Tannakian Categories

J.S. Milne



The proof reader at work.

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Introduction

Origins

André Weil's work on the arithmetic of curves and other varieties over finite fields led him in 1949 to state his famous "Weil conjectures". These had a profound influence on algebraic geometry and number theory in the following decades. In an effort to explain the conjectures, Grothendieck was led to define several different "Weil cohomology theories" and to posit an ur-theory underlying all of them whose objects he called motives. In order to provide a framework for studying these different theories, especially motives, Grothendieck introduced the notion of a tannakian category.

Weil's first insight was that the numbers of points on smooth projective algebraic varieties over finite fields behave as if they were the alternating sums of the traces of an operator acting on a well-behaved homology theory. In particular, the (co)homology groups should be vector spaces over a field of characteristic zero, be functorial, and give the "correct" Betti numbers.[1] However, already in the 1930s, Deuring and Hasse had shown that the endomorphism algebra of an elliptic curve over a field of characteristic p may be a quaternion algebra over $\mathbb{Q}$ that remains a division algebra even when tensored with $\mathbb{Q}_p$ or $\mathbb{R}$, and hence cannot act on a 2-dimensional vector space over $\mathbb{Q}$ (or even $\mathbb{Q}_p$ or $\mathbb{R}$). In particular, no such cohomology theory with $\mathbb{Q}$-coefficients exists.

Grothendieck defined étale cohomology groups with $\mathbb{Q}_\ell$-coefficients for each prime ℓ distinct from the characteristic of the ground field, and in characteristic $p \neq 0$, he defined the crystalline cohomology groups with coefficients in an extension of $\mathbb{Q}_p$. Each cohomology theory is well-behaved. In particular it has a Lefschetz trace formula, and Weil's first insight is explained by realizing the points of the variety in a finite field as the fixed points of the Frobenius operator, and hence, by the trace formula, their cardinality as the alternating sum of the traces of the Frobenius operator acting on the cohomology groups. A striking feature of this is that, while the traces of the Frobenius operator are, by definition, elements of different fields $\mathbb{Q}_l$, they in fact lie in $\mathbb{Q}$ and are independent of l (for smooth projective varieties). This last fact suggested to Grothendieck that there was some sort of $\mathbb{Q}$-theory underlying the different $\mathbb{Q}_l$-theories. To explain what this is, we need the notion of a tannakian category.

Briefly, a tannakian category over a field k is a k-linear abelian category with a tensor product structure having most of the properties of the category of finite-dimensional representations of an affine group scheme over k except one: there need not exist an exact tensor functor to the category of k-vector spaces, and when one does exist there need be

[1] Il me fallut du temps avant de pouvoir même imaginer que les nombres de Betti fussent susceptibles d'une interprétation en géométrie algébrique abstraite. Je crois que je fis un raisonnement heuristique basé sur la formule de Lefschetz. (It took me a while before I could even imagine that the Betti numbers were susceptible to an interpretation in abstract algebraic geometry. I think I made a heuristic argument based on the Lefschetz formula.) Weil, Œuvre, Commentaire [1949b].

a canonical one. Each of the cohomology theories takes values, not just in a category of vector spaces, but in a tannakian category. For example, crystalline cohomology takes values in a category of isocrystals. These are finite-dimensional vector spaces over an extension of $\mathbb{Q}_p$, but only the elements of $\mathbb{Q}_p$ act as endomorphisms in the category. More specifically, if $\mathbb{1}$ is the unit object of the category (the tensor product of the empty set), we have $\mathrm{End}(\mathbb{1}) = \mathbb{Q}_p$. Grothendieck's insight is that there should be a tannakian category Mot over $\mathbb{Q}$ such that the functors to the local tannakian categories defined by the different cohomology theories factor through it. Algebraic correspondences between smooth projective algebraic varieties should define maps between motives, whose traces lie in $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$ and map to the traces on the various cohomology groups, which explains why the latter lie in $\mathbb{Q}$.

Weil's second insight was that an analogue of the Riemann hypothesis should hold for the eigenvalues of Frobenius operators. This suggested that some of the well-known positivities in characteristic zero should persist to characteristic p. To see why, we briefly recall Weil's proof of the Riemann hypothesis for abelian varieties over finite fields.

Consider an abelian variety A over an algebraically closed field of characteristic p. For a prime $\ell \neq p$, we have a finite-dimensional $\mathbb{Q}_\ell$-vector space $V_\ell A$, and, for each polarization of A, we have a pairing $\varphi : V_\ell A \times V_\ell A \to \mathbb{Q}_\ell$. As $\mathbb{Q}_\ell$ is not a subfield of $\mathbb{R}$, it makes no sense to say that φ is positive definite. However, Weil showed that φ induces an involution on the finite-dimensional $\mathbb{Q}$-algebra $\mathrm{End}(A) \otimes \mathbb{Q}$ and that this involution *is* positive.[2] The Riemann hypothesis for the abelian variety follows directly from this. Grothendieck extended Weil's ideas to tannakian categories by introducing the notion of a "Weil form" on an object of a tannakian category and of a "polarization" on a tannakian category.

A tannakian category over k is said to be neutral if it admits an exact tensor functor to the category of k-vector spaces. Neutral tannakian categories are the analogues for affine group schemes of the categories studied by Tannaka and Krein. A classical theorem of Tannaka describes how to recover a compact topological group from its category of finite-dimensional unitary representations, and Krein later characterized the categories arising in this way.

Not all tannakian categories are neutral, and the obstruction to a tannakian category over k having a k-valued fibre functor lies in a nonabelian cohomology group of degree 2, more general than was available in the early 1960s. Grothendieck's student Giraud developed the necessary nonabelian cohomology theory in his thesis (Giraud 1971).

As we have noted, the idea of tannakian categories[3] and their importance for motives, was Grothendieck's.[4] He explained it to Saavedra Rivano, who developed the theory of

[2] Over $\mathbb{C}$, this was known to the Italian geometers as the positivity of the Rosati involution.

[3] The categories are named after Tadao Tannaka. In 1938, only four years after Pontryagin had shown that commutative compact topological groups can be recovered from their characters, Tannaka showed that arbitrary compact topological groups can be recovered from their unitary representations. Tannaka was inspired by an application Weil had made of Pontryagin duality to class field theory.

> …best-known is Tannaka's extension of Pontrjagin's duality theorem from the case of Abelian groups to the case of compact groups…Motivation for the research was that A. Weil applied Pontrjagin's duality theorem to prove an isomorphism in class field theory, and Tannaka, who was researching the theory of number fields, thought that it could be meaningful to extend Pontrjagin's duality theorem to the non-Abelian case. (Sasaki 1984, as quoted in Kümmerle 2024, p. 356.)

[4] Deligne writes: [Grothendieck's] aim was to unify the cohomology theories he had created. That each H is with values in a category with $\otimes$, and that Künneth holds, was a brilliant insight which, like a number of his brilliant ideas, is now part of our subconscious, making it hard to see how deep it was.

tannakian categories in his thesis (Saavedra 1972). Grothendieck later objected to the name, but he himself called them “catégories tannakiennes” in his unpublished writings. Grothendieck considered tannakian categories to be part of a vast “Galois–Poincaré” theory engobalizing Galois theory and the theory of fundamental groups.[5]

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[5] La théorie des catégories tannakiennes est issue d’une idée d’Alexandre Grothendieck, développée par N. Saavedra Rivano dans sa thèse (1972). Elle fut conçue comme un outil indispensable à l’élaboration de la théorie des motifs. Alexandre Grothendieck aurait souhaité que ces catégories fussent nommées "catégories de Galois–Poincaré–Grothendieck” mais l’usage en a décidé autrement. (Bruguières 1994.)

The theory of tannakian categories originated from an idea of Alexandre Grothendieck, developed by N. Saavedra Rivano in his thesis (1972). It was conceived as an indispensable tool for the elaboration of the theory of motives. Alexandre Grothendieck would have wished for these categories to be named "Galois–Poincaré–Grothendieck categories," but usage decided otherwise.

Notation and Conventions

Generally, we follow the conventions of Giraud 1971. We use roman letters for sets, mathcal for internal homs, san-serif for categories, small caps for stacks, and mathscr for 2-categories. Thus,

- $\mathrm{Hom}(x, y)$ is a set,
- $\mathcal{H}om(x, y)$ is an object of the same category as x and y,
- $\mathsf{Hom}(x, y)$ is itself a category,
- $\textsc{Hom}(x, y)$ is a stack,
- $\mathscr{H}om$ is a 2-category.

A partial order is reflexive, antisymmetric, and transitive. Functors between additive categories are assumed to be additive. Natural transformations are sometimes called morphisms of functors. All rings are associative with 1, and are commutative unless indicated otherwise. A strictly full subcategory is a full subcategory containing with any X, all objects isomorphic to X. Isomorphisms are denoted by $\approx$, canonical (or given) isomorphisms by $\simeq$,[6] and equivalences of categories by $\cong$. For a field k, k^{al} denotes an algebraic closure of k and k^{sep} the separable closure of k in k^{al}.

For affine schemes $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, we sometimes identify the sets $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(B, A)$ and use the same letter for a map of affine schemes and the corresponding map of rings.

In a category, the collection of morphisms from one object to a second is required to be small. The category itself is small if, in addition, the collection of objects is small. A category is essentially small if its isomorphism classes of objects form a set. Here "small" means, according to taste, a set in the sense of von Neumann–Bernays–Gödel set theory or an element of some fixed universe (i.e., U-small for a fixed universe U).

An isomorphism of functors is a natural transformation $\alpha : F \to G$ such that $F(X) \to G(X)$ is an isomorphism for all X. We sometimes refer to it as a natural isomorphism $F(X) \to G(X)$. A natural isomorphism is an invertible natural transformation.

A category is pre-additive if the Hom-sets are abelian groups and composition is bi-additive. It is additive if, in addition, binary direct sums and a zero object exist. Functors between additive categories are required to be additive. A category is pseudo-abelian if it is additive and every idempotent has an image. Pseudo-abelian categories are idempotent complete, i.e., for every idempotent $e : A \to A$, there exist morphisms $r : A \to B$ and $s : B \to A$ such that $s \circ r = e$ and $r \circ s = \mathrm{id}_B$.

When R is a commutative ring, a category is R-linear if it is additive, the Hom-sets are R-modules, and composition is R-bilinear.

An algebraic scheme over a field k is a separated scheme of finite type over k. An algebraic variety over k is a geometrically reduced algebraic scheme over k.

Our notation agrees with that of Saavedra 1972 except for some simplifications: what is called a $\otimes$-widget AC unifère by Saavedra here becomes a $\otimes$-widget.

Among the different terminologies,

inductive	direct	colimit	$\varinjlim$
projective	inverse	limit	$\varprojlim$

[6] We emphasize that, when we write $X \simeq Y$, we mean that X and Y are isomorphic by a specific isomorphism, usually canonical, not always explicitly described.

we use the first and last pair. An inductive limit of categories is to be understood in a suitable 2-category sense.

Some alternative terminologies (we use that on the left).

tensor category	$\otimes$-category	symmetric monoidal category
tensor functor	$\otimes$-functor	
tensor category over k	tensor category over k	abelian rigid tensor category + an isomorphism $k \simeq \mathrm{End}(\mathbb{1})$

Some categories (k is a field):

Aff_S, Aff_k	schemes affine over an affine scheme S, over k
$\mathsf{coMod}(C)$	right C-comodules (C a coalgebra)
$\mathsf{coModf}(C)$	right C-comodules finite-dimensional over k (C a coalgebra over k)
$\mathsf{coModf}(L)$	L-comodules finitely generated and projective as right B-modules (L a coalgebroid acting on B)
${}_A\mathsf{Mod}$, Mod_A	left, right A-modules (A a noncommutative ring)
Modf_A	finitely presented right A-modules (A a noncommutative ring)
$\mathsf{Mod}(R)$	R-modules (R a commutative ring)
$\mathsf{Modf}(R)$	finitely presented R-modules (R a commutative ring)
$\mathsf{Projf}(R)$	finitely generated projective R-modules
$\mathsf{Repf}(G)$	linear representations of G on finite-dimensional k-vector spaces (G an affine group scheme over k)
$\mathsf{Repf}(S:G)$	representations of G on locally free $\mathcal{O}_S$-modules of finite rank (G a k-groupoid acting on S)
Set	sets
Sch_k	schemes over k
$\mathsf{Vecf}(k)$	finite-dimensional k-vector spaces

See also the Index.

Chapter I

Symmetric monoidal categories

A monoidal category is one in which every finite ordered set of objects has a well-defined tensor product. A symmetric monoidal category is one in which every finite (unordered) set of objects has a well-defined tensor product. A tensor category over a field k is a k-linear abelian symmetric monoidal category in which all objects have duals and $k \simeq \mathrm{End}(\mathbb{1})$. A tannakian category over k is a tensor category over k that admits a fibre functor with values in a nonzero k-algebra.

1 Monoidal categories

A monoid can be defined to be a set together with an associative multiplication such that there exists an element 1 with the following property: $11 = 1$ and the maps $x \mapsto 1x$ and $x \mapsto x1$ are injective. Then $1x = 11x$, so $x = 1x$, and $x1 = x11$, so $x1 = x$. Therefore, this definition agrees with the usual definition of a monoid. Moreover, the element 1 is unique and the maps $x \mapsto 1x$ and $x \mapsto x1$ are bijective. These observations suggest the definition of a monoidal category in 1.2 below.

Let C be a category and let

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}, \quad (X, Y) \rightsquigarrow X \otimes Y$$

be a functor.

An ***associativity constraint*** for $(\mathsf{C}, \otimes)$ is a natural isomorphism

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

such that, for all objects X, Y, Z, T, the following diagram commutes,

$$\begin{array}{ccc} & W\otimes(X\otimes(Y\otimes Z)) & \\ {\scriptstyle W\otimes\alpha_{X,Y,Z}}\swarrow & & \searrow{\scriptstyle \alpha_{W,X,Y\otimes Z}} \\ W\otimes((X\otimes Y)\otimes Z) & & (W\otimes X)\otimes(Y\otimes Z) \\ {\scriptstyle \alpha_{W,X\otimes Y,Z}}\searrow & & \swarrow{\scriptstyle \alpha_{W\otimes X,Y,Z}} \\ & (W\otimes(X\otimes Y))\otimes Z \xrightarrow{\alpha_{W,X,Y}\otimes Z} ((W\otimes X)\otimes Y)\otimes Z & \end{array} \tag{1}$$

This is the ***pentagon axiom*** (Saavedra 1972, I, 1.1.1.1; Mac Lane 1998, p. 162).[1]

DEFINITION 1.1 A pair (U,u) consisting of an object U of C and an isomorphism $u: U\otimes U\to U$ is a ***unit*** of $(\mathsf{C},\otimes)$ if the functors

$$X \rightsquigarrow U\otimes X: \mathsf{C}\to\mathsf{C}$$
$$X \rightsquigarrow X\otimes U: \mathsf{C}\to\mathsf{C}$$

are fully faithful.

DEFINITION 1.2 A triple $(\mathsf{C},\otimes,\alpha)$ consisting of a category C, a functor $\otimes: \mathsf{C}\times\mathsf{C}\to\mathsf{C}$, and an associativity constraint α satisfying the pentagon axiom is a ***monoidal category*** if there exists a unit (U,u).

Note that the set of isomomorphism classes of objects in an essentially small monoidal category becomes a monoid with the multiplication defined by $\otimes$.

Let (U,u) be a unit in a monoidal category. We wish to define morphisms (left and right unit constraints)

$$\lambda_X: U\otimes X\to X,$$
$$\rho_X: X\otimes U\to X.$$

Because of the fully faithfullness of the functors in 1.1, it suffice to define $U\otimes\lambda_X$ and $\rho_X\otimes U$. These we take to be the composites

$$U\otimes(U\otimes X)\xrightarrow{\alpha_{U,U,X}}(U\otimes U)\otimes X\xrightarrow{u\otimes X}U\otimes X$$
$$(X\otimes U)\otimes U\xrightarrow{\alpha^{-1}_{X,U,U}}X\otimes(U\otimes U)\xrightarrow{X\otimes u}X\otimes U.$$

Note that λ and ρ are natural isomorphisms of the functors $U\otimes -$ and $-\otimes U$ with the identity functor (because they become isomorphisms when tensored with U).

PROPOSITION 1.3 *The following triangles commute,*

$$\begin{array}{ccc} U\otimes(X\otimes Y) & \xrightarrow{\alpha_{U,X,Y}} & (U\otimes X)\otimes Y \\ & {\scriptstyle \lambda_{X\otimes Y}}\searrow \quad \swarrow{\scriptstyle \lambda_X\otimes Y} & \\ & X\otimes Y & \end{array} \qquad \begin{array}{ccc} X\otimes(Y\otimes U) & \xrightarrow{\alpha_{X,Y,U}} & (X\otimes Y)\otimes U \\ & {\scriptstyle X\otimes\rho_Y}\searrow \quad \swarrow{\scriptstyle \rho_{X\otimes Y}} & \\ & X\otimes Y & \end{array} \tag{2}$$

(left and right unity diagrams).

[1] In some sources, the arrow α has the opposite direction. A similar remark applies to other arrows in this chapter.

PROOF To show that the first diagram,

$$\begin{array}{ccc} U \otimes (X \otimes Y) & \xrightarrow{\lambda_{X\otimes Y}} & X \otimes Y \\ \downarrow{\scriptstyle \alpha_{U,X,Y}} & & \| \\ (U \otimes X) \otimes Y & \xrightarrow{\lambda_X \otimes Y} & X \otimes Y, \end{array}$$

commutes, it suffices to show that

$$\begin{array}{ccccc} U \otimes (U \otimes (X \otimes Y)) & \xrightarrow{\alpha_{U,U,X\otimes Y}} & (U \otimes U) \otimes (X \otimes Y) & \xrightarrow{u\otimes(X\otimes Y)} & U \otimes (X \otimes Y) \\ \downarrow{\scriptstyle U\otimes\alpha_{U,X,Y}} & & \downarrow{\scriptstyle \alpha_{U\otimes U,X,Y}} & & \downarrow{\scriptstyle \alpha_{U,X,Y}} \\ U \otimes ((U \otimes X) \otimes Y) & \longrightarrow & ((U \otimes U) \otimes X) \otimes Y & \xrightarrow{(u\otimes X)\otimes Y} & (U \otimes X) \otimes Y \end{array}$$

commutes. The left-hand square commutes because of the pentagon axiom (the unmarked arrow involves two applications of α) and the right-hand square commutes because of the naturality of α. This proves the statement for λ_X, and the proof for ρ_X is similar. □

PROPOSITION 1.4 *For any object X,*

$$\begin{aligned} \lambda_{U\otimes X} &= U \otimes \lambda_X \\ \rho_{X\otimes U} &= \rho_X \otimes U. \end{aligned}$$

PROOF The diagram

$$\begin{array}{ccc} U \otimes (U \otimes X) & \xrightarrow{U\otimes\lambda_X} & U \otimes X \\ \downarrow{\scriptstyle \lambda_{U\otimes X}} & & \downarrow{\scriptstyle \lambda_X} \\ U \otimes X & \xrightarrow{\lambda_X} & X \end{array}$$

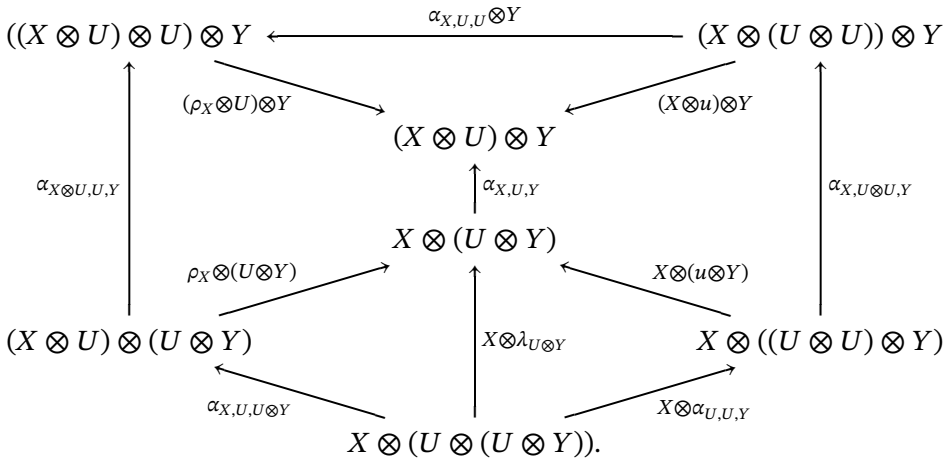
commutes because of the naturality of λ. As λ_X is an isomorphism, this proves the first equality. Similarly, the second equality follows from the naturality of ρ. □

PROPOSITION 1.5 *The following diagram commutes for all X, Y,*

$$\begin{array}{ccc} X \otimes (U \otimes Y) & \xrightarrow{\alpha_{X,U,Y}} & (X \otimes U) \otimes Y \\ & {\scriptstyle X\otimes\lambda_Y}\searrow \quad \swarrow{\scriptstyle \rho_X\otimes Y} & \\ & X \otimes Y & \end{array} \tag{3}$$

(middle unity diagram).

PROOF Consider the diagram,



The triangle at lower left is (3), except with Y replaced by $U \otimes Y$. Because $Y \rightsquigarrow U \otimes Y$ is fully faithful, it suffices to show that this triangle commutes. The outside pentagon is that in the pentagon axiom, and so it suffices to show that each of the remaining subdiagrams commutes. For the two rectangles, this follows from the functoriality of α; for the top triangle, this follows from the definition of ρ_X; and for the lower-right triangle, it follows from the definition of $U \otimes \lambda_Y$ and the equality $\lambda_{U\otimes Y} = U \otimes \lambda_Y$. □

PROPOSITION 1.6 *If (U, u) is a unit, then*

$$\lambda_U = u = \rho_U.$$

PROOF From the left unity diagram (1.3),

$$(\lambda_U \otimes U) \circ \alpha_{U,U,U} = \lambda_{U\otimes U} \overset{1.4}{=} U \otimes \lambda_U.$$

From the middle unity diagram (1.5),

$$(\rho_U \otimes U) \circ \alpha_{U,U,U} = U \otimes \lambda_U.$$

From the definition of λ_U,

$$(u \otimes U) \circ \alpha_{U,U,U} = U \otimes \lambda_U.$$

As $\alpha_{U,U,U}$ is an isomorphism, this implies that

$$\lambda_U \otimes U = u \otimes U = \rho_U \otimes U,$$

which, in turn, implies the required statement. □

PROPOSITION 1.7 *If (U, u) is a unit then*

$$U \otimes u = (u \otimes U) \circ \alpha_{U,U,U} \tag{4}$$

and the functors $X \rightsquigarrow U \otimes X$ and $X \rightsquigarrow X \otimes U$ are equivalences of categories.

PROOF The equality (4) is the special case of (3) with $X = Y = U$. For the second part of the statement, use that λ and ρ are natural isomorphisms of the functors $U \otimes -$ and $- \otimes U$ with the identity functor. □

PROPOSITION 1.8 *If (U,u) and (U',u') are both units, then there is a unique isomorphism $a: U \to U'$ making the following diagram commute,*

$$\begin{array}{ccc} U \otimes U & \xrightarrow{u} & U \\ \downarrow{\scriptstyle a\otimes a} & & \downarrow{\scriptstyle a} \\ U' \otimes U' & \xrightarrow{u'} & U'. \end{array}$$

PROOF Define a to be the isomorphism

$$U \xleftarrow{\rho_U} U \otimes U' \xrightarrow{\lambda_{U'}} U'.$$

Using the left and right unity diagrams, it is easy to show that a makes the above diagram commute. To show that a is unique, it suffices to show that if $b: U \to U$ is an isomorphism such that

$$\begin{array}{ccc} U \otimes U & \xrightarrow{u} & U \\ \downarrow{\scriptstyle b\otimes b} & & \downarrow{\scriptstyle b} \\ U \otimes U & \xrightarrow{u} & U \end{array}$$

commutes, then $b = \mathrm{id}_U$. For any morphism $c: U \to U$, the diagram

$$\begin{array}{ccc} U \otimes U & \xrightarrow{u} & U \\ \downarrow{\scriptstyle c\otimes U} & & \downarrow{\scriptstyle c} \\ U \otimes U & \xrightarrow{u} & U \end{array}$$

commutes because $u = \rho_U$. Therefore, $b \otimes b = b \otimes U$, and hence $b = \mathrm{id}_U$. □

EXAMPLE 1.9 The category Cat of small categories and functors becomes a monoidal category when we take the tensor product of two categories to be their product. Any category with only one arrow is a unit object.

Monoidal functors

DEFINITION 1.10 Let $(\mathsf{C},\otimes)$ and $(\mathsf{D},\otimes)$ be monoidal categories. A ***monoidal functor*** $(\mathsf{C},\otimes) \to (\mathsf{D},\otimes)$ is a pair (F,c) consisting of a functor $F: \mathsf{C} \to \mathsf{D}$ and a natural isomorphism $c_{X,Y}: F(X) \otimes F(Y) \to F(X \otimes Y)$ such that

(a) for all $X,Y,Z \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{array}{ccccc} FX \otimes (FY \otimes FZ) & \xrightarrow{FX\otimes c} & FX \otimes F(Y \otimes Z) & \xrightarrow{c} & F(X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle \alpha_{FX,FY,FZ}} & & & & \downarrow{\scriptstyle F(\alpha_{X,Y,Z})} \\ (FX \otimes FY) \otimes FZ & \xrightarrow{c\otimes FZ} & F(X \otimes Y) \otimes FZ & \xrightarrow{c} & F((X \otimes Y) \otimes Z) \end{array}$$

commutes, and

(b) if $U \otimes U \xrightarrow{u} U$ is a unit in C, then

$$F(U) \otimes F(U) \xrightarrow{c_{U,U}} F(U \otimes U) \xrightarrow{F(u)} F(U)$$

is a unit in D.

1.11 Let $(F, c^F) : \mathsf{C} \to \mathsf{D}$ and $(G, c^G) : \mathsf{D} \to \mathsf{E}$ be monoidal functors. When we define

$$c^{G\circ F}_{X,Y} : (G\circ F)(X) \otimes (G\circ F)(Y) \to (G\circ F)(X \otimes Y)$$

to be the composite of

$$(G\circ F)(X) \otimes (G\circ F)(Y) \xrightarrow{c^G_{FX,FY}} G(FX \otimes FY) \xrightarrow{G(c^F_{X,Y})} GF(X \otimes Y),$$

then

$$(G\circ F, c^{G\circ F}) : \mathsf{C} \to \mathsf{E}$$

is a monoidal functor.

DEFINITION 1.12 Let (F, c^F) and (G, c^G) be monoidal functors $\mathsf{C} \to \mathsf{D}$. A natural transformation $\alpha : F \to G$ is ***monoidal*** if the following diagrams commute for all X, Y and units U, U',

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{c^F_{X,Y}} & F(X \otimes Y) \\ \downarrow{\scriptstyle \alpha_X \otimes \alpha_Y} & & \downarrow{\scriptstyle \alpha_{X\otimes Y}} \\ G(X) \otimes G(Y) & \xrightarrow{c^G_{X,Y}} & G(X \otimes Y) \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{\simeq} & F(U) \\ \| & & \downarrow{\scriptstyle \alpha_U} \\ U' & \xrightarrow{\simeq} & G(U). \end{array}$$

When α_X is an isomorphism for all X, we call α a ***monoidal natural isomorphism***.

1.13 If $\alpha : G \to F$ a monoidal natural transformation, then for all finite families $(X_i)_{i\in I}$ of objects in C, the diagram

$$\begin{array}{ccc} \bigotimes_{i\in I} F(X_i) & \xrightarrow{c} & F(\bigotimes_{i\in I} X_i) \\ \downarrow{\scriptstyle \bigotimes_{i\in I} \lambda_{X_i}} & & \downarrow{\scriptstyle \lambda_{\bigotimes_{i\in I} X_i}} \\ \bigotimes_{i\in I} G(X_i) & \xrightarrow{d} & G(\bigotimes_{i\in I} X_i) \end{array} \tag{5}$$

commutes. The diagrams in 1.12 correspond to the cases $I = \{1, 2\}$ and $I = \emptyset$.

Notes

1.14 In Kock 2008, a unit in our sense is called a Saavedra unit.

1.15 Saavedra 1972, I, 1.3.2, defines a "unité reduit" to be a pair (U, u) consisting of an object U and an isomorphism $u : U \otimes U \to U$ such that the functors $X \rightsquigarrow U \otimes X$ and $X \rightsquigarrow X \otimes U$ are equivalences. According to 1.7, this agrees with our notion of a unit.

1.16 Define an LR-unit to be a triple (U, λ, ρ) such that the middle unity diagram (3) commutes. It follows from Proposition 1.3 that, to give the structure of an LR-unit on an object is the same as giving the structure of a Saavedra unit.

1.17 A monoidal category is classically defined to be a triple $(C, \otimes, \alpha)$ together with an LR-unit (Mac Lane 1998, p. 162). According to 1.16, this is the same as giving a triple together with a Saavedra unit. In our definition of a monoidal category, instead of specifying a unit object we only required it to exist. According to 1.8, this makes little difference.

1.18 Our notion of a monoidal functor agrees with that in most of the literature. There are also weak monoidal functors, in which c need not be an isomorphism, and strong (or strict) monoidal functors, in which c is equality. For example, for a ring R, the forgetful functor $(\mathsf{Mod}(R), \otimes_R) \to (\mathsf{Ab}, \otimes)$ is only a weak monoidal functor.

2 Symmetric monoidal categories

Let C be a category and $(X, Y) \rightsquigarrow X \otimes Y : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ a functor. A ***commutativity constraint*** for $(\mathsf{C}, \otimes)$ is a natural isomorphism

$$\gamma_{X,Y} : X \otimes Y \to Y \otimes X$$

such that, for all objects X, Y,

$$\gamma_{Y,X} \circ \gamma_{X,Y} : X \otimes Y \to X \otimes Y$$

is the identity morphism on $X \otimes Y$ (Saavedra 1972, I, 1.2.1).

An associativity constraint α and a commutativity constraint γ are ***compatible*** if, for all objects X, Y, Z, the following diagram commutes,[2]

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z \\
{\scriptstyle X \otimes \gamma}\searrow & & \swarrow{\scriptstyle \gamma} \\
X \otimes (Z \otimes Y) & & Z \otimes (X \otimes Y) \\
{\scriptstyle \alpha}\searrow & & \swarrow{\scriptstyle \alpha} \\
 & (X \otimes Z) \otimes Y \xrightarrow{\gamma \otimes Y} (Z \otimes X) \otimes Y &
\end{array} \tag{6}$$

This is the ***hexagon axiom*** (Saavedra 1972, I, 2.1.1.1; Mac Lane 1998, p. 184).

DEFINITION 2.1 A ***symmetric monoidal category*** is a system $(\mathsf{C}, \otimes, \alpha, \gamma)$, where $(\mathsf{C}, \otimes, \alpha)$ is a monoidal category and γ is a compatible commutativity constraint.

EXERCISE 2.2 Show that in a symmetric monoidal category, the following diagram commutes

$$\begin{array}{ccc}
X \otimes U & \xrightarrow{\gamma_{X,U}} & U \otimes X \\
{\scriptstyle \rho_X}\searrow & & \swarrow{\scriptstyle \lambda_X} \\
 & X &
\end{array}$$

[2] When we use the associativity constraint to omit parentheses, this becomes the triangle

$$\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{\gamma_{X\otimes Y,Z}} & Z \otimes X \otimes Y. \\
{\scriptstyle X \otimes \gamma_{Y,Z}}\searrow & & \nearrow{\scriptstyle \gamma_{X,Z} \otimes Y} \\
 & X \otimes Z \otimes Y &
\end{array}$$

EXERCISE 2.3 Show that in a symmetric monoidal category, the following diagram commutes,

$$\begin{array}{ccc} X \otimes (U \otimes Y) & \xrightarrow{\alpha_{X,U,Y}} & (X \otimes U) \otimes Y \\ \downarrow{\scriptstyle X\otimes\lambda_Y} & & \downarrow{\scriptstyle \gamma\otimes Y} \\ X \otimes Y & \xleftarrow{\lambda_X\otimes Y} & (U \otimes X) \otimes Y \end{array}$$

We sometimes denote a unit of a symmetric monoidal category by $(\mathbb{1}, e)$ and call $\mathbb{1}$ an ***identity object***.

EXAMPLE 2.4 Let R be a commutative ring. The category $\mathsf{Mod}(R)$ becomes a symmetric monoidal category with the usual tensor product and the obvious constraints. (If one perversely takes α to the negative of the obvious isomorphism, then the pentagon (1) fails to commute by a sign.) A pair (U, e) consisting of a free R-module U of rank 1 and a basis element e determines a unit (U, u) of Mod_R – take u to be the unique isomorphism $U \otimes U \to U$ sending $e \otimes e$ to e. Every unit is of this form. In this case, there is a canonical unit, namely, $(R, R \otimes R \xrightarrow{\text{mult.}} R)$.

EXAMPLE 2.5 The category of complex Hilbert spaces and bounded linear maps becomes a symmetric monoidal category with the completed tensor product $\hat{\otimes}$ as tensor product. The pair $(\mathbb{C}, \mathbb{C} \otimes \mathbb{C} \xrightarrow{\text{mult.}} \mathbb{C})$ is a unit.

EXAMPLE 2.6 The monoidal category of $\mathbb{Z}$-graded vector spaces can be made into a symmetric monoidal category in two different ways, with the usual symmetry

$$\gamma(x \otimes y) = y \otimes x$$

and with the super symmetry

$$\gamma(x \otimes y) = (-1)^{\deg(x)\deg(y)}(y \otimes x).$$

For other examples, see §9 below.

Extending $\otimes$

Let $(\mathbb{C}, \otimes, \alpha)$ be a monoidal category. Any functor $\mathsf{C}^n \to \mathsf{C}$ defined by repeated application of $\otimes$ is called an ***iterate*** of $\otimes$. If $F, F' : \mathsf{C}^n \to \mathsf{C}$ are iterates of $\otimes$, then it is possible to construct an isomorphism of functors $F \to F'$ using only α and α^{-1}. The significance of the pentagon axiom is that it implies that the isomorphism is unique: any two iterates of $\otimes$ to C^n are isomorphic by a unique isomorphism constructed out of α and α^{-1} (Mac Lane 1963, 1998, VII, 2). This means that there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes^n : \mathsf{C}^n \to \mathsf{C}$ for all $n \geq 0$. Similarly, when $(\mathsf{C}, \otimes, \alpha, \gamma)$ is a symmetric monoidal category, there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes_{i\in I} : \mathsf{C}^I \to \mathsf{C}$, where I is any (unordered) finite indexing set. In other words, the tensor product of any finite family of objects of C is well-defined up to a unique isomorphism. We can make this statement more precise.

PROPOSITION 2.7 *The tensor structure on a symmetric monoidal category* $(\mathsf{C}, \otimes)$ *admits an extension as follows: for each finite set I, there is a functor*

$$\bigotimes_{i\in I} : \mathsf{C}^I \to \mathsf{C},$$

and, for each map $a: I \to J$ of finite sets, there is a natural isomorphism

$$\chi(a): \bigotimes_{i\in I} X_i \to \bigotimes_{j\in J}\Big(\bigotimes_{i\mapsto j} X_i\Big)$$

satisfying the following conditions,

(a) *if I is a singleton, then $\bigotimes_{i\in I}$ is the identity functor $X \rightsquigarrow X$; if a is a map between singletons, then $\chi(a)$ is the identity automorphism of the identity functor;*

(b) *the isomorphisms defined by maps $I \xrightarrow{a} J \xrightarrow{b} K$ give rise to a commutative diagram*

$$\begin{array}{ccc} \bigotimes_{i\in I} X_i & \xrightarrow{\chi(a)} & \bigotimes_{j\in J}\Big(\bigotimes_{i\mapsto j} X_i\Big) \\ \downarrow{\scriptstyle \chi(ba)} & & \downarrow{\scriptstyle \chi(b)} \\ \bigotimes_{k\in K}\Big(\bigotimes_{i\mapsto k} X_i\Big) & \xrightarrow{\bigotimes(\chi(a|I_k))} & \bigotimes_{k\in K}\Big(\bigotimes_{j\mapsto k}\Big(\bigotimes_{i\mapsto j} X_i\Big)\Big), \end{array}$$

where $I_k = (ba)^{-1}(k)$.

PROOF Apply the coherence theorems of Mac Lane 1963, 1998. See also Elmendorf 2025. □

By $(\bigotimes_{i\in I}, \chi)$ being an extension of the tensor structure on C, we mean that $\bigotimes_{i\in I} X_i = X_1 \otimes X_2$ when $I = \{1,2\}$ and that the isomorphisms

$$X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$
$$X \otimes Y \to Y \otimes X$$

induced by χ are equal to α and γ respectively. It is automatic that $(\bigotimes_{\emptyset} X_i, \chi(\emptyset \to \{1,2\})$ is a unit and that $\chi(\{2\} \hookrightarrow \{1,2\})$ is $\lambda_X : X \to \mathbb{1} \otimes X$. If $(\bigotimes'_{i\in I}, \chi')$ is a second such extension, then there is a unique system of natural isomorphisms $\bigotimes_{i\in I} X_i \to \bigotimes'_{i\in I} X_i$ compatible with χ and χ' and such that, when $I = \{i\}$, the isomorphism is id_{X_i}.

Whenever a symmetric monoidal category $(\mathsf{C}, \otimes)$ is given, we shall always assume that an extension as in Proposition 2.7 has been made. Note that the constraints imposed are the minimum necessary to force the proposition to hold.

Invertible objects

Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. An object L of C is ***invertible*** if

$$X \rightsquigarrow L \otimes X : \mathsf{C} \to \mathsf{C}$$

is an equivalence of categories. For example, an object L of $\mathsf{Modf}(R)$ is invertible if and only if it is projective of rank 1.

If L is invertible, then there exists an L' such that $L \otimes L'$ is a unit object. The converse assertion is also true: if $L \otimes L' = \mathbb{1}$, then $L \otimes -$ and $- \otimes L'$ are quasi-inverse functors.

An ***inverse*** of L is any pair (L^{-1}, δ) with L^{-1} and object and δ a morphism,

$$\delta: \bigotimes_{i\in\{\pm\}} X_i \xrightarrow{\simeq} \mathbb{1}, \quad X_+ = L, \quad X_- = L^{-1}.$$

Note that this definition is symmetric: (L, δ) is an inverse of L^{-1}. If (L_1, δ_1) and (L_2, δ_2) are both inverses of L, then there is a unique isomorphism $a: L_1 \to L_2$ such that the composite

$$\delta_2 \circ (1 \otimes a): L \otimes L_1 \to L \otimes L_2 \to \mathbb{1}$$

is δ_1 (because the functors $- \otimes L_1$ and $- \otimes L_2$ are both quasi-inverse to $L \otimes -$).

Symmetric monoidal functors

DEFINITION 2.8 Let $(\mathsf{C}, \otimes)$ and $(\mathsf{D}, \otimes)$ be symmetric monoidal categories. A ***symmetric monoidal functor*** $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a monoidal functor (1.10) such that for all $X, Y \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{c_{X,Y}} & F(X \otimes Y) \\ \downarrow{\scriptstyle \gamma_{FX,FY}} & & \downarrow{\scriptstyle F(\gamma_{X,Y})} \\ FY \otimes FX & \xrightarrow{c_{Y,X}} & F(Y \otimes X) \end{array}$$

commutes;

2.9 Let (F, c) be a symmetric monoidal functor $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$. For any finite family $(X_i)_{i \in I}$ of objects in C, c gives rise to a well-defined isomorphism

$$c : \bigotimes_{i \in I} F(X_i) \to F(\bigotimes_{i \in I} X_i).$$

Moreover, for any map $a : I \to J$, the following diagram commutes,

$$\begin{array}{ccccc} \bigotimes_{i \in I} F(X_i) & & \xrightarrow{\qquad\qquad c \qquad\qquad} & & F\left(\bigotimes_{i \in I} X_i\right) \\ \downarrow{\scriptstyle \chi(a)} & & & & \downarrow{\scriptstyle F(\chi(a))} \\ \bigotimes_{j \in J}(\bigotimes_{i \mapsto j} F(X_i)) & \xrightarrow{c} & \bigotimes_{j \in J}(F(\bigotimes_{i \mapsto j} X_i)) & \xrightarrow{c} & F(\bigotimes_{j \in J}(\bigotimes_{i \mapsto j} X_i)). \end{array}$$

For example, if (U, u) and (U', u') are units in C and D, then there exists a unique isomorphism $U' \to FU$ compatible with u' and Fu.

2.10 Let $(F, c^F) : \mathsf{C} \to \mathsf{D}$ and $(G, c^G) : \mathsf{D} \to \mathsf{E}$ be symmetric monoidal functors. Then $(G \circ F, c^{G \circ F}) : \mathsf{C} \to \mathsf{E}$ (see 1.11) is again a symmetric monoidal functor.

DEFINITION 2.11 Let (F, c^F) and (G, c^G) be symmetric monoidal functors $\mathsf{C} \to \mathsf{D}$. A natural transformation $F \to G$ is ***symmetric monoidal*** if it is monoidal.

There being no extra condition to be symmetric, we usually usually refer to symmetric monoidal natural transformations as monoidal natural transformations.

DEFINITION 2.12 A monoidal natural transformation $(F, c) : (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a ***monoidal equivalence*** if $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories.

This definition is justified by the following remark, which shows that such a functor admits a symmetric monoidal functor as a quasi-inverse.

2.13 Let $(F, c) : (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ be a monoidal equivalence. To say that $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories means that there exists a functor $G : \mathsf{D} \to \mathsf{C}$ and natural isomorphisms

$$\eta : \mathrm{id}_{\mathsf{C}} \xrightarrow{\simeq} GF, \quad \epsilon : \mathrm{id}_{\mathsf{D}} \xrightarrow{\simeq} FG.$$

After possibly replacing ϵ with a different natural isomorphism $\mathrm{id}_{\mathsf{D}} \xrightarrow{\simeq} FG$, we obtain a system (F, G, η, ϵ) satisfying the triangle identities (see A.5). There then exists a d such that (G, d) is symmetric monoidal functor and η and ϵ are monoidal natural isomorphisms (Saavedra 1972, I, 4.4).

EXAMPLE 2.14 Let $(\mathsf{C}, \otimes, \alpha)$ be a monoidal category. The triple $(\mathsf{C}, \otimes\circ S, \alpha^{-1})$, where S is the transposition functor

$$(X, Y) \rightsquigarrow (Y, X) : \mathsf{C} \times \mathsf{C} \to \mathsf{C} \times \mathsf{C},$$

is a monoidal category, called the ***opposite*** of $(\mathsf{C}, \otimes, \alpha)$. A compatible commutativity constraint defines an equivalence of $(\mathsf{C}, \otimes, \alpha)$ with its opposite.

2.15 For symmetric monoidal functors (F, c) and (G, d) with the same source and target, we let $\mathrm{Hom}^{\otimes}(F, G)$ denote the collection of monoidal natural transformations $(F, c) \to (G, d)$. For a field k and k-algebra R, we let

$$\phi_R : \mathsf{Vecf}(k) \to \mathsf{Mod}(R),$$

denote the functor $V \rightsquigarrow V \otimes_k R$. When (F, c) and (G, d) are symmetric monoidal functors $\mathsf{C} \to \mathsf{Vecf}(k)$, we define $\mathcal{H}om^{\otimes}(F, G)$ to be the functor of k-algebras such that

$$\mathcal{H}om^{\otimes}(F, G)(R) = \mathrm{Hom}^{\otimes}(\phi_R \circ F, \phi_R \circ G). \tag{7}$$

TERMINOLOGY 2.16 We often abbreviate "symmetric monoidal widget" to "$\otimes$-widget". For example a $\otimes$-category is a symmetric monoidal category, and a $\otimes$-functor is a symmetric monoidal functor.

NOTES

2.17 In Deligne and Milne 1982, symmetric monoidal categories are called tensor categories.

2.18 Our notion of a symmetric monoidal category essentially agrees with that in Mac Lane 1998, p. 184 (see 1.17).

2.19 Our notion of a symmetric monoidal category is the same as that of a "$\otimes$-catégorie AC unifère" in Saavedra 1972 and, because of 1.8, it is essentially the same as the notion of a "$\otimes$-catégorie ACU" in ibid., I, 2.4.1 (cf. ibid., I, 2.4.3).

2.20 Proposition 2.7 suggests the notion of an "unbiased symmetric monoidal category" in which no preference is given to the functor $\otimes : \mathsf{C}^2 \to \mathsf{C}$ and the constraints are canonical. See Leinster 2004 and Galbiati 2026.

2.21 There is a large literature on coherence in monoidal and symmetric monoidal categories, beginning with Mac Lane 1963. For a recent review, see Mimram 2025.

2.22 There is a large literature on monoidal categories satisfying a commutativity constraint weaker than that we have imposed. For example, without the condition $\gamma_{Y,X} = \gamma_{X,Y}^{-1}$, the category is said to be braided.

2.23 In Saavedra 1972, I, 4.2.3, a symmetric monoidal functor is called a "$\otimes$-foncteur AC unifère".

3 Internal Homs; closed symmetric monoidal categories

In this section, $(\mathsf{C}, \otimes)$ is a symmetric monoidal category.

Internal Homs

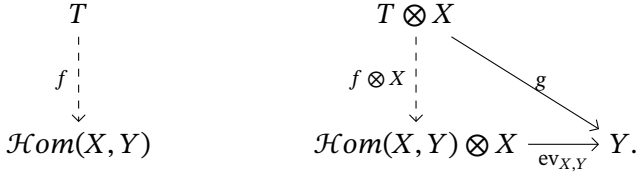
DEFINITION 3.1 Let $X, Y \in \operatorname{ob} \mathsf{C}$. When the functor

$$T \rightsquigarrow \operatorname{Hom}(T \otimes X, Y) : \mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$$

is representable, we let $\mathcal{H}om(X,Y)$ denote the representing object and

$$\mathrm{ev}_{X,Y} : \mathcal{H}om(X,Y) \otimes X \to Y$$

the morphism corresponding to $\mathrm{id}_{\mathcal{H}om(X,Y)}$. Thus, to every morphism $g : T \otimes X \to Y$ there corresponds a unique morphism $f : T \to \mathcal{H}om(X,Y)$ such that $\mathrm{ev}_{X,Y} \circ (f \otimes X) = g$,



(8)

More succinctly,

$$\operatorname{Hom}(T \otimes X, Y) \simeq \operatorname{Hom}(T, \mathcal{H}om(X,Y)). \tag{9}$$

If, for a given X, $\mathcal{H}om(X,Y)$ exists for all Y, then the functor $\mathcal{H}om(X,-)$ is right adjoint to $- \otimes X$.[3]

Note that

$$\operatorname{Hom}(\mathbb{1}, \mathcal{H}om(X,Y)) \overset{(9)}{\simeq} \operatorname{Hom}(\mathbb{1} \otimes X, Y) \simeq \operatorname{Hom}(X,Y). \tag{10}$$

The ***weak dual*** of an object X is defined to be $\mathcal{H}om(X,\mathbb{1})$ when this exists. The morphism

$$\mathrm{ev}_X \overset{\mathrm{def}}{=} \mathrm{ev}_{X,\mathbb{1}} : \mathcal{H}om(X,\mathbb{1}) \otimes X \to \mathbb{1}$$

induces a bijection

$$f \mapsto \mathrm{ev}_X \circ (f \otimes X) : \operatorname{Hom}(T, \mathcal{H}om(X,\mathbb{1})) \to \operatorname{Hom}(T \otimes X, \mathbb{1}), \tag{11}$$

natural in T, and this property characterizes $(X^\vee, \mathrm{ev}_X)$. There is also a coevaluation map

$$\mathrm{coev}_X : \mathbb{1} \to \mathcal{H}om(X,X)$$

corresponding to the identity map under $\operatorname{Hom}(\mathbb{1}, \mathcal{H}om(X,X)) \simeq \operatorname{Hom}(X,X)$.

Closed symmetric monoidal categories

DEFINITION 3.2 A symmetric monoidal category is ***closed*** if internal Homs exist, i.e., if for all objects X, the functor $- \otimes X$ has a right adjoint $\mathcal{H}om(X,-)$.

[3] Strictly speaking, this is the left internal Hom. The right internal Hom is right adjoint to $X \otimes -$, so $\operatorname{Hom}(X \otimes T, Y) \simeq \operatorname{Hom}(T, \mathcal{H}om(X,Y))$. Because of the commutativity constraint, the two notions essentially coincide.

EXAMPLE 3.3 Let R be a commutative ring. For R-modules M, N, $\mathrm{Hom}_R(M,N)$ is again an R-module, and

$$\mathrm{Hom}_R(T \otimes_R M, N) \simeq \mathrm{Hom}_R(T, \mathrm{Hom}_R(M,N)), \quad \theta \leftrightarrow (t \mapsto (m \mapsto \theta(t \otimes m))$$

(Bourbaki A, II, 4.1). Thus, $\mathcal{H}om(M,N)$ exists for all modules M, N, and equals $\mathrm{Hom}_R(M,N)$ (viewed as an R-module). In this case, $\mathrm{ev}_{M,N}$ is

$$f \otimes x \mapsto f(x) : \ \mathrm{Hom}_R(M,N) \otimes M \to N,$$

which explains its name.

In the remainder of this section, $(C, \otimes)$ is a closed symmetric monoidal category.
There is a composition morphism

$$\mathcal{H}om(Y,Z) \otimes \mathcal{H}om(X,Y) \to \mathcal{H}om(X,Z), \tag{12}$$

corresponding to

$$\mathcal{H}om(Y,Z) \otimes \mathcal{H}om(X,Y) \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \mathcal{H}om(Y,Z) \otimes Y \xrightarrow{\mathrm{ev}} Z.$$

From the canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}(T, \mathcal{H}om(Z, \mathcal{H}om(X,Y))) &\overset{(9)}{\simeq} \mathrm{Hom}(T \otimes Z, \mathcal{H}om(X,Y)) \\ &\overset{(9)}{\simeq} \mathrm{Hom}(T \otimes Z \otimes X, Y) \\ &\overset{(9)}{\simeq} \mathrm{Hom}(T, \mathcal{H}om(Z \otimes X, Y)), \end{aligned}$$

and the Yoneda lemma, we get a canonical isomorphism

$$\mathcal{H}om(Z, \mathcal{H}om(X,Y)) \simeq \mathcal{H}om(Z \otimes X, Y). \tag{13}$$

Write $X^\vee$ for the weak dual of X. The map $X \mapsto X^\vee$ can be made into a contravariant functor by sending $f : X \to Y$ to the unique morphism ${}^t f : Y^\vee \to X^\vee$ rendering commutative the diagram

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{{}^t f \otimes X} & X^\vee \otimes X \\ \downarrow{\scriptstyle Y^\vee \otimes f} & & \downarrow{\scriptstyle \mathrm{ev}_X} \\ Y^\vee \otimes Y & \xrightarrow{\mathrm{ev}_Y} & \mathbb{1}. \end{array} \tag{14}$$

In other words, ${}^t f$ is the morphism corresponding to $\mathrm{ev}_Y \circ (Y^\vee \otimes f)$ under the isomorphism (11)

$$g \mapsto \mathrm{ev}_X \circ (g \otimes X) : \ \mathrm{Hom}(Y^\vee, X^\vee) \to \mathrm{Hom}(Y^\vee \otimes X, \mathbb{1}).$$

When f is an isomorphism, we let $f^\vee = ({}^t f)^{-1} : X^\vee \to Y^\vee$, so that

$$\mathrm{ev}_Y \circ (f^\vee \otimes f) = \mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}. \tag{15}$$

EXAMPLE 3.4 In $\mathsf{Mod}(R)$, $M^\vee = \mathrm{Hom}_R(M,R)$ and ${}^t f$ is determined by the equation

$$\langle {}^t f(y), x\rangle_M = \langle y, f(x)\rangle_N, \quad y \in N^\vee \quad x \in M,$$

where we have written $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ for ev_M and ev_N. We have

$$\langle f^\vee(x'), f(x)\rangle_N = \langle x', x\rangle_M, \quad x' \in M^\vee, x \in M.$$

Let $i_X : X \to X^{\vee\vee}$ be the morphism corresponding in (11) to $\mathrm{ev}_X \circ \gamma : X \otimes X^\vee \to \mathbb{1}$. When i_X is an isomorphism, X is said to be ***reflexive***. If X has an inverse

$$(X^{-1}, \delta : X^{-1} \otimes X \xrightarrow{\simeq} \mathbb{1}),$$

then X is reflexive and the morphism $X^{-1} \to X^\vee$ corresponding to δ in (9) is an isomorphism.

For finite families of objects $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$, there is a morphism

$$\textstyle\bigotimes_{i\in I} \mathcal{H}om(X_i, Y_i) \to \mathcal{H}om(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i) \tag{16}$$

corresponding in (9) to

$$\textstyle\left(\bigotimes_{i\in I} \mathcal{H}om(X_i, Y_i)\right) \otimes \left(\bigotimes_{i\in I} X_i\right) \xrightarrow{\simeq} \bigotimes_{i\in I} \left(\mathcal{H}om(X_i, Y_i) \otimes X_i\right) \xrightarrow{\otimes \mathrm{ev}} \bigotimes_{i\in I} Y_i.$$

In particular, there are morphisms

$$\textstyle\bigotimes_{i\in I} X_i^\vee \to \left(\bigotimes_{i\in I} X_i\right)^\vee \tag{17}$$

and

$$X^\vee \otimes Y \to \mathcal{H}om(X, Y) \tag{18}$$

obtained respectively by taking $Y_i = \mathbb{1}$ all i, and $X_1 = X, X_2 = \mathbb{1} = Y_1, Y_2 = Y$.

NONEXAMPLE 3.5 The category of commutative rings is a symmetric monoidal category under tensor product with $\mathbb{Z}$ as the unit object, but it is not complete. If it were, there would be exactly one homomorphism between any two rings because

$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(\mathbb{Z} \otimes A, B) = \mathrm{Hom}(\mathbb{Z}, \mathcal{H}om(A, B)) = \text{1-point set}.$$

EXAMPLE 3.6 The quasicoherent $\mathcal{D}_X$-modules on a smooth algebraic variety over a field k of characteristic zero form a closed symmetric monoidal category (Tang et al. 2026, 3.7).

NOTES For more examples, see Borceux 1994b, 6.1. Later (§12) we study abelian closed symmetric monoidal categories.

4 Dual objects; rigid symmetric monoidal categories

In this section, $(\mathsf{C}, \otimes)$ is a symmetric monoidal category.

Duals

DEFINITION 4.1 Let X be an object in C. A pair $(X', X' \otimes X \xrightarrow{\varepsilon} \mathbb{1})$ is a ***dual***[4] of X if there exists a morphism $\delta : \mathbb{1} \to X \otimes X'$ such that the following equalities hold,

$$\begin{aligned}(X \simeq \mathbb{1} \otimes X \xrightarrow{\delta \otimes X} X \otimes X' \otimes X \xrightarrow{X \otimes \varepsilon} X \otimes \mathbb{1} \simeq X) &= \mathrm{id}_X \\ (X' \simeq X' \otimes \mathbb{1} \xrightarrow{X' \otimes \delta} X' \otimes X \otimes X' \xrightarrow{\varepsilon \otimes X'} \mathbb{1} \otimes X' \simeq X') &= \mathrm{id}_{X'}\,.\end{aligned} \tag{19}$$

Here ε and δ are called the ***evaluation*** and ***coevaluation*** morphisms, and are often denoted ev_X and coev_X. If it exists, coev_X is unique (see 4.5 below).

4.2 Let (X', ev_X) be a dual of X with coevaluation map coev_X. For any $\otimes$-functor F, the pair $(F(X'), F(\mathrm{ev}_X))$ is a dual of $F(X)$ with coevaluation map $F(\mathrm{coev}_X)$. To prove this, apply F to the diagrams (19). In particular, if (X', ev_X) is the dual of X in C, then it is the dual of X in any symmetric monoidal category containing C as a symmetric monoidal subcategory.

EXAMPLE 4.3 Let V be a finite-dimensional vector space over k, and $\phi : V \times V \to k$ a nondegenerate bilinear form. Regard ϕ as a morphism $V \otimes V \to k$. Then (V, ϕ) is the dual of V in the category $\mathsf{Vecf}(k)$. Let (e_i) be a basis for V, and (f_i) the dual basis relative to ϕ. The map $\phi^* : k \to V \otimes V$ sending 1 to $\sum e_i \otimes f_i$ is a coevaluation map. Note that $\dim(V) = \mathrm{Tr}(\phi \circ \phi^* | k)$.

EXAMPLE 4.4 Let M be a free R-module of finite rank, let $N = \mathcal{H}om(M, R)$, and let ε be the evaluation map $f \otimes m \mapsto f(m) : N \otimes M \to R$. Let (e_i) and (e_i') be dual bases for M and N, and let $\delta : R \to M \otimes N$ be the map sending 1 to $\sum e_i \otimes e_i'$. Then δ is independent of the choice the basis (e_i), and the following equalities hold,

$$\begin{aligned}(M \simeq R \otimes M \xrightarrow{\delta \otimes M} M \otimes N \otimes M \xrightarrow{M \otimes \varepsilon} M \otimes R \simeq M) &= \mathrm{id}_M \\ (N \simeq N \otimes R \xrightarrow{N \otimes \delta} N \otimes M \otimes N \xrightarrow{\varepsilon \otimes M} R \otimes N \simeq N) &= \mathrm{id}_N\,.\end{aligned} \tag{20}$$

Thus, (N, ε) is the dual of M.

4.5 Recall (A.3) that to give an adjunction is to give functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$ and natural transformations $\mathrm{id}_{\mathsf{C}} \to GF$ and $FG \to \mathrm{id}_{\mathsf{D}}$ satisfying the triangle identities. If $(X^\vee, \mathrm{ev}_X)$ is the dual of X, then the triple

$$\mathsf{C} \underset{-\otimes X^\vee}{\overset{-\otimes X}{\rightleftarrows}} \mathsf{C}\,, \quad T \xrightarrow{T \otimes \mathrm{coev}_X} T \otimes X \otimes X^\vee, \quad T \otimes X^\vee \otimes X \xrightarrow{T \otimes \mathrm{ev}_X} T$$

satisfies the triangle identities. Therefore $- \otimes X^\vee$ is right adjoint to $- \otimes X$ with unit and counit $- \otimes \mathrm{coev}_X$ and $- \otimes \mathrm{ev}_X$. Moreover, the coevaluation morphism is uniquely determined by $(X^\vee, \mathrm{ev}_X)$.

PROPOSITION 4.6 *Let $\varepsilon : X' \otimes X \to \mathbb{1}$ be a morphism in C. The pair (X', ε) is a dual of X if and only if the map*

$$\Psi_{S,T} : \mathrm{Hom}(S, T \otimes X') \to \mathrm{Hom}(S \otimes X, T)$$

sending $f : S \to T \otimes X'$ to the composite $S \otimes X \xrightarrow{f \otimes X} T \otimes X' \otimes X \xrightarrow{T \otimes \varepsilon} T \otimes \mathbb{1} \simeq T$ is a bijection for all $S, T \in \mathrm{ob}\,\mathsf{C}$.

[4] Strictly speaking, this is a left dual – the right dual is a morphism $X \otimes X' \to \mathbb{1}$ such that there exists a morphism $\mathbb{1} \to X' \otimes X$ making the similar diagrams commute.

PROOF We have functors

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{C} \qquad \begin{aligned} FS &= S \otimes X \\ GT &= T \otimes X', \end{aligned}$$

and a natural transformation

$$\epsilon : FG \to \mathrm{id}_{\mathsf{C}}, \quad T \otimes X' \otimes X \overset{T\otimes\varepsilon}{\longrightarrow} T \otimes \mathbb{1} \simeq T.$$

According to A.4, $\psi_{S,T}$ is bijective for all S, T, i.e., (F, G, ψ^{-1}) is an adjunction, if and only if there exists a natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \to GF$ such that the triangle identities

$$\begin{aligned} (S \otimes X \longrightarrow S \otimes X \otimes X' \otimes X \longrightarrow S \otimes X) &= \mathrm{id}_{S\otimes X} \\ (T \otimes X' \to T \otimes X' \otimes X \otimes X' \to T \otimes X') &= \mathrm{id}_{T\otimes X} \end{aligned}$$

hold for all $S, T \in \mathrm{ob}\,\mathsf{C}$, i.e., (X', ε) is a dual of X. □

Assume that $\mathcal{H}om(X, T)$ exists for all T, and consider

$$\mathrm{ev}_{X,\mathbb{1}} : \mathcal{H}om(X, \mathbb{1}) \otimes X \to \mathbb{1}.$$

As in 4.6, this defines a morphism $\psi_{S,T}$, and from

$$\mathrm{Hom}(S, T \otimes \mathcal{H}om(X, \mathbb{1})) \overset{\psi_{S,T}}{\longrightarrow} \mathrm{Hom}(S \otimes X, T) \simeq \mathrm{Hom}(S, \mathcal{H}om(X, T)), \tag{21}$$

we get (by the Yoneda lemma) a canonical morphism, natural in T,

$$\phi_{X,T} : T \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T). \tag{22}$$

The morphism $\phi_{X,T}$ can also be obtained from the composition morphism (12)

$$\mathcal{H}om(\mathbb{1}, T) \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T)$$

by replacing $\mathcal{H}om(\mathbb{1}, T)$ with T.

PROPOSITION 4.7 *An object X of C admits a dual if and only if, for all $T \in \mathrm{ob}\,\mathsf{C}$, $\mathcal{H}om(X, T)$ exists and $\phi_{X,T}$ is an isomorphism.*

PROOF Assume that $\mathcal{H}om(X, T)$ exists and (22) is an isomorphism for all T. Then the composite of the morphisms in (21) is an isomorphism for all S and T, so $\psi_{S,T}$ is an isomorphism. According to 4.6, this implies that $(\mathcal{H}om(X, T), \mathrm{ev}_{X,\mathbb{1}})$ is a dual of X.

Conversely, if X has a dual $(X^\vee, \varepsilon)$, then

$$\psi_{S,T} : \mathrm{Hom}(S, T \otimes X^\vee) \to \mathrm{Hom}(S \otimes X, T),$$

is an isomorphism for all S (see 4.6). Therefore, $\mathcal{H}om(X, T)$ exists and equals $T \otimes X^\vee$. □

COROLLARY 4.8 *If an object X of C admits a dual $(X^\vee, \mathrm{ev}_X)$, then $\mathcal{H}om(X, T)$ exists for all T, and*

$$\begin{aligned} (X^\vee, \mathrm{ev}_X) &\simeq (\mathcal{H}om(X, \mathbb{1}), \mathrm{ev}_{X,\mathbb{1}}) \\ T \otimes X^\vee &\simeq \mathcal{H}om(X, T). \end{aligned}$$

In particular, if $(X^\vee, \mathrm{ev}_X)$ exists, then it is unique up to a unique isomorphism, and the morphism (8)

$$X^\vee \otimes T \to \mathcal{H}om(X, T)$$

is an isomorphism.

PROOF This was shown in the above proof. □

For an $M \in \operatorname{ob}\mathsf{Mod}(R)$, the morphism (22) becomes

$$T \otimes \operatorname{Hom}(M,R) \to \operatorname{Hom}(M,T), \quad t \otimes f \mapsto (m \mapsto f(m)t). \tag{23}$$

PROPOSITION 4.9 *The following conditions on an R-module M are equivalent:*

(a) *M admits a dual;*

(b) *the map (23) is an isomorphism for all T;*

(c) *M is finitely generated and projective.*

PROOF (a) $\Rightarrow$ (b): Special case of 4.7 (it can also be proved directly).

(b) $\Rightarrow$ (c): In particular, $M \otimes \operatorname{Hom}(M,R) \simeq \operatorname{End}_R(M)$. If $\sum_{i\in I} m_i \otimes f_i$ corresponds to id_M under this isomorphism, so that $\sum_{i\in I} f_i(m)m_i = m$ for all $m \in M$, then

$$M \xrightarrow{m\mapsto(f_i(m))} R^I \xrightarrow{(a_i)\mapsto\sum a_i m_i} M$$

is a factorization of id_M. Therefore M is a direct summand of a free module of finite rank, and so it is finitely generated and projective.

(c) $\Rightarrow$ (a): When M is free of finite rank, we saw in 4.4 that there exists a dual. In the general case, there exists a finite family $(f_i)_{i\in I}$ of elements of R generating R as an ideal and such that, for each i, the R_{f_i}-module M_{f_i} is free. Thus, there exists a δ_i for each i, and the uniqueness assertion in Corollary 4.5 implies that they patch together to give a δ for M. □

For example, a module over a Dedekind domain admits a dual if and only if it is finitely generated and torsion free, and a vector space over a field admits a dual if and only if has finite dimension (for an infinite-dimensional vector space V, there is no coevaluation map $k \to V \otimes V^\vee$).

SUMMARY 4.10 Let $(\mathsf{C},\otimes)$ be a $\otimes$-category in which internal Homs exist. For any object X in C, the following statements are equivalent:

(a) $\phi_{X,T} : T \otimes \mathcal{H}om(X,\mathbb{1}) \to \mathcal{H}om(X,T)$ is an isomorphism for all T;

(b) $\phi_{X,X} : X \otimes \mathcal{H}om(X,\mathbb{1}) \to \mathcal{H}om(X,X)$ is an isomorphism;

(c) X admits a dual, in which case the dual is $(\mathcal{H}om(X,\mathbb{1}), \mathrm{ev}_{X,\mathbb{1}})$.

NOTES This subsection follows Dold and Puppe 1980.

Rigid symmetric monoidal categories

DEFINITION 4.11 A symmetric monoidal category $(\mathsf{C},\otimes)$ is ***rigid*** if every nonzero object of C admits a dual (in the sense of 4.1).

4.12 Equivalent condition (see 4.7):

(a) $\mathcal{H}om(X,Y)$ exists for all objects X and Y, and

(b) the canonical morphism (22)

$$T \otimes \mathcal{H}om(X,\mathbb{1}) \to \mathcal{H}om(X,T)$$

is an isomorphism for all objects X and T.

4.13 Equivalent condition:

(a) $\mathcal{H}om(X,Y)$ exists for all objects X and Y,

(b) every object X is reflexive, and

(c) the canonical morphism (16)

$$\mathcal{H}om(X_1,Y_1)\otimes\mathcal{H}om(X_2,Y_2)\to\mathcal{H}om(X_1\otimes X_2,Y_1\otimes Y_2)$$

is an isomorphism for all objects X_1,Y_2,X_2,Y_2.

This is the definition in Saavedra 1972 (I, 5.1.1).

EXAMPLE 4.14 Let G be a monoid over a field k. The category $\mathsf{Repf}(G)$ has internal Homs if and only if G is a group, in which case it is rigid.

The functor on a closed category sending an object to its weak dual is now a functor sending an object to its dual.

Let $(\mathsf{C},\otimes)$ be a rigid symmetric monoidal category. The opposite category $\mathsf{C}^{\mathrm{opp}}$ has a tensor structure for which $\otimes X_i^{\mathrm{opp}}=(\otimes X_i)^{\mathrm{opp}}$. The $\otimes$-functor

$$\{X,f\}\rightsquigarrow\{X^\vee,{}^tf\}:\ \mathsf{C}^{\mathrm{opp}}\to\mathsf{C}$$

is a contravariant equivalence of categories because its composite with itself is isomorphic to the identity functor. It is therefore a contravariant equivalence of symmetric monoidal categories (2.13). In particular,

$$f\mapsto{}^tf:\ \mathrm{Hom}(X,Y)\to\mathrm{Hom}(Y^\vee,X^\vee) \tag{24}$$

is an isomorphism. There is also a canonical isomorphism

$$\mathcal{H}om(X,Y)\to\mathcal{H}om(Y^\vee,X^\vee), \tag{25}$$

namely, the composite of the isomorphisms

$$\mathcal{H}om(X,Y)\xleftarrow{(4.8)}X^\vee\otimes Y\xrightarrow{X^\vee\otimes i_Y}X^\vee\otimes Y^{\vee\vee}\xrightarrow{\gamma}Y^{\vee\vee}\otimes X^\vee\xrightarrow{(4.8)}\mathcal{H}om(Y^\vee,X^\vee).$$

PROPOSITION 4.15 *Let $(\mathsf{C},\otimes)$ be a symmetric monoidal category. If C is rigid, then $X\rightsquigarrow D(X)\overset{\mathrm{def}}{=}X^\vee$ is a functor equipped with a natural isomorphism*

$$\psi_{X,Y,Z}:\ \mathrm{Hom}(X\otimes Y,Z)\simeq\mathrm{Hom}(X,Z\otimes D(Y)).$$

Conversely, if there exists a functor $D:\ \mathsf{C}\to\mathsf{C}$ and a natural isomorphism ψ, then C is rigid; moreover, $(D(Y),\varepsilon)$, where ε corresponds to the identity map under $\psi_{D(Y),Y,\mathbb{1}}$, is the dual of Y.

PROOF Restatement of 4.6. □

Traces

Let $(\mathsf{C}, \otimes)$ be a rigid symmetric monoidal category.

For any object X of C, there are morphisms

$$\mathcal{H}om(X,X) \xrightarrow[(18)]{\simeq} X^\vee \otimes X \overset{\mathrm{ev}_X}{\to} \mathbb{1}. \tag{26}$$

On applying the functor $\mathrm{Hom}(\mathbb{1}, -)$ to this, we obtain by (10) a morphism

$$\mathrm{Tr}_X : \ \mathrm{End}(X) \to \mathrm{End}(\mathbb{1}) \tag{27}$$

called the ***trace morphism***. More directly, the trace of $f : X \to X$ is the composite of the morphisms

$$\mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{f \otimes X^\vee} X \otimes X^\vee \xrightarrow{\gamma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}. \tag{28}$$

We sometimes write $\mathrm{Tr}(f|X)$ for $\mathrm{Tr}_X(f)$. The ***(categorical) dimension*** (or ***rank***) of X is defined to be $\mathrm{Tr}_X(\mathrm{id}_X)$, so $\dim X$ is the composite of the morphisms

$$\mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{\gamma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}. \tag{29}$$

Note that $\dim(X)$ is an element of the ring $\mathrm{End}(\mathbb{1})$. In particular, it need not be an integer, much less a positive integer (see 8.9 and 8.11 for example). In the $\otimes$-category of finite-dimensional vector spaces over a field k of characteristic p, a vector space of dimension p has categorical dimension 0.

PROPOSITION 4.16 *For any morphisms $f : X \to X$, $g : Y \to Y$ of C,*

$$\begin{aligned} \mathrm{Tr}_{X\otimes Y}(f \otimes g) &= \mathrm{Tr}_X(f) \cdot \mathrm{Tr}_Y(g) \\ \mathrm{Tr}_{X^\vee}(f^\vee) &= \mathrm{Tr}_X(f) \\ \mathrm{Tr}_{\mathbb{1}}(f) &= f. \end{aligned}$$

Moreover, when $X \oplus Y$ exists,

$$\mathrm{Tr}_{X\oplus Y}(f \oplus g) = \mathrm{Tr}_X(f) + \mathrm{Tr}_Y(g).$$

PROOF Straightforward. □

COROLLARY 4.17 *For any objects X, Y of C,*

$$\begin{aligned} \dim(X \otimes Y) &= \dim X \cdot \dim Y \\ \dim(X^\vee) &= \dim(X) \\ \dim(\mathbb{1}) &= \mathrm{id}_{\mathbb{1}}\,. \end{aligned}$$

Moreover, when $X \oplus Y$ exists,

$$\dim(X \oplus Y) = \dim X + \dim Y.$$

PROOF Apply the proposition to the identity morphisms. □

PROPOSITION 4.18 *For any morphisms $X \xrightarrow{f} Y \xrightarrow{g} X$,*

$$\mathrm{Tr}_X(g \circ f) = \mathrm{Tr}_{Y\otimes X}(\gamma_{X,Y} \circ (g \otimes f)) = \mathrm{Tr}_Y(f \circ g),$$

PROOF For a morphism $f : X \to Y$, let $\delta(f) : \mathbb{1} \to Y \otimes X^\vee$ denote the composite

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f \otimes X^\vee} Y \otimes X^\vee.$$

Thus, when $Y = X$, $\mathrm{Tr}_X(f) = \mathrm{ev}_X \circ \gamma_{X,X^\vee} \circ \delta(f)$.

For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the morphism $\delta(g \circ f)$ is the composite

$$\mathbb{1} \simeq \mathbb{1} \otimes \mathbb{1} \xrightarrow{\delta(g) \otimes \delta(f)} Z \otimes Y^\vee \otimes Y \otimes X^\vee \xrightarrow{Z \otimes \mathrm{ev}_Y \otimes X^\vee} Z \otimes \mathbb{1} \otimes X^\vee \simeq Z \otimes X^\vee. \tag{30}$$

Thus, when $Z = X$,

$$\mathrm{Tr}_X(g \circ f) \overset{\text{def}}{=} \mathrm{ev}_X \circ \gamma_{X,X^\vee} \circ \delta(g \circ f) = \mathrm{Tr}(\gamma_{X,Y} \circ (g \otimes f)).$$

Similarly,

$$\mathrm{Tr}_Y(f \circ g) = \mathrm{Tr}(\gamma_{Y,X} \circ (f \otimes g)) = \mathrm{Tr}(\gamma_{X,Y} \circ (g \otimes f)).$$

□

REMARK 4.19 (DELIGNE 1990, 7.2) Consider, as in Proposition 4.18, morphisms $f : X \to Y$ and $g : Y \to X$. The tensor product of the unordered set $\{X, Y\}$ is well-defined up to a canonical isomorphism, as is the morphism $\otimes\{f, g\} : \otimes\{X, Y\} \to \otimes\{X, Y\}$. The proposition says that the trace of this morphism is $\mathrm{Tr}(g \circ f)$, hence, by symmetry, also $\mathrm{Tr}(f \circ g)$.

More generally, consider morphisms

$$X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} X_n \xrightarrow{u_n} X_1.$$

These give rise to a morphism,

$$\bigotimes_{i \in \mathbb{Z}/(n)} u_i : \bigotimes_{i \in \mathbb{Z}/(n)} X_i \longrightarrow \bigotimes_{i \in \mathbb{Z}/(n)} X_i,$$

which we denote by $\otimes u_i$. On iterating (30), we find that $\delta(u_n \circ \cdots \circ u_1)$ is the composite

$$\mathbb{1} \xrightarrow{\delta(u_n) \otimes \cdots \otimes \delta(u_1)} X_1 \otimes X_n^\vee \otimes X_n \otimes \cdots \otimes X_2 \otimes X_1^\vee \xrightarrow{X_1 \otimes \mathrm{ev}_{X_n} \otimes \cdots \otimes \mathrm{ev}_{X_2} \otimes X_1^\vee} X_1 \otimes X_1^\vee,$$

and it follows that

$$\mathrm{Tr}(u_n \circ \cdots \circ u_1) = \otimes_i(\mathrm{ev}_{X_i} \circ \gamma_{X_i,X_i^\vee}) \circ \otimes_i \delta(u_i) = \mathrm{Tr}(\otimes_i u_i). \tag{31}$$

⊗-functors of rigid ⊗-categories

It is clear from Definition 4.1 that a ⊗-functor $F : \mathsf{C} \to \mathsf{D}$ of rigid ⊗-categories preserves duals:

$$F(X^\vee) \simeq F(X)^\vee.$$

Moreover, it induces a homomorphism $F : \mathrm{End}(\mathbb{1}_\mathsf{C}) \to \mathrm{End}(F(\mathbb{1}_\mathsf{C})) \simeq \mathrm{End}(\mathbb{1}_\mathsf{D})$. This preserves traces: for any $f : X \to X$,

$$\mathrm{Tr}_{F(X)} F(f) = F(\mathrm{Tr}_X(f)).$$

In particular, $\dim(F(X)) = F(\dim(X))$.

PROPOSITION 4.20 *Let $(F,c)\colon (\mathsf{C},\otimes)\to(\mathsf{D},\otimes)$ be a $\otimes$-functor of rigid symmetric monoidal categories. For any $X,Y\in \mathrm{ob}(\mathsf{C})$,*

$$F(\mathcal{H}om(X,Y))\simeq \mathcal{H}om(FX,FY). \tag{32}$$

In particular,

$$F(X^\vee)\simeq F(X)^\vee.$$

PROOF The morphism (32) comes by adjunction (9) from

$$F(\mathrm{ev}_{X,Y})\colon F(\mathcal{H}om(X,Y))\otimes FX\to FY.$$

The commutative diagram

$$\begin{array}{ccc} F(\mathcal{H}om(X,Y)) & \longrightarrow & \mathcal{H}om(FX,FY) \\ \simeq\uparrow{\scriptstyle 4.8} & & \simeq\uparrow{\scriptstyle 4.8} \\ F(X^\vee\otimes Y) & \xrightarrow{\simeq} & (FX)^\vee\otimes FY. \end{array}$$

shows that it is an isomorphism. □

PROPOSITION 4.21 *Let (F,c) and (G,d) be $\otimes$-functors $\mathsf{C}\to\mathsf{D}$ of $\otimes$-categories. If C is rigid, then every monoidal natural transformation $u\colon F\to G$ is an isomorphism.*

PROOF The morphism $v\colon G\to F$ making the diagrams

$$\begin{array}{ccc} F(X^\vee) & \xrightarrow{v_{X^\vee}} & G(X^\vee) \\ \downarrow\simeq & & \downarrow\simeq \\ F(X)^\vee & \xrightarrow{{}^t(u_X)} & G(X)^\vee \end{array}$$

commute for all $X\in\mathrm{ob}(\mathsf{C})$, is an inverse for u. Note that v_X is the composite

$$G(X)\simeq G(X^\vee)^\vee \xrightarrow{{}^t u_{X^\vee}} F(X^\vee)^\vee\simeq F(X).$$

To see that $u_X\circ v_X=\mathrm{id}_{GX}$, chase around the outside of the following diagram in two ways, starting from the GX at lower left. The outer diagram commutes because each subdiagram does.

$$\begin{array}{ccccc} FX\otimes F(X^\vee)\otimes GX & \xrightarrow{\mathrm{id}\otimes u_{X^\vee}\otimes\mathrm{id}} & FX\otimes G(X^\vee)\otimes GX & \xrightarrow{\mathrm{id}\otimes\varepsilon_{GX}} & FX \\ \delta_{FX}\otimes\mathrm{id}\uparrow & \searrow^{u_X\otimes u_{X^\vee}\otimes\mathrm{id}} & \downarrow{\scriptstyle u_X\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle u_X} \\ GX & \xrightarrow{\delta_{GX}\otimes\mathrm{id}} & GX\otimes G(X^\vee)\otimes GX & \xrightarrow{\mathrm{id}\otimes\varepsilon_{GX}} & GX \end{array}$$

The proof that $v_X\circ u_X=\mathrm{id}_{GX}$ is similar. □

For a k-algebra R, let ϕ_R denote the functor $-\otimes_k R\colon \mathsf{Vecf}(k)\to\mathsf{Mod}(R)$. Let $\omega\colon \mathsf{C}\to\mathsf{Vecf}(k)$ be a $\otimes$-functor, and define functors $\mathcal{E}nd^\otimes(\omega)$ and $\mathcal{A}ut^\otimes(\omega)$ by by

$$\mathcal{E}nd^\otimes(\omega)(R)\overset{\mathrm{def}}{=}\mathrm{End}(\phi_R\circ\omega)$$
$$\mathcal{A}ut^\otimes(\omega)(R)\overset{\mathrm{def}}{=}\mathrm{Aut}(\phi_R\circ\omega).$$

(cf. (7), p. 17). If C is rigid, then $\mathrm{End}(\phi_R\circ\omega)\simeq\mathrm{Aut}(\phi_R\circ\omega)$ (4.21), and so

$$\mathcal{E}nd^\otimes(\omega)\simeq\mathrm{Aut}(\omega). \tag{33}$$

PROPOSITION 4.22 *Let $F : \mathsf{C} \to \mathsf{D}$ be a $\otimes$-functor between rigid symmetric monoidal categories. The following conditions on F are equivalent:*

(a) *every object of* D *is a subobject of $F(X)$ for some object X of* C*;*

(b) *every object of* D *is a quotient of $F(X)$ for some object X of* C.

PROOF Assume (a), and let Y be an object of D. There exists an object X of C and a monomorphism $i : Y^\vee \to F(X)$. In a rigid symmetric monoidal category, the functor $^\vee$ is a contravariant equivalence with itself as a quasi-inverse, and $\otimes$-functors preserve duals (4.20). Hence

$$F(X^\vee) \simeq F(X)^\vee \xrightarrow{i^\vee} Y^{\vee\vee} \simeq Y$$

is an epimorphism. We have shown that (a) implies (b), and the converse is proved similarly. □

DEFINITION 4.23 A $\otimes$-functor of $\otimes$-categories is said to be ***dominant*** if it satisfies the equivalent conditions of 4.22.

DEFINITION 4.24 Let C' be a strictly full subcategory of a monoidal category C. We say that C' is a ***monoidal subcategory*** of C if it is stable under the formation of finite tensor products (it suffices to check that it contains a unit and that it contains $X \otimes Y$ whenever it contains X and Y). It becomes a monoidal category under the induced tensor structure. If C is symmetric, then C' is symmetric, and if C is symmetric and rigid, then C' is rigid if it contains contains $X^\vee$ whenever it contains X.

ASIDE 4.25 A rigid symmetric monoidal category is sometimes called a ***compact closed category*** or simply a ***compact category***.

NOTES For a proof that the diagram in Proposition 4.20 commutes, see sx4932465. The proof of Proposition 4.21 was extracted from ncatlab.org.

5 Abelian rigid symmetric monoidal categories

Our convention, that functors between additive categories are additive, forces the following definition.

DEFINITION 5.1 A symmetric monoidal category $(\mathsf{C}, \otimes)$ is ***additive*** (resp. ***abelian***) if C is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

If $(\mathsf{C}, \otimes)$ is an additive symmetric monoidal category and $(\mathbb{1}, e)$ is a unit, then $R \overset{\text{def}}{=} \mathrm{End}(\mathbb{1})$ is a ring that acts through the isomorphism $\lambda_X : \mathbb{1} \otimes X \xrightarrow{\simeq} X$ on each object X of C. The action of R on X commutes with endomorphisms of X and so, in particular, R is commutative. If $(\mathbb{1}', e')$ is a second unit, then the unique isomorphism $a : (\mathbb{1}, e) \to (\mathbb{1}', e')$ (see 1.8) defines an isomorphism $R \simeq \mathrm{End}(\mathbb{1}')$. The category C is R-linear in the sense that each Hom-set is equipped with an R-module structure and $\circ$ and $\otimes$ are R-bilinear.

In more detail, we have a bi-additive map

$$(a, f) \mapsto a \cdot f : \ \mathrm{End}(\mathbb{1}) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$$

given by

$$\begin{array}{ccc} \mathbb{1}\otimes X & \xrightarrow{a\otimes f} & \mathbb{1}\otimes Y \\ {\scriptstyle\simeq}\downarrow{\scriptstyle\lambda_X} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\lambda_Y} \\ X & \xrightarrow{a\cdot f} & Y. \end{array}$$

That this defines an action of the ring R on $\mathrm{Hom}(X,Y)$ follows from the middle unity diagram 1.5. That the action is R-bilinear follows from the definition of the action and the functoriality of $\otimes$. That $\otimes$ is R-bilinear follows from the middle unity diagram 1.5, the left unity diagram 1.3, and 2.2. See sx5080743.

When C is rigid and $R = \mathrm{End}(\mathbb{1})$, the trace morphism is an R-linear map

$$\mathrm{Tr}: \ \mathrm{End}(X) \to R.$$

PROPOSITION 5.2 *Let $(\mathsf{C},\otimes)$ be a symmetric monoidal category. If C is closed, then $\otimes$ commutes with inductive limits in each variable. If C is rigid, then $\otimes$ also commutes with projective limits in each variable.*

PROOF If C is closed, then the functor $\mathcal{H}om(X,-)$ is right adjoint to $-\otimes X$ (see (9)), and so $-\otimes X$ commutes with inductive limits. If C is rigid, then $X^\vee\otimes -$ is both a left and a right adjoint of $-\otimes X$. □

COROLLARY 5.3 *In an abelian rigid symmetric monoidal category, $\otimes$ is exact in each variable.*

Note that $\otimes$ is not usually exact in $\mathsf{Mod}(R)$, so "rigid" cannot be omitted.

PROPOSITION 5.4 *Let $(\mathsf{C},\otimes)$ be an abelian symmetric monoidal category. If C is a Grothendieck category, then $(\mathsf{C},\otimes)$ is closed if and only if the $\otimes$ commutes with inductive limits in each variable.*

PROOF The necessity was proved in 5.2, and the sufficiency follows from Freyd's special adjoint functor theorem. □

PROPOSITION 5.5 *Let $(\mathsf{C},\otimes)$ be an abelian rigid symmetric monoidal category. If U is a subobject of $\mathbb{1}$, then $\mathbb{1}\simeq U\oplus U^\perp$, where $U^\perp=\mathrm{Ker}(\mathbb{1}\to U^\vee)$. Consequently, $\mathbb{1}$ is a simple object if $\mathrm{End}(\mathbb{1})$ is a field.*

PROOF Let V be the cokernel of $U\to\mathbb{1}$, so

$$0\longrightarrow U\longrightarrow \mathbb{1}\longrightarrow V\longrightarrow 0$$

is exact. On applying $\otimes$ to the commutative diagram

$$\begin{array}{ccccc} U\times\mathbb{1} & \longrightarrow & \mathbb{1}\times\mathbb{1} & \longrightarrow & V\times\mathbb{1} \\ \uparrow & & \uparrow & & \uparrow \\ U\times U & \longrightarrow & \mathbb{1}\times U & \longrightarrow & V\times U \end{array}$$

in $\mathsf{C}\times\mathsf{C}$, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & \mathbb{1} & \longrightarrow & V & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & \overset{0}{\nearrow} & \uparrow & & \\ 0 & \longrightarrow & U\otimes U & \longrightarrow & U & \longrightarrow & V\otimes U & \longrightarrow & 0, \end{array}$$

in C. The bottom row is (top row) $\otimes U$, so it also is exact. It follows that $V \otimes U = 0$ and that $U \otimes U = U$ as a subobject of $\mathbb{1} \otimes \mathbb{1} = \mathbb{1}$.

For any object T, the morphism $T \otimes U \to T$, obtained from $U \hookrightarrow \mathbb{1}$ by tensoring with T, is a monomorphism. This proves the first equivalence in

$$\begin{aligned} T \otimes U = 0 &\iff \text{the morphism } T \otimes U \to T \text{ is zero} \\ &\iff \text{the morphism } T \to U^\vee \otimes T \text{ is zero,} \end{aligned}$$

and the second follows from the canonical isomorphisms

$$\mathrm{Hom}(T \otimes U, T) \overset{1.6.5}{\simeq} \mathrm{Hom}(T \otimes U \otimes T^\vee, \mathbf{1}) \overset{1.6.5}{\simeq} \mathrm{Hom}(T, U^\vee \otimes T).$$

For an object X of C and subobject T, we deduce from the diagram

$$\begin{array}{ccc} X & \longrightarrow & U^\vee \otimes X \\ \uparrow & & \uparrow \\ T & \longrightarrow & U^\vee \otimes T \end{array}$$

that

$$\mathrm{Ker}(X \to U^\vee \otimes X)$$

is the largest subobject K of X such that $K \otimes U = 0$. On tensoring the exact sequence

$$0 \to U^\perp \to \mathbb{1} \to U^\vee \to 0$$

with X, we see that $K = U^\perp \otimes X$.

We apply this statement to the object V. As $V \otimes U = 0$, we deduce that $U^\perp \otimes V = V$.

We apply the statement to the object U. Let T be a subobject of U. The argument in the first paragraph shows that $T \otimes T = T$, and so

$$T \otimes U = 0 \;\Rightarrow\; T = 0.$$

We deduce that $U^\perp \otimes U = 0$.

