8).

A biequivalence is sometimes called an equivalence between 2-categories (it is the correct analogue of an equivalence of 1-categories).

A.27 When $\mathscr{C}$ and $\mathscr{D}$ are small, the internal equivalences in $\mathcal{P}s\mathcal{F}un(\mathscr{C}, \mathscr{C})$ are invertible pseudo natural transformations (ibid., 6.2.7). In this case, we can define a biequivalence of $\mathscr{C}$ and $\mathscr{D}$ to be a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}. \qquad \mathrm{id}_{\mathscr{C}} \to GF, \quad FG \to \mathrm{id}_{\mathscr{D}}$$

with $\mathrm{id}_{\mathscr{C}} \to GF$ and $FG \to \mathrm{id}_{\mathscr{D}}$ invertible pseudo natural transformations. A pseudo natural transformation $F \to G$ is invertible if and only if each component $F(A) \to G(A)$ is an invertible 1-morphism in $\mathscr{D}$ (ibid., 6.2.16).

THEOREM A.28 (WHITEHEAD THEOREM FOR BIEQUIVALENCES) *Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ is a biequivalence if and only if*

(a) *F_0 is surjective on equivalence classes of objects, and*

(b) *for all objects A, B of $\mathscr{C}$, the component functor*

$$F_{A,B} : \mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(F_0 A, F_0 B)$$

is an equivalence of 1-categories.

PROOF See Johnson and Yau 2021, Theorem 7.4.1 and Definition 7.0.1. □

A.29 The conditions (a) and (b) in the theorem are equivalent to the following conditions:

(a) F is surjective on equivalence classes of objects,

(b) F is surjective on isomorphism classes of 1-morphisms (between any two objects),

(c) F is bijective on 2-morphisms (between any two 1-morphisms).

REMARK A.30 (a) When the F in the theorem is a 2-functor, there need not exist a 2-functor G satisfying the conditions in A.26, only a pseudofunctor (ibid., 6.2.11).

(b) When F is a biequivalence, the underlying functor on 1-categories need not be full, faithful, or essentially surjective.

(c) Let $F\colon \mathscr{C} \to \mathscr{D}$ be a 2-functor of 2-categories. If F is a 2-equivalence, then F_0 is surjective on isomorphism classes of objects; if it is a biequivalence, it only required to be surjective on equivalence classes of objects.

NOTES The Whitehead theorems are folklore. They were named by Johnson and Yau 2021 in analogy with the Whitehead theorem in homotopy. They are also proved in Gabber and Ramero 2018, 2.4.30.

Appendix B

Ind categories

Basic definitions

B.1 A set with a partial order $\leq$ is ***directed*** if, for every pair of elements a, b, there exists an element c such that $a, b \leq c$. A category is ***filtered*** if, for every pair i, j of objects, there exists an object k and morphisms $i \to k$, $j \to k$, and for every parallel pair of morphisms $u, v\colon i \rightrightarrows j$ there exists a morphism $w\colon j \to k$ such that $w \circ u = w \circ v$. A directed set can be viewed as a filtered category in an obvious way.

B.2 Let C be an essentially small category. An ind-object in C is a functor $\alpha \rightsquigarrow X_\alpha\colon \mathsf{A} \to \mathsf{C}$, where A a small filtered category. Given two ind-objects $\alpha \rightsquigarrow X_\alpha$ and $\beta \rightsquigarrow Y_\beta$, we set

$$\mathrm{Hom}((X_\alpha),(Y_\beta)) = \varprojlim_\alpha \varinjlim_\beta \mathrm{Hom}(X_\alpha, Y_\beta). \tag{89}$$

In this way, we obtain the category $\mathrm{Ind}\,\mathsf{C}$ of ind-objects of C. The same category is obtained when A is required to be a directed set (Adámek and Rosický 1994, 1.5). The functor ι_{C} sending an object of C to a constant inductive system embeds C as a full subcategory of $\mathrm{Ind}\,\mathsf{C}$. The category $\mathrm{Ind}\,\mathsf{C}$ is essentialy small (Kashiwara and Schapira 2006, 6.1.2).

B.3 The functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim h_{X_\alpha}$$

is an equivalence of $\mathrm{Ind}\,\mathsf{C}$ with the category of functors $\mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$ that are small filtered inductive limits of representable functors. This allows us to identify “$\varinjlim$”X_α with its limit in $\mathsf{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Set})$, and $\mathrm{Ind}\,\mathsf{C}$ with the full subcategory of $\mathsf{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Set})$ of limits of small filtered inductive systems of representable functors.

When C has finite inductive limits, a functor $\mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$ lies in $\mathrm{Ind}\,\mathsf{C}$ if and only if it is left exact and satisfies a certain smallness condition (Kashiwara and Schapira 2006, 6.1.7).

B.4 A functor $F\colon \mathsf{C} \to \mathsf{D}$ has a unique extension to a functor of ind-objects,

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[d, "\iota_{\mathsf{C}}"] & \mathsf{D} \arrow[d, "\iota_{\mathsf{D}}"] \\ \mathrm{Ind}\,\mathsf{C} \arrow[r, "\mathrm{Ind}\,F"] & \mathrm{Ind}\,\mathsf{D}, \end{tikzcd}$$

commuting with small filtered inductive limits. Moreover,

(a) this construction respects composition,

$$\mathrm{Ind}(G\circ F) = \mathrm{Ind}(G)\circ \mathrm{Ind}(F);$$

(b) if F is faithful (resp. fully faithful), then so also in $\mathrm{Ind}\,F$.

See Kashiwara and Schapira 2006, 6.1.9, 6.1.10, 6.1.11.

B.5 When C has finite inductive limits, $\mathrm{Ind}\,\mathsf{C}$ has small inductive limits and the functor $i_C\colon \mathsf{C} \to \mathrm{Ind}\,\mathsf{C}$ is right exact, i.e., commutes with finite inductive limits. If, in addition, C admits finite projective limits, then small filtered inductive limits are exact in $\mathrm{Ind}\,\mathsf{C}$. See Kashiwara and Schapira 2006, 6.1.6, 6.1.18, 6.1.19.

An object X of a category C with filtered inductive limits is ***finitely presented*** if $\mathrm{Hom}(X,-)$ commutes with filtered inductive limits, i.e.,

$$\mathrm{Hom}(X, \varinjlim_{\alpha} Y_\alpha) \simeq \varinjlim_{\alpha} \mathrm{Hom}(X, Y_\alpha)$$

for every functor $\alpha \rightsquigarrow Y_\alpha\colon \mathsf{A} \to \mathsf{C}$ with A filtered.

Proposition B.6 *Let C be a category with finite inductive and finite projective limits. If C is idempotent complete, then the essential image of ι_C consists of the finite presented objects of* $\mathrm{Ind}\,\mathsf{C}$. *Hence, ι_C defines an equivalence*

$$\mathsf{C} \xrightarrow{\sim} (\mathrm{Ind}\,\mathsf{C})^{\mathrm{fp}}.$$

Proof Almost by definition, the objects of C are finitely presented when regarded as objects of $\mathrm{Ind}\,\mathsf{C}$. For the converse, let X be a finitely presented object of $\mathrm{Ind}\,\mathsf{C}$, and write $X = \varinjlim_\alpha X_\alpha$ with X_α in C. Then

$$\mathrm{Hom}(X,X) = \mathrm{Hom}(X, \varinjlim_{\alpha} X_\alpha) \simeq \varinjlim_{\alpha} \mathrm{Hom}(X, X_\alpha).$$

In particular, id_X factors through some X_α,

$$\mathrm{id}_X = (X \xrightarrow{f} X_\alpha \xrightarrow{g} \varinjlim X_\alpha = X).$$

Let $e = f\circ g\colon X_\alpha \to X_\alpha$. Then e is an idempotent endomorphism of X_α in $\mathrm{Ind}\,\mathsf{C}$, hence in C because ι_C is fully faithful. As C is idempotent complete, e splits in C: there exist morphisms $p\colon X_\alpha \to Y$ and $i\colon Y \to X_\alpha$ in C with $p\circ i = \mathrm{id}_Y$ and $i\circ p = e$. The factorization of id_X through X_α identifies X_α with Y, and so X lies in the essential image of ι_C. □

Ind categories of abelian categories

B.7 Let C be an essentially small abelian category. Then $\mathrm{Ind}\,\mathsf{C}$ is an abelian category and the functor $\iota_\mathsf{C}\colon \mathsf{C} \to \mathrm{Ind}\,\mathsf{C}$ is fully faithful and exact. In particular, C is an abelian subcategory of $\mathrm{Ind}\,\mathsf{C}$ (i.e., closed under taking kernels and cokernels). Small inductive limits exist in $\mathrm{Ind}\,\mathsf{C}$, and small filtered inductive limits are exact. See Kashiwara and Schapira 2006, 8.6.5.

B.8 Let C be an essentially small abelian category. If S is any set of objects of C such that every object of C is isomorphic to one in S, then S generates Ind C. To see this, note that every object X of Ind C can be written $X = \varinjlim_{i\in I} X_i$ with the X_i in S and, almost by definition, $\varinjlim_{i\in I} X_i$ is a quotient of $\bigoplus_{i\in I} X_i$. In particular Ind C has a set of generators, and is therefore a Grothendieck category.

Recall that a ***subobject*** of an object X is an equivalence class of monomorphisms with target X. A category is ***well-powered*** if the subobjects of any object form a set, in which case they form a partially ordered set. An object of such a category is ***artinian*** (resp. ***noetherian***) if the set of its subobjects satisfies the descending chain condition (resp. ascending chain condition).

PROPOSITION B.9 *Let* C *be an essentially small abelian category whose objects are noetherian.*

(a) *Every object X of* Ind C *is of the form "$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is filtered inductive system in* C *whose transition morphisms are monomorphisms.*

(b) *The category* C *is stable under subquotients in* Ind C, *i.e., if X is in* C *and Z is a subquotient of X in* Ind C, *then Z is in* C.

(c) *Let X ="$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is an inductive system as in (a). Each X_α is a subobject of X, and the set of X_α is cofinal in the collection of all subobjects of X.*

(d) *An object of* Ind C *is finitely presented if and only if it is noetherian.*

PROOF (a). Let X ="$\varinjlim$"X_α. We may suppose that the indexing category is a directed set (B.2). For each α, the kernels of the morphisms $X_\alpha \to X_\beta$, $\beta > \alpha$, form an increasing system of subobjects of X_α, which is stationary because X_α is noetherian. Let $K_\alpha = \mathrm{Ker}(X_\alpha \to X_\beta)$ for all β sufficiently large. The canonical morphism

$$\text{“}\varinjlim\text{”} X_\alpha \to \text{“}\varinjlim\text{”}(X_\alpha/K_\alpha)$$

is an isomorphism, and the inductive system $(X_\alpha/K_\alpha)_\alpha$ satisfies (a).

(b) Because C is an abelian subcategory of Ind C, quotients in C agree with quotients in Ind C. Therefore, C is stable under subquotients if it is stable under subobjects. Let $Y \in \mathrm{ob}\,\mathsf{C}$, and let X be a subobject of Y in Ind C. Write X ="$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is an inductive system as in (a). For each α, the inductive limit of the monomorphisms $X_\alpha \to X_\beta$, $\beta > \alpha$, is a monomorphism $X_\alpha \to X$ (exactness of inductive limits in Ind C; see B.7). Thus the X_α are subobjects of Y. As Y is noetherian, they form a stationary system, and so $X = X_\alpha \in \mathrm{ob}\,\mathsf{C}$ for α sufficiently large.

(c) As in the proof of (b), the morphisms $X_\alpha \to X$ are monic, and so the X_α are subobjects of X. For any other subobject Z of X in C, we have

$$\mathrm{Hom}(Z,X) = \varinjlim_{\alpha} \mathrm{Hom}(Z,X_\alpha),$$

and so $Z \to X$ factors through X_α for some α.

(d) Every finitely presented object of Ind C is in the essential image of $\iota_{\mathsf{C}}\colon \mathsf{C} \to \mathrm{Ind}\,\mathsf{C}$ (see B.6), hence is noetherian.

For the converse, let N be a noetherian object of Ind C and $(X_\alpha)_\alpha$ a filtered inductive sytem in Ind C. We have to show that the canonical map

$$\varinjlim_{\alpha} \mathrm{Hom}(N,X_\alpha) \to \mathrm{Hom}(N,\varinjlim_{\alpha} X_\alpha) \qquad (*)$$

is bijective. Let $X = \varinjlim_{\alpha} X_\alpha$.

For the surjectivity, let f be a morphism $N \to X$, and consider the diagram

$$\begin{tikzcd} N \arrow[r, "f"] & X \\ N \times_X X_\alpha \arrow[u] \arrow[r, "f_\alpha"] & X_\alpha. \arrow[u, "i_\alpha"'] \end{tikzcd}$$

Because filtered inductive limits are exact in $\operatorname{Ind} \mathsf{C}$, $f = \varinjlim_{\alpha} f_\alpha$. Clearly,

$$\operatorname{Im}(f) \subset \bigcup_\alpha \operatorname{Im}(i_\alpha \circ f_\alpha) \quad \text{(filtered union of subobjects),}$$

and so $\operatorname{Im}(f) \subset \operatorname{Im}(i_\alpha \circ f_\alpha)$ for some α (because N, hence $\operatorname{Im}(f)$, is noetherian). For such an α, f factors through X_α, and so arises from an element of $\varinjlim \operatorname{Hom}(N, X_\alpha)$. Hence (*) is surjective.

For the injectivity, let $(f_\alpha)_\alpha$ be an inductive system of morphisms $N \to X_\alpha$ such that $\varinjlim_{\alpha} f_\alpha = 0$. Because $\varinjlim_{\alpha} f_\alpha = 0$, the filtered union $\bigcup_\alpha \operatorname{Ker}(f_\alpha) = N$, and because N is noetherian, $\operatorname{Ker}(f_\alpha) = N$ for some α. This means that $(f_\alpha)_\alpha = 0$. Hence (*) is injective. □

PROPOSITION B.10 *Let* D *be an abelian category and* C *a full subcategory stable under finite sums and subquotients in* D. *Assume that* C *is essentially small and that* D *is well-powered. If*

(a) *all objects of* C *are noetherian,*

(b) *small filtered inductive limits exist in* D *and are exact, and*

(c) *every object of* D *is a small filtered inductive limit of objects of* C,

then the functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim X_\alpha \colon \operatorname{Ind} \mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an object X *of* D *to the inductive system of its subobjects in* C.

Loosely, we can say that D is the category of ind-objects of C.

PROOF As in the proof of B.9(a),(c), we can deduce from (c) that each X in D is a small filtered inductive limit of subobjects X_α in C. Moreover, the X_α are cofinal in the collection of all subobjects of X in C. To see this, let $Z \in \operatorname{ob} \mathsf{C}$ be a subobject of X; on passing to the limit in

$$0 \to Z \cap X_\alpha \to X_\alpha \to X/Z,$$

we find that the $Z \cap X_\alpha$ have inductive limit Z; as Z is noetherian, they form a stationary system, and so this means that Z is contained in X_α for all sufficiently large α. We have shown that X is the inductive limit of its subobjects in C, and so the composite

$$\mathsf{D} \to \operatorname{Ind} \mathsf{C} \to \mathsf{D}$$

is naturally isomorphic to the identity functor. Conversely, every X in $\operatorname{Ind} \mathsf{C}$ is a “$\varinjlim$”X_α as in B.9(a) and the X_α are cofinal in the set of subobjects in C of $\varinjlim X_\alpha$; therefore the composite

$$\operatorname{Ind} \mathsf{C} \to \mathsf{D} \to \operatorname{Ind} \mathsf{C}$$

is naturally isomorphic to the identity functor. □

We list some examples where Proposition B.10 implies that D is equivalent to $\operatorname{Ind}\mathsf{C}$. Throughout k is a field.