From the exact sequence

$$0 \to U^\perp \otimes U \to U^\perp \otimes \mathbb{1} \to U^\perp \otimes V \to 0$$

we now deduce that the composite $U^\perp \to \mathbb{1} \to V$ is an isomorphism, so $\mathbb{1} \simeq U^\perp \oplus U$. □

PROPOSITION 5.6 *Let* C *be an abelian symmetric monoidal category with internal Homs. If* $\mathbb{1}$ *is simple, then the tensor product of any two nonzero objects of* C *is nonzero provided one admits a strong dual.*

PROOF Let X and Y be nonzero objects of C with X admitting a dual. As X is nonzero, the equality (4.1)

$$(\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X$$

shows that the evaluation morphism $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ is nonzero, and hence an epimorphism because $\mathbb{1}$ is simple. Now $X^\vee \otimes X \otimes Y \to Y$ is an epimorphism (5.2), which implies that $X \otimes Y$ is nonzero. □

REMARK 5.7 It follows from the Proposition 5.5 that, in an abelian rigid symmetric monoidal category, there is a one-to-one correspondence between subobjects of $\mathbb{1}$ and idempotents in $\mathrm{End}(\mathbb{1})$. Such an idempotent e determines a decomposition of symmetric monoidal categories $\mathsf{C} = \mathsf{C}' \times \mathsf{C}''$ in which the objects of C' (resp. C'') are those on which e (resp. $1 - e$) acts as the identity morphism.

PROPOSITION 5.8 *Let* C *and* C' *be abelian symmetric monoidal categories with identity objects* $\mathbb{1}$ *and* $\mathbb{1}'$*. If* C *is rigid,* $\mathbb{1}$ *is simple, and* $\mathbb{1}' \neq 0$*, then every right exact* $\otimes$*-functor* $F : \mathsf{C} \to \mathsf{C}'$ *is faithful.*

PROOF As F is additive and exact, it suffices to show that $F(X) \neq 0$ when $X \neq 0$. If $X \neq 0$, then $\mathrm{ev} : X^\vee \otimes X \to \mathbb{1}$ is an epimorphism because $\mathbb{1}$ is simple, and so

$$F(X^\vee) \otimes F(X) \simeq F(X^\vee \otimes X) \to F(\mathbb{1}) \simeq \mathbb{1}'$$

is an epimorphism. As $\mathbb{1}' \neq 0$, this implies that $F(X) \neq 0$. □

Note that $\mathbb{1}$ is simple if $\mathrm{End}(\mathbb{1})$ is a field, and $\mathbb{1}' \neq 0$ if $\mathsf{C}' \neq 0$.

PROPOSITION 5.9 (DELIGNE 1990, 2.10) *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a right exact* $\otimes$*-functor between abelian symmetric monoidal categories. If* C *is rigid and the tensor product on* D *is right exact, then* F *is exact.*

PROOF Let $0 \to X \to Y \to Z$ be an exact sequence in C. On applying F to the dual sequence, we get an exact sequence

$$F(Z)^\vee \to F(Y)^\vee \to F(X)^\vee \to 0 \tag{34}$$

in D. Let $U \to V \to W \to 0$ be an exact sequence in D. If $\mathcal{H}om(U, \mathbb{1})$ and $\mathcal{H}om(V, \mathbb{1})$ exist in D, then $\mathcal{H}om(W, \mathbb{1})$ exists in D and the sequence

$$0 \to \mathcal{H}om(W, \mathbb{1}) \to \mathcal{H}om(V, \mathbb{1}) \to \mathcal{H}om(U, \mathbb{1}) \tag{35}$$

is exact. As $F(X)$ is dual to $F(X)^\vee$, it equals $\mathcal{H}om(F(X)^\vee, \mathbb{1})$ (by 4.8) and similarly for Y and Z. For (34), the sequence (35) becomes

$$0 \to F(X) \to F(Y) \to F(Z),$$

which is therefore exact. □

Traces in abelian rigid symmetric monoidal categories

In this subsection, the underlying category is an abelian rigid symmetric monoidal category. We show that traces are additive on short exact sequences.

PROPOSITION 5.10 (DELIGNE) *For any exact commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0, \end{array}$$

$$\mathrm{Tr}_X(f) = \mathrm{Tr}_{X'}(f') + \mathrm{Tr}_{X''}(f'').$$

In particular,

$$\dim(X) = \dim(X') + \dim(X'').$$

PROOF For an object X of such a category, let t_X denote the morphism (26)

$$\mathcal{H}om(X,X) \simeq X^\vee \otimes X \xrightarrow{\text{ev}} \mathbb{1}.$$

Then $\mathrm{Hom}(\mathbb{1}, t_X)$ is the trace map $\mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$.

For a short exact sequence

$$\Sigma:\ 0 \to X' \to X \to X'' \to 0,$$

let

$$\mathcal{H}om(\Sigma,\Sigma) = \mathrm{Ker}\left(\mathcal{H}om(X,X) \to \mathcal{H}om(X',X'')\right).$$

Thus,

$$\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(\Sigma,\Sigma)) \simeq \mathrm{End}(\Sigma) \overset{\text{def}}{=} \{f \in \mathrm{End}(X) \mid f \text{ respects } \Sigma\}.$$

It suffices to show that the diagram

$$\begin{array}{ccc} \mathcal{H}om(\Sigma,\Sigma) & \longrightarrow & \mathcal{H}om(X',X') \oplus \mathcal{H}om(X'',X'') \\ \downarrow & & \downarrow{\scriptstyle t_{X'}+t_{X''}} \\ \mathcal{H}om(X,X) & \xrightarrow{\quad t_X \quad} & \mathbb{1} \end{array} \tag{36}$$

commutes. On tensoring Σ with its dual, we get a diagram with exact rows and columns (5.2),

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X''^\vee \otimes X' & \longrightarrow & X''^\vee \otimes X & \longrightarrow & X''^\vee \otimes X'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X^\vee \otimes X' & \longrightarrow & X^\vee \otimes X & \longrightarrow & X^\vee \otimes X'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X'^\vee \otimes X' & \longrightarrow & X'^\vee \otimes X & \longrightarrow & X'^\vee \otimes X'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

From a diagram chase, we see that $(X''^\vee \otimes X) \oplus (X^\vee \otimes X')$ maps onto the kernel of $X^\vee \otimes X \to X'^\vee \otimes X''$. Hence $\mathcal{H}om(X'',X) \oplus \mathcal{H}om(X,X')$ maps onto $\mathcal{H}om(\Sigma,\Sigma)$, and so it suffices to show that (36) commutes with $\mathcal{H}om(\Sigma,\Sigma)$ replaced by each of $\mathcal{H}om(X'',X)$ and $\mathcal{H}om(X,X')$. On $\mathcal{H}om(X'',X)$, the map to $\mathcal{H}om(X',X')$ vanishes, and so we have to show that

$$\begin{array}{ccc} \mathcal{H}om(X'',X) & \longrightarrow & \mathcal{H}om(X'',X'') \\ \downarrow & & \downarrow{\scriptstyle t_{X''}} \\ \mathcal{H}om(X,X) & \xrightarrow{\quad t_X \quad} & \mathbb{1} \end{array}$$

commutes, and dually for $\mathcal{H}om(X,X')$.

From the identity $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$, we see that the diagram

$$\begin{array}{ccc} \mathcal{H}om(X,X'') \otimes \mathcal{H}om(X'',X) & \longrightarrow & \mathcal{H}om(X'',X'') \\ \downarrow & & \downarrow{\scriptstyle t_{X''}} \\ \mathcal{H}om(X,X) & \xrightarrow{\quad t_X \quad} & \mathbb{1} \end{array}$$

commutes. On composing with the morphism $\mathbb{1} \to \mathcal{H}om(X, X'')$ that corresponds to the given morphism $X \to X''$ under the isomorphism

$$\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X, X'')) \simeq \mathrm{Hom}(X, X''),$$

we obtain the required diagram for $\mathcal{H}om(X'', X)$. □

COROLLARY 5.11 *The trace of any nilpotent endomorphism is zero.*

PROOF If the endomorphism f of X is nilpotent, then there exists a finite decreasing filtration F of X such that $f(F^i) \subset F^{i+1}$, for example, the filtration with $F^i = f^i(X)$. Then $f = 0$ on F^i/F^{i+1}, and so

$$\mathrm{Tr}(f) \overset{5.10}{=} \sum \mathrm{Tr}(f|F^i/F^{i+1}) = 0.$$ □

The corollary fails for categories that are not abelian.

NOTES This subsection follows a letter of Deligne, dated July 11, 2003.

Traces and fibre functors

DEFINITION 5.12 Let C be a k-linear abelian $\otimes$-category. A ***fibre functor*** on C with values in a k-algebra R is a k-linear right exact $\otimes$-functor $\mathsf{C} \to \mathsf{Mod}(R)$.

In the rest of this subsection, C is a k-linear *rigid* abelian $\otimes$-category. Because C is rigid, fibre functors on C are exact (5.9).

5.13 Let $\omega : \mathsf{C} \to \mathsf{Mod}(R)$ be a fibre functor on C, and let (U, u) be a unit in C. There is a unique isomorphism $a_\omega : R \to \omega(U)$ such that the top square in the following diagram commutes,

$$\begin{array}{ccc} R \otimes R & \longrightarrow & R \\ \downarrow{\scriptstyle a_\omega \otimes a_\omega} & & \downarrow{\scriptstyle a_\omega} \\ \omega(U) \otimes \omega(U) & \xrightarrow{\omega(u)} & \omega(U) \\ \downarrow{\scriptstyle \eta_U \otimes \eta_U} & & \downarrow{\scriptstyle \eta_U} \\ \omega'(U) \otimes \omega'(U) & \xrightarrow{\omega'(u)} & \omega'(U) \end{array}$$

(see 1.8). If ω' is a second R-valued fibre functor on C and $\eta : \omega \to \omega'$ is an isomorphism of $\otimes$-functors, then

$$a_{\omega'} = \eta_U \circ a_\omega \tag{37}$$

follows from the above diagram.

5.14 For an object X of C and an R-valued fibre functor ω, we define

$$\mathrm{Tr}_\omega : \mathrm{End}(X) \to R$$

by

$$\mathrm{Tr}_\omega(\alpha) = \mathrm{Tr}(\omega(\alpha) \mid \omega(X)), \quad \alpha \in \mathrm{End}(X).$$

Alternatively, Tr_ω is the composite of

$$\mathrm{End}(X) \xrightarrow{\mathrm{Tr}_X} \mathrm{End}(U) \xrightarrow{\omega} \mathrm{End}(\omega(U)) \simeq R,$$

where the canonical isomorphism is defined by a_ω. From this second description and (37), we see that $\mathrm{Tr}_\omega = \mathrm{Tr}_{\omega'}$ if ω and ω' are isomorphic, or even if they become isomorphic after a faithfully flat homomorphism $R \to R'$.

5.15 Assume now that the fibre functors on C are locally isomorphic for the fpqc topology, i.e., if ω and ω' are R-valued fibre functors on C, then there exists a faithfully flat homomorphism $R \to R'$ such that $\omega \otimes_R R' \approx \omega' \otimes_R R'$.

Then, for any R-valued fibre functor ω,

$$\mathrm{Tr}_\omega(\alpha) \in k, \quad \text{all } X \in \mathrm{ob}\,\mathsf{C} \text{ and } \alpha \in \mathrm{End}(X).$$

To see this, let ω be an R-valued fibre functor on C and consider the exact sequence

$$0 \longrightarrow k \longrightarrow R \underset{i_2}{\overset{i_1}{\rightrightarrows}} R \otimes_k R.$$

Then $\omega_1 \overset{\text{def}}{=} \omega \otimes_{R,i_1} R \otimes_k R$ and $\omega_2 \overset{\text{def}}{=} \omega \otimes_{R,i_2} R \otimes_k R$ are $R \otimes_k R$-valued fibre functors on C. Let α be an endomorphism of an object X of C. Under the maps i_1 and i_2, $\mathrm{Tr}_\omega(\alpha)$ maps to $\mathrm{Tr}_{\omega_1}(\alpha)$ and $\mathrm{Tr}_{\omega_2}(\alpha)$. As ω_1 and ω_2 are locally isomorphic, $\mathrm{Tr}_{\omega_1}(\alpha)$=$\mathrm{Tr}_{\omega_2}(\alpha)$ (by 5.14), and so $\mathrm{Tr}_\omega(\alpha) \in k$.

5.16 If C admits a nonzero fibre functor and any two fibre functors are locally isomorphic, then

$$\mathrm{End}(\mathbb{1}) = k.$$

Apply 5.15 to an endomorphism of $\mathbb{1}$.

6 Tensor categories

In this section, k is a field.

DEFINITION 6.1 A ***tensor category over*** k is a k-linear abelian rigid symmetric monoidal category such that the structure homomorphism $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism.[5]

In other words, a tensor category over k is an abelian rigid symmetric monoidal category equipped with an isomorphism $k \simeq \mathrm{End}(\mathbb{1})$.

DEFINITION 6.2 A ***tensor subcategory*** of a tensor category over k is a strictly full additive subcategory stable under direct sums, tensor products, duals, and subquotients.

In particular, a tensor subcategory is an abelian subcategory (abelian category such that the inclusion map is exact), contains an identity object, and is a tensor category over k.

DEFINITION 6.3 A ***tensor functor*** of tensor categories over k is a right exact k-linear symmetric monoidal functor.

PROPOSITION 6.4 *Tensor functors between tensor categories over k are exact and faithful.*

PROOF This follows from 5.8 and 5.9. □

Let $P \in \mathbb{N}[x,y]$. When X, Y are objects of a symmetric monoidal category, we can define $P(X,Y)$ by interpreting addition as $\oplus$ and multiplication as $\otimes$, so

$$P(x,y) = \textstyle\sum_{i,j} m_{i,j} x^i y^j \;\Rightarrow\; P(X,Y) = \bigoplus_{i,j} m_{i,j} X^{\otimes i} \otimes Y^{\otimes j}.$$

[5] Following Deligne 1990, 2.1. The condition $k \simeq \mathrm{End}(\mathbb{1})$ is necessary for the fibre functors to form a gerbe (5.16). Some authors do not require tensor categories over k to be symmetric and some require them to be essentially small or locally finite.

DEFINITION 6.5 For an object X of an abelian rigid symmetric monoidal category C, we let $\langle X\rangle^{\otimes}$ denote the strictly full subcategory of C whose objects are subquotients of $P(X, X^\vee)$ for some $P \in \mathbb{N}[t, s]$. It is again an abelian rigid symmetric monoidal category. We call X a ***tensor generator*** of C if $\mathsf{C} = \langle X\rangle^{\otimes}$.

In other words, $\langle X\rangle^{\otimes}$ is the strictly full subcategory of C whose objects are those that can be constructed from X by iterated applications of the operations of direct sum, tensor product, dual, passage to a subobject or a quotient object. When C is a tensor category over k, $\langle X\rangle^{\otimes}$ is the smallest tensor subcategory containing X.

DEFINITION 6.6 A tensor category over k is ***monogenic*** if it admits a tensor generator.

Following convention, we usually call a monogenic tensor category a finite generated tensor category. Note that if a tensor category is generated by objects $X_1, \dots, X_n$, i.e., is the smallest tensor subcategory over k containing the X_i, then it equals $\langle X\rangle^{\otimes}$, where $X = X_1 \oplus \cdots \oplus X_n$.

DEFINITION 6.7 A k-linear abelian category is ***locally finite*** if every object has finite length and every k-vector space $\mathrm{Hom}(X, Y)$ has finite dimension.

An object of an abelian category is said to have finite length if it admits a (finite) composition series, in which case all composition series have the same length. Both the Jordan–Hölder and Krull–Schmidt theorems hold in locally finite k-linear abelian categories.

PROPOSITION 6.8 *A tensor category over k is locally finite if every object has finite length.*

PROOF Certainly, $\mathrm{Hom}(\mathbb{1}, \mathbb{1}) = k$ is finite dimensional. Recall ((10), p. 18) that, for objects X, Y,

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X, Y)).$$

As $\mathbb{1}$ is simple (5.5), the morphism $\mathbb{1}^{\oplus n} \to \mathcal{H}om(X, Y)$ defined by linearly independent morphisms $f_1, \dots, f_n : X \to Y$ is a monomorphism. Therefore,

$$\dim_k \mathrm{Hom}(X, Y) \le \mathrm{length}(\mathcal{H}om(X, Y)).$$

□

ASIDE 6.9 Let $F : \mathsf{C} \to \mathsf{D}$ be a k-linear exact $\otimes$-functor of essentially small rigid tensor categories over k. By the special adjoint functor theorem, F has a right adjoint $F_* : \mathrm{Ind}\,\mathsf{D} \to \mathrm{Ind}\,\mathsf{C}$. When C and D are locally finite, the functor F is dominant if and only if F_* is faithful. See Coulembier 2026, 1.3.4.

ASIDE 6.10 A tensor category over k with finite-dimensional Homs need not be locally finite (Coulembier and Sherman 2026, Appendix A).

Extension of scalars

In this subsection, k is a field. Let C be a k-linear abelian category.

6.11 For an X in C and a finite-dimensional vector space V, the functor

$$T \rightsquigarrow \mathrm{Hom}_k(V, \mathrm{Hom}_{\mathsf{C}}(X, T))$$

is representable, and we let $V \otimes X$ denote the object in C representing it, so

$$\mathrm{Hom}_{\mathsf{C}}(V \otimes X, T) \simeq V \otimes_k \mathrm{Hom}_{\mathsf{C}}(X, T).$$

If $(e_i)_{i\in I}$ is a basis for V, then $V\otimes X$ is a direct sum of copies of X indexed by I. Now

$$V, X \rightsquigarrow V\otimes X : \mathsf{Vecf}(k)\times \mathsf{C}\to \mathsf{C}$$

is a functor such $F(V\otimes X)\simeq V\otimes F(X)$ for any k-linear functor $F:\mathsf{C}\to\mathsf{Vecf}(k)$.

When C has small inductive limits, in particular, all small direct sums, this construction works for all vector spaces over k, not necessarily finite-dimensional.

6.12 Let k' be a finite extension of k. We define $\mathsf{C}_{(k')}$ to be the category whose objects are the pairs (X,α), where X is an object of C and α is a k'-module structure on X, i.e., a k-algebra homomorphism $k'\to \mathrm{End}(X)$.[6] A morphism $(X,\alpha)\to(Y,\beta)$ is a morphism $f: X\to Y$ such that $f\circ\alpha(c)=\beta(c)\circ f$ for all $c\in k'$. The category $\mathsf{C}_{(k')}$ is a k'-linear abelian category, said to have been obtained from C by ***extension of scalars***. For X in C, there is a natural k'-module structure on $k'\otimes X$ (tensor product in the sense of 6.11), and in this way we get a functor

$$i_{k'/k}:\mathsf{C}\to\mathsf{C}_{(k')},\quad X\rightsquigarrow k'\otimes X,\quad \text{(extension of scalars)}.$$

The forgetful functor is a functor

$$r_{k'/k}:\mathsf{C}_{(k')}\to\mathsf{C},\quad (X,\alpha)\rightsquigarrow X,\quad \text{(restriction of scalars)},$$

and

$$\mathrm{Hom}_{\mathsf{C}}(X, r_{k'/k}Y)\simeq \mathrm{Hom}_{\mathsf{C}_{(k')}}(i_{k'/k}X, Y),\quad X\in\mathsf{C}, Y\in\mathsf{C}_{(k')}.$$

In particular,

$$\mathrm{Hom}_{\mathsf{C}_{(k')}}(i_{k'/k}X, i_{k'/k}Y)\simeq \mathrm{Hom}_{\mathsf{C}}(X, k'\otimes Y)\simeq k'\otimes_k \mathrm{Hom}_{\mathsf{C}}(X,Y),\quad X,Y\in \mathrm{ob}\,\mathsf{C}.$$

As $r_{k'/k}$ and $i_{k'/k}$ are right and left adjoints, the first is left exact and the second is right exact. In fact, both functors are exact. For example, to see that $r_{k'/k}$ is right exact, let $(X,\alpha)\to(Y,\beta)$ be an epimorphism in $\mathsf{C}_{(k')}$, and let C be the cokernel of $X\to Y$ in C; then C acquires a k'-module structure from those on X and Y, and so it is 0. See Deligne 2014, 5.1, and I, 12.22.

EXAMPLE 6.13 Let A be a finite-dimensional k-algebra and k' a finite extension of k. The k-linear abelian category $\mathsf{Modf}(A)$ is locally finite, and $\mathsf{Modf}(A)_{(k')}$ can be identified with $\mathsf{Modf}(k'\otimes_k A)$. In this case, $i_{k'/k}$ and $r_{k'/k}$ are the usual extension and restriction functors.

PROPOSITION 6.14 *Let C be a locally finite k-linear abelian category.*

(a) *Let X be a simple object of C.*

  i) *$D\overset{\text{def}}{=}\mathrm{End}(X)$ is a division algebra, finite-dimensional over k, and its centre F is a finite extension of k.*

  ii) *$i_{k'/k}(X)$ is semisimple in $\mathsf{C}_{(k')}$ if and only if $k'\otimes_k F$ is a product of fields.*

(b) *If $Y=(X,\alpha)$ is a semisimple object of $\mathsf{C}_{(k')}$, then X is a semisimple object of C.*

(c) *Let X be an object of C. If $i_{k'/k}(X)$ is simple (resp. semisimple), then so also is X.*

[6] Equivalently: let $k'\otimes X$ denote the tensor product in the sense of 6.11; a k'-module structure on X is defined by a morphism $\mu: k'\otimes X\to X$ satisfying the usual conditions (9.4).

PROOF (a) (i). This is obvious.

(ii). The functor $\mathrm{Hom}(X,-)$ is an equivalence to Vecf_D from the strictly full subcategory of C whose objects are the finite direct sums of copies of X. This remains an equivalence when we extend scalars to k', and $i_{k'/k}(X)$ corresponds to the right $k'\otimes_k D$-module $k'\otimes_k D$, which is semisimple if and only if $F\otimes_k k'$ is a product of fields (Bourbaki A, VIII, §7).

(b) We may suppose that Y is simple. The sum of the simple subobjects of X is a nonzero k'-submodule of X, hence equals X (by the simplicity of Y).

(c) If $i_{k'/k}(X)$ is semisimple, then X is semisimple because it is a subobject of $r(i(X)$, which is semisimple by (b). If $i_{k'/k}(X)$ is simple, then X is obviously simple. □

PROPOSITION 6.15 *Let C be a locally finite k-linear abelian category, and let X be an object of C. If X is a sum of simple subobjects, say, $X=\sum_{i\in I}S_i$ (the sum need not be direct), then for every subobject Y of X, there is a subset J of I such that*

$$X = Y\oplus\bigoplus\nolimits_{i\in J}S_i.$$

In particular, X is a finite direct *sum of simple subobjects, and Y is a direct summand of X.*

PROOF Let J be maximal among the subsets of I such that the sum $S_J=\sum_{j\in J}S_j$ is direct and $Y\cap S_J=0$. We claim that $Y+S_J=X$ (hence X is the direct sum of Y and the S_j with $j\in J$). For this, it suffices to show that each S_i is contained in $Y+S_J$. Because S_i is simple, $S_i\cap(Y+S_J)$ equals S_i or 0. In the first case, $S_i\subset Y+S_J$, and in the second $S_J\cap S_i=0$ and $Y\cap(S_J+S_i)=0$, contradicting the maximality of J. □

DEFINITION 6.16 An abelian category is ***semisimple*** if every object is a finite direct sum of simple objects.

PROPOSITION 6.17 *Let C be a locally finite k-linear abelian category, and let k' be a finite extension of k. If $\mathsf{C}_{(k')}$ is semisimple, then so also is C, and the converse is true if k' is separable over k.*

PROOF The necessity follows directly from 6.14(c), so suppose that k' is separable over k. If X in C is semisimple, then $i_{k'/k}(X)$ is semisimple (6.14(a)), and every object Y of $\mathsf{C}_{(k')}$ is a direct factor of such an object. More precisely, the adjunction morphism $(i\circ r)(Y)\to Y$ splits. The object $(i\circ r)(Y)=k'\otimes_k Y$ has two k'-module structures, that provided by k' and that provided by Y, and hence an action of $k'\otimes_k k'$. The adjunction morphism is the natural morphism $k'\otimes Y\to(k'\otimes Y)\otimes_{k'\otimes_k k'}k'$. This is split because $k'\otimes_k k'\to k'$ is projection onto a direct factor. □

Let $(\mathsf{C},\otimes)$ be a tensor category over k and k' a finite extension of k. For objects X and Y of $\mathsf{C}_{(k')}$, let

$$X\otimes_{k'}Y=\mathrm{Coker}(X\otimes k'\otimes Y\rightrightarrows X\otimes Y). \tag{38}$$

The two morphisms are defined by the actions of k' on X and Y respectively.

PROPOSITION 6.18 *Let $(\mathsf{C},\otimes)$ be a tensor category over k and k' a finite extension of k. With the tensor product (38), $\mathsf{C}_{(k')}$ becomes a k'-linear abelian $\otimes$-category, and $i_{k'/k}:\mathsf{C}\to\mathsf{C}_{(k')}$ is a k-linear right exact $\otimes$-functor. Moreover, $i_{k'/k}$ sends unit objects to unit objects and duals to duals. Internal Homs exist in $\mathsf{C}_{(k')}$.*

PROOF The proof is straightforward, If $X' = (X, \mu_X)$ and $Y' = (Y, \mu_Y)$ are objects of $\mathsf{C}_{(k')}$, then $\mathcal{H}om(X', Y')$ is the intersection of the kernels of the morphisms

$$f \mapsto \lambda f - f\lambda : \mathcal{H}om(X, Y) \to \mathcal{H}om(X, Y),$$

where λ runs over a basis for k' as a k-vector space. □

PROPOSITION 6.19 *Let $(\mathsf{C}, \otimes)$ be a tensor category over k and k' a finite extension of k. If k'/k is separable, then $\mathsf{C}_{(k')}$ is a tensor category over k' and $i_{k'/k}$ is exact.*

PROOF If $\mathbb{1}$ is an identity object in C, then $i_{k'/k}(\mathbb{1})$ is an identity object in $\mathsf{C}_{(k')}$, and clearly $\mathrm{End}(i_{k'/k}(\mathbb{1})) \simeq k'$. To show that $\mathsf{C}_{(k')}$ is a tensor category, it remains to show that duals exist. If X in C has dual $X^\vee$, then $i_{k'/k}(X)$ has dual $i_{k'/k}(X^\vee)$, and every object Y of $\mathsf{C}_{(k')}$ is a direct factor of such an object. More precisely, the adjunction morphism $(i \circ r)(Y) \to Y$ splits (see the proof of 6.17). Finally, $i_{k'/k}$ is exact by 5.3 and 5.9, □

NOTES This subsection is based on Deligne 2014, 5.3.

Descent of tensor categories

6.20 Let k'/k be a finite Galois extension with Galois group Γ, and let C' be a tensor category over k'. A ***descent datum*** on C' relative to k'/k is

(a) a family $(\beta_\gamma)_{\gamma\in\Gamma}$ of equivalences of symmetric monoidal categories $\beta_\gamma : \mathsf{C}' \to \mathsf{C}'$, β_γ being semi-linear relative to γ, together with

(b) a family $(\mu_{\gamma',\gamma})$ of isomorphisms of tensor functors $\mu_{\gamma',\gamma} : \beta_{\gamma'\gamma} \xrightarrow{\approx} \beta_{\gamma'} \circ \beta_\gamma$ such that

$$\begin{array}{ccc} \beta_{\gamma''\gamma'\gamma}(X) & \xrightarrow{\mu_{\gamma'',\gamma'\gamma}(X)} & \beta_{\gamma''}(\beta_{\gamma'\gamma}(X)) \\ \downarrow{\scriptstyle \mu_{\gamma''\gamma',\gamma}(X)} & & \downarrow{\scriptstyle \beta_{\gamma''}(\mu_{\gamma'\gamma}(X))} \\ \beta_{\gamma''\gamma'}(\beta_\gamma(X)) & \xrightarrow{\mu_{\gamma''\gamma'}(\beta_\gamma(X))} & \beta_{\gamma''}(\beta_{\gamma'}(\beta_\gamma(X))) \end{array}$$

commutes for all $X \in \mathrm{ob}(\mathsf{C})$.

6.21 For a tensor category C over k, we let $\mathsf{C}_{(k')}$ denote the category of pairs (X, α), where X is an object of C and α is a k-linear homomorphism $k' \to \mathrm{End}(X)$. Then $\mathsf{C}_{(k')}$ is a tensor category over k' (6.18) with a canonical descent datum for which

$$\beta_\gamma(X, \alpha_X) = (X, \alpha_X \circ \gamma^{-1}).$$

Conversely, a tensor category C' over k' together with a descent datum relative to k'/k gives rise to a tensor category $(\mathsf{C}')^\Gamma$ over k whose objects are pairs $(X, (a_\gamma))$, where $X \in \mathrm{ob}(\mathsf{C}')$ and $(a_\gamma : X \to \beta_\gamma(X))_{\gamma\in\Gamma}$ is such that $(\mu_{\gamma',\gamma})_X \circ a_{\gamma'\gamma} = \beta_{\gamma'}(a_\gamma) \circ a_{\gamma'}$, and whose morphisms are morphisms in C' commuting with the a_γ.

These operations are inverse in the sense that there are canonical equivalences of tensor categories

$$\mathsf{C} \to (\mathsf{C}_{(k')})^\Gamma, \quad \left(\mathsf{C}'^\Gamma\right)_{(k')} \to \mathsf{C}'.$$

It follows that to give a tensor category over k amounts to giving a tensor category over k' equipped with a descent datum relative to k'/k.[7] The proofs of these statements are elementary for k-linear abelian categories, and extend without difficulty to tensor categories. See Saavedra 1972, III, 1.2.

[7] More precisely, there is a biequivalence of 2-categories.

7 Tannakian categories

In this section, k is a field.

DEFINITION 7.1 Let $(\mathsf{C}, \otimes)$ be a tensor category over k and R a k-algebra.

(a) A ***fibre functor on*** C ***with values in*** R (or an R-***valued fibre functor on*** C) is a k-linear exact $\otimes$-functor $\omega : \mathsf{C} \to \mathsf{Mod}(R)$.

(b) A ***morphism of fibre functors*** is a monoidal natural transformation (2.11); it is automatically an isomorphism (4.21).

More generally, a ***fibre functor on*** C ***over a*** k-***scheme*** S is an exact k-linear $\otimes$-functor from C to the category of quasi-coherent sheaves on S. Because C is rigid, this agrees with Definition 5.12.

DEFINITION 7.2 A ***tannakian category over*** k is a tensor category over k that admits an R-valued fibre functor for some nonzero k-algebra R. If there exists a fibre functor with values in k itself, then the category is said to be ***neutral***.

In other words, a tannakian category over k is a k-linear abelian rigid symmetric monoidal category C such that the structure map $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism and such that there exists a fibre functor with values in some nonzero k-algebra R.

DEFINITION 7.3 A ***tannakian subcategory*** of a tannakian category over k is a tensor subcategory (6.2). In other words, it is a strictly full abelian subcategory closed under the formation of direct sums, tensor products, duals, and subquotients. It is again a tannakian category over k – any fibre functor restricts to a fibre functor.

There exist strictly full abelian subcategories of tannakian categories closed under the formation of tensor products and duals but not subquotients. See II, 5.5, below.

Let C be a tensor category over k and R a k-algebra.

PROPOSITION 7.4 *An R-valued fibre functor ω on C is faithful if $R \neq 0$.*

PROOF Now assume that $R \neq 0$. As ω is exact, it suffices to show that $\omega(X) \neq 0$ if $X \neq 0$. For any nonzero X, $\mathrm{ev} : X^\vee \otimes X \to \mathbb{1}$ is an epimorphism, so $\omega(X)^\vee \otimes \omega(X)^\vee \to R$ is surjectivc, which implies that $\omega(X) \neq 0$. □

PROPOSITION 7.5 *Let ω be an R-valued fibre functor on C and X a nonzero object of C. Then $\omega(X)$ is finitely generated and projective, hence flat. It is faithfully flat if $R \neq 0$.*

PROOF As X admits a dual, so also does $\omega(X)$, and hence it is finitely generated and projective (4.9). In particular, it is flat. Assume that $R \neq 0$. To show that $\omega(X)$ is faithfully flat, we have to show that, for all maximal ideals $\mathfrak{m}$ of R, $\omega(X) \otimes_R R/\mathfrak{m} \neq 0$. The functor $X \rightsquigarrow \omega(X) \otimes_R R/\mathfrak{m}$ is again a fibre functor, so this follows from 7.4. □

Thus, ω takes values in $\mathsf{Projf}(R)$, but is only exact as a functor to $\mathsf{Mod}(R)$.

PROPOSITION 7.6 *Let ω be an R-valued fibre functor on C. For any R-algebra R', the functor*

$$X \rightsquigarrow \omega_{R'}(X) \stackrel{\text{def}}{=} \omega(X) \otimes_R R'$$

is an R'-valued fibre functor on C.

PROOF Certainly, $\omega_{R'}$ is a k-linear right exact $\otimes$-functor $\mathsf{C} \to \mathsf{Mod}(R')$, hence a fibre functor. Here is a direct proof that ω exact implies $\omega_{R'}$ exact. If

$$0 \to X' \to X \to X'' \to 0$$

is a short exact sequence in C, then

$$0 \to \omega(X') \to \omega(X) \to \omega(X'') \to 0$$

is exact in $\mathsf{Mod}(R)$. As $\omega(X'')$ is projective, the sequence splits, and so it remains exact when tensored with any R-algebra. Thus, $\omega_{R'}$ is exact. □

COROLLARY 7.7 *Every tannakian category C admits a fibre functor with values in a field.*

PROOF Let ω be an R-valued fibre functor on C with $R \neq 0$. For any maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is field containing k and $X \rightsquigarrow \omega(X) \otimes_R R/\mathfrak{m}$ is a fibre functor with values in $R/\mathfrak{m}$. □

PROPOSITION 7.8 *Every exact $\otimes$-functor from a tannakian category to a nonzero abelian $\otimes$-category with $\mathbb{1} \neq 0$ is faithful. In particular, every R-valued fibre functor, $R \neq 0$, is faithful.*

PROOF This is a special case of 5.8. □

7.9 An object of a tannakian category C is said to be ***trivial*** if it is isomorphic to a finite sum of copies of $\mathbb{1}$. As $\mathbb{1}$ is simple (5.5), every such object is a finite *direct* sum of copies of $\mathbb{1}$, and the full subcategory of trivial objects is stable under the formation of subquotients in C. The functors

$$\mathsf{Vecf}(k) \underset{\mathrm{Hom}(\mathbb{1},X)\leftsquigarrow X}{\overset{V\rightsquigarrow V\otimes\mathbb{1}}{\rightleftarrows}} \mathsf{C}$$

are adjoint,

$$\mathrm{Hom}(V \otimes \mathbb{1}, X) \simeq \mathrm{Hom}(V, \mathrm{Hom}(\mathbb{1}, X)).$$

Here $V \otimes \mathbb{1}$ is as in 6.11. The unit of the adjunction

$$V \to \mathrm{Hom}(\mathbb{1}, V \otimes \mathbb{1})$$

(see A.2) is an isomorphism, whereas the counit

$$\mathrm{Hom}(\mathbb{1}, X) \otimes \mathbb{1} \to X$$

is a monomorphism, and it is an isomorphism if and only if X is trivial. Therefore, the functors define an equivalence of $\mathsf{Vecf}(k)$ with the full subcategory of trivial objects in C.

PROPOSITION 7.10 *Let C be a tannakian category over k, and let ω be a fibre functor with values in a nonzero k-algebra R. For all X, Y in C, the canonical map*

$$\mathrm{Hom}(X, Y) \otimes_k R \to \mathrm{Hom}_R(\omega(X), \omega(Y))$$

is injective.

PROOF Recall (10), p. 18, that $\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X,Y)) \simeq \mathrm{Hom}(X,Y)$. In particular, each element of $\mathrm{Hom}(X,Y)$ defines a morphism $\mathbb{1} \to \mathcal{H}om(X,Y)$, and linearly independent elements define linearly independent morphisms. Therefore, for each finite-dimensional subspace V of $\mathrm{Hom}(X,Y)$ we have a canonical monomorphism

$$V \otimes \mathbb{1} \hookrightarrow \mathcal{H}om(X,Y).$$

On applying ω to this monomorphism, we obtain an injective map

$$V \otimes_k R \hookrightarrow \omega(\mathcal{H}om(X,Y)) \simeq \mathrm{Hom}_R(\omega(X), \omega(Y)).$$

On passing to the inductive limit over the subspaces V, we obtain the required injective map. □

PROPOSITION 7.11 *Every tannakian category is locally finite.*

PROOF Let $k = \mathrm{End}(\mathbb{1})$. According to Proposition 7.7, the category has a fibre functor ω with values in a field K containing k. Then $\omega : \mathsf{C} \to \mathsf{Vecf}(K)$ is exact and faithful, and according to Proposition 7.10, the canonical map

$$\mathrm{Hom}(X,Y) \otimes_k K \to \mathrm{Hom}(\omega(X), \omega(Y))$$

is injective, and so $\mathrm{Hom}(X,Y)$ is finite-dimensional as a k-vector space. Subobjects of X are determined by their images in $\omega(X)$, and so X has finite length. □

DEFINITION 7.12 A ***pre-tannakian category*** over k is a locally finite tensor category over k.

Proposition 7.11 may fail for tensor categories over k. See Example 8.11 below.

DEFINITION 7.13 A tannakian category is said to be ***algebraic*** if it is finitely generated, i.e., admits a tensor generator in the sense of I, 6.5.

This definition of algebraic tannakian category agrees with that in Saavedra 1972, III, 3.3.1 (ibid., 3.3.1.1).

EXAMPLE 7.14 We show in 8.5 below that there exists a tannakian category over $\mathbb{R}$ containing an object X of dimension 2 with endomorphism algebra the quaternions. As the quaternions cannot act on a real vector space of dimension 2, there does not exist an $\mathbb{R}$-valued fibre functor, so the category is not neutral.

Extension of scalars

Let $(\mathsf{C}, \otimes)$ be a tensor category over k, and let k' be a finite extension of k. Recall (6.12, 6.18) that the category $\mathsf{C}_{(k')}$ of k'-modules in C is a k'-linear abelian $\otimes$-category with internal Homs. Moreover, the extension functor $i_{k'/k} : \mathsf{C} \to \mathsf{C}_{(k')}$ is an exact k-linear $\otimes$-functor.

THEOREM 7.15 (GROTHENDIECK) *Let k' be a finite extension of k. If $(\mathsf{C}, \otimes)$ is a tannakian category over k, then $(\mathsf{C}_{(k')}, \otimes_{k'})$ is a tannakian category over k'.*

The proof requires several lemmas.

LEMMA 7.16 *If $(\mathsf{C}_{(k')}, \otimes_{k'})$ is rigid, then it is tannakian over k'.*

PROOF We have to show that $\mathsf{C}_{(k')}$ admits a fibre functor. Let ω be a fibre functor on C with values in a nonzero k-algebra R. For any object (X,μ) in $\mathsf{C}_{(k')}$, $\omega(X)$ has the structure of an $R\otimes_k k'$-module. The functor

$$(X,\mu)\rightsquigarrow\omega(X):\ \mathsf{C}_{(k')}\to\mathsf{Mod}(R\otimes_k k')$$

is k'-linear and exact (because $(X,\mu)\rightsquigarrow X$ is exact, 6.12), and the natural isomorphism

$$\omega(X\otimes_{k'}Y)\simeq\omega(X)\otimes_{R\otimes_k k'}\omega(Y),$$

makes it into a symmetric monoidal functor. It is therefore a fibre functor. □

If $k''\supset k'\supset k$ are extensions of k, then $\mathsf{C}_{(k'')}$ can be identified with $(\mathsf{C}_{(k')})_{(k'')}$, and so it suffices to prove that $(\mathsf{C}_{(k')},\otimes_{k'})$ is a tensor category over k' in the two cases, k'/k separable and k'/k purely inseparable of degree p. The first case was treated in 6.19, and so it remains to prove the purely inseparable case. As $(\mathsf{C}_{(k')},\otimes_{k'})$ is a k'-linear abelian symmetric monoidal category and clearly $\mathrm{End}(\mathbb{1}')\simeq k'$, it remains to prove that it is rigid, i.e., that duals exist.

In the next two lemmas, $(\mathsf{C},\otimes)$ is a tannakian category over k and $k'=k(a^{1/p})$, $a\in k$.

LEMMA 7.17 *Let ω be a fibre functor on C with values in an extension K of k, and let X be an object of C. The $k'\otimes_k K$-module structure on $\omega(X)$ defined by any k'-module structure on X makes it into a free $k'\otimes_k K$-module.*

PROOF Let $K'=k'\otimes_k K$. If K' is a field, then the statement is obvious. Otherwise, a is a pth power in K, say, $a=\alpha^p$, and $K'=K[\varepsilon]/(\varepsilon^p)$, where $\varepsilon=\alpha\otimes 1-1\otimes\alpha$. Let d be the dimension of the K-vector space $\omega(X)/\varepsilon\omega(X)$. The K'-module $\omega(X)$ is isomorphic to a direct sum of d modules of the form $K[\varepsilon]/(\varepsilon^j)$, $1\le j\le p$. From this, we see that it is free if and only if $\bigwedge^d_{K'}\omega(X)$ is free (of rank one). Let $\bigwedge^d_{k'}X$ denote the image of the antisymmetrization map

$$\bigotimes^d X\to\bigotimes^d X.$$

Because ω is exact, it maps $\bigwedge^d_{k'}X$ to $\bigwedge^d_{K'}\omega(X)$. After replacing X with $\bigwedge^d_{k'}X$, we may suppose that $\omega(X)$ is a nonzero monogenic K'-module.

The k'-module structure on X defines a morphism

$$k'\otimes\mathbb{1}\to\mathcal{H}om(X,X).\tag{39}$$

As $\mathbb{1}$ is simple (5.5), the kernel of (39) is of the form $A\otimes\mathbb{1}$, where A is a vector subspace of k'. As A is also an ideal in k', it is 0, and so (39) is a monomorphism. On applying ω, we deduce that $\omega(X)$ is a faithful K'-module, and we conclude by noting that every faithful monogenic K'-module is free. □

LEMMA 7.18 *The $\otimes$-category $(\mathsf{C}_{(k')},\otimes_{k'})$ is rigid.*

PROOF Let $\mathbb{1}'$ be the unit object $k'\otimes\mathbb{1}$ of $\mathsf{C}_{(k')}$, and let $X\in\mathrm{ob}\,\mathsf{C}_{(k')}$. We show that the weak dual $\mathcal{H}om_{k'}(X,\mathbb{1}')$ of X is a dual. For this, it suffices to show that for all objects T of $\mathsf{C}_{(k')}$, the canonical map

$$T\otimes_{k'}\mathcal{H}om_{k'}(X,\mathbb{1}')\to\mathcal{H}om_{k'}(X,T)\tag{40}$$

is an isomorphism (see 4.7). Let ω be a K-valued fibre functor on C, and let ω' be the functor $(X,\mu) \rightsquigarrow \omega(X)$ taking values in $k' \otimes_k K$-modules. Then ω' transforms (40) into

$$\omega(T) \otimes_{k'} \operatorname{Hom}_{k'\otimes_k K}(\omega(X), k' \otimes_k K) \to \operatorname{Hom}_{k'\otimes_k K}(\omega(X), \omega(T)). \tag{41}$$

Lemma 7.17 shows that (41) is an isomorphism. The functor ω' is exact and such that

$$\omega'(Z) = 0 \implies Z = 0$$

because ω has these properties. Therefore ω' is conservative and so (40) is an isomorphism. □

7.19 (DELIGNE 2014, 5.10) Let C be tannakian category over k, and let k^a be an algebraic extension of k, for example, an algebraic closure of k. As k' runs over the finite extensions of k in k^a, the categories $\mathsf{C}_{(k')}$ form a 2-inductive system of abelian categories (not an inductive system, because we only have canonical isomorphisms $i_{k'''/k''} \circ i_{k''/k'} \simeq i_{k'''/k'}$ satisfying a cocycle condition). We define $\mathsf{C}_{(k^a)}$ to be the 2-inductive limit of this system. Specifically, an object of $\mathsf{C}_{(k^a)}$ is an object X of $\mathsf{C}_{(k')}$ for some $k' \subset k^a$ finite over k. If X' in $\mathsf{C}_{(k'')}$ is a second such object, then

$$\operatorname{Hom}_{\mathsf{C}_{(k^a)}}(X, X') = \varinjlim \operatorname{Hom}_{\mathsf{C}_{(k''')}}(i_{k'''/k'}X, i_{k'''/k''}X'),$$

where k''' runs over the finite extensions of k in k^a containing k' and k''.

The category $\mathsf{C}_{(k^a)}$ is a tensor category over k^a and $i_{k^a/k} : \mathsf{C} \to \mathsf{C}_{(k^a)}$ is an exact k-linear $\otimes$-functor. An R-valued fibre functor ω on C defines a $k' \otimes_k R$-valued fibre functor on $\mathsf{C}_{(k')}$ for each $k' \subset k^a$ (cf. the proof of 7.16). On passing to the limit, we obtain a $k^a \otimes_k K$-valued fibre functor on $\mathsf{C}_{(k^a)}$. Therefore, $\mathsf{C}_{(k^a)}$ is a tannakian category over k^a.

We list some consquences of Theorem 7.15 (Deligne 2014, 5.7, 5.9, 5.11).

COROLLARY 7.20 *If an object X of a tannakian category C over k admits a k'-module structure, where k' is a extension of k, then $[k' : k]$ is finite and divides* $\dim X$.

PROOF Let X be an object of C admitting a k'-module structure μ. Then (X,μ) is an object of $\mathsf{C}_{(k')}$. Let ω be a K-valued fibre functor on C. Then $(X,\mu) \rightsquigarrow \omega(X)$ is a $k' \otimes K$-valued fibre functor ω' on $\mathsf{C}_{(k')}$. As $\omega'(X,\mu)$ is locally free of rank equal to the (categorical) dimension of (X,μ), we have

$$\dim(X) = \dim_K \omega(X) = [k' \otimes K : K] \cdot \operatorname{rank}_{K'} \omega(X) = [k' : k] \cdot \dim(X,\mu).$$

Therefore, $[k' : k]$ divides $\dim(X)$, with quotient of the dimension of X as an object of $\mathsf{C}_{(k')}$. □

COROLLARY 7.21 *Let X be a simple object of a tannakian category C over a field k of characteristic $p \neq 0$. If $\dim X < p$, then the centre of* $\operatorname{End}(X)$ *is separable extension of k.*

PROOF Indeed, according to 7.20, it is an extension of degree $< p$. □

COROLLARY 7.22 *Let C be a tannakian category over a field k of characteristic p and k^a an algebraic extension of k. Let X be an object of* C.

(a) *If $i_{k^a/k}(X)$ is semisimple in $\mathsf{C}_{(k^a)}$, then X is semisimple in* C.

(b) *If X is semisimple in C and* $\dim X < p$, *then $i_{k^a/k}(X)$ is semisimple in $\mathsf{C}_{(k^a)}$.*

PROOF See Deligne 2014, 5.11. □

NOTES Theorem 7.15 is due to Grothendieck. The exposition follows Deligne 2014, 5.4.

8 Examples

EXAMPLE 8.1 The category $\mathsf{Vecf}(k)$ of finite-dimensional vector spaces over a field k is a tannakian category over k with the identity functor as a k-valued fibre functor. All the preceding definitions take on a familiar meaning when applied to $\mathsf{Vecf}(k)$. For example, $\mathrm{Tr}: \mathrm{End}(X) \to k$ is the usual trace map.

EXAMPLE 8.2 The category $\mathsf{Mod}(R)$ of modules over a commutative ring R is an abelian symmetric monoidal category such that $\mathrm{End}(\mathbb{1}) = R$. In general it is not rigid because not all R-modules will admit duals (in the sense of 4.1). The category $\mathsf{Modf}(R)$ of finitely generated R-modules is an R-linear abelian symmetric monoidal category if R is noetherian. (Otherwise it need not be abelian: if the ideal $I \subset R$ is not finitely generated, then the homomorphism $R \to R/I$ does not have a kernel.)

EXAMPLE 8.3 The category $\mathsf{Projf}(R)$ of finitely generated projective modules over a commutative ring R is an additive symmetric monoidal category such that $\mathrm{End}(\mathbb{1}) = R$. It is rigid (4.9) but not in general abelian.

EXAMPLE 8.4 Let G be an affine group scheme over a field k. The category $\mathsf{Repf}(G)$ of linear representations of G on finite-dimensional k-vector spaces is a tannakian category over k with the forgetful functor as a k-valued fibre functor.

EXAMPLE 8.5 (A NONNEUTRAL TANNAKIAN CATEGORY) We give an example of a tannakian category that is not neutral, hence not of the form $\mathsf{Repf}(G)$ for any affine group scheme G.

Let T be the category of pairs (V, F) consisting of a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space $V = V^0 \oplus V^1$ and an ι-semilinear endomorphism F of V such that $F^2 = (-1)^i$ on V^i. With the obvious tensor structure, T becomes a nonneutral tannakian category.

Let $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ denote the usual quaternion algebra over $\mathbb{R}$. Let $V = \mathbb{H}$ regarded as a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space of degree 1, and let F act on V as

$$a + bj \mapsto j(a + bj) = -\bar{b} + \bar{a}j.$$

Then (V, F) is an object in T of (categorical) dimension 2, and the action of $\mathbb{H}$ on itself by right multiplication defines an isomorphism $\mathbb{H}^{\mathrm{opp}} \to \mathrm{End}(V, F)$. The forgetful functor is a $\mathbb{C}$-valued fibre functor on T. Any $\mathbb{R}$-valued fibre functor on T would send (V, F) to a 2-dimensional $\mathbb{R}$-vector space on which $\mathbb{H}^{\mathrm{opp}}$ acts, which is impossible: there is no homomorphism of $\mathbb{R}$-algebras $\mathbb{H}^{\mathrm{opp}} \to M_2(\mathbb{R})$ because $\mathbb{H}$ is a division algebra.

EXAMPLE 8.6 (MORE NONNEUTRAL TANNAKIAN CATEGORIES) For a central simple algebra A of dimension n^2 over k, we construct a tannakian category T over k containing an object with categorical dimension n and endomorphism algebra A^{opp}. If $A \neq M_n(k)$, then T is not neutral.

As a first attempt, one might try defining T to be the category of finite-dimensional vector spaces over k equipped with a left action of A, but this is not rigid. Instead, we proceed as follows.

Suppose first that A contains a maximal subfield L Galois over k. Let E denote the group of invertible elements a of A such that $aLa^{-1} = L$. Each $a \in E$ defines an element $x \mapsto axa^{-1}$ of $\mathrm{Gal}(L/k)$, and, according to the Noether–Skolem theorem, all elements

of $\mathrm{Gal}(L/k)$ arise this way. Because L is a maximal subfield of A, it is its own centralizer in A, and so the sequence

$$1 \to L^\times \to E \to \mathrm{Gal}(L/k) \to 1$$

is exact.

Define T to be the category of representations of E on finite-dimensional L-vector spaces. Such a representation is a finite-dimensional $\mathbb{Z}$-graded L-vector space $V = \bigoplus_i V^i$ with $a \in L^\times$ acting as a^i on V^i together with an extension of the action of $L^\times$ on V to E.

Let V denote A regarded as a graded L-vector space of degree 1. The action of A on V by left multiplication induces an action r of E on V. The left and right actions of A on V define an isomorphism of k-algebras

$$A \otimes_k A^{\mathrm{opp}} \to \mathrm{End}_k(V).$$

This maps the centralizer A^{opp} of A in $A \otimes_k A^{\mathrm{opp}}$ isomorphically onto $\mathrm{End}(V, r)$.

Consider, for example, the case of a cyclic k-algebra A. Such an algebra equals $L[\gamma]$, where γ is an element such that $a \mapsto \gamma a \gamma^{-1} : L \to L$ is a generator σ of $\mathrm{Gal}(L/k)$ and $\gamma^n = c \in k$. The objects of T are the pairs (V, F) consisting of a finite-dimensional $\mathbb{Z}$-graded L-vector space V and a σ-semilinear endomorphism $F : V \to V$ such that F^n acts as c^i on V^i. For example, let V denote A regarded as a graded L-vector space of degree 1, and let F act on V as left multiplication by γ. Then (V, F) is an object of T and $\mathrm{End}(V, F) \simeq A^{\mathrm{opp}}$.

When no maximal subfield of A is Galois over k, we replace A with a suitable similar algebra. The tannakian category will then contain an object with endomorphism algebra $M_m(A)^{\mathrm{opp}}$, and a suitable direct factor of the object will have endomorphism algebra A^{opp}.

EXAMPLE 8.7 Let V' and V'' be vector spaces over k of the same dimension, each equipped with a nondegenerate quadratic form. Then the categories $\mathsf{Repf}(\mathrm{O}(V'))$ and $\mathsf{Repf}(\mathrm{O}(V''))$ are canonically equivalent as k-linear symmetric monoidal categories. To see this, note that $\mathrm{Isom}(V', V'')$ is an $\mathrm{O}(V')$ torsor with an action of $\mathrm{O}(V'')$. Twisting a representation of $\mathrm{O}(V')$ by the torsor gives the equivalence.

EXAMPLE 8.8 The finite groups D_4 and Q_8 have isomorphic representation rings (over $\mathbb{C}$, say) because they have the same character tables, but the categories $\mathsf{Repf}(D_4)$ and $\mathsf{Repf}(Q_8)$ are not equivalent as $\mathbb{C}$-linear tensor categories because a direct calculation shows that they have different associativity constraints. Alternatively, suppose that there is a tensor equivalence, which we may assume commutes with the forgetful fibre functors. Such an equivalence sends the trivial representation $\mathbb{1}_D$ of D_4 to the trivial representation $\mathbb{1}_Q$ of Q_8 (they are identity objects) and it sends the simple two-dimensional representation V_D of D_4 to the simple two-dimensional representation V_Q of Q_8 (they are unique). In particular, we would get a $\mathbb{C}$-linear isomorphism $g : V_D \to V_Q$. Now

$$g \otimes g : V_D \otimes V_D \to V_Q \otimes V_Q$$

sends the unique identity object $\mathbb{1}_D$ in $V_D \otimes V_D$ to the unique identity object $\mathbb{1}_Q$ in $V_Q \otimes V_Q$. But no such g can exist because the map $v \otimes w \mapsto w \otimes v$ acts as 1 on $\mathbb{1}_D$ and -1 on $\mathbb{1}_Q$.

It may be an open problem to classify the tannakian categories with the same representation ring as Q_8. See mo282292.

EXAMPLE 8.9 (SUPER VECTOR SPACES) Let $\mathsf{sVec}(k)$ denote the category whose objects are pairs (V^0, V^1) of finite-dimensional vector spaces over k, i.e., $\mathbb{Z}/(2)$-graded k-vector spaces. We give $\mathsf{sVec}(k)$ the tensor structure whose commutativity constraint is determined by the Koszul sign rule, i.e., that defined by the isomorphisms

$$v \otimes w \mapsto (-1)^{ij} w \otimes v : V^i \otimes W^j \to W^j \otimes V^i.$$

Then $\mathsf{sVec}(k)$ is a tensor category over k, but it is does not admit a nonzero fibre functor because

$$\dim(V^0, V^1) = \dim(V^0) - \dim(V^1),$$

which need not be positive.

EXAMPLE 8.10 Let C be a rigid additive symmetric monoidal category and T an object of C. We say that C is the ***rigid additive symmetric monoidal category freely generated by*** T if $\mathrm{End}(\mathbb{1})$ is the polynomial ring $\mathbb{Z}[t]$ and

$$F \rightsquigarrow F(T) : \mathsf{Hom}^{\otimes}(\mathsf{C}, \mathsf{C}') \to \mathsf{C}'$$

is an equivalence of categories for all rigid additive symmetric monoidal categories C' (t will turn out to be the categorical dimension of T). We show how to construct such a pair (C, T) – clearly it is unique up to a unique equivalence of symmetric monoidal categories preserving T.

Let V be a free module of finite rank over a commutative ring k and let $T^{a,b}(V)$ be the space of tensors $V^{\otimes a} \otimes V^{\vee \otimes b}$. A morphism $f : T^{a,b}(V) \to T^{c,d}(V)$ can be identified with a tensor "f" in $T^{b+c,a+d}(V)$. When $a + d = b + c$, $T^{b+c,a+d}(V)$ contains a special element, namely, the $(a + d)$th tensor power of "id"$\in T^{1,1}(V)$, and other elements can be obtained by allowing an element of the symmetric group S_{a+d} to permute the contravariant components of this special element. We therefore have a map

$$\epsilon : S_{a+d} \to \mathrm{Hom}(T^{a,b}, T^{c,d}) \qquad (\text{when } a + d = b + c).$$

The induced map $k[S_{a+d}] \to \mathrm{Hom}(T^{a,b}, T^{c,d})$ on the group algebra is injective provided $\mathrm{rank}(V) \geq a + d$. One checks that the composite of two such maps $\epsilon(\sigma) : T^{a,b}(V) \to T^{c,d}(V)$ and $\epsilon(\tau) : T^{c,d}(V) \to T^{e,f}(V)$ is given by a universal formula

$$\epsilon(\tau) \cdot \epsilon(\sigma) = (\mathrm{rank}\, V)^N \cdot \epsilon(\rho) \tag{42}$$

with ρ and N depending only on a, b, c, d, e, f, σ, and τ.

We define C' to be the category having as objects symbols $T^{a,b}$ ($a, b \in \mathbb{N}$), and for which $\mathrm{Hom}(T^{a,b}, T^{c,d})$ is the free $\mathbb{Z}[t]$-module with basis S_{a+d} if $a+d = b+c$ and is zero otherwise. Composition of morphisms is defined to be $\mathbb{Z}[t]$-bilinear and to agree on basis elements with the universal formula (42) but with $\mathrm{rank}(V)$ replaced by the indeterminate t. The associativity law holds for this composition because it does whenever t is replaced by a large enough positive integer (it becomes the associativity law in a category of modules). Tensor products are defined by

$$T^{a,b} \otimes T^{c,d} = T^{a+c,b+d}$$

and by an obvious rule for morphisms. We define T to be $T^{1,0}$.

The category C is deduced from C' by formally adjoining direct sums of objects. Its universality follows from the fact that the formula (42) holds in any rigid additive category.

EXAMPLE 8.11 (GL_t) Let (C, T) be the rigid additive symmetric monoidal category freely generated by T in Example 8.10. Let n be an integer, and use $t \mapsto n : \mathbb{Z}[t] \to \mathbb{C}$ to extend the scalars in C from $\mathbb{Z}[t]$ to $\mathbb{C}$. If V is an n-dimensional complex vector space and if $a + d \leq n$, then the map

$$\mathrm{Hom}(T^{a,b}, T^{c,d}) \otimes_{\mathbb{Z}[t]} \mathbb{C} \to \mathrm{Hom}_{\mathrm{GL}_V}(T^{a,b}(V), T^{c,d}(V))$$

is an isomorphism. For any sum T' of objects $T^{a,b}$ and sufficiently large integer n, $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{C}$ is therefore a product of matrix algebras. This implies that $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t)$ is a semisimple algebra.

After extending the scalars in C to $\mathbb{Q}(t)$, i.e., replacing $\mathrm{Hom}(T', T'')$ with

$$\mathrm{Hom}(T', T'') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t),$$

and passing to the pseudo-abelian envelope (formally adjoining images of idempotents), we obtain a pseudo-abelian rigid symmetric monoidal category GL_t with $\mathrm{End}(\mathbb{1}) = \mathbb{Q}(t)$. As its endomorphism algebras are semisimple, GL_t is semisimple abelian (VI, 1.4 below), hence a semisimple tensor category over $\mathbb{Q}(t)$. The dimension of $T \in \mathrm{ob}(\mathsf{GL}_t)$ is $t \notin \mathbb{N}$ and so GL_t is not tannakian.

For any object X of dimension t of a tensor category T over a field k containing t, there exists an exact $\otimes$-functor from GL_t to T sending the universal object X_t to X. Two such $\otimes$-functors are isomorphic, and the tensor automorphisms of such a $\otimes$-functor are those of X. In particular, we have a symmetric monoidal functor

$$X_t \rightsquigarrow \mathbb{1} \oplus X_{t-1} : \mathsf{GL}_t \to \mathsf{GL}_{t-1}.$$

Iterate this construction, and let T be the inductive limit of the categories GL_{t-n}, $n \geq 0$. This tensor category over a field k is freely generated by an object X_t of dimension t equipped with a decomposition

$$X_t = \mathbb{1} \oplus X_{t-1},\ X_{t-1} = \mathbb{1} \oplus X_{t-2},\ \ldots$$

In GL_{t-n}, $X_t = \mathbb{1}^n \oplus X_{i-n}$ has endomorphism ring $M_n(k) \times k$. Passing to the limit, we find that the ring of endomorphisms of X_t in T is the ring of matrices of the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & \lambda I \end{array}\right)$$

The object X_t of T is not of finite length, and $\mathrm{Hom}(X_t, X_t)$ is not of finite dimension over k.

For more on these categories, see Deligne 2007 and the many articles citing it.

9 Algebraic geometry in a tensor category

Throughout this section, T is an essentially small tensor category over a field k. We explain, following Deligne 1989, how to do algebraic geometry in T.

9.1 For objects S, T, X, Y in T, the canonical morphism (16), p. 20,

$$\mathcal{H}om(S, X) \otimes \mathcal{H}om(T, Y) \to \mathcal{H}om(S \otimes T, X \otimes Y)$$

is an isomorphism. On applying $\mathrm{Hom}(\mathbb{1}, -)$, we obtain an isomorphism of k-vector spaces,

$$\mathrm{Hom}(S,X) \otimes \mathrm{Hom}(T,Y) \simeq \mathrm{Hom}(S \otimes T, X \otimes Y).$$

(The first $\otimes$ is in the category of k-vector spaces.) To give a morphism $X \otimes Y \to Z$ from $X \otimes Y$ to another object Z is the same as giving a homomorphism of k-vector spaces,

$$\mathrm{Hom}(S \otimes T, X \otimes Y) \to \mathrm{Hom}(S \otimes T, Z),$$

natural in S and T (Yoneda lemma). On combining the last two statements, we see that to give a morphism $X \otimes Y \to Z$ is the same as giving a homomorphism of k-vector spaces

$$\mathrm{Hom}(S,X) \otimes \mathrm{Hom}(T,Y) \to \mathrm{Hom}(S \otimes T, Z)$$

9.2 The category $\mathrm{Ind}\,\mathsf{T}$ of ind-objects of T is again a k-linear abelian category (B.8). The tensor product and the associativity and commutativity constraints on T extend to $\mathrm{Ind}\,\mathsf{T}$, making it a tensor category over k. Moreover, the tensor product on $\mathrm{Ind}\,\mathsf{T}$ is exact in each variable (12.18 below). The objects of $\mathrm{Ind}\,\mathsf{T}$ can be identified with the small filtered inductive limits of representable functors $\mathsf{T}^{\mathrm{opp}} \to \mathsf{Vec}(k)$,

$$X = \text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim h_{X_\alpha} = h_X, \quad h_{X_\alpha} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(-, X_\alpha).$$

To give a morphism $X \otimes Y \to Z$ in $\mathrm{Ind}\,\mathsf{T}$ is the same as giving a k-bilinear map

$$h_X(S) \times h_Y(T) \to h_Z(S \otimes T)$$

natural in S and T.

9.3 A ***ring*** A (associative with 1) in $\mathrm{Ind}\,\mathsf{T}$ is an object A equipped with morphisms $m : A \otimes A \to A$ and $1 : \mathbb{1} \to A$ such that the two composed morphisms

$$A \otimes A \otimes A \underset{A \otimes m}{\overset{m \otimes A}{\rightrightarrows}} A \otimes A \xrightarrow{\ m\ } A$$

are equal and the two morphisms

$$A \simeq \mathbb{1} \otimes A \xrightarrow{1 \otimes A} A \otimes A \xrightarrow{m} A$$
$$A \simeq A \otimes \mathbb{1} \xrightarrow{A \otimes 1} A \otimes A \xrightarrow{m} A$$

equal id_A. For example, when $\mathsf{T} = \mathsf{Vecf}(k)$, a ring in $\mathrm{Ind}\,\mathsf{T}$ is a k-algebra (associative with 1) in the usual sense. A ***homomorphism*** of rings in $\mathrm{Ind}\,\mathsf{T}$ is a morphism of objects compatible with the multiplication and the morphisms 1.

The multiplication morphism $m : A \otimes A \to A$ corresponds to a k-bilinear map

$$x, y \mapsto xy : h_A(S) \times h_A(T) \to h_A(S \otimes T),$$

natural in S and T, and the associativity of m becomes the equality

$$(xy)z = x(yz) \quad \text{for } x \in h_A(S),\ y \in h_A(T),\ z \in h_A(U).$$

Here $xy \in h_A(S \otimes T)$, $yz \in h_A(T \otimes U)$, and $(xy)z$, $x(yz) \in h_A(S \otimes T \otimes U)$. That $1 : \mathbb{1} \to A$ is an identity becomes the equalities

$$1x = x = x1 \text{ for } x \in h_A(S).$$

Here $1 \in h_A(\mathbb{1})$, so $1x \in h_A(\mathbb{1} \otimes S) \simeq h_A(S)$ and $x1 \in h_A(S \otimes \mathbb{1}) \simeq h_A(S)$.

9.4 A left A-module is an object M of $\mathrm{Ind}\,\mathsf{T}$ together with a morphism $\mu : A \otimes M \to M$ such that the two morphisms

$$A \otimes A \otimes M \underset{A\otimes\mu}{\overset{m\otimes M}{\rightrightarrows}} A \otimes M \xrightarrow{\ \mu\ } M$$

are equal, and such that

$$M \simeq \mathbb{1} \otimes M \xrightarrow{1\otimes M} A \otimes M \xrightarrow{\mu} M$$

equals id_M. Right A-modules are defined similarly. The left A-modules in $\mathrm{Ind}\,\mathsf{T}$ form an abelian category ${}_A\mathsf{Mod}$ – to form the kernel or cokernel of a morphism in ${}_A\mathsf{Mod}$, first form it in $\mathrm{Ind}\,\mathsf{T}$, and then equip it with the induced A-module structure.

When M is a right A-module and N a left A-module, we define $M \otimes_A N$ to be the coequalizer of the pair of morphisms

$$M \otimes A \otimes N \underset{M\otimes\mu_N}{\overset{\mu_M\otimes N}{\rightrightarrows}} M \otimes N$$

9.5 When A is commutative, $\otimes_A$ makes $\mathsf{Mod}(A) \overset{\text{def}}{=} \mathsf{Mod}_A$ into an A-linear symmetric monoidal category. The unit is (A, m). There is a canonical functor

$$X \rightsquigarrow X \otimes A : \mathsf{T} \to \mathsf{Mod}(A).$$

This sends $X^\vee$ to the dual of $X \otimes A$ in $\mathsf{Mod}(A)$,

$$X^\vee \otimes A = (X \otimes A)^\vee.$$

The categorical dimension of X in T becomes the categorical dimension of $X \otimes A$ in $\mathsf{Mod}(A)$,

$$\dim(X) = \dim_A(X \otimes A), \tag{43}$$

once we identify $k = \mathrm{End}(\mathbb{1})$ with a subring of $\mathrm{End}_A(A) \simeq \mathrm{Hom}(\mathbb{1}, A)$ using the morphism $e : \mathbb{1} \to A$.

DEFINITION 9.6 A homomorphism $f : A \to B$ of commutative rings in $\mathrm{Ind}\,\mathsf{T}$ is ***flat*** (resp. ***faithfully flat***) if the functor

$$M \rightsquigarrow M \otimes_A B : \mathsf{Mod}(A) \to \mathsf{Mod}(B)$$

is exact (resp. exact and faithful).

The usual argument shows that, to check that a flat homomorphism $A \to B$ is faithfully flat, it suffices to show that

$$M \neq 0 \ \Rightarrow\ M \otimes_A B \neq 0.$$

LEMMA 9.7 *Let $f : A \to B$ be faithfully flat. A sequence*

$$(N) : N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$$

of A-modules is exact if (and only if)

$$(N) \otimes B : N' \otimes_A B \xrightarrow{\alpha\otimes 1} N \otimes_A B \xrightarrow{\beta\otimes 1} N'' \otimes_A B$$

is exact.

PROOF Let C be the cokernel of $N' \xrightarrow{\alpha} \mathrm{Ker}(\beta)$, so that

$$N' \xrightarrow{\alpha} \mathrm{Ker}(\beta) \longrightarrow C \longrightarrow 0$$

is exact. As f is flat, the sequence

$$N' \otimes_A B \xrightarrow{\alpha\otimes 1} \mathrm{Ker}(\beta) \otimes_A B \longrightarrow C \otimes_A B \longrightarrow 0$$

is exact, and as $(N) \otimes_A B$ is exact, $C \otimes_A B = 0$. This implies that $C = 0$ because f is faithful, and so (N) is exact. □

LEMMA 9.8 *For any nonzero commutative ring $(A, m, 1)$ in* Ind T*, the morphism $1 : \mathbb{1} \to A$ is faithfully flat.*

PROOF The morphism 1 is a monomorphism because it is nonzero and $\mathbb{1}$ is simple (5.5). As $\otimes$ is exact in each variable (5.2), for any object M, the morphism $M \otimes 1 : M \otimes \mathbb{1} \to M \otimes A$ is a monomorphism. It follows that the functor $- \otimes A$ is exact and that $M \otimes A \neq 0$ whenever $M \neq 0$. The functor is therefore also faithful. □

Faithfully flat descent

PROPOSITION 9.9 *Let $f : A \to B$ be a faithfully flat homomorphism of commutative rings in* Ind T*. For any A-module M, the sequence*

$$M \xrightarrow{d_0} B \otimes_A M \underset{e_1\otimes \mathrm{id}_M}{\overset{e_0\otimes \mathrm{id}_M}{\rightrightarrows}} B \otimes_A B \otimes_A M \tag{44}$$

is exact, i.e., d_0 is the equalizer of the parallel pair of arrows. Here $e_0, e_1 : B \to B \otimes_A B$ are the morphisms

$$B \simeq A \otimes_A B \xrightarrow{f\otimes \mathrm{id}_B} B \otimes_A B, \quad b \mapsto 1_A \otimes b,$$
$$B \simeq B \otimes_A A \xrightarrow{\mathrm{id}_B \otimes f} B \otimes_A B, \quad b \mapsto b \otimes 1_A,$$

and d_0 is the morphism

$$M \simeq A \otimes_A M \xrightarrow{f\otimes \mathrm{id}_M} B \otimes_A M.$$

PROOF Let $d_1 = e_0 \otimes \mathrm{id}_M - e_1 \otimes \mathrm{id}_M$. Clearly $d_1 \circ d_0 = 0$. Assume first that there exists an A-linear section to f, i.e., an A-linear map $g : B \to A$ such that $g \circ f = \mathrm{id}_A$. Define k_0 and k_1 to be the A-linear morphisms

$$\begin{cases} B \otimes_A M \xrightarrow{g\otimes M} A \otimes_A M \simeq M \\ B \otimes_A B \otimes_A M \xrightarrow{g\otimes B\otimes M} A \otimes_A B \otimes_A M \simeq B \otimes_A M. \end{cases}$$

Then $k_0 \circ d_0 = \mathrm{id}_M$, which shows that d_0 is a monomorphism. Moreover,

$$k_1 \circ d_1 + d_0 \circ k_0 = \mathrm{id}_{B\otimes_A M},$$

which shows that d_0 maps onto the kernel of d_1.

We now consider the general case. Because $A \to B$ is faithfully flat, it suffices to prove that the sequence (44) becomes exact after tensoring with B. But the sequence

obtained by tensoring (44) with B is isomorphic to the sequence (44) for the faithfully flat homomorphism $B \simeq A \otimes_A B \xrightarrow{f\otimes B} B \otimes_A B$ and the B-module $B \otimes_A M$ because, for example,

$$B \otimes_A (B \otimes_A M) \simeq (B \otimes_A B) \otimes_B (B \otimes_A M).$$

Now $B \to B \otimes_A B$ has a B-linear section, namely, that defined by multiplication $B \otimes B \to B$, and so we can apply the first case. □

9.10 Let $f : A \to B$ be a faithfully flat homomorphism of commutative rings in $\mathrm{Ind}\,\mathsf{T}$, and let M be an A-module. Set $M' = f_*M \overset{\text{def}}{=} B \otimes_A M$. The modules $e_{0*}M'$ and $e_{1*}M'$ can be identified with $B \otimes_A M$ and $M \otimes_A B$ respectively, with the natural action of $B \otimes_A B$. There is a canonical isomorphism $\phi : e_{1*}M' \to e_{0*}M'$, namely,

$$e_{1*}M' \simeq (e_1 f)_* M = (e_0 f)_* M \simeq e_{0*}M'.$$

Moreover, M can be recovered from the pair (M', ϕ) as the equalizer of

$$M' \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B \otimes_A M', \qquad \begin{cases} \alpha : M' \simeq A \otimes_A M' \xrightarrow{f\otimes M'} B \otimes_A M' \\ \beta : M' \simeq M' \otimes_A A \xrightarrow{M'\otimes f} M' \otimes_A B \xrightarrow{\phi} B \otimes_A M'. \end{cases}$$

Conversely, every pair (M', ϕ), with M' a B-module and ϕ a $B \otimes_A B$-linear morphism $\phi : M' \otimes_A B \to B \otimes_A M'$ satisfying a certain natural condition arises in this way from an A-module. Given ϕ, we construct morphisms

$$\begin{aligned} &\phi_1 : B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M', && \phi_1 = B \otimes \phi \\ &\phi_2 : M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M', && \phi_2 = (\gamma_{B,B} \otimes M') \circ \phi \circ (\gamma_{M',B} \otimes B) \\ &\phi_3 : M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B, && \phi_3 = \phi \otimes B. \end{aligned}$$

On points, ϕ_1, ϕ_2, and ϕ_3 are obtained by tensoring ϕ with id_B in the first, second, and third positions respectively. A pair (M', ϕ) arises from an A-module M as above if and only if $\phi_2 = \phi_1 \circ \phi_3$. The necessity is easy to check. For the sufficiency, define M to be the equalizer of $\alpha, \beta : M' \rightrightarrows B \otimes_A M'$ with α and β as above. There is a canonical morphism $B \otimes_A M \to M'$, and it suffices to show that this is an isomorphism and that the map arising from M is ϕ. The diagram

$$\begin{array}{ccc} M' \otimes_A B & \underset{\beta\otimes B}{\overset{\alpha\otimes B}{\rightrightarrows}} & B \otimes_A M' \otimes_A B \\ \simeq \downarrow \phi & & \simeq \downarrow \phi_1 \\ B \otimes_A M' & \underset{e_1\otimes M'}{\overset{e_0\otimes M'}{\rightrightarrows}} & B \otimes_A B \otimes_A M' \end{array}$$

commutes with either the upper or the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1 \circ \phi_3$), and so ϕ induces an isomorphism on the equalizers. But, by definition of M, the equalizer of the pair $(\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id})$ is $M \otimes_A B$, and, according to Proposition 9.9, the equalizer of the pair $(e_0 \otimes \mathrm{id}, e_1 \otimes \mathrm{id})$ is M'. This completes the proof.

9.11 More precisely, $M \rightsquigarrow (B \otimes_A M, \phi)$ is an equivalence from $\mathsf{Mod}(A)$ to the category of pairs (M', ϕ) satisfying $\phi_2 = \phi_1 \circ \phi_3$. This statement holds also when M and M' are equipped with A-algebra structures preserved by ϕ.

REMARK 9.12 When T is the category of finite-dimensional vector spaces over k, so $\operatorname{Ind}\mathsf{T}$ is the category of all vector spaces over k, then a ring in $\operatorname{Ind}\mathsf{T}$ is a k-algebra (associative with 1) in the usual sense, and the results become familiar, for example, 9.10 is faithfully flat descent in the usual sense (e.g., Waterhouse 1979, 17.1, 17.2).

Affine schemes

9.13 Following Deligne 1989, in order to have a geometric language at our disposal, we define the ***category of affine schemes in*** $\operatorname{Ind}\mathsf{T}$ to be the opposite of that of commutative rings (associative with 1) in $\operatorname{Ind}\mathsf{T}$. We call an object of the category an ***affine scheme in*** $\operatorname{Ind}T$ or an ***affine*** T***-scheme***, and we write $\operatorname{Sp}(A)$ for the affine T-scheme defined by A. Fibre products exist, and correspond to tensor products. An A-module M is called a ***module*** over $\operatorname{Sp}(A)$, and, when $\operatorname{Sp}(B)$ is an affine scheme over $\operatorname{Sp}(A)$, $B\otimes_A M$ is called the ***inverse image*** of M over $\operatorname{Sp}(B)$.

For example, $\operatorname{Sp}(0)$ is the empty scheme and $\operatorname{Sp}(\mathbb{1})$ is the point (pt) – they are the initial and final objects in the category. We say that $S=\operatorname{Sp}(A)$ is nonempty if $A\neq 0$. Every S is either empty or faithfully flat over (pt). For X and S affine schemes in T, the set $X(S)$ of S-***points*** of X is $\operatorname{Hom}(S,X)$.

An ***affine group*** T***-scheme*** is a group object in the category of affine T-schemes. Let H be an affine group scheme in T. An H-***torsor*** is a nonempty affine T-scheme P equipped with a right action $\rho: P\times H\to P$ such that, for all S, $P(S)$ is either empty or a torsor under $H(S)$. The condition "empty or a torsor" means that, for all S, $(\mathrm{pr}_1,\rho): P(S)\times H(S)\to P(S)\times P(S)$ is bijective, i.e., that $(\mathrm{pr}_1,\rho): P\times H\to P\times P$ is an isomorphism.

EXAMPLE 9.14 (VECTORIAL SCHEMES IN T) For M in $\operatorname{Ind}\mathsf{T}$, put $\Gamma(M)=\operatorname{Hom}(\mathbb{1},M)$, the global sections of M over S. When M is a module over $S=\operatorname{Sp}(A)$, we have

$$\Gamma(M)=\operatorname{Hom}(\mathbb{1},M)\simeq \operatorname{Hom}_A(A,M).$$

Note that the functor Γ need not be exact. For example, when $\mathsf{T}=\mathtt{Repf}(G)$, it is the functor of G-invariants.

An object X of T defines for each $S=\operatorname{Sp}(A)$ a module $X_S=A\otimes X$, the inverse image of X by $S\to(\mathrm{pt})$. The functor $S\rightsquigarrow\Gamma(X_S)$ is representable,

$$\operatorname{Hom}(\mathbb{1},A\otimes X)=\operatorname{Hom}(X^\vee,A)=\operatorname{Hom}_{\mathrm{rings}}(\operatorname{Sym}(X^\vee),A).$$

We sometimes denote by X the T-scheme $\operatorname{Sp}(\operatorname{Sym}(X^\vee))$ representing this functor. This is similar, when V is a finite-dimensional k-vector space, to using V to denote the scheme $\operatorname{Spec}(\operatorname{Sym}^*(V^\vee))$, which has V for its k-points.

The functor $S\rightsquigarrow\Gamma(X_S)$ is a functor to groups. The T-scheme X is therefore a group scheme in T. The group structure corresponds to the usual Hopf algebra structure on $\operatorname{Sym}^*(X^\vee)$.

EXAMPLE 9.15 (AN AFFINE k-SCHEME IS AN AFFINE T-SCHEME) Since $\operatorname{End}(\mathbb{1})=k$, the subcategory of T of sums of copies of $\mathbb{1}$ is naturally equivalent to that of vector spaces of finite dimension over k, by a functor $V\rightsquigarrow V\otimes\mathbb{1}$. The choice of a basis $e_1,\dots,e_n$ of V identifies $V\otimes\mathbb{1}$ with $\mathbb{1}^n$. See 7.9.

Passing to the ind-objects, we obtain a functor from the category of (all) vector spaces over k to $\operatorname{Ind}\mathsf{T}$. Under this functor, an affine scheme over k defines a scheme in T. Similarly, for affine group schemes, torsors, and so on. The point $\operatorname{Spec}(k)$ defines the T-scheme (pt).

9.16 Let G be an affine group T-scheme and X an object of T. To give an ***action*** of G on X is to give, for every S, an action of $G(S)$ on the S-module X_S, compatible with base changes S'/S. Such an action is determined by the action of the universal element $\mathrm{id}_G \in G(G)$ on X_G. For $G = \mathrm{Sp}(A)$, this is an A-linear morphism $A \otimes X \to A \otimes X$, which is determined by its restriction to $X \to A \otimes X$. The morphism $X \to A \otimes X$ makes X a comodule over the Hopf algebra A (commutative with 1) in $\mathrm{Ind}\,\mathsf{T}$.

9.17 (THE CASE OF $\mathsf{Repf}(G)$) Let G be an affine group scheme over k and let $\mathsf{T} = \mathsf{Repf}(G)$. The ind-objects of T are the linear representations – not necessarily of finite dimension – of G (B.14). The affine T-schemes are the affine schemes over k equipped with an action of G, an affine group T-scheme H is an affine group scheme over k equipped with an action of G, an H-torsor is an G-equivariant H-torsor (in the usual sense), a vectorial T-scheme is the equivariant affine scheme of a finite-dimensional representation of G, and the inclusion of affine k-schemes into affine T-schemes is "equip with the trivial action of G".

When T is a neutral tannakian category, the choice of a fibre functor identifies T with a category $\mathsf{Repf}(G)$ (see Chapter II), so this interpretation allows us to routinely reduce questions on T-schemes to questions in usual algebraic geometry.

On reversing the arrows in 9.11, we obtain faithfully flat descent for affine schemes.

THEOREM 9.18 *Let $a : V \to U$ be a faithfully flat map of affine schemes in* $\mathrm{Ind}\,\mathsf{T}$. *To give an affine scheme W over U is the same as giving an affine scheme W' over V together with an isomorphism $\phi : \mathrm{pr}_1^* W' \to \mathrm{pr}_2^* W'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi).$$

Here p_{ji} denotes the projection $V \times V \times V \to V \times V$ such that $p_{ji}(w_1, w_2, w_3) = (w_j, w_i)$.

ASIDE 9.19 Let T be a pre-tannakian category over a perfect field. In this case, Deligne (1990, 8.13) constructs a T-group $\pi(\mathsf{T})$ called the ***fundamental group*** of T. In particular, $\pi(\mathsf{T})$ acts on the objects of T. For example, if $\mathsf{T} = \mathsf{Repf}(G)$, $\pi(\mathsf{T})$ is (the Hopf algebra of) G with G acting by conjugation.

When T is tannakian, the construction of the fundamental group is easier – see V, 4.13. For any fibre functor ω of T over a scheme S, $\omega(\pi(\mathsf{T}))$ is an affine group scheme over S, and the action of $\pi(\mathsf{T})$ on an object X defines an action of $\omega(\pi(\mathsf{T}))$ on $\omega(X)$, natural in X and compatible with tensor products, which for varying X gives an isomorphism

$$\omega(\pi(\mathsf{T})) \xrightarrow{\simeq} \mathcal{A}ut_S^{\otimes}(\omega).$$

NOTES This section largely follows Deligne 1989, §5,6.

10 An intrinsic characterization of tannakian categories

We show that a tensor category over a field of characteristic zero is tannakian, i.e., admits a fibre functor, if its objects have categorical dimension an integer ≥ 0.

THEOREM 10.1 (DELIGNE 1990, 7.1) *Let* T *be an essentially small tensor category over a field k of characteristic zero. The following conditions are equivalent:*

(a) T *is tannakian, i.e., there exists a fibre functor with values in a nonzero ring;*

(b) *for all X in T, $\dim X$ is an integer ≥ 0;*

(c) *for all X in T, there exists an integer $n \geq 0$ such that $\bigwedge^n X = 0$.*

The proof will occupy the rest of this section. Throughout, T is a tensor category over a field k of characteristic 0.

The action of the symmetric group on $X^{\otimes n}$

Let $X \in \operatorname{ob}\mathsf{T}$. Because T is symmetric, there is a natural action of the symmetric group S_n on $X^{\otimes n}$. Explicitly, when we let σ_i denote the transposition $(i\ i+1)$ in S_n, then S_n has a presentation

$$S_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i^2 = 1 & 1 \leq i \leq n-1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & 1 \leq i \leq n-2 \end{array} \right\rangle,$$

and there is an action of S_n on $X^{\otimes n}$ for which σ_i acts as $X^{\otimes i-1} \otimes \gamma_{X,X} \otimes X^{\otimes n-i-1}$ (Müger 2007, Lemma 416).

The nth ***exterior power*** $\bigwedge^n X$ of X is defined to be the image of the antisymmetrization map

$$a \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma : X^{\otimes n} \to X^{\otimes n}.$$

As $\operatorname{char} k = 0$, it is also the image of the projector $a/n!$, and so

$$\dim(\textstyle\bigwedge^n X) = \operatorname{Tr}(a/n!) = \operatorname{Tr}(a)/n!. \tag{45}$$

PROPOSITION 10.2 (DELIGNE 1990, 7.1.2) *We have*

$$\dim(\bigwedge^n X) = \binom{\dim X}{n} \stackrel{\text{def}}{=} \frac{(\dim X)(\dim X - 1)\cdots(\dim X - n + 1)}{n \cdot n-1 \cdot \cdots \cdot 1}.$$

PROOF Let σ be a cyclic permutation of order n. It follows from (31), p. 26, applied to $X_i = X$ and $u_i = \mathrm{id}_X$, that

$$\operatorname{Tr}(\sigma \mid X^{\otimes n}) = \dim(X).$$

If σ has $r(\sigma)$ cycles (including cycles of length 1), then

$$\operatorname{Tr}(\sigma \mid X^{\otimes n}) = \dim(X)^{r(\sigma)}.$$

It follows from (45) that there exists a universal polynomial $P \in \mathbb{Q}[T]$ such that $\dim \bigwedge^n X = P(\dim X)$. Taking T to be $\mathsf{Vec}f(k)$, we find that, for all $d \in \mathbb{N}$,

$$P(d) = \binom{d}{n} \stackrel{\text{def}}{=} \frac{d!}{(d-n)!n!}.$$

We deduce that $P(T) = \binom{T}{n}$, and the statement follows. □

EXAMPLE 10.3 If X has dimension $n \in \mathbb{N}$, then $\bigwedge^n X$ has dimension 1.

First step in the proof of Theorem 10.1

It is obvious that (a) of Theorem 10.1 implies (b). We next show that (c) implies (b). If $\bigwedge^n X = 0$, then $\dim \bigwedge^n X = 0$, and 10.2 shows that $\dim X = 0, 1, \dots$ or $n-1$; in particular, it is an integer ≥ 0. It remains to show that (b) $\Rightarrow$ (a), (c).

For the remainder of this subsection we assume that T satisfies (b) of Theorem 10.1, i.e., for all X in T, $\dim X$ is an integer ≥ 0.

LEMMA 10.4 *Let $X \in \operatorname{ob} \mathsf{T}$. If $\dim X = 0$, then $X = 0$.*

PROOF If $X \neq 0$, then $\operatorname{coev}_X : \mathbb{1} \to X \otimes X^\vee$ is not zero because otherwise

$$\operatorname{id}_X \overset{(19)}{=} (\operatorname{id}_X \otimes \operatorname{ev}_X) \circ (\operatorname{coev}_X \otimes \operatorname{id}_X) = 0.$$

As $\mathbb{1}$ is simple (5.5), coev_X is a monomorphism – let Q be its cokernel. Then

$$1 + \dim(Q) = \dim(X \otimes X^\vee) = (\dim X)(\dim X^\vee)$$

by 5.10 and 4.18. As $\dim(Q) \geq 0$, this shows that $\dim X \neq 0$. □

PROPOSITION 10.5 *Let $(A, m, 1)$ be a ring in $\operatorname{Ind} \mathsf{T}$, and let $R = \operatorname{End}_A(A)$. If $A \otimes X \approx A^{\oplus \dim(X)}$ (isomorphism of A-modules) for every X in T, then*

$$X \rightsquigarrow \operatorname{Hom}(\mathbb{1}, A \otimes X) : \mathsf{T} \to \mathsf{Mod}(R)$$

is a faithful $\otimes$-functor.

PROOF Let F denote the functor, so

$$F(X) \overset{\text{def}}{=} \operatorname{Hom}(\mathbb{1}, A \otimes X) \simeq \operatorname{Hom}_A(A, A \otimes X) \approx \operatorname{Hom}_A(A, A^{\oplus \dim X}),$$

which is free of rank $\dim X$. It follows from Lemma 10.4 that F is faithful.