EXAMPLE B.11 Let R be a noetherian ring. Take C to be the category $\mathsf{Modf}(R)$ of finitely generated R-modules and D to be the category $\mathsf{Mod}(R)$ of all R-modules. The conditions of B.10 are obviously satisfied.

EXAMPLE B.12 Let L be a coalgebra over a noetherian ring R. If L is flat over R, then the category $\mathsf{coMod}(L)$ of right L-comodules is abelian. Moreover, every L-comodule is the filtered union of the L-subcomodules finitely generated over R (Serre 1968, Cor. 2). Therefore, $\mathsf{coMod}(L)$ is locally noetherian, and its noetherian objects are those finitely generatd over R.

EXAMPLE B.13 Let G be an affine group scheme over k. Take C to be the category $\mathsf{Repf}(G)$ of representations of G on finite-dimensional k-vector spaces, and D to be the category $\mathsf{Rep}(G)$ of representations of G on arbitrary k-vector spaces. Condition (c) of B.10 says that every representation is the inductive limit of its finite-dimensional subrepresentations (II, 1.17).

EXAMPLE B.14 Take C to be an abelian category whose objects are noetherian, and D to be the category of additive left exact functors from $\mathsf{C}^{\mathrm{opp}}$ to the category of abelian groups.

An abelian category is ***locally noetherian*** if it has small inductive limits, filtered inductive limits are exact, and there exists a family of noetherian generators with small index set.

THEOREM B.15 (GABRIEL 1962) *Let* C *be an essentially small abelian category whose objects are noetherian. There exists a locally noetherian abelian category* D *such that* C *is equivalent to the category of noetherian objects in* D. *Moreover, the conditions determine* D *up to an equivalence preserving* C.

PROOF Indeed, $\operatorname{Ind}\mathsf{C}$ has these properties. □

Extension of scalars

B.16 Let D be a k-linear abelian category with arbitrary small direct sums. For X in D and V a k-vector space, the functor $Y \rightsquigarrow \operatorname{Hom}(V, \operatorname{Hom}(X, Y))$ is representable, and we let $V \otimes_k X$ denote the object in D representing it, so

$$\operatorname{Hom}(V \otimes_k X, Y) \simeq \operatorname{Hom}(V, \operatorname{Hom}(X, Y)).$$

If $(e_i)_{i \in I}$ is a basis for V (possibly infinite), then $V \otimes_k X$ is a direct sum of copies of X indexed by I.

B.17 Let k be a field and k' an extension of k. Suppose that we are given a pair of k-linear abelian categories $\mathsf{C} \subset \mathsf{D}$ satisfying the conditions of Proposition B.10, a pair of k'-linear abelian categories $\mathsf{C}' \subset \mathsf{D}'$ satisfying the same conditions, and an adjoint pair of k-linear functors

$$\mathsf{D} \underset{\text{restriction}}{\overset{\text{extension}}{\rightleftarrows}} \mathsf{D}'$$

(extension of scalars, restriction of scalars). We have a diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[d, hook] & \mathsf{C}' \arrow[d, hook] \\ \mathsf{D} \arrow[r, shift left, "\text{extension}"] & \mathsf{D}' \arrow[l, shift left, "\text{restriction}"] \end{tikzcd}$$

The functor restriction of scalars induces a functor

$$\mathsf{D}' \to \{\text{object } X \text{ of } \mathsf{D} \text{ together with a } k\text{-linear } k'\text{-module structure}\}. \tag{90}$$

Let X be an object of D equipped with a k-linear k'-module structure. Such a structure can be interpreted as a morphism $k' \otimes_k X \to X$ such that certain diagrams commute.We say that $Y \subset X$ generates X as a k'-module if the composite

$$k' \otimes_k Y \to k' \otimes_k X \to X$$

is an epimorphism, i.e., any k'-submodule of X containing Y equals X. We let $\mathsf{C}_{(k')}$ denote the category of objects of $\mathsf{D} \approxeq \operatorname{Ind} \mathsf{C}$ equipped with a k-linear k'-module structure and generated as a k'-module by a subobject in C.

PROPOSITION B.18 *In the above situation, suppose*

(a) *that the functor (90) is an equivalence;*

(b) *the extension of scalars* $\mathsf{D} \to \mathsf{D}'$ *sends* C *into* C'.

Then the functor restriction of scalars induces an equivalence $\mathsf{C}' \to \mathsf{C}_{(k')}$.

PROOF Identify D with $\operatorname{Ind} \mathsf{C}$, and, by (90), D' with the category of Ind-objects of C equipped with a k'-module structure. With these identification, the restriction of scalars functor becomes the forgetful functor. Its left adjoint (extension of scalars) becomes $X \rightsquigarrow k' \otimes_k X$. It remains to determine C'.

If $X \in \operatorname{ob} \mathsf{D}'$ is generated as a k'-module by a D-subobject $Y \subset X$ with $Y \in \operatorname{ob} \mathsf{C}$, then it is a quotient of $k' \otimes_k Y$ and is therefore in C'. Conversely, if $X \in \operatorname{ob} \mathsf{C}'$, we write it in D as a filtered union of its subobjects X_α in C. Since X is noetherian in C', the system of submodule images of the $k' \otimes_k X_\alpha$ in X is stationary. Therefore X is a quotient of $k' \otimes_k X_\alpha$ for α sufficiently large, and so lies in $\mathsf{C}_{(k')}$. □

EXAMPLE B.19 We list some examples where Proposition B.18 implies that $\mathsf{C}' \to \mathsf{C}_{(k')}$.

(a) Let X be a scheme of finite type over k and X' the scheme over k' deduced from X by extension of scalars. Take C and C' to be the categories of coherent sheaves on X and X' and D and D' the categories of quasi-coherent sheaves. In this case, "restriction of scalars" is direct image by the morphism $X' \to X$.

(b) Let G be an algebraic group over k and G' the group scheme over k' deduced from G by extension of scalars. Take $\mathsf{C} = \mathsf{Repf}(G)$ and $\mathsf{C}' = \mathsf{Repf}(G')$, and let D and D' be as in B.13.

Pro categories

B.20 On reversing the arrows, we get the notion of a pro-object. A pro-object in a category C is a small filtered projective system $\alpha \rightsquigarrow X_\alpha$, and

$$\mathrm{Hom}((X_\alpha),(Y_\beta)) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Hom}(X_\alpha, Y_\beta).$$

If C is abelian, so also is $\mathrm{Pro}\,\mathsf{C}$, and the canonical functor $\mathsf{C} \to \mathrm{Pro}\,\mathsf{C}$ is exact. The opposites (duals) of Propositions B.9 and B.10 hold for abelian categories C whose objects are artinian.

EXAMPLE B.21 Let $\mathsf{C} = \mathsf{Vecf}(k)$, the category of finite-dimensional vector spaces over k. Then $\mathrm{Ind}\,\mathsf{C}$ is the category of all vector spaces over k and $\mathrm{Pro}\,\mathsf{C}$ is the category of linearly compact vector spaces over k. Duality is an antiequivalence betuween $\mathrm{Ind}\,\mathsf{C}$ and $\mathrm{Pro}\,\mathsf{C}$.

EXAMPLE B.22 Let k be a field. Take C to be the category of algebraic group schemes over k, and D to be the category of all affine group schemes over k. The functor

$$\text{“}\varprojlim\text{”} G_\alpha \rightsquigarrow \varprojlim G_\alpha : \ \mathrm{Pro}\, C \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an affine group scheme G to the projective system of its algebraic quotients (the system $(G/N)_N$, where runs over the normal subgroup schemes of G such that G/N is algebraic). Every affine group scheme over k is the limit of a filtered projective system of algebraic group schemes over k whose transition morphisms are faithfully flat.