For objects X, Y, let

$$c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

be the map sending $\phi \otimes \psi$ to the composite of the morphisms

$$\mathbb{1} \overset{\psi}{\longrightarrow} A \otimes Y \simeq A \otimes \mathbb{1} \otimes Y \overset{A \otimes \phi \otimes Y}{\longrightarrow} A \otimes A \otimes X \otimes Y \overset{m \otimes X \otimes Y}{\longrightarrow} A \otimes X \otimes Y.$$

When we identify $F(X)$ with $\operatorname{Hom}_A(A, A \otimes X)$, $c_{X,Y}$ becomes the map

$$a_{X,Y} : \operatorname{Hom}_A(A, A \otimes X) \otimes_k \operatorname{Hom}_A(A, A \otimes Y) \to \operatorname{Hom}_A(A, A \otimes X \otimes Y)$$

sending $s \otimes t$ to $s \otimes Y \circ t$.

We now prove that $a_{X,Y}$ (hence $c_{X,Y}$) is bijective. According to (b), for $i = 1, \dots, \dim(X)$, there exist A-module morphisms

$$\begin{cases} s_i : A \to A \otimes X \\ s_i' : A \otimes X \to A \end{cases} \quad \text{such that} \quad \begin{cases} s_i' \circ s_j = \delta_{ij} \operatorname{id}_A \\ \sum_i s_i \circ s_i' = \operatorname{id}_{A \otimes X}, \end{cases}$$

and there exist similar such morphisms t_i, t_i' for Y. The elements $a_{ij} \overset{\text{def}}{=} a_{X,Y}(s_i \otimes t_j)$ of $F(X \otimes Y)$ are linearly independent because there exist elements $a_{i'j'}' \overset{\text{def}}{=} t_{j'}' \circ s_{i'}' \otimes Y \in \operatorname{Hom}(A \otimes X \otimes Y, A)$ such that

$$a_{i'j'}' \circ a_{ij} = \delta_{i'i} \delta_{j'j} \operatorname{id}_A, \quad \text{all } i, j, i', j'.$$

This shows that $a_{X,Y}$ is bijective because its domain and codomain both have dimension $(\dim X)(\dim Y)$.

The rest of the verification that F is a $\otimes$-functor is straightforward. See Müger 2007, Proposition 413. □

Linear algebra in a tensor category

PROPOSITION 10.6 (DELIGNE 1990, 7.14) *Let*

$$0 \to N \xrightarrow{i} E \longrightarrow M \to 0$$

be an exact sequence in T. *There exists a ring* P *in* $\operatorname{Ind}\mathsf{T}$ *such that*

$$0 \to N \otimes P \to E \otimes P \to M \otimes P \to 0$$

is split as a sequence of P*-modules.*

PROOF Suppose first that $N = \mathbb{1}$. Then i is a monomorphism $\mathbb{1} \to E$. Let $S^n = \operatorname{Sym}^n(E)$, and let

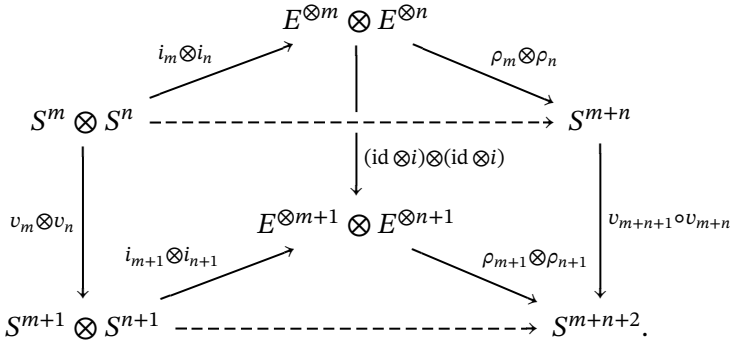
$$S^n \xrightarrow{i_n} E^{\otimes n} \xrightarrow{\rho_n} S^n$$

be the canonical factorization of id_{S^n}. Let $v_n : S^n \to S^{n+1}$ be the composite of the morphisms

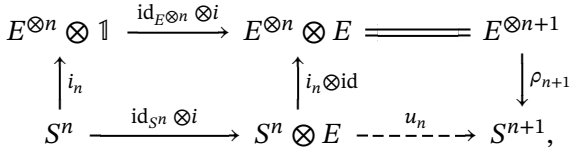
$$S^n \xrightarrow{i_n} E^{\otimes n} \simeq E^{\otimes n} \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{E^{\otimes n}} \otimes i} E^{\otimes n+1} \xrightarrow{\rho_{n+1}} S^{n+1},$$

and let $P = \varinjlim_n (S^n, v_n)$. Then v_n is a monomorphism with cokernel $\operatorname{Sym}^{n+1}(M)$. In particular, $P \neq 0$.

The next diagram defines a ring structure on P,



Consider the diagram



where u_n is defined to make the right-hand square commute. From the diagram, we see that

$$u_n \circ (\mathrm{id}_{S^n} \otimes i) = v_n.$$

On passing to the limit, we obtain a morphism $u : P \otimes E \to E$ such that

$$u \circ (\mathrm{id}_P \otimes v) = \mathrm{id}_P .$$

Moreover, u is a P-module homomorphism because the diagram

$$\begin{array}{ccc} S^m \otimes S^n \otimes E & \xrightarrow{u_n} & S^m \otimes S^{n+1} \\ \downarrow & & \downarrow \\ S^{m+n} \otimes E & \xrightarrow{u_{m+n}} & S^{m+n+1} \end{array}$$

is obviously commutative as it does not involve i. This completes the proof of the lemma in the case $N = \mathbb{1}$.

Now consider an arbitrary N. In the next diagram, the bottom row is the pushout of the top row by $\mathrm{ev} \circ \gamma_{N,N^\vee}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes N^\vee & \xrightarrow{i\otimes \mathrm{id}_{N^\vee}} & E \otimes N^\vee & \longrightarrow & M \otimes N^\vee & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev} \circ \gamma_{N,N^\vee}} & & \downarrow{\scriptstyle h} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{1} & \xrightarrow{i'} & E' & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

The top row is exact because internal Homs are exact in T. The first part of the proof gives us a morphism $u' : E' \to \mathbb{1}$ such that $u' \circ h \circ (i \otimes \mathrm{id}_{N^\vee}) = \mathrm{ev}_N \circ \gamma_{N,N^\vee}$. Consider the morphism

$$u : E \to N, \quad u = (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_E \otimes \delta_N).$$

Then

$$\begin{aligned} u \circ i &= (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_E \otimes \delta_N) \circ i \\ &= (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(i \otimes \mathrm{id}_{N\otimes N^\vee})(\mathrm{id}_N \otimes \delta_N) \\ &= (\mathrm{id}_N \otimes u' \circ h)(\mathrm{id}_N \otimes i \otimes \mathrm{id}_{N^\vee})(\gamma_{N,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_N \otimes \gamma_N) \\ &= (\mathrm{id}_N \otimes \mathrm{ev}_N \gamma_{N,N^\vee})(\gamma_{N,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_N \otimes \delta_N) \\ &= (\mathrm{id}_N \otimes \mathrm{ev}_N)(\mathrm{id}_N \otimes \delta_N) \\ &= \mathrm{id}_N, \end{aligned}$$

where in the third and fifth equations, we used the naturality of γ. This completes the proof. □

COROLLARY 10.7 *Let $F : \mathsf{C} \to \mathsf{D}$ be a k-linear $\otimes$-functor of tensor categories over a field k of characteristic 0. If F is faithful, then it is exact.*

PROOF Extend F to a functor $\mathrm{Ind}\,\mathsf{C} \to \mathrm{Ind}\,\mathsf{D}$, again denoted F – it is again a faithful k-linear $\otimes$-functor. Let $\xi : 0 \to N \to E \to M \to 0$ be an exact sequence in C, and let P be as in 10.6. Then $\xi \otimes P$ is split-exact, and so $F(\xi) \otimes P \simeq F(\xi) \otimes F(P)$ is exact because $\mathrm{Ind}\,F$ is additive. Therefore $F(\xi)$ is exact (9.7, 9.8). □

COROLLARY 10.8 *Let T be a tensor category over a field k of characteristic 0, let R be a k-algebra, and let $F : \mathsf{T} \to \mathsf{Modf}(R)$ be a k-linear $\otimes$-functor. If F is faithful, then it is exact.*

PROOF Extend F to a functor $\mathrm{Ind}\,\mathsf{T} \to \mathsf{Mod}(R)$, again denoted F – it is again a faithful k-linear $\otimes$-functor (B.4). Let $\xi : 0 \to N \to E \to M \to 0$ be an exact sequence in T, and let P be as in 10.6. Then $F(E)$ is a faithfully flat R-module (7.5), so $F(P) \overset{\text{def}}{=} \varinjlim \mathrm{Sym}^n(F(E))$ is a faithfully flat R-algebra. As $\xi \otimes P$ is split-exact and F is additive, the sequence $F(\xi \otimes P)$ is exact. But $F(\xi \otimes P) \simeq F(\xi) \otimes F(P)$, and so $F(\xi)$ is exact. □

Corollaries 10.7 and 10.8 also hold for tensor categories over fields k of nonzero characteristic. See 12.37 below.

Decomposition of $X^{\otimes n}$ under the action of the symmetric group

10.9 Let T be a tensor category over a field k of characteristic zero. The simple representations of S_n on finite-dimensional vector spaces are classified by the partitions of n. For each partition λ, choose a simple representation V_λ in the isomorphism class corresponding to λ, and define the Schur functor S_λ by

$$S_\lambda(X) = \mathrm{Hom}(V \otimes \mathbb{1}, X)^{S_n}, \quad X \in \mathrm{ob}\,\mathsf{T}.$$

Then

$$\bigoplus_\lambda V_\lambda \otimes S_\lambda(X) \simeq X^{\otimes n}$$

(Deligne 2002, 1.3, 1.4). Here $V \otimes \mathbb{1}$ is as in 7.9.

Let X be an object of T such that $\bigwedge^{n+1} X = 0$ and $\bigwedge^n X$ is invertible. Let $\mathrm{GL}_n(k)$ act on $V \overset{\text{def}}{=} k^n$ according to the standard representation. Every simple $\mathrm{GL}_n(k)$-module has the form $V_\lambda \otimes \left(V_{(1,\dots,1)}^\vee\right)^{\otimes m}$ for a partition λ of length at most n and an $m \in \mathbb{N}$. Define F_X to be the functor sending $V_\lambda \otimes \left(V_{(1,\dots,1)}^\vee\right)^{\otimes m}$ to $X_\lambda \otimes \left(X_{(1,\dots,1)}^\vee\right)^{\otimes m}$. Note that $X_{(1,\dots,1)} = \bigwedge^n X$, so $\left(X_{(1,\dots,1)}^\vee\right)^{\otimes m} = (\bigwedge^n X)^{\otimes -m}$ is defined. Then F_X is an exact $\otimes$-functor $\mathsf{Repf}(\mathrm{GL}_n) \to \mathsf{T}$ sending the standard representation to X.

Completion of the proof of Theorem 10.1

LEMMA 10.10 *Let A be a commutative ring in* $\mathrm{Ind}\,\mathsf{T}$ *and let M be an A-module that is a direct summand (as an A-module) of $A \otimes X$ for some $X \in \mathrm{ob}\,\mathsf{T}$.*

(a) *If $\dim_A M = 0$, then $M = 0$.*

(b) *If $\dim_A M > 0$, then there exists an algebra P, faithfully flat over A, such that $M \otimes_A P = P \oplus N$ as P-modules.*

(c) *If $\dim_A M = d > 0$, then there exists a faithfully flat extension B of A such that $M \otimes_A B \approx B^{\oplus d}$.*

PROOF (a) Let $n = \dim X$ and $A \otimes X = M \oplus N$. Then

$$\bigwedge_A^{n+1}(A \otimes X) = A \otimes \bigwedge^{n+1} X = 0.$$

On the other hand, as

$$\bigwedge^m (U \oplus V) \simeq \bigoplus_{p+q=m} \bigwedge^p U \otimes \bigwedge^q U,$$

we see that $M \otimes_A \bigwedge_A^n N$ is a direct summand of $\bigwedge_A^{n+1}(A \otimes X)$; hence is zero. As

$$\dim_A(N) = \dim_A(A \otimes X) = \dim X = d,$$

we have $\dim_A(\bigwedge_A^n N) = 1$. Let $Q = \bigwedge_A^n N$. Then A is a direct summand of $Q \otimes_A Q^\vee$ because the dimension of Q over A is 1. Consequently,

$$M \otimes_A Q \otimes_A Q^\vee = 0 \;\Rightarrow\; M = 0.$$

(b) We can now argue as in 10.9 to obtain an exact faithful functor from $\mathsf{Repf}(\mathrm{GL}_n)$ to $\mathsf{Mod}(A)$ sending V to $A \otimes X$. Let $e_1, \dots, e_n$ be a basis for V and $f_1, \dots f_n$ the dual basis for $V^\vee$. The group GL_n acts on the polynomial algebra $k[e_i, f_j]$, which can therefore be viewed as an algebra in $\mathrm{Ind}(\mathsf{Repf}(\mathrm{GL}_n))$. The relation $\sum e_i f_i = 1$ is invariant under the action of GL_n, and hence the quotient algebra

$$C = \frac{k[e_i, f_i]}{(\sum e_i f_i - 1)}$$

is also an algebra in $\mathsf{Repf}(\mathrm{GL}_n)$.

The image of P of C under the above functor is the required algebra in $\mathsf{Mod}(A)$. Indeed, as M is a direct summand of $A \otimes X$, which is flat over A, M is flat over A. Therefore P is also flat over A, and it is faithfully flat because it contains A as a direct summand (as C contains k as a direct summand). Finally, there exists a decomposition $V \otimes C = C \oplus Q$ in $\mathsf{Repf}(\mathrm{GL}_d(k))$: the projection $V \otimes C \to C$ is $v \otimes p \mapsto v \cdot p$ and the embedding $C \to V \otimes C$ is $p \mapsto \sum e_i \otimes f_i(p)$, $p \in C$.

(c) Applying (b) d times, we get a faithfully flat A-algebra B such that $M \otimes_A B \approx B^{\oplus d} \oplus N$ with $\dim_B N = 0$. □

We now prove the theorem. For each object X of C, we have constructed a commutative ring A_X such that $A_X \otimes X \approx A_X^{\oplus \dim X}$ as A_X-modules. Define a partial order on the isomorphism classes of objects of C by $[X] \le [Y]$ if and only if $\langle X \rangle \subset \langle Y \rangle$, and apply transfinite induction to obtain an algebra A in $\mathrm{Ind}\,\mathsf{T}$ such that

(a) for all X in T, $A \otimes X \approx A^{\oplus \dim X}$;

(b) for all exact sequences $0 \to M \to N \to P \to 0$ in T, the sequence of A-modules $0 \to A \otimes M \to A \otimes N \to A \otimes P \to 0$ is split exact.

Let $R = \mathrm{End}_A(A)$. The functor

$$X \rightsquigarrow \mathrm{Hom}(\mathbb{1}, A \otimes X) \simeq \mathrm{Hom}_A(A, A \otimes X)$$

is an R-valued fibre functor on T (10.5).

QUESTION 10.11 Let T be a tensor category over a field k of characteristic zero. Assume that, for all X in T, $\dim X$ is an integer ≥ 0. We then showed that there exists a fibre functor on T. If T is algebraic, i.e., $\mathsf{T} = \langle X \rangle^\otimes$ for some X in T, is it possible modify the argument to show that there exists a fibre functor on T with values in a finite extension of k?

NOTES

10.12 When k has nonzero characteristic, condition (c) of Theorem 10.1 no longer implies that the category is tannakian – one needs an additional hypothesis (Coulembier 2020, Theorem B).

10.13 For an analogue of Theorem 10.1 in super mathematics (i.e., $\mathbb{Z}/2\mathbb{Z}$-graded mathematics), see Deligne 2002.

NOTES The exposition in this section follows Deligne 1990, §7, and Hái 2002.

11 Tensor products of tensor categories

In this section, we explain a construction that will be needed in later chapters. All categories are essentially small.

Tensor products of abelian categories: definition and preliminaries

In this subsection, k is a commutative ring. Unadorned tensor products are over k.

DEFINITION 11.1 (DELIGNE 1990, 5.1) Let A and B be k-linear abelian categories. A k-linear abelian category $\mathsf{A}\boxtimes\mathsf{B}$ together with a k-bilinear functor

$$\boxtimes\colon \mathsf{A}\times\mathsf{B}\to\mathsf{A}\boxtimes\mathsf{B},$$

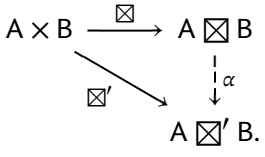
right exact in each variable, is the ***tensor product*** of A and B if, for all small k-linear abelian categories D, the functor

$$F \rightsquigarrow F\circ\boxtimes\colon \mathsf{Rex}_k(\mathsf{A}\boxtimes\mathsf{B},\mathsf{D})\to\mathsf{Rex}_k(\mathsf{A}\times\mathsf{B},\mathsf{D}) \tag{46}$$

is an equivalence of categories. On the left, Rex_k is the category of k-linear right exact functors, and on the right, it is the category of k-bilinear functors $\mathsf{A}\times\mathsf{B}\to\mathsf{D}$ right exact in each variable.

LEMMA 11.2 *If it exists, $(\mathsf{A}\boxtimes\mathsf{B},\boxtimes)$ is unique up to an equivalence, which itself is unique up to isomorphism.*

PROOF If $(\mathsf{A}\boxtimes\mathsf{B},\boxtimes)$ and $(\mathsf{A}\boxtimes'\mathsf{B},\boxtimes')$ are two tensor products of A and B, then there exists an equivalence $\alpha\colon \mathsf{A}\boxtimes\mathsf{B}\to\mathsf{A}\boxtimes'\mathsf{B}$ such that $\alpha\circ\boxtimes\approx\boxtimes'$,



and any two such α are isomomorphic. □

Tensor products of abelian categories in the sense of 11.1 have become known in the literature as ***Deligne tensor products*** (see, for example, ncatlab.org).

We shall prove the existence for locally finite k-linear abelian categories. Without some finiteness condition, the tensor product need not exist, even for $k=\mathbb{Q}$ (see 11.16 below).

Let A and R be rings (not necessarily commutative), and let M be an (A,R)-module. The functor

$$F_M\colon \mathsf{Mod}_A\to\mathsf{Mod}_R,\quad X\rightsquigarrow X\otimes_A M$$

is right exact and commutes with all direct sums. It has been known since about 1960 (Eilenberg, Gabriel, Watts) that every such functor arises in this way. We shall prove a more precise statement. Note that F_M is natural in M: a homomorphism $f\colon M\to N$ of (A,R)-bimodules defines a natural transformation $F_M\to F_N$ whose value on X is

$$X\otimes f\colon X\otimes_A M\to X\otimes_A N.$$

PROPOSITION 11.3 *The functor*

$$M \rightsquigarrow F_M : {}_A\mathsf{Mod}_R \to \mathsf{Rex}(\mathsf{Modf}_A, \mathsf{Mod}_R)$$

is an equivalence with quasi-inverse $F \rightsquigarrow F(A_A)$.

PROOF From a natural transformation $u : F_M \to F_N$, we get a morphism

$$M \simeq A_A \otimes_A M \xrightarrow{u_A} A_A \otimes_A N \simeq N.$$

When applied to $u = F_f$, where $f : M \to N$, this gives back f. Thus, the functor $M \rightsquigarrow F_M$ is faithful.

Let $u : F_M \to F_N$ be a natural transformation. We shall show that, for all X in Modf_A,

$$u_X = X \otimes_A u_A \quad (\text{as maps } X \otimes_A M \to X \otimes_A N).$$

Let $\mathcal{C}$ be the collection of X for which this is true. Certainly $\mathcal{C}$ contains A_A, and it is closed under finite direct sums. If $X, Y \in \mathcal{C}$ and $X \to Y \to Z \to 0$ is exact, then the exact commutative diagram

$$\begin{array}{ccccccc} X \otimes_A M & \longrightarrow & Y \otimes_A M & \longrightarrow & Z \otimes_M & \longrightarrow & 0 \\ {\scriptstyle u_X}\downarrow{\scriptstyle X\otimes u_A} & & {\scriptstyle u_Y}\downarrow{\scriptstyle Y\otimes u_A} & & \downarrow & & \\ X \otimes_A N & \longrightarrow & Y \otimes_A N & \longrightarrow & Z \otimes_A N & \longrightarrow & 0 \end{array}$$

shows that $Z \in \mathcal{C}$. By definition, every X in Modf_A arises as a cokernel

$$A^{\oplus m} \to A^{\oplus n} \to X \to 0,$$

and so it lies in $\mathcal{C}$, completing the proof.

From $u_X = X \otimes u_A$ we deduce that $u = F(u_A)$. In particular, the functor $M \rightsquigarrow F_M$ is full.

Let $F : \mathsf{Modf}_A \to \mathsf{Mod}_R$ be right exact, and let $M = F(A_A)$. Then M is a right R-module by definition, and it becomes a left A-module through the map

$$A = \mathrm{End}_A(A_A) \xrightarrow{F} \mathrm{End}_R(F(A_A)) = \mathrm{End}_R(M).$$

Note that, for $a \in A$, $r \in R$, $m \in M$,

$$(am)r \overset{\text{def}}{=} (F(a)(m))r = F(a)(mr) = a(mr),$$

and so $M \in {}_A\mathsf{Mod}_R$. We shall show that F is isomorphic to F_M.

For X in Modf_A, we regard $Hom_R(M, FX)$ as a right A-module by

$$(f \cdot a)(m) = f(am), \quad f : M \to FX,\ a \in A,\ m \in M.$$

Then

$$f_X \overset{\text{def}}{=} \big(X \simeq \mathrm{Hom}_A(A, X) \xrightarrow{F} \mathrm{Hom}_R(M, FX)\big)$$

is A-linear: for $a \in A$, $m \in M$, $t : A \to X$,

$$F(ta)(m) = F(t \circ a)(m) = F(t)(F(a)(m)) = F(t)(am) = (F(t)a)(m).$$

Let $g_X : X \otimes M \to FX$ be the R-linear map corresponding to f_X under the canonical isomorphism

$$\mathrm{Hom}_A(X, \mathrm{Hom}_R(M, FX)) \simeq \mathrm{Hom}_R(X \otimes M, FX),$$

and let $\mathcal{C}$ be the collection of X such that g_X is an isomorphism. Then $\mathcal{C}$ contains A_A, is closed under finite direct sums, and contains the cokernel of $X \to Y$ if it contains X and Y. As before, this implies that $\mathcal{C} = \mathrm{ob}\,\mathsf{Mod}_A$, and so g is an isomorphism $F_M \simeq F$. □

EXAMPLE 11.4 Let k be a field and A a finite-dimensional k-algebra. Every left A-module M defines a right exact k-linear functor

$$F_M : \mathsf{Modf}_A \to \mathrm{Vec}(k), \quad X \rightsquigarrow X \otimes_A M.$$

The functor

$$M \rightsquigarrow F_M : {}_A\mathsf{Modf} \to \mathsf{Rex}_k(\mathsf{Modf}_A, \mathrm{Vecf}(k))$$

is an equivalence of categories, with quasi-inverse $F \rightsquigarrow F(A_A)$.

Tensor products of abelian categories: construction

In this subsection k is a field. Let A and R be k-algebras (not necessarily commutative) with A finite-dimensional, and let ${}_A\mathsf{Mod}_R$ denote the category of (A,R)-bimodules on which the two actions of k coincide.

For M in ${}_A\mathsf{Mod}_R$, F_M is k-linear, and so we have the following k-linear variant of Proposition 11.3.

PROPOSITION 11.5 *The functor*

$$M \rightsquigarrow F_M : {}_A\mathsf{Mod}_R \to \mathsf{Rex}_k(\mathsf{Modf}_A, \mathsf{Mod}_R)$$

is an equivalence of categories with quasi-inverse $F \rightsquigarrow F(A_A)$

PROPOSITION 11.6 *Let A, B be finite-dimensional k-algebras and R a k-algebra. The functor*

$$F \rightsquigarrow F_{(A,B)} : \mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{Mod}_R) \to {}_{A\otimes B}\mathsf{Mod}_R$$

is an equivalence of categories.

PROOF The proof of Proposition 11.3 generalizes. □

PROPOSITION 11.7 *Let A and B be finite-dimensional k-algebras. Then*

$$\otimes : \mathsf{Modf}_A \times \mathsf{Modf}_B \to \mathsf{Modf}_{A\otimes_k B}, \quad (M,N) \rightsquigarrow M \otimes_k N,$$

is the Deligne tensor product of the categories Modf_A *and* Modf_B.

PROOF We have to show that, for any small k-linear abelian category D, the functor

$$F \rightsquigarrow F \circ \otimes : \mathsf{Rex}_k(\mathsf{Modf}_{A\otimes B}, \mathsf{D}) \to \mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{D}) \tag{47}$$

is an equivalence of categories. If $\mathsf{D} \to \mathsf{D}'$ is fully faithful and exact, and the statement is true for D', then it is true for D. Now a k-linear variant of the full embedding theorem (Mitchell 1965, VI, 7.2) allows us to replace D with Mod_R for some k-algebra R. In this case, (47) has quasi-inverse

$$\mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{Mod}_R) \xrightarrow{11.6} {}_{A\otimes B}\mathsf{Mod}_R \xrightarrow{11.5} \mathsf{Rex}_k(\mathsf{Modf}_{A\otimes B}, \mathsf{Mod}_R).$$

□

THEOREM 11.8 (DELIGNE 1990, 5.13) *Let* A *and* B *be essentially small k-linear abelian categories (k a field). If* A *and* B *are locally finite (6.7), then their tensor product exists and is locally finite.*

PROOF Write each of A and B as filtered unions of subcategories $\mathsf{A} = \bigcup_\alpha \mathsf{A}_\alpha$ and $\mathsf{B} = \bigcup_\beta \mathsf{B}_\beta$ as in (II, 3.18). Now $\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta$ exists for each α, β (by 11.7), and the transition map

$$\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta \to \mathsf{A}_{\alpha'} \boxtimes \mathsf{B}_{\beta'}, \quad \alpha \le \alpha, \quad \beta \le \beta',$$

is fully faithful and exact because it can be identified with the map

$$\mathsf{Mod}_{A_\alpha \otimes B_\alpha} \to \mathsf{Mod}_{\bar{A}_\alpha \otimes \bar{B}_\alpha}$$

defined by a surjective map of k-algebras $A_\alpha \otimes B_\beta \to \bar{A}_\alpha \otimes \bar{B}_\beta$ (see II, 3.6). Define $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ to be the inductive 2-limit

$$\varinjlim_{\alpha,\beta} (\boxtimes : \mathsf{A}_\alpha \times \mathsf{B}_\beta \to \mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta).$$

Specifically, an object of $\mathsf{A} \boxtimes \mathsf{B}$ is an object X of $\mathsf{A}_{\alpha_0} \boxtimes \mathsf{B}_{\beta_0}$ for some (α_0, β_0). Such an X defines an object $X_{(\alpha,\beta)}$ of $\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta$ for all $(\alpha, \beta) \ge (\alpha_0, \beta_0)$. If Y is a second object of $\mathsf{A} \boxtimes \mathsf{B}$, then

$$\mathrm{Hom}_{\mathsf{A}\boxtimes\mathsf{B}}(X, Y) = \varinjlim \mathrm{Hom}_{\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta}(X_{(\alpha,\beta)}, Y_{(\alpha,\beta)}),$$

where the limit is over the pairs (α, β) such that $X_{(\alpha,\beta)}$ and $Y_{(\alpha,\beta)}$ are defined. It is straightforward to verify that $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ has the required properties. □

In summary, the proof of the theorem is a straightforward consequence of (a) existence of tensor products for module categories (11.7); (b) the description of finite-dimensional abelian categories as module categories (Mitchell); (c) inductive 2-limits.

PROPOSITION 11.9 *Let* A, B, *and* C *be locally finite k-linear abelian categories. There are canonical equivalences of categories*

$$\mathsf{A} \boxtimes (\mathsf{B} \boxtimes \mathsf{C}) \xleftrightarrow{\approx} (\mathsf{A} \boxtimes \mathsf{B}) \boxtimes \mathsf{C}$$
$$\mathsf{A} \boxtimes \mathsf{B} \xleftrightarrow{\approx} \mathsf{B} \boxtimes \mathsf{A}.$$

PROOF Obvious from the definitions. □

PROPOSITION 11.10 *Let* A *and* B *be locally finite k-linear categories, and let* $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ *be their tensor product.*

(a) *The functor* $\boxtimes : \mathsf{A} \times \mathsf{B} \to \mathsf{A} \boxtimes \mathsf{B}$ *is exact in each variable.*

(b) *The functor* $\boxtimes$ *induces isomorphisms*

$$\mathrm{Hom}_{\mathsf{A}}(X_1, X_2) \otimes \mathrm{Hom}_{\mathsf{B}}(Y_1, Y_2) \simeq \mathrm{Hom}(X_1 \boxtimes Y_1, X_2 \boxtimes Y_2),$$

all $X_1, X_2 \in \mathrm{ob}\,\mathsf{A}$, $Y_1, Y_2 \in \mathrm{ob}\,\mathsf{B}$.

(c) *The functor* $\boxtimes$ *makes* $(\mathsf{A} \boxtimes \mathsf{B})^{\mathrm{opp}}$ *the tensor product of* $\mathsf{A}^{\mathrm{opp}}$ *and* $\mathsf{B}^{\mathrm{opp}}$.

(d) *For all small k-linear abelian categories* D, *the functor* $\boxtimes$ *induces an equivalence of categories*

$$\mathsf{Lex}(\mathsf{A} \boxtimes \mathsf{B}, \mathsf{D}) \xrightarrow{\approx} \mathsf{Lex}(\mathsf{A} \times \mathsf{B}, \mathsf{D}),$$

where Lex *denotes the category of k-linear functors left exact (in each variable).*

PROOF It suffices to prove each statement for $\mathsf{A} = \mathsf{Modf}_A$ and $\mathsf{B} = \mathsf{Modf}_B$, where A and B are finite-dimensional k-algebras (see the proof of 11.8).

(a) As $\boxtimes = \otimes_k$, this is obvious.

(b) For finitely generated A and B modules, we have a bijection

$$\begin{aligned}\mathrm{Hom}_k(M_1, M_2) \otimes \mathrm{Hom}_k(N_1, N_2) &\simeq \mathrm{Hom}_k(M_1 \otimes N_1, M_2 \otimes N_2)\\ f \otimes g &\mapsto (m \otimes n \mapsto f(m) \otimes g(n)).\end{aligned}$$

Under the bijection, maps that are A and B linear correspond to maps that are $A \otimes B$-linear.

(c) For a finite-dimensional k-algebra A, the functor

$$M \rightsquigarrow M^\vee \stackrel{\text{def}}{=} \mathrm{Hom}_k(M, k) : \ (\mathsf{Modf}_A)^{\mathrm{opp}} \to \mathsf{Modf}_{A^{\mathrm{opp}}}$$

is an equivalence of categories because $M^{\vee\vee} \simeq M$ and $(A \otimes B)^{\mathrm{opp}} \simeq A^{\mathrm{opp}} \otimes B^{\mathrm{opp}}$. Therefore,

$$\mathsf{Modf}_A^{\mathrm{opp}} \boxtimes \mathsf{Modf}_B^{\mathrm{opp}} \sim \mathsf{Modf}_{A^{\mathrm{opp}} \otimes B^{\mathrm{opp}}} \cong \mathsf{Modf}_{(A \otimes B)^{\mathrm{opp}}} \cong (\mathsf{Modf}_A \boxtimes \mathsf{Modf}_B)^{\mathrm{opp}}.$$

This proves the statement.

(d) Under $\mathsf{A} \mapsto \mathsf{A}^{\mathrm{opp}}$, right exact functors go to left-exact functors. Therefore, this follows from (c). □

PROPOSITION 11.11 *Suppose that the functors $F : \mathsf{A} \times \mathsf{B} \to \mathsf{D}$ and $F' : \mathsf{A} \boxtimes \mathsf{B} \to \mathsf{D}$ correspond under (46), p. 60. If F is exact in both variables and k is perfect, then F' is exact.*

PROOF Again, it suffices to prove this in the key case 11.13 below. □

11.12 Let A and B be finite-dimensional k-algebras and S a simple $A \otimes_k B$-module. There exist a simple A-module M and a simple B-module N, both finite-dimensional over k, such that S is a quotient of $M \otimes_k N$ (Bourbaki A, 7.7, Pptn 8). The centres k_M and k_N of $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$ are finite field extensions of k. If k is perfect, they are separable over k, and hence $k_M \otimes k_N$ is a semisimple k-algebra, which implies that $M \otimes_k N$ is a semisimple $A \otimes_k B$-algebra (Bourbaki A, 7.4, Thm 2). In this case, S is a direct summand of $M \otimes_k N$.

LEMMA 11.13 *Let A and B be finite-dimensional k-algebras, and suppose that the functors $F : \mathsf{Modf}_A \times \mathsf{Modf}_B \to \mathsf{Mod}_R$ and $F' : \mathsf{Modf}_{A \otimes B} \to \mathsf{Mod}_R$ correspond under (47), p. 62. If F is exact in both variables and k is perfect, then F' is exact.*

PROOF Let $M = F(A_A, B_B) = F'((A \otimes B)_{A \otimes B})$. Then

$$\begin{aligned}F(X, Y) &= (X \otimes Y) \otimes_{A \otimes B} M\\ F'(Z) &= Z \otimes_{A \otimes B} M.\end{aligned}$$

We choose projective resolutions $X^\bullet \to X$ and $Y^\bullet \to Y$ for X and Y, and form the complex $(X^\bullet \otimes Y^\bullet) \otimes_{A \otimes B} M$. Let $\mathrm{Tor}_n^{A,B}((X, Y), M)$ denote the nth homology group of this complex. After our assumption, $\mathrm{Tor}_n^{A,B}((X, Y), M) = 0$ for $n > 0$. For $Z \in \mathrm{ob}\,\mathsf{Modf}_{A \otimes B}$, we define $\mathrm{Tor}_n^{A \otimes B}(Z, M)$ similarly. We have to show that $\mathrm{Tor}_n^{A \otimes B}(Z, M) = 0$ for $n > 0$. If $Z = X \otimes Y$ as above, then $X^\bullet \otimes Y^\bullet \to Z$ is a projective resolution, and therefore

$$\mathrm{Tor}_n^{A \otimes B}(X \otimes Y, M) = \mathrm{Tor}_n^{A,B}((X, Y), M) = 0.$$

An exact sequence

$$0 \to Z' \to Z \to Z'' \to 0$$

in $\mathsf{Modf}_{A\otimes B}$ gives a long exact sequence

$$\cdots \to \mathrm{Tor}_n^{A\otimes B}(Z', M) \to \mathrm{Tor}_n^{A\otimes B}(Z, M) \to \mathrm{Tor}_n^{A\otimes B}(Z'', M).$$

As every finitely generated $A \otimes B$-module has finite length, it suffices to show that $\mathrm{Tor}_n^{A\otimes B}(Z, M)$ for all $n > 0$ and all simple $A \otimes B$-modules Z. Because

$$\mathrm{Tor}_n^{A\otimes B}(Z \otimes Z', M) \simeq \mathrm{Tor}_n^{A\otimes B}(Z, M) \otimes \mathrm{Tor}_n^{A\otimes B}(Z', M)$$

(Mac Lane 1963, V, §§7,8) this follows from 11.12. □

NOTES The exposition in this subsection follows the original in Deligne 1990, §5, except that we have been more careful in the passage to the inductive limits. See also Lattermann 1989, 3.5.

Tensor products of abelian categories: an alternative approach

Let k be a commutative ring.

11.14 Let A and B be k-linear categories with finite inductive limits. There exists a k-linear category $\mathsf{A} \bullet \mathsf{B}$ with finite inductive limits and a k-bilinear functor

$$\bullet : \mathsf{A} \times \mathsf{B} \to \mathsf{A} \bullet \mathsf{B},$$

right exact in each variable, with the following universal property: for all k-linear categories D with finite inductive limits, the functor

$$F \rightsquigarrow F \circ \bullet : \mathsf{Rex}(\mathsf{A} \bullet \mathsf{B}, \mathsf{D}) \to \mathsf{Rex}(\mathsf{A} \times \mathsf{B}, \mathsf{D})$$

is an equivalence of categories (Kelly 1982b, Kelly 1982a, §6.5). The category $\mathsf{A} \bullet \mathsf{B}$ is called the ***Kelly tensor product*** of A and B.

11.15 Let k be a field and A a k-linear category. For a field K containing k, we let A_K denote the category K-modules in A.

If K/k is a finite and A has finite inductive limits, then the pairing (6.11)

$$K, X \rightsquigarrow K \otimes X : \mathsf{Vecf}(K) \times \mathsf{A} \to \mathsf{A}_K$$

is universal, and so $\mathsf{Vecf}(K) \bullet \mathsf{A} \approxeq \mathsf{A}_K$.

More generally, the Kelly tensor product can be defined relative to any class of inductive limits. If A is a k-linear category with small inductive limits and K is an arbitrary extension of k, then

$$\mathsf{Vec}(K) \bullet \mathsf{A} \approxeq \mathsf{A}_K.$$

See Schäppi 2018, p. 6.

11.16 If k is a field, and A and B are locally finite k-linear abelian categories, then $\mathsf{A} \bullet \mathsf{B}$ is abelian and $(\mathsf{A} \bullet \mathsf{B}, \bullet)$ is the Deligne tensor product of A and B (López Franco 2013, Theorem 3). This gives a second proof of Theorem 11.8.

More precisely, Lopez Franco (ibid., Theorem 1) shows that two small abelian categories have a Deligne tensor product if and only if their Kelly tensor product is abelian,

in which case the two tensor products coincide. This can be used to give examples where the Deligne tensor product does not exist.

A k-algebra R is said to be ***left coherent*** if the category of finitely presented left R-modules ${}_R\mathsf{Modf}$ is abelian (for this it suffices to check that the left ideals in R are finitely presented). If R and S are left coherent k-algebras, then the Kelly tensor product of ${}_R\mathsf{Modf}$ and ${}_S\mathsf{Modf}$ exists and equals ${}_{R\otimes_k S}\mathsf{Modf}$. If $R\otimes_k S$ is not left coherent, then this category is not abelian, and so the Deligne tensor product does not exist. For example, $\mathbb{Q}^{\mathbb{N}}[[T,U]]$ and $\mathbb{Q}[X]$ are coherent $\mathbb{Q}$-algebras, but $\mathbb{Q}^{\mathbb{N}}[[T,U]]\otimes_{\mathbb{Q}}\mathbb{Q}[X]\simeq\mathbb{Q}^{\mathbb{N}}[[T,U]][X]$ is not coherent (ibid. Example 21).

Tensor products of tensor categories

Let $(\mathsf{C},\otimes)$ be a locally finite k-linear abelian rigid symmetric monoidal category. Then $\otimes$ is k-bilinear and exact in each variable (I, 5.2), so it factors through $\mathsf{C}\boxtimes\mathsf{C}$,

$$\begin{array}{ccc}\mathsf{C}\times\mathsf{C} & \xrightarrow{\boxtimes} & \mathsf{C}\boxtimes\mathsf{C}\\ & {\scriptstyle\otimes}\searrow & \downarrow{\scriptstyle T}\\ & & \mathsf{C}\end{array}$$

When k is perfect, T is exact (11.11). After 11.10(a), $\boxtimes$ is k-linear and exact in each variable.

Let C_1 and C_2 be k-linear abelian rigid categories.

PROPOSITION 11.17 *Let k be a perfect field. If C_1 and C_2 are tensor categories over k, then so also is $\mathsf{C}_1\boxtimes\mathsf{C}_2$.*

PROOF Let $\mathsf{C}=\mathsf{C}_1\boxtimes\mathsf{C}_2$. For $i=1,2$, we have a tensor $T_i:\mathsf{C}_i\boxtimes\mathsf{C}_i\to\mathsf{C}_i$. On taking their exterior tensor product, we get a natural transformation T

$$\mathsf{C}\boxtimes\mathsf{C}=(\mathsf{C}_1\boxtimes\mathsf{C}_2)\boxtimes(\mathsf{C}_1\boxtimes\mathsf{C}_2)\simeq(\mathsf{C}_1\boxtimes\mathsf{C}_1)\boxtimes(\mathsf{C}_2\boxtimes\mathsf{C}_2)\xrightarrow[T_1\boxtimes T_2]{T}\mathsf{C},$$

which is k-linear and exact (11.11). The middle isomorphism comes from the canonical isomorphisms in Proposition 11.9. On composing T with $\boxtimes:\mathsf{C}\times\mathsf{C}\to\mathsf{C}\boxtimes\mathsf{C}$, we get a functor

$$\otimes:\mathsf{C}\times\mathsf{C}\to\mathsf{C}.$$

As unit object, we choose $\mathbb{1}\boxtimes\mathbb{1}$. It follows from (11.10(b)) that $\mathrm{End}(\mathbb{1}\otimes\mathbb{1})\simeq k$.

It remains to prove that C is rigid. The equivalences (I, §5)

$$X_i\rightsquigarrow X_i^\vee:\mathsf{C}_i^{\mathrm{opp}}\to\mathsf{C}_i$$

induce an equivalence

$$\mathsf{C}^{\mathrm{opp}}\simeq\mathsf{C}_1^{\mathrm{opp}}\boxtimes\mathsf{C}_2^{\mathrm{opp}}\to\mathsf{C},$$

denoted $X\rightsquigarrow X^\vee$, which is characterized by

$$(X_1\boxtimes X_2)^\vee=X_1^\vee\boxtimes X_2^\vee,\quad X_i\in\mathsf{C}_i.$$

We shall show that internal Homs exist by constructing a natural isomorphism

$$\mathrm{Hom}(X\otimes Y,Z)\simeq\mathrm{Hom}(Y,X^\vee\otimes Z),\quad X,Y,Z\in\mathrm{ob}\,\mathsf{C}.$$

Let $F_1, F_2 : \mathsf{C}^{\mathrm{opp}} \times \mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \mathsf{Vecf}(k)$, denote the two functors. Both are left exact in each variable, so this is equivalent to constructing an isomorphism between the functors

$$\widetilde{F}_1, \widetilde{F}_2 : \mathsf{C}_1^{\mathrm{opp}} \times \mathsf{C}_2^{\mathrm{opp}} \times \mathsf{C}_1^{\mathrm{opp}} \times \mathsf{C}_2^{\mathrm{opp}} \times \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{Vecf}(k),$$

where

$$\begin{aligned}\widetilde{F}_1(X_1, X_2, Y_1, Y_2, Z_1, Z_2) &= \mathrm{Hom}((X_1 \boxtimes X_2) \boxtimes (Y_1 \boxtimes Y_2), Z_1 \boxtimes Z_2) \\ &\simeq \mathrm{Hom}((X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2), Z_1 \boxtimes Z_2) \\ &\simeq \mathrm{Hom}(X_1 \otimes Y_1, Z_1) \boxtimes \mathrm{Hom}(X_2 \otimes Y_2, Z_2)\end{aligned}$$

and

$$\begin{aligned}\widetilde{F}_2(X_1, X_2, Y_1, Y_2, Z_1, Z_2) &= \mathrm{Hom}((Y_1 \boxtimes Y_2), (X_1^\vee \boxtimes X_2^\vee) \otimes (Z_1 \boxtimes Z_2)) \\ &\simeq \mathrm{Hom}(Y_1, X_1^\vee \otimes Z_1) \boxtimes \mathrm{Hom}(Y_2, X_2^\vee \otimes Z_2).\end{aligned}$$

For this, combine the isomorphisms (I, 4.6)

$$\begin{aligned}\mathrm{Hom}(X_1 \otimes Y_1, Z_1) &\simeq \mathrm{Hom}(Y_1, X_1^\vee \otimes Z_1), \quad X_1, Y_1, Z_1 \in \mathrm{ob}\,\mathsf{C}_1 \\ \mathrm{Hom}(X_2 \otimes Y_2, Z_2) &\simeq \mathrm{Hom}(Y_2, X_2^\vee \otimes Z_2), \quad X_2, Y_2, Z_3 \in \mathrm{ob}\,\mathsf{C}_2.\end{aligned}$$

It remains to show that the canonical morphism (22), p. 22,

$$\phi : \mathcal{H}om(X, \mathbb{1}) \otimes Y \to \mathcal{H}om(X, Y)$$

is an isomorphism for all $X, Y \in \mathrm{ob}(\mathsf{C})$. We regard ϕ as a morphism of k-bilinear left exact functors $\mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \mathsf{C}$. Then it suffices to show that $\phi_{X,Y}$ is an isomorphism for $X = X_1 \boxtimes X_2$ and $Y = Y_1 \boxtimes Y_2$ with $X_i, Y_i \in \mathrm{ob}\,\mathsf{C}_i$. When X is of the form $X_1 \boxtimes X_2$, it has a dual, namely, $(X_1^\vee \boxtimes X_2^\vee, \mathrm{ev}_{X_1} \boxtimes \mathrm{ev}_{X_2})$ and $\delta_{X_1} \boxtimes \delta_{X_2}$. From this the proposition follows. □

COROLLARY 11.18 *Let k be a perfect field. If C_1 and C_2 are tannakian categories over k, then so also is $\mathsf{C}_1 \boxtimes \mathsf{C}_2$.*

PROOF There exist fibre functors $\omega_i : \mathsf{C}_i \to \mathsf{Mod}(B_i)$, where B_i is a nonzero commutative k-algebra. The functor

$$(X_1, X_2) \rightsquigarrow \omega_1(X_1) \otimes \omega_2(X_2) : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{Mod}(B_1 \otimes B_2)$$

is k-linear and left exact in both variables, and so it induces a k-linear left exact functor

$$X_1 \boxtimes X_2 \rightsquigarrow \omega_1(X_1) \otimes \omega_2(X_2) : \mathsf{C}_1 \boxtimes .\mathsf{C}_2 \to \mathsf{Mod}(B_1 \otimes B_2).$$

We wish to define a natural isomorphism between $\omega(X \otimes Y)$ and $\omega X \otimes_{B_1 \otimes B_2} \omega Y$. It suffices to do this with $X = X_1 \boxtimes X_2$ and $Y = Y_1 \boxtimes Y_2$. In this case, we have

$$\begin{aligned}\omega((X_1 \boxtimes X_2) \otimes (Y_1 \boxtimes Y_2)) &\simeq \omega((X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2)) \\ &\simeq \omega_1(X_1 \otimes Y_1) \boxtimes \omega_2(X_2 \otimes Y_2) \\ &\simeq (\omega_1 X_1 \otimes_{B_1} \omega_1 Y_1) \boxtimes (\omega_2 X_2 \otimes_{B_2} \omega_2 Y_2) \\ &\simeq (\omega_1 X_1 \otimes \omega_2 X_2) \otimes_{B_1 \otimes B_2} (\omega_1 Y_1 \otimes \omega_2 Y_2) \\ &\simeq \omega(X_1 \boxtimes X_2) \otimes_{B_1 \otimes B_2} \omega(Y_1 \boxtimes Y_2).\end{aligned}$$

We have shown that $\mathsf{C}_1 \boxtimes \mathsf{C}_2$ has a fibre functor with values in the nonzero commutative k-algebra $B_1 \otimes B_2$. □

ASIDE 11.19 Over nonperfect fields, it seems not to be known whether tensor products of tensor categories always exist. For tannakian categories, this is is proved in Deligne 1990, 5.18. It is also an immediate consequence of V, Theorem 1.1, whose proof, however, uses Corollary 11.18; see V, §5.

EXAMPLE 11.20 Let T be a tensor category over k, and let k' be a finite extension of k. For a finite-dimensional k'-vector space V and object X of T, we define $V \otimes X$ to represent the functor

$$\mathsf{T} \rightsquigarrow V \otimes \mathrm{Hom}(X, T)$$

(cf. 6.11). Then $V \otimes X$ has a natural action of k', and so lies in $\mathsf{T}_{(k')}$. The functor

$$\mathsf{Vecf}(k') \times \mathsf{T} \to \mathsf{T}_{(k')}$$

induces a functor

$$\mathsf{Vecf}(k') \boxtimes \mathsf{T} \to \mathsf{T}_{(k')},$$

which is an equivalence of tensor categories over k'.

Similarly, when k' is a possibly infinite extension, there is an equivalence of k'-linear $\otimes$-categories

$$\mathsf{Vec}(k') \bullet \mathrm{Ind}\,\mathsf{T} \approxeq (\mathrm{Ind}\,\mathsf{T})_{k'},$$

where $(\mathrm{Ind}\,\mathsf{T})_{k'}$ is the category of k'-modules in $\mathrm{Ind}\,\mathsf{T}$. See 11.15.

12 Complements

In this section, we add some complements to earlier results.

Closed abelian $\otimes$-categories

This subsection collects basic results about abelian $\otimes$-categories with internal Homs.

Let $(\mathsf{C}, \otimes)$ be an abelian $\otimes$-category with internal Homs (see §3). Recall (3.1) that this means that, for all objects X, the functor $- \otimes X$ has a right adjoint $\mathcal{H}om(X, -)$,

$$\mathrm{Hom}(T \otimes X, Y) \simeq \mathrm{Hom}(T, \mathcal{H}om(X, Y)),$$

so $- \otimes X$ is right exact and $\mathcal{H}om(X, -)$ is left exact. Let $X^\vee = \mathcal{H}om(X, \mathbb{1})$ (weak dual).

PROPOSITION 12.1 *If $X \to Y \to Z \to 0$ is exact, then the sequence of weak duals*

$$0 \to Z^\vee \to Y^\vee \to X^\vee$$

is exact.

PROOF For all objects T, $T \otimes X \to T \otimes Y \to T \otimes Z \to 0$ is exact, so

$$0 \to \mathrm{Hom}(T \otimes Z, \mathbb{1}) \to \mathrm{Hom}(T \otimes Y, \mathbb{1}) \to \mathrm{Hom}(T \otimes X, \mathbb{1})$$

is exact. By adjointness, it follows that

$$0 \to \mathrm{Hom}(T, Z^\vee) \to \mathrm{Hom}(T, Y^\vee) \to \mathrm{Hom}(T, X^\vee)$$

is exact (all T), and so $0 \to Z^\vee \to Y^\vee \to X^\vee$ is exact. □

DEFINITION 12.2 An object X of C is ***flat*** if the functor $X \otimes -$ is left exact (hence exact).

DEFINITION 12.3 An object X of C is ***rigid*** if the pair $(X^\vee, X^\vee \otimes X \xrightarrow{\mathrm{ev}_{X,\mathbb{1}}} \mathbb{1})$ is a dual of X, i.e., if there exists a morphism $\mathrm{coev} : \mathbb{1} \to X \otimes X^\vee$ satisfying (19), p. 21.

If X is rigid, then $X^\vee$ is also rigid, and $X \simeq X^{\vee\vee}$.

EXAMPLE 12.4 In the category $\mathsf{Rep}(G)$ of representations of an affine group scheme G on k-vector spaces, the rigid objects are those in $\mathsf{Repf}(G)$. Similarly, if T is a tensor category over k, then the rigid objects in $\mathrm{Ind}\,\mathsf{T}$ are those in T.

PROPOSITION 12.5 *Every rigid object X of C is flat.*

PROOF Let $0 \to A \to B \to C$ be exact. As X is rigid, the functors $- \otimes X$ and $- \otimes X^\vee$ are adjoint (4.5), and so, for all objects T, the sequence

$$0 \to \mathrm{Hom}(T, A \otimes X) \to \mathrm{Hom}(T, B \otimes X) \to \mathrm{Hom}(T, C \otimes X)$$

is isomorphic to the sequence

$$0 \to \mathrm{Hom}(T \otimes X^\vee, A) \to \mathrm{Hom}(T \otimes X^\vee, B) \to \mathrm{Hom}(T \otimes X^\vee, C),$$

which is obviously exact. Therefore $0 \to A \otimes X \to B \otimes X \to C \otimes X$ is exact. □

PROPOSITION 12.6 *The tensor product of a projective object with a rigid object is again projective. In particular, if the unit object is projective, then all rigid objects are projective.*

PROOF Let X be rigid and P projective. As X is rigid, $\mathrm{Hom}(P \otimes X, T) \simeq \mathrm{Hom}(P, T \otimes X^\vee)$. Let $B \to C$ be an epimorphism, and consider a morphism $P \otimes X \to C$,

$$\begin{array}{ccc} & & P \otimes X \\ & & \downarrow \\ B & \xrightarrow{\text{epi}} & C. \end{array}$$

On using the adjointness, we get a diagram

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ B \otimes X^\vee & \xrightarrow{\text{epi}} & C \otimes X^\vee. \end{array}$$

As P is projective, there is a morphism $P \to B \otimes X^\vee$ making the diagram commute. By adjointness again, this corresponds to a morphism $X \to B$ making the top diagram commute. □

EXAMPLE 12.7 The unit object is projective if every epimorphism $X \to \mathbb{1}$ splits. In $\mathsf{Mod}(R)$, the unit object is R, which is projective. In $\mathsf{Repf}(G)$, $\mathbb{1}$ being projective means that the functor $V \rightsquigarrow V^G$ is exact. In general, it means that the functor $A \rightsquigarrow \mathrm{Hom}(\mathbb{1}, A)$ is exact.

PROPOSITION 12.8 *If X is rigid and I is injective, then $I \otimes X$ is injective.*

PROOF As X is rigid,

$$\mathrm{Hom}(T, I \otimes X) \simeq \mathrm{Hom}(T \otimes X^\vee, I).$$

The object $X^\vee$ is rigid, hence flat (12.5), and so the right-hand term is exact as a functor of T . Therefore, the left-hand term is also exact as a functor of T, which implies that $I \otimes X$ is injective. □

PROPOSITION 12.9 *Assume that every object of* C *is a quotient of a flat object, and let Z be a flat object of* C. *If* $0 \to X \to Y \to Z \to 0$ *is exact, then*

$$0 \to X \otimes T \to Y \otimes T \to Z \otimes T \to 0$$

is exact for all objects T.

PROOF Let T be an object of C. By assumption, there exists an exact sequence

$$0 \to R \to P \to T \to 0$$

with P flat. There then exists an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & X \otimes R & \longrightarrow & Y \otimes R & \longrightarrow & Z \otimes R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \otimes P & \longrightarrow & Y \otimes P & \longrightarrow & Z \otimes P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X \otimes T & \longrightarrow & Y \otimes T & \longrightarrow & Z \otimes T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where the zeros at right and bottom come from the right exactness of the tensor product and the zeros at left and top come from the flatness of P and Z. Now the snake lemma shows that the bottom row is exact with a zero at left. □

LEMMA 12.10 *In an abelian category with enough projectives, a sequence* $B \to C \to 0$ *is exact if and only if* $\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$ *is exact for all projectives P.*

PROOF Certainly, if P is projective, then

$$B \to C \to 0 \quad \text{exact} \;\Rightarrow\; \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0 \quad \text{exact}.$$

For the converse, suppose that $\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$ is exact when P is projective. The sequence $B \to C \to C/B \to 0$ is exact, so

$$\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to \mathrm{Hom}(P, C/B) \to 0$$

is exact for all projectives P. As the first map is surjective, $\mathrm{Hom}(P, C/B) = 0$, but, by assumption, there exists an epimorphism $P \to C/B$ with P projective, which is zero only if C/B is zero. □

PROPOSITION 12.11 *Assume that every object is a quotient of a flat object. Let* $0 \to A \to B \to C \to 0$ *be exact with* C *flat. For all objects* I*, the sequence*

$$0 \to \mathcal{H}om(C,I) \to \mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$$

is left exact. If I *is injective and there are enough projectives, then the sequence is also right exact.*

PROOF The hypotheses imply that the sequence

$$0 \to T \otimes A \to T \otimes B \to T \otimes C \to 0$$

is exact for all T (12.9). Therefore

$$0 \to \mathrm{Hom}(T \otimes C, I) \to \mathrm{Hom}(T \otimes B, I) \to \mathrm{Hom}(T \otimes A, I) \to 0$$

is left exact, and right exact if I is injective. This sequence is isomorphic to the sequence

$$0 \to \mathrm{Hom}(T, \mathcal{H}om(C,I)) \to \mathrm{Hom}(T, \mathcal{H}om(B,I)) \to \mathrm{Hom}(T, \mathcal{H}om(A,I)) \to 0.$$

The first assertion follows. Now assume that I is injective. Then

$$\mathrm{Hom}(P, \mathcal{H}om(B,I)) \to \mathrm{Hom}(P, \mathcal{H}om(A,I)) \to 0$$

is exact for all projective P, which, by Lemma 12.10, implies that $\mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$ is exact. □

PROPOSITION 12.12 [8] *Assume that every object of* C *is a quotient of a flat object, and that there are enough projectives. If* $0 \to A \to B \to C \to 0$ *is exact and* C *is rigid, then*

$$0 \to \mathcal{H}om(C,I) \to \mathcal{H}om(B,I) \to \mathcal{H}om(A,I) \to 0$$

is exact for all objects I*.*

PROOF Under the hypotheses, projective objects are flat because they are direct summands of flat objects. It suffices to prove that

$$0 \to \mathrm{Hom}(P, \mathcal{H}om(C,I)) \to \mathrm{Hom}(P, \mathcal{H}om(B,I)) \to \mathrm{Hom}(P, \mathcal{H}om(A,I)) \to 0$$

is exact with P projective (12.10). By adjunction (9), this is isomorphic to the sequence

$$0 \to \mathrm{Hom}(P \otimes C, I) \to \mathrm{Hom}(P \otimes B, I) \to \mathrm{Hom}(P \otimes A, I) \to 0.$$

As P is flat,

$$0 \to P \otimes A \to P \otimes B \to P \otimes C \to 0$$

is exact. From this, we get an exact sequence

$$0 \to \mathrm{Hom}(P \otimes C, I) \to \mathrm{Hom}(P \otimes B, I) \to \mathrm{Hom}(P \otimes A, I) \to \mathrm{Ext}^1(P \otimes C, I).$$

Note that $\mathrm{Hom}(P \otimes C, -) \simeq \mathrm{Hom}(P, - \otimes C^\vee)$. As $C^\vee$ is flat, the second functor is exact, so $P \otimes C$ is projective. Hence $\mathrm{Ext}^1(P \otimes C, -) = 0$. □

[8] I thank Kevin Coulembier for this result.

PROPOSITION 12.13 *Assume that every object is a quotient of a flat object. Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence such that the sequence of weak duals

$$0 \to C^\vee \to B^\vee \to A^\vee \to 0$$

is exact. If A and C are rigid, then B is rigid.

PROOF Recall (4.7) that an object X is rigid if and only if the canonical morphism

$$T \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T)$$

is bijective for all T. If A is rigid, so also is $A^\vee$, hence it is flat (12.5), so, for all T, the top row of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \otimes C^\vee & \longrightarrow & T \otimes B^\vee & \longrightarrow & T \otimes A^\vee & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{H}om(C,T) & \longrightarrow & \mathcal{H}om(B.T) & \longrightarrow & \mathcal{H}om(A,T) & & \end{array}$$

is exact (12.9). Now the five-lemma shows that the middle vertical arrow is bijective. □

REMARK 12.14 (a) Not all duals of exact sequences are exact. For example, the dual of the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ in $\mathsf{Mod}(\mathbb{Z})$ is $0 \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$, which is not exact (in this example, C is not rigid).

(b) Let $0 \to A \to B \to C \to 0$ be exact. If A and C are rigid and C is projective, then $B \approx A \oplus C$, and so it is rigid.

PROPOSITION 12.15 *Assume that C has enough injectives and that $\mathcal{H}om(-, I)$ is right exact when I is injective. Then every projective object P of C is flat.*

PROOF Let $0 \to A \to B \to C$ be exact. By assumption, for any injective object I, $\mathcal{H}om(C, I) \to \mathcal{H}om(B, I) \to \mathcal{H}om(A, I) \to 0$ is exact. As P is projective,

$$\mathrm{Hom}(P, \mathcal{H}om(C, I)) \to \mathrm{Hom}(P, \mathcal{H}om(B, I)) \to \mathrm{Hom}(P, \mathcal{H}om(A, I)) \to 0$$

is exact. By adjunction (9), this implies that

$$\mathrm{Hom}(P \otimes C, I) \to \mathrm{Hom}(P \otimes B, I) \to \mathrm{Hom}(P \otimes A, I) \to 0$$

is exact. Hence, the kernel K of $P \otimes A \to P \otimes B$ has the property that $\mathrm{Hom}(K, I) = 0$ for all injectives. As C has enough injectives, this implies that $K = 0$. □

The category of ind-objects of a tensor category

Let T be an essentially small tensor category over a field k whose objects are noetherian.

12.16 Recall (B.8) that $\mathrm{Ind}\,\mathsf{C}$ is an abelian category and the fully faithful functor $\iota_{\mathsf{C}} : \mathsf{C} \to \mathrm{Ind}\,\mathsf{C}$ is exact (B.8). In particular, C is closed under taking kernels and cokernels in $\mathrm{Ind}\,\mathsf{C}$. Small inductive limits exist in $\mathrm{Ind}\,\mathsf{C}$, and small filtered inductive limits are exact.

The finitely presentable objects of Ind T are exactly the objects in the essential image of ι_T (B.6).

Any set of representatives for the isomorphism clases of T generates Ind T, and so Ind T is a Grothendieck category (B.9). In particular, Ind T has enough injectives.

As the objects of T are noetherian, Ind T is locally noetherian (B.16). An object of Ind T is noetherian if and only if it is finitely presentable (B.10(d)).

12.17 For $X = \varinjlim_i X_i$ and $Y = \varinjlim_j Y_j$, we set

$$X \otimes Y = \varinjlim_{i,j} X_i \otimes Y_j.$$

With this tensor functor, Ind T becomes a k-linear $\otimes$-category, and ι_T becomes a $\otimes$-functor. The functor $\otimes$ commutes with inductive limits in each variable, and so internal Homs exist in Ind T (5.4). Therefore, we can apply the results of the preceding section to Ind T.

12.18 The functor $\otimes : \operatorname{Ind} T \otimes \operatorname{Ind} T \to \operatorname{Ind} T$ is exact in each variable. To see this, let X be an object of Ind T and ξ a short exact sequence in Ind T. Write X as a filtered inductive limit $X = \varinjlim X_i$ of objects of T. As X_i is rigid in T, it is rigid in Ind T, and hence flat (12.5). Therefore, each sequence $X_i \otimes \xi$ is exact, and so $X \otimes \xi \simeq \varinjlim(X_i \otimes \xi)$ is exact.

12.19 Let I be an injective object of Ind T and X an object of T. Then X is rigid in T, hence also in Ind T, and so $I \otimes X$ is injective (12.8).

Extension of scalars for tensor categories

In this subsection, we extend some earlier definitions and results from finite field extensions to arbitrary field extensions. In particular, we prove the following generalization of Theorem 7.15.

THEOREM 12.20 *Let* C *be an essentially small tannakian category over a field* k*, and let* k' *be a field containing* k*. Let* D' *be the category of* k'*-modules in* $\mathsf{D} \overset{\text{def}}{=} \operatorname{Ind} \mathsf{C}$*, and let* $\mathsf{C}_{(k')}$ *be the full subcategory of* D' *whose objects are those generated as a* k'*-module by an object in* C*. Then* $\mathsf{C}_{(k')}$ *is a tannakian category over* k' *and*

$$\operatorname{Ind} \mathsf{C}_{(k')} \approxeq \mathsf{D}'.$$

COROLLARY 12.21 *Let* C *be an essentially small tannakian category over* k*, and let* k' *be a field containing* k*. If* C *has a* k'*-valued fibre functor, then* $\mathsf{C}_{(k')}$ *is a neutral tannakian category over* k'*.*

PROOF Let ω be a k'-valued fibre functor on C. Then $(X, \mu) \rightsquigarrow \omega(X)$ is a $k' \otimes_k k'$-valued fibre functor ω' on $\mathsf{C}_{(k')}$ (cf. the proof of 7.16). The multiplication map $k' \times k' \to k'$ defines a k-algebra homomorphism $k' \otimes_k k' \to k'$. Now $\omega' \otimes_{k' \otimes_k k'} k'$ is a k'-valued fibre functor on $\mathsf{C}_{(k')}$. □

PRELIMINARIES

12.22 Let k be a commutative ring and C a k-linear category. For a commutative k-algebra R, we let $\mathsf{Mod}(R,\mathsf{C})$ denote the category of R-modules in C, i.e., the category of pairs (X,μ) with X an object of C and μ a homomorphism $R \to \mathrm{End}(X)$ of k-algebras. Then $\mathsf{Mod}(R,\mathsf{C})$ is an R-linear category, and the forgetful functor $U:\ \mathsf{Mod}(R,\mathsf{C}) \to \mathsf{C}$ is k-linear and faithful. If C is abelian, then $\mathsf{Mod}(R,\mathsf{C})$ is abelian, and the forgetful functor is exact. See Kashiwara and Schapira 2006, 8.5.1, 8.5.2, 8.5.5.

12.23 Let k be field, and let D be a k-linear category with small inductive limits.[9] For K a field containing k, we let D_K denote the category of K-modules in D. The following statements hold.

(a) The forgetful functor $U:\ \mathsf{D}_K \to \mathsf{D}$ is faithful and reflects isomorphisms.

(b) For each object X of D, there is an object $K \otimes X$ of D_K (external tensor product), and the functor $X \rightsquigarrow K \otimes X$ is left adjoint to U

$$\mathrm{Hom}_{\mathsf{D}_K}(K \otimes X, Y) \simeq \mathrm{Hom}_{\mathsf{D}}(X, U(Y)), \quad X \in \mathrm{ob}\,\mathsf{D},\ Y \in \mathrm{ob}\,\mathsf{D}_K.$$

(c) If D admits some type of limits, then D_K admits the same type of limits. and they are preserved by U.

(d) If C is abelian, then so also is C_K, and U is exact.

These statements can be proved directly as in the finite case (6.12), or by recognizing D_K as the category of modules for a monad on C and applying Borceux 1994b, 4.1.4, 4.1.6, and 4.3.1. Specifically, let (T,ε,μ) be the monad on D with $T(X) = K \otimes X$, ε the natural morphism $X \simeq k \otimes X \to K \otimes X$ defined by $k \to K$, and $T^2(X) \to T(X)$ the natural morphism induced by the product map $K \otimes K \to K$. A module for this monad is a pair (X,ξ) with X an object of D and ξ a morphism $K \otimes X \to X$ restricting to the identity on X, i.e., k-linear K-module structure on X.

12.24 Let k be a field, and let C an essentially small tensor category over k whose objects are noetherian. Then, as we saw in the last subsection, $\mathsf{D} \overset{\text{def}}{=} \mathrm{Ind}\,\mathsf{C}$ is a k-linear abelian $\otimes$-category with internal Homs. The canonical functor $\iota_{\mathsf{C}}:\ \mathsf{C} \to \mathsf{D}$ is fully faithful with essential image the subcategory of finitely presentable objects, or, equivalently, noetherian objects (B.10(d)).

For X in D and V a k-vector space, we let $V \otimes X$ denote the object of D such that

$$\mathrm{Hom}_{\mathsf{D}}(V \otimes X, T) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_{\mathsf{D}}(X,T)) \tag{48}$$

for all objects T of D (cf. 6.11).

Let k' be a field extension of k, and let $\mathsf{D}' \overset{\text{def}}{=} \mathsf{Mod}(k',\mathsf{D})$ be the category of k'-modules in D. Let

$$X \otimes_{k'} Y = \mathrm{Coker}(X \otimes k' \otimes Y \rightrightarrows X \otimes Y),$$

where the two morphisms are defined by the actions of k' on X and Y. Then D' is a k'-linear abelian $\otimes$-category, and the forgetful functor $\mathsf{D}' \to \mathsf{D}$ is k-linear, faithful, and exact (12.23).

PROPOSITION 12.25 *The forgetful functor $\mathsf{D}' \to \mathsf{D}$ has a left adjoint $X \rightsquigarrow k' \otimes X$, so*

$$\mathrm{Hom}_{\mathsf{D}'}(k' \otimes X, T) \simeq \mathrm{Hom}_{\mathsf{D}}(X, T).$$

[9] If C only admits finite inductive limits, the same statement holds provided K is a finite extension of k.

PROOF Take V in 12.24 to be k'. Then $k' \otimes X$ has a structure of a k'-module, and (48) induces the required isomorphism. □

12.26 Let C' be the full subcategory of finitely presentable objects of D'. Every object X of C' is generated as a k'-module by a subobject in C, i.e., there exist a $Y \subset X$, $Y \in \mathrm{ob}\,\mathsf{C}$, for which the composite

$$k' \otimes_k Y \to k' \otimes_k X \to X$$

is an epimorphism. To see this, write X in C as a filtered union of its subobjects X_α in C. Since X is noetherian in C', the system of submodule images of the $k' \otimes_k X_\alpha$ in X is stationary. Therefore X is a quotient of $k' \otimes_k X_\alpha$ for α sufficiently large.

Recall (11.14) that $\mathsf{A} \bullet \mathsf{B}$ denotes the Kelly tensor product of two k -linear categories A and B with (finite) inductive limits.

PROPOSITION 12.27 (SCHÄPPI 2018, 2.2.1) *The pairing*

$$(V, X) \rightsquigarrow V \otimes_k X : \mathsf{Vecf}(k') \times \mathsf{C} \to \mathsf{C}'$$

is universal, and so induces an equivalence

$$\mathsf{Vecf}(k') \bullet \mathsf{C} \approx \mathsf{C}'.$$

PROOF It follows from the definitions that, for any two k-linear categories A and B with finite inductive limits,

$$\mathrm{Ind}(\mathsf{A} \bullet \mathsf{B}) \approx \mathrm{Ind}(\mathsf{A}) \bullet \mathrm{Ind}(\mathsf{B}).$$

In particular,

$$\mathrm{Ind}(\mathsf{Vecf}(k') \bullet \mathsf{C}) \approx \mathsf{Vec}(k') \bullet \mathsf{D}.$$

But $\mathsf{Vec}(k') \bullet \mathsf{D} \approx \mathsf{D}_{k'}$ (11.15), and so

$$\mathrm{Ind}(\mathsf{Vecf}(k') \bullet \mathsf{C}) \approx \mathsf{D}_{k'}.$$

On passing to the finitely presented objects and using (B.6), we obtain an equivalence

$$\mathsf{Vecf}(k') \bullet \mathsf{C} \approx \mathsf{C}'.$$

□

PROPOSITION 12.28 *Every object of C' is a quotient of a rigid object.*

PROOF Every object of C' is a quotient of an object $k' \otimes Y$ with Y in C. As Y is rigid in C, $i_{k'/k}(Y) = k' \otimes Y$ is rigid in D' (see 4.2). □

PROPOSITION 12.29 *If k is perfect, then every simple object X of $\mathsf{C}_{(k')}$ is rigid.*

PROOF Let Y generate X. As X is simple, any simple subobject of Y also generates X. But, if Y is simple in C, then $k' \otimes Y$ is semisimple in $\mathsf{C}_{(k')}$ (I, 6.14(a)), and so the epimorphism $k' \otimes Y \to X$ maps some simple direct summand of $k' \otimes Y$ isomorphically onto X. □

An intermediate result

Let C be a tannakian category over k, and let k' be a field containing k. Let $\mathsf{D} = \operatorname{Ind}\mathsf{C}$, let $\mathsf{D}' = \mathsf{Mod}(k', \mathsf{D})$, and let C' be the full subcategory of D' of finitely presentable objects.

Proposition 12.30 (Schäppi 2018, 3.2.4.) *Suppose that there exists a right exact k'-linear faithful $\otimes$-functor*

$$\omega : \mathsf{C}' \to \mathsf{Modf}(R)$$

such that $\operatorname{Ind}\omega$ *is exact and R is a filtered union of principal ideal domains. Then the category C' is rigid.*

Recall (4.7) that an object X of C' admits a dual if and only if the canonical morphism

$$\phi_{X,T} : T \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, T).$$

is an isomorphism for all objects T. As ω reflects isomorphisms, it suffices to show that, for all X in C', $\omega(X)$ is a finitely generated projective R-module (4.9).

As X is a quotient of rigid object, $\omega(X)$ is a quotient of a finitely generated projective R-module. In particular, it is finitely generated. The assumption on R therefore implies that $\omega(X) = F_n \oplus T$, where F_n is free of rank n and T is torsion. It remains to show that $T = 0$.

If not

$$\omega(\textstyle\bigwedge^{n+1} X) = \bigwedge^{n+1}(F_n \oplus T),$$

is a nonzero torsion module (see Lemma 12.31 below). After replacing X with $\bigwedge^{n+1} X$, we may suppose that $\omega(X)$ itself is a nonzero torsion module.

By assumption, there exists an epimorphism $p : V \to X$ in C' with V rigid. Let $\varphi : V' \to V$ denote its kernel in $\operatorname{Ind}\mathsf{C}'$. From the assumption on the ring R, we see that $\omega(p) : \omega(V) \to \omega(X)$ arises from a homomorphism over a principal ideal subdomain of R. As R is flat over the subdomain, the kernel of $\omega(p)$ also arises from the subdomain, and hence is finitely generated and free. This kernel equals $(\operatorname{Ind}\omega)(V')$, so V' is rigid, and, in particular, lies in C'. On passing to the field of fractions of R, we see that $\omega(V)$ and $\omega(V')$ are free of the same rank.

It remains to show that $\varphi : V' \to V$ is an isomorphism, and for this it suffices to show that $\omega(\varphi)$ is an isomorphism. As it is a map between free R-modules of the same rank, n say, it suffices to check that the determinant

$$E^n(\omega(\varphi)) \simeq \textstyle\bigwedge^n(\omega(\varphi)) \simeq \omega(\bigwedge^n(\varphi))$$

is invertible (E^n denotes the exterior power). As $\omega(V')$ and $\omega(V)$ are flat, $\varphi^{\otimes n} : (V')^{\otimes n} \to V^{\otimes n}$ is a monomorphism. Thus, $\bigwedge^n \varphi$ is a monomorphism between two ind-objects whose images in $\mathsf{Mod}(R)$ are free of rank one. But any such ind-object is invertible (in the sense of §2) in $\operatorname{Ind}\mathsf{C}'$ (so lies in C'). Moreover, as the unit object of C' is simple, all invertible objects of C' are simple. Thus $\bigwedge^n \varphi$ is indeed an isomorphism. This shows that $\omega(X) = 0$, giving a contradiction.

Lemma 12.31 *Let R be an integral domain and T a nonzero torsion module. Then $\bigwedge^{n+1}(R^{\oplus n} \oplus T)$ is a nonzero torsion module.*

Proof In the decomposition

$$\textstyle\bigwedge^{n+1}(R^{\oplus n} \oplus T) \simeq \bigoplus_{i=0}^{n+1}(\bigwedge^{n+1-i} R^{\oplus n}) \otimes_R (\bigwedge^i T),$$

the term for $i = 1$ is

$$\bigwedge^n R^{\oplus n} \otimes_R T \simeq R \otimes_R T \simeq T,$$

so T occurs as a direct summand of $\bigwedge^{n+1}(R^{\oplus n} \oplus T)$, which is therefore nonzero. It remains to show that it is torsion. As $\bigwedge^{n+1}(R^{\oplus n}) = 0$, every element of $\bigwedge^{n+1}(R^{\oplus n} \oplus T)$ is a sum of terms $m_1 \wedge \cdots \wedge m_{n+1}$ with each m_i in $R^{\oplus n}$ or T, and at least one m_i in T. It is therefore torsion. □

PROOF OF THEOREM 12.20.

Let C be an essentially small tannakian category over k, and let k' be a field containing k. Let $\mathsf{D} = \operatorname{Ind}\mathsf{C}$, let $\mathsf{D}_{k'} = \mathsf{Mod}(k', \mathsf{D})$, and let

$$\mathsf{C}_{k'} = (\mathsf{D}_{k'})^{\mathrm{fp}} \approxeq \mathsf{Vecf}(k') \bullet \mathsf{C}.$$

We first prove that $\mathsf{C}_{k'}$ is rigid. In proving the this, we may suppose that k' is finitely generated over k. This statement can be checked directly, or by using that the Kelly tensor product commutes with filtered bicategorical inductive limits. Then, arguing as in the finite case (7.15), we may suppose that k' is either a finite extension of k or equal to $k(t)$. The first case was proved in 7.15. For the second case, recall (7.7) that C has a fibre functor ω with values in some field K, and

$$(X, \mu) \rightsquigarrow \omega(X) : \mathsf{C}_{(k(t))} \to \mathsf{Mod}(k(t) \otimes_k K)$$

is then a $k(t)$-linear exact faithful functor As

$$k(t) \otimes K = \bigcup_f k[t]_f \otimes K \simeq \bigcup_f K[t]_f,$$

where f runs over the nonzero polynomials in $k[t]$, the statement now follows Proposition 12.30 below applied to $\mathsf{C}_{k(t)}$.

The category $\mathsf{C}_{k'}$ because any rigid category with cokernels has kernels, which implies that it is abelian because $\mathsf{D}_{k'}$ is abelian and the inclusion of $\mathsf{C}_{k'}$ into $\mathsf{D}_{k'}$ preserves kernels.

PROPOSITION 12.32 *The functor "$\varinjlim X_\alpha$"$\rightsquigarrow \varinjlim X_\alpha : \operatorname{Ind}\mathsf{C}' \to \mathsf{D}'$ is an equivalence of categories with quasi-inverse the functor sending an object of D' to the inductive system of its noetherian subobjects.*

PROOF All objects of C are noetherian, small filtered inductive limits exist in D and are exact, and every object of D' is a small filtered inductive limit of objects in C'. Therefore the statement follows from Proposition B.11 of the Appendix. □

NOTES

12.33 Corollary 12.21, when combined with the results of the next chapter, shows that any two fibre functors on a tannakian category are locally isomorphic for the fpqc topology (II, 8.7).

12.34 Deligne (1990) proved Theorem 12.20 as a consequence of his description of tannakian categories in terms of affine groupoid schemes (see IV, 9.1). The elementary proof given above is from Schäppi 2018.

12.35 Here is a third proof of Theorem 12.20.

(a) Case k' is algebraic over k. This case follows from Theorem 7.15 (cf. 7.19).

(b) Case k is algebraically closed. In this case, it is known that, if C is a tensor category over k whose objects are noetherian, then $\mathsf{C}_{(k')}$ is a tensor category (see 12.36 below).[10]

(c) General case. Let $\bar{k}$ be an algebraic closure of k and K a field containing both k' and $\bar{k}$. Then $\mathsf{C}_{(K)} \sim \left(\mathsf{C}_{(\bar{k})}\right)_{(K)}$ so it is tannakian by (a) and (b), and it follows that $\mathsf{C}_{(k')}$ is tannakian (because $\left(\mathsf{C}_{(k')}\right)_{(K)} \approxeq \mathsf{C}_{(K)}$ is tannakian).

12.36 One can ask whether Theorem 12.20 holds for tensor categories: does a tensor category T over k remain a tensor category under extension of the base field?

This is known when k is perfect and the objects of C are noetherian. As simple objects in $\mathsf{C}_{(k')}$ are rigid and the objects of $\mathsf{C}_{(k')}$ are noetherian, it suffices to prove that an extension of rigid objects in $\mathsf{C}_{(k')}$ is again rigid. Propositions 12.12 and 12.13 applied to $(\operatorname{Ind} C)_{k'}$ prove this under the hypothesis that there are enough projectives. The same argument shows that it holds under the following weaker hypothesis: there exists a family of inverse systems $(P_i^\alpha)_{i\in I_\alpha}$, $\alpha \in S$, in C such that the functors

$$F_\alpha = \varinjlim_i \operatorname{Hom}(P_i^\alpha, -), \quad \alpha \in S,$$

are exact and jointly faithful. It is possible to construct such a family as follows. Let S be the set of isomomorphism classes of simple objects in C. For $\alpha \in S$, let L_α represent α and write the injective hull I^α of $L_\alpha^\vee$ in $\operatorname{Ind}\mathsf{C}$ as an inductive limit $\varinjlim I_i^\alpha$ of objects of C. The family of projective systems $(K \otimes (I_i^\alpha)^\vee)_i$ has the required property. See Coulembier and Sherman 2026, 4.2.

Faithful $\otimes$-functors of tensor categories are exact

We prove that Corollaries 10.7 and 10.8 hold over all fields.

PROPOSITION 12.37 *Let $F : \mathsf{C} \to \mathsf{D}$ be a k-linear $\otimes$-functor of tensor categories over k. If F is faithful, then it is exact.*

PROPOSITION 12.38 *Let T be a tensor category over k and R a nonzero k-algebra. Every faithful k-linear $\otimes$-functor $\omega : \mathsf{T} \to \mathsf{Mod}(R)$ is exact (hence a fibre functor).*

Therefore, in both cases, F is faithful if and only if it is exact (5.8).

FIRST PROOF OF PROPOSITION 12.37

Let

$$\xi : 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

be an exact sequence in C, and let I be a nonzero injective object of $\operatorname{Ind}\mathsf{T}$, which exists because $\operatorname{Ind}\mathsf{T}$ is a Grothendieck category (12.16). Then

$$\xi \otimes I : 0 \to A \otimes I \to B \otimes I \to C \otimes I \to 0$$

[10] This is stated without proof in Etingof and Ostrik 2021, 2.2, and Coulembier et al. 2023, 5.1.3(2).

is exact (12.18), and even split-exact because $A \otimes I$ is injective (12.8). As F is additive, $F(\xi \otimes I)$ is exact, and hence the isomorphic sequence

$$F(\xi) \otimes F(I) : 0 \to F(A) \otimes F(I) \to F(B) \otimes F(I) \to F(C) \otimes F(I) \to 0$$

is also exact. Let H be some homology object of the complex

$$F(\xi) : 0 \to F(A) \to F(B) \to F(C) \to 0.$$

Because $- \otimes F(I)$ is exact (12.18), $H \otimes F(I)$ is the corresponding homology object of the complex $F(\xi) \otimes F(I)$, which is zero. As H is rigid, this implies that $H = 0$ (by 5.6).

Proof of Propositions 12.37 and 12.38

The proofs of the propositions are based on the following lemma.

Lemma 12.39 *Let* T *be a tensor category over* k*. A sequence*

$$\xi : 0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$$

is exact in T *if and only if there exists a nonzero morphism* $u : \mathbb{1} \to U$ *such that the map of chain complexes*

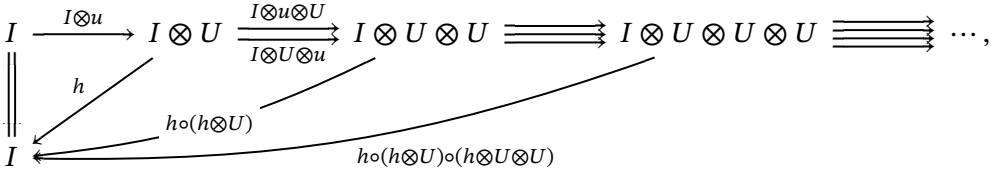
$$\begin{array}{ccccccccc} \xi & & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle u\otimes\xi} & & & & \downarrow & & \downarrow & & \downarrow & & \\ U \otimes \xi & & 0 & \longrightarrow & U \otimes X & \longrightarrow & U \otimes Y & \longrightarrow & U \otimes Z & \longrightarrow & 0 \end{array}$$

is null homotopic.

Proof For the sufficiency, let $u : \mathbb{1} \to U$ be a nonzero morphism such that $u \otimes \xi$ is null homotopic. Define the ring A_u in $\operatorname{Ind}\mathsf{T}$ to be the ind-object

$$\mathbb{1} \xrightarrow{u} U \underset{U\otimes u}{\overset{u\otimes U}{\rightrightarrows}} U \otimes U \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; U \otimes U \otimes U \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

(indexed by a category whose objects are the natural numbers). Let I be a nonzero injective object in $\operatorname{Ind}\mathsf{T}$ (exists by 12.16). Then $I \otimes u$ is a monomorphism of injective objects (12.17), so there exists a morphism $h : I \otimes U \to I$ such that $h \circ (I \otimes u) = \mathrm{id}_I$. The morphism h can be extended to a morphism $f : I \otimes A_u \to I$,



such that $f \circ (I \otimes u) = \mathrm{id}_I$. Therefore $A_u \neq 0$

Let $v : \mathbb{1} \to A_u$ be the composite $I \xrightarrow{u} U \to A_u$. As $u \otimes \xi$ is null homotopic, the map of chain complexes $v \otimes \xi : \xi \to A_u \otimes \xi$ is also null homotopic, which implies that the identity map $A_u \otimes \xi \to A_u \otimes \xi$ is null homotopic. This last statement means that

$A_u \otimes \xi$ is split-exact in $\mathsf{Mod}(A_u)$. As the morphism $\mathbb{1} \to A_u$ is faithfully exact (9.8), this implies that ξ is exact.

Now assume that ξ is exact. Form the pushout diagram,

$$\begin{array}{ccc} X \otimes X^\vee & \xrightarrow{a\otimes X^\vee} & Y \otimes X^\vee \\ \downarrow{\scriptstyle \mathrm{ev}_{X^\vee}} & & \downarrow \\ \mathbb{1} & \xrightarrow{\ e\ } & E. \end{array}$$

On applying adjunction, we see that the left hand triangle in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ a\ } & Y \\ {\scriptstyle e\otimes X}\downarrow & \swarrow & \downarrow{\scriptstyle e\otimes Y} \\ E \otimes X & \xrightarrow{E\otimes a} & E \otimes Y \end{array}$$

commutes. Hence the difference between the two morphisms $Y \to E \otimes Y$ restricts to zero when composed with a. As the sequence ξ is exact, this difference factors through $b : Y \to Z$. This gives the required homotopy. □

PROOF (12.37) Let ξ be a short exact sequence in C. By Lemma 12.39, there exists a nonzero morphism $u : \mathbb{1} \to U$ such that $u \otimes \xi$ is null homotopic, As F is faithful

$$\mathbb{1}_{\mathsf{D}} \simeq F(\mathbb{1}_{\mathsf{C}}) \xrightarrow{F(U)} F(U)$$

is nonzero, and it induces a null homotopic morphism $F(\xi) \to F(U) \otimes \xi$ because F preserves homotopies. By Lemma 12.39 again, $F(\xi)$ is exact. □

PROOF (12.38) Let ξ be a short exact sequence in T and let $u : \mathbb{1} \to U$ be a nonzero morphism such that $u \otimes \xi$ is null homotopic. Let A_u be as in the proof of Lemma 12.39. Then, as in the proof of the lemma, $A_u \neq 0$ and $A_u \otimes \xi$ is split-exact in $\mathsf{Mod}(A_u)$, so $F(A_u \otimes \xi)$ is exact (because F is additive). The R-module $F(U)$ faithfully flat (7.5). Therefore $F(A_u)$, being an inductive limit of faithfully flat R-modules, is also faithfully flat. As $F(A_u \otimes \xi) \simeq F(A_u) \otimes F(\xi)$, we see that $F(\xi)$ is exact. □

NOTES In characteristic zero, Proposition 12.37 is an immediate consequence of statements in Deligne 1990 (see 10.7). It is proved in general in Coulembier 2020. The above proofs follow Coulembier et al. 2023, 2.4.1, 2.4.2.

Abelian envelopes

In this section, all pseudo-tensor and tensor categories over k are essentially small.

12.40 When we relax the condition of being abelian to being pseudo-abelian, we arrive at the notion of a pseudo-tensor category. Thus, a k-linear $\otimes$-category D is a ***pseudo-tensor category over*** k if

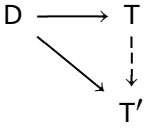
(a) $k \simeq \mathrm{End}(\mathbb{1})$,

(b) D is rigid (all objects have duals),

(c) D is pseudo-abelian (additive and idempotent complete).

From a k-linear $\otimes$-category satisfying (a) and (b), we can obtain a pseudo-tensor category over k by passing to the pseudo-abelian envelope. In this subsection, we examine the possibility of obtaining a tensor category over k by passing to an abelian envelope.