If $G = \varprojlim G_\alpha$, we view its Lie algebra

$$\mathrm{Lie}(G) = \varprojlim \mathrm{Lie}(G_\alpha)$$

as a pro Lie algebra. If $G = \mathrm{Spec}(A)$ and I is ideal of $f \in A$ zero at the origin, the linearly compact vector space underlying $\mathrm{Lie}\, G$ is the dual of I/I^2.

NOTES The exposition in this section is partly based on Deligne 1989 §4.

Appendix C

Nonabelian cohomology

We review some definitions from Giraud 1971. Throughout, we work with affine schemes. This allows us to sheafify (basically bounded) presheaves for the fpqc topology without having to pass to a larger universe.

1 Fibred categories

Let A be a small category and $\phi\colon \mathsf{F} \to \mathsf{A}$ a functor. For an object S of A, the ***fibre*** of F at S is the category F_S whose objects are the X in F such that $\phi(X) = S$ and whose arrows are the f in F such that $\phi(f) = \mathrm{id}_S$. For an arrow $a\colon \phi(Y) \to \phi(X)$, we let $\mathrm{Hom}_a(Y,X)$ denote the set of $f\colon Y \to X$ such that $\phi(f) = a$; also $\mathrm{Hom}_T(Y,X) = \mathrm{Hom}_{\mathrm{id}_T}(Y,X)$.

Let $a\colon T \to S$ be an arrow in A and let $f \in \mathrm{Hom}_a(Y,X)$. We say f is ***cartesian***, and that (Y,f) is an ***inverse image*** X relative to a, written $Y = a^*X$, if, for every $Z \in \mathrm{ob}\,\mathsf{F}_T$ and $g \in \mathrm{Hom}_a(Z,X)$, there exists a unique $h \in \mathrm{Hom}_T(Z,Y)$ such that $f \circ h = g$:

$$\begin{tikzcd} Z \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ Y \arrow[r, "f"] & X \\ T \arrow[r, "a"] & S. \end{tikzcd}$$

In other words, if composition with f is an isomorphism $\mathrm{Hom}_{\mathrm{id}_T}(Z, a^*X) \simeq \mathrm{Hom}_a(Z,X)$, natural in $Z \in \mathrm{ob}\,\mathsf{F}_T$. When it exists, the inverse image is uniquely determined up to a unique isomorphism.

The functor $\phi\colon \mathsf{F} \to \mathsf{A}$ is a ***fibred category*** [1] if

(a) (existence of inverse images) for every arrow $a\colon T \to S$ in A and $X \in \mathrm{ob}(\mathsf{F}_S)$, an inverse image a^*X exists, and

(b) (transitivity of inverse images) given arrows $U \xrightarrow{b} T \xrightarrow{a} S$ in A and $X \in \mathrm{ob}\,\mathsf{F}_S$, $b^*(a^*X)$ is an inverse image of X relative to $a \circ b$.

When we choose an inverse image a^*X of each X in F_S, a^* becomes a functor $F_S \to F_T$.

Let $\phi\colon \mathsf{F} \to \mathsf{A}$ and $\phi'\colon \mathsf{F}' \to \mathsf{A}$ be fibred categories over A. A functor $u\colon \mathsf{F} \to \mathsf{F}'$ such that $\phi' \circ u = \phi$ is ***cartesian*** if it preserves cartesian morphisms, When u, u' are

[1] Literally, a category furnished with fibres. Also called a (Grothendieck) ***fibration*** over A.

cartesian functors $\mathsf{F} \to \mathsf{F}'$, a ***vertical natural transformation*** is a natural transformation $m: u \to u'$ such that $\mathrm{id}_{\phi'} * m = \mathrm{id}_\phi$,

$$\begin{tikzcd} \mathsf{F} \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow m"] & \mathsf{F}' \arrow[r, bend left, "\phi'"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\phi'}"] & \mathsf{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathsf{F} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_\phi"] & \mathsf{A}. \end{tikzcd}$$

We write $\mathsf{Cart}(\mathsf{F}, \mathsf{F}')$ for the category whose objects are the cartesian functors $\mathsf{F} \to \mathsf{F}'$ and whose morphisms are the vertical natural transformations.

There is a 2-category $\mathcal{Fib}(\mathsf{A})$ whose objects are the fibred categories $\phi: \mathsf{F} \to \mathsf{A}$ with F small, whose 1-morphisms are the cartesian functors, and whose 2-morphisms are the vertical natural transformations (Johnson and Yau 2021, 9.1.18).

A ***cleavage*** of a fibred category $\phi: \mathsf{F} \to \mathsf{A}$ is the choice of a cartesian lifting $Y \to X$ for every arrow $a \to b$ in A and X with $\phi(X) = a$. A cleavage is a ***splitting*** if it contains the identity morphisms and is closed under composition. A ***split*** fibred category is one equipped with a splitting. In general, a fibred category does not admit a splitting, but it is always equivalent to one that does (Vistoli 2005, 3.14, 3.45).

Let $\phi: \mathsf{F} \to \mathsf{A}$ be a fibred category. When we choose a cleavage, every morphism $a: T \to S$ in A defines a functor $a^*: \mathsf{F}_S \to \mathsf{F}_T$, and by universality, there are functorial isomorphisms

$$\mathrm{id}_S \to \mathrm{id}_S^*, \quad b^* \circ a^* \to (a \circ b)^*$$

for all objects S of A and composable morphisms a, b in A. In this way, we get a pseudofunctor $\mathsf{A}^{\mathrm{opp}} \to \mathcal{Cat}$. Every such pseudofunctor arises from a fibred category, and every fibred category over A is isomorphic to one defined by a pseudofunctor. More precisely, there are a canonical 2-equivalences of 2-categories

$$\mathcal{Fib}(\mathsf{A}) \xleftarrow{\text{forget}} \mathcal{Fib}^{\mathrm{cl}}(\mathsf{A}) \longrightarrow \mathcal{PsFun}(\mathsf{A}^{\mathrm{opp}}, \mathcal{Cat})$$

under which split fibred categories correspond to 2-functors (Grothendieck's construction; Johnson and Yau 2021, 9.1.9, 10.1.11, 10.6.16). Here $\mathcal{Fib}^{\mathrm{cl}}(\mathsf{A})$ has objects the fibred categories equipped with a cleavage.

Thus, the theory of fibred categories sometimes allows us to deal with pseudofunctors without having to keep track of the invertible natural transformations that measure the failure of certain diagrams to commute.

NOTES Fibred categories were introduced by Grothendieck in SGA 1 and developed in Giraud 1971. Our definitions are those of Giraud. For more recent accounts, see Vistoli 2005 and Johnson and Yau 2021.

TODO 22 Look at the alternative definitions. aax.

2 Sheaves for the fpqc topology

Let $S = \mathrm{Spec}(R)$ be an affine scheme, and let Aff_S denote the category of affine schemes over S. The ***fpqc topology*** on Aff_S is that for which the coverings are finite surjective families of flat morphisms $U_i \to U$ of affine S-schemes. A ***presheaf of sets***[2] on S is a

[2] In order to be sure that associated sheaves exist, we should consider only basically bounded presheaves; see Waterhouse 1975.

contravariant functor $F: \mathsf{Aff}_S \to \mathsf{Set}$. A ***sheaf of sets*** on S is a presheaf satisfying the sheaf condition: for all coverings $(U_i \to U)_{i\in I}$, the sequence

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I\times I} F(U_i \times U_j)$$

is exact, i.e., the first arrow is the equalizer of the parallel pair. More concretely, a sheaf of sets for the fpqc topology on Aff_S is a functor $F: \mathrm{Alg}_R \to \mathsf{Set}$ such that

(a) $F(R_1 \times R_2) = F(R_1) \times F(R_2)$, and

(b) for any faithfully flat homomorphism $R \to R'$, the arrow $F(R) \to F(R')$ is the equalizer of the parallel pair of arrows $F(R') \rightrightarrows F(R' \otimes_R R')$ defined by $a \mapsto a \otimes 1, 1 \otimes a$.

For any S-scheme X, the functor $T \rightsquigarrow h_X(T) \stackrel{\text{def}}{=} \mathrm{Hom}_S(T, X)$ is a sheaf. Indeed, (a) is obvious, and (b) follows from the exactness of $R \to R' \rightrightarrows R' \otimes_R R'$ (Waterhouse 1979, 13.1).

Let G be a sheaf of groups. A G-***torsor over*** S (or S-***torsor under*** G) is a sheaf P of sets together with a right action of G such that

(a) for all maps $U \to S$ such that $P(U)$ is nonempty, $G(U)$ acts simply transitively on $P(U)$;

(b) for some covering $(U_i \to S)_i$, every set $P(U_i)$ is nonempty.

A ***morphism*** of G-torsors is a morphism of sheaves of sets with an action of G.

3 Stacks (Champs)

Let S be an affine scheme, and let $\phi: \mathsf{F} \to \mathsf{Aff}_S$ be a fibred category over Aff_S. In the following, we implicitly choose a cleavage.

Descent data

Let $a: V \to U$ be a faithfully flat morphism of affine S-schemes, and let $F \in \mathrm{ob}(\mathsf{F}_U)$. A ***descent datum*** on F relative to a is an isomorphism

$$u: \mathrm{pr}_1^*(F) \to \mathrm{pr}_2^*(F)$$

over $V \times_U V$ satisfying the "cocycle" condition

$$\mathrm{pr}_{31}^*(u) = \mathrm{pr}_{32}^*(u) \circ \mathrm{pr}_{21}^*(u)$$

over $V \times_U V \times_U V$,

$$\begin{tikzcd} \mathrm{pr}_1^* F \arrow[r, "\mathrm{pr}_{21}^*(u)"'] \arrow[rr, bend left, "\mathrm{pr}_{31}^*(u)"] & \mathrm{pr}_2^* F \arrow[r, "\mathrm{pr}_{32}^*(u)"'] & \mathrm{pr}_3^* F. \end{tikzcd}$$

Here pr_i is the projection onto the ith factor and pr_{ji} is the projection

$$V \times_U V \times_U V \to V \times_U V$$

onto the (i, j)th factor. With the obvious notion of morphism, the pairs (F, u) form a category $\mathsf{Desc}(V/U)$.

There is a functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ sending an object F of F_U to (a^*F, u) with u the canonical isomorphism

$$\mathrm{pr}_1^*(a^*F) \simeq (a \circ \mathrm{pr}_1)^* F = (a \circ \mathrm{pr}_2)^* F \simeq \mathrm{pr}_2^*(a^*F).$$

Definition

A ***stack*** is a fibred category $\phi : \mathsf{F} \to \mathsf{Aff}_S$ such that, for all faithfully flat morphisms $a : V \to U$ in Aff_S, the functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ is an equivalence of categories.

Explicitly, this means the following:

(a) for an affine S-scheme U and objects F, G in F_U, the functor sending $a : V \to U$ to $\mathrm{Hom}_V(a^*F, a^*G)$ is a sheaf of sets on U (for the fpqc topology);

(b) for every faithfully flat morphism $V \to U$ of affine S-schemes, descent is effective (that is, every descent datum for V/U is isomorphic to the descent datum defined by an object of F_U).

In other words, a fibred category is a stack if both morphisms and objects, given locally for the fpqc topology, patch uniquely to global objects.

The stacks over Aff_S form a 2-category with 1-morphisms the cartesian functors, with the 2-morphisms the vertical natural transformations, and with

$$\mathsf{Hom}(\mathsf{F}, \mathsf{F}') = \mathsf{Cart}(\mathsf{F}, \mathsf{F}').$$

Examples

Let S be an affine scheme.

- There is a fibred category $\phi :$ MOD$\to \mathsf{Aff}_S$ such that MOD$_U$ is the category of $\Gamma(U, \mathcal{O}_U)$-modules. Descent theory shows that this is a stack: if $R \to R'$ is faithfully flat, then $R' \otimes_R -$ is an equivalence from R-modules to R'-modules equipped with a descent datum (Waterhouse 1979, 17.2).
- There is a fibred category $\phi :$ PROJ$\to \mathsf{Aff}_S$ such that PROJ$_U$ is the category of finitely generated projective $\Gamma(U, \mathcal{O}_U)$-modules. Descent theory shows that this is a stack: if $R \to R'$ is faithfully flat, then an R-module M is finitely generated and projective if and only if $R' \otimes_R M$ is.
- There is a fibred category $\phi :$ AFF$\to \mathsf{Aff}_S$ such that AFF$_U = \mathsf{Aff}_U$, i.e., the fibre over U is the category of affine U-schemes. Descent theory shows that this is a stack (Waterhouse 1979, 17.3).
- There is a fibred category $\phi :$ SHGR$\to \mathsf{Aff}_S$ such that SHGR_U is the category of sheaves of groups on U (for the fpqc) topology. This is a split stack, the inverse image of a sheaf being its restriction.
- Let G be a sheaf of groups on S (for the fpqc topology). There is stack $\phi :$ TORS$(G) \to \mathsf{Aff}_S$ such that the fibre over U is the category of right torsors under $G|U$ over U.

Prestacks and the associated stack

A fibred category is a ***prestack*** if it satisfies the condition (a) to be a stack, i.e., for all $F, G \in \mathrm{ob}\,\mathsf{F}_U$, the functor

$$\left(V \xrightarrow{a} U\right) \rightsquigarrow \mathrm{Hom}(a^*F, a^*G)$$

is a sheaf on U for the fpqc topology.

Let $\phi : \mathsf{F} \to \mathsf{Aff}_S$ be a prestack. The ***associated stack*** $\phi : \mathsf{F}' \to \mathsf{Aff}_S$ of F contains it as a full subcategory and is characterized by having the property that every object of

F' is locally in F. For any stack H over Aff_S, the inclusion functor $i: \mathsf{F} \to \mathsf{F}'$ induces an equivalence of categories

$$\mathsf{Hom}(\mathsf{F}', \mathsf{H}) \xrightarrow{\approx} \mathsf{Hom}(\mathsf{F}, \mathsf{H}), \tag{91}$$

compatible with base change. To construct F', we have to add an object over U for each faithfully flat morphism $V \to U$ and object over V equipped with a descent datum. We can do this by defining $\mathrm{ob}\,\mathsf{F}'_U = \varinjlim \mathrm{Desc}(V/U)$, where $V \to U$ runs over a suitably large collection faithfully flat morphisms. See Giraud 1971, II, §2.

For example, let G be a sheaf of groups on S. There is a fibred category $\mathsf{G} \to \mathsf{Aff}_S$ such that the fibre over U is $G|U$ regarded as a category with one object. This is a prestack, but not a stack. The functor $\mathsf{G} \to \mathrm{TORS}$ sending the unique object of G_U to the trivial $G|U$-torsor realizes $\mathrm{TORS}(G)$ as the associated stack of G: G is a full subcategory of $\mathrm{TORS}(G)$ and every object of $\mathrm{TORS}(G)$ is locally in G.

4 Gerbes

Let S be an affine scheme. A ***gerbe*** over S is a stack $\mathsf{G} \to \mathsf{Aff}_S$ such that,

(a) for all U, the category G_U is a groupoid (all morphisms are isomorphisms);

(b) there exists a faithfully flat morphism $U \to S$ such that G_U is nonempty;

(c) any two objects of a fibre G_U are locally isomorphic (i.e., their inverse images under some faithfully flat morphism $V \to U$ of affine S-schemes are isomorphic).