Let D be a pseudo-tensor category over k. Roughly speaking, we say that a fully faithful $\otimes$-functor from D to a tensor category T over k is an abelian envelope of D if every faithful $\otimes$-functor from D to a $\otimes$-category T' extends essentially uniquely to an exact $\otimes$-functor from T,



DEFINITION 12.41 (COULEMBIER ET AL. 2023, 2.1.1, 2.5.1) Let D be a pseudo-tensor category over k. A fully faithful k-linear $\otimes$-functor $F : \mathsf{D} \to \mathsf{T}$ from D to a tensor category T over k is an ***abelian envelope*** of D if, for every tensor category T' over k, composition with F defines an equivalence from

- the category of k-linear exact $\otimes$-functors $\mathsf{T} \to \mathsf{T}'$ to
- the category of k-linear faithful $\otimes$-functors $\mathsf{D} \to \mathsf{T}'$.

This definition makes sense because k-linear exact $\otimes$-functors of tensor categories are faithful (6.4).

By a ***monoidal category over*** k, we mean a k-linear monoidal category such that $k \simeq \mathrm{End}(\mathbb{1})$.

PROPOSITION 12.42 (COULEMBIER 2022, 4.4.4) *Consider a k-linear faithful monoidal functor $u : \mathsf{A} \to \mathsf{T}$, where A is a rigid monoidal category over k and T is an abelian rigid monoidal category over k. Assume that*

(a) *every object in T is a quotient of $u(X)$ for some object X in A, and*

(b) *for every morphism $a : u(X) \to u(Y)$ in T, there exists a morphism $q : X' \to X$ in A such that $u(q)$ is an epimorphism and $a \circ u(q)$ is in the image of u.*

Then, for every abelian rigid monoidal T' over k, u defines an equivalence from the category of k-linear exact tensor functors $\mathsf{T} \to \mathsf{T}'$ to the category of k-linear faithful $\otimes$-functors $\mathsf{A} \to \mathsf{T}'$.

COROLLARY 12.43 *Let D be a pseudo-tensor category over k and T a tensor category over k. A fully faithful $\otimes$-functor $F : \mathsf{D} \to \mathsf{T}$ is an **abelian envelope** of D if every object of T is a quotient of $F(X)$ for some X in D.*

PROOF Immediate consequence of the proposition. □

PROPOSITION 12.44 (COULEMBIER ET AL. 2023, THEOREM A) *A pseudo-abelian tensor category D over k admits an abelian envelope if and only if the following two conditions hold:*

(a) *every nonzero morphism $U \to \mathbb{1}$ in D, is the coequalizer of the pair of arrows*

$$U \otimes U \rightrightarrows U.$$

(b) *for every arrow $X \to Y$ in D, there exists a nonzero M in $\mathrm{Ind}\,\mathsf{D}$ such that the morphism $M \otimes X \to M \otimes Y$ is split in $\mathrm{Ind}\,\mathsf{D}$.*

A morphism in an additive category is ***split*** if it is a direct sum of an isomorphism and a zero morphism. In a pseudo-abelian category, a morphism $f : X \to Y$ is split if and only if there exists a $g : Y \to X$ such that $f = f \circ g \circ f$. For example, an endomorphism f of X is split if and only if it is von Neumann regular in $\mathrm{End}(X)$.

EXAMPLE 12.45 Let T be a tensor category over k, and let K be a field containing k. Let $K \otimes \mathsf{T}$ be the category with the same objects as T but with Hom replaced by $K \otimes_k \mathrm{Hom}$. Its pseudo-abelian envelope T_K is a K-linear pseudo-abelian rigid $\otimes$-category with $K \simeq \mathrm{End}(\mathbb{1})$.

Now let T be a tannakian category over k, and let K be a field containing k. Then $\mathsf{T}_{(K)}$ is the abelian envelope of T_K. Moreover, $\mathrm{Ind}(\mathsf{T}_{(K)})$ is equivalent to $\mathrm{Ind}(\mathsf{T})_K$.

EXAMPLE 12.46 (COULEMBIER ET AL. 2023, 5.1.1, 5.1.2, 5.1.4) Let K be a field containing k and D a pseudo-tensor category over k. The naive extension of scalars $K \otimes \mathsf{D}$ is the category with the same objects as D but with Hom replace by $K \otimes_k \mathrm{Hom}$. Its pseudo-abelian envelope is a pseudo-tensor category, denoted $K\dot{\otimes}\mathsf{C}$. If the K-linear $\otimes$-category T_K is a tensor category over K, then it is the abelian envelope of $K\dot{\otimes}\mathsf{T}$; moreover, $\mathrm{Ind}(\mathsf{T}_{(K)}) \sim (\mathrm{Ind}\,\mathsf{T})_K$.

13 Some 2-categories

We prove that the the monoidal categories form a (strict) 2-category $\mathscr{M}on\mathscr{C}at$, which enables us to deduce that other collections of categories, for example, the tannakian categories over a fixed field also form a 2-category.

The 2-category $\mathscr{M}on\mathscr{C}at$

- The objects of $\mathscr{M}on\mathscr{C}at$ are the small monoidal categories.
- The 1-morphisms of $\mathscr{M}on\mathscr{C}at$ are the monoidal functors.
- The 2-morphisms of $\mathscr{M}on\mathscr{C}at$ are the monoidal natural transformations.

Thus, a 1-morphism $\mathsf{C} \to \mathsf{D}$ is a functor $F : \mathsf{C} \to \mathsf{D}$ together with a natural isomorphism $c^F_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ such that certain diagrams commute (1.10) and F maps unit objects to unit objects. A 2-morphism

$$\alpha : F \Rightarrow G$$

is a natural transformation $F \Rightarrow G$ such that certain diagrams commute (1.12).

Recall (A.9) that the small categories form a 2-category $\mathscr{C}at$. The forgetful functor $\mathscr{M}on\mathscr{C}at \to \mathscr{C}at$ sends a monoidal category to its underlying category, a monoidal functor to its underlying functor, and a monoidal natural transformation to the natural transformation. The associativity and interchange laws follow from those in $\mathscr{C}at$ once we show that the various constructions preserve monoidal functors and monoidal natural transformations.

VERTICAL COMPOSITION

Given monoidal categories C, D, the Hom category

$$\mathsf{Hom}(\mathsf{C}, \mathsf{D})$$

is the category whose objects are the monoidal functors $F : \mathsf{C} \to \mathsf{D}$ and whose morphisms are the monoidal natural transformations. We check that this is indeed a category,

Given a monoidal functor F, the identity natural transformation $F \Rightarrow F$ is clearly monoidal. Given monoidal natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, we see from the commutative diagram

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{c^F} & F(X\otimes Y) \\ \downarrow{\scriptstyle \alpha_X\otimes\alpha_Y} & & \downarrow{\scriptstyle \alpha_{X\otimes Y}} \\ G(X)\otimes G(Y) & \xrightarrow{c^G} & G(X\otimes Y) \\ \downarrow{\scriptstyle \beta_X\otimes\beta_Y} & & \downarrow{\scriptstyle \beta_{X\otimes Y}} \\ H(X)\otimes H(Y) & \xrightarrow{c^H} & H(X\otimes Y) \end{array}$$

with $(\beta\circ\alpha)_X\otimes(\beta\circ\alpha)_Y$ on the left and $(\beta\circ\alpha)_{X\otimes Y}$ on the right, that $\beta\circ\alpha$ is monoidal. Therefore vertical composition is well-defined, and it is associative and unital because natural transformations have these properties.

Composition of 1-morphisms

Let $(F, c^F) : \mathsf{C} \to \mathsf{D}$ and $(G, c^G) : \mathsf{D} \to \mathsf{E}$ be monoidal functors. Their composite is the monoidal functor (1.11)

$$(G\circ F, c^{G\circ F}) : \mathsf{C} \to \mathsf{E}.$$

Strict associativity of 1-morphism composition

Let

$$F : \mathsf{A} \to \mathsf{B}, \quad G : \mathsf{B} \to \mathsf{C}, \quad H : \mathsf{C} \to \mathsf{D}$$

be monoidal functors. Then

$$(H\circ G)\circ F = H\circ(G\circ F)$$

as functors, and $c^{(HG)F}_{X,Y}$ and $c^{H(GF)}_{X,Y}$ both equal

$$HG(c^F_{X,Y})\circ H(c^G_{F(X),F(Y)})\circ c^H_{GF(X),GF(Y)}.$$

Horizontal composition

Let F, G be monoidal functors $\mathsf{C} \to \mathsf{D}$ and H, I functors $\mathsf{D} \to \mathsf{E}$. The horizontal composite of monoidal natural transformations $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow I$,

$$\mathsf{C} \;\overset{F}{\underset{G}{\Downarrow\alpha}}\; \mathsf{D} \;\overset{H}{\underset{I}{\Downarrow\beta}}\; \mathsf{E},$$

is the natural transformation $\beta * \alpha : H\circ F \to I\circ G$ such that $(\beta * \alpha)_X$ is the diagonal map in the commutative square

$$\begin{array}{ccc} (H\circ F)(X) & \xrightarrow{\beta_{F(X)}} & (I\circ F)(X) \\ \downarrow{\scriptstyle H(\alpha_X)} & & \downarrow{\scriptstyle I(\alpha_X)} \\ (H\circ G)(X) & \xrightarrow{\beta_{G(X)}} & (I\circ G)(X). \end{array}$$

Thus

$$I(\alpha_X)\circ\beta_{F(X)} \stackrel{\text{def}}{=} (\beta * \alpha)_X \stackrel{\text{def}}{=} \beta_{G(X)}\circ H(\alpha_X).$$

It remains to see that $\beta * \alpha$ is monoidal. The required compatibility diagram follows by pasting the monoidality squares for α and β together with the naturality square for the tensor constraint of I. Explicitly, one checks that

$$(\beta * \alpha)_{X\otimes Y}\circ c^{HF}_{X,Y} = c^{IG}_{X,Y}\circ((\beta * \alpha)_X \otimes (\beta * \alpha)_Y).$$

We omit the verification.

THE INTERCHANGE LAW

Since horizontal and vertical composition of monoidal natural transformations This follows directly from the interchange law in $\mathscr{C}at$.

Now that we have shown $\mathscr{M}on\mathscr{C}at$ to be a 2-category, it follows easily that the following are also 2-categories.

The 2-category of symmetric monoidal categories

- The objects are the small symmetric monoidal categories.
- The 1-morphisms are the symmetric monoidal functors.
- The 2-morphisms are the monoidal natural transformations.

If monoidal functors F and G are symmetric, so also is $G\circ F$.

Now fix a field k.

The 2-category of tensor categories over k.

- The objects are the small tensor categories over k.
- The 1-morphisms are the exact k-linear symmetric monoidal functors.
- The 2-morphisms are the k-linear monoidal natural transformations.

If symmetric monoidal functors F and G are k-linear and exact, so also is $G\circ F$.

The 2-category of tannakian categories over k, $\mathscr{T}ann_k$.

- The objects are the small tannakian categories over k.
- The 1-morphisms are the exact k-linear symmetric monoidal functors.
- The 2-morphisms are the k-linear monoidal natural transformations.

NOTES That monoidal categories form a (strict) 2-category is well-known to the experts, but I was unable to find a published proof.

Chapter II

Neutral tannakian categories

In this chapter, we describe neutral tannakian categories and their fibre functors in terms of affine group schemes and their torsors.

Throughout, k is a field except when indicated otherwise. Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$.

1 Affine group schemes and their representations

We review the basic theory of affine group schemes and their representations. For more details, see Demazure and Gabriel 1970, Milne 2017, or Waterhouse 1979. After 1.10, all bialgebras and Hopf algebras are commutative.

Affine monoid schemes and bialgebras

In this subsection, we allow k to be a commutative ring.

1.1 Let $G = \operatorname{Spec} A$ be an affine scheme over k, and $m: G\times G \to G$ and $e: \operatorname{Spec} k \to G$ morphisms. The triple (G, m, e) is an ***affine monoid scheme*** over k if $(G(R), m(R), e(R))$ is a monoid for all k-algebras R. This condition can be expressed in terms of diagrams: the associativity condition requires that the two composed morphisms

$$G\times G\times G \underset{\mathrm{id}_G\times m}{\overset{m\times\mathrm{id}_G}{\rightrightarrows}} G\times G \xrightarrow{\ m\ } G \tag{49}$$

are equal and the condition that $e(R)$ is a neutral element requires that the two morphisms

$$\begin{aligned} G &\simeq G\times\operatorname{Spec} k \xrightarrow{\mathrm{id}_G\times e} G\times G \xrightarrow{m} G \\ G &\simeq \operatorname{Spec} k\times G \xrightarrow{e\times\mathrm{id}_G} G\times G \xrightarrow{m} G \end{aligned} \tag{50}$$

equal id_G.

1.2 An ***algebra*** over k (associative with 1) is a k-module V together with k-linear maps $m: A\otimes A\to A$ and $e: k\to A$ such that the two composed maps

$$A\otimes A\otimes A \underset{\mathrm{id}_A\otimes m}{\overset{m\otimes\mathrm{id}_A}{\rightrightarrows}} A\otimes A \xrightarrow{\ m\ } A \tag{51}$$

are equal and the two maps

$$\begin{aligned} A \simeq k \otimes A &\xrightarrow{e\otimes \mathrm{id}} A \otimes A \xrightarrow{m} A \\ A \simeq A \otimes k &\xrightarrow{\mathrm{id}\otimes e} A \otimes A \xrightarrow{m} A \end{aligned} \tag{52}$$

equal id_A.

1.3 A ***coalgebra*** over k (co-associative with co-identity) is a k-module C together with k-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ such that the two composed maps

$$C \xrightarrow{\Delta} C \otimes C \underset{\mathrm{id}\otimes\Delta}{\overset{\Delta\otimes\mathrm{id}}{\rightrightarrows}} C \otimes C \otimes C$$

are equal and the two maps

$$\begin{aligned} C \xrightarrow{\Delta} C \otimes C &\xrightarrow{\epsilon\otimes\mathrm{id}} k \otimes C \simeq C \\ C \xrightarrow{\Delta} C \otimes C &\xrightarrow{\mathrm{id}\otimes\epsilon} C \otimes k \simeq C \end{aligned}$$

equal id_C.

Thus, "algebra" and "coalgebra" are opposite (dual) notions.

1.4 A ***bialgebra*** over k is a quintuple $(A, m, e, \Delta, \epsilon)$ such that

(a) (A, m, e) is an algebra over k,

(b) (A, Δ, ϵ) is a coalgebra over k,

(c) Δ and ϵ are algebra homomorphisms,

(d) m and e are coalgebra homomorphisms.

In the presence of (a) and (b), the conditions (c) and (d) are equivalent (Milne 2017, 9.40). A bialgebra $(A, m, e, \Delta, \epsilon)$ is said to be ***commutative*** if the underlying algebra (A, m, e) is commutative.

1.5 Let $G = \operatorname{Spec} A$ be an affine scheme over k. Then A is a commutative k-algebra, and the monoid structures (m, e) on G correspond exactly to the coalgebra structures (Δ, ϵ) on A given by algebra homomorphisms, i.e., to the bialgebra structures on A. The functor $A \rightsquigarrow \operatorname{Spec} A$ defines a contravariant equivalence between the category of commutative bialgebras over k and the category of affine monoid schemes over k.

1.6 Let (C, Δ, ϵ) and (C', Δ', ϵ') be coalgebras over k. Then $C \otimes C'$ becomes a coalgebra over k with the comultiplication

$$C \otimes C' \xrightarrow{\Delta\otimes\Delta'} (C \otimes C) \otimes (C' \otimes C') \xrightarrow{C\otimes\gamma_{C,C'}\otimes C'} (C \otimes C') \otimes (C \otimes C'),$$

where $\gamma_{C,C'}(c \otimes c') = c' \otimes c$, and co-identity

$$\epsilon \otimes \epsilon' : C \otimes C' \to k \otimes k \simeq k.$$

If G and G' are affine monoid schemes over k with associated coalgebras C and C', then $G \times G'$ is an affine monoid scheme over k with associated coalgebra $C \otimes C'$.

Affine group schemes and Hopf algebras

In this subsection, we allow k to be a commutative ring.

1.7 Let G be an affine scheme over k and $m: G \times G \to G$ a k-morphism. The pair (G, m) is an ***affine group scheme*** over k if $(G(R), m(R))$ is a group for all k-algebras R. In terms of diagrams, this condition says that there exist morphisms $e: \operatorname{Spec} k \to G$ and $\mathrm{inv}: G \to G$ (necessarily unique) such that (G, m, e) is a monoid scheme over k and

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G \times G & \xleftarrow{(\mathrm{id},\mathrm{inv})} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e} & G & \xleftarrow{e} & \operatorname{Spec} k \end{array}$$

commutes.

1.8 Let A be a k-algebra (not necessarily commutative) and $\Delta: A \to A \otimes A$ a homomorphism of k-algebras. The pair (A, Δ) is a ***Hopf algebra*** over k if there exist k-algebra homomorphisms $\epsilon: A \to k$ and $S: A \to A$ (necessarily unique) such that (A, Δ, ϵ) is a bialgebra over k, and

$$(S, \mathrm{id}_A) \circ \Delta = \epsilon = (\mathrm{id}_A, S) \circ \Delta.$$

Such an S is called an ***antipode***. Thus, a bialgebra over k is a Hopf algebra if and only if there exists an antipode. A Hopf algebra is said to be ***commutative*** if the algebra A is commutative.

1.9 Let $G = \operatorname{Spec} A$ be an affine scheme over k. To give the structure of a group scheme on G is the same as giving the structure of a commutative Hopf algebra on A. The functor $A \rightsquigarrow \operatorname{Spec} A$ defines a contravariant equivalence between the category of commutative Hopf algebras over k and the category of affine group schemes over k.

1.10 An affine group scheme $G = \operatorname{Spec} A$ over k is of finite type over k if A is finitely generated as a k-algebra. By an ***algebraic group over*** k we mean an affine group scheme of finite type over k.

Henceforth, all bialgebras and Hopf algebras are commutative.

1.11 Let $A \subset B$ be Hopf algebras over a field. Then B is faithfully flat over A. See Waterhouse 1979, 14.1.

Representations

In this subsection, k is a field.

1.12 Let G be an affine group (or monoid) scheme over k. A ***representation*** of G on a k-vector space V is a homomorphism

$$G(R) \to \mathrm{Aut}_{R\text{-linear}}(V(R))$$

natural in R. In other words, it is a family of homomorphisms $G(R) \to \mathrm{GL}(V(R))$, indexed by the k-algebras R, compatible with extension of scalars. When V is finite-dimensional, this is the same as a homomorphism $G \to \mathrm{GL}_V$ of affine group (or monoid) schemes over k. We let $\mathsf{Repf}(G)$ denote the category of representations of G on finite-dimensional k-vector spaces.

1.13 A ***(right) comodule*** over a k-coalgebra C is a vector space V over k together with a k-linear map $\rho: V \to V \otimes_k C$ such that the two composed maps

$$V \xrightarrow{\rho} V \otimes C \underset{\rho\otimes \mathrm{id}_C}{\overset{\mathrm{id}_V \otimes \Delta}{\rightrightarrows}} V \otimes C \otimes C$$

are equal and

$$V \xrightarrow{\rho} V \otimes C \xrightarrow{\mathrm{id}_V \otimes \epsilon} V \otimes k \simeq V$$

equals id_V. For example, Δ defines an C-comodule structure on C.

PROPOSITION 1.14 *Let $G = \operatorname{Spec} A$ be an affine group (or monoid) scheme and V a k-vector space. To give an A-comodule structure on V is the same as giving a linear representation of G on V.*

PROOF A representation r of G on V is determined by its action on the "universal" element

$$\mathrm{id}_G \in \operatorname{Hom}(G, G) = G(A).$$

Indeed, $r(\mathrm{id}_G)$ is an A-linear isomorphism $V \otimes A \to V \otimes A$ whose restriction to $V = V \otimes k \subset V \otimes A$ determines it and is an A-comodule structure ρ on V. Conversely, a comodule structure ρ on V determines a representation of G on V such that, for every k-algebra R and $g \in G(R) \overset{\text{def}}{=} \operatorname{Hom}_k(A, R)$, the restriction of $g_V: V \otimes R \to V \otimes R$ to $V \otimes k \subset V \otimes R$ is

$$(\mathrm{id}_V \otimes g)\circ\rho: V \to V \otimes A \to V \otimes R.$$

For more details, see Milne 2017, 4.1. □

Let $G = \operatorname{Spec} A$. The representation of G on A defined by the A-comodule structure $\Delta: A \to A \otimes A$ is called the ***regular representation*** of G.

PROPOSITION 1.15 *Let C be a k-coalgebra and (V, ρ) a comodule over C. Every finite subset S of V is contained in a sub-comodule of V having finite dimension over k.*

PROOF Let $\{c_i\}$ be a basis for C over k (possibly infinite). For each v in S, write $\rho(v) = \sum v_i \otimes c_i$ (finite sum). The k-subspace spanned by the v and the v_i occurring in such a sum is a sub-comodule over C containing S (see Milne 2017, 4.7). □

COROLLARY 1.16 *Every C-comodule is a filtered union of finite-dimensional sub-comodules.*

PROOF Let C be a k-coalgebra, and V a C-comodule. The set of all sub-comodules of V finite-dimensional over k is partially ordered by inclusion, filtered (any two are contained in a third), and has union V (by the proposition). □

COROLLARY 1.17 *Every representation of an affine group scheme (or monoid scheme) on a vector space is a filtered union of finite-dimensional subrepresentations.*

PROOF This is a restatment of Corollary 1.16 (by Proposition 1.14). □

A representation $r: G \to \mathrm{GL}_V$ is ***faithful*** if $r(R): G(L) \to \operatorname{Aut}_{R-\text{linear}}(V(R))$ is injective for all k-algebras R. When V is finite dimensional, this is equivalent to r being a closed immersion (Milne 2017, 3.35).

PROPOSITION 1.18 *An affine group scheme G is algebraic if and only if it has a faithful finite-dimensional representation over k.*

PROOF If G admits a faithful representation on a finite-dimensional vector space, then it admits a closed immersion into scheme of finite type over k, and so it is of finite type over k.

For the converse, let $G = \operatorname{Spec} A$ with A a finitely generated k-algebra, and let (Δ, ϵ) be the corresponding k-coalgebra structure on A. A sub-comodule V of A provides a faithful representation of G if it contains a set of generators for the k-algebra A (Milne 2017, proof of 4.9). According to 1.15, we can choose V to be finite-dimensional over k. □

PROPOSITION 1.19 *Let C be a coalgebra over k. Every finite subset of C is contained in a finite-dimensional k-subcoalgebra.*

PROOF According to 1.15, the finite subset is contained in a finite-dimensional k-subspace V of A such that $\Delta(V) \subset V \otimes_k C$. Let $\{v_j\}$ be a basis for V, and let $\Delta(v_j) = \sum v_i \otimes a_{ij}$ (finite sum). Then $\Delta(a_{ij}) = \sum a_{il} a_{lj}$, and so the k-subspace V' spanned by the v_i and a_{ij} satisfies $\Delta(V') \subset V' \otimes_k V'$. Now V' is the required k-coalgebra. □

PROPOSITION 1.20 *Let A be a Hopf algebra over k. Every finite subset of A is contained in a Hopf subalgebra B that is finitely generated as a k-algebra.*

PROOF According to 1.19, the finite subset is contained in a finite-dimensional k-subspace C of A such that $\Delta(C) \subset C \otimes_k C$. If $\Delta a = \sum b_i \otimes c_i$, then $\Delta(Sa) = \sum Sb_i \otimes Sc_i$, and so the k-subspace V spanned by C and SC satisfies $\Delta(V) \subset V \otimes_k V$ and $S(V) \subset V$. Let B be the k-algebra generated by V. Then

$$1 \in B, \quad \Delta(B) \subset B \otimes B, \quad \epsilon(B) \subset k, \quad S(B) \subset B.$$

□

COROLLARY 1.21 *Let G be an affine group scheme over k. Then G is a filtered projective limit $G = \varprojlim G_i$ of (affine) algebraic groups over k in which the transition maps $G_j \to G_i$, $i \le j$, are faithfully flat.*

PROOF Let A be a the Hopf algebra over G. According to the proposition, A is a filtered union $A = \bigcup A_i$ of Hopf subalgebras A_i that are finitely generated as k-algebras. The functor Spec transforms the inductive limit $A = \varinjlim A_i$ into an projective limit $G = \varprojlim G_i$. As Hopf algebras are always faithfully flat over Hopf subalgebras (1.11), A_j is faithfully flat over A_i, and so the transition map $G_j \to G_i$ is faithfully flat. □

More precisely, if G is an affine group scheme over k, then $G = \varprojlim G/N$, where N runs over the set of normal affine subgroup schemes of G such that G/N is algebraic. Projective limits of affine group schemes are again affine group schemes. See Demazure and Gabriel 1970, III, §3, n° 7.

PROPOSITION 1.22 *Let G be an algebraic group over k and (V, r) a faithful finite-dimensional representation of G. Every finite-dimensional representation of G can be constructed from V by iterated applications of the operations of direct sum, tensor product, dual, passage to a subobject or quotient object.*

PROOF See Milne 2017, 4.14. □

In other words, (V, r) is a tensor generator (I, 6.5) for the rigid symmetric monoidal category $\mathsf{Repf}(G)$.

THEOREM 1.23 (CHEVALLEY) *Let G be an algebraic group over k. Every algebraic subgroup of H of G arises as the stabilizer of a one-dimensional subspace in a finite-dimensional representation of G.*

PROOF See Milne 2017, 4.27. □

2 Recovering a group scheme from its representations

Let G be an affine group scheme over k. Let R be a k-algebra, and let $g \in G(R)$. For any finite-dimensional representation (V, r_V) of G over k, we have an R-linear isomorphism

$$\lambda_V \stackrel{\text{def}}{=} r_V(g) : V_R \to V_R.$$

These maps satisfy the following conditions:

(a) for all representations V and W, $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$;

(b) $\lambda_{\mathbb{1}}$ is the identity map (here $\mathbb{1} = k$ with the trivial action);

(c) for all G-equivariant maps $u : V \to W$, $\lambda_W \circ u_R = u_R \circ \lambda_V$.

THEOREM 2.1 *Let G be an affine group scheme over k, and let R be a k-algebra. Suppose that, for every finite-dimensional representation (V, r_V) of G over k, we are given an R-linear map $\lambda_V : V_R \to V_R$. If the family $(\lambda_V)_V$ satisfies the conditions (a, b, c), then there exists a unique $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all V. In particular, the maps λ_V are isomorphisms.*

PROOF Let $A(G, R)$ denote the set of families $\lambda = (\lambda_V)_V$ satisfying the conditions (a,b,c). Suppose first that G is algebraic, and let (W, r) be a faithful representation of G. It follows from Proposition 1.22 that an element $(\lambda_V)_V$ of $A(G, R)$ is determined by λ_W, and so we have inclusions

$$G(R) \hookrightarrow A(G, R) \hookrightarrow \mathrm{GL}_W(R), \quad g \mapsto (r_V(g))_V \mapsto r_W(g).$$

According to Chevalley's theorem (1.23), there exists a representation (W, r_W) of G, which we may choose to be faithful, and a one-dimensional subspace L in W such that G is the stabilizer of L in GL_W. Let $u : L \to W$ be the inclusion map, and let $\lambda \in A(G, R)$. As $\lambda_W \circ u_R = u_R \circ \lambda_L$, we see that λ_W stabilizes L, and so it lies in $G(R)$, as required.

In the general case, let $V \in \mathsf{Repf}(G)$, and let $\langle V\rangle^{\otimes}$ be the full subcategory of $\mathsf{Repf}(G)$ of objects isomorphic to a subquotient of $P(V, V^\vee)$ for some $P \in \mathbb{N}[t, s]$. Let G_V be the image of G in GL_V. It is an algebraic quotient of G acting faithfully on V, and so

$$G_V(R) \simeq A(G_V, R) \hookrightarrow \mathrm{GL}_V(R).$$

Define a partial ordering on the set of isomorphism classes of objects of $\mathsf{Repf}(G)$ by the rule

$$[V] \le [V'] \iff \langle V\rangle^{\otimes} \subset \langle V'\rangle^{\otimes}.$$

Note that $[V], [V'] \le [V \oplus V']$, and so the set is filtered. If $[V] \le [V']$, then restriction gives a commutative diagram

$$\begin{array}{ccc} G_{V'}(R) & \xrightarrow{\simeq} & A(G_{V'}, R) \\ \downarrow & & \downarrow \\ G_V(R) & \xrightarrow{\simeq} & A(G_V, R) \end{array}$$

Now $G = \varprojlim G_V$ so, on passing to the projective limit, we get bijections

$$G(R) \simeq \varprojlim G_V(R) \simeq \varprojlim A(G_V, R) \simeq A(G, R).$$

□

COROLLARY 2.2 *Let G be an affine group scheme over k and let ω be the forgetful functor on $\mathsf{Repf}(G)$. Then the canonical morphism*

$$G \to \mathcal{A}ut^{\otimes}(\omega)$$

is an isomorphism of functors (of k-algebras).

PROOF For any k-algebra R,

$$\mathcal{A}ut^{\otimes}(\omega)(R) \overset{\text{def}}{=} \mathrm{End}(\phi_R \circ \omega) = A(G,R). \qquad \square$$

Let $f : G \to H$ be a homomorphism of affine group schemes over k. Using f, we can regard an H-module as a G-module. In this way, we get a $\otimes$-functor

$$\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G) \tag{53}$$

such that $\omega^G_{\text{forget}} \circ \omega^f = \omega^H_{\text{forget}}$. Our next result shows that all such functors arise in this fashion.

COROLLARY 2.3 *Let G and H be affine k-group schemes, and let $F : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ be a $\otimes$-functor such that $\omega^G_{\text{forget}} \circ \omega^f = \omega^H_{\text{forget}}$. There exists a unique homomorphism $f : G \to H$ such that $F = \omega^f$.*

PROOF Such an F defines a homomorphism (functorial in the k-algebra R)

$$F^* : \mathcal{A}ut^{\otimes}(\omega^G_{\text{forget}})(R) \to \mathcal{A}ut^{\otimes}(\omega^H_{\text{forget}})(R), \quad F^*(\lambda)_V = \lambda_{F(V)}.$$

Corollary 2.2 allows us to regard F^* as a functorial homomorphism $G(R) \to H(R)$. According to the Yoneda lemma this arises from a unique homomorphism $f : G \to H$. Clearly, the maps $F \mapsto f$ and $f \mapsto \omega^f$ are inverse. $\square$

REMARK 2.4 (a) Theorem 2.1 holds also for affine monoid schemes, but requires a different proof as we cannot use Chevalley's theorem (see Milne 2017, 9.2).

(b) Proposition 2.2 shows that G is determined by the triple $(\mathsf{Repf}(G), \otimes, \omega^G)$. In fact, the coalgebra of G is already determined by $(\mathsf{Repf}(G), \omega^G)$ (see the proof of Theorem 3.1 below).

EXERCISE 2.5 Let G be an algebraic group over k, and let ω be the forgetful functor $\mathsf{Repf}(G) \to \mathsf{Vecf}(k)$. We have seen that $G(k)$ can be identified with set of natural automorphisms $(\lambda_V)_V$ of ω such that $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$ and $\lambda_{\mathbb{1}} = \mathrm{id}$. Show that the Lie algebra of G can be identified with the set of natural endomorphisms $(\lambda_V)_V$ of ω such that $\lambda_{V\otimes W} = \lambda_V \otimes \mathrm{id} + \mathrm{id} \otimes \lambda_W$ and $\lambda_{\mathbb{1}} = 0$.

3 The main theorem

Recall that a neutral tannakian category over k is a k-linear rigid abelian symmetric monoidal category such that there exists a fibre functor ω to $\mathsf{Vecf}(k)$. These conditions imply that $k \simeq \mathrm{End}(\mathbb{1})$ and that fibre functors are faithful.

Statements

The main theorem shows that every neutral tannakian category is equivalent (in possibly many different ways) to the category of finite-dimensional representations of an affine group scheme.

THEOREM 3.1 *Let* $(\mathsf{C}, \otimes)$ *be an essentially small neutral tannakian category over* k *and* ω *a* k*-valued fibre functor.*

(a) *The functor* $\mathcal{A}ut^{\otimes}(\omega)$ *of* k*-algebras is represented by an affine group scheme* G*.*

(b) *The functor* $\mathsf{C} \to \mathsf{Repf}(G)$ *defined by* ω *is an equivalence of* $\otimes$*-categories.*

The proof will occupy the rest of this section. After a series of preliminaries on abelian categories, we construct the coalgebra A of G without using the tensor structure on C (Theorem 3.20). The tensor structure then allows us to define an algebra structure on A, and the rigidity of C implies that A is a Hopf algebra (so that G is a group scheme rather than a monoid scheme).

COROLLARY 3.2 *A neutral tannakian category* $(\mathsf{C}, \otimes)$ *is monogenic if and only if, for one (hence every)* k*-valued fibre functor* ω*, the affine group scheme* $\mathcal{A}ut^{\otimes}(\omega)$ *is algebraic.*

PROOF If $(\mathsf{C}, \otimes)$ is monogenic, with tensor generator X, then $\omega(X)$ is a faithful finite-dimensional representation of $\mathcal{A}ut^{\otimes}(\omega)$ for any fibre functor ω, which is therefore algebraic (1.18).

Conversely, suppose that for some k-valued fibre functor ω on C, the affine group scheme $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}(\omega)$ is algebraic. Then $\mathsf{Repf}(G)$ has a tensor generator (1.22), which corresponds to a tensor generator of $(\mathsf{C}, \otimes)$ under the tensor equivalence $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$ defined by ω. □

Abelian categories as module categories

Let A be an abelian category. An object P of A is a ***generator*** if the functor $h^P \overset{\text{def}}{=} \mathrm{Hom}(P, -)$ is faithful and it is ***projective*** if h^P is exact. Thus, P is a ***projective generator*** if and only if h^P is exact and faithful.

LEMMA 3.3 *Let* A *be an abelian category and* P *a generator* A*. For any noetherian object* X *of* A*, there exists an epimorphism* $P^{\oplus n} \to X$*, some* n*.*

PROOF If X is nonzero, then there exists a nonzero morphism $\phi_1 : P \to X$ (otherwise P would not separate the identity morphism $X \to X$ from the zero morphism). Similarly, if ϕ_1 is not an epimorphism, then there exists a morphism $\phi_2 : P \to X$ such that the image of $(\phi_1, \phi_2) : P^2 \to X$ is strictly larger than that of ϕ_1 (otherwise P would not separate the projection $X \to X/\operatorname{Im}(\phi_1)$ from zero). Continuing in this fashion, and using that X is noetherian, we obtain an epimorphism $(\phi_1, \dots, \phi_n) : P^{\oplus n} \to X$. □

PROPOSITION 3.4 *Let* A *be an abelian category whose objects are noetherian. Assume that* A *has a projective generator* P*, and let* $A = \mathrm{End}(P)$*. Then* A *is a right noetherian ring, and* h^P *is an equivalence from* A *to the category of finitely generated right* A*-modules.*

PROOF We first show that A is right noetherian. Let $\mathfrak{a}$ be a right ideal in A, and let $\mathfrak{a}P$ be the subobject of P generated by the images of the maps $\alpha : P \to P$ with $\alpha \in \mathfrak{a}$. It suffices to show that $\mathfrak{a} = \mathrm{Hom}(P, \mathfrak{a}P)$ because then the ascending chain condition on the submodules of P implies it for the right ideals of A. Certainly $\mathfrak{a} \subset \mathrm{Hom}(P, \mathfrak{a}P)$.

Let $\alpha \in \mathrm{Hom}(P, \mathfrak{a}P)$. There exist $\phi_1, \dots, \phi_n \in \mathfrak{a}$ such that $(\phi_1, \dots, \phi_n) \colon P^n \to X$ is an epimorphism. Now the diagram

$$\begin{array}{ccccc} & & & & P \\ & & \overset{h}{\swarrow} & & \downarrow \alpha \\ P^n & \xrightarrow[(\phi_1,\dots,\phi_n)]{} & X & \longrightarrow & 0 \end{array}$$

shows that $\alpha = \sum \phi_i \circ h \in \mathfrak{a}$.

The left action of A on P defines a right action of A on $h^P(X)$, natural in X, and so h^P is a functor $\mathsf{A} \to \mathsf{Mod}_A$. Because P is a projective generator, h^P is exact and faithful.

Let X an object of A. For some m, n, we have an exact sequence

$$P^{\oplus m} \to P^{\oplus n} \to X \to 0.$$

On applying h^P to this sequence, we get an exact sequence

$$A^{\oplus m} \to A^{\oplus n} \to h^P(X) \to 0,$$

which shows that $h^P(X)$ has finite presentation. Conversely, because A is right noetherian, every finitely generated right A-module M is finitely presented, say,

$$A^{\oplus m} \xrightarrow{u} A^{\oplus n} \to M \to 0,$$

where u is an $m \times n$ matrix with coefficients in A. This matrix defines a morphism $P^{\oplus m} \to P^{\oplus n}$ whose cokernel X has the property that $h^P(X) \simeq M$. This shows that h^P is essentially surjective, and it remains to show that it is full.

Let X, Y be objects of A, and choose an exact sequence $P^{\oplus m} \to P^{\oplus n} \to X \to 0$. Then

$$\mathrm{Hom}(P^{\oplus m}, Y) \simeq h^P(Y)^{\oplus m} \simeq \mathrm{Hom}(A^{\oplus m}, h^P(Y)) \simeq \mathrm{Hom}(h^P(P^{\oplus m}), h^P(Y)),$$

and the composite of these maps is that defined by h^P. From the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(X, Y) & \longrightarrow & \mathrm{Hom}(P^n, Y) & \longrightarrow & \mathrm{Hom}(P^m, Y) \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & \mathrm{Hom}(h^P(X), h^P(Y)) & \longrightarrow & \mathrm{Hom}(h^P(P^n), h^P(Y)) & \longrightarrow & \mathrm{Hom}(h^P(P^m), h^P(Y)), \end{array}$$

we see that $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(h^P(X), h^P(Y))$ is an isomorphism, and so h^P is full. □

EXAMPLE 3.5 Let A be a locally finite k-linear abelian category. Assume that A has a projective generator P, and let $A = \mathrm{End}(P)$. Then A is a finite-dimensional k-algebra, and h^P is an equivalence from A to the category of finitely generated right A-modules.

EXAMPLE 3.6 Let A be abelian with noetherian objects, and let B be a full subcategory of A stable under finite direct sums and subquotients. For X in A, let $q(X) = X / \bigcap \mathrm{Ker}(\alpha)$, where α runs over the epimorphisms $X \to Y$ with $Y \in \mathrm{ob}\,\mathsf{B}$. As X is noetherian,

$$q(X) = X / \textstyle\bigcap_{\alpha \in F} \mathrm{Ker}(\alpha)$$

for some finite set F of α. Note that $q(X)$ is a subobject of $\bigoplus_{\alpha \in F} X/\mathrm{Ker}(\alpha)$, and so it lies in B. Clearly, $q(X)$ is the largest quotient of X in B – any other epimorphism

$X \to Y$ factors uniquely through $q(X)$ – and so q is a left adjoint for the inclusion functor $i : \mathsf{B} \to \mathsf{A}$,

$$\mathrm{Hom}_{\mathsf{B}}(q(X), Y) \simeq \mathrm{Hom}_{\mathsf{A}}(X, i(Y)), \quad X \in \mathrm{ob}\,\mathsf{A}, \quad Y \in \mathrm{ob}\,\mathsf{B}.$$

Now assume that A has a projective generator P, and let $A = \mathrm{End}(P)$. Then $Q \overset{\text{def}}{=} q(P)$ is a projective generator for B, and $B \overset{\text{def}}{=} \mathrm{End}(Q)$ is a quotient $A/\mathfrak{a}$ of A.

According to Proposition 3.4, the functor $\mathrm{Hom}(P, -)$ identifies A with the category Modf_A. In this model, P is A_A, Q is B_B, and B consists of the right A-modules killed by $\mathfrak{a}$. In summary:

$$\begin{array}{ccc} \mathsf{B} & \xrightarrow{\ i\ } & \mathsf{A} \\ {\scriptstyle h^Q}\downarrow{\scriptstyle \sim} & & {\scriptstyle h^P}\downarrow{\scriptstyle \sim} \\ \mathsf{Mod}_B & \xrightarrow{\ j\ } & \mathsf{Mod}_A \end{array} \qquad \left\{ \begin{array}{l} A = \mathrm{End}(P), \quad Q = q(P), \\ B = \mathrm{End}(Q), \\ j \text{ defined by } A \twoheadrightarrow B. \end{array} \right.$$

NOTES Proposition 3.4 and its infinite version (without the noetherian hypothesis) have been variously credited to Freyd, Gabriel, Lam, and Mitchell.

Abelian categories with a fibre functor

EXAMPLE 3.7 Let $\mathsf{A} = \mathsf{Modf}_A$, where A is a finite-dimensional k-algebra. Let Q be a projective generator of A, and let $\omega = h^Q$. Then ω is an exact faithful k-linear functor $\mathsf{A} \to \mathsf{Vecf}(k)$. If $\mathrm{End}(\omega) = B$, then $\mathrm{End}(Q) = B^{\mathrm{opp}}$, and, according to 3.4, ω is an equivalence

$$\mathsf{Modf}_A \xrightarrow{\ \omega\ } \mathsf{Modf}_{B^{\mathrm{opp}}} = {}_B\mathsf{Modf}.$$

PROPOSITION 3.8 *Let A be a k-linear abelian category, and let $\omega : \mathsf{A} \to \mathsf{Vecf}(k)$ be an exact faithful k-linear functor. Assume that A has a projective generator. Then $B \overset{\text{def}}{=} \mathrm{End}(\omega)$ is a finite-dimensional k-algebra, and ω is an equivalence $\mathsf{A} \to {}_B\mathsf{Modf}$.*

PROOF Note first that the existence of ω implies that the k-linear category A is locally finite: certainly $\mathrm{Hom}(X, Y) \subset \mathrm{Hom}(\omega(X), \omega(Y))$ is finite dimensional, and a subobject Z of X is determined by the subspace $\omega(Z)$ of $\omega(X)$, and so the set of subobjects of X can be identified with a subset of the lattice of subspaces of $\omega(X)$.

The left action of B on ω defines a left action of B on $\omega(X)$, natural in X, and so ω is a functor $\mathsf{A} \to {}_B\mathsf{Modf}$. By definition, it is exact and faithful, and it remains to show that it is full and essentially surjective.

Let P be a projective generator for A, and let $A = \mathrm{End}(P)$, so h^P is an equivalence $\mathsf{A} \to \mathsf{Modf}_A$. Then A acts on $\omega(P)$ on the left, and the map

$$\alpha \otimes p \mapsto \omega(\alpha)(p) : \ \mathrm{Hom}(P, X) \otimes_A \omega(P) \to \omega(X)$$

is natural in $X \in \mathrm{ob}\,\mathsf{A}$. This map is obviously an isomorphism when $X = P$. As both functors commute with finite direct sums and are right exact, it follows (as in the proof of Proposition 3.4) that it is an isomorphism for all X in A, and so

$$h^P(-) \otimes \omega(P) \simeq \omega(-).$$

In other words, up to a natural isomorphism, we have a factorization

$$\mathsf{A} \xrightarrow[h^P]{\sim} \mathsf{Modf}_A \xrightarrow[-\otimes\omega(P)]{} {}_B\mathsf{Modf}, \qquad \mathsf{A} \xrightarrow{\ \omega\ } {}_B\mathsf{Modf}.$$

As ω is exact, it follows from the diagram, that the A-module $\omega(P)$ is flat, and hence projective (because it is finitely presented). Let $Q = \mathrm{Hom}_A(\omega(P), {}_AA)$. Then $h^Q(-) \simeq - \otimes \omega(P)$, and so h^Q is exact and faithful. Thus, Q is a projective generator for Mod_A, and so

$$h^Q : \mathsf{Mod}_A \xrightarrow{\approx} \mathsf{Modf}_{B^{\mathrm{opp}}} = {}_B\mathsf{Modf},$$

where $B^{\mathrm{opp}} = \mathrm{End}(Q) \simeq \mathrm{End}(h^Q)^{\mathrm{opp}} \simeq \mathrm{End}(\omega)^{\mathrm{opp}}$. □

REMARK 3.9 Let $\mathsf{A} = \mathsf{Modf}_A$, where A is a finitely generated k-algebra. Then every exact faithful k-linear functor $\omega : \mathsf{A} \to \mathsf{Vecf}(k)$ is of the form h^Q for some projective generator Q. Indeed, on applying the proof of 3.8 with $\mathsf{A} = \mathsf{Modf}_A$ and $P = A_A$, we find that $- \otimes_A \omega(A_A) = \omega(-)$, and so $\omega = h^Q$ with Q the A-linear dual of $\omega(A_A)$.

Existence of a projective generator

Let A be an abelian category. For an object X of A, we let $\langle X \rangle$ denote the strictly full subcategory of A whose objects are subquotients of a finite direct sum of copies of X. It is the smallest strictly full abelian subcategory of A containing X.

An object of A is ***simple*** if it is nonzero and contains no proper nonzero subobject. A ***composition series*** for an object X of A is a finite decreasing filtration

$$X = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

with simple successive quotients F^i/F^{i+1}. An object is said to have ***finite length*** if it admits a composition series, in which case any two composition series have the same length and multiset of quotients (taken up to isomorphism). We let $\lg(X)$ denote the common length of the composition series for X, and, for a simple object S, we let $\lg_S(X)$ denote the number of $i$such that F^i/F^{i+1} is isomorphic to S. In the Grothendieck group of A,

$$[X] = \sum \lg_S(X) \cdot [S],$$

where $[S]$ runs over the isomorphism classes of simple objects in A.

LEMMA 3.10 *Let* A *be an abelian category whose objects have finite length. If*

(a) *there are only finitely many isomorphism classes of simple objects, and*

(b) *every simple object is a quotient of a projective object,*

then there exists a projective generator for A.

PROOF Let $S_1, \dots, S_m$ be a set of representatives for the simple objects, and, for each i, let $P_i \to S_i$ be an epimorphism with P_i projective. We claim that $P \overset{\mathrm{def}}{=} P_1 \oplus \cdots \oplus P_m$ is a projective generator. Certainly, it is projective, and to show that h^P is faithful, it suffices to show that $M \neq 0 \implies h^P(M) \neq 0$. Every nonzero M has a simple quotient, so there exists an epimorphism $M \to S_i$ for some i. The epimorphism $P_i \to S_i$ extends (by zero) to an epimorphism $P \to S_i$, which lifts to a nonzero morphism $P \to M$. □

3.11 An ***essential extension*** of an object Y is an epimorphism $\alpha : E \to Y$ such that no subobject $E' \subset E$, distinct from E, maps onto Y. When Y is simple, this just says that the kernel of α contains every $E' \subset E$ distinct from E.

LEMMA 3.12 *Assume* $\mathsf{A} = \langle X \rangle$, *and let* $E \twoheadrightarrow S$ *be an essential extension of a simple object* S *in* A. *For all* $Y \in \operatorname{ob} \mathsf{A}$,

$$\dim_k \operatorname{Hom}(E, Y) \leq \lg_S(Y) \cdot \dim_k \operatorname{End}(S). \tag{54}$$

If equality holds for X, *then it holds for all objects of* A, *and* E *is projective.*

PROOF When Y is simple, every nonzero morphism $E \to Y$ factors through $E \twoheadrightarrow S$ (because its kernel is contained in $\operatorname{Ker}(E \twoheadrightarrow S)$) and induces an isomorpism $S \to Y$. Thus, $\operatorname{Hom}(E, Y) \simeq \operatorname{Hom}(S, Y)$, and so

$$\begin{cases} \operatorname{Hom}(E, Y) \approx \operatorname{End}(S) \text{ if } Y \approx S \\ \operatorname{Hom}(E, Y) = 0 \text{ otherwise.} \end{cases}$$

In both cases, (54) holds with equality.

An exact sequence

$$0 \to Y' \to Y \to Y'' \to 0 \tag{55}$$

gives an exact sequence

$$0 \to \operatorname{Hom}(E, Y') \to \operatorname{Hom}(E, Y) \to \operatorname{Hom}(E, Y''), \tag{56}$$

from which it follows that

$$\dim_k \operatorname{Hom}(E, Y) \leq \dim_k \operatorname{Hom}(E, Y') + \dim_k \operatorname{Hom}(E, Y'').$$

As the right-hand side of (54) is additive on short exact sequences, the inequality (54) now follows by induction on the length of Y. If equality holds for Y, then equalities hold throughout the following diagram

$$\begin{array}{ccccc} \operatorname{RHS}(Y) & = & \operatorname{RHS}(Y') & + & \operatorname{RHS}(Y'') \\ \| & & \vee | & & \vee | \\ \operatorname{LHS}(Y) & \leq & \operatorname{LHS}(Y') & + & \operatorname{LHS}(Y''), \end{array}$$

so equality holds for each of Y' and Y'' and the sequence (56) is exact with a 0 on the right.

If equality holds in (54) for $Y = X$, then it holds for $X^{\oplus m}$, and so also for all subquotients of $X^{\oplus m}$. Moreover, the sequence (56) is exact with a 0 on the right, which says that E is projective. □

PROPOSITION 3.13 (GABBER) *Let* A *be a locally finite* k*-linear abelian category. For any* X *in* A, *the subcategory* $\langle X \rangle$ *of* A *has a projective generator.*

PROOF The quotients of any composition series for X represent the isomorphism classes of simple objects in $\langle X \rangle$, and so $\langle X \rangle$ satisfies the condition (a) of 3.10. We shall complete the proof by showing that every simple object S of $\langle X \rangle$ admits an essential extension $P(S) \twoheadrightarrow S$ with $P(S)$ projective.

Let

$$X = F^0 \supset \cdots \supset F^i \supset F^{i+1} \supset \cdots \supset F^r = 0$$

be a composition series for X. We construct, by induction on i, an essential extension $P_i \twoheadrightarrow S$ of S such that

$$\dim_k \operatorname{Hom}(P_i, X/F^i) = \lg_S(X/F^i) \cdot \dim_k \operatorname{End}(S). \tag{57}$$

We can take $P_1 = S$. Supposing P_i to have been constructed, we now construct P_{i+1}. Let $f_1, \dots, f_s$ span $\mathrm{Hom}(P_i, X/F^i)$. Define Q_j by the fibre product diagram

$$\begin{array}{ccc} Q_j & \longrightarrow & X/F^{i+1} \\ \downarrow & & \downarrow \\ P_i & \xrightarrow{f_j} & X/F^i, \end{array}$$

and let Q be a subobject of $Q_1 \times_{P_i} \cdots \times_{P_i} Q_s$ minimal among those mapping onto P_i. Then Q is an essential extension of P_i, hence also of S. As Q maps onto P_i, we have an inclusion

$$\mathrm{Hom}(P_i, X/F^i) \hookrightarrow \mathrm{Hom}(Q, X/F^i),$$

but

$$\dim_k \mathrm{Hom}(Q, X/F^i) \overset{(54)}{\leq} \lg_S(X/F^i) \cdot \dim_k \mathrm{End}(S) \overset{(57)}{=} \dim_k \mathrm{Hom}(P_i, X/F^i),$$

and so

$$\mathrm{Hom}(P_i, X/F^i) \simeq \mathrm{Hom}(Q, X/F^i). \tag{58}$$

From the diagram

$$\begin{array}{ccccccccc} Q & \hookrightarrow & Q_1 \times_{P_i} \cdots \times_{P_i} Q_s & \longrightarrow & Q_1 \times \cdots \times Q_s & \xrightarrow{\text{project}} & Q_j & \longrightarrow & X/F^{i+1} \\ & \searrow^{\text{onto}} & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & P_i & \xrightarrow{\text{diagonal}} & P_i \times \cdots \times P_i & \xrightarrow{\text{project}} & P_i & \xrightarrow{f_j} & X/F^i \\ & & & \searrow_{f_j} & & & & \nearrow & \end{array}$$

we see that f_j, hence every element of $\mathrm{Hom}(P_i, X/F^i)$, lifts to an element of $\mathrm{Hom}(Q, X/F^{i+1})$, and so the map

$$\mathrm{Hom}(Q, X/F^{i+1}) \to \mathrm{Hom}(P_i, X/F^i) \overset{(58)}{\simeq} \mathrm{Hom}(Q, X/F^i)$$

is surjective. Therefore, the sequence

$$0 \to \mathrm{Hom}(Q, F^{i+1}/F^i) \to \mathrm{Hom}(Q, X/F^{i+1}) \to \mathrm{Hom}(Q, X/F^i) \to 0$$

is exact. The dimensions of the end terms are respectively $\lg_S(F^i/F^{i+1}) \cdot \dim_k \mathrm{End}(S)$ (because F^{i+1}/F^i is simple) and $\lg_S(X/F^i) \cdot \dim_k \mathrm{End}(S)$ (use (58) and the induction hypothesis). Therefore,

$$\dim_k \mathrm{Hom}(Q, X/F^{i+1}) = \lg_S(X/F^{i+1}) \cdot \dim_k \mathrm{End}(S),$$

and we can take $P_{i+1} = Q$.

The induction ends with an essential extension $P(S) \overset{\text{def}}{=} P_r$ of S such that (54) is an equality for $E = P(S)$ and $Y = X$. Lemma 3.12 now shows that $P(S)$ is projective, as required. □

COROLLARY 3.14 *A locally finite k-linear abelian category A has a projective generator if and only if $\mathsf{A} = \langle X \rangle$ for some object X.*

PROOF The sufficiency follows from the proposition. For the necessity, if A admits a projective generator P, then $\mathsf{A} \approx \mathsf{Modf}_A$ with $A = \mathrm{End}(P)$ (see 3.5), and $\mathsf{Modf}_A = \langle A_A \rangle$. □

COROLLARY 3.15 *Let A be a k-linear abelian category, and let $\omega : \mathsf{A} \to \mathsf{Vecf}(k)$ be an exact faithful k-linear functor. For any X in A, $B \overset{\text{def}}{=} \mathrm{End}(\omega|\langle X\rangle)$ is a finite-dimensional k-algebra, and ω is an equivalence $\langle X\rangle \xrightarrow{\approx} {}_B\mathsf{Modf}$.*

PROOF The hypotheses imply that A is locally finite, and hence $\langle X\rangle$ has a projective generator. We can now apply Proposition 3.8. □

NOTES Proposition 3.13 is Deligne 1990, 2.14, where Deligne writes "Le résultat suivant m'a été signalé par O. Gabber". There are many similar results in the literature, but this seems to have been new in 1990.

Alternative proof of 3.15

In this section, we give a more direct proof of 3.15, not using Gabber's result 3.13. Let A be a k-linear abelian category, and let $\omega : \mathsf{A} \to \mathsf{Vecf}(k)$ be an exact faithful k-linear functor. We wish to find a ring A_X such that ω defines an equivalence $\langle X\rangle \to \mathsf{Modf}(A_X)$. As a first attempt, we might try $A_X = \mathrm{End}(\omega(X))$, but this ring does not preserve subobjects in $\langle X\rangle$. Instead, we take the largest k-subalgebra of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$.

For X in A and a finite-dimensional k-vector space, we let $\mathcal{H}om(V,X) = V^\vee \otimes X$. If $W \subset V$ and $Y \subset X$, then the ***transporter of*** W ***to*** Y is

$$(Y : W) = \mathrm{Ker}(\mathcal{H}om(V,X) \to \mathcal{H}om(W, X/Y)).$$

For any k-linear functor F, $F(\mathcal{H}om(V,X)) = \mathcal{H}om(V,FX)$, and if F is exact, then $F(Y : W) = (FY : W)$.

If $W \subset V$ and $Y \subset X$ all lie in $\mathsf{Vecf}(k)$, then

$$(Y : W) = \{\alpha : V \to X \mid \alpha(Y) \subset W\},$$

which explains the notation and name.

LEMMA 3.16 *Let C be a k-linear abelian category and let $\omega : \mathsf{C} \to \mathsf{Vecf}_k$ be a k-linear exact faithful functor. Then, for any object $X \in \mathrm{ob}(\mathsf{C})$, the following two objects are equal:*

(a) *the largest subobject P of $\mathcal{H}om(\omega(X), X))$ whose image in $\mathcal{H}om(\omega(X)^n, X^n)$ (embedded diagonally) is contained in $(Y : \omega(Y))$ for all $Y \subset X^n$, $n \geq 1$;*

(b) *the smallest subobject P' of $\mathcal{H}om(\omega(X), X)$ such that the subspace*

$$\omega(P') \subset \mathrm{Hom}(\omega(X), \omega(X))$$

contains $\mathrm{id}_{\omega(X)}$.

PROOF Clearly $\omega(X) = 0$ implies $\mathrm{End}(X) = 0$, which implies $X = 0$. Thus, if $X \subset Y$ and $\omega(X) = \omega(Y)$, then $X = Y$, and it follows that all objects of C are both artinian and noetherian. The objects P and P' therefore exist.

The functor ω maps $\mathcal{H}om(V,X)$ to $\mathrm{Hom}(V, \omega X)$ and $(Y : W)$ to $(\omega Y : W)$ for all $W \subset V \in \mathrm{ob}(\mathsf{Vecf}_k)$ and $Y \subset X \in \mathrm{ob}(\mathsf{C})$. It therefore maps

$$P \overset{\text{def}}{=} \bigcap (\mathcal{H}om(\omega X, X) \cap (Y : \omega Y))$$

to

$$\bigcap (\mathrm{End}(\omega X) \cap (\omega Y : \omega Y)).$$

This means ωP is the largest subring of $\mathrm{End}(\omega X)$ stabilizing ωY for all $Y \subset X^n$, $n \geq 1$. Hence $\mathrm{id}_{\omega X} \in \omega P$ and $P \supset P'$.

Let V be a finite-dimensional vector space over k. There is an obvious map

$$\mathcal{H}om(\omega X, X) \to \mathcal{H}om(\omega(V \otimes X), V \otimes X)$$

which, after the application of ω, becomes

$$f \mapsto \mathrm{id}_V \otimes f : \ \mathrm{End}(\omega X) \to \mathrm{End}(V \otimes \omega(X)).$$

By definition, $\omega P \subset \mathrm{End}(\omega X)$ stabilizes ωY for all $Y \subset V \otimes X$. On applying this remark to a subobject

$$Q \subset \mathcal{H}om(\omega X, X) = (\omega X)^\vee \otimes X,$$

we find that ωP, when acting by left multiplication on $\mathrm{End}(\omega X)$, stabilizes ωQ. Therefore, if ωQ contains $\mathrm{id}_{\omega X}$, then $\omega P \subset \omega Q$, and $P \subset Q$. On applying this statement with $Q = P'$, we find that $P \subset P'$. □

Let $P_X \subset \mathcal{H}om(\omega(X), X)$ be the subobject defined in (a) (equivalently (b)) of the lemma, and let $A_X = \omega(P_X)$ — it is the largest k-subalgebra of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$. Let $\langle X \rangle$ be the strictly full subcategory of C whose objects are those isomorphic to a subquotient of X^n for some $n \in \mathbb{N}$. Then $\omega|\langle X \rangle : \langle X \rangle \to \mathsf{Vec}_k$ factors through Mod_{A_X}.

LEMMA 3.17 *Let C, ω be as in (3.16). There is a natural action of the ring A_X on $\omega(Y)$, $Y \in \langle X \rangle$, and ω defines an equivalence of categories $\langle X \rangle \to \mathsf{Mod}(A_X)$ carrying $\omega|\langle X \rangle$ to the forgetful functor. Moreover $A_X = \mathrm{End}(\omega|\langle X \rangle)$.*

PROOF The right action $f \mapsto f \circ a$ of A_X on $\mathcal{H}om(\omega X, X)$ stabilizes P_X because obviously,

$$(Y : \omega Y)(\omega Y : \omega Y) \subset (Y : \omega Y).$$

If M is an A_X-module, we define

$$P_X \otimes_{A_X} M = \mathrm{Coker}(P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M).$$

Then

$$\omega(P_X \otimes_{A_X} M) \simeq \omega(P_X) \otimes_{A_X} M = A_X \otimes_{A_X} M \simeq M.$$

This shows that ω is essentially surjective. A similar argument shows that $\langle X \rangle \to \mathsf{Mod}(A_X)$ is full.

Clearly any element of A_X defines an endomorphism of $\omega|\langle X \rangle$. On the other hand an element λ of $\mathrm{End}(\omega|\langle X \rangle)$ is determined by $\lambda_X \in \mathrm{End}(\omega(X))$; thus

$$\mathrm{End}(\omega(X)) \supset \mathrm{End}(\omega|\langle X \rangle) \supset A_X.$$

But λ_X stabilizes $\omega(Y)$ for all $Y \subset X^n$, and so $\mathrm{End}(\omega|\langle X \rangle) \subset A_X$. This completes the proof of the lemma. □

Locally finite k-linear abelian categories as unions of module categories

PROPOSITION 3.18 *Every locally finite k-linear abelian category A is a filtered union of strictly full subcategories A_α such that*

- *each A_α is stable under finite direct sums and subquotients,*
- *each A_α is equivalent to Modf_{A_α} for some finite-dimensional k-algebra A_α (not necessarily commutative).*

PROOF The category A is a union of the subcategories of the form $\langle X\rangle$. The union is filtered because $\langle X\rangle, \langle Y\rangle \subset \langle X \oplus Y\rangle$). Each category $\langle X\rangle$ satisfies the first condition by definition, and it satisfies the second by 3.5 and 3.13. □

Locally finite k-linear abelian categories as comodule categories

The next proposition allows us to express the results of the last subsection in terms of coalgebras, which are more convenient for passage to the limit.

Recall that $V^\vee$ denotes the linear dual $\mathrm{Hom}_{k\text{-linear}}(V,k)$ of a k-vector space V. Note that $V^\vee \otimes V^\vee \subset (V\otimes V)^\vee$, with equality if (and only if) V is finite-dimensional.

PROPOSITION 3.19 (a) *If (C,Δ,ϵ) is a coalgebra over k, then $(C^\vee, \Delta^\vee|_{C^\vee\otimes C^\vee}, \epsilon^\vee)$ is an algebra over k.*

(b) *If (A,m,e) is a finite-dimensional algebra over k, then $(A^\vee, m^\vee, e^\vee)$ is a coalgebra over k.*

(c) *Let C be a coalgebra over k. Every right C-comodule is a left $C^\vee$-module.*

(d) *Let A be an algebra over k. If A is finite dimensional, then every left A-module is a right $A^\vee$-comodule.*

PROOF For (a) and (b), compare the definitions 1.2 and 1.3.

For (c), let (M,ρ) be a right C-comodule. For $m\in M$, write $\rho(m)=\sum_i m_i\otimes c_i$, and for $f\in C^\vee$, define

$$f\cdot m = \sum\nolimits_i f(c_i)\cdot m_i.$$

This makes M into a left $C^\vee$-module.

For (d), let M be a left A-module. For $m\in M$, let $\{m_1,\dots,m_n\}$ be a basis of $A\cdot m$. Then there exist $f_i\in A^\vee$ such that $a\cdot m=\sum f_i(a)m_i$ for all $a\in A$. Now

$$\rho(m)=\sum m_i\otimes f_i$$

defines a co-action of $A^\vee$ on M. □

The operations in (c) and (d) are inverse, so, when C is finite-dimensional, to give a right co-action of C on a k-vector space V is the same as giving a left action of $C^\vee$ on V.

Let C be an essentially small k-linear abelian category and $\omega:\mathsf{C}\to\mathsf{Vecf}(k)$ an exact faithful k-linear functor. Recall that the existence of ω implies that C is locally finite. For an object X of C, let $A_X=\mathrm{End}(\omega|\langle X\rangle)$, and let $C_X=A_X^\vee$. For any Y in $\langle X\rangle$, $\omega(Y)$ is a left A_X-module, hence also a right C_X-comodule. The functor $Y\rightsquigarrow\omega(Y)$ is an equivalence of categories (3.8, 3.13, 3.19)

$$\langle X\rangle \xrightarrow{\approx} \mathsf{coModf}(C_X).$$

Define a partial ordering on the set of isomorphism classes of objects in C by the rule

$$[X] \leq [Y] \text{ if } \langle X\rangle \subset \langle Y\rangle.$$

Note that $[X], [Y] \leq [X \oplus Y]$, so the set is filtered, and that if $[X] \leq [Y]$, then restriction defines a homomorphism $A_Y \to A_X$. On passing to the inductive limit over the isomorphism classes, we obtain the following statement.

THEOREM 3.20 *Let C be an essentially small k-linear abelian category and $\omega : \mathsf{C} \to \mathsf{Vecf}_k$ an exact faithful k-linear functor. Let $C(\omega)$ denote the k-coalgebra $\varinjlim_{[X]} \mathrm{End}(\omega|\langle X\rangle)^\vee$. Then ω is an equivalence $\mathsf{C} \xrightarrow{\approx} \mathsf{coModf}(C(\omega))$.*

In more detail, the natural actions of $C(\omega)$ on the objects $\omega(Y)$, $Y \in \mathsf{C}$, allow us to regard ω as a functor

$$\mathsf{C} \to \mathsf{coModf}(C(\omega)).$$

This functor is an equivalence of categories carrying ω into the forgetful functor.

EXAMPLE 3.21 Let A be a finite-dimensional k-algebra (not necessarily commutative) and ω the forgetful functor ${}_A\mathsf{Mod} \to \mathsf{Vecf}(k)$. For R a commutative k-algebra, let ϕ_R denote the functor $R \otimes - : \mathsf{Vecf}(k) \to \mathsf{Mod}(R)$. The action of $R \otimes A$ on $R \otimes \omega(M)$ defines a map

$$\alpha : R \otimes A \to \mathrm{End}(\phi_R \circ \omega),$$

which we shall show to be an isomorphism by describing an inverse β. For $\lambda \in \mathrm{End}(\phi_R \circ \omega)$, set $\beta(\lambda) = \lambda_A(1 \otimes 1)$. Clearly $\beta \circ \alpha = \mathrm{id}$, and so we need only show that $\alpha \circ \beta = \mathrm{id}$. For $M \in \mathrm{ob}(\mathsf{Mod}_A)$, let $M_0 = \omega(M)$. The A-module $A \otimes M_0$ is a direct sum of copies of A, and the additivity of λ shows that $\lambda_{A \otimes M_0} = \lambda_A \otimes \mathrm{id}_{M_0}$. The map $a \otimes m \mapsto am : A \otimes M_0 \to M$ is A-linear, and hence

$$\begin{array}{ccc} R \otimes A \otimes M_0 & \longrightarrow & R \otimes M \\ \downarrow{\scriptstyle \lambda_A \otimes \mathrm{id}_{M_0}} & & \downarrow{\scriptstyle \lambda_M} \\ R \otimes A \otimes M_0 & \longrightarrow & R \otimes M \end{array}$$

is commutative. Therefore

$$\lambda_M(1 \otimes m) = \lambda_A(1) \otimes m = (\alpha \circ \beta(\lambda))_M(1 \otimes m) \text{ for } m \in M,$$

i.e., $\alpha \circ \beta = \mathrm{id}$.

We have shown that $A \simeq \mathrm{End}(\omega)$, and it follows that, if in (3.20) we take $\mathsf{C} = {}_A\mathsf{Mod}$, so that $\mathsf{C} = \langle {}_AA\rangle$, then the equivalence of categories obtained sends a left A-module to the associated right $A^\vee$-comodule (3.19).

NOTES

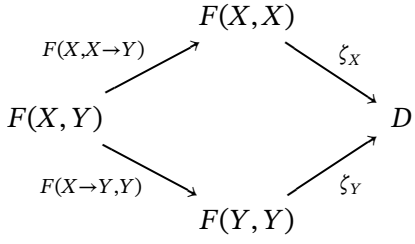
3.22 It is necessary in Theorem 3.20 that C be essentially small, because otherwise the underlying "set" of $C(\omega)$ may be a proper class. For example, let S be a proper class, and let C be the category of finite-dimensional k-vector spaces graded by S. In this case $C(\omega)$ contains an idempotent for each element of S, and so cannot be a set.

In terms of Grothendieck universes, we can say that C is always equivalent to a category of comodules, but possibly only in a larger universe.

3.23 It is possible to realize the category in Theorem 3.20 as a category of modules over k-algebra, but for this it is necessary to introduce topologies. The category of coalgebras over k is the ind-category of the category of finite coalgebras over k. Hence, its opposite is the pro-category of the category of finite k-algebras, i.e., the category of profinite k-algebras. The category of right comodules over a coalgebra is equivalent, as a k-linear category, to the category of finite continuous modules over the corresponding profinite k-algebra. See Serre 1993, 2.2.

3.24 Theorem 3.24 can be expressed in terms of coends. Let C be a small category and $F: \mathsf{C}\times\mathsf{C}^{\mathrm{opp}} \to \mathsf{D}$ a functor. A ***coend*** for F is an object D of D together a family of morphisms $\zeta_X: F(X,X)\to D$ indexed by the objects X of C such that

(a) for all morphisms $X\to Y$ in C,



commutes;

(b) the pair (D,ζ) is universal with respect to (a).

When it exists, the coend is uniquely determined up to a unique isomorphism. The definition can be extended to essentially small categories by choosing a skeleton for the category.

Now consider a category C and functor $\omega: \mathsf{C}\to \mathsf{Vecf}(k)$, as in 3.20. Let

$$C(\omega) = \varinjlim_{[X]} \mathrm{End}(\omega|\langle X\rangle)^\vee.$$

For each X in C, we have an isomorphism

$$\omega(X)\otimes\omega(X)^\vee \to \mathrm{End}(\omega(X))^\vee, \quad (a\otimes f)(\alpha) = f(\alpha(a)),$$

and hence morphism $\zeta_X: \omega(X)\otimes\omega(X)^\vee\to C(\omega)$. The pair $(C(\omega),\zeta)$ is the coend for the functor $\omega\otimes\omega^\vee: \mathsf{C}\times\mathsf{C}^{\mathrm{opp}}\to\mathsf{Vecf}(k)$.

NOTES Theorem 3.20 is proved Takeuchi 1977, but was known earlier to Bourbaki (Serre 1993, §2, Thm 3).

Categories of comodules

Let (C,Δ,ϵ) be a coalgebra over the field k (co-associative with co-identity), and let ω be the forgetful functor $\mathsf{coModf}(C)\to\mathsf{Vecf}(k)$.

By definition, an object of $\mathsf{coModf}(C)$ is a pair (M,ρ_M) with M a finite-dimensional k-vector space and $\rho_M: M\to M\otimes C$ a k-linear map satisfying certain conditions. On varying M, we obtain a natural transformation $\rho:\omega\to\omega\otimes C$. On combining ρ with a k-linear map $a: C\to V$, where V is a k-vector space (not necessarily finite-dimensional), we get a natural transformation $\alpha:\omega\to\omega\otimes V$.

PROPOSITION 3.25 *The map of k-vector spaces*

$$a \mapsto \alpha : \ \mathrm{Hom}_{k\text{-linear}}(C, V) \to \mathrm{Nat}(\omega, \omega \otimes V) \tag{59}$$

is an isomorphism.

PROOF We construct an inverse. Let α be natural transformation $\omega \to \omega \otimes V$. Let $c \in C$, and let M be a finite-dimensional subcomodule containing c (which exists by 1.15). Then $\alpha_M(c) \in M \otimes V$, and its image under $\epsilon \otimes \mathrm{id}_V$ lies in V. The map $\alpha \mapsto (c \mapsto (\epsilon \otimes \mathrm{id}_V)(\alpha_M(c))$ is well-defined, and is the required inverse. □

The map (59) is natural in V, and so we have a natural isomorphism of functors

$$\mathrm{Hom}_{k\text{-linear}}(C, -) \simeq \mathrm{Nat}(\omega, \omega \otimes -). \tag{60}$$

REMARK 3.26 The comultiplication map $\Delta : \ C \to C \otimes C$ corresponds under (59) to a natural transformation

$$\omega \to \omega \otimes (C \otimes C).$$

This can be shown to be the composite

$$\omega \xrightarrow{\rho} \omega \otimes C \xrightarrow{\rho \otimes C} (\omega \otimes C) \otimes C \simeq \omega \otimes (C \otimes C).$$

The coidentity map $\epsilon : \ C \to k$ corresponds under (59) to the natural isomorphism $\omega \to \omega \otimes k$. As the underlying vector space of C represents the functor $\mathrm{Nat}(\omega, \omega \otimes -)$, we see that (C, Δ, ϵ) can be recovered from the pair $(\mathsf{coModf}(C), \omega)$ (uniquely, up to a unique isomorphism).

A homomorphism $m : \ C \otimes C \to C$ of k-coalgebras defines a C-coaction $\rho^m_{M,N}$

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes C \otimes N \otimes C \xrightarrow{M \otimes \gamma_{C,N} \otimes N} M \otimes N \otimes C \otimes C \xrightarrow{M \otimes N \otimes m} M \otimes N \otimes C$$

on $M \otimes N$ for any C-comodules M, N. On varying M and N in $\mathsf{coModf}(C)$, we get a natural transformation

$$\rho^m : \ \omega \otimes \omega \to \omega \otimes \omega \otimes C.$$

PROPOSITION 3.27 *The map*

$$m \mapsto \rho^m : \ \mathrm{Hom}_{k\text{-coalgebra}}(C \otimes C, C) \to \mathrm{Nat}(\omega \otimes \omega, \omega \otimes \omega \otimes C)$$

is an isomorphism of k-vector spaces.

PROOF Construct an inverse, as in 3.25. □

Define

$$\phi^m : \ \mathsf{coModf}_C \times \mathsf{coModf}_C \to \mathsf{coModf}_C$$

to be the functor sending a pair of C-comodules (M, ρ_M), (N, ρ_N) to $(M \otimes N, \rho^m_{M,N})$.

PROPOSITION 3.28 *The map $m \mapsto \phi^m$ defines a one-to-one correspondence between the set of k-coalgebra homomorphisms $m : \ C \otimes_k C \to C$ and the set of k-bilinear functors*

$$\phi : \ \mathsf{coModf}_C \times \mathsf{coModf}_C \to \mathsf{coModf}_C$$

with the property that $\phi(M, N) = M \otimes N$ as k-vector spaces.

(a) *The homomorphism m is associative (51) if and only if, for all M, N, P in $\mathsf{coModf}(C)$, the canonical isomorphism of k-vector spaces*

$$M \otimes (N \otimes P) \simeq (M \otimes N) \otimes P$$

is an isomorphism of C-comodules.

(b) *There exists a k-coalgebra homomorphism $e : k \to C$ satisfying (52), p. 86, if and only if there exists a C-comodule U with underlying k-vector space of dimension 1 such that the canonical isomorphism of k-vector spaces*

$$U \otimes U \simeq U$$

is an isomorphism of C-comodules.

(c) *The homomorphism m is commutative (i.e., $m(a \otimes b) = m(b \otimes a)$ for all $a, b \in C$) if and only if, for all M, N in $\mathsf{coModf}(C)$, the canonical isomorphism of k-vector spaces*

$$M \otimes N \simeq N \otimes M$$

is an isomorphism of C-comodules.

PROOF The first assertion is a restatement 3.27.

(a) Similar to (c).

(b) The map

$$M \simeq M \otimes k \xrightarrow{\rho_M \otimes e} M \otimes C \otimes C \xrightarrow{M \otimes m} M \otimes C, \quad M \in \mathsf{coModf}(C),$$

corresponds under (60) to $c \mapsto m(c \otimes e) : C \to C$. Therefore the right identity property (52) holds if and only if the above map is ρ_M (see **??**). This means precisely that the canonical isomorphism $M \otimes k \simeq M$ of k-vector spaces is a C-comodule map.

(c) Note that m is commutative if and only if $m = m \circ \gamma$, where $\gamma : C \otimes C \to C \otimes C$ is $c \otimes d \mapsto d \otimes c$. This holds if and only if

$$(\rho^m : M \otimes N \to M \otimes N \otimes C) = (\rho^{m \circ \gamma} : M \otimes N \to M \otimes N \otimes C).$$

From the definition of ρ^m, we see that this is the case if and only if the canonical isomorphism of k-vector spaces $M \otimes N \simeq N \otimes M$ is an isomorphism of C-comodules. □

For an alternative proof of Proposition 3.28, see Saavedra 1972, II, 1.3.3.3.

Summary

3.29 Let C be an essentially small k-linear abelian category, and let $\omega : \mathsf{C} \to \mathsf{Vecf}_k$ be an exact faithful k-linear functor.

- The k-coalgebra

$$C(\omega) \overset{\text{def}}{=} \varinjlim_{[X]} \mathrm{End}(\omega|\langle X \rangle)^\vee$$

acts on each object $\omega(Y)$, $Y \in \mathrm{ob}\,\mathsf{C}$, and so ω defines a functor $\mathsf{C} \to \mathsf{coModf}(C(\omega))$. This is an equivalence of categories.

- If C is a monoidal category and ω is a monoidal functor, then $C(\omega)$ is a bialgebra. Specifically, the monoidal structure on C provides $C(\omega)$ with a multiplication.
- If, in addition, C is symmetric, then $C(\omega)$ is a commutative bialgebra.
- If, in addition, C is rigid, then $C(\omega)$ is a Hopf algebra.
- The k-coalgebra $C(\omega)$ is equal to the coend of the functor

$$\omega \otimes \omega^\vee : \mathsf{C} \times \mathsf{C}^{\mathrm{opp}} \to \mathsf{Vecf}(k).$$

Completion of the proof of the main theorem 3.1

THEOREM 3.30 *Let* $(\mathsf{C},\otimes)$ *be an essentially small* k*-linear abelian* $\otimes$*-category and* $\omega:\mathsf{C}\to\mathsf{Vec}(k)$ *an exact faithful* k*-linear* $\otimes$*-functor.*

(a) *The functor* $\mathcal{E}nd^{\otimes}(\omega)$ *of* k*-algebras is represented by an affine monoid scheme* G *over* k*.*

(b) *The functor* $\mathsf{C}\to\mathsf{Repf}(G)$ *defined by* ω *is an equivalence of* $\otimes$*-categories.*

PROOF Because of the correspondence between monoid schemes and bialgebras (and their representations) (see §1), this is simply a restatement of part of 3.29. □

If $(\mathsf{C},\otimes)$ is rigid, ω is automatically faithful (I, 5.8). Moreover, $\mathcal{E}nd^{\otimes}(\omega)=\mathcal{A}ut^{\otimes}(\omega)$ (I, 4.21), and so $G(R)$ is a group for all k-algebras R, which implies that G is a group scheme. This completes the proof of Theorem 3.1.

REMARK 3.31 Let (C,ω) be $(\mathsf{Repf}(G),\omega^G)$. On following through the proof of Theorem 3.1 in this case one recovers Corollary 2.2: $\mathcal{A}ut^{\otimes}(\omega^G)$ is represented by G.

ASIDE 3.32 If in Theorem 3.30, $(\mathsf{C},\otimes)$ is a monoidal rather than a symmetric monoidal category, then the Hopf algebra C may be neither commutative nor cocommutative, and this is an interesting way of constructing such Hopf algebras. For more on this theme, see Breen 1994, 1.5.

4 Nori's criterion to be a neutral tannakian category

The following is a sometimes useful criterion for a category with a tensor product and forgetful functor to be tannakian.

PROPOSITION 4.1 *Let* C *be an essentially small* k*-linear abelian category and* $\otimes:\mathsf{C}\times\mathsf{C}\to\mathsf{C}$ *a functor,* k*-linear in each variable. Suppose that there are given an exact faithful* k*-linear functor* $F:\mathsf{C}\to\mathsf{Vec}(k)$ *and natural isomorphisms*

$$\alpha_{X,Y,Z}:X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$
$$\gamma_{X,Y}:X\otimes Y\to Y\otimes X$$

with the following properties,

(a) $F\circ\otimes=\otimes\circ(F\times F)$;

(b) $F(\alpha_{X,Y,Z})$ *is the usual associativity constraint in* $\mathsf{Vec}(k)$*;*

(c) $F(\gamma_{X,Y})$ *is the usual commutativity constraint in* $\mathsf{Vec}(k)$*;*

(d) *there exists a unit object* (U,u) *in* $(\mathsf{C},\otimes)$ *such that* (FU,Fu) *is a unit object in* $\mathsf{Vec}(k)$*;*

(e) *if* $F(L)$ *has dimension* 1*, then there exists an object* L^{-1} *in* C *and an isomorphism* $L\otimes L^{-1}\to U$*.*

Then $(C,\otimes,\alpha,\gamma)$ *is a tannakian category over* k*, and* F *is a* k*-valued fibre functor.*

PROOF Certainly $(\mathsf{C}, \otimes, \alpha, \gamma)$ is a k-linear abelian $\otimes$-category, and Theorem 3.30 shows that F defines an equivalence of $\otimes$-categories $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$, where G is the affine monoid scheme over k representing $\mathcal{E}nd^{\otimes}(F)$. Thus, we may assume $\mathsf{C} = \mathsf{Repf}(G)$ and that F is the forgetful functor. Let (U, u) be as in (d). Because it is a unit object in $\mathsf{Repf}(G)$, U can be identified with k (trivial action of G). Let $\lambda \in G(R)$. If L in $\mathsf{Repf}(G)$ has dimension 1, then $\lambda_L : L \to L$ is invertible, as follows from the existence of a G-isomorphism $L \otimes L^{-1} \to U$. It follows that λ_X is invertible for all X in $\mathsf{Repf}(G)$, because

$$\det(\lambda_X) \overset{\text{def}}{=} \bigwedge^d \lambda_X = \lambda_{\bigwedge^d X}, \quad d = \dim X,$$

is invertible. Thus, G is an affine group scheme. □

NOTES Nori (1976, §1) adopts the statement of 4.1 as a definition of tannakian category.

5 The functor defined by a homomorphism of group schemes

Let $f : G \to H$ be a homomorphism of affine group schemes over k. Using f, we can regard an H-module as a G-module. In this way, we get a k-linear exact $\otimes$-functor

$$\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$$

compatible with the forgetful functors.

THEOREM 5.1 *Let $f : G \to H$ be a homomorphism of affine group schemes over k.*

(a) *The homomorphism f is faithfully flat if and only if the functor ω^f is fully faithful and its essential image is closed under subobjects.*[1]

(b) *The homomorphism f is a closed immersion if and only if every object of $\mathsf{Repf}(G)$ is a subquotient of an object in the image of ω^f.*

Sometimes it is possible to omit the condition on subobjects in (a).

COROLLARY 5.2 *A homomorphism $f : G \to H$ of affine group schemes is faithfully flat if ω^f is fully faithful and at least one of the following conditions holds:*

(a) *$\mathsf{Repf}(G)$ is semisimple (for example, G reductive and $\mathrm{char}(k) = 0$);*

(b) *G or H is profinite.*

PROOF We have to show that the essential image is closed under subobjects.

(a) When $\mathsf{Repf}(G)$ is semisimple, every subobject of $\omega^f(X)$ is of the form $e(\omega^f(X))$ with e an idempotent in $\mathrm{End}(\omega^f(X))$. As ω^f is fully faithful, $e = \omega^f(e')$ with e' an idempotent in $\mathrm{End}(X)$, and $e(\omega^f(X)) \simeq \omega^f(e'X)$.

(b) See Biswas et al. 2021, Lemma 2.1. □

[1] Let $F : \mathsf{A} \to \mathsf{B}$ be an exact fully faithful functor of abelian categories. We say that its essential image is ***closed under subobjects*** if, for all objects X of A, every subobject of $F(X)$ is isomorphic to $F(Y)$ for some subobject Y of X.

Proof of (a) *of Theorem 5.1*

For a k-algebra A (not necessarily commutative), we let ${}_A\mathsf{Modf}_k$ denote the category of left A-modules finite-dimensional over k.

Let $f : A \to B$ be a homomorphism of k-algebras. Using f, we can regard a B-module as an A-module and ${}_B\mathsf{Modf}_k$ as a subcategory of ${}_A\mathsf{Modf}_k$.

LEMMA 5.3 *Assume that B is finite-dimensional over k. The homomorphism $f : A \to B$ is surjective if and only if ${}_B\mathsf{Modf}_k$ is a full subcategory of ${}_A\mathsf{Modf}_k$ stable under forming submodules.*

PROOF If f is surjective, then the subcategory ${}_B\mathsf{Modf}_k$ certainly has the claimed properties. For the converse, let $\bar{A}$ denote the image of A in B. Then $\bar{A}$ is an A-submodule of B, and hence also a B-submodule. As it contains the identity element 1 of B, it equals B. □

Let $f : C \to D$ be a homomorphism of k-coalgebras. Using f, we can regard a C-comodule as a D-comodule and $\mathsf{coModf}(C)$ as a subcategory of $\mathsf{coModf}(D)$.

LEMMA 5.4 *The homomorphism $f : C \to D$ is injective if and only if $\mathsf{coModf}(C)$ is a full subcategory of $\mathsf{coModf}(D)$ closed under subobjects.*

PROOF If C is finite-dimensional over k, this follows from 5.3 applied to $f^\vee : D^\vee \to C^\vee$ (see 3.19). In the general case, we can write C as a union $C = \bigcup C_i$ of finite-dimensional k-subcoalgebras (1.19), and correspondingly $\mathsf{coModf}(C) = \bigcup_i \mathsf{coModf}(C_i)$. Now the statement for C follows from the statement for the C_i. □

We now prove (a) of Proposition 5.1.

If $f : G \to H$ is faithfully flat, and therefore an epimorphism, then $\mathsf{Repf}(H)$ can be identified with the subcategory of $\mathsf{Repf}(G)$ of representations of G factoring through H. It is therefore obvious that ω^f has the stated properties. Conversely, if ω^f has the stated properties, then the homomorphism $\mathcal{O}(H) \to \mathcal{O}(G)$ of Hopf algebras over k is injective (5.4), and so faithfully flat (1.11).

Proof of (b) *of Proposition 5.1*

Let $f : G \to H$ be a homomorphism of affine group schemes over k. Let C be the strictly full subcategory of $\mathsf{Repf}(G)$ whose objects are subobjects of objects of the form of $\omega^f(Y)$, $Y \in \mathrm{ob}(\mathsf{Repf}(H))$. The functors

$$\mathsf{Repf}(H) \to \mathsf{C} \to \mathsf{Repf}(G)$$

correspond (see 3.20) to homomorphisms of k-coalgebras

$$\mathcal{O}(H) \to B \to \mathcal{O}(G).$$

As C is closed under taking subobjects in $\mathsf{Repf}(G)$, we see that $B \to \mathcal{O}(G)$ is injective (Lemma 5.4). Moreover, for $Y \in \mathrm{ob}(\mathsf{Repf}(H))$,

$$\mathrm{End}(\omega_G|\langle\omega^f(Y)\rangle) \to \mathrm{End}(\omega_H|\langle Y\rangle)$$

is injective, where ω_G and ω_H are the forgetful functors, and so $\mathcal{O}(H) \to B$ is surjective.

We now prove (b) of Proposition 5.1.

If f is a closed immersion, then $\mathcal{O}(H) \to \mathcal{O}(G)$ is surjective, and it follows that $B \simeq \mathcal{O}(G)$ and $\mathsf{C} = \mathsf{Repf}(G)$.

Conversely, if $\mathsf{C} = \mathsf{Repf}(G)$, then $B = \mathcal{O}(G)$, and $\mathcal{O}(H) \to \mathcal{O}(G)$ is surjective, i.e., f is a closed immersion.

Complements

5.5 Let $f : G \to H$ be a homomorphism of affine algebraic groups over k. If f is an epimorphism in the category of affine group schemes, then $\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ is fully faithful, and so $\mathsf{Repf}(H)$ is a strictly full abelian subcategory of $\mathsf{Repf}(G)$ closed under the formation of direct sums, tensor products, and duals, but not subquotients unless f is faithfully flat. For example, the inclusion $G \hookrightarrow H$ of a proper parabolic subgroup G into a smooth connected algebraic group H is an epimorphism but is not faithfully flat. See Milne 2017, 5.44, and Brion 2017, Theorem 1.

5.6 Let G be an affine group scheme over k and N a normal subgroup scheme of G. Then $\mathsf{Repf}(G/N)$ is the full subcategory of $\mathsf{Repf}(G)$ whose objects are the representations of G on which N acts trivially. In this way, we get a one-to-one correspondence between the normal subgroup schemes of G and the tannakian subcategories of $\mathsf{Repf}(G)$ closed under subobjects.