A ***morphism of gerbes*** over S is a morphism of stacks whose domain and codomain are gerbes, and similarly for a morphism of morphisms of gerbes. Thus the gerbes over Aff_k form a 2-category such that

$$\mathsf{Hom}(\mathsf{F}, \mathsf{F}') = \mathsf{Cart}(\mathsf{F}, \mathsf{F}').$$

A gerbe $\mathsf{G}\to \mathsf{Aff}_S$ is ***neutral*** if G_S is nonempty.

Example: torsors

The stack $\mathrm{TORS}(G) \to \mathsf{Aff}_S$ defined by a sheaf G of groups on S is a gerbe. It is neutral because $\mathrm{TORS}(G)_S$ contains the trivial torsor (G acting on itself on the right).

Conversely, let G be a neutral gerbe, and choose a $Q \in \mathrm{ob}(\mathsf{G}_S)$. Then $G \stackrel{\text{def}}{=} \mathcal{A}ut(Q)$ is a sheaf of groups on S, and, for any $a: U \to S$ and $P \in \mathrm{ob}(\mathsf{G}_U)$, the sheaf $\mathcal{I}som(a^*Q, a^*P)$ is a torsor under G over U. The functor

$$P \rightsquigarrow \mathcal{I}som_U(a^*Q, a^*P): \mathsf{G} \to \mathrm{TORS}(G)$$

is an isomorphism of gerbes.

5 Bands (Liens)

In this section, S is an affine scheme equipped with the fpqc topology.

C.1 Roughly speaking, we pass from groups to bands by systematically ignore inner automorphisms. For example, when two objects X and Y are isomorphic, the choice of an isomorphism $X \to Y$ determines an isomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(Y)$, well-defined up to an inner automorphism. In other words, $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ as bands. We make this precise in the category of sheaves of groups on S.

C.2 Let LI_U denote the category whose objects are sheaves of groups on U and whose morphisms $G \to H$ are the sections of the quotient sheaf

$$H\backslash\mathcal{H}om(G,H)/G,$$

where G and H act on $\mathcal{H}om(G,H)$ by inner automorphisms. Note that, as $f\circ \mathrm{inn}_g = \mathrm{inn}_{f(g)}\circ f$,

$$H\backslash\mathcal{H}om(G,H) \simeq H\backslash\mathcal{H}om(G,H)/G.$$

On varying U, we get a fibred category $\mathrm{LI} \to \mathsf{Aff}_S$. It is, in fact, a prestack, and the associated stack

$$\mathrm{LIEN} \to \mathsf{Aff}_S$$

is called the ***stack of bands***. Thus LI is a full subcategory of LIEN, and every object of LIEN is locally in LI. A ***band over*** S is an object of LIEN_S.

C.3 There are obvious morphisms of fibred categories

$$\mathrm{S{\scriptstyle HGR}} \to \mathrm{LI} \to \mathrm{LIEN}$$

over Aff_S. We write $\mathrm{Bd}(H)$ for the band defined by a sheaf of groups H. For sheaves of groups G,H on Aff_S, the functor Bd induces an isomorphism of sheaves of homomorphisms (Giraud 1971, IV, 1.1.7)

$$H\backslash\mathcal{H}om(G,H) \to \mathcal{H}om(\mathrm{Bd}(G),\mathrm{Bd}(H)).$$

Let

$$\mathcal{I}sex(G,H) \overset{\mathrm{def}}{=} H\backslash\mathcal{I}som(G,H) = \mathcal{I}som(\mathrm{Bd}(G),\mathrm{Bd}(H))$$
$$\mathrm{Isex}(G,H) \overset{\mathrm{def}}{=} \Gamma(S,\mathcal{I}sex(G,H)) = \mathrm{Isom}(\mathrm{Bd}(G),\mathrm{Bd}(H)).$$

C.4 Every band B over S is locally the band of a sheaf of groups. This means it comes by descent from a triple (U,G,u), where U is faithfully flat over S, G is a sheaf of groups on U, and $u \in \mathrm{Isex}(\mathrm{pr}_1^* G, \mathrm{pr}_2^* G)$ is such that

$$\mathrm{pr}_{31}^*(u) = \mathrm{pr}_{32}^*(u)\circ \mathrm{pr}_{21}^*(u).$$

If V is also a faithfully flat affine S-scheme, and $a\colon V \to U$ is an S-morphism, then (U,G,u) and $(V,a^*(G),(a\times a)^*(u))$ define the same band. If B_1 and B_2 are the bands defined by (U,G_1,u_1) and (U,G_2,u_2), then an element $\psi \in \mathrm{Isex}(G_1,G_2)$ such that $\mathrm{pr}_2^*(\psi)\circ u_1 = u_2\circ \mathrm{pr}_1^*(\psi)$ defines an isomorphism $B_1 \to B_2$.

C.5 Recall that every sheaf of groups G on S defines a band $\mathrm{Bd}(G)$ over S, and that

$$\mathrm{Isom}(\mathrm{Bd}(G),\mathrm{Bd}(H)) = \mathrm{Isex}(G,H).$$

Thus, $\mathrm{Bd}(G)$ and $\mathrm{Bd}(H)$ are isomorphic if and only if H is an inner form of G, i.e., H becomes isomorphic to G on some faithfully flat S-scheme T, and the class of H in $H^1(S,\mathcal{A}ut(G))$ comes from $H^1(S,\mathrm{Inn}(G))$, where $\mathrm{inn}(G)$ is the image of the homomorphism of sheaves

$$\mathrm{inn}\colon \mathrm{Inn}(G) \to \mathrm{Aut}(G), \quad \mathrm{inn}(g)(x) = gxg^{-1}.$$

C.6 The fibred category LI contains the stack of sheaves of commutative groups as a full subcategory. Thus, we have an equivalence from the stack of sheaves of commutative groups to the stack of commutative bands.

C.7 The ***centre*** $Z(B)$ of the band B defined by (U,G,u) is defined by $(U,Z,u|\,\mathrm{pr}_1^* Z)$, where Z is the centre of G. The restriction $u|\,\mathrm{pr}_1^* Z$ lifts to an element $u_1 \in \mathrm{Isom}(\mathrm{pr}_1^* Z, \mathrm{pr}_2^* Z)$, and one checks immediately that $\mathrm{pr}_{31}^*(u_1) = \mathrm{pr}_{32}^*(u_1)\circ \mathrm{pr}_{21}^*(u_1)$. Thus $(U,Z,u|\,\mathrm{pr}_1^* Z)$ arises from a sheaf of groups on S, which we identify with $Z(B)$.

Affine and algebraic bands

C.8 A band is said to be ***affine*** if it is represented by a triple (U, G, u) with G an affine group scheme over U. When G is algebraic, the band is said to be ***algebraic***

C.9 Let k^{al} be an algebraic closure of k. Later, we shall see that every affine algebraic band over k is defined by a pair (G, u), where G is an algebraic group over k^{al} and u is an isomorphism $\mathrm{pr}_1 G \to \mathrm{pr}_2 G$.

The band of a gerbe

Let G be a gerbe on Aff_S. By definition, there exists an object $Q \in \mathsf{G}_U$ for some U faithfully flat over S. Let $G = \mathcal{Aut}(Q)$; it is a sheaf of groups on U. Again, by definition, $\mathrm{pr}_1^* Q$ and $\mathrm{pr}_2^* Q$ are locally isomorphic on $U \times_S U$, and the locally-defined isomorphisms determine an element $u \in \mathrm{Isex}(\mathrm{pr}_1^* G, \mathrm{pr}_2^* G)$. The triple (U, G, u) defines a band B which is uniquely determined up to a unique isomorphism. This is the band of the gerbe G.