5.7 Statement (a) of Theorem 5.1 generalizes. A homomorphism $f : G \to H$ of flat affine group schemes over a noetherian ring R is faithfully flat if and only if the functor $\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ on the categories of representations on finitely generated R-modules is fully faithful and its essential image is closed under subobjects. See Hai et al. 2025, Theorem 7.2.

5.8 Let

$$1 \to N \xrightarrow{f} G \xrightarrow{g} Q \to 1$$

be a sequence of homomorphisms of affine group schemes over k. Assume that f is a closed immersion and that g is faithfully flat. The sequence is exact if and only if the following three statements hold.

(a) Let $V \in \mathsf{Repf}(G)$. Then $\omega^f(V)$ is trivial if and only if $V \approx \omega^g(W)$ for some $W \in \mathsf{Repf}(Q)$.

(b) Let $V \in \mathsf{Repf}(G)$. There exists a subobject $V_0 \subset V$ such that $\omega^f(V_0)$ is the largest trivial subobject of $\omega^f(V)$.

(c) Every object of $\mathsf{Repf}(N)$ is a subobject of $\omega^f(V)$ for some V in $\mathsf{Repf}(G)$ (i.e., ω^f is dominant, 4.22).

See Esnault et al. 2008, Appendix A . For a different criterion, see dos Santos 2015, §4.2.

6 Properties of G reflected in $\mathsf{Repf}(G)$

In view of the previous theorems, it is natural to ask how properties of G are reflected in $\mathsf{Repf}(G)$.

Finiteness and connectedness

PROPOSITION 6.1 *An affine group scheme G over k is finite if and only if $\mathsf{Repf}(G) = \langle X \rangle$ for some representation X, i.e., there exists an X such that the objects of $\mathsf{Repf}(G)$ are subquotients of a direct sum of copies of X.*

PROOF If G is finite, then the regular representation X of G is finite-dimensional and has the required property. Conversely if $\mathsf{Repf}(G) = \langle X \rangle$, then $G = \operatorname{Spec} B$, where B is the linear dual of the finite k-algebra A_X in the proof of 3.20. □

PROPOSITION 6.2 *An algebraic group G has no nontrivial finite quotients if and only if, for every representation X on which G acts nontrivially, the subcategory $\langle X\rangle$ is not stable under $\otimes$.*

PROOF According to 5.1(a) and Proposition 6.1, the finite quotients of G correspond to the subcategories of $\mathsf{Repf}(G)$ of the form $\langle X\rangle$ stable under tensor products (hence tannakian). □

COROLLARY 6.3 *In characteristic zero, an algebraic group G is connected if $\mathsf{Repf}(G)$ has no $\otimes$-subcategory with only finitely many simple objects up to isomorphism.*

PROOF In characteristic zero, G is disconnected $\Rightarrow$ G has a nontrivial finite quotient $\Rightarrow$ $\mathsf{Repf}(G)$ has a tensor subcategory of the form $\langle X\rangle$. Such a category has only finitely many simple objects (the quotients of any composition series for X represent the isomorphism classes of simple objects in $\langle X\rangle$). □

The converse to the corollary is false: in characteristic zero, every unipotent algebraic group G is connected, but $\mathsf{Repf}(G)$ has a single simple object (up to isomorphism).

Algebraicity

PROPOSITION 6.4 *An affine group scheme G over k is algebraic if and only if $\mathsf{Repf}(G)$ admits a tensor generator, i.e., $\mathsf{Repf}(G) = \langle X\rangle^{\otimes}$ for some object X.*

PROOF Restatement of 3.2. □

This helps explain the definition 7.13: a tannakian category is algebraic if it is monogenic.

Smoothness; reducedness

6.5 Let G be an affine group scheme over a field k. Is there a criterion on $\mathsf{Repf}(G)$ for G to be smooth or reduced? In characteristic zero, every algebraic group is smooth, and over a perfect field of characteristic p, an algebraic group is smooth if and only if it is reduced (equivalently, geometrically reduced) (Milne 2017, 1.26). Note that $\mathsf{Repf}(\mu_p)$ is semisimple even though μ_p is not reduced (hence not smooth).

6.6 Let G be an affine group scheme over a perfect field k of characteristic p. Then G is reduced if and only if the relative Frobenius $\mathrm{Frob}_{G/k}$ is faithfully flat. Thus Proposition 5.1(a) applies.

6.7 Let T be a neutral tannakian category over a perfect field k of characteristic p. Define $\mathsf{T}^{(p)}$ to be category with the same objects as T but with

$$\mathrm{Hom}_{\mathsf{T}^{(p)}}(X,Y) = \mathrm{Hom}_{\mathsf{T}}(X,Y) \otimes_{k,x\mapsto x^p} k.$$

There is a k-linear exact $\otimes$-functor $\mathrm{Fr}_+ : \mathsf{T}^{(p)} \to \mathsf{T}$. Let ω be a k-valued fibre functor, and let $G = \mathcal{A}ut^{\otimes}(\omega)$.

(a) The group scheme G is reduced if and only if Fr_+ is fully faithful and its essential image is closed under subobjects.

(b) The group scheme G is perfect if and only if Fr_+ is an equivalence of categories.

See Deninger and Wibmer 2026, Corollary 4.2, and mo356131.

Unipotent groups

6.8 An algebraic group G over k is ***unipotent*** if its only simple representations are the one-dimensional representations with G acting trivially. Thus, if G is unipotent, then every nonzero representation has a nonzero fixed vector, and an easy induction argument shows that, for any (V,r) in $\mathsf{Repf}(G)$, there exists a basis of V for which

$$r(G)\subset \mathbb{U}_n \overset{\text{def}}{=} \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & * \\ & & \ddots & \ddots & \\ & 0 & & 1 & * \\ & & & & 1 \end{pmatrix}.$$

i.e., there exists a flag such that G acts trivially on the quotients. Conversely, every algebraic subgroup of $\mathbb{U}_n$ is unipotent (see, for example, Milne 2017, 14.5). An algebraic group over k is unipotent if and only if every nontrivial algebraic subgroup of it admits a nontrivial homomorphism to $\mathbb{G}_a$ (ibid., 14.22). In characteristic zero, unipotent algebraic groups are connected.

Trigonalizable groups

6.9 An algebraic group G over k is ***trigonalizable*** if its only simple representations are those of dimension 1. An easy induction argument shows that, for any (V,r) in $\mathsf{Repf}(G)$, there exists a basis of V for which

$$r(G)\subset \mathbb{T}_n \overset{\text{def}}{=} \begin{pmatrix} * & * & * & \cdots & * \\ & * & * & & * \\ & & \ddots & \ddots & \\ & 0 & & * & * \\ & & & & * \end{pmatrix}.$$

Conversely, every algebraic subgroup of $\mathbb{T}_n$ is trigonalizable (see, for example, Milne 2017, 16.2). A smooth connected algebraic group over an algebraically closed field is solvable if and only if it is trigonalizable (Lie–Kolchin theorem, ibid., 16.30).

Reductive groups

6.10 A smooth connected algebraic group G over k is ***reductive*** if it has no nontrivial smooth connected normal unipotent algebraic subgroup and this condition continues to hold under extension of the base field k. When k is perfect, the condition has to be checked only over k, and a smooth connected algebraic group is reductive if and only if it has a faithful semisimple representation (see Milne 2017, 6.46, 19.17).

6.11 Let G be an affine group scheme over a field of characteristic zero. Then G is a reductive if and only if

(a) $\mathsf{Repf}(G)$ has a tensor generator (so G is algebraic; 6.4),

(b) $\mathsf{Repf}(G)$ contains no nontrivial object X such that $\langle X\rangle$ is stable under $\otimes$ (so G is connected; 6.2), and

(c) $\mathsf{Repf}(G)$ is semisimple (so G is reductive; 6.15 below).

Semisimple groups

6.12 A smooth connected algebraic group G over k is ***semisimple*** if it has no nontrivial smooth connected normal solvable algebraic subgroup and this condition continues to hold under extension of the base field k. When k is perfect, the condition has to be checked only over k. See Milne 2017, 6.44.

6.13 A reductive group G is semisimple if and only if its centre is finite. The centre of G is reflected in the gradations on $\mathsf{Repf}(G)$ (see 9.2 below). For example, let D be a diagonalizable algebraic group with character group M. To give a homomorphism $D \to Z(G)$ is the same as giving an M-gradation on $\mathsf{Repf}(G)$.

6.14 Let G be a reductive group. Then G/G^{der} is a torus, which is trivial if and only if G is semisimple. It follows that G is semisimple if and only if there do not exist nontrivial representations V and W of G such that $V \otimes W$ is trivial.

Semisimple tannakian categories

In this subsection, k is a field of characteristic 0 and $\bar{k}$ is an algebraic closure of k.

THEOREM 6.15 *Let G be a connected affine group scheme over k. The category $\mathsf{Repf}(G)$ is semisimple if and only if G is pro-reductive (i.e., a projective limit of reductive groups).*

The theorem fails (both implications are false) if k has nonzero characteristic. The theorem will be proved as a consequence of a series of propositions.

PROPOSITION 6.16 *The category $\mathsf{Repf}(G)$ is semisimple if and only if $\mathsf{Repf}_{\bar{k}}(G_{\bar{k}})$ is semisimple.*

PROOF Let G be an affine group scheme over k, and let k' be a finite extension of k. A representation of $G_{k'}$ on a finite-dimensional k'-vector space induces a representation of G on the underlying k-vector space together with a k-linear action of k' on the representation. In this way, we get an equivalence of categories

$$\mathsf{Repf}_{k'}(G_{k'}) \to \mathsf{Repf}_k(G)_{(k')}$$

(see B.21(b)). Now I, 6.17, shows that $\mathsf{Repf}_k(G)$ is semisimple if and only if $\mathsf{Repf}_{k'}(G_{k'})$ is semisimple (because k' is separable over k). The proposition follows. □

PROPOSITION 6.17 *Let (V, r) be a representation of a connected algebraic group G over k. A subspace $W \subset V$ is stable under G if and only if it is stable under* $\mathrm{Lie}(G)$.

PROOF We have

$$W \text{ is stable under } G \iff \mathrm{Stab}_G(W) = G.$$

As $\mathrm{Stab}_G(W)$ is smooth and G is connected,

$$\mathrm{Stab}_G(W) = G \iff \mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Lie}(G)$$

(Milne 2017, 10.15). On the other hand,

$$W \text{ is stable under } \mathrm{Lie}(G) \iff \mathrm{Stab}_{\mathrm{Lie}(G)}(W) = \mathrm{Lie}(G).$$

As $\mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Stab}_{\mathrm{Lie}(G)}(W)$ (ibid., 10.31), the statement follows. □

PROPOSITION 6.18 (WEYL) *Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Every finite-dimensional representation of $\mathfrak{g}$ is semisimple.*

PROOF For an algebraic proof, see Humphreys 1972, 6.3. We sketch Weyl's original proof. We can assume that $k = \mathbb{C}$. Let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$, and let G_0 be a connected simply-connected real Lie group with Lie algebra $\mathfrak{g}_0$. Then G_0 is compact, so every finite-dimensional representation (V, r) of it carries a $\mathfrak{g}_0$-invariant positive-definite form, namely, $\langle x, y\rangle_0 = \int_{G_0} \langle x, y\rangle dg$, where $\langle\ ,\ \rangle$ is any positive-definite form on V. It follows that every finite-dimensional (real or complex) representation of G_0 is semisimple, but, for any complex vector space V, the restriction map is an isomorphism

$$\mathrm{Hom}(G, \mathrm{GL}_V) \simeq \mathrm{Hom}(G_0, \mathrm{GL}_V),$$

and so every complex representation of G is semisimple. □

COROLLARY 6.19 *Let G be an algebraic group over k. If G is semisimple, then* $\mathsf{Repf}(G)$ *is semisimple.*

PROOF After 6.16, we may suppose that k is algebraically closed. The Lie algebra of G is semisimple (Milne 2017, 23.68), and so this follows from 6.17 and 6.18. □

PROPOSITION 6.20 *Let G be an algebraic group over an algebraically closed field k, and let N be a normal algebraic subgroup of G. If (V, r) is a semisimple representation of G, then $(V, r|N)$ is a semisimple representation of N.*

PROOF We may suppose that (V, r) is simple. Let W be a nonzero simple N-submodule of V. For any $g \in G(k)$, gW is an N-module and it is simple because $g \mapsto g^{-1}S$ maps N-submodules of gW to N-submodules of W. The sum $\sum gW$, $g \in G(k)$, is G-stable and nonzero, and therefore equals V. Thus V, being a sum of simple N-submodules, is semisimple (I, 6.15). □

We now prove Theorem 6.15. After 6.16, we may suppose that k is algebraically closed. As every finite-dimensional representation $G \to \mathrm{GL}_V$ of G factors through an algebraic quotient of G, and (by definition) G is pro-reductive if and only its algebraic quotients are reductive, we may suppose that G is algebraic.

If G is reductive, then $G = Z \cdot G'$, where Z is the centre of G and G' is the derived subgroup of G (Milne 2017, 19.25). Let $r : G \to \mathrm{GL}_V$ be a finite-dimensional representation of G. The algebraic group Z is a diagonalizable, so $r|Z$ is diagonalizable: $V = \bigoplus_i V_i$, where Z acts on the V_i through distinct characters χ_i (ibid., 12.12). Each V_i is stable under G'. The algebraic group G' is semisimple, so each V_i is a sum of simple G'-modules (6.19). Each of these simple submodules is stable under $G = Z \cdot G'$, and so V is a sum of simple G-modules.

Conversely, suppose that $\mathsf{Repf}(G)$ is semisimple and choose a faithful representation V of G. Let N be a normal unipotent algebraic subgroup of G. We have to show that $N = 1$. According to 6.20, V is semisimple as a representation of N: it is a sum of its simple subrepresentations. As N is unipotent, it acts trivially on each simple subrepresentation, hence on V, but V was chosen to be faithful, so this implies that $N = 1$, as required.

REMARK 6.21 The proposition can be strengthened as follows: the identity component G° of an affine group scheme G over k is pro-reductive if and only if $\mathsf{Repf}(G)$ is semisimple.

To prove this, we have to show that the category $\mathsf{Repf}(G)$ is semisimple if and only if $\mathsf{Repf}(G^\circ)$ is semisimple. We may suppose that G is algebraic. As G° is a normal algebraic subgroup of G, the necessity follows from 6.20. For the sufficiency, let V be a representation of G. Replace G with its image in GL_V. Let W be a G-stable subspace of V. By assumption, there is a G°-equivariant map $p: V \to W$ such that $p|W = \mathrm{id}$. Define

$$q: \bar{k} \otimes V \to \bar{k} \otimes W, \quad q = \frac{1}{n} \sum_g g_W p g_V^{-1},$$

where $n = (G(\bar{k}): G^\circ(\bar{k}))$ and g runs over a set of coset representatives for $G^\circ(\bar{k})$ in $G(\bar{k})$. One checks easily that q has the following properties:

(a) it is independent of the choice of the coset representatives;

(b) for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, $\sigma(q) = q$;

(c) for all $y \in \bar{k} \otimes W$, $q(y) = q$;

(d) for all $g \in G(\bar{k})$, $g_W \cdot q = q \cdot g_V$.

Thus, q is defined over k, restricts to the identity map on W, and is G-equivariant.

REMARK 6.22 An algebraic group G is said to be ***linearly reductive*** if $\mathsf{Repf}(G)$ semisimple. Thus, in characteristic zero, G is linearly reductive if and only if G° is reductive. An algebraic group G over a field of characteristic $p \neq 0$ is linearly reductive if and only if G° is of multiplicative type and p does not divide the index $(G: G^\circ)$. This was proved by Nagata in 1961 for smooth algebraic groups, and is often referred to as Nagata's theorem. See Demazure and Gabriel 1970, IV, §3, 3.6.

Tannakian categories with the Chevalley property

A tensor category over k is said to have the ***Chevalley property*** if the tensor product of any two semisimple objects is semisimple, or, in other words, if the subcategory of semisimple objects is a tensor subcategory. Chevalley's theorem says that the category of finite-dimensional representations of an abstract group over a field of characteristic zero has the Chevalley property (Milne 2017, 22.45). We can extend this to nonneutral tannakian categories.

THEOREM 6.23 *Let T be a tannakian category over a field k of characteristic zero. Then T has the Chevalley property.*

PROOF We may suppose that T is algebraic. Then there exists a finite extension k' of k such that $\mathsf{T}_{(k')}$ is a neutral tannakian category (IV, 10.1). Let V and W be semisimple objects in T, and let V' and W' be their images in $\mathsf{T}_{(k')}$. Then V' and W' are semisimple (I, 6.14(a)), and so $V' \otimes W'$ is semisimple (as noted above). As $(V \otimes W)' \simeq V' \otimes W'$, it follows that $V \otimes W$ is semisimple (I, 6.14(c)). □

In nonzero characteristic, there is only the following theorem of Deligne and Serre.

THEOREM 6.24 *Let T be a tannakian category over a field of characteristic $p \neq 0$, and let $V_1, \dots, V_m$ be objects of T. If the V_i are semisimple and $\sum_{i=1}^m (\dim V_i - 1) < p$, then $V_1 \otimes \cdots \otimes V_m$ is semisimple.*

PROOF As for the preceding theorem, it suffices to prove this for $\mathsf{Repf}(G)$. In that case, it is proved in Serre 1994b when G is smooth and in Deligne 2014 in general. □

The bound $\sum(\dim V_i - 1) < p$ in the theorem is optimal, as the following example shows. Let $G = \mathrm{SL}_2$, and for $d \in \mathbb{N}$, let $V(d)$ denote the k-vector space of homogeneous polynomials of degree d in two symbols, so $\dim_k V(d) = d+1$. There is a canonical action of G on $V(d)$ for which $V(d)$ is simple if $d < p$ and nonsemisimple if $d = p$. Let $d_1, \dots, d_m$ be integers such that $1 \le d_i \le p-1$ and $\sum d_i = p$. Then

$$\sum(\dim V(d_i) - 1) = p,$$

and the homomorphism

$$(f_1, \dots, f_m) \mapsto f_1 \cdots f_m : V(d_1) \otimes \cdots \otimes V(d_m) \to V(p)$$

is surjective. As $V(p)$ is not semisimple, neither is $V(d_1) \otimes \cdots \otimes V(d_m)$.

EXERCISE 6.25 Let k be a field of characteristic p and G a finite group. Show that $\mathsf{Repf}(G)$ has the Chevalley property if and only if G has only one Sylow p-subgroup.

ASIDE 6.26 Let k be an algebraically closed field. A k-linear abelian rigid monoidal category is a ***fusion category*** if it is semisimple, has only finitely many isomorphism classes of simple objects, and $k \simeq \mathrm{End}(\mathbb{1})$. Note that these conditions imply that the category equals $\langle X \rangle$ for some object X. When G is an affine group scheme over k, $\mathsf{Repf}(G)$ is a fusion category if and only if G is a finite discrete group whose order is prime to the characteristic exponent of k (6.22, 6.1). Every symmetric fusion category such that $\dim(X) \in \mathbb{N}$ for all objects X is of this form (10.1).

7 Torsors

Basic definitions

In this subsection, we allow k to be a commutative ring. Unadorned tensor products are over k, and unadorned products are over $\mathrm{Spec}\, k$.

7.1 Let G be an affine group scheme flat[2] over k, and let X be an affine scheme over k. An ***action*** of G on X is a morphism $X \times G \to X$ of k-schemes such that, for every k-algebra R, $X(R) \times G(R) \to X(R)$ is an action of the group $G(R)$ on the set $X(R)$. In other words, the following two diagrams commute,

$$\begin{array}{ccc} X\times G\times G & \xrightarrow{\mu\times G} & X\times G \\ \downarrow{\scriptstyle X\times m} & & \downarrow{\scriptstyle \mu} \\ X\times G & \xrightarrow{\mu} & X \end{array} \qquad\qquad \begin{array}{ccc} X\times \mathrm{Spec}(k) & \xrightarrow{X\times e} & X\times G \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mu} \\ & & X. \end{array}$$

The action is said to be ***simply transitive*** if, for every R and pair of points (x_1, x_2) in $X(R)$, there is a unique g in $G(R)$ such that $gx_1 = x_2$. In other words, for all R, the map

$$(t, g) \mapsto (t, tg) : X(R) \times G(R) \to X(R) \times X(R)$$

is a bijection. This is equivalent to the morphism

$$X \times G \to X \times X$$

being an isomorphism.

[2] Hence faithfully flat, because of the existence of the identity section.

7.2 Let $\mu : X \times G \to X$ be a simply transitive action of G on an affine k-scheme X. We say that (X, μ) is a ***torsor under*** G ***over*** k (for the fpqc topology) if $X(R) \neq \emptyset$ for some faithfully flat k-algebra R. For example, G acting on itself by right translation is a torsor under G over k (this is the ***trivial torsor***). There is an obvious notion of a morphism of torsors under G over k.

7.3 Let (X, μ) be a torsor under G over k. By assumption, there exists a $P \in X(R)$ for some R faithfully flat over k. For all R-algebras R',

$$g \mapsto Pg : G(R') \to X(R')$$

is a bijection compatible with the actions of $G(R')$, and so $X_R \simeq G_R$ as G_R-torsors. In other words, (X, μ) is locally trivial for the fpqc topology on $\operatorname{Spec} k$. Note that X is faithfully flat over k because it becomes faithfully flat over the faithfully flat k-algebra R.

We sometimes write "G-torsor over k" for "torsor under G over k".

7.4 A ***sheaf-torsor under*** G ***over*** k is a sheaf F for the fpqc topology on $\operatorname{Spec}(k)$ together with an action of G such that, locally, F is G-equivariantly isomorphic to G. When regarded as a sheaf, a torsor under G over k becomes a sheaf-torsor under G over k. In this way, we get an equivalence from the category of torsors under G over k to the category of sheaf-torsors under G over k. That the functor is essentially surjective follows from descent theory: let F be a sheaf-torsor on $\operatorname{Spec}(k)$; by assumption, there exists a faithfully flat k-algebra R such that the restriction of F to $\operatorname{Spec} R$ is representable, and this implies that F is representable over $\operatorname{Spec} k$ (V, 1.9).

7.5 Let G be an affine group scheme over k, and suppose that G acts on a sheaf X in such a way that $G(R)$ acts simply transitively on $X(R)$ for all k-algebras X. Then X is torsor under G over k if it satisfies any one of the following equivalent conditions:

(a) for some faithfully flat k-algebra R, $X(R) \neq \emptyset$;

(b) X is represented by an affine scheme faithfully flat over k.

7.6 Let $G \to H$ be a homomorphism of affine group schemes flat over k, and let X be a torsor under G over k. The quotient of the sheaf $X \times H$ by the diagonal action $(x, h)g = (xg, g^{-1}h)$ of G is a sheaf torsor, hence a torsor, under H over k. The torsor is denoted by $X \wedge^G H$, and called the ***contracted product***. If $G \to H$ is faithfully flat, then so also is $X \to X \wedge^G H$.

ASIDE 7.7 The correspondence between affine group schemes over a ring and commutative Hopf algebras over the ring extends to a correspondence between torsors and Hopf-Galois extensions. For noncommutative Hopf algebras this gives a noncommutative notion of torsor. See ncatlab. We leave it as an exercise to the reader to interpret the construction in 7.6 in terms of Hopf algebras.

Projective limits

In this subsection, k is a field. We show that torsors for affine group schemes over algebraically closed fields are trivial.

THEOREM 7.8 *Let X be a torsor under an affine group scheme G over k. Let $G = \varprojlim G_i$, as in 1.21, and let $X_i = X \wedge^G G_i$. For all i, the map $X(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ is surjective; in particular, $X(k^{\mathrm{al}}) \neq \emptyset$.*

We first need a lemma from topology. Let $(X_i)_{i\in I}$, $(\phi_{i,j})_{i\le j}$, be a filtered projective system of topological spaces and continuous maps. If the X_i are non-empty and compact (i.e., quasi-compact and T_2), then, according to a standard theorem, $\varprojlim X_i$ is nonempty. The next lemma shows that we may weaken T_2 to T_1 in this statement provided that we require the transition maps to be closed.

LEMMA 7.9 *Let $(X_i)_{i\in I}$, $(\phi_{i,j})_{i\le j}$ be a filtered projective system of topological spaces and continuous maps. If*

(a) *the X_i are non-empty, quasi-compact, and T_1, and*

(b) *the $\phi_{i,j}$ are closed maps,*

then $\varprojlim X_i$ is nonempty. Furthermore, if, for some fixed i, the maps $\phi_{i,j}: X_j \to X_i$ are surjective for all $j \ge i$, then the map $\varprojlim X_j \to X_i$ is surjective.

PROOF Let $\mathcal{S}$ be the set of families $(A_i)_{i\in I}$ such that A_i is a nonempty closed subset of X_i and $\phi_{i,j}(A_j) \subset A_i$ for all $i, j \in I$ with $j \ge i$. Define a partial ordering on $\mathcal{S}$ by setting $(A_i) \le (B_i)$ if $A_i \subset B_i$ for all $i \in I$. By quasicompactness $(\mathcal{S}, \ge)$ satisfies the hypotheses of Zorn's lemma, and so there exists an element $(A_i)_{i\in I}$ of $\mathcal{S}$ that is minimal with respect to $\le$. By (b), $B_i \overset{\text{def}}{=} \bigcap_{j\ge i} \phi_{i,j}(A_j)$ is closed in A_i, and it is easy to see that $(B_i)_{i\in I} \in \mathcal{S}$. By minimality, $B_i = A_i$ for all i. We have shown that, for a minimal family (A_i), $\phi_{i,j}$ maps A_j onto A_i for all $j \ge i$.

For some fixed i, let $x_i \in A_i$, and define

$$C_j = \begin{cases} \phi_{ij}^{-1}(x_i) \cap A_j & \text{if } j \ge i \\ A_j & \text{otherwise.} \end{cases}$$

The condition T_1 implies that C_j is closed, hence $(C_j)_{j\in I} \in \mathcal{S}$. By minimality, $A_i = C_i = \{x_i\}$. As this is true for all i, we see that $(x_i)_{i\in I} \in \varprojlim X_i$. This proves the first statement, that $\varprojlim X_i \ne \emptyset$, and the second statement follows from the first applied to the projective system with $Y_j = \phi_{ij}^{-1}(x_i)$ for $j \ge i$, where x_i is any element of X_i. □

PROOF (OF THEOREM 7.8) We first remark, that if X is an affine scheme of finite type over k, say, $X = \operatorname{Spec} A$ with A a nonzero finitely generated k-algebra, then $X(k^{\mathrm{al}}) \ne \emptyset$. Indeed, for any maximal ideal $\mathfrak{m}$ of A, $k' \overset{\text{def}}{=} A/\mathfrak{m}$ is a finite extension of k (Zariski's lemma), and $X(k') \ne \emptyset$. Similarly, if $Y \to X$ is a faithfully flat map of affine schemes of finite type over k, then $Y(k^{\mathrm{al}}) \to X(k^{\mathrm{al}})$ is surjective.

In the situation of the theorem, if I contains a countable cofinal subset, then we may suppose that $I = \mathbb{N}$. The maps

$$\cdots \to X_j(k^{\mathrm{al}}) \to \cdots \to X_{i+1}(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$$

are surjective, and so the statement is obvious in this case.

For more general I, we want to apply Lemma 7.9. However, the transition maps $X_j(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ are not closed for the Zariski topology, and so we need to define a new "orbit topology".

Let Y be a torsor under an algebraic group H over k, and consider the collection $\mathcal{C}$ of subsets of $Y(k^{\mathrm{al}})$ that are finite unions of orbits $yH'(k^{\mathrm{al}})$, where y ranges over $Y(k^{\mathrm{al}})$ and H' ranges over the algebraic subgroups of H. These sets are closed for the Zariski topology on $Y(k^{\mathrm{al}})$, which is noetherian, and so any infinite intersection of such subsets is actually a finite intersection. As $y_1H_1(k^{\mathrm{al}}) \cap y_2H_2(k^{\mathrm{al}})$ is either empty or equal to

$z(H_1 \cap H_2)(k^{\mathrm{al}})$ for any element z of the intersection, we see that every finite intersection of sets in $\mathcal{C}$, hence every intersection, lies in $\mathcal{C}$. It follows that the elements of $\mathcal{C}$ are the closed sets of a topology on $X(k^{\mathrm{al}})$ – this is the ***orbit topology***. As every $y \in H(k^{\mathrm{al}})$ is an orbit of the trivial group, the topology is T_1. It is quasi-compact because of the property we proved for infinite intersections.

We now prove Theorem 7.8. The transition maps $\phi_{ij} : X_j(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ are surjective. When we endow each set $X_i(k)$ with its orbit topology, they are continuous because, if H is an algebraic subgroup of G_i and $x \in X_i(k^{\mathrm{al}})$, then $\phi_{ij}^{-1}(xH(k^{\mathrm{al}})) = x'H'(k^{\mathrm{al}})$, where x' is any preimage of x in $X_j(k^{\mathrm{al}})$ and H' is the preimage of H in G_j. They are also closed because, if H is an algebraic subgroup of G_j and $x \in X_j(k^{\mathrm{al}})$, then $\phi_{ji}(xH(k^{\mathrm{al}})) = \phi_{ji}(x)H'(k^{\mathrm{al}})$, where H' is the image of H in G_i (an algebraic subgroup). Thus, the theorem follows from Lemma 7.9. □

COROLLARY 7.10 *Let G be an affine k-group scheme, and write G as a projective limit $\varprojlim G_i$, as in 1.21. For all i, the map $G(k^{\mathrm{al}}) \to G_i(k^{\mathrm{al}})$ is surjective.*

PROOF Apply Theorem 7.8 to $X = G$. □

REMARK 7.11 For any faithfully flat homomorphism $f : G \to H$ of affine group schemes over k, the map $f(k^{\mathrm{al}}) : G(k^{\mathrm{al}}) \to H(k^{\mathrm{al}})$ is surjective (Demazure and Gabriel 1970, III, §3, 7.6).

REMARK 7.12 Let $G \to H$ be a faithfully flat homomorphism of algebraic groups over a field k and $X \to Y$ an equivariant morphism of homogeneous spaces. When $X(k^{\mathrm{al}})$ and $Y(k^{\mathrm{al}})$ are endowed with the orbit topology, the map $X(k^{\mathrm{al}}) \to Y(k^{\mathrm{al}})$ is closed and continuous. Using this, it is possible to prove Theorem 7.8 for homogeneous spaces (Stewart 1975, §2).

NOTES The orbit topology and Lemma 7.9 are used to prove Corollary 7.10 in Hochschild and Mostow 1957. In Stewart 1975 they are used to study projective limits of coset varieties.

8 Classification of the fibre functors

Let C be a neutral tannakian category over k. By definition, there exists a fibre functor ω with values in k and we proved (3.1) that the functor $\mathcal{A}ut^{\otimes}(\omega)$ of k-algebras is represented by an affine group scheme G over k. Moreover, in a natural way, ω takes values in $\mathsf{Repf}(G)$ and defines an equivalence $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$. For any fibre functor η with values in a k-algebra R, composition defines a pairing

$$\mathcal{H}om^{\otimes}(\omega_R, \eta) \times \mathcal{A}ut^{\otimes}(\omega_R) \to \mathcal{H}om^{\otimes}(\omega_R, \eta)$$

of functors of R-algebras. According to I, 4.21, $\mathcal{H}om^{\otimes}(\omega_R, \eta) = \mathcal{I}som^{\otimes}(\omega_R, \eta)$, so $\mathcal{A}ut^{\otimes}(\omega)(R')$ acts simply transitively on $\mathcal{H}om^{\otimes}(\omega, \eta)(R')$ for all R-algebras R'. To prove that $\mathcal{H}om^{\otimes}(\omega, \eta)$ is a G-torsor, it remains to show that it is representable by an affine scheme over k (7.5).

THEOREM 8.1 *Let C be a neutral tannakian category over k, let ω be a k-valued fibre functor, and let $G = \mathcal{A}ut^{\otimes}(\omega)$.*

(a) *For any fibre functor η on C with values in R, $\mathcal{H}om^{\otimes}(\omega_R, \eta)$ is representable by an affine scheme faithfully flat over* Spec R*; it is therefore a G-torsor.*

(b) *The functor $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega_R, \eta)$ determines an equivalence between the category of fibre functors on C with values in R and the category of G-torsors over R.*

PROOF Let $X \in \mathrm{ob}(\mathsf{C})$, and, with the notation of the proof of 3.1, let

$$\begin{cases} A_X \subset \mathrm{End}(\omega X), & A_X = \bigcap_Y (\omega Y : \omega Y), \quad Y \subset X^n, \quad n \geq 1 \\ P_X \subset \mathrm{End}(\omega X, X), & P_X = \bigcap_Y (Y : \omega Y), \quad Y \subset X^n, \quad n \geq 1. \end{cases}$$

Then $\omega(P_X) = A_X$ and $P_X \in \mathrm{ob}(\langle X \rangle)$. For any R-algebra R', $\mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R')$ is the subspace of $\mathrm{Hom}(\omega(P_X) \otimes_k R', \eta(P_X) \otimes_R R')$ of maps respecting all $Y \subset X^n$, some n. It therefore equals $\eta(P_X) \otimes R'$. Thus

$$\mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \xrightarrow{\simeq} \mathrm{Hom}_{R\text{-linear}}(\eta(P_X^\vee), R').$$

Let Q be the ind-object $(P_X^\vee)_X$, and let $B = \varinjlim A_X^\vee$. We saw in 3.28, that the tensor structure on C defines an algebra structure on B. Because ω is an equivalence $\mathsf{C} \to \mathsf{Repf}(G)$, it also defines a ring structure on Q, i.e., a map $Q \otimes Q \to Q$ in $\mathrm{Ind}(\mathsf{C})$ making $\omega(Q) \to B$ into an isomorphism of k-algebras. We have

$$\begin{aligned} \mathcal{H}om(\omega, \eta)(R') &= \varprojlim \mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \\ &= \varprojlim \mathrm{Hom}_{R\text{-linear}}(\eta(P_X^\vee), R') \\ &= \mathrm{Hom}_{R\text{-linear}}(\eta(Q), R), \end{aligned}$$

where $\eta(Q) \overset{\text{def}}{=} \varinjlim \eta(P_X^\vee)$. Under this correspondence,

$$\mathcal{H}om^{\otimes}(\omega, \eta)(R') = \mathrm{Hom}_{R\text{-algebra}}(\eta(Q), R'),$$

and so $\mathcal{H}om^{\otimes}(\omega, \eta)$ is represnted by $\eta(Q)$. We know (I, 7.5) that $\eta(P_X^\vee)$ is a projective R-module, in particular flat, and so $\eta(Q) = \varinjlim \eta(P_X^\vee)$ is flat over R. For each X, there is an epimorphism $P_X \twoheadrightarrow \mathbb{1}$, and the exact sequence

$$0 \to \mathbb{1} \to P_X^\vee \to P_X^\vee/\mathbb{1} \to 0$$

gives rise to an exact sequence

$$0 \to \eta(\mathbb{1}) \to \eta(P_X^\vee) \to \eta(P_X^\vee/\mathbb{1}) \to 0.$$

As $\eta(\mathbb{1}) = R$ and $\eta(P_X^\vee/\mathbb{1})$ is flat, this shows that $\eta(P_X^\vee)$ is a faithfully flat R-module. Hence $\eta(Q)$ is faithfully flat over R, which completes the proof that $\mathcal{H}om^{\otimes}(\omega, \eta)$ is a G-torsor.

To show that $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega, \eta)$ is an equivalence, we construct a quasi-inverse. Let T be a G-torsor over R. For a fixed X, define $R' \rightsquigarrow \eta_T(X)(R')$ to be the sheaf associated with

$$R' \rightsquigarrow T(R') \wedge^{G(R')} \left(\omega(X) \otimes R'\right).$$

Then $X \rightsquigarrow \eta_T(X)$ is a fibre functor on C with values in R. □

COROLLARY 8.2 *Any two fibre functors on a neutral tannakian category over k are locally isomorphic for the fpqc topology.*

PROOF Let η be an R-valued fibre functor and ω a k-valued fibre functor. Then $\mathcal{H}om^{\otimes}(\omega_R, \eta)$ is a torsor over $\operatorname{Spec} R$ for the fpqc topology, and so becomes trivial over some faithfully flat R-algebra R'. This means that ω and η become isomorphic over R'. If η_1 and η_2 are fibre functors with values in R_1 and R_2 respectively, then they become isomorphic to ω over some common ring faithfully flat over R_1 and R_2. □

COROLLARY 8.3 *Let* C *be a neutral tannakian category over* k*. Any two* k*-valued fibre functors* ω *and* η *of* C *become isomorphic over* k^{al} *(and over a finite extension of* k *if* C *is algebraic).*

PROOF Let ω and η be k-valued fibre functors on C. If C is algebraic, then $\mathcal{H}om(\omega, \eta)$ is represented by a scheme $X = \operatorname{Spec} A$, where A is a nonzero finitely generated k-algebra. For any maximal ideal of A, $A/\mathfrak{m}$ is a finite extension of k (Zariski's lemma), and $X(A/\mathfrak{m}) \neq \emptyset$.

In the general case, $\mathcal{H}om^{\otimes}(\omega, \eta)$ is represented by a torsor under the affine group scheme $\mathcal{A}ut^{\otimes}(\omega)$ over k, and so has a k^{al}-point by Theorem 7.8. □

A ***nonassociative algebra***[3] in a symmetric monoidal category is a pair (X, t) consisting of an object X and a morphism $t : X \otimes X \to X$ (no conditions). Let $A = (V, t)$ be a nonassociative algebra in $\mathsf{Vecf}(k)$. The functor of commutative k-algebras

$$R \rightsquigarrow \operatorname{Aut}(A \otimes R) \quad \text{(automorphisms of } R\text{-algebras)}$$

is represented by an algebraic subgroup of GL_V, denoted $\mathcal{A}ut(A)$ or $\mathcal{A}ut(V, t)$.

COROLLARY 8.4 *Let* $(\mathsf{C}, \otimes)$ *be a neutral algebraic tannakian category over* k*. There exists a nonassociative algebra* (X, t) *in* C *such that, for any fibre functor over an extension* k' *of* k*,*

$$\mathcal{A}ut^{\otimes}(\omega)_{k'} = \mathcal{A}ut(\omega(X), \omega(t)).$$

PROOF As C is neutral, there exists a k-valued fibre functor ω_0, and ω_0 defines an equivalence of symmetric monoidal categories $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$, where $G = \mathcal{A}ut^{\otimes}(\omega_0)$. According to Milne 2020a, Theorem 1, $G = \mathcal{A}ut(A)$ for some nonassociative algebra $A = (V, t_V)$ in $\mathsf{Repf}(G)$. There exists a nonassociative algebra (X, t) in C and an isomorphism $\omega_0(X, t) \simeq (V, t_V)$ (unique up to a unique isomorphism). For any fibre functor ω with values in an extension k' of k,

$$\mathcal{A}ut^{\otimes}(\omega)_{k'} \subset \mathcal{A}ut(\omega(X), \omega(t)),$$

but ω becomes isomorphic to ω_0 over some extension of k', and so the inclusion is an equality. □

REMARK 8.5 Let (X, t) be as in 8.4, and let $A = \omega(X, t)$ for some k-valued fibre functor ω. For any k-algebra R, the functor $\eta \rightsquigarrow \eta(X, t)$ is an equivalence from the category of fibre functors over R to the category of forms of A over R (i.e., nonassociative algebras over R that become isomorphic to A over some faithfully flat R-algebra).

EXAMPLE 8.6 Let V and V' be vector spaces of the same dimension, each equipped with a nondegenerate quadratic form. There is a canonical equivalence between $\mathsf{Repf}(O(V))$ and $\mathsf{Repf}(O(V'))$ given by the $O(V)$-torsor of isomorphisms $V \to V'$.

[3] Of course, this is shorthand for "possibly nonassociative algebra".

By using Theorem 12.20 of Chapter I, we can extend Corollary 8.2 to nonneutral tannakian categories.

COROLLARY 8.7 *Any two fibre functors on a tannakian category are locally isomorphic for the fpqc topology.*

PROOF Let T be a tannakian category over k, and let ω_1 and ω_2 be R_1- and R_2- valued fibre functors. Then T has a fibre functor ω over a field K (I, 7.7), and $\mathsf{T}_{(K)}$ is a neutral tannakian category (I, 12.21) with fibre functor ω',

$$(X,\mu) \rightsquigarrow \omega(X)\otimes_{K\otimes_k K} K.$$

The functor ω_1 defines an $R_1\otimes_k K$-valued fibre functor ω_1' on $\mathsf{T}_{(K)}$,

$$(X,\mu) \rightsquigarrow \omega_1(X),$$

and similarly for ω_2. According to 8.2, there exists a K-algebra R, faithfully flat over both $R_1\otimes_k K$ and $R_2\otimes_k K$ such that $\omega_1'\otimes_{R_1\otimes K} R \approx \omega_2'\otimes_{R_2\otimes K} R$. Such an isomorphism defines an isomorphism $\omega_1\otimes_{R_1} R \to \omega_2\otimes_{R_2} R$. □

EXERCISE 8.8 Rewrite the proof of Theorem 8.1 in terms of Hopf–Galois extensions.

NOTES

8.9 Let G be an affine group scheme over k. Let $\mathsf{C} = \mathsf{Repf}(G)$, and let ω be a fibre functor on C. If ω is the forgetful functor, then $G \simeq \mathcal{A}ut^{\otimes}(\omega)$, but otherwise $\mathcal{A}ut^{\otimes}(\omega)$ is the inner twist of G by the G-torsor $\mathcal{H}om(\omega_{\text{forget}}, \omega)$. Thus, except when k is algebraically closed, C only determines G up to an inner twist (i.e., it determines the band of G).

8.10 We noted in 8.9 that, over an algebraically closed field, an affine group scheme G is determined up to isomorphism by the pair $(\mathsf{Repf}(G),\otimes)$. Without $\otimes$, we have only the following results.

Let G be a connected reductive group over an algebraically closed field of characteristic zero. Then G is determined up to isomorphism by the set of isomorphism classes of its finite-dimensional semisimple representations endowed with the obvious sum and product (i.e., by the Grothendieck semiring of $\mathsf{Repf}(G)$). See, for example, Kazhdan et al. 2014.

A connected split reductive group G over a field of characteristic 0 is determined up to isomorphism by specifying a maximal torus T of G, the set of isomorphism classes of simple representations of G, and the character homomorphism from the Grothendieck ring of G to that of T. More precisely, it is possible to determine all isomorphisms compatible with the specified data.

9 Examples

Graded vector spaces

9.1 Let C be the category whose objects are the families $(V^n)_{n\in\mathbb{Z}}$ of vector spaces over k with finite-dimensional direct sum $V = \bigoplus V^n$. There is an obvious rigid symmetric monoidal structure on C for which $\mathrm{End}(\mathbb{1}) = k$ and $\omega : (V^n) \rightsquigarrow \bigoplus V^n$ is a fibre functor. Thus, according to Theorem 3.1, ω defines an equivalence of symmetric monoidal

categories $\mathsf{C} \xrightarrow{\approx} \mathsf{Repf}(G)$ for some affine k-group scheme G. This equivalence is easy to describe: let $G = \mathbb{G}_m$ and send (V^n) to the representation of $\mathbb{G}_m$ on $V = \bigoplus V^n$ for which $\mathbb{G}_m$ acts on V^n through the character $\lambda \mapsto \lambda^n$.

Gradations on tannakian categories

9.2 Let M be a set. An M-***gradation*** on an object X of an additive category is a decomposition $X = \bigoplus_{m\in M} X^m$. An M-***gradation*** on an additive functor $u : \mathsf{C} \to \mathsf{C}'$ is an M-gradation on each $u(X)$, $X \in \mathrm{ob}(\mathsf{C})$, depending functorially on X.

Suppose now that M is an abelian group, and let D be the diagonalizable group over k whose character group is M. For example, if $M = \mathbb{Z}$, then $D = \mathbb{G}_m$, and if $M = \mathbb{Z}/n\mathbb{Z}$, then $D = \mu_n$.

An M-***gradation*** on a tannakian category C over k can be variously described as follows:

(a) an M-gradation, $X = \bigoplus X^m$, on each object X of C that depends functorially on X and is compatible with tensor products in the sense that

$$(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes Y^s;$$

(b) an M-gradation on the identity functor id_{C} of C that is compatible with tensor products;

(c) a homomorphism $D \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$;

(d) a central homomorphism $D \to G$, where $G = \mathcal{A}ut^{\otimes}(\omega)$, for one (or every) fibre functor ω.

Definition (a) is simply a restatement of (b). By a central homomorphism in (d), we mean a homomorphism from D into the centre of G defined over k. Although G need not be defined over k, its centre is, and equals $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$, from which the equivalence of (c) and (d) follows. Finally, a homomorphism $w : D \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ defines a gradation $X = \bigoplus X^m$ on every $X \in \mathrm{ob}\,\mathsf{C}$: let X^m be the subobject on which $w(d)$ acts as $m(d) \in k$.

Note that, when regarded as a map into the centre of $\mathcal{A}ut^{\otimes}(\omega)$, the homomorphism in (d) is independent of ω (in fact, it is equal to the homomorphism in (c)).

NOTES The results in this subsection are from Saavedra 1972, IV, 1.1.

Representations of groups of multiplicative type

9.3 Let $\bar{k}$ be a separable algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. An algebraic group G over k is of ***multiplicative type*** if every representation of G becomes diagonalizable over $\bar{k}$. In characteristic zero, this is equivalent to G being commutative and its identity component being a torus. The character group $X^*(G) \overset{\text{def}}{=} \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_m)$ of a group G of multiplicative type is a finitely generated abelian group on which Γ acts continuously. Let $M = X^*(G)$, and let $k' \subset \bar{k}$ be a Galois extension of k over which all elements of M are defined. For any finite-dimensional representation V of G, we have a decomposition

$$V \otimes_k k' = \bigoplus_{m\in M} V_m, \quad V_m \overset{\text{def}}{=} \{v \in V \otimes_k k' \mid gv = m(g)v \text{ all } g \in G(k)\}.$$

A finite-dimensional vector space V over k together with a decomposition

$$k' \otimes V = \bigoplus_{m\in M} V_m$$

arises from a representation of G if and only if $V^{\sigma(m)} = \sigma V^m$ for all $m \in M$ and $\sigma \in \Gamma$. Thus, an object of $\mathsf{Repf}(G)$ can be identified with a finite-dimensional vector space V over k together with an M-gradation on $V \otimes k'$ compatible with the action of Γ. See III, 5.26 below.

Filtrations of $\mathsf{Repf}(G)$

Let V be a vector space. A homomorphism $\lambda : \mathbb{G}_m \to \mathrm{GL}_V$ defines a filtration

$$\cdots \supset F^nV \supset F^{n+1}V \supset \cdots, \quad F^nV = \bigoplus\nolimits_{i \geq n} V_i$$

of V, where $V = \bigoplus_i V_i$ is the gradation defined by λ.

Let G be an algebraic group over a field k. A homomorphism $\lambda : \mathbb{G}_m \to G$ defines a filtration $F^\bullet$ on V for every representation (V, r) of G, namely, that corresponding to $r \circ \lambda$. These filtrations are compatible with the formation of tensor products and duals, and they are exact in the sense that the functor $V \rightsquigarrow \mathrm{Gr}^\bullet(V)$ is exact. A functor $(V, r) \rightsquigarrow (V, F^\bullet)$ from representations of G to filtered vector spaces satisfying these conditions is called a ***filtration*** $F^\bullet$ of $\mathsf{Repf}(G)$, and a homomorphism $\lambda : \mathbb{G}_m \to G$ defining $F^\bullet$ is said to ***split*** $F^\bullet$. We write $\mathrm{Filt}(\lambda)$ for the filtration defined by λ.

Define F^0G to be the algebraic subgroup of G respecting the filtration on each representation of G, and, for $n \geq 1$, define F^nG to be the algebraic subgroup of F^0G acting trivially on the graded module $\bigoplus_i F^iV/F^{i+n}V$ attached to each representation V of G. Clearly, F^nG is unipotent for $n \geq 1$.

THEOREM 9.4 *Let G be a reductive group over a field k, and let $F^\bullet$ be a filtration of $\mathsf{Repf}(G)$. From the adjoint action of G on $\mathfrak{g}$, we acquire a filtration of $\mathfrak{g}$.*

(a) *There exists a cocharacter λ of G splitting the filtration $F^\bullet$.*

(b) *F^0G is a parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$.*

(c) *F^1G is the unipotent radical of F^0G, and $\mathrm{Lie}(F^1G) = F^1\mathfrak{g}$.*

(d) *The centralizer $Z(\lambda)$ of any cocharacter λ splitting $F^\bullet$ is a connected algebraic subgroup of F^0G such that the quotient map $q : F^0G \to F^0G/F^1G$ induces an isomorphism $Z(\lambda) \to F^0G/F^1G$, so*

$$F^0G = F^1G \rtimes Z(\lambda),$$

and the composite $q \circ \lambda$ of λ with q is central.

(e) *Two cocharacters λ and λ' of G define the same filtration of G if and only if they define the same group F^0G and $q \circ \lambda = q \circ \lambda'$; the cocharacters λ and λ' are then conjugate under F^1G.*

PROOF Choose a faithful representation V of G, and let P be the algebraic subgroup of G preserving the filtration on V. Then P is obviously parabolic, and so $P = P(\lambda)$ for some cocharacter λ of G, i.e., P is the unique smooth algebraic subgroup of G such that

$$P(k^{\mathrm{al}}) = \left\{ g \in G(k^{\mathrm{al}}) \;\middle|\; \lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} \text{ exists in } G(k^{\mathrm{al}}) \right\}$$

see Milne 2017, 25.1. Now λ splits the filtration, and so (a), (b), and (c) follow from ibid., 13.3, 25.6. □

REMARK 9.5 It sometimes more convenient to work with ascending filtrations. To turn a descending filtration $F^\bullet$ into an ascending filtration $W_\bullet$, set $W_i = F^{-i}$; if μ splits $F^\bullet$, then $z \mapsto \mu(z)^{-1}$ splits $W_\bullet$. With this terminology, we have $W_0G = W_{-1}G \rtimes Z(\mu)$.

NOTES See Saavedra 1972, especially IV, 2.2.5.

Filtered fibre functors

Let T be a tannakian category (not necessarily neutral) over a field k, and let R be a commutative k-algebra.

A ***graded R-module*** is an R-module M together with a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M^n$. With the obvious tensor structure, the graded R-modules become an R-linear abelian symmetric monoidal category. A graded R-module admits a dual if and only if it is finitely generated and projective. To give a gradation on an R-module M is the same as giving a representation of $\mathbb{G}_m$ on M.

An R-valued ***graded fibre functor*** on T is an exact k-linear $\otimes$-functor from T to the category of graded R-modules. It takes values in the finitely generated projective graded R-modules.

A ***filtered R-module*** is an R-module M together with a family $(F^nM)_{n\in\mathbb{Z}}$ of submodules

$$\cdots \supset F^nM \supset F^{n+1}M \supset \cdots$$

such that

$$\bigcup_{n\in\mathbb{Z}} F^nM = M, \qquad \bigcap_{n\in\mathbb{Z}} F^nM = 0.$$

A ***morphism*** $(M,(F^nM)) \to (N,(F^nN))$ of filtered R-modules is an R-linear map $f: M \to N$ such that $f(F^nM) \subset F^nN$ for all n. With the obvious tensor structure, the filtered R-modules become an R-linear symmetric monoidal category. A filtered R-module $(M,(F^nM))$ admits a dual if and only if M is finitely generated and projective and the submodules F^nM are direct summands of M locally for the Zariski topology on $\operatorname{Spec} R$. There is a $\otimes$-functor

$$M \rightsquigarrow \operatorname{Gr}^\bullet(M) \stackrel{\text{def}}{=} \bigoplus\nolimits_n F^nM/F^{n+1}M$$

from filtered R-modules to graded R-modules.

An R-valued ***filtered fibre functor*** on T is a k-linear $\otimes$-functor ω from T to the category of filtered R-modules such that the functor $X \rightsquigarrow \operatorname{Gr}^\bullet(\omega(X))$ is exact. For example, if ω is an R-valued graded fibre functor on T, then

$$X \rightsquigarrow (M,(F^nM)), \quad M \stackrel{\text{def}}{=} \bigoplus\nolimits_{i\in\mathbb{Z}} \omega(X)^i, \quad F^nM \stackrel{\text{def}}{=} \bigoplus\nolimits_{i\geq n} \omega(X)^i,$$

is a filtered fibre functor on T. Filtered fibre functors of this form are said to be ***splittable***.

In the last subsection, we defined a filtration on $\mathsf{Repf}(G)$ to be a filtration on its forgetful functor, and we saw that such a filtration is splittable if G is reductive.

THEOREM 9.6 *Let ω be a filtered fibre functor on T with values in a k-algebra R.*

(a) *There exists a faithfully flat R-algebra R' such that $\omega \otimes_R R'$ is splittable.*

(b) *Let ω' be the composite of ω with the forgetful functor to $\mathsf{Mod}(R)$. If the affine group scheme over R representing $\mathcal{A}ut^\otimes(\omega')$ is pro-smooth (i.e., a projective limit of smooth algebraic groups), then ω is splittable.*

PROOF See Ziegler 2015, Theorems 1.2 and 1.3. □

NOTES Saavedra (1972, IV, 2.2.1) states 9.6(a) as an open problem. In ibid., IV, 2.4, he gives proofs (due to Deligne) of the theorem under various additional hypotheses, for example, if T is neutral and k has characteristic zero.

Real envelopes; compact groups

9.7 Let K be a topological group. The category $\mathsf{Repf}_{\mathbb{R}}(K)$ of continuous representations of K on finite-dimensional $\mathbb{R}$-vector spaces is, in a natural way, a neutral tannakian category with the forgetful functor as an $\mathbb{R}$-valued fibre functor. There is therefore an affine group scheme $\widetilde{K}$ over $\mathbb{R}$, called the ***real envelope*** of K, and an equivalence of categories $\mathsf{Repf}_{\mathbb{R}}(\widetilde{K}) \xrightarrow{\approx} \mathsf{Repf}_{\mathbb{R}}(K)$ compatible with the forgetful fibre functors. This equivalence arises from a homomorphism $K \to \widetilde{K}(\mathbb{R})$. When K is compact, $K = \widetilde{K}(\mathbb{R})$ and $\widetilde{K}$ is of finite type if and only if K is a Lie group. See Serre 1993, §5.

An algebraic group G over $\mathbb{R}$ is said to be ***compact*** if $G(\mathbb{R})$ is compact and the canonical functor $\mathsf{Repf}_{\mathbb{R}}(G) \longrightarrow \mathsf{Repf}_{\mathbb{R}}(G(\mathbb{R}))$ is an equivalence. The second condition is equivalent to each connected component of $G(\mathbb{C})$ containing a real point (or to $G(\mathbb{R})$ being Zariski dense in G).

ASIDE 9.8 For the original Tannaka duality, see Tannaka 1938. For a contemporary review (by van der Waerden), see zbMATH 0020.00904. In modern language:

> Let G be a topological group and let $\Pi(G)$ be the symmetric monoidal category of representations of G on finite-dimensional $\mathbb{C}$-vector spaces. Let $F: \Pi(G) \to \mathsf{Vecf}(\mathbb{C})$ be the forgetful functor. Endow the collection of natural transformations $\alpha: F \to F$ with the coarsest topology such that the projections $\alpha \mapsto \alpha_V: \mathrm{End}(F) \to \mathrm{End}(V)$ are continuous. We say that a natural transformation α is tensor-preserving if it is the identity map on the trivial representation and the equality $\alpha_{V\otimes W} = \alpha_V \otimes \alpha_W$ holds, and that it is self-conjugate if it equals its complex conjugate. The set $\mathcal{T}(G)$ of all tensor-preserving self-conjugate natural transformations of F is a compact subgroup of $\mathrm{End}(F)$. If G has sufficiently many almost-periodic functions, then every element of G defines an element of $\mathcal{T}(G)$ and Tannaka shows that the embedding $G \to \mathcal{T}(G)$ has dense image. When G is compact, this map is an isomorphism of topological groups. The main tool in the proof is an examination of the prime ideals in the ring of almost-periodic functions on G.

For a historical study of Tannaka's 1938 paper, see Kümmerle 2024.

Representations of Lie algebras

9.9 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. The category $\mathsf{Repf}(\mathfrak{g})$ of representations of $\mathfrak{g}$ on finite-dimensional k-vector spaces is a tannakian category over k with the forgetful functor ω as a k-valued fibre functor (apply 4.1). We examine this category in two cases.

(a) Let $\mathfrak{g}$ be one-dimensional and assume that k is algebraically closed. The affine group scheme $\mathcal{A}ut_k^{\otimes}(\omega)$ is $D(M) \times \mathbb{G}_a$, where $D(M)$ is the diagonalizable group scheme with character group $M = (k, +)$, i.e., k viewed as an abelian group under addition. This can be deduced from Iwahori 1954.

(b) Let $\mathfrak{g}$ be semisimple, and assume that k has characteristic zero. Then $\mathcal{A}ut_k^{\otimes}(\omega)$ is the simply connected semisimple algebraic group over k with Lie algebra $\mathfrak{g}$ (Cartier 1956; Milne 2007b).

9.10 Recall that a diagram is a root system (V, R) together with a lattice X such that $Q(R) \subset X \subset P(R)$. It is possible to attach a tannakian category to a diagram (V, R, X) without mentioning algebraic groups. Indeed, let $\mathfrak{g}$ be a split Lie algebra over a field k of characteristic zero with root system (V, R), and let $\mathsf{Repf}(\mathfrak{g})^X$ be the full subcategory of $\mathsf{Repf}(\mathfrak{g})$ of representations whose simple components have highest weight in X. This is a tannakian category over k, and the group attached to the forgetful fibre functor is a split semisimple algebraic group over k with diagram (V, R, X). See Milne 2007b, 8.3.

Nori's (true) fundamental group

9.11 Let S be a scheme over a field k. By a ***vector sheaf*** on S, we mean a locally free sheaf of finite rank (equivalently, a vector bundle). A vector sheaf E on S is ***finite*** if there exist polynomials $g, h \in \mathbb{N}[t]$, $g \neq h$, such that $g(E) \approx h(E)$. For example, an invertible L sheaf on S is finite if and only of $L^{\otimes m} \approx \mathcal{O}_S$ for some m. With the obvious structures, the finite vector sheaves form k-linear rigid symmetric monoidal category with $\mathcal{O}_S$ as the unit object, but it is not necessarily abelian.

Define a vector sheaf E on a curve to be ***semi-stable*** if for every vector subsheaf $E' \subset E$,

$$\frac{\deg(E')}{\operatorname{rank}(E')} \leq \frac{\deg(E)}{\operatorname{rank}(E)} \stackrel{\text{def}}{=} \mu(E).$$

Let S be a complete connected reduced scheme over a field k. Following Nori 1976, we say that a vector sheaf on S is ***semi-stable*** if for every nonconstant morphism $f : C \to S$ with C a complete connected normal curve, f^*E is semi-stable with slope $\mu(f^*E) = 0$. Let $\mathsf{C}(S/k)$ denote the category of semi-stable vector sheaves on S that are subquotients of finite vector sheaves. If S has a k-rational point s, then $\mathsf{C}(S/k)$ is a tannakian category over k (in particular, abelian) with a k-valued fibre functor $\omega_s : E \rightsquigarrow E_s$. The affine group scheme attached to (C, ω_s) is called the ***fundamental group scheme*** $\pi_1^N(S, s)$ of S. It is a projective limit of finite group schemes over k. Note that $\mathsf{C}(S/k)$ also has a tautologous fibre functor η over S such that $\eta_s = \omega_s$.

Fix an $s \in S(k)$, and consider the triples (G, P, p), where G is a finite group scheme over k, P is a torsor under G over S, and $p \in P(k)$ maps to $s \in S(k)$. They form a category $N(S/k, s)$ in an obvious way. From a homomorphism $\pi_1^N(S, s) \to G$ from $\pi_1^N(S, s)$ to a finite group scheme G over k, we get an exact $\otimes$-functor $\phi : \mathsf{Repf}(G) \to \mathsf{C}(S/k)$, and hence a fibre functor $\eta = \eta \circ \phi$ over S, a fibre functor $\omega = \omega_s \circ \phi$ over k, and an isomorphism $\eta_s \to \omega$. Now $\mathcal{H}om(\eta, \omega_S)$ is a torsor under G over S together with a k-rational point lying over s. In this way, we obtain an equivalence of categories

$$\operatorname{Hom}(\pi_1^N(S, s), -) \to N(S/k, s), \tag{61}$$

where $\operatorname{Hom}(\pi_1^N(S, s), -)$ is the category whose objects are finite group schemes over k equipped with a homomorphism from $\pi_1^N(S, s)$ and whose morphisms are the homomorphisms of k-group schemes compatible with the morphism from $\pi_1^N(S, s)$.

Let $\pi_1^{\text{ét}}(S, s)$ denote the largest pro-étale quotient of $\pi_1^N(S, s)$. When k is algebraically closed, $\pi_1^{\text{ét}}(S, s)(k)$ is the usual étale fundamental group $\pi_1(S, s)$ of S since they classify the same objects. Let k be a perfect field, and let $\bar{S} = S \times \operatorname{Spec}(\bar{k})$, where $\bar{k}$ is an algebraic closure of k. The point $s \in X(k)$ defines a point $\bar{s} \in X(\bar{k})$. Because $\bar{s}$ comes from a k-rational point, $\operatorname{Gal}(\bar{k}/k)$ acts continuously on the étale fundamental group $\pi_1(\bar{S}, \bar{s})$. The associated pro-étale group scheme over k is $\pi_1^{\text{ét}}(\bar{S}, \bar{s})$. For example, both groups are trivial if $S = \operatorname{Spec}(k)$.

See Nori 1976 for the original account and Szamuely 2009, 6.7, for a more recent account. There is, by now, a large literature on the Nori fundamental group scheme and various extensions of it.

The Galois theory of linear differential equations.

A ***differential field*** is a field K, which we shall always assume to have characteristic zero, equipped with derivation, i.e., an additive map $\partial : K \to K$ such that $\partial(ab) =$

$\partial(a)b + a\partial(b)$ for all $a, b \in K$. We sometimes write a' for ∂a. For example, $(\mathbb{C}(T), \frac{d}{dT})$, is a differential field.

Let (K, ∂) be a differential field. A ***differential module*** (V, ∇) is a finite-dimensional K-vector space V together with an additive map ∇ such that $\nabla(fm) = f'm + f\nabla m$ for all $f \in K$, $m \in V$. The choice of a basis for V gives rise to a ***matrix differential equation***

$$y' = Ay, \quad A \in M_n(K), \quad y \in K^n.$$

Roughly speaking, the Picard–Vessiot field for a differential module is the differential field extension generated by the solutions of the matrix differential equation. The ***differential Galois group*** is then the group of differential k-algebra automorphisms of the Picard–Vessiot field. It has a natural structure of an algebraic group. The classical theory provides a Galois correspondence between the algebraic subgroups of the differential Galois group and the differential subfields of the Picard–Vessiot ring. The basic problem in the theory is how to compute the differential Galois group of a given differential module.

There is a tannakian interpretation of the above theory, which provides new insights, and which we now briefly describe.

Let (K, ∂) be a differential field. The kernel of ∂ is a subfield k of K, called the ***constant field***. Let (V, ∇) be a differential module over (K, ∂). The subset $V^\nabla \stackrel{\text{def}}{=} \text{Ker}(\nabla)$ is k-subspace, whose elements are called the ***horizontal vectors.*** Given an extension $(L, \partial) \subset (K, \partial)$ of differential fields, define (V_L, ∇_L) to be the differential module over (L, ∂) with $V_L = V \otimes_K L$ and such that, relative to a basis for V, ∇_L gives rise to a differential equation with the same matrix A as ∇.

The differential modules over (K, ∂) form a tensor category over k. For example, the tensor product of (V_1, ∇_1) and (V_2, ∇_2) is $(V_1 \otimes V_2, \nabla_1 \otimes \nabla_2)$, where

$$(\nabla_1 \otimes \nabla_2)(v_1 \otimes v_2) = \nabla_1(v_1) \otimes v_2 + v_1 \otimes \nabla_2(v_2),$$

and the dual of (V, ∇) is $(V^\vee, \nabla^\vee)$, where

$$\nabla^\vee(\phi)(v) = \partial(\phi(v)) - \phi(\nabla(v)).$$

The forgetful functor $(V, \nabla) \rightsquigarrow V$ is a K-valued fibre functor, and so the category is even tannakian.

DEFINITION 9.12 Let (V, ∇) be a differential module over a differential field (K, ∂). A differential field extension $(L, \partial) \supset (K, \partial)$ is a ***Picard–Vessiot extension*** for (V, ∇) if it has the following properties:

(a) (L, ∂) has the same constant field k as (K, ∂);

(b) the k-subspace $V_L^{\nabla_L}$ of horizontal vectors in V_L spans V_L as an L-vector space;

(c) the coordinates of the horizontal vectors of V_L, relative to an L-basis of V_L coming from a K-basis of V, generate the field L as an extension of K.

Note that if (L, ∂) is a Picard–Vessiot extension for (V, ∇), then the condition (b) holds for all differential modules in the tannakian subcategory $\langle (V, \nabla) \rangle^\otimes$ generated by (V, ∇).

THEOREM 9.13 *Let (V, ∇) be a differential module over a differential field (K, ∂).*

(a) *Let (L,∂) be a Picard–Vessiot extension for (V,∇). The functor*

$$\omega_L : \langle (V,\nabla)\rangle^{\otimes} \to \mathsf{Vecf}(k), \quad (W,\nabla) \rightsquigarrow (W_L)^{\nabla_L},$$

is a fibre functor on $\langle (V,\nabla)\rangle^{\otimes}$, and $\mathcal{A}ut^{\otimes}(\omega_L)$ is canonically isomorphic to the differential Galois group of (V,∇).

(b) *Every k-valued fibre functor on $\langle (V,\nabla)\rangle^{\otimes}$ arises, as in (a), from a Picard–Vessiot extension.*

In particular, when k is algebraically closed, (IV, 10.1) implies that there exists a Picard–Vessiot extension for (V,∇) (unique, up to a nonunique isomorphism).

NOTES For more, see Deligne 1990, §9, van der Put and Singer 2003, and Szamuely 2009, 6.6.

Real Hodge structures

9.14 A ***real Hodge structure*** is a finite-dimensional vector space V over $\mathbb{R}$ together with a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus\nolimits_{p,q} V^{p,q}$$

such that $V^{p,q}$ and $V^{q,p}$ are conjugate complex subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$. There is an obvious rigid symmetric monoidal structure on the category $\mathsf{Hod}_{\mathbb{R}}$ of real Hodge structures, and

$$\omega : (V,(V^{p,q})) \rightsquigarrow V$$

is a fibre functor. The group corresponding to $\mathsf{Hod}_{\mathbb{R}}$ and ω is the real algebraic group $\mathbb{S}$ obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$, that is, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ (the Deligne torus). The real Hodge structure $(V,(V^{p,q}))$ corresponds to the representation $h : \mathbb{S} \to \mathrm{GL}_V$ on V such that an element $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ acts on $V^{p,q}$ as $h(\lambda) = \lambda^{-p}\bar{\lambda}^{-q}$. There is a ***weight gradation*** $V = \bigoplus V_m$, where $V_m \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q}$. It is defined by a character $w_h : \mathbb{G}_m \to G_{\mathbb{C}}$. The functor $(V,(V^{p,q})) \rightsquigarrow (V_m)$ from $\mathsf{Hod}_{\mathbb{R}}$ to the category of graded real vector spaces corresponds to the homomorphism $\mathbb{G}_m \to \mathbb{S}$ that, on real points, is

$$t \mapsto t^{-1} : \mathbb{R}^{\times} \to \mathbb{C}^{\times}.$$

Rational Hodge structures

9.15 A ***rational Hodge structure*** is a $\mathbb{Q}$-vector space V together with a real Hodge structure on $V \otimes \mathbb{R}$ such that the weight gradation is defined over $\mathbb{Q}$. Thus, to give a rational Hodge structure on V is the same as giving a gradation $V = \bigoplus_m V_m$ on V together with a real Hodge structure of weight m on $V_m \otimes \mathbb{R}$ for each m. The ***Tate Hodge structure*** $\mathbb{Q}(m)$ is defined to be the $\mathbb{Q}$-subspace $(2\pi i)^m\mathbb{Q}$ of $\mathbb{C}$ with $h(z)$ acting as $(z\bar{z})^m$. It has weight $-2m$ and type $(-m,-m)$.

For a real Hodge structure (V,h), the $\mathbb{R}$-linear map $C = h(i)$ is called the ***Weil operator***. It acts as i^{q-p} on $V^{p,q}$, and $C^2 = h(-1)$ acts as $(-1)^m$ on V_m.

Let (V,h) be a rational Hodge structure, and let $\langle V,h\rangle^{\otimes}$ be the tannakian subcategory generated by (V,h). The ***Mumford–Tate group*** of (V,h) is the algebraic group attached to $\langle V,h\rangle^{\otimes}$ and the forgetful fibre functor. It is the smallest algebraic subgroup of GL_V such that $G_{\mathbb{R}} \supset h(\mathbb{S})$. We usually regard the Mumford-Tate group as a pair (G,h). Note that G is connected, because otherwise we could replace it with its neutral component.

Let $\mathbb{S}^1$ be the one-dimensional torus over $\mathbb{R}$ such that $\mathbb{S}^1(\mathbb{R}) = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\}$. The ***special Mumford–Tate group*** of (V, h) is smallest algebraic subgroup of GL_V such that $G^1_{\mathbb{R}} \supset h(\mathbb{S}^1)$.

PROPOSITION 9.16 *A pair (G, h) is the Mumford–Tate group of a rational Hodge structure if and only if h it safisfies the following conditions:*

(a) *the weight $w_h : \mathbb{G}_{m\mathbb{R}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$ and $w_h(\mathbb{G}_m) \subset Z(G)$;*

(b) *h generates G (i.e., any algebraic subgroup of G containing $\mathrm{Im}(h)$ over $\mathbb{R}$ equals G).*

PROOF Let (G, h) be the Mumford–Tate group of a rational Hodge structure (V, h). Then h generates G by definition. The weight homomorphism w_h is defined over $\mathbb{Q}$ because (V, h) is a rational Hodge structure. Let $Z(w_h)$ denote the centralizer of w_h in G. For any $a \in \mathbb{R}^\times$, $w_h(a) : V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a morphism of real Hodge structures, and so it commutes with the action of $h(\mathbb{S})$. Hence $h(\mathbb{S}) \subset Z(w_h)_{\mathbb{R}}$. As h generates G, this implies that $Z(w_h) = G$. Conversely, suppose that (G, h) satisfies the conditions. For any faithful representation $\rho : G \to \mathrm{GL}_V$ of G, the pair $(V, h\circ\rho)$ is a rational Hodge structure, and (G, h) is its Mumford–Tate group. □

9.17 A ***polarization*** of a real Hodge structure (V, h) of weight m is a morphism of Hodge structures

$$\psi : V \otimes V \to \mathbb{R}(-m), \quad m \in \mathbb{Z},$$

such that

$$(x, y) \mapsto (2\pi i)^m \psi(x, Cy) : V \times V \to \mathbb{R}$$

is symmetric and positive definite. A ***polarization*** of a rational Hodge structure V is a morphism of rational Hodge structures $\psi : V \otimes V \to \mathbb{Q}(-m)$ such that $\psi \otimes \mathbb{R}$ is a polarization of $V \otimes \mathbb{R}$. A rational Hodge structure is polarizable, i.e., admits a polarization, if and only if $V \otimes \mathbb{R}$ is polarizable.

PROPOSITION 9.18 *Let (G, h) be the Mumford–Tate group of a rational Hodge structure (V, h). Then (V, h) is polarizable if and only if* $\mathrm{ad}\, h(i)$ *is a Cartan involution of $(G/w_h(\mathbb{G}_m))_{\mathbb{R}}$.*

PROOF Let $C = h(i)$. For notational convenience, assume that (V, h) has a single weight m. Let G^1 be the special Mumford–Tate group of (V, h). Then $C \in G^1(\mathbb{R})$, and a pairing $\psi : V \times V \to \mathbb{Q}(-m)$ is a polarization of the Hodge structure (V, h) if and only if $(2\pi i)^m\psi$ is a C-polarization of V for G^1 in the sense of VI, 1.19. It follows from VI, 1.20, and VI, 1.21, that a polarization ψ for (V, h) exists if and only if $\mathrm{ad}(C)$ is a Cartan involution of $G^1_{\mathbb{R}}$. Now $G^1 \subset G$ and the quotient map $G^1 \to G/w_h(\mathbb{G}_m)$ is an isogeny, and so $\mathrm{ad}(C)$ is a Cartan involution of G^1 if and only if it is a Cartan involution of $G/w_h(\mathbb{G}_m)$. □

9.19 The polarizable rational Hodge structures form a tannakian category $\mathsf{Hdg}_{\mathbb{Q}}$ with the forgetful functor as a fibre functor. Let G denote the associated affine group scheme over $\mathbb{Q}$. Then,

(a) all algebraic quotients of G are reductive (in particular, connected);

(b) the quotient of G by its derived group is the (well-known) Serre protorus;

(c) the derived group of G is simply connected (hence a product of simply connected almost-simple algebraic groups over $\mathbb{Q}$);

(d) the simple factors of the adjoint group of G are the groups of the form $\mathrm{Res}_{F/\mathbb{Q}}(H)$, where F is a totally real number field and H is a geometrically simple algebraic group over F such that, for all embeddings σ of F in $\mathbb{R}$, the real algebraic group σH contains a compact maximal torus.

See Milne 2020b.

Hodge–Tate modules

9.20 Let K be a field of characteristic zero, complete with respect to a discrete valuation, whose residue field is perfect of characteristic $p \neq 0$. Let C be the completion of an algebraic closure K^{al} of K, and let $G = \mathrm{Gal}(K^{\mathrm{al}}/K)$. The cyclotomic character $\chi : G \to \mathbb{Z}_p^\times$ is defined by the action G on the p^nth roots of 1 in K^{al},

$$\sigma\zeta = \zeta^{\chi(\sigma)}, \quad \sigma \in G, \quad \zeta^n = 1 \text{ some } n.$$

Let V be a finite-dimensional C-vector space equipped with a semilinear action of G,

$$\sigma(cv) = \sigma(c)\sigma(v), \quad \sigma \in G, \quad c \in C, \quad v \in V.$$

For $i \in \mathbb{Z}$, let V^i denote the subset of V on which G acts through χ^i,

$$V^i = \{v \in V \mid \sigma v = \chi(\sigma)^i v, \text{ all } \sigma \in G\}.$$

It is a K-subspace of V, and we set $V(i) = V^i \otimes_K C$. The inclusion $V^i \hookrightarrow V$ extends to a C-linear map $V(i) \to V$, and Tate showed that the map

$$\bigoplus_i V(i) \to V$$

is injective. If it is also surjective, then V is said to be Hodge–Tate.

Now assume that the residue field is algebraically closed. An action of G on a finite-dimensional K-vector space V extends to a semilinear action of G on $V \otimes C$, and we say that V is Hodge–Tate if $V \otimes C$ is Hodge–Tate in the above sense. The Hodge-Tate modules for K form a tannakian category (Serre 1979).

10 Neutral tannakian categories over a base ring

In this section, we explain how to extend some of the results in this chapter to more general base rings. Throughout, R is a commutative ring. By a ***finite flat*** R-module, we mean an R-module that is finitely presented and flat (equivalently, finitely generated and projective, or locally free of finite rank). All Hopf algebras are commutative.

Affine group schemes and their representations

The correspondence between affine group schemes and Hopf algebras described in §1 holds over any commutative ring R. Let G be the affine group scheme corresponding to the Hopf algebra A, and let V be an R-module. To give a representation of G on V is the same as giving an A-comodule structure on V. In this way, we obtain an isomorphism of categories

$$\mathsf{Rep}(G) \approx \mathsf{coMod}(A)$$

When A is flat over R, it is possible to endow the kernel and cokernel of a homomorphism of A-comodules with an A-comodule structure in such a way that $\mathsf{coMod}(A)$ becomes an abelian category (Serre 1968, 1.3). If, in addition, R is noetherian, then the category $\mathsf{coModf}(A)$ of A-comodules finitely generated over R is also an abelian category.

PROPOSITION 10.1 *When R is noetherian, every comodule over a flat R-coalgebra is a filtered union of its finitely generated subcomodules.*

PROOF See Serre 1993, 1.4. □

Correspondingly, when G is a affine group scheme flat over a noetherian ring R, every G-module is a filtered union of finite generated G-submodules.

PROPOSITION 10.2 *Let G be an affine group scheme flat and of finite type over a Dedekind domain R. There exists a finite flat R-submodule V of $\mathcal{O}(G)$, stable under G, such that the homomorphism $G \to \mathrm{GL}_V$ is a closed immersion.*

PROOF There exists a finitely generated R-submodule V of $\mathcal{O}(G)$, stable under G, containing a set of generators for $\mathcal{O}(G)$ (by 10.1). Now G flat over $R \implies \mathcal{O}(G)$ is torsion-free (as an R-module) $\implies V$ is torsion-free $\implies V$ is flat (because R is a Dedekind domain). It remains to show that the homomorphism $\alpha: \mathcal{O}(\mathrm{GL}_V) \to \mathcal{O}(G)$ defined by the action of G on V is surjective.

We may suppose that V is free. Let $\Delta: \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ be the comultiplication map and $\epsilon: \mathcal{O}(G) \to R$ the co-identity map. Let $(e_i)_{1\le i\le n}$ be a basis for V, and write $\Delta(e_j) = \sum_i e_i \otimes a_{ij}$, $a_{ij} \in \mathcal{O}(G)$. The image of α contains the a_{ij} (the choice of the basis (e_i), determines an isomorphism $\mathcal{O}(\mathrm{GL}_V) \simeq R[T_{ij}]$, and α maps T_{ij} to a_{ij}). As $\epsilon: \mathcal{O}(G) \to R$ is the co-identity,

$$e_j = (\epsilon \otimes \mathrm{id}_A)\Delta(e_j) = \sum\nolimits_i \epsilon(e_i)a_{ij},$$

and so the image of α contains V, which we chose to generate $\mathcal{O}(G)$. □

Thus, if R is a Dedekind domain and G is flat, then there is an embedding of G into GL_V for some finite flat R-module V. Such a V is a direct summand $F = V \oplus W$ of a free R-module F of finite rank. Extend the action of G on V to F by letting it act trivially on W and choose a basis for F. Now G is a closed algebraic subgroup of GL_n, $n = \operatorname{rank} F$.

Let S be a noetherian scheme and $\mathcal{E}$ a locally free $\mathcal{O}_S$-module of finite rank. The functor sending an S-scheme $f: T \to S$ to $\mathrm{Aut}_{\mathcal{O}_T}(f^*\mathcal{E})$ is representable by an affine group scheme $\mathrm{GL}_{\mathcal{E}}$ over S.

DEFINITION 10.3 A flat group scheme G over a noetherian scheme S is said to be ***linear*** if there exists a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of finite rank and an S-morphism $G \to \mathrm{GL}_{\mathcal{E}}$ that is both a closed immersion and a homomorphism.

Grothendieck showed that an algebraic group scheme of multiplicative type over a connected noetherian scheme S is linear if and only it is isotrivial, i.e., split by a finite étale covering $S' \to S$ of S. Gille extended this to reductive groups as follows. A group scheme G smooth and affine over a scheme S is said to be ***reductive*** if its geometric fibres are reductive. For such a group scheme, $\mathrm{rad}(G)$ is defined to be the largest central torus of G.

PROPOSITION 10.4 (GILLE 2022, 1.1) *A reductive group over a connected noetherian scheme S is linear if and only if* rad(G) *is isotrivial.*

Let S a noetherian connected affine scheme, and let $\bar{s}$ be a geometric point of S. Let T be a torus over S, and let T^f be the quotient of T corresponding to the submodule of $X^*(T_{\bar{s}}) \overset{\text{def}}{=} \mathrm{Hom}_{\bar{s}}(T_{\bar{s}}, \mathbb{G}_m)$ consisting of the elements with finite $\pi_1(S, \bar{s})$-orbits. Then T^f is the universal isotrivial quotient of T.

COROLLARY 10.5 *Let G be a reductive group over S (affine, connected, and noetherian), and let G^f be the quotient of G by the kernel of* $\mathrm{rad}(G) \to \mathrm{rad}(G)^f$ *. Then G^f is linear.*

PROOF Indeed, $\mathrm{rad}(G^f) = \mathrm{rad}(G)^f$, which is isotrivial. □

Let V be a finite flat R-module. By a ***line*** in V, we mean a finite flat submodule L of V of rank 1 that, locally on $\operatorname{Spec} R$, is a direct summand of V. Given a line L in a finite flat R-module V, we let $\mathcal{A}ut(V, L)$ denote the functor of R-algebras

$$R' \rightsquigarrow \{\alpha \in \mathrm{Aut}_{R'}(V \otimes_R R') \mid \alpha(L_{R'}) = LR'\}.$$

PROPOSITION 10.6 *Let G be a group scheme flat of finite type over a Dedekind domain R. There exists a finite flat R-module V and a line L in V such that*

$$G = \mathcal{A}ut(V, L).$$

PROOF The usual proof over fields extends to Dedekind domains (Broshi 2013, 1.1). □

The statement (1.11), that a Hopf algebra over a field is faithfully flat over any Hopf subalgebra, has an extension to base rings.

THEOREM 10.7 *Let $A \subset B$ be Hopf algebras flat over a noetherian ring R. The algebra B is faithfully flat over A if and only if B/A is a flat R-module.*

PROOF From the criterion,[4]

> a homomorphism $A \to B$ of commutative rings is faithfully flat if and only if it is injective and B/A is a flat A-module,

we see that it suffices to show that $A \to B$ is faithfully flat if B/A is a flat R-module. Note that this last condition implies that[5]

$$0 \to A \otimes_R \kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p}) \to (B/A) \otimes_R \kappa(\mathfrak{p}) \to 0$$

is exact for all prime ideals $\mathfrak{p}$ of R. Now the original theorem 1.11 shows that $A \otimes_R \kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p})$ is faithfully flat for all $\mathfrak{p}$, and the next criterion shows that $A \to B$ is faithfully flat.

> Let R be a noetherian ring and $f : A \to B$ a homomorphism of flat R-algebras. Then f is faithfully flat if $f \otimes_R \kappa(\mathfrak{p})$ is faithfully flat for all prime ideals $\mathfrak{p}$ of R (Hai et al. 2025, 3.3). □

NOTES When R is a Dedekind domain, 10.7 is proved in Duong and Hai 2018, 4.1.1. For a noetherian ring, it is Hai et al. 2025, 7.1.

[4] If $A \to B$ is faithfully flat, then $A \otimes M \to B \otimes M$ is faithfully flat, hence injective, for all A-modules M. Now

$$0 \to A \otimes M \to B \otimes M \to (B/A) \otimes M \to 0$$

is exact for all A-modules M, which implies that B/A is flat (because $\mathrm{Tor}_1^A(B/A, M) = 0$). Conversely, if $A \to B$ is injective and B/A is flat, then B is flat (because it is an extension of flat modules) and faithfully flat (because $M \to B \otimes M$ is injective for all A-modules M).

[5] $\kappa(\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$.

Recovering G from Rep(*G*)

PROPOSITION 10.8 *Let G be an affine group scheme flat over a Dedekind domain R, and let ω be the forgetful functor* $\mathsf{Repf}(G) \to \mathsf{Mod}(R)$. *Then ω defines an isomorphism $G \to \mathcal{A}ut^{\otimes}(\omega)$.*

PROOF Once 10.6 has been acquired, the proof of Theorem 2.1 applies. □

PROPOSITION 10.9 *Let G be a reductive group over scheme S, assumed to be affine noetherian and connected, and let G^f be the quotient of G by the kernel of* $\mathrm{rad}(G) \to \mathrm{rad}(G)^f$. *Then the homomorphism $G \to \mathcal{A}ut^{\otimes}(\omega)$ defined by the forgetful functor ω induces an isomorphism*

$$G^f \simeq \mathcal{A}ut^{\otimes}(\omega).$$

PROOF See Zhao 2022, 3.2.3. □

For example, the map $G \to \mathcal{A}ut^{\otimes}(\omega)$ is an isomorphism when G is a reductive group over a Dedekind domain and it need not be an isomorphism when G is a torus of dimension 2 over a curve with a node.

Tannakian duality over a Dedekind domain

When no longer working over a field, we have to take account of the fact that not all finitely generated modules are projective. For a Dedekind domain, there is the following result.

PROPOSITION 10.10 *Let L be a flat coalgebra over a Dedekind domain, and let M be a finitely generated L-comodule. Then M is a quotient of a finite flat L-comodule,*

PROOF See Serre 1968, 2.2. □

Hence, for any finitely generated L-comodule, there exists an exact sequence of L-comodules

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0 and P_1 finite flat L-comodules.

PROPOSITION 10.11 *Let R be a Dedekind domain. Let* A *be an R-linear abelian category and* $\omega : \mathsf{A} \to \mathsf{Modf}(R)$ *an exact R-linear faithful R-linear functor. Assume that there exists a full subcategory* A° *of* A *such that*

(a) *every object of* A *is a quotient of an object of* A° *and*

(b) *$\omega(X)$ is a finite flat R-module for all X in* A°.

Then the coalgebra $L(\omega) \overset{\text{def}}{=} \mathrm{Coend}(\omega \otimes \omega^\vee)$ is flat over R, acts naturally on each object of A, *and ω defines an equivalence* $\mathsf{A} \to \mathsf{coModf}(L)$.

PROOF Saavedra 1972, II, 2.3.5, 2.6.1, or Duong and Hai 2018, 1.2.2. □

Recall (12.3) that an object of a symmetric monoidal category is said to be rigid if it admits a dual.

DEFINITION 10.12 Let R be a Dedekind domain. A ***neutral tannakian category*** over R is an R-linear abelian $\otimes$-category T such that

(a) every object is a quotient of a rigid object, and

(b) there exists an exact faithful R-linear $\otimes$-functor $\omega : \mathsf{T} \to \mathsf{Mod}(R)$.

EXAMPLE 10.13 Let L be a flat Hopf algebra over a Dedekind domain R. Then $\mathsf{coModf}(L)$ is a neutral tannakian category over R. Proposition 10.10 shows that it satisfies condition (a), and the forgetful functor satisfies (b).

THEOREM 10.14 (SAAVEDRA 1972, II, 4.1.1) *Let R be a Dedekind domain. Let T be a neutral tannakian category over R, and let ω be an exact faithful R-linear $\otimes$-functor $\omega : \mathsf{T} \to \mathsf{Mod}(R)$.*

(a) *The functor $\mathcal{E}nd^{\otimes}(\omega)$ of R-algebras is represented by an affine group scheme G flat over R.*

(b) *The functor $\mathsf{T} \to \mathsf{Repf}(G)$ defined by G is an equivalence of tensor categories.*

PROOF Let T° denote the full subcategory of T of rigid objects. For X in T°, $\omega(X)$ is rigid, and hence a finite flat R-module (I, 4.9). Therefore, T° satisfies the hypotheses of 10.11, and so ω defines an equivalence $\mathsf{T} \to \mathsf{coModf}(L(\omega))$, where $L(\omega)$ is the coend of $\omega \otimes \omega^\vee$. Now, as in the proof of Theorem 3.1, the tensor structure on T defines a bialgebra structure on $L(\omega)$, so $G = \mathrm{Spec}(L(\omega))$ is an affine monoid scheme flat over R such that $G \simeq \mathcal{E}nd^{\otimes}(\omega)$ and $\mathsf{T} \to \mathsf{Repf}(G)$ is an equivalence. Now

$$\mathcal{E}nd^{\otimes}(\omega) \simeq \mathcal{E}nd^{\otimes}(\omega|\mathsf{T}^\circ) \simeq \mathcal{A}ut^{\otimes}(\omega|\mathsf{T}^\circ),$$

so G is a group scheme. □

The functor defined by a homomorphism of group schemes

Let L be a coalgebra over a noetherian ring R. A right L-comodule is an R-module V together with an R-linear map $\rho : V \to V \otimes L$ satisfying the conditions in 1.13. We write $\mathsf{coMod}(L)$ for the category of such comodules, $\mathsf{coModf}(L)$ for the full subcategory of those such that V is finitely generated as an R-module, and $\mathsf{coModf}^\circ(L)$ for the full subcategory of those such that V is a finite flat R-module.