A gerbe is said to be ***affine*** (resp. ***algebraic***) if it is banded by an affine (resp. algebraic) band.

Bands over a perfect field

Now let $S = \operatorname{Spec} k$, where k is a perfect field, and let G be a smooth affine group scheme over k^{al}. Recall that, if k has characteristic zero, then all algebraic group schemes are smooth.

C.10 *To give G the structure of a band over k is the same as giving an isomorphism $u\colon\ \mathrm{pr}_1 G \to \mathrm{pr}_2 G$ satisfying the cocycle condition modulo inner automorphisms.*

This is esentially the definition. Applying Galois descent (III, §5) to C.10, we obtain the following statement.

C.11 *To give G the structure of a band over k is the same as giving a family $(\phi_\sigma)_{\sigma\in\mathrm{Gal}(k^{\mathrm{al}}/k)}$ of isomorphisms $\phi_\sigma\colon \sigma G \to G$ satisfying the following two conditions:*

(a) *(cocycle) for all $\sigma, \tau \in \mathrm{Gal}(k^{\mathrm{al}}/k)$,*

$$\phi_\sigma \circ (\sigma\phi_\tau) = \phi_{\sigma\tau} \quad \textit{modulo inner automorphisms}$$

(b) *(continuity) (ϕ_σ) is split by some model of G over a subfield K of k^{al} finite over k.*

Definition C.12 (Springer 1966, 1.12) Let G be an algebraic group over k^{al}. A k^{al}/k-***kernel*** in G is a homomorphism

$$\kappa\colon\ \mathrm{Gal}(k^{\mathrm{al}}/k) \to \mathrm{Out}(G(k^{\mathrm{al}})) \stackrel{\text{def}}{=} \frac{\mathrm{Aut}(G(k^{\mathrm{al}}))}{\mathrm{Inn}(G(k^{\mathrm{al}}))}$$

such that

(a) every automorphism $\tilde{\kappa}(\sigma)$ of $G(k^{\mathrm{al}})$ lifting $\kappa(\sigma)$ is σ-linear (III, 6.1),

(b) for some finite extension $K \subset k^{\mathrm{al}}$ of k, the restriction of κ to $\mathrm{Gal}(\bar{k}/K)$ is defined by a model of G over k.

PROPOSITION C.13 *To give G the structure of a band over k is the same as giving it the structure of a k^{al}/k-kernel in G.*

PROOF Let κ be a k^{al}/k-kernel. For each σ, choose a lifting $\widetilde{\kappa}(\sigma)$ of $\kappa(\sigma)$. The condition (a) says that

$$\sigma P \mapsto \widetilde{\kappa}(\sigma)(P) \colon (\sigma G)(k^{\mathrm{al}}) \to G(k^{\mathrm{al}})$$

is defined by an isomorphism of algebraic varieties $\sigma G \to G$. The family (ϕ_σ) satisfies the cocycle and continuity conditions. Conversely, a family $(\phi_\sigma)_{\sigma\in\Gamma}$ satisfying these conditions defines a k^{al}/k-kernel by

$$\kappa(\sigma)(P) = \phi_\sigma(\sigma P).$$

□

C.14 Let G and H be algebraic group schemes over k. To give a morphism $\mathrm{Bd}(G) \to \mathrm{Bd}(H)$ of bands over k is the same as giving an $H(k^{\mathrm{al}})$-conjugacy class of homomorphisms $G_{k^{\mathrm{al}}} \to H_{k^{\mathrm{al}}}$ stable under the action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$.

TODO 23 Perhaps rethink the definition of kernel, aax.

6 Cohomology

Preliminaries

C.15 Let S be a site. For a sheaf F of abelian groups on S, let $\mathcal{H}^i(F)$ denote the presheaf $U \rightsquigarrow H^i(U,F)$. The Čech and derived cohomologies F are related by a spectral sequence

$$\check{H}^i(S,\mathcal{H}^j(F)) \Rightarrow H^{i+j}(S,F).$$

Moreover, $\check{H}^0(S,\mathcal{H}^j(F)) = 0$ for all $j > 0$, and so there is an exact sequence

$$0 \to \check{H}^2(S,F) \to H^2(S,F) \to \check{H}^1(S,\mathcal{H}^1(F)) \to \check{H}^3(S,F). \tag{92}$$

Note that the equality $\check{H}^0(S,\mathcal{H}^j(F)) = 0$, $j > 0$, says that for all U and $s \in H^j(U,F)$, $j > 0$, there exists a covering $(U_i \to U)$ such that s maps to zero in $H^j(U_i,F)$ for all i. See Milne 1980, III, 2.7, 2.9, 2.10, 2.11.

The fpqc cohomology versus the fppf cohomology

Let S be an affine scheme. Recall that the fpqc topology on Aff_S is that for which the coverings are finite surjective families of flat morphisms of affine S-schemes. For the ***fppf topology*** on Aff_S, the morphisms are required to be flat of finite presentation. The identity map on S is a continuous map $f\colon S_{\mathrm{fpqc}} \to S_{\mathrm{fppf}}$ of sites, and so, for any sheaf F of abelian groups on S_{fpqc} we have a canonical homomorphisms

$$H^i(S_{\mathrm{fppf}}, f_*F) \to H^i(S_{\mathrm{fpqc}}, F), \quad i \geq 0.$$

We now let $S = \operatorname{Spec} k$, k a field.

For a presheaf of abelian groups F on Aff_k and a k-algebra R, define $H^i(R/k,F)$ to be the ith cohomology group of the complex

$$F(R) \to F(R^{\otimes 2}) \to \cdots \to F(R^{\otimes i}) \to \cdots.$$

Then, by definition, the Čech groups

$$\check{H}^i(k_{\mathrm{fpqc}},F) = \varinjlim H^i(R/k,F) \quad \text{(limit over all k-algebras)}$$
$$\check{H}^i(k_{\mathrm{fppf}},F) = \varinjlim H^i(R/k,F) \quad \text{(limit over finitely generated k-algebras).}$$

PROPOSITION C.16 *Let F be a presheaf of abelian groups on Aff_k. If F transforms filtered projective limits of affine k-schemes into inductive limits, the canonical maps*

$$\check{H}^i(k_{\mathrm{fppf}}, F) \to \check{H}^i(k_{\mathrm{fpqc}}, F)$$

are isomorphisms for all i.

PROOF For any k-algebra R,

$$R^{\otimes i} = \varinjlim_{R'} R'^{\otimes i},$$

where the limit is over all finitely generated subalgebras R' of R, and so similarly

$$H^i(R/k, F) = \varinjlim_{R' \subset F,\ \text{finitely generated}} H^i(R'/k, F).$$

□

PROPOSITION C.17 *Let U be an affine scheme and G an affine group scheme flat of finite presentation over U. Then the canonical map*

$$H^1(U_{\mathrm{fppf}}, G) \to H^1(U_{\mathrm{fpqc}}, G)$$

is a bijection.

PROOF The sets classify the isomorphisms classes of torsors under G for the two topologies over U, and the functor

$$\mathsf{Tors}(U_{\mathrm{fppf}}, G) \to \mathsf{Tors}(U_{\mathrm{fpqc}}, G)$$

is an equivalence (even an isomorphism) of categories. Indeed, under the hypotheses, the torsors are representable by affine schemes flat and of finite presentation over U. The functor is obviously fully faithful, and every fpqc torsor T under G is also an fppf torsor because it has a point in an affine scheme flat and of finite presentation over U, namely, in T itself. □

PROPOSITION C.18 *Let Z be a commutative algebraic group scheme over k. The canonical map*

$$H^2(k_{\mathrm{fppf}}, Z) \to H^2(k_{\mathrm{fpqc}}, Z)$$

is an isomorphism.