A subcomodule M of an L-comodule N is said to be ***special*** if N/M is flat over R. A ***special subquotient*** M of N is a special subcomodule of a quotient of N, or, equivalently, a quotient of a special subcomodule.

Let $f : L' \to L$ be a homomorphhism of coalgebras over R. From an L-module (V, ρ) we get an L'-module by composing $\rho : V \to V \otimes L'$ with $\mathrm{id}_V \otimes f : V \otimes L' \to V \otimes L$. In this way, we get a functor

$$\omega^\circ : \mathsf{coModf}^\circ(L) \to \mathsf{coModf}^\circ(L').$$

PROPOSITION 10.15 *Let $f : L' \to L$ be a homomorphism of flat coalgebras over a Dedekind domain R. The homomorphism f is surjective if and only if every object of $\mathsf{coModf}^\circ(L')$ is a special subquotient of an object in the image of ω°.*

PROOF See Duong and Hai 2018, 3.2.5. □

Let G be an affine group scheme over a noetherian ring R. We write $\mathsf{Repf}^\circ(G)$ for the category of representations G on finite flat R-modules. Let $f : G \to H$ be a homomorphism of affine group schemes over R. Using f, we can regard an H-module as a G-module. In this way, we get a R-linear exact $\otimes$-functor

$$\omega^\circ : \mathsf{Repf}^\circ(H) \to \mathsf{Repf}^\circ(G)$$

compatible with the forgetful functors.

THEOREM 10.16 (DUONG AND HAI 2018, 4.1.2) *Let $f : G \to H$ be a homomorphism of affine group schemes flat over a Dedekind domain R. The homomorphism f is a closed immersion if and only if every object of $\mathsf{Repf}^\circ(G)$ is a special subquotient of an object in the image of ω^f.*

PROOF Restatement of 10.15. □

PROPOSITION 10.17 *Let $f : L' \to L$ be a homomorphism of flat coalgebras over a noetherian ring R. The following are equivalent:*

- *the homomorphism f is injective and L/L' is a flat R-module;*
- *ω° is fully faithful and its essential image is closed under subobjects.*

PROOF See Duong and Hai 2018, 3.2.1, when R is a Dedekind domain, Hai et al. 2025, 7.3, in the general case. □

THEOREM 10.18 *Let $f : G \to H$ be a homomorphism of affine group schemes flat over a noetherian ring R. The homomorphism f is faithfully flat if and only if the functor ω° is fully faithful and its essential image is closed under subobjects.*

PROOF Combine 10.7 and 10.17. □

NOTES Theorem 10.18 is Duong and Hai 2018, 4.1.2(i), when R is a Dedekind domain, Hai et al. 2025, 7.2, in the general case.

Classification of the fibre functors

Let R be a commutative ring and R' an R-algebra. A ***fibre functor*** on an R-linear abelian $\otimes$-category T with values in R' is an exact faithful R-linear $\otimes$-functor $\omega : \mathsf{T} \to \mathsf{Mod}(R')$.

THEOREM 10.19 *Let T be a neutral tannakian category over a Dedekind domain R, let ω be an R-valued fibre functor, and let $G = \mathcal{A}ut^\otimes(\omega)$.*

(a) *For any fibre functor η on T over an R-algebra R', $\mathcal{H}om^\otimes(\omega_{R'}, \eta)$ is representable by an affine scheme faithfully flat over $\operatorname{Spec} R'$; it is therefore a torsor under $G_{R'}$ over R'.*

(b) *The functor $\eta \rightsquigarrow \mathcal{H}om^\otimes(\omega_{R'}, \eta)$ determines an equivalence between the category of fibre functors on T with values in R' and the category of torsors under $G_{R'}$ over R'.*

PROOF After Theorem 10.14, it suffices to prove this in the case $\mathsf{T} = \mathsf{Repf}(G)$ with G a flat affine group. For this, see Broshi 2013. See also Saavedra 1972, II, 3.2.3. □

Tannakian lattices over Dedekind domains

There are situations where one would like to use Saavedra's theorem 10.14 to construct an affine group schemes but the tensor category does not have all the properties of Saavedra's definition. One therefore wants to reconstruct an affine group scheme from a smaller part of its representation category. There are at least three approaches to this problem, one in Wedhorn 2004, one in Bruguiéres 2004 following an idea of M. Nori, which is used in dos Santos 2009, and one in Duong and Hai 2018. Here we follow the last approach. First, a little category theory.

10.20 A morphism is a ***regular epimorphism*** if it is the coequalizer of some parallel pair of morphisms. In an additive category, regular epimorphisms are just cokernels. See Borceux 1994b, 2.1.1.

10.21 A category is ***regular*** if

(a) it is finitely complete (i.e., has finite products and equalizers),

(b) every kernel pair admits a coequalizer,

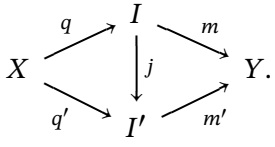
(c) regular epimorphisms are stable under pullbacks.

10.22 In a regular category, every morphism factors into a regular epimorphism followed by a monomorphism, and this factorization is unique up to a unique isomorphism.

In more detail, every morphism $f : A \to B$ in a regular category factors into a composite

$$A \xrightarrow{q} I \xrightarrow{m} B \tag{62}$$

with q a regular epimorphism and m a monomorphism. If $f = m' \circ q'$ with m' a monomorphism, then there exists a unique morphism j such that $j \circ q = q'$ and $m = m' \circ j$,



When q' is a regular epimorphism, j is an isomorphism. See Borceux 1994b, 2.1.3.

10.23 In an additive regular category, every cokernel $p : X \to Y$ has a kernel, namely, its pullback along $0 \to Y$. Let $f : X \to Y$ be morphism in an additive regular category, and let $f = m \circ p$ be its cokernel-monomorphism factorization. Then $\operatorname{Ker} p = \operatorname{Ker} f$, so f admits a kernel. Moreover, $p = \operatorname{Coker}(\operatorname{Ker} f)$ so all kernels have cokernels. See Henrard et al. 2023, 4.4.

10.24 Every additive regular category C admits an full embedding into an abelian category A, characterized by the following properties

(a) C is closed under subobjects in A,

(b) every object of A is a quotient of an object of C.

See Henrard et al. 2023, 1.3.

EXAMPLE 10.25 Let G be an affine group scheme flat over a noetherian ring R. Then $\mathsf{Repf}(G)$ is an abelian R-linear category. If R is a Dedekind domain, then the subcategory $\mathsf{Repf}^\circ(G)$ of $\mathsf{Repf}(G)$ is closed under subobjects, and hence regular.

Let R be a Dedekind domain with field of fractions K.

DEFINITION 10.26 A ***tensor lattice*** over R is an R-linear regular rigid $\otimes$-category T such that $R \simeq \operatorname{End}(\mathbb{1})$ and T_K is abelian. An R'-valued ***fibre functor*** on T is a faithful R-linear $\otimes$-functor $\omega : \mathsf{T} \to \mathsf{Mod}(R')$ preserving kernels and cokernel-monomorphism factorizations and such that $R' \simeq \operatorname{End}(\omega(\mathbb{1}))$.

DEFINITION 10.27 A ***neutral tannakian lattice*** over R is a tensor lattice over R that admits an R-valued fibre functor.

THEOREM 10.28 *Let T be neutral tannakian lattice over R and ω an R-valued fibre functor.*

(a) *The functor $\mathcal{A}ut^{\otimes}(\omega)$ is represented by an affine group scheme flat over R, and ω induces an equivalence $\mathsf{T} \to \mathsf{Repf}^\circ(G)$.*

(b) *For any fibre functor η on T with values in an R-algebra R', $\mathcal{I}som^{\otimes}(\omega_{R'}, \eta)$ is a torsor under G over R'.*

PROOF See Duong and Hai 2018, 2.3.2, 2.4.8. □

Chapter III

Tannakian categories and Galois groupoids

In this chapter, we describe tannakian categories over k with a $\bar{k}$-valued fibre functor in terms of Galois groupoids (following Langlands and Rapoport 1987).

Throughout, k is a field and $\bar{k}$ is a fixed separable algebraic closure of k. We let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Finite extensions of k are required to be subfields of $\bar{k}$. All algebraic groups over k are affine and smooth, and all affine group schemes are pro-smooth. When k has characteristic zero, which is the most important case, $\bar{k}$ is an algebraic closure of k and the smoothness condition is automatic.

1 Galois groupoids

We assume that the reader is familiar with Galois descent, as described in §6 below.

DEFINITION 1.1 Let G be an algebraic group over $\bar{k}$. A $\bar{k}/k$-***Galois groupoid with kernel*** G is an exact sequence of groups

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1, \tag{63}$$

together with a section $\sigma \mapsto s_\sigma : \mathrm{Gal}(\bar{k}/K) \to \mathcal{G}$ for a suitable finite extension K of k, such that the automorphisms

$$\gamma_\sigma : G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma \overset{\text{def}}{=} (g \mapsto s_\sigma \cdot g \cdot s_\sigma^{-1}), \quad \sigma \in \mathrm{Gal}(\bar{k}/K),$$

define a K-structure on G. It is to be understood that K can be replaced by a larger finite extension, so that s is actually a germ of a section for the Krull topology on $\mathrm{Gal}(\bar{k}/k)$.

In more detail, consider an extension $\mathcal{G}$ of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. Let $K \subset \bar{k}$ be a finite extension of k and s a homomorphism such that

$$\begin{array}{ccc} & & \mathrm{Gal}(\bar{k}/K) \\ & \overset{s}{\swarrow} & \big\downarrow \\ \mathcal{G} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) \end{array}$$

commutes. To say that the automorphisms $\gamma_\sigma : G(\bar{k}) \to G(\bar{k})$, $\sigma \in \mathrm{Gal}(\bar{k}/K)$, define a K-structure on G means that there exists an algebraic group G_0 over K and an isomorphism

$\varphi: G_{0\bar{k}} \to G$ such that

$$\varphi(\sigma g) = \gamma_\sigma(\varphi(g)), \quad \text{all } \sigma \in \mathrm{Gal}(\bar{k}/K), \quad g \in G_0(\bar{k}).$$

A necessary and sufficient condition for this is that the homomorphisms γ_σ be σ-linear and that the action of $\mathrm{Gal}(\bar{k}/K)$ on $G(\bar{k})$ they define be continuous (Appendix, 6.4).

Two sections $s: \mathrm{Gal}(\bar{k}/K) \to \mathcal{G}$ and $s': \mathrm{Gal}(\bar{k}/K') \to \mathcal{G}$ are said to be ***equivalent*** if they agree on $\mathrm{Gal}(\bar{k}/K'')$ for some field K'' containing K and K' and finite over k. A ***germ*** of sections is an equivalence class. Now a $\bar{k}/k$-Galois groupoid with kernel G can be defined to be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$ together with a germ of sections satisfying the above conditions.

1.2 A ***homomorphism*** of Galois groupoids $\mathcal{G} \to \mathcal{H}$ is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \overset{s}{\curvearrowleft} & & & & \\
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi}\ \text{homomorphism} & & \| & & \\
1 & \longrightarrow & H(\bar{k}) & \longrightarrow & \mathcal{H} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
 & & & & \underset{s'}{\curvearrowleft} & & & &
\end{array}$$

such that ϕ preserves the germs of sections and such that the restriction of ϕ to $G(\bar{k})$ is regular (i.e., defined by a homomorphism of algebraic groups).

From a different point-of-view, a homomorphism of Galois groupoids is a homomorphism of algebraic groups $\varphi: G \to H$ together with an extension of $\varphi(\bar{k})$ to a homomorphism $\mathcal{G} \to \mathcal{H}$ compatible with the germs of sections and inducing the identity map on the Galois groups.[1]

For example, conjugation by an element g of $G(\bar{k})$ defines an automorphism $\mathrm{ad}(g)$ of $\mathcal{G}$,

$$\begin{array}{ccccccccc}
 & & & & \overset{s}{\curvearrowleft} & & & & \\
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{ad}\,g} & & \downarrow{\scriptstyle \mathrm{ad}\,g} & & \| & & \\
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1. \\
 & & & & \underset{\mathrm{ad}(g)\circ s}{\curvearrowleft} & & & &
\end{array}$$

1.3 Let $\mathcal{G}$ and $\mathcal{H}$ be Galois groupoids with kernels G and H, and let ϕ and ϕ' be homomorphisms $\mathcal{G} \to \mathcal{H}$. A ***morphism*** $\phi \to \phi'$ is an element g of $H(\bar{k})$ such that $\mathrm{ad}(g)\circ\phi = \phi'$, i.e., such that

$$g \cdot \phi(x) \cdot g^{-1} = \phi'(x), \quad \text{all } x \in \mathcal{G}.$$

With this structure, the Galois groupoids form a 2-category.

EXAMPLE 1.4 Let G be an algebraic group over k. The extension

$$1 \to G(\bar{k}) \to G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

[1] Recall that if V is a geometrically reduced scheme of finite type over a separably closed field $\bar{k}$, then $V(\bar{k})$ is schematically dense in V. As G is smooth over $\bar{k}$, it is geometrically reduced, and so the homomorphism $\varphi: G \to H$ is determined by $\varphi(\bar{k})$

together with the section $\sigma \mapsto (1, \sigma)$ defines Galois gerbe $\mathcal{G}_G$ with kernel G. A $\bar{k}/k$-Galois groupoid $\mathcal{G}$ is ***split*** if it is isomorphic to $\mathcal{G}_G$ for some G. Equivalently, if there exists a section s to the homomorphism $\pi : \mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$, compatible with the germ of sections, such that the automorphisms

$$\gamma_\sigma : G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

define a k-structure on G.

1.5 Let

$$1 \to G(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

be a $\bar{k}/k$-Galois groupoid with kernel G. The choice of a section s in the germ determines a model (G_0, φ) of G over some finite extension K of k and a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_0(\bar{k}) & \longrightarrow & G_0(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/K) & \longrightarrow & \mathrm{Gal}(\bar{k}/K) & \longrightarrow & 1 \\
 & & \simeq\downarrow{\scriptstyle \varphi(\bar{k})} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \xrightarrow{\pi} & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1.
\end{array}$$

We call such a diagram a ***splitting*** of $\mathcal{G}$ over K.

1.6 Let E be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. When k' is a subfield of $\bar{k}$ containing k, we can form an extension of $\mathrm{Gal}(\bar{k}/k')$ by $G(\bar{k})$ by pullback. It is uniquely determined, up to a unique isomorphism, by the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & E' & \longrightarrow & \mathrm{Gal}(\bar{k}/k') & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & E & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1.
\end{array}$$

When E is a $\bar{k}/k$-Galois groupoid, E' becomes a $\bar{k}/k'$-Galois groupoid with the obvious germ of sections.

1.7 Let E be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. When $G \to H$ is a homomorphism of algebraic groups over $\bar{k}$, we can form an extension E' of $\mathrm{Gal}(\bar{k}/k)$ by $H(\bar{k})$ by pushout. It is uniquely determined, up to a unique isomorphism, by the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(\bar{k}) & \longrightarrow & E & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & H(\bar{k}) & \longrightarrow & E' & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1.
\end{array}$$

When E is a $\bar{k}/k$-Galois groupoid with kernel G, E' becomes a $\bar{k}/k$-Galois groupoid with kernel H and the obvious germ of sections.

1.8 Let

$$1 \to G(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

be a $\bar{k}/k$-Galois groupoid with G is algebraic, and let $s : \mathrm{Gal}(\bar{k}/k) \to \mathcal{G}$ be a set-theoretic section to π such that the restriction of s to $\mathrm{Gal}(\bar{k}/K)$ lies in the given germ for some $K \subset \bar{k}$ finite over k.[2] Then $\mathcal{G}$ is determined by the following data:

[2] For example, choose a group-theoretic section $\mathrm{Gal}(\bar{k}/K) \to \mathcal{G}$ in the given germ, and extend it to a set-theoretic section on $\mathrm{Gal}(\bar{k}/k)$.

(a) the family $(\gamma_\sigma)_\sigma$ of automorphisms of G (as an algebraic group over $\bar{k}$) given by

$$\gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k), \quad g \in G(\bar{k});$$

(b) the family $(a_{\sigma,\tau})_{\sigma,\tau}$ of elements of $G(\bar{k})$ given by

$$s_\sigma s_\tau = a_{\sigma,\tau} s_{\sigma\tau}, \quad \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

Indeed, every element of $\mathcal{G}$ can be written uniquely

$$g \cdot s_\sigma, \quad g \in G(\bar{k}), \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

and

$$(g \cdot s_\sigma)(h \cdot s_\tau) = (g\ \gamma_\sigma(h)\ a_{\sigma,\tau}) \cdot s_{\sigma\tau}.$$

PROPOSITION 1.9 *A homomorphism $\phi : \mathcal{G} \to \mathcal{G}'$ of $\bar{k}/k$-Galois groupoids is an isomorphism if and only if it is an isomorphism on the kernels.*

PROOF If ϕ has an inverse ϕ', then $\phi'|G'$ is an inverse for $\phi|G$. For the converse, suppose that $\phi|G$ is an isomorphism, and choose a section s as in 1.8. Then $\sigma \mapsto s'_\sigma \overset{\text{def}}{=} \phi \circ s_\sigma$ is a section for $\mathcal{G}'$ with similar properties, and clearly $g \cdot s_\sigma \mapsto \phi(g \cdot s_\sigma) = \phi(g) \cdot s'_\sigma$ is a bijection. Alternatively, this follows from a diagram chase. □

The band of a Galois groupoid

1.10 Let $\mathcal{G}$ be a Galois groupoid with kernel an algebraic group G. Let s be a set-theoretic section to π, as in 1.8. The automorphisms

$$\gamma_\sigma : G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

form a descent datum modulo inner automorphisms on G, and so define the structure of a band on G (Appendix C, §5). This structure is independent of the choice of the section s. The band $B(\mathcal{G})$ of $\mathcal{G}$ is well-defined up to a unique isomorphism.

When the kernel is commutative, the family of automorphisms $(\gamma_\sigma)_\sigma$ is a descent datum on G, and so G acquires a model over k, well-defined up to a unique isomorphism.

The cohomology class of a Galois groupoid

1.11 We fix a band B over k, and consider $\bar{k}/k$-Galois groupoids equipped with an isomorphism $B \simeq B(\mathcal{G})$. The cohomology set $H^2(k, B)$ is the set of isomorphism classes of such systems. This agrees with the definition in Appendix C, §6.

1.12 When B is commutative, we can identify it with a commutative algebraic group G over k. When G is of algebraic over k, $H^2(k, G)$ becomes the familiar Galois cohomology group $H^2(\mathrm{Gal}(\bar{k}/k), G(\bar{k}))$.

Galois groupoids admitting a special section

In this subsection, we study an important class of Galois groupoids, which includes all those with commutative kernels.

Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with algebraic kernel G, and let s be a set-theoretic section to π, as in 1.8. We say that $(s_\sigma)_\sigma$ is ***special*** if the family $(\gamma_\sigma)_\sigma$ satisfies the cocycle condition (6.3),

$$\gamma_{\sigma\tau} = \gamma_\sigma \circ (\sigma\gamma_\tau), \quad \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

Then $(\gamma_\sigma)_\sigma$ is a continuous cocycle and so defines a model G_0 of G over k. Moreover,

$$a_{\sigma,\tau} \overset{\text{def}}{=} \gamma_\sigma \cdot \sigma\gamma_\tau \cdot \gamma_{\sigma\tau}^{-1}$$

is a continuous 2-cocycle on $\mathrm{Gal}(\bar{k}/k)$ with values in $Z_G(\bar{k})$, where Z_G is the centre of G.

Consider the category whose objects are the pairs (G, a), where G is an algebraic group over k and $a = (a_{\sigma,\tau})_{\sigma,\tau\in\Gamma}$ is a continuous 2-cocycle on Γ with values in $Z_G(\bar{k})$. A morphism $(G', a') \to (G, a)$ is a pair (φ, f), where $\varphi : G' \to G$ is a homomorphism of algebraic groups over $\bar{k}$ and $f = (f_\sigma)_{\sigma\in\Gamma}$ is a continuous 1-cochain with values in $G(\bar{k})$ such that, for all $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$,

$$\begin{cases} a_{\sigma,\tau} \cdot f_\sigma \sigma(f_\tau) f_{\sigma\tau}^{-1} = \varphi(a'_{\sigma,\tau}) \\ \mathrm{ad}(f_\sigma) \circ \sigma^*(\varphi) = \varphi. \end{cases}$$

Composition of morphisms is given by

$$(\varphi, f) \circ (\varphi', f') = (\varphi \circ \varphi', f''),$$

where $f''_\sigma = \varphi(f'_\sigma) f_\sigma$, $\sigma \in \Gamma$.

For an object (G, a), we let $\mathcal{G}$ be the extension

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

defined by the 2-cocycle a: the elements of $\mathcal{G}$ are the pairs $(g, \sigma) \in G(\bar{k}) \times \mathrm{Gal}(\bar{k}/k)$, and

$$(g, \sigma)(h, \tau) = (g \cdot \sigma h \cdot a_{\sigma,\tau}, \sigma\tau).$$

For example, if $a_{\sigma,\tau} = 1$ for all σ, τ, then $\mathcal{G}$ is the split Galois groupoid $G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$.

PROPOSITION 1.13 *The functor $(G, a) \rightsquigarrow \mathcal{G}$ from pairs (G, a) to $\bar{k}/k$-Galois groupoids is fully faithful; its essential image consists of the Galois groupoids admitting a special section.*

PROOF Routine verification using descent theory. □

REMARK 1.14 Let $\mathcal{P}$ be a commutative $\bar{k}/k$-groupoid with kernel P. Let $\varphi : P \to G$ be a homomorphism of algebraic groups over k, and let Z_φ be the centralizer of $\varphi(P)$ in G. Assume that φ extends to a homomorphism $\phi : \mathcal{P} \to \mathcal{G}_G$, and let $I_\phi = \mathcal{A}ut(\phi)$. Then I_ϕ is an inner form of Z_ϕ whose cohomology class can be described as follows. Choose special section s, and let $(d_{\rho,\tau})$ be the corresponding 2-cocycle. When we write $\phi(s(\rho)) = (c_\rho, \rho)$, we obtain a 1-cochain (c_ρ) splitting the cocycle $(\varphi(d_{\rho,\tau}))$:

$$c_\rho \cdot \rho c_\tau = \varphi(d_{\rho,\tau}) \cdot c_{\rho\tau}.$$

For $p \in P(\bar{k})$ we have

$$\rho\varphi(p) = \varphi(\rho p) = \varphi(s(\rho) \cdot p \cdot s(\rho)^{-1}) = (c_\rho, \rho) \cdot \varphi(p) \cdot (c_\rho, \rho)^{-1} = c_\rho \cdot \rho\varphi(p) \cdot c_\rho^{-1},$$

and so $c_\rho \in Z_\varphi(\bar{k})$. The formula displayed above shows that the image of (c_ρ) in $Z_\varphi/\varphi(P)$ is a cocycle. Its class in $H^1(k, Z_\varphi/\varphi(P))$ depends only on the isomorphism class of φ, and it is the cohomology class of I_φ.

The category of Galois groupoids

1.15 When G is an affine group scheme over k, we define a $\bar{k}/k$-***Galois groupoid with kernel*** G to be an extension of groups

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

such that, for all algebraic quotients $G \to G_\alpha$ of G, the pushout

$$1 \to G_\alpha(\bar{k}) \to \mathcal{G}_\alpha \to \mathrm{Gal}(\bar{k}/k) \to 1$$

is a $\bar{k}/k$-Galois groupoid with kernel G_α. In particular, this means that we are given a compatible system of germs of sections on the $\mathcal{G}_\alpha$.

PROPOSITION 1.16 *Let* D *be the category of* $\bar{k}/k$*-Galois groupoids and* C *the full subcategory of* $\bar{k}/k$*-Galois groupoids with algebraic kernel.*

(a) The objects of C *are artinian.*

(b) The functor

$$\text{“}\varprojlim\text{”}G_\alpha \rightsquigarrow \varprojlim G_\alpha : \ \mathrm{Pro}\, C \to \mathsf{D}$$

is an equivalence of categories, with quasi-inverse the functor sending an affine group scheme G *to the projective system of its algebraic quotients.*

PROOF Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G. The subobjects of $\mathcal{G}$ are in one-to-one correspondence with the subgroup schemes of G (see 1.9), and so $\mathcal{G}$ is artinian if G is algebraic. The rest of the proof is opposite to that of Appendix B, B.11. □

In other words, the category of $\bar{k}/k$-Galois groupoids is the category of pro-objects in the category of $\bar{k}/k$-Galois groupoids with algebraic kernel.

Galois groupoids and groupoid schemes

1.17 Let $S_0 = \mathrm{Spec}\, k$ and $S = \mathrm{Spec}\, \bar{k}$. Let $(G, (t,s), \circ)$ be a k-groupoid scheme acting transitively on S (see IV, 2.8). Then $(S(\bar{k}), G(\bar{k}), (t,s), \circ)$ is a transitive groupoid (in sets). In particular, $(t,s) : G(\bar{k}) \to S(\bar{k}) \times S(\bar{k})$ is surjective, so

$$G(\bar{k}) = \coprod_{(b,a)\in S(\bar{k})\times S(\bar{k})} G(\bar{k})_{b,a} = \coprod_{(b,a)\in S(\bar{k})\times S(\bar{k})} \mathrm{Hom}(a,b),$$

where $G(\bar{k})_{b,a}$ is the fibre of $G(\bar{k})$ over (b,a). Note that $\mathrm{Gal}(\bar{k}/k)$ acts simply transitively on $S(\bar{k}) = \mathrm{Hom}(\bar{k}, \bar{k})$. Let a denote the identity element in S, and let

$$E = \{f \in G \mid t(f) = a\} = \coprod_{\sigma\in \mathrm{Gal}(\bar{k}/k)} G_{a,\sigma a} = \coprod_{\sigma\in \mathrm{Gal}(\bar{k}/k)} \mathrm{Hom}(\sigma a, a).$$

Write π for the map $E \to \mathrm{Gal}(\bar{k}/k)$ sending $f \in G_{a,\sigma a}$ to σ. Define the product of $f_1 : \sigma_1 a \to a$ and $f_2 : \sigma_2 a \to a$ by

$$f_1 \cdot f_2 = f_1 \circ \sigma_1 f_2.$$

Then $f_1 \cdot f_2 \in G_{a,\sigma_1\sigma_2 a}$, and so

$$\pi(f_1 \cdot f_2) = \sigma_1\sigma_2 = \pi(f_1)\pi(f_2).$$

Moreover,

$$\begin{cases} (f_1 \cdot f_2) \cdot f_3 = f_1 \circ \sigma_1 f_2 \circ \sigma_1\sigma_2 f_3 = f_1 \cdot (f_2 \cdot f_3) \\ f \cdot \mathrm{id}_a = f = \mathrm{id}_a \cdot f \\ f \cdot \sigma^{-1} f^{-1} = \mathrm{id}_a = \sigma^{-1} f^{-1} \cdot f, \end{cases}$$

so E is a group. We have constructed an exact sequence

$$1 \to G_{a,a} \to E \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

of abstract groups and group homomorphisms. This is a $\bar{k}/k$-Galois groupoid. This explains our choice of name. See IV, 2.23, for more details.

2 Representations of Galois groupoids

2.1 Let V be a vector space over $\bar{k}$ and let $\sigma \in \mathrm{Gal}(\bar{k}/k)$. An additive map $\alpha : V \to V$ is said to be σ-***linear*** if $\alpha(c \cdot v) = \sigma c \cdot \alpha(v)$ for all $c \in \bar{k}$ and $v \in V$. Note that if α_1 is σ_1-linear and α_2 is σ_2-linear, then $\alpha_2 \circ \alpha_1$ is $\sigma_2\sigma_1$-linear,

$$(\alpha_2 \circ \alpha_1)(c \cdot v) = \alpha_2(\sigma_1 c \cdot \alpha_1(v)) = \sigma_2\sigma_1 c \cdot \alpha_2 \circ \alpha_1(v).$$

2.2 Let V be a finite-dimensional vector space over $\bar{k}$. Let $\mathcal{G}_V$ be the collection of all additive isomorphisms $g : V \to V$ that are σ-linear for some $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Then $\mathcal{G}_V$ becomes a group under composition, and there is an exact sequence

$$1 \to \mathrm{GL}(V) \longrightarrow \mathcal{G}_V \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

in which π sends a σ-linear map to σ. The choice of a basis $(v_i)_i$ for V over $\bar{k}$ determines a section $s : \mathrm{Gal}(\bar{k}/k) \to \mathcal{G}_V$ to π,

$$s(\sigma)(\textstyle\sum c_i \cdot v_i) = \sum \sigma c_i \cdot v_i.$$

A different basis determines an equivalent section and so, in this way, we get a Galois groupoid $\mathcal{G}_V$ with kernel GL_V. Note that the choice of a $\bar{k}$-basis for V determines a $\mathbb{Q}$-structure on V and a splitting of $\mathcal{G}_V$ over k.

2.3 Let $\mathcal{G}$ be a Galois groupoid with kernel G. A ***representation*** of $\mathcal{G}$ on a finite dimensional $\bar{k}$-vector space V is a homomorphism of $\bar{k}/k$-Galois groupoids $\rho : \mathcal{G} \to \mathcal{G}_V$. In other words, a representation of $\mathcal{G}$ on V is a representation $\rho_0 : G \to \mathrm{GL}_V$ of G on V together with an extension ρ of $\rho_0(\bar{k})$ to a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_0(\bar{k})} & & \downarrow{\scriptstyle \rho} & & \| & & \\ 1 & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathcal{G}_V & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \end{array}$$

compatible with the germs of sections.

2.4 A ***morphism*** of representations (V, ρ_V) and (W, ρ_W) of $\mathcal{G}$ is a $\bar{k}$-linear map $\varphi : V \to W$ such that

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow{\scriptstyle \rho_V(g)} & & \downarrow{\scriptstyle \rho_W(g)} \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes for all $g \in \mathcal{G}$ (equality of σ-linear maps).

2.5 Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid. The category $\mathsf{Repf}(\mathcal{G})$ of representations of $\mathcal{G}$ is on finite-dimensional $\bar{k}$-vector spaces is abelian and k-linear. It has a natural structure of a symmetric monoidal category. For example, the trivial representation

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \xrightarrow{\pi} & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \rho_{\text{trivial}}} & & \| & & \\ 1 & \longrightarrow & \mathrm{GL}(\bar{k}) & \longrightarrow & \mathcal{G}_{\bar{k}} & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1. \end{array}$$

in which $\rho_{\text{trivial}}(g) = \pi(g) = \sigma$ (viewed as a σ-linear isomorphism $\bar{k} \to \bar{k}$) is a unit object. Duals exist, and obviously $\mathrm{End}(\mathbb{1}) = k$, so $\mathsf{Repf}(\mathcal{G})$ is a tensor category over k. The functor sending a representation to its underlying vector space is a $\bar{k}$-valued fibre functor, so $\mathsf{Repf}(\mathcal{G})$ is tannakian category over k.

2.6 Let G be an algebraic group scheme over k, and let $\mathcal{G}$ be a $\bar{k}/k$ Galois groupoid with kernel $G_{\bar{k}}$. We suppose given a section s to $\mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$ in the germ such that $\mathbb{Q}$-structure on $G_{\bar{k}}$ defined by s is G (see 1.4). For example, $\mathcal{G}$ could be $\mathcal{G}_G \overset{\text{def}}{=} G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$ with its canonical section.

A representation $\rho : G \to \mathrm{GL}_V$ of G on a finite-dimensional k-vector space defines a representation $\bar{\rho} \overset{\text{def}}{=} \rho(\bar{k}) : G(\bar{k}) \to \mathrm{GL}(V(\bar{k}))$ of $G(\bar{k})$ on $V(\bar{k})$ with the property that

$$\bar{\rho}(\sigma g) = \sigma\rho(g), \quad g \in G(\bar{k}), \quad \sigma \in \mathrm{Gal}(\bar{k}/k).$$

Every element of $\mathcal{G}$ can be written uniquely $g \cdot s(\sigma)$ with $g \in G(\bar{k})$ and $\sigma \in \mathrm{Gal}(\bar{k}/k)$, and we extend $\bar{\rho}$ to a representation $\widetilde{\rho}$ of $\mathcal{G}$ on $V(\bar{k})$ by setting

$$\widetilde{\rho}(g \cdot s(\sigma)) = \bar{\rho}(g)\sigma.$$

Thus, a representation of G defines a representation of $\mathcal{G}$. Conversely, a representation $\mathcal{G} \to \mathcal{G}_V$ restricts to a representation $G(\bar{k}) \to V$, which commutes with the action of $\mathrm{Gal}(\bar{k}/k)$ (for a suitable $\mathbb{Q}$-structure on V), and so arises from a representation of G on V. In this way, we get an equivalence of tensor categories over k,

$$\mathsf{Repf}(G) \to \mathsf{Repf}(\mathcal{G}).$$

In the next three paragraphs, we generalize this statement.

2.7 Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G. The choice of a section s,

$$\begin{array}{ccc} & & \mathrm{Gal}(\bar{k}/K) \\ & \overset{s}{\swarrow} & \downarrow \\ \mathcal{G} & \longrightarrow & \mathrm{Gal}(\bar{k}/k), \end{array}$$

in the germ of $\mathcal{G}$ determines a model of G over K, which we again denote G, and a homomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\bar{k}) & \longrightarrow & G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/K) & \longrightarrow & \mathrm{Gal}(\bar{k}/K) & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G(\bar{k}) & \longrightarrow & \mathcal{G} & \xrightarrow{\pi} & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1, \end{array}$$

The group-theoretic section s can be extended to a set-theoretic section s of $\mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$ with the property that

$$s(\sigma\tau) = s(\sigma)s(\tau), \quad \text{all } \sigma \in \mathrm{Gal}(\bar{k}/K), \quad \tau \in \mathrm{Gal}(\bar{k}/k).$$

See Lattermann 1989, 4.2.6.

2.8 In the situation of 2.7, we define a functor

$$\mathsf{Repf}_K(G) \to \mathsf{Repf}(\mathcal{G})_{(K)}.$$

A representation $\rho : G \to \mathrm{GL}_V$ of G on a finite-dimensional K-vector space V defines a representation $\bar{\rho} \overset{\text{def}}{=} \rho(\bar{k})$ of $G(\bar{k})$ on $\bar{V} \overset{\text{def}}{=} V \otimes \bar{k}$ with the property that

$$\bar{\rho}(\sigma g) = \sigma\,(\bar{\rho}(g)), \quad g \in G(\bar{k}) \quad \sigma \in \mathrm{Gal}(\bar{k}/K).$$

Because G is smooth, ρ can be recovered from $\bar{\rho}$ and the action of $\mathrm{Gal}(\bar{k}/K)$. Write gv for $\bar{\rho}(g)v$, and let

$$\mathrm{Ind}\,\bar{V} = \{f : \mathrm{Gal}(\bar{k}/k) \to \bar{V} \mid f(\sigma\tau) = \sigma f(\tau), \text{ all } \sigma \in \mathrm{Gal}(\bar{k}/K)\}.$$

Then $\mathrm{Ind}\,\bar{V}$ is a $\bar{k}$-vector space by

$$(\lambda f)(\tau) = (\tau\lambda)(f(\tau)), \quad \lambda \in \bar{k}, \quad f \in \mathrm{Ind}\,\bar{V}, \quad \tau \in \mathrm{Gal}(\bar{k}/k),$$

and a $\mathrm{Gal}(\bar{k}/k)$-module by

$$(\sigma f)(\tau) = f(\tau\sigma), \quad f \in \mathrm{Ind}\,\bar{V}, \quad \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

Every element x of $\mathcal{G}$ can be written uniquely $x = g \cdot s(\sigma)$ with $g \in G(\bar{k})$ and $\sigma \in \mathrm{Gal}(\bar{k}/k)$, and we define an action of $\mathcal{G}$ on $\bar{V}$ by setting

$$(xf)(\tau) = (s(\tau) \cdot g \cdot s(\sigma)s(\tau\sigma)^{-1})f(\tau\sigma).$$

Then $g \cdot s(\sigma)$ acts σ-linearly on $\mathrm{Ind}\,\bar{V}$, and this way we get a representation of $\mathcal{G}$ on $\mathrm{Ind}\,\bar{V}$. There is an action of K on $\mathrm{Ind}\,\bar{V}$,

$$(\lambda f)(\tau) = \lambda f(\tau), \quad \lambda \in K, \quad f \in \mathrm{Ind}\,\bar{V}, \quad \tau \in \mathrm{Gal}(\bar{k}/k),$$

and so we have defined an object of $\mathsf{Repf}(\mathcal{G})_{(K)}$. See Lattermann 1989, 4.2.7.

2.9 In the situation of 2.7, we define a functor

$$\mathsf{Repf}(\mathcal{G})_{(K)} \to \mathsf{Repf}_K(G).$$

Let V be a finite-dimensional $\bar{k}$-vector space on which $\mathcal{G}$ and K act compatibly, and let V_1 be the subspace of V on which the two actions of K coincide. Then V_1 is a finite-dimensional $\bar{k}$-vector space on which $G(\bar{k})$ acts compatibly with the actions of $\mathrm{Gal}(\ddot{k}/K)$, and so this action arises from a representation of G on a finite-dimensional K-vector space. See Lattermann 1989, 4.2.8.

2.10 The functors

$$\mathsf{Repf}_K(G) \to \mathsf{Repf}(\mathcal{G})_{(K)}$$
$$\mathsf{Repf}(\mathcal{G})_{(K)} \to \mathsf{Repf}_K(G)$$

defined in 2.8 and 2.9 are quasi-inverse. See Lattermann 1989, 4.2.9.

3 Tannakian categories and Galois groupoids

Preliminaries

PROPOSITION 3.1 *Let* T *be a tannakian category over* k*, and let* k' *be a finite separable extension of* k*. If* T *has a* k'*-valued fibre functor, then* $\mathsf{T}_{(k')}$ *is neutral.*

PROOF Let ω be a k'-valued fibre functor on T. For any (X,μ) in $\mathsf{T}_{(k')}$, $\omega(X)$ has a natural $k' \otimes_k k'$-structure, and $(X,\mu) \rightsquigarrow \omega(X)$ is a $k' \otimes_k k'$-valued fibre functor (see the proof of I, 7.16). The multiplication map $k' \times k' \to k'$ defines a homomorphism of k-algebras $k' \otimes_k k' \to k'$. Now $(X,\mu) \rightsquigarrow \omega(X) \otimes_{k' \otimes_k k'} k'$ is a k'-valued fibre functor on $\mathsf{T}_{(k')}$ (I, 7.6). □

COROLLARY 3.2 *Let* ω' *be the fibre functor* $(X,\mu) \rightsquigarrow \omega(X) \otimes_{k' \otimes_k k'} k'$ *on* $\mathsf{T}_{(k')}$*. Then* ω' *defines an equivalence*

$$\mathsf{T}_{(k')} \to \mathsf{Repf}(\mathcal{A}ut^{\otimes}(\omega')),$$

of tensor categories over k'*.*

PROOF Combine the proposition with II, 3.1. □

PROPOSITION 3.3 *Let* T *be a tannakian category over* k*. If* T *has a* $\bar{k}$*-valued fibre functor* ω*, then* $\mathsf{T}_{(\bar{k})}$ *is neutral.*

PROOF As in the above proof, for any subfield k' of $\bar{k}$ finite over k, the functor

$$(X,\mu) \rightsquigarrow \omega(X) \otimes_{k' \otimes_k \bar{k}} \bar{k} \otimes_k \bar{k}$$

is a $\bar{k} \otimes_k \bar{k}$-valued functor on $\mathsf{T}_{(k')}$. Using the descripion of $\mathsf{T}_{(\bar{k})}$ in I, 7.19, we deduce that there exists a $\bar{k} \otimes_k \bar{k}$-valued fibre functor $\widetilde{\omega}$ on $\mathsf{T}_{(k)}$ agreeing with this fibre functor on each subcategory $\mathsf{T}_{(k')}$. Now choose a homomorphism $\bar{k} \otimes_k \bar{k} \to \bar{k}$, and set $\bar{\omega} = \widetilde{\omega} \otimes_{\bar{k} \otimes_k \bar{k}} \bar{k}$. □

PROPOSITION 3.4 *Let* T *be a tannakian category over* k*. Any two* $\bar{k}$*-valued fibre functors on* T *are isomorphic over* $\bar{k}$*.*

PROOF We saw in the last proof that every $\bar{k}$-valued fibre functor ω on T, extends to a $\bar{k} \otimes_k \bar{k}$-valued fibre functor $\widetilde{\omega}$ on $\mathsf{T}_{(\bar{k})}$. If ω' is a second $\bar{k}$-valued fibre functor, then, because $\mathsf{T}_{(\bar{k})}$ is neutral, $\widetilde{\omega}$ and $\widetilde{\omega}'$ are isomorphic, and this implies that ω and ω' are isomorphic. □

PROPOSITION 3.5 *Let* T *be a tannakian category over* k*, and let* ω *be a* $\bar{k}$*-valued fibre functor on* T*. For any algebraic tannakian subcategory* T_0 *of* T*, there exists a finite extension* $K \subset \bar{k}$ *of* k *such that* ω *restricts to a* K*-valued fibre functor on* T_0*.*

PROOF Let $\mathsf{T}_0 = \langle X \rangle^{\otimes}$, and let $V = \omega(X)$. There exists a family t of tensors for V such that $\mathcal{A}ut^{\otimes}(\omega) = \mathcal{A}ut(V,t)$. Now the pair (V,t) has a model over a finite extension $K \subset \bar{k}$ of k and $\omega|\langle X \rangle^{\otimes}$ is K-valued. □

Let T be a tannakian category over k, and let ω be a $\bar{k}$-valued fibre functor on T (assumed to exist). For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, define ${}^{\sigma}\omega$ to be the fibre functor $X \rightsquigarrow \omega(X) \otimes_{\bar{k},\sigma} \bar{k}$. Then

$$ {}^{\sigma_2}({}^{\sigma_1}\omega) \simeq {}^{\sigma_2\sigma_1}\omega. $$

Define

$$ \mathcal{G}(\omega) = \coprod_{\sigma\in\Gamma} \mathrm{Isom}^{\otimes}({}^{\sigma}\omega, \omega), $$

and let $\pi : \mathcal{G}(\omega) \to \mathrm{Gal}(\bar{k}/k)$ be the map sending the elements of $\mathrm{Isom}^{\otimes}({}^{\sigma}\omega, \omega)$ to σ. From an isomorphism $f : {}^{\sigma}\omega \to \omega$ and an element ρ of $\mathrm{Gal}(\bar{k}/k)$, we obtain an isomorphism ${}^{\rho}f : {}^{\rho\sigma}\omega \to {}^{\rho}\omega$ by applying the functor $- \otimes_{\bar{k},\rho} \bar{k}$. We define the product of the elements $f_1 : {}^{\sigma_1}\omega \to \omega$ and $f_2 : {}^{\sigma_2}\omega_2 \to \omega$ of $\mathcal{G}$ by the rule

$$ f_1 \cdot f_2 = f_1 \circ {}^{\sigma_1}f_2, \qquad {}^{\sigma_1\sigma_2}\omega \xrightarrow[{}^{\sigma_1}f_2]{} {}^{\sigma_1}\omega \xrightarrow[f_1]{} \omega \quad (f_1\cdot f_2). $$

Then

$$ \pi(f_1 \cdot f_2) = \sigma_1\sigma_2 = \pi(f_1)\pi(f_2), $$

and

$$ \begin{cases} (f_1 \cdot f_2) \cdot f_3 = f_1 \circ {}^{\sigma_1}f_2 \circ {}^{\sigma_1\sigma_2}f_3 = f_1 \cdot (f_2 \cdot f_3) \\ f \cdot \mathrm{id}_{\omega} = f = \mathrm{id}_{\omega} \cdot f \\ f \cdot {}^{\pi(f)^{-1}}(f^{-1}) = \mathrm{id}_{\omega} = {}^{\pi(f)^{-1}}(f^{-1}) \cdot f, \end{cases} $$

and so $\mathcal{G}$ has a group structure for which π is a homomorphism. We have an exact sequence

$$ 1 \longrightarrow \mathrm{Aut}^{\otimes}(\omega) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \longrightarrow 1. $$

of abstract groups. The homomorphism π is surjective because any two fibre functors on T with values in $\bar{k}$ are isomorphic. Every algebraic tannakian subcategory admits a fibre functor with values in a finite extension K of k in $\bar{k}$, which can be used to define a section of π over $\mathrm{Gal}(\bar{k}/K)$. From C and ω we have constructed a $\bar{k}/k$-Galois groupoid $\mathcal{G} \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_k(\omega)$ with kernel $\mathcal{A}ut^{\otimes}_{\bar{k}}(\omega)$.

THEOREM 3.6 *Let T be an essentially small tannakian category over k and ω a $\bar{k}$-valued fibre functor on T.*

(a) *The extension $\mathcal{G}(\omega)$ is a $\bar{k}/k$-Galois groupoid with kernel $\mathcal{A}ut^{\otimes}_{\bar{k}}(\omega)$.*

(b) *The functor $\mathsf{T} \to \mathsf{Repf}(\mathcal{G}(\omega))$ defined by ω is an equivalence of tensor categories over k.*

Conversely, if $\mathcal{G}$ is a $\bar{k}/k$-Galois groupoid, then $\mathsf{Repf}(\mathcal{G})$ is a tannakian category over k, the forgetful functor is a fibre functor, and $\mathcal{G} \simeq \mathcal{A}ut^{\otimes}_k(\omega_{\mathrm{forget}})$.

PROOF Statement (a) is proved above. In proving (b), we may suppose that T is algebraic. Then T has a fibre functor ω over a finite exension k', and

$$ \mathsf{T} \xrightarrow{\omega} \mathsf{Repf}(\mathcal{G}) $$

is an equivalence because it becomes an equivalence after we have extended scalars to k',

$$ \mathsf{T}_{(k')} \xrightarrow[3.2]{\sim} \mathsf{Repf}(\mathcal{A}ut^{\otimes}(\omega')) \xrightarrow[2.10]{\sim} \mathsf{Repf}(\mathcal{G})_{(k')}. $$

The final statement is proved in 2.5. □

Notes

3.7 In 1.3 we defined a 2-category $\mathcal{GGrpd}$ of $\bar{k}/k$-Galois groupoids. Similarly (IV, 7.2) we can define a 2-category $\mathcal{Tann}^{\bullet}$ of essentially small tannakian categories over k equipped with a $\bar{k}$-valued fibre functor ω. There are 2-functors

$$\mathcal{Tann}^{\bullet\mathrm{opp}} \to \mathcal{GGrpd}, \quad (\mathsf{T}, \omega) \rightsquigarrow \mathcal{Aut}_k^{\otimes}(\omega)$$
$$\mathcal{GGrpd} \to \mathcal{Tann}^{\bullet\mathrm{opp}}, \quad \mathcal{G} \rightsquigarrow \mathsf{Repf}(\mathcal{G}).$$

These define biequivalence.

We have classified tannakian categories over k equipped with a $\bar{k}$-valued fibre functor. Later (IV, 10.1) we shall see that all essentially small tannakian categories over k with smooth band admit a $\bar{k}$-valued fibre functor.

3.8 Let k' be a finite extension of k. Essentially small tannakian categories over k equipped with a fibre functor over k' correspond to $\bar{k}/k$-Galois groupoids equipped with a splitting over k'.

3.9 To give a tannakian category T over k together with a $\bar{k}$-valued fibre functor ω is essentially the same as giving a tannakian category $\bar{\mathsf{T}}$ over $\bar{k}$ together with a $\bar{k}$-valued fibre functor $\bar{\omega}$ and a descent datum. If $\rho : \mathcal{Aut}_{\bar{k}}^{\otimes}(\omega) \to \mathrm{GL}_V$ is the representation attached to $(\bar{\mathsf{T}}, \bar{\omega})$, then giving the descent datum amounts to extending ρ to a representation $\mathcal{Aut}_{\bar{k}}^{\otimes}(\omega) \to \mathcal{G}_V$ of the Galois gerbe attached to (T, ω).

4 Examples

Galois groupoids for $\mathbb{C}/\mathbb{R}$

4.1 Let

$$1 \to G(\mathbb{C}) \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

be a $\mathbb{C}/\mathbb{R}$-Galois groupoid with kernel G. Choose an $s \in \mathcal{G}$ such that $\pi(s) = \iota$ (complex conjugation). Let $\sigma : G(\mathbb{C}) \to G(\mathbb{C})$ be conjugation by s, and let $c = s^2$. Then σ is ι-linear, $c \in G(\mathbb{C})$, and

$$\sigma^2 = \mathrm{ad}(c), \quad \sigma(c) = c. \tag{*}$$

Every triple (G, σ, c) satisfying these conditions arises in this way from a $\mathbb{C}/\mathbb{R}$-Galois groupoid. When we replace the section s with sm, $m \in G(\mathbb{C})$, the pair (σ, c) is replaced by $(\sigma \circ \mathrm{ad}(m), \sigma(m)cm)$.

4.2 Let (G, σ, c) be a triple satisfying the conditions in 4.1, and let $\mathcal{G}$ be the corresponding $\mathbb{C}/\mathbb{R}$-Galois groupoid. Then $\mathsf{Repf}(\mathcal{G})$ is a tannakian category over $\mathbb{R}$ with a $\mathbb{C}$-valued (forgetful) fibre functor ω such that $G = \mathcal{Aut}^{\otimes}(\omega)$. Together with Theorem 3.6, this gives a description of tannakian categories over $\mathbb{R}$ that we exploit in Chapter VI.

5 Tannakian categories with band of multiplicative type

Let T be a tannakian category over a field k. If $\mathcal{Aut}^{\otimes}(\omega)$ is commutative for one fibre functor ω, then there exists an affine group scheme G over k such that $G_R \simeq \mathcal{Aut}^{\otimes}(\omega_R)$

for any fibre functor ω_R over a k-algebra R. We then say that T has commutative band G. In this section we study tannakian categories with band a group scheme of multiplicative type. They form an important class – for example, the category of motives over a finite field is conjecturally of this type.

Throughout, $\bar{k}$ is a separable algebraic closure of a field k, and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. We implicitly assume that tannakian categories are essentially small.

Review of the Brauer group

5.1 Two central simple algebras A and B over k are ***similar*** if $M_m(A) \approx M_n(B)$ for some m, n. This is an equivalence relation, and we let $[A]$ denote the similarity class of A. The ***Brauer group*** $\mathrm{Br}(k)$ of k is the set of similarity classes of central simple algebras over k with the multiplication defined by

$$[A] \cdot [B] = [A \otimes B].$$

It is a commutative group with identity element $[k]$. Let V denote A regarded as a vector space over k. Then $A \otimes_k A^{\mathrm{opp}}$ acts on V by $(a \otimes b)(v) = avb$, and $A \otimes_k A^{\mathrm{opp}} \simeq \mathrm{End}_{k\text{-linear}}(V)$. Therefore $[A]^{-1} = [A^{\mathrm{opp}}]$.

5.2 For a subfield L of $\bar{k}$, let $\mathrm{Br}(L/k)$ denote the kernel of

$$[A] \mapsto [A \otimes L] : \ \mathrm{Br}(k) \to \mathrm{Br}(L).$$

Then $\mathrm{Br}(k) = \bigcup \mathrm{Br}(L/k)$, where L runs over the subfields of $\bar{k}$ finite over k.

5.3 Let $L \subset \bar{k}$ be a finite Galois extension of k. Every class in $\mathrm{Br}(L/k)$ is represented by a central simple algebra A over k containing L as a maximal subfield (so $[A : k] = [L : k]^2$). Let E be the group of invertible elements $\alpha \in A$ such that $\alpha L \alpha^{-1} = L$. Then each $\alpha \in E$ defines an element $x \mapsto \alpha x \alpha^{-1}$ of $\mathrm{Gal}(L/k)$, and the Noether–Skolem theorem implies that every element of $\mathrm{Gal}(L/k)$ arises from an $\alpha \in E$. Because L is a maximal subfield of A, the centralizer of L in A is L itself, and so the sequence

$$1 \to L^\times \longrightarrow E \xrightarrow{\pi} \mathrm{Gal}(L/k) \to 1$$

is exact. Let $s : \ \mathrm{Gal}(L/k) \to E$ be a section to π, and write

$$s(\sigma)s(\tau) = a_{\sigma,\tau}s(\sigma\tau).$$

Then $a_{\sigma,\tau}$ is a 2-cocycle, whose class in $H^2(L/k, L^\times)$ we denote by $\gamma(A)$.

THEOREM 5.4 *The map $A \mapsto \gamma(A)$ induces an isomorphism*

$$\mathrm{Br}(L/k) \to H^2(L/k, \mathbb{G}_m).$$

On passing to the limit over the finite Galois extensions of k in $\bar{k}$, we obtain an isomorphism

$$\mathrm{Br}(k) \simeq H^2(k, \mathbb{G}_m) \quad \textit{(Galois cohomology group).}$$

PROOF See Serre 2001. □

Numeric study of abelian categories

In this section, we define some numeric invariants of abelian and tannakian categories. Later we determine these invariants for tannakian categories with band of multiplicative type.

5.5 Let k be a field and A a locally finite k-linear abelian category. Recall that an object of an abelian category is simple if it is nonzero and contains no proper nonzero subobject. An object is ***isotypic*** if it is isomorphic to a direct sum of copies of a single simple object, whose isomorphism class is then called the ***type*** of the isotypic object. Two isotypic objects M and N have the same type if and only if $M^{\oplus m}$ is isomorphic to $N^{\oplus n}$ for some $m, n > 0$.

5.6 For a finite extension k' of k, we let $\mathsf{A}_{(k')}$ denote the category whose objects are the pairs consisting of an object of A together with an action of k'. There is an exact functor

$$i_{k'/k} : \mathsf{A} \to \mathsf{A}_{(k')}, \quad X \rightsquigarrow k' \otimes_k X,$$

and $\mathrm{Hom}_{\mathsf{A}_{(k')}}(k' \otimes X, k' \otimes Y) \simeq k' \otimes \mathrm{Hom}_{\mathsf{A}}(X, Y)$. The forgetful functor

$$r_{k'/k} : \mathsf{A}_{(k')} \to \mathsf{A}, \quad (X, \mu) \rightsquigarrow X,$$

is also exact, and is right adjoint to $i_{k'/k}$. See I, 6.12.

5.7 If M is a simple object of A, then the smallest abelian subcategory $\langle M \rangle$ of A containing M is equivalent to the category of finite-dimensional vector spaces over the division algebra $D = \mathrm{End}(M)$. The object M remains semisimple under an extension k'/k if and only if $D \otimes_k k'$ is semisimple, i.e., a product of matrix algebras over division algebras. This is true for all k' if it is true for one perfect field containing k, or, equivalently, if the centre of D is separable over k. One then says that M is ***absolutely semisimple***.

5.8 The Grothendieck group $K(\mathsf{A})$ of A is the free abelian group generated by the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects of A. If k' is an extension of k and M is a simple object of A such that $k' \otimes M$ is semisimple, then the decomposition of $k' \otimes M$ can be read off from the structure of $k' \otimes_k D = D'$: if

$$D' = \prod\nolimits_{1 \le i \le r} M_{n_i}(D_i), \quad D_i \text{ a division algebra}, \quad n_i \ge 1,$$

then $k' \otimes M$ decomposes into r isotypic components each of the form $S_i^{\oplus n_i}$ with S_i simple. If k' is Galois over k with Galois group Π, then the $[S_i]$, $1 \le i \le r$, are conjugate under the action of Π, and so there is a canonical bijection

$$\Sigma(\mathsf{A}) \simeq \Sigma(\mathsf{A}_{(k')})/\Pi.$$

5.9 Now assume that A is semisimple. Recall that this means that every object is a sum of its simple subobjects, and hence a finite direct sum of simple objects (I, 6.15). If e is simple, then every nonzero endomorphism of e is an isomorphism. Therefore, $\mathrm{End}(e)$ is a division algebra. It contains k in its centre, and is finite-dimensional over k. Let re denote the direct sum of r copies of e. Then $\mathrm{End}(re) \simeq M_r(\mathrm{End}(e))$. If e' is a second simple object, then either $e \approx e'$ or $\mathrm{Hom}(e, e') = 0$. Therefore, if $x = \sum r_i e_i$ ($r_i \ge 0$) and $y = \sum s_i e_i$ ($s_i \ge 0$) are two objects of A expressed as sums of copies of simple objects e_i such that $e_i \not\approx e_j$ if $i \ne j$, then

$$\mathrm{Hom}(x, y) \simeq \prod M_{s_i, r_i}(\mathrm{End}(e_i)).$$

Thus, the category A is determined up to equivalence by

(a) the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects in A;

(b) for each $\sigma \in \Sigma$, the isomorphism class $[D_\sigma]$ of the endomorphism algebra of a representative of σ.

We call $\Sigma(\mathsf{A})$ and $([D_\sigma])_{\sigma\in\Sigma(\mathsf{A})}$ the ***numeric characters*** of A.

Let k_σ denote the centre of D_σ. The isomorphism class of D_σ as a k-algebra is determined by the isomorphism class of k_σ as an extension of k and the class ζ_σ of D_σ in $\mathrm{Br}(k_\sigma)$.

5.10 Suppose in addition that A has a k-linear tensor structure. We call the ***numeric characters*** of A the following data:

(a) the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects of A;

(b) the multiplication in $K(\mathsf{A}) = \mathbb{Z}^{(\Sigma)}$ defined by the tensor product; this amounts to giving, for all $\sigma, \tau \in \Sigma$, the natural numbers $c^\rho_{\sigma,\tau}$ such that

$$\sigma\tau = \sum\nolimits_\rho c^\rho_{\sigma,\tau}\rho;$$

(c) the map

$$\sigma \mapsto (k_\sigma, \zeta_\sigma), \quad \zeta_\sigma \in \mathrm{Br}(k_\sigma),$$

considered in (5.9);

(d) the function

$$\dim : \mathsf{A} \to \mathbb{Z}$$

when A is assumed to be tannakian.

5.11 Note that to give (a) and (b) is equivalent to giving the commutative ring $K(\mathsf{A})$ together with its "effective subset"

$$\begin{aligned} K(\mathsf{A})^{\mathrm{eff}} &= \text{submonoid of } K(\mathsf{A}) \text{ generated by } [M], \quad M \in \mathrm{ob}\, A, \\ &= \mathbb{N}^{(\Sigma)}. \end{aligned}$$

Indeed, Σ can be recovered from $K(\mathsf{A})^{\mathrm{eff}} \subset K(\mathsf{A})$ as the set of minimal nonzero elements of $K(\mathsf{A})^{\mathrm{eff}}$. The datum (c) then allows us to recover, as described above, the functorial variance of the ring $K(\mathsf{A}_{(k')})$ with respect to k'. Finally, the datum in (d) allows us to recover in principle the characteristic polynomial (in particular, the trace and determinant) of a semisimple object M of A in terms of the reduced characteristic polynomial (resp. reduced trace, reduced norm) in the central simple algebra $\mathrm{End}(M_i)$ corresponding to the isotypic components M_i of M. Let M be isotypic of rank n, and suppose that $D = \mathrm{End}(M)$ has centre Z of rank r over k; then $n = n'r$ where n' is the rank of M over Z, and D is of rank d^2 over Z with $d|n'$. Moreover,

$$\begin{aligned} \det_M f &= \mathrm{Nm}_{Z/k}(\mathrm{rNm}_{D/Z} f)^{n'/d} \\ \mathrm{Tr}_M f &= \mathrm{Tr}_{Z/k}(\mathrm{rTr}_{D/Z} f)\frac{n'}{d} \\ \mathrm{Pol}_M(f,t) &= \mathrm{Nm}_{Z/k}(\mathrm{rPol}_{D/Z}(f,t))^{n'/d}, \quad n'/d = n/dr. \end{aligned}$$

5.12 The knowledge of the numeric characters of a tannakian category A does not allow us to reconstruct it up to equivalence, even if k is an algebraically closed field of characteristic zero (in which case the datum (c) is vacuous) and A is semisimple, even

in the particular case where A is the category of representations of a finite group.[3] As an exception to this remark, we shall show that a tannakian category with diagonalizable band is determined by its numeric characters.

Neutral tannakian categories with diagonalizable band

Let G be an affine group scheme over k.

5.13 A k-***character*** of G is a homomorphism $\chi : G \to \mathbb{G}_m$ (over k). The k-characters of G form an abelian group $X(G)$.

Let $(\mathcal{O}(G), \Delta)$ be the Hopf algebra of G. A ***group-like element*** of G is an element $m \in \mathcal{O}(G)^\times$ such that $\Delta(m) = m \otimes m$.

A k-character χ of G corresponds to a homomorphism of k-algebras $k[T, T^{-1}] \to \mathcal{O}(G)$ compatible with the comultiplications. Such a homomorphism sends T to a group-like element $a(\chi)$ of $\mathcal{O}(G)$ and is determined by it. Therefore $\chi \leftrightarrow a(\chi)$ is a one-to-one correspondence between the characters of G and the group-like elements of $\mathcal{O}(G)$.

5.14 Let $r : G \to \mathrm{GL}_V$ be a representation of G on a finite-dimensional k-vector space V and $\rho : V \to V \otimes \mathcal{O}(G)$ the corresponding co-action. For a k-character χ of G, we let

$$V_\chi = \{v \in V \mid \rho(v) = v \otimes a(\chi)\},$$

and call it the ***eigenspace for*** G ***with character*** χ.

The following conditions on G are equivalent:

(a) $\mathcal{O}(G)$ is spanned (as a k-vector space) by its group-like elements;

(b) for every finite-dimensional representation (V, r) of G,

$$V = \bigoplus_{\chi \in X(G)} V_\chi.$$

An affine group scheme satisfying these equivalent conditions is said to be ***diagonalizable.*** See Milne 2017, 12.12.

5.15 Let M be an abelian group written multiplicatively, and let $k[M]$ be the k-vector space with basis M. When equipped with the multiplication extending that of M and the comultiplication determined by

$$\Delta(m) = m \otimes m, \quad m \in M,$$

$k[M]$ becomes a Hopf algebra. We let $D(M)$ denote the corresponding affine group scheme over k. Then $D(M)$ is diagonalizable because the elements of M are group-like. Conversely, if G is diagonalizable, then $G \simeq D(M)$ with M the group of group-like elements in $\mathcal{O}(G)$. The functor D is an equivalence from the category of abelian groups to the category of affine group schemes of multiplicative type.

For example, $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}/p\mathbb{Z}) = \mu_p$, which is an infinitesimal group scheme if $\mathrm{char}(k) = p$ and a finite étale group scheme otherwise.

[3] The first order for which there are two nonisomorphic noncommutative groups is 8, and their character groups do not distinguish them. But see II, 8.10.

5.16 Let G be diagonalizable, and let $M = X(G)$, so $G \simeq D(M)$. Then $\mathsf{Rep}(G)$ is the category of finite-dimensional k-vector spaces V equipped with an M-gradation,

$$V = \bigoplus\nolimits_{m\in M} V_m$$

(cf. II, 9.2).

THEOREM 5.17 *A neutral tannakian category has diagonalizable band if and only if*

(a) *it is semisimple,*

(b) *the tensor product of any two simple objects is simple.*

PROOF Let C be a neutral tannakian category, let ω be k-valued fibre functor, and let $G = \mathcal{A}ut^{\otimes}(\omega)$.

If G is diagonalizable, say, $G = D(M)$, then ω defines an equivalence of C with the category $\mathsf{Repf}(D(M))$, which has the properties (a) and (b).

For the converse, suppose that C has the properties (a) and (b). If S is a simple object of C, then $\mathrm{ev}_S : S^\vee \otimes S \to \mathbb{1}$ is a nonzero morphism between simple objects, hence an isomorphism. Therefore $\omega(S)^\vee \otimes \omega(S) \simeq k$, and so $\omega(S)$ has dimension 1. Every finite-dimensional representation of G is a sum of one-dimensional representations, which implies that G satisfies condition 5.14(b) to be diagonalizable. □

Tannakian categories with diagonalizable band

Let $G = D(M)$ be a diagonalizable algebraic group over k. The simple objects of $\mathsf{Repf}(G)$ are the one-dimensional k-vector spaces on which G acts through a character. The abelian group M can be recovered from $\mathsf{Repf}(G)$ as the set of isomorphism classes of simple objects with the addition defined by tensor product. Alternatively, it is the group of types of the isotypic objects.

PROPOSITION 5.18 *Let T be a tannakian category over k admitting a fibre functor over $\bar{k}$. Then T has diagonalizable band if and only if*

(a) *it is semisimple, and*

(b) *the tensor product of any two isotypic objects is isotypic.*

In this case, the set of types of isotypic objects M forms a group under tensor product, and the band of T is $D(M)$.

PROOF If T has diagonalizable band, then $\mathsf{T}_{(\bar{k})}$ is equivalent to $\mathsf{Repf}(D(M))$ with M set of types of isotypic classes in $\mathsf{T}_{(\bar{k})}$. Therefore $\mathsf{T}_{(\bar{k})}$ satisfies (a) and (b) (5.17), and it follows that T satisfies these conditions (see I, 6.17 and 5.7)

Conversely, suppose that (a) and (b) hold, and let M be the set of types of isotypic objects. As the tensor product of two isotypic objects is again isotypic, M has the structure of an abelian group, and the isotypic objects are the homogeneous objects for an M-gradation on T. The gradation defines a homomorphism

$$D(M) \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{T}}) = Z(\text{band of } \mathsf{T})$$

(II, 9.2) and it remains to prove that this induces an isomorphism of $D(M)$ onto the band of T. It suffices to do this after an extension of the base field, and so we may suppose that there is a k-valued fibre functor ω on T. Once we have shown that the simple objects of T have (categorical) dimension 1, the functor ω will define an equivalence from T to

the tannakian category of M-graded finite-dimensional k-vector spaces, and hence to $\mathrm{Repf}(D(M))$, as required. If S is a simple object of dimension r, then $\dim(S \otimes S^\vee) = r^2$ by I, 4.17. As $S \otimes S^\vee$ is isotypic, the coevaluation morphism shows that it is trivial, and so $\mathrm{Hom}(\mathbb{1}, S \otimes S^\vee)$ has dimension r^2. Recall that $\mathrm{End}(S) = \mathrm{Hom}(\mathbb{1}, S \otimes S^\vee)$; see (10), p. 18. As ω induces an isomorphism

$$\mathrm{End}(S) \to \mathrm{End}(\omega(S)) \simeq M_r(k)$$

and $\mathrm{End}(S)$ is a division algebra, we see that $r = 1$. □

EXAMPLE 5.19 Let A be a central simple algebra over k. After possibly replacing A with a similar algebra, we may assume that A contains a maximal field L Galois over k. Consider the extension

$$1 \to L^\times \to E \to \mathrm{Gal}(L/k) \to 1 \qquad (*)$$

defined in 5.3. For each $\sigma \in \mathrm{Gal}(L/k)$, let $s(\sigma)$ be an element of E mapping to σ. Then

$$s(\sigma) \cdot \sigma s(\tau) = a_{\sigma,\tau} s(\sigma\tau),$$

where $(a_{\sigma,\tau})_{\sigma,\tau}$ is a 2-cocycle representing the cohomology class of A in $\mathrm{Br}(k)$.

Let V denote A regarded a k-vector space, and let L act on it by left multiplication. The action of A on V defines an action r of E on V, and

$$\mathrm{End}(V, r) \simeq A^{\mathrm{opp}}$$

(see I, 8.6).

Let ${}^\sigma V$ equal V as a k-vector space, but with $c \in L$ acting as σc. Then left multiplication by $s(\sigma)$ is an L-linear map ${}^\sigma V \to V$, which we again denote by $s(\sigma)$. Now

$$s(\sigma) \circ \sigma s(\tau) = s(\sigma\tau)$$

as L-linear maps ${}^{\sigma\tau}V \to V$.

Let T be the tannakian category of representations of E on finite-dimensional k-vector spaces, and let ω be the forgetful fibre functor. Note that E acts on V through the action of A, and we have $\omega(V) = V$ and ${}^\sigma\omega(V) = {}^\sigma V$. It follows that the cohomology class of T is the image of the class of the extension (*) under the canonical map

$$H^2(L/k, L^\times) \to H^2(\bar{k}/k, \bar{k}^\times) = H^2(k, \mathbb{G}_m).$$

EXERCISE 5.20 Show that a semisimple tannakian category has diagonalizable band if and only if, for all isotypic objects X,

$$\dim_k(\mathrm{End}(X)) = \dim(X)^2.$$

Let T be a tannakian category whose band is the diagonalizable group $D(M)$. An element m of M determines a homomorphism $\mathbb{Z} \to M$ and hence a homomorphism

$$\varphi_m : H^2(k, D(M)) \to H^2(k, D(\mathbb{Z})) \simeq \mathrm{Br}(k).$$

PROPOSITION 5.21 *For any isotypic object X of type m, the homomorphism φ_m sends the class of T in $H^2(k, D(M))$ to the class of $\mathrm{End}(X)^{\mathrm{opp}}$ in $\mathrm{Br}(k)$.*

PROOF We prove this using Galois groupoids, which requires us to assume M has no p-torsion if k has characteristic $p \neq 0$. For simplicity, we also assume that M is finitely generated. For a proof using gerbes, see Saavedra 1972, VI, 3.5.3.

On choosing a fibre functor over $\bar{k}$, we get a Galois groupoid

$$1 \to D(M)(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

whose cohomology class in $H^2(k, D(M))$ is the cohomology class of T.

Let X be an isotypic object of type m, and let $A = \mathrm{End}(X)$. After possibly replacing X with a different isotypic object of type m, we may suppose that A contains a maximal subfield $L \subset \bar{k}$ Galois over k. Let

$$1 \to L^\times \to E \to \mathrm{Gal}(L/k) \to 1$$

the extension attached to A as in 5.3. The category $\langle X \rangle^\otimes$ is equivalent to the category of the representations of this extension. Therefore the cohomology class of A^{opp} in $H^2(k, \mathbb{G}_m)$ is equal to the cohomology class of $\langle X \rangle^\otimes$ (5.19). Since φ_m sends the class of T in $H^2(k, D(M))$ to the class of $\langle X \rangle^\otimes$ in $H^2(k, \mathbb{G}_m)$, this completes the proof. □

From φ_m, we get a pairing

$$H^2(k, D(M)) \times M \to \mathrm{Br}(k), \quad x, m \mapsto \varphi_m(x).$$

PROPOSITION 5.22 *Assume that M is finitely generated. The homomorphism*

$$H^2(k, D(M)) \to \mathrm{Hom}(M, \mathrm{Br}(k)) \tag{64}$$

defined by the above pairing is an isomorphism.

PROOF When $M = \mathbb{Z}$, the homomorphism $\varphi_1 : H^2(k, \mathbb{G}_m) \to \mathrm{Br}(k)$ is the canonical isomorphism $H^2(k, \mathbb{G}_m) \simeq \mathrm{Br}(k)$. It follows that the proposition holds for $\mathbb{Z}$, hence for $\mathbb{Z}^{(I)}$, where I is any finite set, because both sides of (64) transform sums into products.

As M is finitely generated M, there is an exact sequence

$$0 \to \mathbb{Z}^{(I)} \to \mathbb{Z}^{(J)} \to M \to 0$$

with I and J finite, and hence an exact sequence

$$0 \to D(M) \to \mathbb{G}_m^J \to \mathbb{G}_m^I \to 0.$$

As for H^2, we have $H^1(k, \mathbb{G}_m^I) \simeq H^1(k, \mathbb{G}_m)^I$, which is zero, and so we have an exact sequence

$$0 \to H^2(k, D(M)) \to H^2(k, \mathbb{G}_m)^J \to H^2(k, \mathbb{G}_m)^I.$$

On comparing this with the exact sequence

$$0 \to \mathrm{Hom}(M, \mathrm{Br}(k)) \to \mathrm{Hom}(\mathbb{Z}^{(J)}, \mathrm{Br}(k)) \to \mathrm{Hom}(\mathbb{Z}^{(I)}, \mathrm{Br}(k)),$$

we obtain the statement for M. □

The statement is true also when M is not finitely generated provided one interprets $H^2(k, D(M))$ as an fpqc cohomology group (the fppf groups need not commute with infinite products).

COROLLARY 5.23 *If* $\mathrm{Br}(k) = 0$*, for example, if* k *is separably closed or finite, then every tannakian category over* k *with diagonalizable band is neutral.*

SUMMARY 5.24 Let T be tannakian category over k with band $D(M)$. Then T is determined up to tensor equivalence by a homomorphism

$$u : M \to \mathrm{Br}(k).$$

The category T is graded of type M. The homogeneous objects are the isotypic objects. For each $m \in M$, there is exactly one simple object (up to isomorphism) S_m homogeneous of degree m, and $\mathrm{End}(S_m)$ is a division algebra with centre k and invariant $u(m)$ in $\mathrm{Br}(k)$.

Let $d(m)$ be the order of $u(m)$ in $\mathrm{Br}(k)$. When k is a local or global field,

$$d(m) = \sqrt{\dim(\mathrm{End}(S_m))},$$

but not in general for other fields (period-index problem).

The Grothendieck ring of T is the subring $K(\mathsf{T}) \subset \mathbb{Z}[M]$ determined by

$$\sum n_m[S_m] \in K(\mathsf{T}) \iff n_m \equiv 0 \bmod d(m) \text{ for all } m \in M.$$

A canonical basis of $K(\mathsf{T})$ is formed by the elements $d(m)[S_m]$.

5.25 When $M = \mathbb{Z}^{(I)}$, i.e., $G = \mathbb{G}_m^I$, to give u amounts to giving an element $\xi_i \in \mathrm{Br}(k)$ for each $i \in I$. Choose, for each i, a nonzero isotypic object E_i of degree i, and let $A_i = \mathrm{End}(E_i)$ be a central simple k-algebra with invariant ξ_i. It is possible to reconstruct T (up to tensor equivalence) from the A_i.

Consider first the additive subcategory having as objects the elements $\gamma \in \mathbb{N}^{(I)}$, ($\gamma = (\gamma_i)_{i\in I}$ being interpreted as $\otimes E_i^{\gamma_i}$), with

$$\mathrm{Hom}(\gamma, \gamma') = \begin{cases} \otimes A_i^{\otimes \gamma_i} & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma'. \end{cases}$$

Composition of morphisms is obvious. Define in the obvious way the functor $\otimes$ (addition on $\mathbb{N}^{(I)}$ on the objects, tensor product on the homomorphisms) and the constraints so that one obtains a tensor subcategory. Its karoubian envelope is an abelian symmetric monoidal category, which can be identified with the full subcategory of T formed of objects whose nonzero homogeneous components have degree in $\mathbb{N}^{(I)}$. There is a fibre functor over k^{al}. On the other hand, for all i, if A_i has of rank d_i^2 (therefore E_i is of rank d_i), let $L_i = \det E_i \overset{\text{def}}{=} \bigwedge^{d_i} E_i$ (where E_i denotes the object of the category corresponding to the ith element of the canonical basis of $\mathbb{Z}^{(I)}$, say l_i ...); it is a pre-invertible object ($X \mapsto L_i \otimes X$ is fully faithful), and on inverting the L_i, one obtains (up a tensor equivalence) the category T. When M is not free, one finds an analogous construction in terms of suitable systems of generators $(\gamma_i)_{i\in I}$, but "putting equal to 1" certain elements. These constructions are well adapted for describing the $\otimes$-functors of T into a Tannakian category, and to describing base change. For this last reason notably, it would be maladroit in the general case to take the E_i to be simple.

Algebraic groups of multiplicative type and their representations

An algebraic group scheme G over k is of ***multiplicative type*** if it becomes diagonalizable over some extension of k, in which case it becomes diagonalizable over a finite separable extension of k. We let

$$X^*(G) = X(G_{\bar{k}}) \stackrel{\text{def}}{=} \text{Hom}(G_{\bar{k}}, (\mathbb{G}_m)_{\bar{k}}).$$

The functor $G \rightsquigarrow X^*(G)$ is a contravariant equivalence from the category of algebraic groups of multiplicative type over k to the category of finitely generated $\mathbb{Z}$-modules equipped with a continuous action of $\Gamma \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$.

PROPOSITION 5.26 *Let G be an algebraic group of multiplicative type over k. Then* $\mathsf{Repf}(G)$ *is a k-linear semisimple abelian category whose isomorphism classes are classified by the orbits of Γ acting on $X^*(G)$. Let (V, r) be the simple representation corresponding to an orbit Ξ, and let $\chi \in \Xi$; then* $\text{End}(V, r) \simeq k(\chi)$*, where $k(\chi)$ is the subfield of $\bar{k}$ fixed by the subgroup of Γ fixing χ.*

PROOF The group G is split by a finite Galois extension K of k – let $\bar{\Gamma} = \text{Gal}(K/k)$. Then $\bar{\Gamma}$ acts on $\mathcal{O}(G_K) \simeq K \otimes \mathcal{O}(G)$ through its action on K. Let (V, r) be a representation of G_K, and let ρ be the corresponding co-action. By a semilinear action of $\bar{\Gamma}$ on (V, r), we mean a semilinear action of $\bar{\Gamma}$ on V fixing ρ. It follows from descent theory (see §6) that the functor $V \rightsquigarrow V_K$ from $\mathsf{Rep}_k(G)$ to the category of objects of $\mathsf{Rep}_K(G_K)$ equipped with a semilinear action of $\bar{\Gamma}$ is an equivalence of categories. Let V be a representation of G over k. Then $K \otimes V$ is a representation of G_K equipped with a semilinear action of $\bar{\Gamma}$. As a representation of G_K, it decomposes into a direct sum

$$K \otimes V = \bigoplus\nolimits_{\chi \in X(G_K)} V_\chi.$$

An element γ of $\bar{\Gamma}$ acts on $K \otimes V$ by mapping V_χ isomorphically onto $V_{\gamma\chi}$. Thus, we see that the set of χ occurring in the sum is stable under the action of $\bar{\Gamma}$. Conversely, if Ξ is an orbit of $\bar{\Gamma}$ in $X(G_K)$, then $\bigoplus_{\chi \in \Xi} V_\chi$ has a natural semilinear action of G_K, and so arises from a (simple) representation of G over k. From this the statement follows easily. □

Tannakian categories with band of multiplicative type

Let T be an essentially small tannakian category over a field k of characteristic zero, and let ω be a fibre functor on T over $\bar{k}$. Assume that $G \stackrel{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\bar{k}}(\omega)$ is an algebraic group of multiplicative type. Then

$$1 \to G(\bar{k}) \to \mathcal{G} \to \Gamma \to 1$$

with

$$\mathcal{G} = \mathcal{A}ut^{\otimes}_k \omega \stackrel{\text{def}}{=} \bigsqcup_{\sigma \in \Gamma} \text{Isom}^{\otimes}({}^{\sigma}\omega, \omega)$$

is a $\bar{k}/k$-Galois groupoid and the functor

$$\mathsf{T} \to \mathsf{Repf}(\mathcal{G})$$

is an equivalence of tensor categories (3.6).

Let $\chi \in X^*(G)$, let $\Gamma(\chi)$ be the subgroup of Γ fixing χ, and let $k(\chi) = \bar{k}^{\Gamma(\chi)}$. Then χ is defined over $k(\chi)$, and from $\chi : G_{k(\chi)} \to \mathbb{G}_m$ (homomorphism of commutative algebraic groups over $k(\chi)$), we get a homomorphism

$$H^2(k(\chi), G_{k(\chi)}) \to H^2(k(\chi), \mathbb{G}_m).$$

On combining this with the restriction map $H^2(k, G) \to H^2(k(\chi), G_{k(\chi)})$ and the isomorphism $H^2(k(\chi), \mathbb{G}_m) \simeq \mathrm{Br}(k(\chi))$, we get a homomorphism

$$H^2(k, G) \to \mathrm{Br}(k(\chi)). \qquad (*)$$

PROPOSITION 5.27 *Let $V(\chi) = \bigoplus_{\chi' \in \Gamma\chi} V_{\chi'}$. Then $V(\chi)$ is simple,* $\mathrm{End}(V(\chi))$ *is a division algebra with centre $k(\chi)$, and the homomorphism (*) sends the class of T in $H^2(k, G)$ to the class of* $\mathrm{End}(V(\chi))$ *in* $\mathrm{Br}(k(\chi))$.

PROOF This is a straightforward consequence of the above definitions (see the case of a torus). □

THEOREM 5.28 *The category T is a locally finite semisimple k-linear abelian category, and the fibre functor ω defines a bijection*

$$\Sigma(\mathsf{T}) \simeq \Gamma\backslash X^*(G),$$

where $G = \mathcal{A}ut^{\otimes}_{\bar{k}}(\omega)$. If $V(\chi)$ is the simple representation corresponding to the orbit $\Gamma\chi$, then $\mathrm{End}(V(\chi))$ *is a division algebra with centre $k(\chi)$, and its class in* $\mathrm{Br}(k(\chi))$ *is the image of the class of T under the homomorphism $*$.*

PROOF This summarizes previous results. □

When do the numeric characters determine a tannakian category up to a numerical equivalence?

The question of deciding whether the numeric characters determine the Tannakian category up to equivalence comes down, for a fixed G, to determining whether an element $\xi \in H^2(k, G)$ is known when $u_\sigma(\xi)$ is known for all $u : G \to (\mathbb{G}_m)_{Z_\sigma/k}$ as before. This is true when G is diagonalizable. We look at some other examples.

5.29 Let G be a one-dimensional torus, hence either $\mathbb{G}_m$ or $\mathbb{G}_m$ twisted by a quadratic extension Z of k. In the second case, we have an exact sequence

$$0 \longrightarrow G \xrightarrow{u} (\mathbb{G}_m)_{Z/k} \xrightarrow{\mathrm{Nm}_{Z/k}} \mathbb{G}_m \longrightarrow 0$$

and we conclude from the exact cohomology sequence and Hilbert's Theorem 90, that $\mathrm{Ker}(u : H^2(k, G) \to H^2(k, (\mathbb{G}_m)_{Z/k})$ is zero. In this case, the numeric characters determine T up to equivalence.

5.30 Let P be the group of multiplicative type conjecturally attached to the category of motives over $\mathbb{F}$. An element of $H^2(\mathbb{Q}, P)$ is zero if its image in $H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\chi]/\mathbb{Q}})$ is zero for all characters χ of P. See the statement (*) and its proof following Lemma 3.15 in Milne 1994a.

The category of F-isocrystals

See Saavedra, VI, §3, p331. Let R_p be the category of F-isocrystals over $\mathbb{F}$. Thus, an object of R_p is a pair (V, F) consisting of a finite-dimensional vector space V over $B(\mathbb{F})$ and a σ-semilinear isomorphism $F: V \to V$. With the obvious tensor structure, R_p becomes a tannakian category over $\mathbb{Q}_p$ whose fundamental group is the universal covering group $\mathbb{G}$ of $\mathbb{G}_m$ (so $X^*(\mathbb{G}) = \mathbb{Q}$). The objects fixed by $\mathbb{G}$ are those of slope zero. If (V, F) is of slope zero, then

$$V^F \stackrel{\text{def}}{=} \{v \in V \mid Fv = v\}$$

is a $\mathbb{Q}_p$-structure on V. The functor $V \rightsquigarrow V^F$ is a $\mathbb{Q}_p$-valued fibre functor on $\mathsf{R}_p^{\mathbb{G}}$.

The category of Tate modules

We let $\mathsf{R}_\ell(\mathbb{F}_q)$ denote the category of finite-dimensional $\mathbb{Q}_\ell$-vector spaces equipped with a continuous action of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. It is a tannakian category over $\mathbb{Q}_\ell$ equipped with a (forgetful) fibre functor. We let $\mathsf{R}_l = \varinjlim R_l(\mathbb{F}_q)$ (see Saavedra 1972, Appendice, 4.3). Let $G = \mathcal{A}ut^{\otimes}(\omega_\ell)$. There is an exact sequence

$$1 \to G^\circ \to G \to \Gamma \to 1.$$

Let $\mathrm{Tate}(\mathbb{F}_{p^m})$ be the Tannakian category over $\mathbb{Q}_\ell$ of all continuous semisimple ℓ-adic representations of the Galois group $T_m = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$; it is equipped with a natural fibre functor

$$\omega_m : \mathrm{Tate}(\mathbb{F}_{p^m}) \to \mathsf{Vecf}(\mathbb{Q}_\ell).$$

We form the direct limit (restriction)

$$\mathrm{Tate}(\mathbb{F}) \to \varinjlim \mathrm{Tate}(\mathbb{F}_{p^m})$$

and so obtain a neutralized semisimple Tannakian category over $\mathbb{Q}_\ell$. The Frobenius element in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$ defines an isomorphism $T_m \xrightarrow{\simeq} \mathbb{Z}$. The proalgebraic group $\mathbf{T}_m = \mathrm{Aut}^{\otimes}(\omega_m)$ is the algebraic hull of T_m over $\mathbb{Q}_\ell$, and there is a map

$$\mathrm{RepCont}_{\mathbb{Q}_\ell}(T_m) \to \mathsf{Repf}_{\mathbb{Q}_\ell}(\mathbf{T}_m)$$

(see Saavedra 1972, V, 0.3.1). In particular, $\mathbf{T}_m$ is abelian, and it follows (ibid., 2.23) that it is a projective limit of algebraic groups of multiplicative type because $\mathsf{Repf}_{\mathbb{Q}_\ell}(\mathbf{T}_m)$ is semisimple. We shall determine the character group of $\mathbf{T}_m$. For this, we must extend scalars from $\mathbb{Q}_\ell$ to $\mathbb{Q}_\ell^{\mathrm{al}}$, so that

$$\mathrm{RepCont}_{\mathbb{Q}_\ell}(T_{m\mathbb{Q}^{\mathrm{al}}}) \xrightarrow{\simeq} \mathsf{Repf}_{\mathbb{Q}_\ell}(\mathbf{T}_{m\mathbb{Q}_\ell^{\mathrm{al}}}) = \mathsf{Repf}_{\mathbb{Q}_\ell^{\mathrm{al}}}(\mathbf{T}_m).$$

(For the last identification, see ibid., 3.12.). One sees easily that the left category is simply the category of diagonalizable representations of T_m on $\mathbb{Q}_\ell^{\mathrm{al}}$-vector spaces such that the eigenvalues of the Frobenius elements are ℓ-adic integers. These form a semisimple Tannakian category over $\mathbb{Q}_\ell^{\mathrm{al}}$ in which the simple objects (ibid., 1.7.3) have rank one. Further, the simple objects are parametrized by the group $\mathcal{O}^\times_{\mathbb{Q}_\ell^{\mathrm{al}}}$ of ℓ-adic units. We conclude (Saavedra 1972, VI, 3.5.1) that

$$X^*(T_m) = \mathcal{O}^\times_{\mathbb{Q}_\ell^{\mathrm{al}}}.$$

The transition map $X^*(\mathbf{T}_m) \to X^*(\mathbf{T}_{m'})$ for $m|m'$ sends π to $\pi^{m'/m}$.

NOTES This section is partly based on Saavedra 1972, VI, 3.5, and an unpublished manuscript of Grothendieck.

6 Appendix: Galois descent

Let V be an algebraic scheme over a field k_1 and let $\sigma: k_1 \to k_2$ be a homomorphism of fields. We let σV denote the algebraic scheme over k_2 obtained by applying σ to the coefficients of the equations defining V, and for $P \in V(k_1)$, we let σP denote the point on σV obtained by applying σ to the coordinates of P.

6.1 Let V be an affine algebraic variety over $\bar{k}$. Let $A = \Gamma(V, \mathcal{O}_V)$, and let $\sigma \in \Gamma$. We say that a bijection $\lambda: V(\bar{k}) \to V(\bar{k})$ is σ-***linear*** if there exists a σ-linear automorphism λ' of the $\bar{k}$-algebra A such that

$$(\lambda' f)(\lambda P) = \sigma(f(P)), \quad \text{all } f \in A,\ P \in V(\bar{k})\,(= \mathrm{Hom}_{\bar{k}}(A, \bar{k})). \tag{65}$$

To give such a λ is the same as giving an isomorphism $\Lambda: \sigma V \to V$ of algebraic varieties over $\bar{k}$. Indeed, such an isomorphism defines an automorphism λ of $V(\bar{k})$,

$$V(\bar{k}) \xrightarrow{\mathrm{can}_\sigma} (\sigma V)(\bar{k}) \xrightarrow{\Lambda(\bar{k})} V(\bar{k}),$$

and a $\bar{k}$-algebra automorphism λ' of A,

$$\Gamma(V, \mathcal{O}_V) \xrightarrow{\mathrm{can}_\sigma} \Gamma(\sigma V, \mathcal{O}_{\sigma V}) \xrightarrow{\Gamma(V,\Lambda)^{-1}} \Gamma(V, \mathcal{O}_V),$$

which are related by (65). Moreover, Λ is uniquely determined by λ, and a λ arises from a Λ if and only if there exists a λ' satisfying (65).

An action $*: \Gamma \times V(\bar{k}) \to V(\bar{k})$ of Γ on $V(\bar{k})$ is said to be ***regular*** if, for all $\sigma \in \Gamma$, the map

$$\sigma P \mapsto \sigma * P: (\sigma V)(\bar{k}) \to V(\bar{k})$$

is regular, i.e., defined by a morphism $\sigma V \to V$ of algebraic varieties over $\bar{k}$. An action $*$ of Γ on $V(\bar{k})$ is regular if and only if the map

$$P \mapsto \sigma * P: V(\bar{k}) \to V(\bar{k})$$

is σ-linear for all $\sigma \in \Gamma$. There is a one-to-one correspondence between regular actions $*$ of Γ on $V(\bar{k})$ and actions $(\sigma, f) \mapsto {}^\sigma f: \Gamma \times A \to A$ of Γ on A by semilinear $\bar{k}$-algebra homomorphisms, given by

$$({}^\sigma f)(\sigma * P) = \sigma(f(P)), \quad \text{all } P \in V(\bar{k}).$$

For example, if $W = \mathrm{Spec}\, B$ is a variety over k, then the natural action of Γ on $W(\bar{k})$ is regular, and corresponds to the action of Γ on $B \otimes_k \bar{k}$ through its action on $\bar{k}$.

6.2 Let V be an algebraic variety over $\bar{k}$. An action $*$ of Γ on $V(\bar{k})$ is said to be ***continuous*** if there exists a subfield K of $\bar{k}$ finite over k and a model $(W, \varphi: V \xrightarrow{\approx} W_{\bar{k}})$ of V over K such that the action of $\mathrm{Gal}(\bar{k}/K)$ is that defined by W. This means that the bijection

$$\varphi(\bar{k}): V(\bar{k}) \to W(\bar{k})$$

is Γ-equivariant, that is,

$$\varphi(\bar{k})(\sigma * P) = \sigma\left(\varphi(\bar{k})(P)\right)$$

Two such models, over K and K' say, become isomorphic over some finite extension of $K \cdot K'$.

Suppose that V is affine, and let $A = \Gamma(V, \mathcal{O}_V)$. Call an action of Γ on A trivial if there exists a model A_0 of A over k such that the action is induced through the isomorphism $A_0 \otimes \bar{k} \simeq A$ from the action of Γ on $\bar{k}$. The action of Γ on V is continuous if and only if it becomes trivial on an open subgroup of Γ.

It is easy to write down actions that are not continuous, but, in practice, those "occurring in nature" are continuous.

6.3 Let V be an algebraic scheme over $\bar{k}$. A $\bar{k}/k$-***descent system*** on V is a family of isomorphisms $\varphi_\sigma : {}^\sigma V \to V$, $\sigma \in \mathrm{Gal}(\bar{k}/k)$, satisfying the ***cocycle condition***

$$\varphi_{\sigma\tau} = \varphi_\sigma \circ ({}^\sigma\varphi_\tau) \quad \text{for all } \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

A model (V_0, φ) of V over a subfield K of $\bar{k}$ ***splits*** the descent system if $\varphi_\sigma = \varphi^{-1} \circ {}^\sigma\varphi$ for all σ fixing K. A descent system is ***continuous*** if it is split by some subfield K finite over k. A ***descent datum*** is a continuous descent system. A descent datum is ***effective*** if it is split by a model over k. When V is quasi-projective, every descent datum is effective (see, for example, Milne 2024, 7.3).

THEOREM 6.4 *The functor $V \rightsquigarrow (\bar{V}, *)$ sending an affine algebraic variety V over k to an affine algebraic variety over $\bar{V}$ over $\bar{k}$ equipped with an action of Γ on $\bar{V}(\bar{k})$ is fully faithful with essential image the pairs $(\bar{V}, *)$ such that $*$ is continuous and regular.*

PROOF For the fully faithfulness, see, for example, Milne 2024, 4.5. For the description of the essential image, let V be an affine algebraic variety over $\bar{k}$ equipped with a continuous regular action of Γ. For each $\sigma \in \Gamma$, there is an isomorphism of algebraic varieties $\varphi_\sigma : {}^\sigma V \to V$ sending ${}^\sigma P$ to $\sigma * P$. The family $(\varphi_\sigma)_\sigma$ obviously satisfies the cocycle condition, and it is continuous because the action is continuous. Therefore, it is a descent datum on V. As V is affine, this descent datum is effective. □

In particular, algebraic groups over k correspond to algebraic groups over $\bar{k}$ equipped with a continuous regular action of Γ on their $\bar{k}$-points. This gives a description of algebraic groups over k that is close to the classical (pre-scheme) description. In a step that some may consider retrograde, we extended this to a description of k-groupoids acting transitively on $\mathrm{Spec}\,\bar{k}$.

NOTES Langlands and Rapoport (1987) introduced Galois groupoids (which they called Galois-gerbes) as a down-to-earth tool for studying tannakian categories over a field of characteristic zero equipped with a fibre functor over the algebraic closure of the field. This chapter is based on their article and on the exposition in Lattermann 1989.

Chapter IV

Tannakian categories and groupoid schemes

In this chapter, we describe tannakian categories in terms of affine groupoid schemes (following Deligne 1990).

Throughout, k is the base commutative ring (usually a field). Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$.

In §1, we state Deligne's theorem relating tannakian categories with a fibre functor over a scheme to groupoids acting transitively on the scheme

$$(\mathsf{T}, \omega) \leftrightarrow (S, G).$$

In §2, we define groupoid schemes; in §3, we describe their representations; in §4, we interpret the representations as comodules over a coalgebroid; in §5, we prove the comonadicity theorem; in §6, we construct the coalgebroid of endomorphisms of a fibre functor and prove Deligne's theorem; in §7, we restate the results in terms of 2-categories.

1 Statement of the main theorem

In this section, k is a field.

A groupoid of sets is a small category in which all morphisms are isomorphisms. Thus, giving a groupoid amounts to giving a set S (of objects), a set G (of arrows), maps $t, s : G \rightrightarrows S$ (sending an arrow to its target and source respectively), and a partial law of composition satisfying certain conditions. When S has only a single element, G is just a group, and so we can think of a groupoid as being a "group with many objects".

A groupoid scheme is a groupoid in the category Aff_k of affine schemes over k. Thus, it consists of an affine k-scheme S, an affine k-scheme G, morphisms $t, s : G \rightrightarrows S$, and a partial law of composition $\circ$ such that $(S(T), G(T), t, s, \circ)$ is a groupoid of sets for all affine k-schemes T. We usually refer to G as a k-groupoid acting on S. The morphism $(t, s) : G \to S \times S$ allows us to regard G as an $S \times S$-scheme. We say that G acts transitively on S if there exists a faithfully flat morphism $S' \to S \times S$ such that $\operatorname{Hom}_{S\times S}(S', G) \neq \emptyset$. Later we shall see that this is equivalent to G itself being faithfully flat over $S \times S$.

We present a detailed description of groupoids in §2. In §3, we define the category $\mathsf{Repf}(S : G)$ of representations of G on locally free sheaves of finite rank on S. This is a k-linear symmetric monoidal category.

When $(\mathsf{T}, \otimes)$ is a tannakian category over k and ω is a fibre functor on T over a nonempty k-scheme S, we let $\mathcal{A}ut_k^{\otimes}(\omega)$ denote the functor sending an $S \times S$-scheme $(b, a) : T \to S \times S$ to the set of isomorphisms of $\otimes$-functors $a^*\omega \to b^*\omega$.

THEOREM 1.1 (DELIGNE 1990, 1.12) *Let T be an essentially small tannakian category over k and ω a fibre functor on T over a nonempty affine k-scheme S.*

(a) *The functor $\mathcal{A}ut_k^{\otimes}(\omega)$ of $S \times S$-schemes is represented by a k-groupoid G acting transitively on S.*

(b) *The functor $\mathsf{T} \to \mathsf{Repf}(S : G)$ defined by ω is a $\otimes$-equivalence.*

Conversely, if G is a k-groupoid acting transitively on a nonempty k-scheme S, then $\mathsf{Repf}(S : G)$ is a tannakian category over k and $G \simeq \mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

Thus, the theorem provides a dictionary between tannakian categories over k equipped with a fibre functor over S and k-groupoids acting transitively on S.

EXAMPLE 1.2 When $S = \operatorname{Spec} k$, Theorem 1.1 becomes Theorem 3.1 of Chapter II.

NOTATION 1.3 Let T be a tannakian category over k. When ω_1 and ω_2 are fibre functors over S, we let $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$ denote the functor of S-schemes

$$(T \xrightarrow{u} S) \rightsquigarrow \operatorname{Isom}^{\otimes}(u^*\omega_1, u^*\omega_2).$$

When ω_1 and ω_2 are fibre functors over S_1 and S_2 respectively, we let

$$\mathcal{I}som_k^{\otimes}(\omega_2, \omega_1) = \mathcal{I}som_{S_1 \times S_2}^{\otimes}(\operatorname{pr}_2^*\omega_2, \operatorname{pr}_1^*\omega_1),$$

so, as a functor of $S_1 \times S_2$-schemes, $\mathcal{I}som_k^{\otimes}(\omega_2, \omega_1)$ is

$$(T \xrightarrow{(b,a)} S_1 \times S_2) \rightsquigarrow \operatorname{Isom}^{\otimes}(a^*\omega_2, b^*\omega_1).$$

For a fibre functor ω over S, we put

$$\begin{aligned} \mathcal{A}ut_S^{\otimes}(\omega) &= \mathcal{I}som_S^{\otimes}(\omega, \omega) \\ \mathcal{A}ut_k^{\otimes}(\omega) &= \mathcal{I}som_k^{\otimes}(\omega, \omega) \overset{\text{def}}{=} \mathcal{I}som_{S \times S}^{\otimes}(\operatorname{pr}_2^*\omega, \operatorname{pr}_1^*\omega). \end{aligned}$$

Thus, $\mathcal{A}ut_S^{\otimes}(\omega)$ is the functor of S-schemes

$$(T \xrightarrow{u} S) \rightsquigarrow \operatorname{Aut}^{\otimes}(u^*\omega)$$

and $\mathcal{A}ut_k^{\otimes}(\omega)$ is the functor of $S \times S$-schemes

$$(T \xrightarrow{(b,a)} S \times S) \rightsquigarrow \operatorname{Isom}^{\otimes}(a^*\omega, b^*\omega).$$

As a functor of k-schemes, $\mathcal{A}ut_k^{\otimes}(\omega)$ sends a k-scheme T to

$$\{(b, a, \varphi) \mid b, a : T \to S, \quad \varphi : a^*\omega \xrightarrow{\approx} b^*\omega\}.$$

According to Theorem 1.1, $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by an $S \times S$-scheme $(t, s) : G \longrightarrow S \times S$ and the partial law of composition on $\mathcal{A}ut_k^{\otimes}(\omega)$ makes $(S, G, t, s, \circ)$ into a k-groupoid acting on S.

Before sketching the proof of the theorem, we prove an important corollary.

COROLLARY 1.4 *Let* T *be as in the theorem. Any two fibre functors on* T *over an affine k-scheme S become isomorphic over some faithfully flat covering of S.*

PROOF Let ω_1 and ω_2 be fibre functors on T over k-schemes S_1 and S_2 respectively. The functor sending a fibre functor over $S \stackrel{\text{def}}{=} S_1 \sqcup S_2$ to its restrictions to S_1 and S_2 is an equivalence of categories. Thus, ω_1 and ω_2 arise from a fibre functor ω over S, unique up to a unique isomorphism. On applying the theorem to (T, ω), we find that $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by a k-groupoid G acting transitively on S. The transitivity implies that there exists a faithfully flat morphism $S' \to S \times S$ such that $\mathrm{Hom}_{S\times S}(S', G) \neq \emptyset$, i.e., such that $\mathcal{A}ut_k^{\otimes}(\omega)(S') \neq \emptyset$. Note that

$$S \times S = (S_1 \times S_1) \sqcup (S_1 \times S_2) \sqcup (S_2 \times S_1) \sqcup (S_2 \times S_2)\,.$$

We now take $S_1 = S_2 = S_0$, so ω_1 and ω_2 are fibre functors over S_0. The restriction of $\mathcal{A}ut_k^{\otimes}(\omega)$ to the diagonal S_0 in $S_2 \times S_1$ is $\mathcal{I}som_{S_0}^{\otimes}(\omega_1, \omega_2)$, and so $\mathcal{I}som_{S_0}^{\otimes}(\omega_1, \omega_2)(S'') \neq \emptyset$, where $S'' = S_0 \times_{S\times S} S'$. As S'' is faithfully flat over S_0, this completes the proof. □

We prove Theorem 1.1 in §6, after presenting various preliminaries in §2–§5. Before outlining the proof, we review the proof in the neutral case.

Let T be a tannakian category over k and ω a k-valued fibre functor on T. We want to realize (T, ω) as $(\mathsf{Repf}(G), \omega_{\text{forget}})$ for some affine group scheme G. Initially, we forget the tensor structure on T and simply regard it as an abelian category. A result of Gabber (II, 3.13) allows us to write T as a union of categories of the form Mod_A with A a finite-dimensional k-algebra. An elementary argument then allows us to replace Mod_A with coMod_C, where C is the dual coalgebra, and realize the whole of (the abelian category) T as the category of comodules over a k-coalgebra C. Now the tensor structure on T provides C with an algebra structure, and the existence of duals implies the existence of an antipode.

The proof in the general case is similar except that, at each stage, we must replace an object by its more complicated "-oid" generalization, and to realize T (as an abelian category) as a category of comodules, we appeal to the comonadic theorem in category theory.

In §2 we develop the basic theory of affine groupoid schemes, and in §3 we show that the category of representations $\mathsf{Repf}(S:G)$ is tannakian. In §4 we explain how to interpret representations of affine k-groupoids as comodules over coalgebroids in the same way that representations of affine k-groups can be interpreted as comodules over coalgebras.

In §5, we prove the comonadic theorem of Barr and Beck. This is a result in category theory that provides a solution to the following problem: given a faithful functor $F : \mathsf{C} \to \mathsf{B}$, use F to define a structure on B with the property that C can be recovered from B and the structure.

After these preparations, in §6 we prove Theorem 1.1. Let T be a tannakian category over k and ω a fibre functor on T over an affine k-scheme S. We first use the results on abelian categories proved in Chapter II and the comonadicity theorem to show that the abelian category T is equivalent to the category of comodules over the coalgebroid $L(\omega)$ of "endomorphisms of ω". Now the tensor structure on T allows us to provide $L(\omega)$ with an algebra structure, and the rigidity of T implies that $\mathrm{Spec}\, L(\omega)$ is a k-groupoid G acting on S (rather than a k-monoidoid).

It remains to show that G is faithfully flat over $S \times S$. For this, we use the statement (I, 9.8) that for a ring (A, m, e) in a tensor category, the morphism $e : \mathbb{1} \to A$ is faithfully flat. In §10 of Chapter II, we constructed a tensor category $\mathsf{T} \boxtimes \mathsf{T}$ with the property that a fibre functor ω on T over $S = \operatorname{Spec} B$ defines a fibre functor $\omega \times \omega$ on $\mathsf{T} \boxtimes \mathsf{T}$ over $S \times S$. The $B \otimes B$-algebra $L(\omega)$ is faithfully flat because it is the image by $\omega \times \omega$ of an Ind-object containing $\mathbb{1}$ of $\mathsf{T} \otimes \mathsf{T}$.

The remaining sections §7–§10 add various complements.

REMARK 1.5 When one defines a groupoid scheme to be a groupoid in the category of all (not necessarily affine) schemes over k and allows fibre functors over nonaffine schemes, one arrives at the following statement.

> Let T be an essentially small tannakian category over k and ω a fibre functor on T over a nonempty k-scheme S.
>
> (a) The functor $\mathcal{A}ut_k^{\otimes}(\omega)$ of $S \times S$-schemes is represented by a k-groupoid G acting transitively on S.
>
> (b) The functor $\mathsf{T} \to \mathsf{Repf}(S : G)$ defined by ω is an equivalence of tensor categories.
>
> Conversely, if G is a k-groupoid acting transitively on a nonempty affine k-scheme S, then $\mathsf{Repf}(S : G)$ is a tannakian category over k and $G \simeq \mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

This is Deligne's original statement (Deligne 1990, 1.12). As he remarks (ibid., 1.13), it suffices to prove the statement with S affine. Moreover, $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by a groupoid G affine over $S \times S$ (see IV, 1.18 below). Thus, requiring everything to be affine changes little. The reader can consult the original works of Deligne for the more general statements.

2 Groupoid schemes

Throughout this section, k is a field.

Groupoids (in Set*)*

A ***groupoid*** (in Set) is a small category in which every morphism is an isomorphism. Thus, giving a groupoid amounts to giving a set S (of objects), a set G (of arrows), two maps[1] $t, s : G \rightrightarrows S$ (sending an arrow to its target and source respectively), and a partial law of composition,

$$(g, h) \mapsto g \circ h : G \underset{s,S,t}{\times} G \to G, \quad \text{where } G \underset{s,S,t}{\times} G = \{(g, h) \in G \times G \mid s(g) = t(h)\},$$

satisfying the following conditions: composition of arrows is associative; each object has an identity arrow; each arrow has an inverse. We often refer to (S, G) as a groupoid or to G as a ***groupoid acting on*** S. The map $(t, s) : G \to S \times S$ allows us to regard G as a set over $S \times S$.

EXAMPLE 2.1 A group G defines a groupoid as follows: take S to be any singleton, t and s to be the unique maps to S, and $\circ$ to be multiplication on G. Conversely, if (S, G) is a groupoid with S a singleton, then $(G, \circ)$ is a group.

[1] In French, t and s become b and s (but and source), and in German z and q (Ziel and Quelle).

A groupoid (S, G) is said to be ***transitive*** if the map

$$(t, s) : G \to S \times S,$$

is surjective, i.e., if for every pair of objects (b, a) of S there exists an arrow $a \to b$.

Let G be a transitive groupoid. Write $G_{b,a}$ for the fibre of G over (b, a); thus

$$G_{b,a} = \{g \in G \mid s(g) = a,\ t(g) = b\} = \{g \mid g : a \to b\} = \mathrm{Hom}(a, b).$$

There is a law of composition

$$\begin{gathered} G_{c,b} \times G_{b,a} \to G_{c,a} \\ \mathrm{Hom}(b, c) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c). \end{gathered}$$

This law makes $G_a \stackrel{\text{def}}{=} G_{a,a}$ into a group, called the ***vertex*** or ***isotropy group***, and $G_{b,a}$ into a right principal homogeneous space under G_a. The choice of an element $u_{b,a} \in G_{b,a}$ defines an isomorphism $\mathrm{ad}\, u_{b,a} : G_a \to G_b$, which is independent of $u_{b,a}$ up to an inner automorphism. The ***kernel*** G^Δ of G is the family $(G_a)_{a \in S}$. It can be viewed as a relative group over S.

If G is transitive and G_a is commutative for one (hence all) $a \in S$, then G is said to be ***commutative***. In this case the isomorphism $\mathrm{ad}\, u_{b,a} : G_a \to G_b$ is independent of the choice of $u_{b,a}$, and so there is a canonical isomorphism $G_o \times S \to G^\Delta$ for any $o \in S$. Therefore, G^Δ is a constant group over S.

EXAMPLE 2.2 Let S be a topological space. The ***fundamental groupoid*** Π of S is the groupoid acting on S for which $\Pi_{b,a}$ is the set of paths from a to b taken up to homotopy. The law of composition is the usual composition of paths. The group Π_a is the fundamental group $\pi_1(S, a)$. The fundamental groupoid Π acts transitively on S if S is path-connected.

This is the archetype that should be kept in mind when thinking of groupoids.

EXAMPLE 2.3 Let S be a set and k a field. Let $V = (V_a)_{a \in S}$ be a family of k-vector spaces indexed by S. For $a, b \in S$, let

$$G_{b,a} = \mathrm{Isom}(V_a, V_b).$$

Then $G(V) \stackrel{\text{def}}{=} \bigsqcup_{a,b \in S} G_{b,a}$ becomes a groupoid acting on S with $s(G_{b,a}) = a$, $t(G_{b,a}) = b$, and $G_{c,b} \times G_{b,a} \to G_{c,a}$ the composition of isomorphisms. If the V_a all have the same finite dimension, then $G(V)$ acts transitively on S.

A ***morphism of groupoids acting on*** S is a map $f : G \to H$ that, together with the identity map $S \to S$, forms a functor. A ***morphism*** $\alpha : f \to f'$ of morphisms $f, f' : G \rightrightarrows H$ of groupoids acting on S is simply a natural transformation. Thus, it is a family of arrows $\alpha_a : a \to a$ in H, indexed by the elements of S, such that the diagrams

$$\begin{array}{ccc} a & \xrightarrow{\alpha_a} & a \\ {\scriptstyle f(g_{b,a})}\downarrow & & \downarrow{\scriptstyle f'(g_{b,a})} \\ b & \xrightarrow{\alpha_b} & b \end{array}$$

commute for all $g_{b,a} \in G_{b,a}$. In this way, the groupoids acting on a fixed S form a 2-category.

DEFINITION 2.4 Let G be a groupoid acting on S, and let $V = (V_a)_{a\in S}$ be a family of k-vector spaces of the same finite dimension. A ***representation*** of G on V is a morphism $\rho : G \to G(V)$. Thus, for each $g \in G$, we have an isomorphism $\rho(g) : V_{s(g)} \to V_{t(g)}$, such that

(a) for the identity element e_a of G_a, $\rho(e_a) : V_a \to V_a$ is the identity map,

(b) $\rho(g\circ h) = \rho(g)\circ\rho(h)$ if $s(g) = t(h)$.

REMARK 2.5 Let (S, G) be a transitive groupoid. For any $a \in S$, the group G_a is a skeleton of the category (S, G). This example should discourage readers from considering equivalent categories, even with a given equivalence, as being "the same".

Groupoids internal to a category

A ***groupoid internal to a category*** is a diagram

$$G \underset{s,S,t}{\times} G \;\underset{\mathrm{pr}_1}{\overset{\mathrm{pr}_2}{\rightrightarrows}}\; G \;\underset{t}{\overset{s}{\rightleftarrows}}\; S, \qquad \mathrm{inv} : G \to G$$

in the category such that the following equalities hold,

$$\begin{cases} s\circ m = s\circ \mathrm{pr}_2,\ \ t\circ m = t\circ \mathrm{pr}_1, \\ s\circ e = \mathrm{id}_S = t\circ e, \\ m\circ(\mathrm{id}_G \times_S m) = m\circ(m \times_S \mathrm{id}_G), \\ m\circ(\mathrm{id}_G \times_S e)\circ(\mathrm{id}_G, s) = \mathrm{id}_G = m\circ(e \times_S \mathrm{id}_G)\circ(s, \mathrm{id}_G), \end{cases}$$

$$\begin{cases} s\circ \mathrm{inv} = t,\ \ t\circ \mathrm{inv} = s, \\ m\circ(\mathrm{inv}, \mathrm{id}_G) = e\circ s, \\ m\circ(\mathrm{id}_G, \mathrm{inv}) = e\circ t \end{cases}$$

The first collection of equalities expresses that the diagram is an internal category (s and t send an arrow to its source and target respectively, e sends an object to its identity morphism, and m sends a pair of composable arrows to their composite), and the second collection expresses that inv sends an arrow to its inverse (its existence means that inverses exist).

EXAMPLE 2.6 The fundamental groupoid of a topological space S has a natural structure of a groupoid in the category of topological spaces if S is locally path connected and semilocally simply connected. It is Hausdorff if and only if S is Hausdorff.

EXAMPLE 2.7 A ***Lie groupoid*** is a groupoid $t, s : G \rightrightarrows S$ internal to the category of smooth manifolds, except that we require the maps t and s to be submersions so that $G \underset{s,S,t}{\times} G$ is a submanifold of $G \times G$.

Groupoid schemes

A groupoid scheme over k is a groupoid internal to the category Aff_k of affine schemes over k.

DEFINITION 2.8 A ***groupoid scheme over*** k consists of

- affine k-schemes S and G,
- a pair of morphisms $t, s : G \rightrightarrows S$ of k-schemes (making G into an $S \times S$-scheme),
- a morphism $\circ : G \underset{s,S,t}{\times} G \to G$ of $S \times S$-schemes

such that, for all affine k-schemes T, the system

$$S(T), \quad G(T), \quad t, s : G(T) \rightrightarrows S(T), \quad \circ : G(T) \underset{s,S(T),t}{\times} G(T) \to G(T)$$

is a groupoid (in Set). We usually call (S, G) a k-***groupoid acting on*** S or an S/k-***groupoid***.

2.9 The condition in the definition can be expressed in terms of diagrams.

(a) The associativity of composition says that the two morphisms

$$G \underset{s,S,t}{\times} G \underset{s,S,t}{\times} G \underset{\mathrm{id}\times\circ}{\overset{\circ\times\mathrm{id}}{\rightrightarrows}} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

are equal.

(b) The existence of identity maps says that there exists a morphism $e : S \to G$ of $S \times S$-schemes (regarding S as an $S \times S$-scheme by $\Delta : S \to S \times S$) such that both morphisms

$$G \simeq G \underset{s,S,\mathrm{id}}{\times} S \xrightarrow{\mathrm{id}\times e} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

$$G \simeq S \underset{\mathrm{id},S,t}{\times} G \xrightarrow{e\times\mathrm{id}} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

equal id_G.

(c) The existence of inverses says that there exists a morphism $\mathrm{inv} : G \to G$ of k-schemes such that

$$\begin{cases} s \circ \mathrm{inv} = t \\ t \circ \mathrm{inv} = s \end{cases}$$

and the diagrams

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G \underset{s,S,t}{\times} G \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \circ} \\ S & \xrightarrow{e} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(\mathrm{id},\mathrm{inv})} & G \underset{s,S,t}{\times} G \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \circ} \\ S & \xrightarrow{e} & G \end{array}$$

commute.

The morphisms e and inv, when they exist, are uniquely determined by $(S, G, t, s, \circ)$.

REMARK 2.10 Without the condition (c), we get the notion of a ***monoidoid scheme over*** k, i.e., a small category internal to the category Aff_k.

DEFINITION 2.11 Let S be an affine k-scheme, and let G and H be k-groupoids acting on S. A morphism $f : G \to H$ of k-schemes is a ***morphism*** of k-groupoids acting on

S if $f(T)\colon G(T) \to H(T)$ is a morphism of groupoids acting on $S(T)$ for every affine k-scheme T. This condition can also be expressed by saying that the diagrams

$$\begin{array}{ccc} G \underset{s,S,t}{\times} G & \xrightarrow{f\times f} & H \underset{s,S,t}{\times} H \\ \downarrow{\scriptstyle \circ_G} & & \downarrow{\scriptstyle \circ_H} \\ G & \xrightarrow{f} & H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{f} & H \\ & {\scriptstyle e_G}\nwarrow \quad \nearrow{\scriptstyle e_H} & \\ & S & \end{array} \tag{66}$$

commute.

DEFINITION 2.12 We say that an S/k-groupoid G is ***transitive***, or that G ***acts transitively on*** S, if the morphism $(t,s)\colon G \to S\times S$ is covering for the fpqc topology, i.e., there exists a commutative diagram

$$\begin{array}{ccc} T & & \\ \downarrow & \searrow^{\text{faithfully flat}} & \\ G & \xrightarrow{(t,s)} & S\times S. \end{array}$$

Obviously, G is transitive if $(t,s)\colon G \to S\times S$ itself is faithfully flat. Below (2.18), we show that the converse is true.

EXAMPLE 2.13 When $S = \operatorname{Spec} k$, a k-groupoid acting on S is nothing but an affine group scheme over k. It is automatically transitive.

Let G be a k-groupoid acting on S. For a scheme $(b,a)\colon T \to S\times S$ over $S\times S$, we write $G_{b,a}$ for $(b,a)^*G$. Note that b and a are objects of the category $S(T)$, and $G_{b,a}$ can be thought of as the scheme of arrows $a \to b$,

$$G_{b,a} = \text{“}\operatorname{Hom}(a,b)\text{”}.$$

The law of composition provides morphisms (of schemes over T)

$$G_{c,b} \times_T G_{b,a} \to G_{c,a}.$$

This law makes $G_a \overset{\text{def}}{=} G_{a,a} \overset{\text{def}}{=} (a,a)^*G$ into an affine group scheme over T, which is flat if G is transitive.

EXAMPLE 2.14 Let S be an affine k-scheme and V a locally free $\mathcal{O}_S$-module of constant[2] finite rank. There exists a k-groupoid $G(V)$, acting transitively on S such that, for any scheme $(b,a)\colon T \to S\times S$ over $S\times S$,

$$G(V)(T) = \operatorname{Isom}_{\mathcal{O}_T}(a^*V, b^*V).$$

For $a = b = s \in S$, we have

$$G(V)_s = \mathrm{GL}_{V_s},$$

where V_s is the fibre of V over s (a finite-dimensional $\kappa(s)$-vector space).

[2] If the rank is not constant, $G(V)$ may fail to act transitively.

Groupoids regarded as fibred categories

2.15 Let $(G, S, (t,s), \circ)$ be a k-groupoid acting on S. By definition, for any affine k-scheme T, the quadruple

$$(S(T), G(T), (t,s), \circ)$$

is a groupoid (in sets). For varying T, these categories form a fibred category over Aff_k, which we denote by $\mathsf{G}^0(S:G)$, or just G^0.

Thus, for an affine k-scheme T,

$$\operatorname{ob} \mathsf{G}^0_T = S(T) \stackrel{\text{def}}{=} \operatorname{Hom}_k(T, S),$$

and, for $a, b \in \operatorname{ob} \mathsf{G}^0_T = S(T)$,

$$\operatorname{Hom}_{\mathsf{G}^0_T}(a,b) = \{h \in G(T) \mid s\circ h = a,\ t\circ h = b\}.$$

For a morphism $f : T' \to T$ of affine k-schemes, the inverse image functor f^* sends an object $a \in S(T)$ to $a\circ f \in S(T')$ and a morphism $h \in G(T)$ to $h\circ f$.

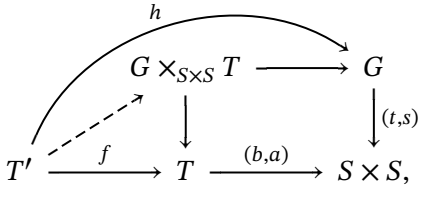
A morphism $f : G \to H$ of groupoids is a cartesian functor $\mathsf{G}^0(S : f) : \mathsf{G}^0(S:G) \to \mathsf{G}^0(S : H)$, i.e., a family of functors compatible with base change.

2.16 Let T be an affine k-scheme, and let a and b be objects of the fibre of G^0 over T, so $a, b \in S(T)$. Define $\mathcal{I}som_T(a,b)$ to be the functor

$$(T' \xrightarrow{f} T) \rightsquigarrow \operatorname{Isom}_{\mathsf{G}^0_{T'}}(f^*a, f^*b) : \mathsf{Aff}_T \to \mathsf{Grp}.$$

Note that

$$\begin{aligned}\operatorname{Isom}_{\mathsf{G}^0_{T'}}(f^*a, f^*b) &\stackrel{\text{def}}{=} \{h : T' \to G \mid s\circ h = a\circ f,\ t\circ h = b\circ f\}\\ &= \operatorname{Isom}_{S\times S}(T', G)\\ &\simeq \operatorname{Isom}_T(T', G \times_{S\times S} T),\end{aligned}$$



so $\mathcal{I}som_T(a,b)$ is represented by the T-scheme $G \times_{S\times S} T$. In particular, $\operatorname{Isom}_T(a,b)$ is a sheaf on T for the fpqc topology.

For example, let Q be the tautological object of G^0 over S, namely, $Q = \mathrm{id}_S$. Then $\mathrm{pr}_1^* Q$ and $\mathrm{pr}_2^* Q$ are objects of the fibre of G^0 over $S \times S$, and the sheaf

$$\mathcal{I}som_{S\times S}(\mathrm{pr}_2^* Q, \mathrm{pr}_1^* Q)$$

on $S \times S$ is represented by the $S \times S$-scheme $(t,s) : G \to S \times S$.

Transitive groupoids

PROPOSITION 2.17 *Let G be a k-groupoid acting on S. The following are equivalent:*

(a) *G is transitive;*

(b) *the objects $\mathrm{pr}_1, \mathrm{pr}_2 : S\times S \to S$ of $\mathsf{G}^0_{S\times S}$ are locally isomorphic;*

(c) *any two objects $a, b : T \to S$ of G^0_T, where T is an affine k-scheme, are locally isomorphic.*

PROOF To say that the objects pr_1 and pr_2 in $\mathsf{G}^0_{S\times S}$ are locally isomorphic means that there exists a faithfully flat map $f : T \to S\times S$ and an $h : T \to G$ such that

$$s\circ h = \mathrm{pr}_1 \circ f, \quad t\circ h = \mathrm{pr}_2 \circ f,$$

i.e., such that $h \in \mathrm{Hom}_{S\times S}(T, G)$. Thus the condition for pr_1 and pr_2 to be locally isomorphic is the definition of "transitive" (2.12).

Assume that pr_1 and pr_2 are locally isomorphic. We show that any two $a, b \in \mathrm{ob}\,\mathsf{G}^0_T = S(T)$ are locally isomorphic. From

$$T \xrightarrow[a]{\overset{b}{\xrightarrow{(a,b)}}} S\times S \underset{\mathrm{pr}_1}{\overset{\mathrm{pr}_2}{\rightrightarrows}} S, \qquad \begin{cases} \mathrm{pr}_1 \circ (a,b) = a \\ \mathrm{pr}_2 \circ (a,b) = b \end{cases}$$

we see that $b = (a,b)^*\,\mathrm{pr}_2$ and $a = (a,b)^*\,\mathrm{pr}_1$. If pr_1 and pr_2 become isomorphic on a faithfully flat covering $U \to S\times S$ of $S\times S$, then a and b become isomorphic on the faithfully flat covering $T\times_{S\times S} U \to T$. □

PROPOSITION 2.18 *A k-groupoid G acting on S is transitive if and only if $(t,s) : G \to S\times S$ is faithfully flat.*

PROOF The necessity if obvious. For the sufficiency, let x be a point of $S\times S$, and let $T \to S\times S$ be a morphism with image x. On pulling back the diagram in 2.11 to T, we see that x is in the image of (t,s), and hence (t,s) is surjective. It remains to prove that it is flat.

Let $Q = \mathrm{id}_S \in S(S) = \mathrm{ob}\,\mathsf{G}^0_S$. We saw in 2.16 that that G, regarded as an $S\times S$-scheme, represents the sheaf $\mathcal{I}som_{S\times S}(\mathrm{pr}_2^* Q, \mathrm{pr}_1^* Q)$ on $S\times S$. On pulling back by the diagonal $\Delta : S \to S\times S$, we deduce that $G\times_{S\times S,\Delta} S$ represents the sheaf $\mathcal{A}ut_S(Q)$ on S. As a first step, we show that $\mathcal{A}ut_S(Q)$ (by which we mean the scheme representing it) is flat over S.

Let $a : P \to S$ be a morphism of affine k-schemes with P the spectrum of a field, and consider the diagram

$$\begin{array}{ccc} S\times_{\mathrm{Spec}(k)} P & \xrightarrow{\mathrm{pr}_P} & P \\ \downarrow{\scriptstyle \mathrm{pr}_S} & \swarrow{\scriptstyle a} & \downarrow \\ S & \longrightarrow & \mathrm{Spec}(k). \end{array}$$

As G is transitive, the two objects $\mathrm{pr}_S^* Q$ and $\mathrm{pr}_P^*(a^*Q)$ of $\mathsf{G}^0_{S\times P}$ become isomorphic on some affine U faithfully flat over $S\times P$. Therefore $\mathcal{A}ut_S(Q)$ and $\mathcal{A}ut_P(a^*Q)$ become isomorphic when restricted to U. As $\mathcal{A}ut_P(a^*Q)$ is obviously flat over P, this implies

that the restriction of $\mathcal{A}ut_S(Q)$ to U is flat over U, which implies that $\mathcal{A}ut_S(Q)$ is flat over S because U is faithfully flat over S.

Recall that, as a scheme over $S \times S$, G represents $\mathcal{I}som_{S\times S}(\mathrm{pr}_2^* Q, \mathrm{pr}_1^* Q)$. As G is transitive, there is a faithfully flat morphism $f : U \to S \times S$ such that $\mathrm{pr}_2^* Q$ and $\mathrm{pr}_1^* Q$ become isomorphic over U. On choosing an isomorphism, we get an isomorphism of sheaves over U,

$$\mathcal{I}som_U(f^* \mathrm{pr}_2^* Q, f^* \mathrm{pr}_1^* Q) \approx \mathcal{A}ut_U(f^* \mathrm{pr}_1^* Q).$$

As the scheme representing the right-hand sheaf is flat over U, so is the scheme representing the left-hand sheaf. This is $G \times_{S\times S} U$, and so G is flat over $S \times S$. □

Pullbacks of groupoid schemes

Let G be a k-groupoid acting on S, and let $u : T \to S$ be a morphism of affine k-schemes. The ***pullback*** of G by $u \times u : T \times T \to S \times S$ is a k-groupoid acting on T, which we denote by G_T.

For example, let G be an affine group scheme over k, viewed as a k-groupoid acting on $\operatorname{Spec} k$. For any affine k-scheme S, we get a k-groupoid scheme

$$G_S \stackrel{\text{def}}{=} G \times (S \times S)$$

acting on S, which is called the ***neutral groupoid scheme*** defined by G. In the special case that G is the trivial group, $G_S = S \times S$ and is called the ***trivial*** S/k-groupoid.

ASIDE 2.19 There is a more abstract version of the above theory. Let E be a Grothendieck topos, i.e., the category of sheaves of sets on some small site, and let 1 be a terminal object of E. A groupoid $(S, G, (s,t), \circ)$ in E is a ***bouquet*** if

(a) the unique morphism $S \to 1$ is an epimorphism ((S,G) is said to be nonempty), and

(b) the morphism $(t,s) : G \to S \times S$ is an epimorphism, ((S,G) is said to be connected),

(Duskin 1982, 2013). A k-groupoid scheme acting transitively on an affine k-scheme S defines a bouquet in the topos of sheaves of sets on Aff_k for the fpqc topology.

Groupoids and Galois groupoids

2.20 Let $S_0 = \operatorname{Spec} k$ and $S = \operatorname{Spec} \bar{k}$. The left action of $\Gamma \stackrel{\text{def}}{=} \mathrm{Gal}(\bar{k}/k)$ on $\bar{k}$ defines a right action of Γ on S, and the map

$$S \times \Gamma \to S \times_{S_0} S, \quad (s, \sigma) \mapsto (s, s \cdot \sigma) \tag{67}$$

is an isomorphism of schemes. Here Γ is to be interpreted as a pro-finite scheme over k.

2.21 Let G be a k-groupoid scheme acting transitively on S, and assume that the kernel G^Δ is of finite type over $\bar{k}$. When we view (67) as a morphism of S-schemes (through projection on the first factor), it identifies $(S \times_{S_0} S)(S)$ with $\Gamma = \mathrm{Gal}(\bar{k}/k)$, and the morphism $(t,s) : G \to S \times_{S_0} S$ defines a map

$$\pi : G(\bar{k}) = G(S) \to (S \times_{S_0} S)(S) \simeq \mathrm{Gal}(\bar{k}/k) \stackrel{\text{def}}{=} \Gamma.$$

More precisely, $G(\bar{k})$ is the set of sections to the morphism t in the diagram

$$\begin{array}{ccccc} G & \xrightarrow{(t,s)} & S \times_{S_0} S & \xleftarrow{\simeq} & S \times \Gamma \\ & \searrow^{t} & \downarrow{\scriptstyle \mathrm{pr}_1} & \swarrow_{\mathrm{pr}_1} & \\ & & S = \operatorname{Spec}(\bar{k}). & & \end{array}$$

The map $G(\bar{k}) \to \Gamma$ is surjective because (t,s) is faithfully flat and $\bar{k}$ is algebraically closed.

2.22 Now $(G(\bar{k}), S(\bar{k}), (s,t), \circ)$ is a transitive groupoid (in Set) with a canonical point $a \in S(\bar{k})$, namely, the identity map. Therefore (see 1.17) there is a well-defined group structure on $G(\bar{k})$ for which the sequence

$$1 \to G^{\Delta}(\bar{k}) \to G(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

is exact.

2.23 In this way, a k-groupoid G acting transitively on S with kernel G^{Δ} defines a $\bar{k}/k$-Galois groupoid,

$$1 \to G^{\Delta} \to G(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

3 Representations of groupoid schemes

In this section, k is a field.

DEFINITION 3.1 Let G be a k-groupoid acting on an affine k-scheme S, and let V be a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. A ***representation*** of G on V is a morphism of S/k-groupoids $G \to G(V)$.

Explicitly, this means that for every affine k-scheme T and $g \in G(T)$, we have a morphism $\rho(g) : V_{g\circ s} \to V_{g\circ t}$ between the inverse images of V with respect to $g\circ s, g\circ t : T \rightrightarrows S$; these satisfy the following conditions (cf. 2.4),

(a) for the element e of $G(S)$, $\rho(e)$ is the identity map of $V = V_{s\circ e} = V_{t\circ e}$;

(b) $\rho(g\circ h) = \rho(g)\rho(h)$ if $s(g) = t(h)$;

(c) the formation of $\rho(g)$ commutes with base change $T' \to T$.

REMARK 3.2 We can use the explicit description to define a representation of G on any quasi-coherent sheaf V on S. If G is transitive and for some $s \in S$, the fibre V_s of V is a vector space of dimension n over the residue field at s, then V is locally free of rank n (see V, 1.19).

3.3 Let $G = \operatorname{Spec} L$, let $S = \operatorname{Spec} B$, and let V be a finitely generated projective B-module (=locally free sheaf of finite rank on S). For $g \in G(R)$, let $s(g)$ denote the image of g under $s(R) : G(R) \to S(R) = \mathrm{Hom}(B,R)$, and similarly for $t(g)$. To give a representation of G on V is the same as giving, for every k-algebra R and $g \in G(R) = \mathrm{Hom}(L,R)$, a homomorphism of R-modules

$$\rho(g) : V \otimes_{B,s(g)} R \to V \otimes_{B,t(g)} R,$$

satisfying the following conditions,

(a) $\rho(e) = \mathrm{id}_V$,

(b) $\rho(g\circ h) = \rho(g)\circ\rho(h)$ whenever $g\circ h$ is defined,

(c) ρ is compatible with base change, i.e., for any homomorphism of k-algebras $u : R \to R'$, the following diagram commutes

$$\begin{array}{ccc} V \otimes_{B,s(g)} R & \xrightarrow{\rho(g)} & V \otimes_{B,t(g)} R \\ \downarrow{\scriptstyle V\otimes u} & & \downarrow{\scriptstyle V\otimes u} \\ V \otimes_{B,s(g')} R' & \xrightarrow{\rho(g')} & V \otimes_{B,t(g')} R', \end{array}$$

where $g' = G(u)(g) \in G(R')$.[3]

As G is a groupoid, all $\rho(g)$ are isomorphisms.

A ***morphism*** of representations (V, ρ_V) and (W, ρ_W) of G is a homomorphism of B-modules $\varphi : V \to W$ such that

$$\begin{array}{ccc} V \otimes_{B,s(g)} R & \xrightarrow{\rho_V(g)} & V \otimes_{B,t(g)} R \\ \downarrow{\scriptstyle \varphi\otimes R} & & \downarrow{\scriptstyle \varphi\otimes R} \\ W \otimes_{B,s(g)} R & \xrightarrow{\rho_W(g)} & W \otimes_{B,t(g)} R \end{array}$$

commutes for all k-algebras R and $g \in G(R)$.

Let $\mathtt{Repf}(S:G)$ denote the category of representations of G on finitely generated projective B-modules. We define the tensor product of two such representations by

$$(V, \rho_V) \otimes (W, \rho_W) = (V \otimes_B W, \rho_{V\otimes_B W}),$$

where $\rho_{V\otimes W}(g)$ is determined by the commutative diagram

$$\begin{array}{ccc} (V \otimes_{B,s(g)} R) \otimes_R (W \otimes_{B,s(g)} R) & \xrightarrow{\rho_V(g)\otimes_R\rho_W(g)} & (V \otimes_{B,t(g)} R) \otimes_R (W \otimes_{B,t(g)} R) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ (V \otimes_B W) \otimes_{B,s(g)} R & \xrightarrow{\rho_{V\otimes W}(g)} & (V \otimes_B W) \otimes_{B,t(g)} R. \end{array}$$

There are obvious associativity and commutativity constraints, and B, equipped with the trivial action, is an identity object. We define the dual of a representation (V, ρ) to be $(V^\vee, \rho^\vee)$, where $V^\vee$ is the B-linear dual of V and $\rho^\vee$ is determined by the commutative diagram

$$\begin{array}{ccc} V^\vee \otimes_{B,s(g)} R & \xrightarrow{\rho^\vee(g)} & V^\vee \otimes_{B,t(g)} R \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ (V \otimes_{B,s(g)} R)^\vee & \xrightarrow{(\rho_V(g)^{-1})^\vee} & (V \otimes_{B,t(g)} R)^\vee. \end{array}$$

In this way, $\mathtt{Repf}(S:G)$ becomes a k-linear rigid symmetric monoidal category.

[3]When we use the same symbol for a map of affine schemes and the corresponding map of rings, we have

$$\begin{aligned} s(g) &\stackrel{\text{def}}{=} g\circ s : B \to L \to R \\ g' &\stackrel{\text{def}}{=} u\circ g : B \to R \to R' \\ s(g') &\stackrel{\text{def}}{=} g'\circ s = u\circ s(g) : B \to R'. \end{aligned}$$

PROPOSITION 3.4 *Let G be a k-groupoid acting transitively on S, and let $u : T \to S$ be a morphism of affine k-schemes with $T \neq \emptyset$. The pullback G_T of G to T is a k-groupoid acting transitively on T, and u induces an equivalence of categories*

$$\mathsf{Repf}(S:G) \xrightarrow{\approx} \mathsf{Repf}(T:G_T).$$

PROOF This is most conveniently proved in terms of gerbes. A groupoid G acting transitively on a scheme S defines a gerbe $\mathsf{G}(S:\ G)$, and there is a canonical equivalence of categories,

$$\mathsf{Repf}(\mathsf{G}(S:\ G)) \xrightarrow{\approx} \mathsf{Repf}(S:\ G).$$

As the morphism of gerbes $\mathsf{G}(S:\ G) \to \mathsf{G}(T:\ G_T)$ defined by $(S:\ G) \to (T:\ G_T)$ is an equivalence, the statement follows. For the details, see the first section of the next chapter (especially 1.20), which is independent of the rest of this chapter – it could have been inserted here. □

For example, if there exists an $s \in S(k)$, then

$$\mathsf{Repf}(S:G) \xrightarrow{\approx} \mathsf{Repf}(G_s),$$

where G_s is the affine group scheme over k fixing s (the fibre of G over (s, s)).

For example, let G be an affine group scheme over k, and let $\bar{G}$ be the neutral k-groupoid scheme $G \times \mathrm{Spec}(\bar{k} \otimes \bar{k})$ acting on $\mathrm{Spec}(\bar{k})$. Then the restriction functor $\mathsf{Repf}(\bar{G}) \xrightarrow{\approx} \mathsf{Repf}(G)$ is an equivalence of categories.

PROPOSITION 3.5 *Let G be a k-groupoid acting transitively on S. Then $\mathsf{Repf}(S:G)$ is a tannakian category over k with the forgetful functor as a fibre functor over S.*

PROOF After 3.4, we may suppose that $S = \mathrm{Spec}\, B$, where B is a field. We know that $\mathsf{Repf}(S:G)$ is a k-linear rigid symmetric monoidal category.

We next show that it is abelian. Obviously, it is additive. Let

$$(V, \rho_V) \xrightarrow{f} (W, \rho_W)$$

be a morphism in $\mathsf{Repf}(S:G)$. There is an exact sequence

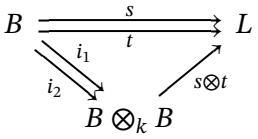
$$0 \to \mathrm{Ker}\, f \to V \to W \to \mathrm{Coker}\, f \to 0$$

of B-vector spaces, and, for each $g \in G(R)$, a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ker}(f) \otimes_{B,s(g)} R & \to & V \otimes_{B,s(g)} R & \to & W \otimes_{B,s(g)} R & \to & \mathrm{Coker}(f) \otimes_{B,s(g)} R \to 0 \\
 & & \vdots & & \simeq \downarrow \rho_V(g) & & \simeq \downarrow \rho_W(g) & & \vdots \\
0 & \to & \mathrm{Ker}(f) \otimes_{B,t(g)} R & \to & V \otimes_{B,t(g)} R & \to & W \otimes_{B,t(g)} R & \to & \mathrm{Coker}(f) \otimes_{B,t(g)} R \to 0
\end{array}$$

(here we use that B is a field). The dashed arrows define an action of G on $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ making them the kernel and cokernel of f in $\mathsf{Repf}(S:G)$. Obviously, the canonical morphism $\mathrm{Coker}(\mathrm{Ker}(f)) \to \mathrm{Ker}(\mathrm{Coker}(f))$ is an isomorphism, and so $\mathsf{Repf}(S:G)$ is abelian. Moreover, the forgetful functor $\omega : \mathsf{Repf}(S:G) \to \mathsf{Vecf}_B$ is an exact faithful $\otimes$-functor, and so it remains to show that $\mathrm{End}(\mathbb{1}) = k$.

Let $G = \operatorname{Spec} L$, and denote the maps $B \rightrightarrows L$ defined by $s, t : G \to S$ by the same letter. An endomorphism of $\mathbb{1} = B$ is given by an element a of B, and $\mathrm{End}(\mathbb{1})$ is the equalizer of the pair of arrows $s, t : B \rightrightarrows L$. The diagram



commutes with $i_1(a) = a \otimes 1$ and $i_2(a) = 1 \otimes a$. As $k \to B$ is faithfully flat, it is the equalizer of the parallel pair (i_1, i_2). The map $s \otimes t$ is faithfully flat, hence injective, and it follows that $\mathrm{End}(\mathbb{1}) = k$. □

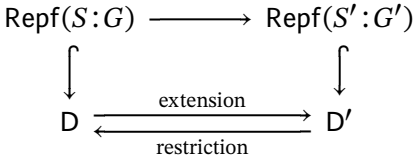
REMARK 3.6 Let G be an affine k-groupoid acting transitively on an affine k-scheme S. Let C be the category $\mathsf{Repf}(S : G)$ of representations of G on locally free sheaves of finite rank on S, and D the category of representations of G on quasi-coherent sheaves on S. Then C and D satisfy the conditions (a), (b), and (c) of Proposition B.11. When S is the spectrum of a field, condition (c) follows from 6.6. The general case then follows from 3.4. Therefore, the functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim X_\alpha : \ \mathrm{Ind}\,\mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an object X of D to the inductive system of its subobjects in C.

If there is an $s \in S(k)$, then the functor “fibre at s” is an equivalence of $\mathsf{Repf}(S{:}G)$ with $\mathsf{Repf}(G_{s,s})$ (6.6), and we are back in the situation of Example B.14.

REMARK 3.7 Let G be an affine groupoid acting transitively on an affine scheme S over k, let $(S' : G')$ be the pair over k' deduced from $(S : G)$ by extension of scalars, and let D and D' be as in a 3.6. Then the system



satisfies the conditions of B.20. Therefore the functor restriction of scalars induces an equivalence $\mathsf{Repf}(S' : G') \to \mathsf{Repf}(S : G)_{(k')}$. See 1.20 for a related result.

4 Representations of groupoids as comodules

In this section, k is a commutative ring. From 4.6, it is a field.

Just as representations of affine group schemes can be realized as comodules over coalgebras (II, §1), representations of groupoid schemes can be realized as comodules over coalgebroids.

TERMINOLOGY 4.1 Let R and S be rings (not necessarily commutative). An (R, S)-***bimodule*** is an abelian group M together with a left action of R and a commuting right action of S:

$$(r \cdot m) \cdot s = r \cdot (m \cdot s), \qquad r \in R,\ m \in M,\ s \in S.$$

Such a module is sometimes denoted ${}_RM_S$. If R, T, S are rings and ${}_RM_T$ and ${}_TM_S$ are bimodules, then

$$ {}_RM_T \underset{T}{\otimes} {}_TM_S $$

is a well-defined (R,S)-bimodule. An (R,R)-bimodule is also called an R-bimodule. When the rings are k-algebras (k a commutative ring), we require that the various actions of k coincide.

DEFINITION 4.2 Let B be a ring, not necessarily commutative.[4] A B-***coalgebroid*** (or ***coalgebroid acting on*** B) is a B-bimodule L equipped with two B-bimodule homomorphisms (comultiplication and coidentity)

$$ c : L \to L \underset{B}{\otimes} L, \quad \text{i.e., } {}_BL_B \to {}_BL_B \underset{B}{\otimes} {}_BL_B $$
$$ \epsilon : L \to B $$

such that the two composed maps

$$ L \xrightarrow{c} L \otimes_B L \underset{L\otimes c}{\overset{c\otimes L}{\rightrightarrows}} L \otimes_B L \otimes_B L $$

are equal and the two maps

$$ L \xrightarrow{c} L \otimes_B L \xrightarrow{\epsilon\otimes L} B \otimes_B L \simeq L $$
$$ L \xrightarrow{c} L \otimes_B L \xrightarrow{L\otimes\epsilon} L \otimes_B B \simeq L $$

equal the id_L.

A ***morphism of*** B-***coalgebroids*** $(L, c, \epsilon) \to (L', c', \epsilon')$ is a homomorphism $f : L \to L'$ of B-bimodules such that the following diagrams commute,

$$ \begin{array}{ccc} L \otimes_B L & \xrightarrow{f\otimes f} & L' \otimes_B L' \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c'} \\ L & \xrightarrow{f} & L' \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{f} & L' \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle\epsilon'} & \\ & B. & \end{array} $$

When B is an algebra over a commutative ring k, a B/k-***coalgebroid*** (or k-***coalgebroid acting on*** B) is a B-coalgebroid L such that the two k-module structures on L coincide.

DEFINITION 4.3 Let L be a B-coalgebroid. A ***representation*** of L is a right B-module M equipped with a ***coaction*** of L, i.e., a homomorphism $\rho : M \to M \otimes_B L$ of right B-modules such that the two composed maps

$$ M \xrightarrow{\rho} M \otimes_B L \underset{M\otimes c}{\overset{\rho\otimes L}{\rightrightarrows}} M \otimes_B L \otimes_B L \tag{68} $$

are equal and the map

$$ M \xrightarrow{\rho} M \otimes_B L \xrightarrow{M\otimes\epsilon} M \otimes_B B \simeq M \tag{69} $$

[4] We shall only need commutative B, but allowing noncommutative rings forces us to distinguish left from right correctly.

equals id_M. We call (M,ρ) an L-***comodule***.

A ***morphism of L-comodules*** $(M,\rho_M)\to(N,\rho_N)$ is a homomorphism of B-modules $f: M\to N$ such that the following diagram commutes

$$\begin{array}{ccc} M\otimes_B L & \xrightarrow{f\otimes L} & N\otimes_B L \\ \uparrow{\scriptstyle \rho_M} & & \uparrow{\scriptstyle \rho_N} \\ M & \xrightarrow{\quad f\quad} & N. \end{array}$$

When L is a B-coalgebroid, we let $\mathtt{coModf}(L)$ denote the category of L-comodules that are finitely generated and projective as right B-modules.

EXAMPLE 4.4 If B is commutative and the two B-module structures on L agree, then L is a B-coalgebra (see II, §1), which helps explains the terminology. The notions of a B-comodule agree in the two cases.

EXAMPLE 4.5 Let L be a B-coalgebroid. If L is flat as a left B-module, then the category of L-comodules is abelian and the forgetful functor is exact. The proof of this is the same as the proof of the similar result for $\mathtt{Repf}(S:G)$ (see 3.5).

Now let k be a field.

EXAMPLE 4.6 Let G be a k-groupoid acting on S, and let $G=\operatorname{Spec}L$ and $S=\operatorname{Spec}B$. From the morphism $(t,s): G\to S\times S$, we get the structure of a $B\otimes_k B$-module on L, i.e., a B-bimodule structure such that the two k-module structures coincide. We write the B-module structure defined by t (resp. s) on the left (resp. right). The composition law

$$G\underset{s,S,t}{\times}G\to G$$

corresponds to a map of B-bimodules

$$c: L\to L\underset{B}{\otimes}L$$

and the identity $e: S\to G$ corresponds to a map of B-bimodules

$$\epsilon: L\to B.$$

A comparison of the diagrams in 2.9 and 4.2 shows that these structures make L into a B/k-coalgebroid, which helps explains the terminology.

PROPOSITION 4.7 *Let $G=\operatorname{Spec}L$ be a k-groupoid acting transitively on $S=\operatorname{Spec}B$ (so L is a B/k-coalgebroid), and let M be a right B-module. There is a canonical one-to-one correspondence between the representations of G on M and the representations of L on M.*

PROOF A representation ρ of G on M is determined by its action on the "universal" element

$$u\overset{\text{def}}{=}\mathrm{id}_G\in\operatorname{Hom}(G,G)=G(L)$$

because, according to 3.3(c), the diagram

$$\begin{array}{ccc} M\otimes_{B,s}L & \xrightarrow{\rho(u)} & M\otimes_{B,t}L \\ \downarrow{\scriptstyle M\otimes g} & & \downarrow{\scriptstyle M\otimes g} \\ M\otimes_{B,s(g)}R & \xrightarrow{\rho(g)} & M\otimes_{B,t(g)}R \end{array}$$

commutes for all $g \in G(R) = \mathrm{Hom}(L, R)$. In turn, the L-linear map

$$\rho(u) : M \otimes_{B,s} L \to M \otimes_{B,t} L,$$

is determined by its restriction to a B-linear map

$$\rho_M : M \to M \otimes_{B,t} L.$$

In the other direction, given a coaction ρ_M of L on a B-module M and a $g \in G(R) = \mathrm{Hom}(L, R)$, we define $\rho(g)$ by the following diagram

$$\begin{array}{ccc} M \otimes_{B,s(g)} R & \xrightarrow{\rho_M \otimes R} & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \\ \Big\downarrow{\scriptstyle \rho(g)} & & \Big\downarrow{\scriptstyle M\otimes g\otimes R} \\ M \otimes_{B,t(g)} R & \xleftarrow[M\otimes \mathrm{mult.}]{} & M \otimes_{B,t(g)} R \underset{s(g),B,s(g)}{\otimes} R. \end{array} \tag{70}$$

If ρ_M is obtained from a ρ as in the first paragraph, then the action of id_G given by (70) returns ρ_M. Conversely, let ρ be a representation of G on M. We get a coaction ρ_M as in the first paragraph, and from ρ_M a representation $\widetilde{\rho}$ of G. Now $\rho = \widetilde{\rho}$ because they agree on $u = \mathrm{id}_G$.

It remains to show that, under the correspondence $\rho \leftrightarrow \rho_M$, ρ satisfies the axioms for a representation if and only if ρ_M satisfies the axioms for a comodule.

If $g, h \in G(R)$ with $t(g) = s(h)$, then $\rho(h \circ g)$ is given by the diagram

$$\begin{array}{ccccc} M \otimes_{B,s(g)} R & \xrightarrow{\rho_M\otimes R} & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R & \xrightarrow{M\otimes c\otimes R} & M \otimes_{B,t} L \underset{s,B,t}{\otimes} L \underset{s,B,s(g)}{\otimes} R \\ & \searrow^{\rho(h\circ g)} & & & \Big\downarrow{\scriptstyle M\otimes h\otimes g\otimes R} \\ & & M \otimes_{B,t(h)} R & \xleftarrow[M\otimes R\otimes \mathrm{mult.}]{} & M \otimes_{B,t(h)} R \underset{s(h),B,t(g)}{\otimes} R \underset{s(h),B,s(g)}{\otimes} R. \end{array}$$

and $\rho(h) \circ \rho(g)$ is given by

$$\begin{array}{ccccc} M \otimes_{B,s(g)} R & \xrightarrow{\rho_M\otimes R} & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R & \xrightarrow{M\otimes g\otimes R} & M \otimes_{B,t(g)} R \underset{s(g),B,s(g)}{\otimes} R \\ & & & & \Big\downarrow{\scriptstyle M\otimes \mathrm{mult.}} \\ \Big\downarrow{\scriptstyle \rho(h)\circ\rho(g)} & & & & M \otimes_{B,t(g)} R \\ & & & & \Big\downarrow{\scriptstyle \rho_M\otimes R} \\ M \otimes_{B,t(h)} R & \xleftarrow[M\otimes \mathrm{mult.}]{} & M \otimes_{B,t(h)} R \underset{s(h),B,s(h)}{\otimes} R & \xleftarrow{M\otimes h\otimes R} & M \otimes_{B,t} L \underset{s,B,s(h)}{\otimes} R. \end{array}$$

When we write $\rho_M(m) = \sum m_i \otimes \ell_i$, then these homomorphisms agree if and only if

$$(1 \otimes h \otimes g)\Big(\sum m_i \otimes c(\ell_i)\Big) = (1 \otimes h \otimes g)\Big(\sum \rho(m_i) \otimes \ell_i\Big)$$

for all k-algebra homomorphisms $h, g : L \to R$ with $t(g) = s(h)$. This is obviously equivalent to the maps in (68) agreeing.

Let ρ be a representation of G on M, and let ρ_M be the associated coaction. Then the action of $\rho(\epsilon)$ is given by the top row of the following commutative diagram

$$\begin{array}{ccccccc} M \otimes_B B & \longrightarrow & M \otimes_{B,t} L \otimes_{s,B} B & \xrightarrow{M\otimes\epsilon\otimes B} & M \otimes_B B \otimes_B B & \xrightarrow{\simeq} & M \otimes_B B \\ \downarrow\simeq & & \downarrow\simeq & & & \nearrow_{\mathrm{id}\otimes\epsilon} & \\ M & \xrightarrow{\rho_M} & M \otimes_{B,t} L & & & & \end{array}$$

It follows that the map in (69) is the identity if and only if ϵ acts trivially on M. □

PROPOSITION 4.8 *Let $G = \operatorname{Spec} L$ be a groupoid acting transitively on $S = \operatorname{Spec} B$. The functor*

$$(M, \rho) \rightsquigarrow (M, \rho_M) : \mathsf{Repf}(S : G) \to \mathsf{coModf}(L)$$

is an equivalence of categories

PROOF After Proposition 4.7, it remains to show that a B-module homomorphism $f : M \to N$ is G-equivariant if and only if it is a morphism of L-comodules, but this is straighforward from the definitions. □

REMARK 4.9 If we define the tensor product of L-comodules $M, \rho_M)$ and (N, ρ_N) to be the L-comodule $(M \otimes N, \rho_{M\otimes N})$, where $\rho_{M\otimes N}$ is given by

$$\begin{array}{ccc} M \otimes_B N & \xrightarrow{\rho_M\otimes\rho_N} & (M \otimes_{B,t} L) \underset{B\otimes B}{\otimes} (N \otimes_{B,t} L) \\ \downarrow{\scriptstyle \rho_{M\otimes N}} & & \downarrow\simeq \\ (M \otimes_B N) \otimes_{B,t} L & \xleftarrow[\mathrm{id}\otimes m]{} & (M \otimes_B N) \otimes_{B,t} (L \underset{B\otimes B}{\otimes} L), \end{array}$$

then the equivalence

$$\mathsf{Repf}(S : G) \xrightarrow{\approx} \mathsf{coModf}(L)$$

respects tensor products.

NOTES The exposition of the proof of 4.7 follows that in Lattermann 1989, 1.4.7.

5 The comonadicity theorem and applications

A faithful functor $F : \mathsf{C} \to \mathsf{B}$ creates a "shadow" of C on B, and it is sometimes possible to recover C from B and the shadow. For example, in (II, Theorem 3.20), we were able to recover C from its shadow (a coalgebra) in Vecf_k. In the case we are interested in here, the shadow is a "comonad" on B, and a standard result in category theory (Theorem 5.12) describes C as the category of "G-modules" in B.

Comonads

DEFINITION 5.1 A ***comonad*** on a category B consists of

- a functor $G : \mathsf{B} \to \mathsf{B}$,
- a natural transformation $c : G \to G \circ G$ (the comultiplication),
- a natural transformation $\epsilon : G \to \mathrm{id}_{\mathsf{B}}$ (the counit)

such that the two natural transformations

$$G \xrightarrow{c} G\circ G \underset{Gc}{\overset{cG}{\rightrightarrows}} G\circ G\circ G$$

are equal and the two natural transformations

$$G \xrightarrow{c} G\circ G \underset{G\epsilon}{\overset{\epsilon G}{\rightrightarrows}} G$$

equal id_G. The counit, if it exists, is uniquely determined by (G, c).

EXAMPLE 5.2 Fix a set E, and let $G\colon \mathrm{Set} \to \mathrm{Set}$ be the functor $X \rightsquigarrow X \times E$. Then

$$(x, e) \mapsto (x, e, e)\colon X \times E \to X \times E \times E$$

is a natural transformation $c\colon G \to G\circ G$, and

$$(x, e) \mapsto e\colon X \times E \to X$$

is a natural transformation $G \to \mathrm{id}$. The triple (G, c, ϵ) is a comonad.

REMARK 5.3 Let C be a category. There is a monoidal category whose objects are the functors $\mathsf{C} \to \mathsf{C}$, whose morphisms are the natural transformations, and whose tensor product is $\circ$. A comonad is a comonoid in this category (monoid in the opposite category), which explains the similarity of the above diagrams to earlier diagrams.

Comonads from adjunctions

We refer the reader to A.2 for our notation concerning adjoint pairs.

PROPOSITION 5.4 *Let* $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ *be an adjoint pair with unit* $\eta\colon \mathrm{id}_{\mathsf{C}} \to U\circ F$ *and counit* $\epsilon\colon F\circ U \to \mathrm{id}_{\mathsf{B}}$. *Then* $F\circ U$ *is a comonad on* B *with comultiplication*

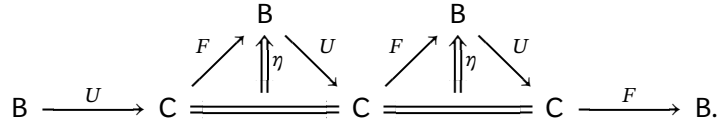
$$F\eta U\colon F\circ U \longrightarrow F\circ U\circ F\circ U$$

and counit ϵ.

PROOF The two composites in

$$FU \xrightarrow{F\eta U} FU\circ FU \underset{FUF\eta U}{\overset{F\eta UFU}{\rightrightarrows}} FU\circ FU\circ FU$$

agree because they both equal the horizontal natural transformation in



The two natural transformations

$$F\circ U \xrightarrow{F\eta U} F\circ U\circ F\circ U \underset{FU\epsilon}{\overset{\epsilon FU}{\rightrightarrows}} F\circ U$$

equal id_{FU} by the triangle identities for the adjunction. □

EXAMPLE 5.5 For the standard adjoint pair

$$\mathsf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Ab}, \qquad \begin{cases} F = \text{forget the group structure} \\ U = \text{form the free abelian group} \end{cases}$$

the endofunctor $F \circ U$ sends an abelian group A to the free abelian group on the underlying set of A.

Statement of the comonadicity theorem

DEFINITION 5.6 Let (G, c, ϵ) be a comonad. A ***coaction*** of G on an object X of B is a morphism $\rho_X : X \to GX$ such that (coassociativity) the two morphisms

$$X \xrightarrow{\rho_X} GX \underset{c_X}{\overset{G\rho_X}{\rightrightarrows}} G \circ GX$$

are equal and (counit) the morphism

$$X \xrightarrow{\rho_X} G_X \xrightarrow{\epsilon_X} X$$

is the identity. We call such a pair (X, ρ_X) a G-***comodule***.

A ***morphism*** of G-comodules $(B, \rho_B) \to (B', \rho_{B'})$ is a morphism $f : B \to B'$ such that the following diagram commutes,

$$\begin{array}{ccc} GB & \xrightarrow{Gf} & GB' \\ \uparrow{\scriptstyle \rho_B} & & \uparrow{\scriptstyle \rho_{B'}} \\ B & \xrightarrow{f} & B'. \end{array}$$

The category B^G of G-modules in B is called the ***Eilenberg–Moore category*** of the comonad (G, c, ϵ).

DEFINITION 5.7 Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and (G, c, ϵ) the associated comonad. The (Eilenberg–Moore) ***comparison functor***

$$\Phi : \mathsf{C} \to \mathsf{B}^G$$

is defined to be

$$\begin{cases} \Phi C = (FC, F\eta c) \\ \Phi f = Ff. \end{cases}$$

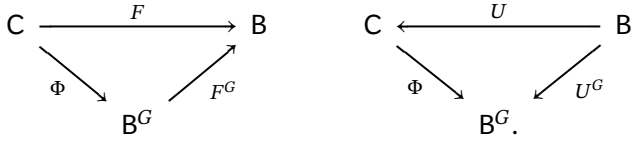
It follows from the triangle identities and the naturality of η that ΦC is a G-comodule and Φf is a morphism of G-comodules.

DEFINITION 5.8 An adjoint pair $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ is ***comonadic*** if the comparison functor $\Phi : \mathsf{C} \to \mathsf{B}^G$ is an equivalence of categories.

Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and define functors

$$\mathsf{B}^G \underset{U^G}{\overset{F^G}{\rightleftarrows}} \mathsf{B} \qquad \begin{cases} F^G(B,\rho_B) = B \\ F^G f = f \end{cases} \qquad \begin{cases} U^G B = (GB, F\eta_{UB}) \\ U^G f = Gf. \end{cases}$$

Then the following diagrams commute,



LEMMA 5.9 *The functors* $\mathsf{B}^G \underset{U^G}{\overset{F^G}{\rightleftarrows}} \mathsf{B}$ *are an adjoint pair.*

PROOF Consider the morphisms $\eta : \mathrm{id}_{\mathsf{B}^G} \to U^G \circ F^G$, $\epsilon : F^G \circ U^G = G \to \mathrm{id}_{\mathsf{B}}$, and $\rho_B : (B,\rho_B) \to (GB, F\eta_{UB})$. The triangle identities for (η,ϵ) must be proven, which means that the compositions

$$(GB, F\eta_{UB}) \overset{F\eta_{UB}}{\longrightarrow} (GGB, F\eta_{UGB}) \overset{G\epsilon_B}{\longrightarrow} (GB, F\eta_{UB})$$

and

$$B \overset{\rho_B}{\longrightarrow} GB \overset{\epsilon_B}{\longrightarrow} B$$

are the identity for all $B \in \mathrm{ob}\,\mathsf{B}$, resp. $(B,\rho_B) \in \mathrm{ob}(\mathsf{B}^G)$. In the first case, this follows from the triangle identities (145), p. 338, and in the second from the co-identity axiom. □

DEFINITION 5.10 A ***split equalizer diagram*** consists of morphisms

$$A \underset{s}{\overset{h}{\rightleftarrows}} B \underset{t}{\overset{f}{\overset{g}{\rightleftarrows}}} C \tag{71}$$

such that $f \circ h = g \circ h$, $s \circ h = \mathrm{id}_A$, $t \circ g = \mathrm{id}_B$, $t \circ f = h \circ s$.

Note that split equalizer diagrams are preserved by functors.

LEMMA 5.11 *In the diagram (71), h is the equalizer of f and g (hence the universal equalizer).*

PROOF The first condition says that h equalizes f and g. Suppose that $w : X \to B$ also equalizes f and g, so $f \circ w = g \circ w$. Let $j = s \circ w$. Then $h \circ j = h \circ s \circ w = t \circ f \circ w = t \circ g \circ w = w$, and so w factors through $A \to B$. The uniqueness of the factorization follows from the condition $s \circ h = \mathrm{id}_A$. □

THEOREM 5.12 (BECK'S COMONADICITY THEOREM) *An adjoint pair*

$$\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$$

is comonadic if and only if the following conditions hold:

(a) *F reflects isomorphisms, i.e., f is an isomorphism whenever Ff is.*

(b) *if the image under F of a parallel pair f, g of morphisms in* C *embeds in a split equalizer diagram in* B*, then the pair f, g has an equalizer in* C *that is preserved by F.*

In the terminology introduced below, (b) says that F-split equalizer pairs have equalizers in C, preserved by F.

We present the proof in the next subsection. For the proof of the opposite (dual) statement, see Borceux 1994b, Theorem 4.4.4, Mac Lane 1998, pp. 147–150, or Riehl 2016, 5.5.1.

COROLLARY 5.13 *Let $F : \mathsf{C} \to \mathsf{B}$ be an exact faithful functor of abelian categories. If F admits a right adjoint functor, then the comparison functor*

$$\Phi : \mathsf{C} \to \mathsf{B}^G, \quad \Phi(C) = (FC, F\eta c)$$

is an equivalence of categories.

PROOF Exact faithful functors of abelian categories reflect isomorphisms,[5] and *every* parallel pair in C has an equalizer in C, which is preserved by F. Therefore, F is comonadic. □

Proof of the comonadicity theorem

Before beginning the proof, we need some definitions.

DEFINITION 5.14 A morphism is a ***regular monomorphism*** if it is the equalizer of some parallel pair of morphisms.

As the name suggests, regular monomorphisms are monomorphisms.

DEFINITION 5.15 Let $F : \mathsf{C} \to \mathsf{B}$ be a functor. An F-***split equalizer pair*** is a parallel pair $f, g : A \rightrightarrows B$ in C together with an extension of $Ff, Fg : FA \rightrightarrows FB$ to a split equalizer diagram in B. If, in addition, there exists a $t : B \to A$ such that $t \circ f = t \circ g = \mathrm{id}_A$, then the pair $f, g : A \to B$ is said to be ***reflexive***.

Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and $\Phi : \mathsf{C} \to \mathsf{B}^G$ the comparison functor.

LEMMA 5.16 *For all C, C' in* C*, Φ induces a bijection*

$$\mathrm{Hom}_{\mathsf{C}}(C, UFC') \simeq \mathrm{Hom}_{\mathsf{B}^G}(\Phi C, \Phi UFC').$$

PROOF We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(C, UFC') &\simeq \mathrm{Hom}_{\mathsf{B}}(FC, FC') \\
&\simeq \mathrm{Hom}_{\mathsf{B}}(F^G(FC, F\eta_C), FC') \\
&\simeq \mathrm{Hom}_{\mathsf{B}^G}((FC, F\eta_C), U^G FC') \\
&\simeq \mathrm{Hom}_{\mathsf{B}^G}((FC, F\eta_C), (GFC', F\eta_{UFC'})) \\
&\simeq \mathrm{Hom}_{\mathsf{B}^G}(\Phi C, \Phi UFC').
\end{aligned}$$

. □

[5] Let $f : A \to B$ be a morphism in C, and consider the exact sequence

$$0 \to K \to A \xrightarrow{f} B \to C \to 0.$$

This sequence remains exact when we apply F. If Ff is an isomorphism, then $FK = 0 = FC$, so $K = 0 = C$, and f is an isomorphism.

LEMMA 5.17 *For all* (B, ρ_B) *in* B^G,

$$(GB, c_B) \underset{G\rho_B}{\overset{c_B}{\rightrightarrows}} (GGB, c_{GB})$$

is a reflexive F^G*-split equalizer pair.*

PROOF The naturality of c implies that $G\rho_B$ and c_B are morphisms of comodules. Their common left inverse is $G\epsilon_B$ (which is a comodule morphism for the same reason). We have

$$G\epsilon_B \circ c_B = FU\epsilon_B \circ F\eta_{UB} = F(U\epsilon_B \circ \eta_{UB}) = \mathrm{id}$$

because of the triangle identities (p. 338), and $G\epsilon_B \circ c_B = \mathrm{id}$ because of the co-identity axiom. It remains to show that

$$B \underset{\epsilon_B}{\overset{\rho_B}{\rightleftarrows}} B \underset{G\rho_B}{\overset{c_B}{\rightrightarrows}} C \tag{72}$$

is a split equalizer diagram in B. However, this follows directly from the triangle equalities and the comodule axioms. □

LEMMA 5.18 *In the situation of Lemma 5.17,*

$$(B, \rho_B) \xrightarrow{\rho_B} (GB, c_B) \underset{G\rho_B}{\overset{c_B}{\rightrightarrows}} (GGB, c_{GB})$$

is an equalizer in B^G.

PROOF Let $f : (B', \rho_{B'}) \to (GB, c_B)$ be a morphism of modules with $c_B \circ f = G\rho_B \circ f$. After 5.17 and 5.11, there exists a unique map $g : B' \to B$ such that $\rho_B \circ g = f$, and it remains to show that g is a morphism of comodules. For this, we consider

$$\begin{array}{ccccc} GB' & \xrightarrow{Gg} & GB & \xrightarrow{G\rho_B} & GGB \\ {\scriptstyle \rho_{B'}}\uparrow & & {\scriptstyle \rho_B}\uparrow & & {\scriptstyle c_B}\uparrow \\ B' & \xrightarrow{g} & B & \xrightarrow{\rho_B} & GB. \end{array}$$

The outer rectangle commutes by assumption and the right-hand rectangle commutes because of coassociativity. It follows that

$$G\rho_B \circ Gg \circ \rho_{B'} = G\rho_B \circ \rho_B \circ g.$$

Since $G\rho_B$ has left inverse $G\epsilon_B$, the assertion follows. □

LEMMA 5.19 *For all* C *in* C,

$$UFC \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} UFUFC$$

is a reflexive F*-split equalizer pair.*

PROOF The triangle identities show that the two arrows have $U\xi_{FC}$ as a common left inverse. The assertion now follows from Lemma 5.17, applied to the comodule (FC, η_C). □

LEMMA 5.20 *If $\eta_C : C \to UFC$ is a regular monomorphism for all C in C, then*

$$C \xrightarrow{\eta_C} UFC \underset{\eta_{UFC}}{\overset{UF\eta_C}{\rightrightarrows}} UFUFC$$

is an equalizer for all C.

PROOF By assumption, η_C is the equalizer of a pair $A \underset{d^1}{\overset{d^0}{\rightrightarrows}} B$. We have to show that, for every morphism $w : X \longrightarrow UFC$,

$$UF\eta_C \circ w = \eta_{UFC} \circ w \iff d^0 \circ w = d^1 \circ w.$$

First of all $\eta_{UFC} \circ w = UF\eta_C \circ w$, and it follows

$$UF\eta_C \circ UFw = UFUF\eta_C \circ UFw.$$

We claim that

$$UFC \underset{U\epsilon_{FC}}{\overset{UF\eta_C}{\rightleftarrows}} UFUFC \underset{UFUF\eta_C}{\overset{UF\eta_{UFC}}{\rightrightarrows}} UFUFUFC \quad (U\epsilon_{FUFC})$$

is a split kernel pair in C. This follows from (72) applied to $(FC, F\eta_C)$ and (5.11). From the claim it follows that UFw factors through UFC. Let $h : X \to UFC$ be a morphism with $UFw = UF\eta_C \circ h$. It follows that

$$\begin{aligned} UFd^0 \circ UFw &= UFd^0 \circ UF\eta_C \circ h \\ &= UFd^1 \circ UF\eta_C \circ h \\ &= UFd^1 \circ UFw. \end{aligned}$$

Because of the naturality of η, the following diagram commutes

$$\begin{array}{ccccc} X & \xrightarrow{w} & UFC & \underset{d^1}{\overset{d^0}{\rightrightarrows}} & C' \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{UFC}} & & \downarrow{\scriptstyle \eta_{C'}} \\ UFX & \xrightarrow{UFw} & UFUFC & \underset{UFd^1}{\overset{UFd^0}{\rightrightarrows}} & UFC'. \end{array}$$

It follows that $\eta_{C'} \circ d^0 \circ w = \eta_{C'} \circ d^1 \circ w$, and from this the assertion follows because $\eta_{C'}$ is a monomorphism. Now the converse applies, $d^0 \circ w = d^1 \circ w$. Then $w = \eta_C \circ g$ for a $g : X \to C$. From this follows

$$\eta_{UFC} \circ w = \eta_{UFC} \circ \eta_C \circ g = UF\eta_C \circ \eta_C \circ g = UF\eta_C \circ w. \qquad \square$$

LEMMA 5.21 *If η_C is a regular monomorphism for all $C \in \operatorname{ob} \mathsf{C}$, then Φ is fully faithful.*

PROOF After Lemma 5.20,

$$C \xrightarrow{\eta_C} UFC \underset{\eta_{UFC}}{\overset{UF\eta_C}{\rightrightarrows}} UFUFC$$

is an equalizer for all $C \in \operatorname{ob}\mathsf{C}$, and after Lemma 5.18,

$$\Phi C \xrightarrow{\Phi\eta_C} \Phi UFC \underset{\Phi\eta_{UFC}}{\overset{\Phi UF\eta_C}{\rightrightarrows}} \Phi UFUFC$$

is an equalizer in B^G. So, for all $A \in \operatorname{ob}\mathsf{C}$, the rows in the commutative diagram

$$\begin{array}{ccccc} \operatorname{Hom}(A,C) & \longrightarrow & \operatorname{Hom}(A,UFC) & \rightrightarrows & \operatorname{Hom}(A,UFUFC) \\ \downarrow\Phi & & \downarrow\Phi & & \downarrow\Phi \\ \operatorname{Hom}(\Phi A,\Phi C) & \longrightarrow & \operatorname{Hom}(\Phi A,\Phi UFC) & \rightrightarrows & \operatorname{Hom}(\Phi A,\Phi UFUFC) \end{array}$$

are equalizers. According to Lemma 5.16, the middle and right vertical arrows are bijective, and so the left is also. □

PROOF (OF THEOREM 5.12) Let $C \in \operatorname{ob}\mathsf{C}$. We first show that η_C is a regular homomorphism. According to Lemma 5.19,

$$UFC \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} UFUFC$$

is a reflexive F-split kernel pair in C. According to hypothesis (b) of the theorem, there exists an equalizer $K \xrightarrow{d} UFC$. Because of the naturalness of η, the morphism η_C equalizes η_{UFC} and $UF\eta_C$, and so it factors through K,

$$\begin{array}{ccccc} K & \xrightarrow{d} & UFC & \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} & UFUFC \\ {\scriptstyle f}\uparrow & \nearrow_{\eta_C} & & & \\ C & & & & \end{array}$$

According to (72) with $B = FC$, when we apply F to the bottom row we get an equalizer, and after hypothesis (b), when we apply F to the top row, we get an equalizer in B. Therefore is Ff an isomorphism, which implies that f is an isomorphism. Thus η_C is a regular monomorphism. According to Lemma 5.21, Φ is fully faithful. Now let $(B,\rho_B) \in \operatorname{ob}\mathsf{B}^G$. According to Lemma 5.17

$$(FUB, F\eta_{UB}) \underset{FU\rho_B}{\overset{F\eta_{UB}}{\rightrightarrows}} (FUFUB, F\eta_{UFUB})$$

is a reflexive F^G-split kernel pair in B^G. On the other hand, this pair is the image of

$$UB \underset{U\rho_B}{\overset{\eta_{UB}}{\rightrightarrows}} UFUB$$

under Φ. Therefore this is a reflexive F-split kernel pair in C, and the common left inverse of both arrows is $U\epsilon_B$. After hypothesis (b), there exists an equalizer $d : K \to UB$ and

$$FK \xrightarrow{Fd} FUB \underset{FU\rho_B}{\overset{F\eta_{UB}}{\rightrightarrows}} B$$

is an equalizer in B. After (72), $\rho_B : B \to FUB$ is the equalizer of the same pair, and so there exists a unique isomorphism $g : B \to FK$ such that $Fd \circ g = \rho_B$. It remains to show that g is a morphism of comodules. Consider

$$\begin{array}{ccccc} FUB & \xrightarrow{FUg} & FUFK & \xrightarrow{FUFd} & FUFUB \\ {\scriptstyle\rho_B}\uparrow & & {\scriptstyle F\eta_K}\uparrow & & {\scriptstyle F\eta_{UB}}\uparrow \\ B & \xrightarrow{g} & FK & \xrightarrow{Fd} & FUB. \end{array}$$

The right-hand square commutes because of the naturalness of η, and the outer one because of the coassociativity. The map ρ_B is left invertible, and so Fd is also left invertible. It follows that the left square commutes. Thus $\Phi K \simeq (B, \rho_B)$, and we have shown that Φ is essentially surjective. □

NOTES As noted, this section is standard category theory, although usually expressed for monads rather than comonads. Our proof of the comonadicity theorem follows that in Lattermann 1989, 2.3.

Application to modules and comodules

5.22 In this subsection, A and B are rings (not necessarily commutative), and $M = {}_AM_B$ is an (A, B)-bimodule, finitely generated and projective as a right B-module, and faithfully flat as an A-module. The B-dual of M,

$$M' = {}_BM'_A \stackrel{\text{def}}{=} \mathrm{Hom}_B(M, B),$$

is a (B, A)-bimodule with the actions

$$\begin{cases} (b \cdot m')(m) = bm'(m) \\ (m' \cdot a)(m) = m'(am) \end{cases} \qquad a \in A, b \in B, m \in M, m' \in M'.$$

LEMMA 5.23 *For any right B-module Y, the homomorphism*

$$\big(y \otimes m'\big) \mapsto \big(m \mapsto ym'(m)\big) : Y_B \underset{B}{\otimes} {}_BM'_A \to \mathrm{Hom}_B(M, Y),$$

is an isomorphism.

PROOF After replacing B with B_f for f in a suitable finite set of elements of B, we may suppose that M is free as a B-module. For a fixed Y, both sides commute with finite direct sums, and so it suffices to check this for $M = B$, where it is obvious. □

When $Y = M$, the lemma says that

$${}_AM_B \underset{B}{\otimes} {}_BM'_A \simeq \mathrm{End}_B(M), \qquad m \otimes m' \leftrightarrow (x \mapsto m \cdot m'(x)). \tag{73}$$

5.24 We have functors

$$\mathsf{Mod}_A \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Mod}_B \qquad \begin{cases} FX = X \otimes_A M \\ UY = Y \otimes_B M'. \end{cases} \tag{74}$$

For $X \in \mathrm{ob}(\mathsf{Mod}_A)$, $Y \in \mathrm{ob}(\mathsf{Mod}_B)$, there are canonical isomorphisms

$$\mathrm{Hom}_B(X \otimes_A M, Y) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y)) \simeq \mathrm{Hom}_A(X, Y \otimes_B M'),$$

natural in X and Y. The first is defined by the pairing

$$(f,x)\mapsto(m\mapsto f(x\otimes m))\,:\ \mathrm{Hom}_B(X\otimes_A M,Y)\times X\to\mathrm{Hom}_B(M,Y),$$

and the second is induced by the isomorphism in Lemma 5.23. Hence (74) is an adjunction. The corresponding unit and counit are

$$\begin{aligned}&\eta\,:\,X\to X\otimes_A M\otimes_B M',\quad \eta(x)=x\otimes\delta\\&\epsilon\,:\,Y\otimes_B M'\otimes_A M\to Y,\quad \epsilon(y\otimes m'\otimes m)=ym'(m),\end{aligned}$$

where $\delta\in M\otimes_B M'$ corresponds to the identity map under the isomorphism (73).