PROOF For the fppf and fpqc topologies, the exact sequence (92) becomes

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^2(k_{\mathrm{fppf}}, Z) \arrow[r] \arrow[d, "a"] & H^2(k_{\mathrm{fppf}}, Z) \arrow[r] \arrow[d, "b"] & \check{H}^1(k_{\mathrm{fppf}}, \mathcal{H}^1(k_{\mathrm{fppf}}, Z)) \arrow[r] \arrow[d, "c"] & \check{H}^3(k_{\mathrm{fppf}}, Z) \arrow[d, "d"] \\
0 \arrow[r] & \check{H}^2(k_{\mathrm{fpqc}}, Z) \arrow[r] & H^2(k_{\mathrm{fpqc}}, Z) \arrow[r] & \check{H}^1(k_{\mathrm{fpqc}}, \mathcal{H}^1(k_{\mathrm{fpqc}}, Z)) \arrow[r] & \check{H}^3(k_{\mathrm{fpqc}}, Z).
\end{tikzcd}$$

The maps a and d are isomorphisms by C.16. The canonical map $\mathcal{H}^1(k_{\mathrm{fppf}}, Z) \to \mathcal{H}^1(k_{\mathrm{fpqc}}, Z)$ is an isomorphism by C.17, and the two functors transform filtered projective limits of affine schemes to inductive limits, so c is an isomorphism by C.16. Now the five-lemma shows that b is an isomorphism. □

ASIDE C.19 Flat algebras need not be filtered inductive limits of finitely presented flat algebras. For example, let p be a prime number, let $A = \mathbb{F}_p[X_1, ..., X_n]$, and let B be the integral closure of A in an algebraic closure of the field of fractions of A. Then B is flat over A by a theorem of Hochster and Huneke, but it is not a filtered direct limit of finitely presented flat A-algebras if $n \geq 3$ (Bhargav Bhatt).

NOTES This subsection is based on Saavedra 1972, III, 3.1.

The fppf cohomology versus the étale cohomology

THEOREM C.20 *Let G be a smooth affine group scheme over a scheme S. The canonical map*

$$H^i(S_{\mathrm{et}}, G) \to H^i(S_{\mathrm{fppf}}, G)$$

is an isomorphism for $i \leq 1$, and for all i if G is commutative.

PROOF Grothendieck 1968, 11.7, or Milne 1980, III, 3.9. □

Application to bands and gerbes

Let $S = \operatorname{Spec} k$, k a field. If a presheaf on S (i.e., a contravariant functor fom Aff_S to groups) is a sheaf for the fpqc topology, then it is also a sheaf for the fppf topology (obviously). This extends to a commutative diagram of fibred categories over Aff_S,

$$\begin{tikzcd} \mathrm{SHGR}_{\mathrm{fppf}} \arrow[r] \arrow[d] & \mathrm{SHGR}_{\mathrm{fpqc}} \arrow[d] \\ \mathrm{LIEN}_{\mathrm{fppf}} \arrow[r] & \mathrm{LIEN}_{\mathrm{fpqc}}. \end{tikzcd}$$

We write $B \rightsquigarrow B^\sharp$ for the functor from bands on S_{fpqc} to bands on S_{fppf}.

Let $\mathsf{L}_{\mathrm{fpqc}}$ (resp. $\mathsf{L}_{\mathrm{fppf}}$) denote the category whose objects are the affine algebraic bands on S_{fpqc} (resp. S_{fppf}) and whose morphisms are the isomorphisms between bands.

THEOREM C.21 *The functor $B \rightsquigarrow B^\sharp : \mathsf{L}_{\mathrm{fpqc}} \to \mathsf{L}_{\mathrm{fppf}}$ is an equivalence of categories.*

PROOF Saavedra 1972, III, 3.1.4.4, p. 191. □

THEOREM C.22 *Let G be a fibred category over Aff_S. Then G is an affine algebraic gerbe for the fpqc topology if and only if it is for the fppf topology.*

PROOF Saavedra 1972, III, 3.1.5, p. 191. □

COROLLARY C.23 *Let B be an affine algebraic band over k. Then the map*

$$H^2(S_{\mathrm{fppf}}, B) \to H^2(S_{\mathrm{fpqc}}, B)$$

is bijective, and induces a bijection

$$H^2(S_{\mathrm{fppf}}, B)' \to H^2(S_{\mathrm{fpqc}}, B)'.$$

COROLLARY C.24 *Let G be an affine algebraic gerbe over Aff_k. There exists a finite extension k' of k such that the fibre of G over $\operatorname{Spec} k'$ is nonempty.*

PROOF We may suppose that G is an affine algebraic gerbe for the fppf topology. Then $G(R) \neq \emptyset$ for some finitely generated k-algebra R. For any maximal ideal $\mathfrak{m}$ in R, $G(R/\mathfrak{m}) \neq \emptyset$ and $R/\mathfrak{m}$ is a finite field extension of k (Zariski's lemma). □

Alternative approach

C.25 If k is algebraically closed, then every algebraic band over k is the band of an algebraic group over k. To see this, let B be such a band. For some affine k-scheme U, B defines an element of $\check{H}^1(U/k, B^{\mathrm{ad}})$. There is an exact sequence of pointed sets

$$H^1(k,B) \to H^1(k,B^{\mathrm{ad}}) \to H^2(k,Z(B))$$

(fpqc cohomology groups). Now $H^2(k,Z(B))$ is equal to the fppf cohomology group (C.18), hence it is zero because k is algebraically closed. Thus, the class of B in $\check{H}^1(U/k,B^{\mathrm{ad}})$ lifts to a class in $\check{H}^1(U/k,B)$, which defines an algebraic group scheme over k.

C.26 Let B be a band on Aff_S, and let G and H be gerbes banded by B. Every morphism $m\colon \mathsf{G} \to \mathsf{H}$ banded by id_B is an equivalence (IV, **??**). We say either that m is a B-morphism or a B-equivalence, since the two are the same. The cohomology set $H^2(S,B)$ is defined to be the set of B-equivalence classes of B-gerbes. If Z is the centre of B, then $H^2(S,Z)$ is equal to the cohomology group of Z in the usual sense of the fpqc topology on S, and either $H^2(S,B)$ is empty or $H^2(S,Z)$ acts simply transitively on it (Giraud 1971, IV, 3.3.3).

PROPOSITION C.27 *Let G be an affine algebraic gerbe over Aff_k. There exists a finite extension k' of k such that the fibre of G over* Spec k' *is nonempty.*

PROOF By assumption, the band B of G is defined by a triple (U,G,u) with G a group scheme of finite presentation over U. Let $U = \operatorname{Spec} R$. The k-algebra R can be replaced by a finitely generated subalgebra, and then by a quotient modulo a maximal ideal, and so we may suppose that $U = \operatorname{Spec} k'$, where k' is a finite field extension of k. We shall show that the gerbes G and TORS(G) become B-equivalent over some finite field extension of k'. The statement preceding the proposition shows that we have to prove that an element of $H^2(U_{\mathrm{fpqc}}, Z)$, where Z is the centre of B, is killed by a finite field extension of k'. But this assertion is obvious for $H^2(U_{\mathrm{fppf}}, Z)$ (C.15), and we can apply C.18. □

REMARK C.28 The same argument (using C.20) shows that for a gerbe over Aff_k with smooth affine band, there exists a finite separable extension k' of k such that the fibre over Spec k' is nonempty. Deduce that a tannakian category over k with prosmooth band has a fibre functor over k^{sep}.

THEOREM C.29 *Let G be an affine gerbe over Aff_k. For any algebraically closed field K containing k, the fibre of G over* Spec K *is nonempty.*

PROOF We may suppose that k is algebraically closed, and then have to show that G is neutral. According to V, **??**, G is equivalent (as a fibred category over Aff_k) to the gerbe of fibre functors on a tannakian category over k. Now the statement follows from IV, Theorem 10.1. □

EXERCISE C.30 Find a direct proof of Theorem C.29 (possibly using 7.9).

1

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Index of definitions and symbols