5.25 We can now apply Proposition 5.4. Let L denote the B-bimodule

$${}_BM'_A\underset{A}{\otimes}{}_AM_B,$$

The functor $G\overset{\text{def}}{=}F\circ U$ sends a right B-module Y to $Y\otimes_B L$, and $G\circ G$ sends Y to $Y\otimes_B{}_BL_B\otimes_B{}_BL_B$. We have B-bimodule homomorphisms

$$\begin{aligned}&c\,:\,L\to L\otimes_B L,\quad m'\otimes m\mapsto m'\otimes\delta\otimes m\\&\epsilon\,:\,L\to B,\quad m'\otimes m\mapsto m'(m),\end{aligned}$$

which make L a B-coalgebroid (with $k=\mathbb{Z}$). The corresponding natural transformations

$$\begin{aligned}&c\,:\,G\to G\circ G\\&\epsilon\,:\,G\to\mathrm{id}\end{aligned}$$

make G into a comonad. The functor F is faithful and exact because ${}_AM_B$ is faithfully flat over A.

In the next theorem, we construct a commutative diagram

$$\begin{array}{ccc}\mathsf{Mod}_A & \xrightarrow{\ \omega\ } & \mathsf{coMod}_L\\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \text{forget}} & \\ & \mathsf{Mod}_B. & \end{array}$$

THEOREM 5.26 (DELIGNE 1990, 4.4) *Let $M={}_AM_B$ be an (A,B)-bimodule, finitely generated and projective as a B-module, and faithfully flat as an A-module; let $M^\vee$ be its B-dual and L the coalgebroid $M^\vee\otimes_A M$ defined above. The functor*

$$\omega\,:\,\mathsf{Mod}_A\longrightarrow\mathsf{coMod}_L,\quad X_A\rightsquigarrow(X_A\underset{A}{\otimes}{}_AM_B,\rho_X),$$

where

$$\rho_X(x\otimes m)=x\otimes\delta\otimes m\in(X\otimes_A M)\otimes(M^\vee\otimes_A M),$$

is an equivalence of categories. Under the equivalence, A-modules of finite type (resp. of finite presentation) correspond to comodules of finite type (resp. of finite presentation) as B-modules.

PROOF The first statement follows from Corollary 5.13 applied to the above discussion. It remains to prove the second statement.

If X is finitely generated over A, then there exists a surjection $A^n \to X$. On tensoring this with M, we get a surjection $M^n \to X \otimes_A M$. As M is finitely generated over B, so also are M^n and $X \otimes_A M$.

If $X \otimes_A M$ is finitely generated as a B-module, then it has a finite set of generators

$$x_i \otimes m_i, \quad x_i \in X, \quad m_i \in M, \quad i \in I.$$

The A-submodule X' of X generated by the x_i has the property that $X' \otimes_A M \simeq X \otimes_A M$. As M is faithfully flat over A, this implies that $X' = X$.

Let X be finitely generated over A. Then $X = A^n/R$ for some $n \geq 0$, and X is finitely presented if and only if R is finitely generated, and $\omega(X) \simeq {}_AM_B^n/\omega(R)$ is finitely presented if and only if $\omega(R)$ is finitely generated. Now use that R is finitely generated if and only if $\omega(R)$ is. □

REMARK 5.27 When A and B are k-algebras (k a commutative ring) and the two k-module structures on ${}_AM_B$ coincide, the coalgebroid $L \overset{\text{def}}{=} {}_BM_A^\vee \underset{A}{\otimes} {}_AM_B$ is a k-coalgebroid.

EXAMPLE 5.28 In the case $A = k$ is a field and B a k-algebra, we obtain from a finitely generated projective B-module M, a k-coalgebroid $L \overset{\text{def}}{=} M^\vee \otimes_k M$ whose coidentity is the evaluation map. This coalgebroid coacts on M (the image of k by $- \otimes_k M$) by

$$\rho_0 : M \to M \otimes_B L, \quad m \mapsto \delta \otimes m \in M \otimes_B M' \otimes_B M \simeq M \otimes_B L.$$

ASIDE 5.29 From the comonadicity theorem, it is possible to deduce a faithfully flat descent theorem for noncommutative rings, and from there the faithfully flat descent theorem for commutative rings. However, we do not need the first theorem and this approach to the second theorem is not easier than the direct approach (Waterhouse 1979, 17.2), even for those familiar with the comonadicty theorem. See Lattermann 1989, 2.4.10, 2.4.11.

6 Proof of the main theorem

After these preliminaries, we are ready to prove Theorem 1.1

The coalgebroid of endomorphisms of a fibre functor

In this subsection, we attach a coalgebroid $L(\omega)$ to a fibre functor ω on a tannakian category. Recall that when k is a commutative ring and B_1 and B_2 are k-algebras, we always require the two actions of k on a (B_1, B_2)-module to be equal.

6.1 We begin with a general definition. Let C be a small category and $F : \mathsf{C}^{\text{opp}} \times \mathsf{C} \to \mathsf{D}$ a functor. A ***cowedge*** $e : F \to w$ is an object w of D and a family of morphisms $e_c : F(c,c) \to w$, indexed by the objects c of C, such that, for all morphisms $f : c' \to c$, the following diagram commutes,

$$\begin{array}{ccc} F(c,c') & \xrightarrow{F(f,c')} & F(c',c') \\ \downarrow{\scriptstyle F(c,f)} & & \downarrow{\scriptstyle e_{c'}} \\ F(c,c) & \xrightarrow{e_c} & w. \end{array}$$

Given a cowedge $e: F \to w$ and a morphism $h: w \to v$, we obtain a cowedge $h \circ e: F \to v$ by composition. A cowedge $e: F \to w$ is a ***coend*** if it is universal, i.e., any other cowedge $e': F \to w'$ factors uniquely through a morphism $h: w \to w'$. When C has direct sums, a wedge can be viewed as a morphism

$$e: \bigoplus_{c \in \operatorname{ob}\mathsf{C}} F(c,c) \to w,$$

and such a morphism is a coend if and only if it is the coequalizer of the pair of morphisms

$$\bigoplus_{(f:\, c' \to c) \in \operatorname{ar}\mathsf{C}} F(c,c') \underset{e_c \circ F(c,f)}{\overset{e_{c'} \circ F(f,c')}{\rightrightarrows}} \bigoplus_{c \in \operatorname{ob}\mathsf{C}} F(c,c).$$

6.2 Let B_1, B_2 be k-algebras (k a commutative ring) and ω_1, ω_2 functors from a small category C to the categories of finitely generated projective right modules over B_1, B_2,

$$\begin{cases} \omega_1 : \mathsf{C} \to \operatorname{Proj}_{B_1} \\ \omega_2 : \mathsf{C} \to \operatorname{Proj}_{B_2}. \end{cases}$$

Let $\omega_1(Y)^\vee$ denote the B_1-dual of $\omega_1(Y)$ – it is a left B_1-module (5.22). We define $L_k(\omega_1, \omega_2)$ to be the coend of the functor

$$(Y, X) \rightsquigarrow \omega_1(Y)^\vee \otimes_k \omega_2(X) : \mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \{(B_1, B_2)\text{-bimodules}\}.$$

Thus, $L_k(\omega_1, \omega_2)$ is a (B_1, B_2)-bimodule equipped with a morphism of (B_1, B_2)-bimodules

$$\omega_1(X)^\vee \otimes_k \omega_2(X) \to L_k(\omega_1, \omega_2) \tag{75}$$

for each $X \in \operatorname{ob}\mathsf{C}$ such that certain diagrams commute and $L_k(\omega_1, \omega_2)$ is universal. More concretely, it is the coequalizer of the parallel pair of morphisms

$$\bigoplus_{(f:\, X \to Y) \in \operatorname{ar}\mathsf{C}} \omega_1(Y)^\vee \otimes \omega_2(X) \rightrightarrows \bigoplus_{X \in \operatorname{ob}\mathsf{C}} \omega_1(X)^\vee \otimes_k \omega_2(X).$$

When it causes no confusion, we drop the k from $L_k(\omega_1, \omega_2)$.

To give the morphisms (75) is equivalent to giving morphisms of B_2-modules

$$\lambda(X): \omega_2(X) \to \omega_1(X) \otimes_{B_1} L(\omega_1, \omega_2), \qquad X \in \operatorname{ob}\mathsf{C}. \tag{76}$$

The commutativity of the diagrams means that $\lambda(X)$ is functorial in X, and the universal property of $L(\omega_1, \omega_2)$ says that, for all (B_1, B_2)-bimodules U, the map

$$f \mapsto (\mathrm{id}_{\omega_1(X)} \otimes f \circ \lambda(X))_X$$

sending a (B_1,B_2)-bimodule homomorphism $f: L(\omega_1, \omega_2) \to U$ into the system of morphisms $u(X)$, natural in X,

$$\omega_2(X) \xrightarrow[\lambda(X)]{} \omega_1(X) \otimes_{B_1} L(\omega_1, \omega_2) \xrightarrow[\mathrm{id}_{\omega_1(X)} \otimes f]{} \omega_1(X) \otimes_{B_1} U, \qquad u(X) = (\mathrm{id}_{\omega_1(X)} \otimes f) \circ \lambda(X)$$

is a bijection, so

$$\operatorname{Hom}_{(B_1,B_2)}(L(\omega_1, \omega_2), U) \simeq \operatorname{Nat}(\omega_2(-), \omega_1(-) \otimes_{B_1} U).$$

6.3 For three functors $\omega_1, \omega_2, \omega_3$, iterating (76), we obtain a morphism

$$\omega_3(X) \to \omega_2(X) \otimes_{B_2} L(\omega_2, \omega_3) \to \omega_1(X) \otimes_{B_1} L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3),$$

natural in X, and hence a morphism

$$L(\omega_1, \omega_3) \to L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3). \tag{77}$$

The coproduct (77) is coassociative: for four functors $\omega_1, \omega_2, \omega_3, \omega_4$, the diagram

$$\begin{array}{ccc} L(\omega_1, \omega_4) & \longrightarrow & L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_4) \\ \downarrow & & \downarrow \\ L(\omega_1, \omega_3) \otimes_{B_3} L(\omega_3, \omega_4) & \longrightarrow & L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3) \otimes_{B_3} L(\omega_3, \omega_4) \end{array}$$

is commutative. The evaluation maps $\omega_j(X)^\vee \otimes_{B_j} \omega_j(X) \to B_j$ define a counit $\epsilon : L(\omega_j, \omega_j) \to B_j$: the two maps

$$L(\omega_1, \omega_2) \to L(\omega_1, \omega_2) \otimes_{B_2} L(\omega_2, \omega_3) \xrightarrow{1 \otimes \epsilon} L(\omega_1, \omega_2)$$
$$L(\omega_1, \omega_2) \to L(\omega_1, \omega_1) \otimes_{B_1} L(\omega_1, \omega_2) \xrightarrow{\epsilon \otimes 1} L(\omega_1, \omega_2)$$

equal the identity map.

6.4 The important case for us is $B_1 = B_2$ and $\omega_1 = \omega_2$. Let $B = B_1 = B_2$. Then the map (77) for $\omega_1 = \omega_2 = \omega_3 = \omega$ makes

$$L_k(\omega) \stackrel{\text{def}}{=} L_k(\omega, \omega)$$

into a k-coalgebroid acting on B. Note that $L_k(\omega)$ is the coequalizer of

$$\bigoplus_{(f\,:\,X \to Y) \in \mathrm{ar}\,\mathsf{C}} \omega(Y)^\vee \otimes \omega(X) \rightrightarrows \bigoplus_{X \in \mathrm{ob}\,\mathsf{C}} \omega(X)^\vee \otimes_k \omega(X).$$

The map (76)

$$\lambda(X) : \omega(X) \to \omega(X) \otimes_B L(\omega)$$

is a coaction of $L(\omega)$ on $\omega(X)$, functorial in X. The universal property of $L(\omega)$ says that, for any k-coalgebroid L' acting on B, to give an action of L' on $\omega(X)$, natural in X, is the same as giving a morphism of coalgebroids $L(\omega) \to L'$. We call $L(\omega) = L_k(\omega)$ the ***coalgebroid of*** k***-endomorphisms*** of ω.

EXAMPLE 6.5 When C consists of a single object and its identity morphism, to give ω is the same as giving a finitely generated projective right B-module M. The coalgebroid $L(\omega)$ is $M^\vee \otimes_k M$ and its coaction on M is that in 5.28. The universal property of $L(\omega)$ says that, for any k-coalgebroid L' acting on B, the canonical isomorphism

$$\mathrm{Hom}_B(M, M \otimes_k L') \simeq \mathrm{Hom}_{(B,B)}(M^\vee \otimes_k M, L')$$

makes coactions of L' on M correspond to morphisms of coalgebroids $M^\vee \otimes_k M \to L'$.

PROPOSITION 6.6 *Let L be a coalgebroid acting on a division algebra B.*

(a) *Every representation V of L is a filtered union of subrepresentations of finite dimension over B.*

(b) *The coalgebroid L is a filtered union of subcoalgebras of finite type as $B \otimes B$-modules.*

PROOF (a) Let $\rho : V \to V \otimes_B L$ be a coaction. Each $a \in V \otimes_B L$ is contained in a subspace $V_1 \otimes_B L$ with V_1 finite-dimensional, and the smallest V_1 is the set of elements $\lambda(a)$, where λ runs over the morphisms $L \to B$ of left B-vector spaces. For $a = \rho(v)$, this V_1 contains v (take λ to be the coidentity) and is stable: if $\rho(v) = \sum v_i \otimes \ell_i$ with the ℓ_i linearly independent, then V_1 is generated by the v_i; the axiom for coactions gives

$$\sum \rho(v_i) \otimes \ell_i = \sum v_i \otimes c(\ell_i) \in V_1 \otimes L \otimes L,$$

and so $\rho(v_i) \in V_1 \otimes L$ for all i.

(b) The comultiplication $c : L \to L \otimes_B L$ is a coaction of L on L (the regular representation). After (a), L is a filtered union of the subrepresentations V_i of finite dimension over B. The coaction of L on V_i corresponds, after 6.5, to a morphism of coalgebroids $f_i : V_i^\vee \otimes V_i \to L$. The coidentity e of L induces a linear form on V_i, and $f_i(e|V_i \otimes x) = x$. The image of f_i therefore contains V_i. Because B is a division algebra, the image of f_i is a subcoalgebroid of L. The coalgebroid L is a filtered union of the images of the f_i, which completes the proof. □

FUNCTORIALITIES

6.7 (a) Consider homomorphisms $f : B_j \to B'_j$ and the corresponding extensions of scalars $\omega_j \mapsto \omega'_j$. We have

$$L(\omega_1, \omega_2) \otimes_{B_1 \otimes_k B_2} (B'_1 \otimes_k B'_2) \xrightarrow{\simeq} L(\omega'_1, \omega'_2). \tag{78}$$

(b) Consider a functor $T : \mathsf{D} \to \mathsf{C}$. The morphisms

$$\omega_2(T(D)) \to \omega_1(T(D)) \otimes L(\omega_1, \omega_2), \quad D \in \operatorname{ob} \mathsf{D},$$

define a morphism of coalgebroids

$$L(\omega_1 \circ T, \omega_2 \circ T) \to L(\omega_1, \omega_2). \tag{79}$$

(c) If C is a filtered inductive limit of categories C_i and T_i is the natural functor $\mathsf{C}_i \to \mathsf{C}$, then the morphisms (79) induce an isomorphism

$$\varinjlim L(\omega_1 \circ T_i, \omega_2 \circ T_i) \to L(\omega_1, \omega_2). \tag{80}$$

(d) Assume that B_1, B_2 are commutative. Let $(\mathsf{C}_i)_{i \in I}$ be a finite family of categories, and let

$$\begin{cases} \omega_1^i : \mathsf{C} \to \operatorname{Proj}_{B_1} \\ \omega_2^i : \mathsf{C} \to \operatorname{Proj}_{B_2} \end{cases} \quad i \in I$$

be functors. Define

$$\begin{cases} \bigotimes_i \omega_1^i : \mathsf{C} \to \operatorname{Proj}_{B_1}, & \left(\bigotimes_i \omega_1^i\right)(C) = \bigotimes_i \omega_1^i(C) \\ \bigotimes_i \omega_2^i : \mathsf{C} \to \operatorname{Proj}_{B_2}, & \left(\bigotimes_i \omega_2^i\right)(C) = \bigotimes_i \omega_2^i(C). \end{cases}$$

Then

$$L(\otimes \omega_1^i, \otimes \omega_2^i) \xrightarrow{\simeq} \bigotimes_{B_1 \otimes_k B_2} L(\omega_1^i, \omega_2^i).$$

As in 5.22 et seq., let ${}_AM_B$ be an (A,B)-module, finitely generated and projective as a right B-module, let ${}_BM_A^\vee$ be its B-dual, and let L be the B coalgebroid ${}_BM_A^\vee \otimes_A {}_AM_B$. We suppose that, for some commutative ring k, A and B are k-algebras and that the two k-module structures on ${}_AM_B$ coincide. For example, we could have $k = \mathbb{Z}$, in which case the hypothesis is automatic.

LEMMA 6.8 *Let* C *be a full subcategory of the category of right A-modules, containing A_A, and such, if $E \in \mathrm{ob}\,\mathsf{C}$, then $E \otimes_A {}_AM_B$ is finitely generated and projective over B. Let ω be the functor $E \rightsquigarrow E \otimes_A {}_AM_B$. By 5.26, L coacts on the $\omega(E)$, $E \in \mathrm{ob}\,\mathsf{C}$, and hence we have a morphism*

$$L(\omega) \to L. \tag{81}$$

This morphism is an isomorphism.

PROOF When C consists only of the A-module A, this is the definition of the tensor product over A. We now give the general proof.

Let D be the full subcategory of C having A_A as its only object. The functoriality 6.7(b) gives a map $L(\omega|\mathsf{D}) \to L(\omega)$. As we just noted, we have $L(\omega|\mathsf{D}) \xrightarrow{\simeq} L$, and the triangle

$$\begin{array}{ccc} L(\omega|\mathsf{D}) & \xrightarrow{6.7(b)} & L(\omega) \\ & {\scriptstyle (81)}\searrow \quad \swarrow{\scriptstyle (81)} & \\ & L & \end{array}$$

commutes. The morphism (79) therefore admits a retraction, and it suffices to show that $L(\omega|\mathsf{D})$ maps onto $L(\omega)$, i.e., that for all C in C, the image in $L(\omega)$ of $\omega(C)^\vee \otimes_k \omega(C)$ is contained in $\omega(A)^\vee \otimes_k \omega(A)$. Every element of $\omega(C)^\vee \otimes \omega(C)$ is a finite sum $\sum \alpha_i \otimes x_i$ and each $x_i \in C \otimes_A {}_AM_B$ is a finite sum $\sum \alpha_{ij} \otimes m_j$, and therefore a sum of elements of the form $f(y)$ with $f : A \to C$ and $y \in \omega(\mathsf{A})$. It therefore suffices to show that an element of $\omega(C)^\vee \otimes_k \omega(C)$ of the form $\alpha \otimes f(y)$, $f : A \to C$, $y \in \omega(A)$, has image in $L(\omega)$ contained in that of $\omega(A)^\vee \otimes_k \omega(A)$. By definition of $L(\omega)$, the diagram

$$\begin{array}{ccc} \omega(C)^\vee \otimes_k \omega(A) & \xrightarrow{{}^tf\otimes 1} & \omega(A)^\vee \otimes_k \omega(A) \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow \\ \omega(C)^\vee \otimes_k \omega(C) & \longrightarrow & L \end{array}$$

commutes. Applying this to $\alpha \otimes y \in \omega(C)^\vee \otimes \omega(A)$, we find that the image of $\alpha \otimes f(y)$ in L is also the image of the element $f^t(\alpha) \otimes y$ of $\omega(A)^\vee \otimes_k \omega(A)$. □

Let B be a division k-algebra and L a k-coalgebroid acting on B. Let $\mathsf{coModf}(L)$ denote the category of L-comodules of finite dimension as B-vector spaces, and let ω be the forgetful functor. Since L coacts on each $\omega(X)$, the universal property of $L(\omega)$ furnishes a morphism u from $L(\omega)$ to L.

PROPOSITION 6.9 *The morphism $u : L(\omega) \to L$ is an isomorphism.*

PROOF By construction, a coaction of L on a finite-dimensional V has a natural lift to $L(\omega)$. Applying 6.6 and passing to the inductive limit, we see that the restriction "finite-dimensional" is unnecessary. Taking $V = L$ and the coaction $c : L \to L \otimes_B L$, we obtain $c_1 : L \to L \otimes_B L(\omega)$. Let $a = (\mathrm{counit}\otimes 1)\circ c_1 : L \to L(\omega)$. As $(1 \otimes u)\circ c_1 = c$, we have $ua = \mathrm{id}$.

Let V be equipped with a coaction ρ, which lifts to a coaction $\widetilde{\rho}$ of $L(\omega)$. Since $\rho : V \to V \otimes L$ is a morphism of vector spaces with coaction, the diagram

$$\begin{array}{ccccc} V & & \xrightarrow{\quad\rho\quad} & & V \otimes L \\ \downarrow{\scriptstyle \bar{\rho}} & & & & \downarrow{\scriptstyle 1\otimes c_1} \\ V \otimes L\omega & \xrightarrow{\rho\otimes 1} & V \otimes L \otimes L\omega & \xrightarrow{1\otimes\text{counit}\otimes 1} & V \otimes L\omega \end{array}$$

commutes, and so $\widetilde{\rho} = (1 \otimes a)\rho$. The morphism deduced from $\rho : V^\vee \otimes V \to L\omega$ admits the factorization $V^\vee \otimes V \to L \xrightarrow{a} L\omega$. The definition of $L\omega$ shows that a is surjective, and therefore u is an isomorphism. □

Realizing (T, ω) *as a category of comodules*

The next theorem generalizes (II, 3.20).

THEOREM 6.10 (DELIGNE 1990, 6.2) *Let B be an algebra over a field k. Let A be an essentially small, locally finite, k-linear abelian category and $\omega : \mathsf{A} \to \mathrm{Proj}(B)$ an exact faithful k-linear functor. Let $L(\omega)$ be the k-coalgebroid of k-endomorphisms of ω (6.4). Then ω defines an equivalence of categories $\mathsf{A} \xrightarrow{\approx} \mathsf{coModf}(L(\omega))$ carrying ω into the forgetful functor.*

PROOF Recall (II, 3) that, for an object X of an abelian category, $\langle X\rangle$ denotes the strictly full subcategory whose objects are subquotients of a finite direct sum of copies of X. It is an abelian subcategory containing X.

For X in A, the category $\langle X\rangle$ admits a projective generator P (II, 3.13). Let $A = \mathrm{End}(P)$. Then the functor $Y \rightsquigarrow \mathrm{Hom}(P, Y)$ is an equivalence of $\langle X\rangle$ with the category Modf_A of right A-modules of finite type. Under this equivalence, P corresponds to A_A. Put ${}_AM_B = \omega(P)$. By (the proof of) 3.8, the right exact functor $\omega|\langle X\rangle$ can be identified with the functor $E \rightsquigarrow E \otimes_A {}_AM_B : \mathsf{Modf}_A \to \mathsf{Mod}_B$. Note that, as ω is k-linear, the two k-module structures on ${}_AM_B$ coincide.

After 6.8, $L(\omega|X)$ is the k-coalgebroid ${}_BM_A^\vee \otimes_A {}_AM_B$ of 5.26. By hypothesis, $\omega|\langle X\rangle$ is exact and faithful. The A-module ${}_AM_B$ is therefore faithfully flat over A. After 5.26 and 5.27, ω induces an equivalence of $\langle X\rangle$ with the category of right B-modules of finite type equipped with a coaction of $L(\omega|\langle X\rangle)$.

The category $\mathrm{Ind}\langle X\rangle$ of Ind-objects of $\langle X\rangle$ can be identified with that of all right A-modules Mod_A (see B.12). The extension of $L(\omega|\langle X\rangle)$ to Ind-objects, $\omega(\text{“}\varinjlim\text{”}X_i) = \varinjlim \omega(X_i)$ is again $E \rightsquigarrow E \otimes_A {}_AM_B$. By 5.26, this extension is an equivalence of $\mathrm{Ind}\langle X\rangle$ with the category of right B-modules equipped with a coaction of $L(\omega|\langle X\rangle)$. This, and the assumed properties of ω, show that any right B-module of finite type, equipped with a coaction of $L(\omega|\langle X\rangle)$, is finitely generated projective, and that any right B-module with a coaction is a inductive limit of finitely generated projective right B-modules, and, in particular, is flat.

The exactness of ω ensures that ${}_AM_B$ is A-flat. Moreover, for any right A-module N, the right B-module $N \otimes_A {}_AM_B$ is flat. The formula

$$L(\omega|\langle X\rangle) = {}_BM_A^\vee \otimes_A {}_AM_B$$

then shows that $L(\omega|\langle X\rangle)$ is flat for the two B-module structures.

If Y is in $\langle X\rangle$, i.e., $\langle Y\rangle \subset \langle X\rangle$, and $\mathfrak{a}$ is the 2-sided ideal of A such that $\langle Y\rangle$ corresponds to the A-modules killed by $\mathfrak{a}$ (see 3.6), we have

$$\begin{aligned} L(\omega|\langle Y\rangle) &= (A/\mathfrak{a} \otimes_A {}_AM_B)^\vee \otimes_{A/\mathfrak{a}} (A/\mathfrak{a} \otimes_A {}_AM_B) \\ &= (A/\mathfrak{a} \otimes_A {}_AM_B)^\vee \otimes_{A/\mathfrak{a}} M_B, \end{aligned}$$

and $(A/\mathfrak{a} \otimes_A {}_AM_B)^\vee$ is the kernel of the epimorphism ${}_BM_A^\vee \to (\mathfrak{a} \otimes_A {}_AM_B)^\vee$, and so there is an exact sequence

$$0 \to L(\omega|\langle Y\rangle) \to L(\omega|\langle X\rangle) \to (\mathfrak{a} \otimes_A {}_AM_B)^\vee \otimes_A {}_AM_B \to 0.$$

The morphism 6.7(b) of $L(\omega|\langle Y\rangle)$ into $L(\omega|\langle X\rangle)$ is therefore injective, with kernel flat as a left and as a right B-module.

A B-module of finite type with a coaction of $L(\omega|\langle X\rangle)$ corresponds to an object of $\langle Y\rangle$ if the coaction

$$N \to N \otimes L(\omega|\langle X\rangle)$$

factors through $N \otimes L(\omega|\langle Y\rangle)$.

The category A is the filtered union of the subcategories $\langle X\rangle$ and $L(\omega)$ is the inductive limit of the $L(\omega|\langle X\rangle)$ (6.7(c)). Passing to the limit, we obtain the theorem. □

EXAMPLE 6.11 Let A be a small locally finite k-linear abelian category, B an extension field of k, and $\omega : \mathsf{C} \to \mathsf{Vecf}_B$ an exact faithful k-linear functor. Then ω factors into

$$\mathsf{C} \xrightarrow{i} \mathsf{coModf}_{L(\omega)} \xrightarrow{\text{forget}} \mathsf{Vecf}_B.$$

The functor i is an equivalence of categories.

Proof of the main theorem, except for the faithful flatness

6.12 Let B be a commutative ring. We shall use the construction of $L(\omega_1, \omega_2)$ in 6.2 for $B_1 = B_2 = k = B$. Later, we shall need to consider two commutative k-algebras B_1, B_2 and we shall take B to be the commutative algebra $B_1 \otimes_k B_2$.

Suppose that we have three categories $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ and, for each category, two functors

$$\begin{cases} \omega_1^i : \mathsf{A}_i \to \mathsf{Projf}(B) \\ \omega_2^i : \mathsf{A}_2 \to \mathsf{Projf}(B). \end{cases}$$

Suppose also that we have a functor $\otimes : \mathsf{A}_1 \times \mathsf{A}_2 \to \mathsf{A}_3$ and isomorphisms of functors

$$\begin{cases} \omega_1^1(X_1) \otimes_B \omega_1^2(X_2) \to \omega_1^3(X_1 \otimes X_2) \\ \omega_2^1(X_1) \otimes_B \omega_j^2(X_2) \to \omega_j^3(X_1 \otimes X_2). \end{cases}$$

In 6.2, we defined B-modules $L_B(\omega_1^i, \omega_2^i)$, and 6.7(b) and 6.7(d) furnish a B-bilinear product

$$L_B(\omega_1^1, \omega_2^1) \otimes_B L_B(\omega_1^2, \omega_2^2) \to L_B(\omega_1^3, \omega_2^3). \tag{82}$$

For any X_1 and X_2 in A_1 and A_2, the morphism (76)

$$\omega_2^3(X_1 \otimes X_2) \to \omega_1^3(X_1 \otimes X_2) \otimes_B L_B(\omega_1^3, \omega_2^3)$$

can be deduced from the analogous morphisms for $\omega_1^j(X_j)$ and $\omega_2^j(X_j)$ by (82).

Let A be a symmetric monoidal category and ω_1 and ω_2 two $\otimes$-functors $\mathsf{A} \to \mathsf{Projf}(B)$ (see Chapter I). When we take $\mathsf{A}_i = \mathsf{A}$ and $\omega_j^i = \omega_j$ $(i = 1, 2, 3)$, the product (82) becomes a product

$$L_B(\omega_1, \omega_2) \otimes_B L_B(\omega_1, \omega_2) \to L_B(\omega_1, \omega_2). \tag{83}$$

PROPOSITION 6.13 *The product (83) makes $L_B(\omega_1, \omega_2)$ into a commutative B-algebra.*

PROOF For $j = 1, 2$, we have essentially commutative diagrams,

$$\begin{array}{ccccc} \mathsf{A}\times\mathsf{A} & \xrightarrow{(X,Y)\mapsto(Y,X)} & \mathsf{A}\times\mathsf{A} & \xrightarrow{\otimes} & \mathsf{A} \\ \downarrow{\scriptstyle \omega_j\otimes\omega_j} & & \downarrow{\scriptstyle \omega_j\otimes\omega_j} & & \downarrow{\scriptstyle \omega_j} \\ \mathsf{Mod}(B) & = & \mathsf{Mod}(B) & = & \mathsf{Mod}(B) \end{array}$$

The left-hand square is rendered commutative by the isomorphism of functors

$$\omega_j(X) \otimes_B \omega_j(Y) \to \omega_j(Y) \otimes_B \omega_j(X),$$

and the right-hand square by the isomorphism

$$\omega_j(X \otimes Y) \to \omega_j(X) \otimes_B \omega_j(Y).$$

By the definition of a $\otimes$-functor, the isomorphism of composed functors makes commutative the boundary of the diagram, and also the left-hand square once we identify $\omega(Y \otimes X)$ with $\omega(X \otimes Y)$ using the commutativity of $\otimes$ in A. Applying $L(\cdot, \cdot)$ to the diagram, we obtain the commutativity in (83). The associativity is obtained the same way. If $\{\mathbb{1}\}$ is the subcategory of A consisting only of the identity object and the identity arrow, then we obtain, by definition of $\otimes$-functor, that $\omega_j(1) \xrightarrow{\sim} B$ and $L_B(\omega_1|\{\mathbb{1}\}, \omega_2|\{\mathbb{1}\}) = B$. The identity $B \to L_B(\omega_1, \omega_2)$ of $L_B(\omega_1, \omega_2)$ is defined by 6.7(b). □

6.14 Recall (1.3) that when A is a symmetric monoidal category and ω_1 and ω_2 are $\otimes$-functors from A to the quasi-coherent sheaves on an affine scheme S, we define $\mathcal{H}om_S^{\otimes}(\omega_1, \omega_2)$ (resp. $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$) to be the functor on Aff_S sending $u : T \to S$ to the set of morphisms (resp. isomorphisms) of tensor functors $u^*\omega_1 \to u^*\omega_2$. If ω_1 and ω_2 take values in the category of locally free modules of finite rank, for example, if A is rigid (I, 7.5), then these functors are representable by an affine scheme over S and

$$\mathcal{H}om_S^{\otimes}(\omega_1, \omega_2) \simeq \mathcal{I}som_S^{\otimes}(\omega_1, \omega_2).$$

6.15 As in 1.3, when ω_i has values in the category of quasi-coherent sheaves on S_i, we put

$$\mathcal{H}om_k^{\otimes}(\omega_2, \omega_1) = \mathcal{H}om_{S_1\times S_s}^{\otimes}(\mathrm{pr}_2^*\,\omega_2, \mathrm{pr}_1^*\,\omega_1),$$

and similarly for $\mathcal{I}som$. For $\omega_1 = \omega_2$, we let

$$\mathcal{E}nd(\omega) = \mathcal{H}om(\omega, \omega)$$
$$\mathcal{A}ut(\omega) = \mathcal{I}som(\omega, \omega).$$

PROPOSITION 6.16 *Let* A *be a symmetric monoidal category and ω_1 and ω_2 $\otimes$-functors* $\mathsf{A} \to \mathsf{Projf}(B)$. *Put* $S = \operatorname{Spec} B$. *The scheme $L_B(\omega_1, \omega_2)$ represents the functor $\mathcal{H}om_S^{\otimes}(\omega_2, \omega_1)$.*

PROOF Let $u : T \to S$, $T = \operatorname{Spec} C$, be an affine scheme over S. By definition (6.2), a morphism $f : L_B(\omega_1, \omega_2) \to C$ of B-modules can be identified with a functorial system of B-modules

$$f_X : \omega_2(X) \to \omega_1(X) \otimes_B C.$$

Giving the f_X is equivalent to giving C-linear morphisms

$$f'_X : \omega_2(X) \otimes_B C \to \omega_1(X) \otimes_B C,$$

functorial in X, i.e., a morphism f' of functors $u^*\omega_1$ to $u^*\omega_1$. It can be checked that f is a morphism of algebras if and only if f' is a morphism of $\otimes$-functors. □

6.17 Let k be a commutative ring, and let B_1 and B_2 be two commutative k-algebras. Let A be a k-linear symmetric monoidal category and ω_1 and $\omega_2\otimes$-functors from A to $\mathsf{Projf}(B_1)$ and $\mathsf{Projf}(B_2)$. By extension of scalars, ω_1 and ω_2 define $\otimes$-functors $\mathsf{A} \to \mathsf{Projf}(B_1 \otimes_k B_2)$, which we denote $\omega_1 \otimes 1$ and $1 \otimes \omega_2$ respectively. We have

$$L_k(\omega_1, \omega_2) = L_{B_1 \otimes_k B_2}(\omega_1 \otimes 1, 1 \otimes \omega_2).$$

After 6.16, $\operatorname{Spec} L_k(\omega_1, \omega_2)$ represents the functor $\mathcal{H}om_k^{\otimes}(\omega_2, \omega_1)$, which is equal to $\mathcal{I}som_k^{\otimes}(\omega_1, \omega_2)$ if A is rigid (see 6.14).

For three B_j and ω_j, composition of morphisms

$$\mathcal{H}om_k^{\otimes}(\omega_3, \omega_2) \times_{S_2} \mathcal{H}om_k^{\otimes}(\omega_2, \omega_1) \to \mathcal{H}om_k^{\otimes}(\omega_3, \omega_1)$$

corresponds to a morphism of k-algebras

$$c : L_k(\omega_1, \omega_3) \to L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3).$$

By definition, the morphism of $L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3)$-modules deduced by extension of scalars (by c) from

$$\omega_3(X) \otimes_{B_3} L_k(\omega_1, \omega_3) \to \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_3)$$

is the composite of the morphisms deduced by extension of scalars from

$$\omega_3(X) \otimes_{B_3} L_k(\omega_2, \omega_3) \to \omega_2(X) \otimes_{B_2} L_k(\omega_2, \omega_3) \quad \text{and}$$
$$\omega_2(X) \otimes_{B2} L_k(\omega_1, \omega_2) \to \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2).$$

This comes down to the commutativity of

$$\begin{array}{ccc} \omega_3(X) & \longrightarrow & \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2) \\ \downarrow & & \downarrow \\ \omega_2(X) \otimes_{B_2} L_k(\omega_2, \omega_3) & \longrightarrow & \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3) \end{array}$$

and c is therefore (77).

6.18 Let k be a field. Let T be a tensor category over k and ω a fibre functor on T over $S = \operatorname{Spec} B$, B a nonzero k-algebra. We prove that ω induces an equivalence of T with $\mathsf{Repf}(S\!:\!G)$, where G is the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$.

In the above, we take $\mathsf{A} = \mathsf{T}$, $B_1 = B_2 = B$, $\omega_1 = \omega_2 = \omega$. According to (I, 7.11), T is locally finite, and so we can apply 6.10. After 6.17, $\mathcal{A}ut_k^{\otimes}(\omega)$ is the spectrum of $L(\omega) = L_k(\omega_1, \omega_2)$. The action of $\mathcal{A}ut_k^{\otimes}(\omega)$ on $\omega(X)$ is defined by the morphisms

$$\omega(X) \to \omega(X) \otimes_B L(\omega, \omega)$$

which defined L (76). By 6.17 again, the law of composition of the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$ is defined by the comultiplication of $L(\omega)$, and 6.10 is equivalent to the required statement.

Proof of the faithful flatness

PROPOSITION 6.19 *Let* C *be a tannakian category over* k *and let* ω *be a fibre functor with values in a (commutative) nonzero* k*-algebra. Let* $\operatorname{Ind}\omega:\ \operatorname{Ind}\mathsf{C}\to\mathsf{Mod}(B)$ *denote the extension of* ω *to* Ind C*. For all nonzero objects* T *in* Ind C*,* $\operatorname{Ind}(\omega)(T)$ *is faithfully flat over* B*.*

PROOF The category C is abelian and its objects are noetherian (I, 7.11). On applying Proposition B.10, we see that $T=\varinjlim X_i$ with $X_i\in\operatorname{ob}\mathsf{C}$, $X_i\subset T$. Fix an $X_j\neq 0$, and consider the exact sequence

$$0\to X_j\to T\to T/X_j\to 0$$

in Ind C. Because ω is right exact, $\operatorname{Ind}\omega$ commutes with arbitrary inductive limits (B.8), and so $(\operatorname{Ind}\omega)(T)=\varinjlim\omega(X_i)$. We deduce an exact sequence

$$0\to\omega X_j\to(\operatorname{Ind}\omega)(T)\to(\operatorname{Ind}\omega)(T/X_j)\to 0.$$

Now $\operatorname{Ind}(\omega)(T)$ is an inductive limit of finitely generated projective (hence flat) modules, and so is a flat B-module. Similarly, $\operatorname{Ind}(\omega)(T/X_j)$ is a flat B-module. Moreover, the B-module $\omega(X_j)$ is faithfully flat (7.5). If M is an arbitrary nonzero B-module, then the sequence

$$0\to\omega(X_j)\otimes_B M\to(\operatorname{Ind}\omega)(T)\otimes_B M\to(\operatorname{Ind}\omega)(T/X_j)\otimes_B M\to 0$$

is exact because $(\operatorname{Ind}\omega)(T/X_j)$ is flat.[6] As $\omega(X_j)\otimes_B M\neq 0$, we have $(\operatorname{Ind}(\omega)(T))\otimes_B M\neq 0$, and so $\operatorname{Ind}(\omega)(T)$ is a faithfully flat B-module. □

COROLLARY 6.20 *Let* C *be a tannakian category over a perfect field* k*, and let* ω *be a fibre functor with values in a (commutative)* k*-algebra* B*. Then* $L_k(\omega)$ *(see 6.4) is a faithfully flat* $B\otimes_k B$*-algebra.*

PROOF Define T to be the coequalizer of the parallel pair of morphisms

$$\bigoplus_{(f:\ X\to Y)\in\operatorname{ar}\mathsf{C}} Y^\vee\otimes X\rightrightarrows\bigoplus_{X\in\operatorname{ob}\mathsf{C}} X^\vee\otimes X$$

in $\operatorname{Ind}(\mathsf{C}\boxtimes\mathsf{C})$ (cf. 6.2). According to (11.18), we have a fibre functor $\omega\boxtimes\omega$ on $\mathsf{C}\boxtimes\mathsf{C}$, and clearly $\operatorname{Ind}(\omega\boxtimes\omega)(T)=L(\omega)$. If $B\neq 0$, then $L(\omega)\neq 0$, and so $T\neq 0$. Now Proposition 6.19 shows that $L(\omega)$ is faithfully flat. □

COROLLARY 6.21 *Let* C *be a tannakian category over a perfect field* k*, and let* ω *be a fibre functor with values in a* k*-algebra* $B\neq 0$*. Then there exists a faithfully flat homomorphism* $f:\ B\to B'$ *such that* $f^*\omega\approx f^*\omega'$*.*

PROOF See 1.4. □

[6] Let $0\to M'\to M\to M''\to 0$ be an exact sequence of right A-modules. If M'' is flat, then

$$0\to M'\otimes_A N\to M\otimes_A N\to M''\otimes_A N\to 0$$

is exact for all left A-modules N. This is most naturally proved using the Tor functor, but, for a proof without the Tor functor, see 12.9.

This completes the proof of Theorem 1.1 when k is perfect. For the case of a nonperfect base field, we refer the reader to Deligne 1990.

REMARK 6.22 We used the hypothesis that k is perfect to show that the functor $\mathsf{A} \boxtimes \mathsf{B} \to \mathsf{D}$ defined by a functor $\mathsf{A} \times \mathsf{B} \to \mathsf{D}$, exact in both variables, is exact (I, 11.11). This was used in the proof (I, 11.17) that a tensor product of tensor categories over a perfect field is tensor and hence in the proof of the similar statement (I, 11.18) for tannakian categories. For an explanation of how to remove the perfectness hypothesis in I, 11.18, hence in the proof of Theorem 1.1, see Deligne 1990, 5.18.

NOTES The exposition of the proof Theorem 1.1 and of its preliminaries largely follows the original (Deligne 1990, §§1–6). See also Lattermann 1989.

7 Restatement for 2-categories

Let S be a nonempty affine scheme over k. Theorem 1.1 can be interpreted as saying that the 2-category of k-groupoids acting transitively on S is biequivalent (not 2-equivalent) to the category of tannakian categories equipped with a fibre functor over S.

DEFINITION 7.1 The 2-category $\mathscr{G}rpd_S$ has

- objects the affine k-groupoids acting transitively on S;
- a 1-morphism from G to H is a morphism $f: G \to H$ of groupoids acting on S (2.11).
- a 2-morphism $f \to g$ is a natural transformation from f to g (viewing f and g as functors of affine $S \times S$-schemes).

DEFINITION 7.2 The 2-category $\mathscr{T}ann_S^{\bullet}$ of S-pointed tannakian categories over Aff_k has

- objects the pairs (T, ω), where T is an essentially small tannakian category over k and ω is a fibre functor on T over S;
- a 1-morphism from (T, ω) to (T', ω') is an exact k-linear $\otimes$-functor from T to T' carrying ω into ω';
- a 2-morphism is a morphism of $\otimes$-functors.

The functors we defined in §1 extend in an obvious way to 2-functors

$$\Phi: \mathscr{T}ann_S^{\bullet\mathrm{opp}} \to \mathscr{G}rpd_S, \quad (\mathsf{T}, \omega) \rightsquigarrow \mathscr{A}ut^{\otimes}(\omega)$$
$$\Psi: \mathscr{G}rpd_S \to \mathscr{T}ann_S^{\bullet\mathrm{opp}}, \quad G \rightsquigarrow \mathsf{Repf}(S{:}G),$$

and we also defined functors $\eta: \mathrm{id} \to \Psi\circ\Phi$ and $\epsilon: \Phi\circ\Psi \to \mathrm{id}$.

THEOREM 7.3 *The system*

$$\mathscr{T}ann_S^{\bullet\mathrm{opp}} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathscr{G}rpd_S, \qquad \eta: \mathrm{id} \to \Psi\circ\Phi, \qquad \epsilon: \Phi\circ\Psi \to \mathrm{id},$$

is a biequivalence of $\mathscr{T}ann_S^{\bullet\mathrm{opp}}$ *and* $\mathscr{G}rpd_S$.

PROOF This is little more than a restatement of Theorem 1.1. □

COROLLARY 7.4 *Let* T_1 *and* T_2 *be tannakian categories over* k*, and let* ω_1 *and* ω_2 *be fibre functors on* T_1 *and* T_2 *over* S*. Then*

$$\mathsf{Hom}((\mathsf{T}_1,\omega_1),(\mathsf{T}_2,\omega_2)) \to \mathsf{Hom}(\mathcal{A}ut^{\otimes}(\omega_2),\mathcal{A}ut^{\otimes}(\omega_1))$$

is an equivalence of categories.

PROOF Immediate consequence of the theorem. □

8 Properties of a tannakian category reflected in its band

As in (II, §5, §6), there is a dictionary relating the properties of tannakian categories and their morphisms to those of their bands. Here are some examples.

8.1 Let C be a tannakian category over k with band L.

(a) L is finite (i.e., locally represented by a finite group scheme) if and only if $\mathsf{C} = \langle X\rangle$ for some object X.

(b) When k has characteristic 0, L is pro-reductive if and only if C is semisimple.

8.2 Let $\omega : \mathsf{C}' \to \mathsf{C}$ be a morphism of tannakian categories over k, bound by a morphism of bands $u : L \to L'$.

(a) u is faithfully flat (i.e., an epimorphism of bands) if and only if ω is fully faithful and its essential image is closed under subobjects.

(b) u is injective (i.e., locally represented by a monomorphism of groups) if and only if every object of C is a subquotient of an object in the image of ω.

See Saavedra 1972, III, 3.3.3, p. 205.

9 Extension of scalars for tannakian categories

Let $(\mathsf{T},\otimes)$ be a tannakian category over k, and let ω be a fibre functor on T with values in a field K. Consider a diagram of fields

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k'. \end{array}$$

Let X be an object of $\mathrm{Ind}(\mathsf{T})$ equipped with a homomorphism $i : k' \to \mathrm{End}_k(X)$ of k-algebras. Recall (I, §13) that we say that $Y \subset X$ ***generates*** (X,i) ***as a*** k'***-module*** if it is not contained in any proper k'-module, and we define $\mathsf{T}_{(k')}$ to be the category whose objects are the k'-modules in $\mathrm{Ind}(\mathsf{T})$ that are generated as k'-modules by subobjects in T.

PROPOSITION 9.1 *The category* $\mathsf{T}_{(k')}$ *is a tannakian category over* k'*, the* K*-valued fibre functor* ω *extends to a* K'*-valued fibre functor* ω' *on* $\mathsf{T}_{(k')}$*, and the* K'/k'*-groupoid* $\mathcal{A}ut^{\otimes}_{k'}(\omega')$ *is the pullback of the* K/k*-groupoid* $\mathcal{A}ut^{\otimes}_{k}(\omega)$*.*

PROOF After Theorem 1.1, we may suppose that $\mathsf{T} = \mathsf{Repf}(S:G)$, where $S = \operatorname{Spec} K$, and that ω is the forgetful functor. Then the statement follows from 3.7. □

EXAMPLE 9.2 Take $k' = K' = K$, so T is a tannakian category over k and ω is a k'-valued fibre functor. The proposition then shows that $\mathsf{T}_{(k')}$ is a neutral tannakian category over k', and that ω extends to a k'-valued fibre functor ω' on $\mathsf{T}_{(k')}$. The affine group scheme attached to $(\mathsf{T}_{(k')}, \omega')$ is the kernel of the groupoid attached to (T, ω),

$$\mathcal{A}ut^{\otimes}_{k'}(\omega') \simeq \mathcal{A}ut^{\otimes}_{k}(\omega)^{\Delta}.$$

10 Existence of a fibre functor over k^{al}

In this section, we show that essentially small tannakian categories over algebraically closed fields are neutral.

THEOREM 10.1 (DELIGNE) *Let T be an essentially small tannakian category over k and k^{al} an algebraic closure of k. Then T has a fibre functor with values in k^{al}.*

We deduce this from the following more precise statements.

THEOREM 10.2 *Let T be an essentially small tannakian category over k. If T is algebraic, then it has a fibre functor with values in some finite extension of k.*

THEOREM 10.3 *Let T be an essentially small tensor category over k. If every finitely generated tensor subcategory has a k^{al}-valued fibre functor, then T has an k^{al}-valued fibre functor. In particular, T is tannakian.*

Proof of Theorem 10.2.

We first state a result from commutative algebra.

LEMMA 10.4 *Let A_0 be a ring and let $A = \varinjlim_{i\in I} A_i$ be the limit of a filtered inductive system of A_0-algebras. Let B_0 be an A_0-algebra of finite presentation, and set $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$.*

(a) *If $\operatorname{spec} B \to \operatorname{spec} A$ is surjective, then there exists an i such that $\operatorname{spec} B_i \to \operatorname{spec} A_i$ is surjective.*

(b) *If B is flat over A, then there exists an i such that B_i is flat over A_i.*

PROOF (a) EGA IV, 8.10.5.

(b) EGA IV, 11.2.6.1. □

10.5 We now prove Theorem 10.2. By assumption, T has a fibre functor over a nonempty k-scheme S, which we may suppose to be the spectrum of a field, and then we may suppose that $\mathsf{T} = \mathsf{Repf}(S:G)$, where G is a groupoid of finite presentation and faithfully flat over $S \times S$ (Theorem 1.1). Let $S = \operatorname{Spec} B$, and write B as a union $B = \bigcup_\alpha B_\alpha$ of finite generated k-algebras B_α. For some α, G is the pullback of a groupoid of finite presentation over $S_\alpha \times S_\alpha$, where $S_\alpha = \operatorname{Spec} B_\alpha$. As G is faithfully flat over $S \times S$, G_α is faithfully flat over $S_\alpha \times S_\alpha$ for α sufficiently large (10.4). Now $\mathsf{T} = \mathsf{Repf}(S_\alpha : G_\alpha)$ (see 1.20). In particular, T has a fibre functor over S_α. As S_α has a K-point for some finite extension K of k (Zariski's lemma), it follows that T has a fibre functor with values in K.

10.6 The above proof of Theorem 10.2, using groupoid schemes, is from Deligne 1990, 6.20. Here is another proof, using gerbes, from Saavedra 1972, III, 3.3.1.1, p. 204. Let T be a tannakian category over a field k. The fibre functors on T form a fibred category over Aff_k, and even a stack for the fpqc topology. It is nonempty by definition, so to show that it a gerbe it remains to show that any two fibre functors are locally isomorphic for the fpqc topology. For a proof of this, independent of the results of this chapter, see II, 8.7. We now use that a fibred category over k is a gerbe for the fppf topology on Aff_k if it is a gerbe for the fpqc topology (C.22). Therefore T has a fibre functor with values in a finitely generated k-algebra R, hence in $R/\mathfrak{m}$ for $\mathfrak{m}$ a maximal ideal of R, and $R/\mathfrak{m}$ is a finite extension of k (Zariski's lemma).

Proof of Theorem 10.3

Let A_f denote the set of all subcategories T_α of T of the form $\langle X\rangle^\otimes$ for some $X \in \mathrm{ob}\,\mathsf{T}$, partially ordered by inclusion, and let $\mathrm{Fib}(T_\alpha)$ denote the set of all k^{al}-valued fibre functors of T_α. If $\mathsf{T}_\alpha \subset \mathsf{T}_\beta$, then the restriction map $\mathrm{Fib}(T_\beta) \to \mathrm{Fib}(T_\alpha)$ is surjective, and we have to show that $\varprojlim_\alpha \mathrm{Fib}(T_\alpha)$ is nonempty. If A_f contains a countable cofinal subset, then this follows from the axiom of dependent choice. For the general case, we need the full axiom of choice (in the form of Zorn's lemma). After I, 7.19, we may suppose that $k = k^{\mathrm{al}}$. To avoid set theoretical difficulties, we consider only fibre functors taking values in a suitable small category of vector spaces over k, for example, in the category of all vector spaces k^n, $n \in \mathbb{N}$. Also, we replace T with an equivalent small category.

Let A be the set of all tensor subcategories of T over k. For $\alpha \in A$, we denote by T_α the corresponding subcategory (so $\mathsf{T}_\alpha = \alpha$). The set A is partially ordered by inclusion. We already know the existence and uniqueness up to isomorphism of fibre functors for the T_α, $\alpha \in A_f$.

If $\mathsf{T}_\alpha \subset \mathsf{T}_\beta$, it makes sense to say that a fibre functor ω_β on T_β extends a fibre functor ω_α on T_α. If an extension up to isomorphism exists, then an actual extension exists too.

We consider the set $\mathcal{S}$ of pairs (α, ω_α) with $\alpha \in A$ and ω_α a fibre functor on T_α, partially ordered by

$$(\alpha, \omega_\alpha) \le (\beta, \omega_\beta) \iff \mathsf{T}_\alpha \subset \mathsf{T}_\beta \text{ and } \omega_\beta \text{ extends } \omega_\alpha.$$

The partially ordered set $\mathcal{S}$ is inductive: if I is a totally ordered subset, then

$$\mathsf{T} \overset{\text{def}}{=} \bigcup \{\mathsf{T}_\alpha \mid (\alpha, \omega_\alpha) \in I\}$$

lies in A and has a fibre functor ω characterized by $\omega|\mathsf{T}_\alpha = \omega_\alpha$. The pair (T, ω) is an upper bound for I. By Zorn's lemma $\mathcal{S}$ has a maximal element (T_1, ω_1). To show that $\mathsf{T}_1 = \mathsf{T}$, it suffices to prove the following statement.

10.7 *Let T' be in A and T'' in A_f. Then, every fibre functor ω on T' extends to $\langle \mathsf{T}', \mathsf{T}''\rangle^\otimes$, the smallest tensor subcategory of T containing T' and T''*

We first prove a lemma.

LEMMA 10.8 *If T' is also in A_f, then restriction is an equivalence of categories*

$$\{\textit{fibre functors } \omega \textit{ on } \langle \mathsf{T}', \mathsf{T}''\rangle^\otimes\} \longrightarrow \left\{\begin{array}{l}\textit{triples } (\omega', \omega'', \tau) \textit{ with } \omega' \textit{ and } \omega'' \textit{ fibre func-}\\ \textit{tors on } \mathsf{T}' \textit{ and } \mathsf{T}'' \textit{ and } \tau \textit{ an isomorphism of}\\ \textit{the restrictions of } \omega' \textit{ and } \omega'' \textit{ to } \mathsf{T}' \cap \mathsf{T}''\end{array}\right\}.$$

Note that both $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$ and $\mathsf{T}' \cap \mathsf{T}''$ lie in A_f.

PROOF We use that the Noether isomomorphism theorems hold for affine group schemes over a field (Milne 2017, Chapter 5). After I, 3.1, and 10.4, we may suppose that $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$ is the category of representations of an algebraic group G over k. There exist normal algebraic subgroups A and B of G such that T' (resp. T'') is the subcategory of representations on which A (resp. B) acts trivially. To say that T' and T''generate $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$ means that $A \cap B = \{e\}$. The intersection $\mathsf{T}' \cap \mathsf{T}''$ is the category of representations on which AB acts trivially.

We claim that the triples $(\omega', \omega'', \tau)$ are all isomorphic. As all ω' (resp. all ω'') are isomorphic, it suffices to show that $(\omega', \omega'', \tau_1)$ and $(\omega', \omega'', \tau_2)$ are isomorphic, but τ_1 and τ_2 differ by an automorphism of $\omega'|\mathsf{T}' \cap \mathsf{T}''$, and such an automorphism lifts to an automorphism of ω': the homomorphism

$$(G/A)(k) \to (G/AB)(k)$$

is surjective.

Thus, the categories in the lemma each have just one isomorphism class of objects, and so it becomes a question of comparing their endomorphism rings (automorphism groups in this case). But

$$\begin{array}{ccc} G & \longrightarrow & G/A \\ \downarrow & & \downarrow \\ G/B & \longrightarrow & G/AB \end{array}$$

is a pullback diagram because $A \cap B = e$, and so the same is true when we take k-valued points. □

We now prove the statement 10.7. Let B be the set of T_α ($\alpha \in A_f$) contained in T'. We have

$$\mathsf{T}' = \bigcup_{\beta \in B} \mathsf{T}_\beta, \qquad \langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes} = \bigcup_{\beta \in B} \langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes},$$

and equivalences

$$\begin{aligned} \{\text{fibre functors on } \mathsf{T}'\} &\xrightarrow{\approx} \left\{ \begin{array}{l} \text{fibre functors } \omega_\beta \text{ on the } \mathsf{T}_\beta, \text{ plus a compatible system} \\ \text{of isomorphisms } \omega_\beta|\mathsf{T}_\gamma \xrightarrow{\sim} \omega_\gamma \text{ for } \mathsf{T}_\gamma \subset \mathsf{T}_\beta, \text{ plus a} \\ \text{compatibility condition for } \mathsf{T}_\delta \subset \mathsf{T}_\gamma \subset \mathsf{T}_\beta \end{array} \right\} \\ \{\text{fibre functors on } \langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}\} &\xrightarrow{\approx} \left\{ \begin{array}{l} \text{fibre functors on the } \langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes} \text{ plus similar} \\ \text{conditions.} \end{array} \right\} \end{aligned}$$

There is a largest $\mathsf{T}_\beta \cap \mathsf{T}''$: if $\mathsf{T}'' = \mathsf{Repf}(G'')$, then each $\mathsf{T}_\beta \cap \mathsf{T}''$ corresponds to a normal algebraic subgroups N_β of G'', and among the N_β, there is a smallest one, namely, $\bigcap_\beta N_\beta$. If β_0 is such that $\mathsf{T}_{\beta_0} \cap \mathsf{T}''$ is the largest $\mathsf{T}_\beta \cap \mathsf{T}''$, then, for any $\beta > \beta_0$, extending $\omega_\beta \overset{\text{def}}{=} \omega|\mathsf{T}_\beta$ to $\langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes}$ amounts to extending $\omega_\beta|\mathsf{T}_\beta \cap \mathsf{T}''$ to T'' (Lemma 10.8), that is, to extending ω_{β_0} from $\mathsf{T}_{\beta_0} \cap \mathsf{T}''$ to T''. If we choose one such extension, then we get, up to *unique* isomorphisms, a system of extensions of the ω_β to $\langle \mathsf{T}_\beta, \mathsf{T}'' \rangle^{\otimes}$, and by gluing them an extension of ω to $\langle \mathsf{T}', \mathsf{T}'' \rangle^{\otimes}$.

This completes the proof of 10.7, which shows that T_1 contains $\langle X \rangle^{\otimes}$ for all X in T.

NOTES

10.9 Only the proof of 10.5 requires Theorem 1.1 – the rest of the proof depends only on the first two chapters.

10.10 Without the essentially small hypotheses, the fibre functor may only exist after the ambient universe has been enlarged.

NOTES Theorem 10.1 and its proof are from a letter of Deligne, dated November 30, 2011. Apparently, they were known to Deligne earlier (see the sentence following 0.6 in Deligne 2002).

Chapter V

Tannakian categories and gerbes

In this chapter, we describe tannakian categories and their fibre functors in terms of gerbes (following Saavedra 1972).

Throughout, k is a field unless indicated otherwise. Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$. We let Aff_k denote the category of affine k-schemes and Aff_S the category of schemes affine over an affine scheme S.

1 Gerbes

In this section, we introduce gerbes and explain their relation to groupoids schemes.

Gerbes

Let k be a field and Aff_k the category of affine schemes over k.

1.1 We refer the reader to Appendix C for the notion of a fibred category $\phi : \mathsf{F} \to \mathsf{Aff}_k$ over Aff_k. A fibred category is a stack if both morphisms and objects, given locally for the fpqc topology on Aff_k, patch to global objects. A stack $\mathsf{G} \to \mathsf{Aff}_k$ is a gerbe if the fibres are groupoids, some fibre is nonempty, and any two objects of a fibre are locally isomorphic for the fpqc topology on Aff_k. A morphism of gerbes is a cartesian functor.

1.2 Let $\mathsf{G} \to \mathsf{Aff}_k$ be a gerbe. A ***representation*** of G is a cartesian functor $\phi : \mathsf{G} \to$MOD, where MOD is the stack over Aff_k whose fibre over a scheme U is the category of $\Gamma(U, \mathcal{O}_U)$-modules. Thus, to give ϕ amounts to giving, for each affine k-scheme S, a functor from G_S to the category of quasi-coherent sheaves on S, these functors being required to be compatible with base change. A ***morphism*** between two representations is a cartesian natural transformation of cartesian functors (Appendix C, §1).

We let $\mathsf{Repf}(\mathsf{G})$ denote the category of cartesian functors $\phi : \mathsf{G} \to$PROJ, where PROJ is the stack over Aff_k whose fibre over a scheme U is the category of finitely generated projective $\Gamma(U, \mathcal{O}_U)$-modules. Thus, to give an object of $\mathsf{Repf}(\mathsf{G})$ is to give a functor $\phi_R : \mathsf{G}_R \to \mathsf{Projf}(R)$, one for each k-algebra R, and a natural isomorphisms

$$\phi_{R'}(g^*Q) \simeq \phi_R(Q) \otimes_R R', \quad Q \in \operatorname{ob}(\mathsf{G}_R),$$

defined whenever $g : R \to R'$ is a homomorphism of k-algebras. With the obvious tensor product structure, $\mathsf{Repf}(\mathsf{G})$ becomes a rigid tensor category over k.

1.3 Let $f : \mathsf{F} \to \mathsf{Aff}_k$ and $g : \mathsf{G} \to \mathsf{Aff}_k$ be fibred categories over Aff_k. The category $\mathsf{Cart}(F, G)$ has objects the cartesian functors $u : \mathsf{F} \to \mathsf{G}$ and morphisms the natural transformations $m : u \to u'$ such that $\mathrm{id}_g * m = \mathrm{id}_f$, i.e., the cartesian natural transformations.

There is a 2-category $\mathcal{F}ib_k$ with objects the small fibred categories over Aff_k and

$$\mathsf{Hom}(\mathsf{F}, \mathsf{G}) = \mathsf{Cart}(\mathsf{F}, \mathsf{G})$$

for all objects F, G (Giraud 1971, 0, 1.8). Similarly, there is a 2-category $\mathcal{G}erb_k$ with objects the small gerbes over Aff_k and the same Hom categories.

1.4 Let G be an affine group scheme over k, and let $\textsc{Tors}(G)$ be the gerbe over Aff_S such that $\textsc{Tors}(G)_U$ is the category of torsors under G over U. Let G_r denote G viewed as a right G-torsor, and let Φ be an object of $\mathsf{Repf}(\textsc{Tors}(G))$. The isomorphism $G \xrightarrow{\simeq} \mathcal{A}ut(G_r)$ defines a representation of G on the vector space $\Phi_k(G_r)$, and the functor $\Phi \rightsquigarrow \Phi_k(G_r)$ is an equivalence of categories

$$\mathsf{Repf}(\textsc{Tors}(G)) \to \mathsf{Repf}(G).$$

To see this, let G be the fibred category over Aut_k such that the fibre over U is the category with one object G_U. This is a prestack, but not a stack. Its associated stack is $\textsc{Tors}(G)$, and so the inclusion $\mathsf{G} \to \textsc{Tors}(G)$ gives an equivalence

$$\mathsf{Repf}(G) \approxeq \mathsf{Repf}(\mathsf{G}) \approxeq \mathsf{Repf}(\textsc{Tors}(G)).$$

1.5 Let T be a tannakian category over k. There is a fibred category $\textsc{Fib}(\mathsf{T}) \to \mathsf{Aff}_k$ whose fibre over S is the category of fibre functors on T over S. It is a stack. Moreover, its fibres are groupoids (I, 4.21), and, by definition, some fibre is nonempty. When T is neutral, any two fibre functors are locally isomorphic (II, 8.2), and so $\textsc{Fib}(\mathsf{T})$ is a gerbe. The functor sending an object X of T to the representation $\omega \rightsquigarrow \omega(X)$ of $\textsc{Fib}(\mathsf{T})$ is an equivalence of tensor categories

$$\mathsf{T} \to \mathsf{Repf}(\textsc{Fib}(\mathsf{T})).$$

Indeed, the choice of a fibre functor ω determines an equivalence

$$\mathsf{T} \approxeq \mathsf{Repf}(G),$$

where $G = \mathcal{A}ut^{\otimes}(\omega)$ (II, 3.1), and the functor $\eta \rightsquigarrow \mathcal{I}som^{\otimes}(\omega, \eta)$ is an equivalence

$$\textsc{Fib}(\mathsf{T}) \approxeq \textsc{Tors}(G),$$

(II, 8.1), and so the required equivalence can be identified with that in 1.4.

Descent within gerbes

Let G be a gerbe over Aff_k for the fpqc topology. Since any two objects are locally isomorphic, if one object in G has a certain property and the property is local for the fpqc topology, then all objects will have the property. We make this explicit.

1.6 Let P be a property of affine schemes. We say that P is ***local for the fpqc topology*** if, for any finite surjective family of flat morphisms $U_i \to U$ of affine schemes,

$$P \text{ holds for } U \iff P \text{ holds for each } U_i.$$

This is equivalent to saying that P is local for the Zariski topology and, for any faithfully flat morphism $U' \to U$,

$$P \text{ holds for } U \iff P \text{ holds for } U'.$$

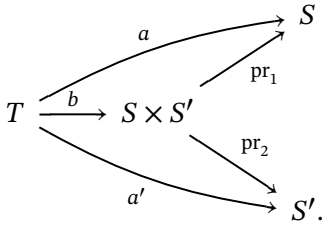
There is a similar definition for other objects. For example, a property P of quasi-coherent sheaves is local for the fpqc topology if it is local for the Zariski topology and, for any faithfully flat morphism $a : U' \to U$ of affine schemes and quasi-coherent sheaf M on U,

$$P \text{ holds for } M \text{ on } U \iff P \text{ holds for } a^*M \text{ on } U'.$$

For quasi-coherent sheaves, the properties "finite type", "finitely presented", "flat", and "locally free of finite rank" are all local for the fpqc topology.

LEMMA 1.7 *Let* G *be a gerbe over* Aff_k. *Let* $Q \in \operatorname{ob}\mathsf{G}_S$ *and* $Q' \in \operatorname{ob}\mathsf{G}_{S'}$, *where* S *and* S' *are nonempty affine* k*-schemes. Then there exists an affine* k*-scheme* T *and faithfully flat maps* $a : T \to S$ *and* $a' : T \to S'$ *such that* $a^*Q \approx a'^*Q'$.

PROOF Note that $\operatorname{pr}_1^* Q$ and $\operatorname{pr}_2^* Q'$ are both objects in the fibre of G over the nonempty scheme $S \times_k S'$, and so there exists a faithfully flat map $b : T \to S \times S'$ such that $b^* \operatorname{pr}_1^* Q \approx b^* \operatorname{pr}_2^* Q'$. We can take $a = \operatorname{pr}_1 \circ b$ and $a' = \operatorname{pr}_2 \circ b$,



The projection maps are faithfully flat because the structure maps $S \to \operatorname{Spec} k$ and $S' \to \operatorname{Spec} k$ are. □

1.8 Similarly, if $Q_1, Q_2 \in \operatorname{ob}\mathsf{G}_S$ and $Q_1', Q_2' \in \operatorname{ob}\mathsf{G}_{S'}$, then there exist faithfully flat maps $a : T \to S$ and $a' : T \to S'$ such that $a^*Q_1 \approx a'^*Q_1'$ and $a^*Q_2 \approx a'^*Q_2'$.

LEMMA 1.9 *Let* U *be an affine scheme and* $\mathcal{F}$ *a sheaf on* U *for the fpqc topology. If for some* V *affine and faithfully flat over* U, *the restriction of* $\mathcal{F}$ *to* V *is representable, then* $\mathcal{F}$ *is representable on* U *(by an affine scheme over* U*).*

PROOF Let X be an affine scheme over V representing the restriction of $\mathcal{F}$ to V. The canonical isomorphism $\operatorname{pr}_1^*(\mathcal{F}|V) \to \operatorname{pr}_2^*(\mathcal{F}|V)$ (over $V \times_U V$) satisfies the cocycle condition (Appendix C, §3). By the Yoneda embedding, this defines an isomorphism $\operatorname{pr}_1^* X \to \operatorname{pr}_2^* X$ (over $V \times_U V$) satisfying the cocycle condition, i.e., a descent datum on X relative to V/U. This descent datum is effective by faithfully flat descent. Thus, we get a scheme X_0 affine over U such that X_0 and $\mathcal{F}$ define the same sheaf over V. This implies that they define the same sheaf on U. □

1.10 Let G be a gerbe over Aff_k. For an affine k-scheme S and $Q_1, Q_2 \in \operatorname{ob}\mathsf{G}_S$, we let $\mathcal{I}som_S(Q_1, Q_2)$ denote the functor of affine S-schemes

$$\left(T \xrightarrow{a} S\right) \rightsquigarrow \operatorname{Isom}_{\mathsf{G}_T}(a^*Q_1, a^*Q_2).$$

For $Q \in \operatorname{ob}\mathsf{G}_S$, we let $\mathcal{A}ut_S(Q) = \mathcal{I}som_S(Q, Q)$, so it is the functor

$$\left(T \xrightarrow{a} S\right) \rightsquigarrow \operatorname{Isom}_{\mathsf{G}_T}(a^*Q, a^*Q).$$

It follows from 1.8 and 1.9, that if there exist an $S \neq \emptyset$ and $Q_1, Q_2 \in \operatorname{ob}\mathsf{G}_S$ such that $\mathcal{I}som_S(Q_1, Q_2)$ is representable by an affine scheme over S, then the same is true for every S' and $Q_1', Q_2' \in \operatorname{ob}\mathsf{G}_{S'}$. We then say that G ***has affine band*** or, more simply, that it is an ***affine gerbe***. Thus, a gerbe G is affine if and only if for one (hence every) $S \neq \emptyset$ and $Q \in \mathsf{G}_S$, $\mathcal{A}ut_S(Q)$ is representable by an affine group scheme over S.

The gerbe attached to a groupoid

1.11 Let $(G, S, (t, s), \circ)$ be a k-groupoid acting on S (in the sense of IV, 2.8; in particular, affine). By definition, for any affine k-scheme T, the quadruple $(S(T), G(T), (t, s), \circ)$ is a groupoid (in sets). For varying T, these categories form a fibred category over Aff_k, which we denote by $\mathsf{G}^0(S:G)$, or just G^0. Thus, for an affine k-scheme T,

$$\operatorname{ob}\mathsf{G}^0_T = S(T) \overset{\text{def}}{=} \operatorname{Hom}_k(T, S),$$

and, for $a, b \in \operatorname{ob}\mathsf{G}^0_T = S(T)$,

$$\operatorname{Hom}_{\mathsf{G}^0_T}(a, b) = \{h \in G(T) \mid s\circ h = a,\ t\circ h = b\}.$$

For a morphism $f : T' \to T$ of affine k-schemes, the inverse image functor f^* sends an object $a \in S(T)$ to $a\circ f \in S(T')$ and a morphism $h \in G(T)$ to $h\circ f$. See IV, 2.15.

LEMMA 1.12 *For any two objects a, b of $\mathsf{G}^0_T \overset{\text{def}}{=} S(T)$, the presheaf $\mathcal{H}om_T(a, b)$ on T,*

$$(T' \xrightarrow{f} T) \rightsquigarrow \operatorname{Hom}_{\mathsf{G}^0_{T'}}(f^*a, f^*b),$$

is a sheaf for the fpqc topology. Hence $\mathsf{G}^0(S:G)$ is a prestack.

PROOF We have to show that, for $a, b \in S(T)$ and $f : T' \to T$ faithfully flat, the map

$$\operatorname{Hom}_{\mathsf{G}^0_T}(a, b) \to \operatorname{Hom}_{\mathsf{Desc}(T'/T)}(f^*a, f^*b)$$

is bijective. But the left-hand side is

$$\{h : T \to G \mid s\circ h = a,\ t\circ h = b\}$$

whereas the right-hand side is

$$\{h' : T' \to G \mid s\circ h' = a,\ t\circ h' = b \text{ and } h'\circ \operatorname{pr}_1 = h'\circ \operatorname{pr}_2\}.$$

The mapping from the first to the second is composition with f. Now f is faithfully flat, in particular, an epimorphism, so

$$\begin{cases} s\circ h = a \iff s\circ h\circ f = a\circ f, \\ t\circ h = b \iff t\circ h\circ f = b\circ f. \end{cases}$$

It therefore suffices to show that the sequence

$$\mathrm{Hom}(T,G) \longrightarrow \mathrm{Hom}(T',G) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathrm{Hom}(T'\times_T T',G)$$

is exact, but this follows directly from the exactness of

$$T'\times_T T' \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} T' \xrightarrow{f} T$$

for faithfully flat f (see, for example, Waterhouse 1979, 13.1).

Alternatively, this follows from IV, 2.16. □

1.13 Let $\mathsf{G}(S:G)$ be the associated stack of $\mathsf{G}^0(S:G)$ (Appendix C, §3). Because $\mathsf{G}^0(S:G)$ is a prestack, $\mathsf{G}(S:G)$ contains it as a full subcategory and is characterized by the property that every object of $\mathsf{G}(S:G)$ is locally in $\mathsf{G}^0(S:G)$. It follows that the fibres of $\mathsf{G}(S:G)$ are groupoids. For any stack H over Aff_k, the inclusion functor $i:\ \mathsf{G}^0 \to \mathsf{G}$ induces an equivalence of categories

$$\mathsf{Hom}(\mathsf{G},\mathsf{H}) \xrightarrow{\cong} \mathsf{Hom}(\mathsf{G}^0,\mathsf{H}), \tag{84}$$

compatible with base change, i.e., an equivalence of stacks over Aff_k,

$$\mathsf{HOM}(\mathsf{G},\mathsf{H}) \xrightarrow{\cong} \mathsf{HOM}(\mathsf{G}^0,\mathsf{H}). \tag{85}$$

1.14 Let G be a k-groupoid acting on S. From (84) with $\mathsf{H}=$PROJ we get an equivalence of categories

$$\mathsf{Repf}(\mathsf{G}(S:G)) \xrightarrow{\cong} \mathsf{Repf}(\mathsf{G}^0(S:G)).$$

Let R be a representation of G^0. For each k-scheme T and object $a:\ T\to S$ of G^0_T, we get an isomorphism

$$R(a) \to a^*(R(\mathrm{id}_S)),$$

and R is determined by the sheaf $R_0 \overset{\mathrm{def}}{=} R(\mathrm{id}_S)$ on S and the isomorphisms

$$R(g) \overset{\mathrm{def}}{=} (a^*R_0 \to b^*R_0)$$

defined by the arrows $a\to b$ in in G^0_T. These $R(g)$ form a representation of G, and the functor

$$R \rightsquigarrow (R(\mathrm{id}_S),(R(g))_g)$$

is an equivalence of categories

$$\mathsf{Repf}(\mathsf{G}^0) \xrightarrow{\cong} \mathsf{Repf}(S:G). \tag{86}$$

PROPOSITION 1.15 *Let G be a k-groupoid acting on $S\neq\emptyset$. The stack $\mathsf{G}(S:G)$ is a gerbe if and only if G is transitive.*

PROOF We know that $\mathsf{G}(S:G)$ satisfies the conditions (a) and (b) to be a gerbe (1.1), and so it remains to check that any two objects of $\mathsf{G}(S:G)$ are locally isomorphic. Because all elements of $\mathsf{G}(S:G)$ are locally in $\mathsf{G}^0(S:G)$, it suffices to prove this for objects of $\mathsf{G}^0(S:G)$, but this we did in IV, 2.17. □

PROPOSITION 1.16 *Let G be a k-groupoid acting transitively on $S \neq \emptyset$, and let $\mathsf{G}(S:G)$ be the associated gerbe. The canonical functor*

$$\mathsf{Repf}(S:G) \to \mathsf{Repf}(\mathsf{G}(S:G))$$

is an equivalence of categories.

PROOF A representation of G defines a cartesian functor $\mathsf{G}^0(S:G) \to$PROJ, which extends uniquely to $\mathsf{G}(S:G)$. In this way, we get a functor $\mathsf{Repf}(S:G) \to \mathsf{Repf}(\mathsf{G}(S:G))$, and "restriction" provides a quasi-inverse. □

The groupoid attached to a pointed gerbe

1.17 Let G be an affine gerbe over Aff_k, and let $Q \in \mathrm{ob}\,\mathsf{G}_S$ for some $S \neq \emptyset$. Consider the presheaf on Aff_k,

$$\mathcal{A}ut_k(Q): T \rightsquigarrow \{(b,a,\varphi) \mid b,a: T \to S,\ \varphi: a^*Q \xrightarrow{\simeq} b^*Q\}.$$

For an $S\times S$-scheme $(b,a): T \to S\times S$, we have

$$\mathcal{A}ut_k(Q)(T) = \mathrm{Isom}_{\mathsf{G}_T}(a^*Q, b^*Q) = \mathcal{I}som_{S\times S}(\mathrm{pr}_2^*Q, \mathrm{pr}_1^*Q)(T).$$

Because G is affine, $\mathcal{I}som_{S\times S}(\mathrm{pr}_2^*Q, \mathrm{pr}_1^*Q)$ is represented by an affine $S\times S$-scheme G (1.10). The universal element in $\mathcal{A}ut_k(Q)(G)$ is a triple (t,s,φ) with $t,s: G \rightrightarrows S$ and $\varphi: s^*Q \to t^*Q$ an isomorphism. To an arbitrary $f: T \to G$, there corresponds a triple

$$(s\circ f, t\circ f, (s\circ f)^*Q \xrightarrow{f^*\varphi} (t\circ f)^*Q).$$

Composition of isomorphisms provides a natural transformation

$$\mathcal{A}ut_k(Q) \underset{s,S,t}{\times} \mathcal{A}ut_k(Q) \to \mathcal{A}ut_k(Q),$$

which corresponds to a morphism of $S\times S$-schemes

$$m: G \underset{s,S,t}{\times} G \to G.$$

The identity automorphism of Q corresponds to a morphism $e: S \to G$ with $s\circ e = t\circ e = \mathrm{id}_S$. The conditions (a) and (b) of (IV, 2.9), are satisfied because composition of isomorphisms is associative, and the identity acts as a neutral element. We therefore obtain a groupoid (G,m,e) over S. Moreover, because the objects pr_1^*Q and pr_2^*Q of $\mathsf{G}_{S\times S}$ are locally isomorphic for the fpqc topology, G is transitive (see the proof of 1.15). We call G the ***groupoid of*** *k-**automorphisms*** of Q.

This construction is inverse to that in 1.11. If G is a k-groupoid acting transitively on $S \neq \emptyset$, then $\mathsf{G}(S:G)$ is an affine gerbe with distinguished object $Q = \mathrm{id}_S$ in $\mathsf{G}(S:G)_S$, and $G \simeq \mathcal{A}ut_k(Q)$. On the other hand, if G is an affine gerbe over k and Q is an object of G_S, some $S \neq \emptyset$, then $\mathcal{A}ut_k(Q)$ is represented by a k-groupoid G acting transitively on S, and the canonical functor $\mathsf{G}^0(S:G) \to \mathsf{G}$ induces an equivalence of gerbes

$$\mathsf{G}(S:G) \xrightarrow{\approx} \mathsf{G}. \tag{87}$$

We describe this correspondence in more detail in 1.28.

REMARK 1.18 Let $(G,S,(t,s),\circ)$ be a groupoid in the category of schemes over k, not necessarily affine. If S is affine and the kernel G^Δ is affine, then G is affine. Indeed, the condition implies that the gerbe $\mathsf{G}(S:G)$ is affine, and G is isomorphic to the groupoid of k-automorphisms of the object id_S of $\mathsf{G}(S:G)_S$.

Some applications of gerbes

The next result is an almost trivial consequence of the definition of a gerbe, but has important applications.

PROPOSITION 1.19 *Let $u: \mathsf{G}_1 \to \mathsf{G}_2$ be a morphism of gerbes over Aff_k. If for some $S \neq \emptyset$ and $Q \in \mathrm{ob}\,\mathsf{G}_{1S}$, the map $\mathcal{A}ut_S(Q) \to \mathcal{A}ut_S(u(Q))$ defined by u is an isomorphism, then u is an equivalence of categories.*

PROOF If follows from 1.7 that if $\mathcal{A}ut_S(Q) \to \mathcal{A}ut_S(u(Q))$ is an isomorphism for one nonempty S and object Q of G_{1S}, then it is an isomorphism for every nonempty S and object Q of G_{1S}.

Let $Q, Q' \in \mathrm{ob}\,\mathsf{G}_{1S}$. We shall show that the map $\mathrm{Isom}_{G_{1S}}(Q, Q') \to \mathrm{Isom}_{G_{2S}}(uQ, uQ')$ defined by u is a bijection. This will show that u is fully faithful. After possibly passing to a faithfully flat cover, we may suppose that Q and Q' are isomorphic in G_{1S}. Then $\mathrm{Isom}_{G_{1S}}(Q, Q')$ is a principal homogeneous space for the group $\mathrm{Aut}_S(Q)(S)$, and similarly $\mathrm{Isom}_{G_{2S}}(uQ, uQ')$ is a principal homogeneous space for $\mathrm{Aut}_S(u(Q))(S)$. As $\mathrm{Aut}_S(Q)(S) \simeq \mathrm{Aut}_S(u(Q))(S)$, this implies that the map $\mathrm{Isom}_{G_{1S}}(Q, Q') \to \mathrm{Isom}_{G_{2S}}(uQ, uQ')$ is bijective.

It follows from 1.7 again that, because one object $u(Q)$ of G_{2S} is in the image of u, every object of G_2 is locally in the image of u.

Thus, u is fully faithful and essentially surjective, and hence an equivalence of categories. □

Here are some applications of the proposition.

1.20 Let G be a k-groupoid acting transitively on S, and let G_T be the pullback of G by $u: T \to S$, where $T \neq \emptyset$ (see p. 173). The morphism of prestacks over Aff_k,

$$u: \mathsf{G}^0(T:G) \to \mathsf{G}^0(S:G),$$

induces an isomorphism of the sheaf of automorphisms of $\mathrm{id}_T \in \mathrm{ob}\,\mathsf{G}^0(T:G_T)$ with the sheaf of automorphisms of $u \in \mathrm{ob}\,\mathsf{G}^0(S:G)$. The induced morphism of gerbes $u: \mathsf{G}(T:G) \to \mathsf{G}(S:G)$ has the same property, and therefore is an equivalence of categories by the proposition. Applying (86), we obtain an equivalence of categories

$$\mathsf{Repf}(S:G) \xrightarrow{\approx} \mathsf{Repf}(T:G_T). \tag{88}$$

1.21 A morphism of transitive groupoids is an isomorphism if its restriction to the kernels is an isomorphism. In more detail, let $u: G_1 \to G_2$ be a morphism of k-groupoids acting transitively on S, and let $u^\Delta: G_1^\Delta \to G_2^\Delta$ be its inverse image under the diagonal morphism $\Delta: S \to S \times S$ (so u^Δ is a homomorphism of affine group schemes over S). If u^Δ is an isomorphism, then u induces an equivalence of gerbes $\mathsf{G}(S:G_1) \to \mathsf{G}(S:G_2)$, and it follows from 1.17 that u is an isomorphism. (See III, 1.9, for the similar statement for Galois groupoids.)

1.22 Let G be a gerbe with affine band. There exists a spectrum S of a field such that G_S is not empty. Let $Q \in \mathrm{ob}\,\mathsf{G}_S$ and $G = \mathcal{A}ut_k(Q)$ (so G is a k-groupoid acting transitively on S). The gerbe G is equivalent to $\mathsf{G}(S:G)$ (1.17), and every gerbe with affine band is equivalent to a gerbe $\mathsf{G}(S:G)$ with S the spectrum of a field and G a k-groupoid acting transitively on S.

1.23 Let G be a k-groupoid acting transitively on S. It follows from 1.22 that the subobjects of G (in the sense of category theory p. 348) are in one-to-one correspondence with the affine subgroup schemes of G^Δ. Therefore, G is artinian if G^Δ is of finite type over k.

1.24 Here is a more precise form of 1.19: a morphism of gerbes is faithful (resp. covering, resp. fully faithful) if and only if the morphism on bands is injective (resp. surjective, i.e., an epimorphism, resp. an isomorphism). See Giraud 1971, IV, 2.2.6, p. 216.

Comparison of the 2-categories of groupoids and pointed gerbes

Let S be a nonempty affine scheme over k. We show that the 2-category of k-groupoids acting transitively on S is biequivalent (not 2-equivalent) to the category of pointed gerbes.

DEFINITION 1.25 The 2-category $\mathcal{G}rpd_S$ of k-groupoids acting transitively on S has

- objects the affine k-groupoids acting transitively on S;
- 1-morphism is a morphism of k-groupoids acting on S (2.11;
- a 2-morphism $f \to g$ is a natural transformation from f to g, viewing f and g as functors of affine $S \times S$-schemes.

DEFINITION 1.26 The 2-category $\mathcal{G}erbe_S^{\bullet}$ of S-pointed gerbes over Aff_k has

- objects the pairs (G, Q), where G is an affine gerbe over Aff_k and $Q \in \operatorname{ob}\mathsf{G}_S$;
- a 1-morphism from (G, Q) to (H, R) is a pair (F, μ), where F is a cartesian functor $F : \mathsf{G} \to \mathsf{H}$ and μ is an isomorphism $F_S Q \to R$;
- a 2-morphism $F \to G$ is a cartesian natural transformation.

1.27 We define a 2-functor $\Phi : \mathcal{G}rpd_S \to \mathcal{G}erbe_S^{\bullet}$. Let $(G, S, (t, s), \circ)$ be a k-groupoid acting on S. For any affine k-scheme T, $(S(T), G(T), (t, s), \circ)$ is a groupoid, and for varying T these categories form a prestack over Aff_k (1.11, 1.19). We let $\Phi(G)$ denote the associated stack $\mathsf{G}(S : G)$. Because G is transitive, $\mathsf{G}(S : G)$ is a gerbe (1.15). We take $\mathrm{id}_S \in S(S)$ to be the distinguished object of G_S.

Let $f : G \to H$ be a morphism of k-groupoids acting transitively on S. For any affine k-scheme T, f defines a functor

$$\mathsf{G}^0(S:G)_T = (S(T), G(T), (t, s), \circ) \to \mathsf{G}^0(S : H)_T = (S(T), H(T), (t, s), \circ).$$

These are compatible with base change, and so, for varying T they define a cartesian functor

$$F^0 : \mathsf{G}^0(S:G) \to \mathsf{G}^0(S : H),$$

which, by the universality of the associated stacks, extends uniquely to a cartesian functor $F : \mathsf{G}(S:G) \to \mathsf{G}(S : H)$. We set $\Phi(f) = F$. Then $F_S(\mathrm{id}_S) = \mathrm{id}_S$, and we set $\mu : F_S(\mathrm{id}_S) \to \mathrm{id}_S$ equal to the identity map.

Let $f, g : G \to H$ be morphisms of k-groupoids acting transitively on S. Then f and g define functors $F^0, G^0 : \mathsf{G}^0(S:G) \to \mathsf{G}^0(S : H)$, and, by definition, a 2-morphism from $u : f \to g$ is a natural transformation $U^0 : F^0 \to G^0$. This extends to a natural transformation $U : F \to G$, which is a cartesian natural transformation. We set $\Phi(u) = U$.

THEOREM 1.28 *The 2-functor $\Phi : \mathcal{G}rpd_S \to \mathcal{G}erbe_S^{\bullet}$ is a biequivalence.*

We check that Φ satisfies the conditions (a), (b), (c) of the version A.30 of the Whitehead Theorem.

(a) Φ_0 *is surjective on equivalence classes of objects.* Let $(\mathsf{G}, Q) \in \mathrm{ob}(\mathcal{G}erb^{\bullet}_S)$. The functor $\mathcal{A}ut_k(Q)$ is represented by an affine k-groupoid G acting transitively on S. From G we get a prestack $\mathsf{G}^0(S:G)$ with $\mathrm{ob}\,\mathsf{G}^0(S:G)_T = S(T)$ and $\mathrm{ar}\,\mathsf{G}^0(S:G)_T = G(T)$. We define a cartesian functor

$$\lambda^0 : \mathsf{G}^0(S:G) \to \mathsf{G}$$

by setting $\lambda^0_T : \mathsf{G}^0(S:G)_T \to \mathsf{G}_T$ equal to

$$\begin{cases} a \in S(T) \mapsto a^*Q \\ f \in G(T) \mapsto ((s\circ f)^*Q \xrightarrow{f} (t\circ f)^*Q). \end{cases}$$

By construction, λ^0 is faithful. We have a stack $\mathsf{G}(S:G)$, a faithful cartesian functor i, and a cartesian functor λ such that the following diagram commutes up to isomorphism,

$$\begin{array}{ccc} \mathsf{G}^0(S:G) & \xrightarrow{i} & \mathsf{G}(S:G) \\ & \searrow^{\lambda^0} & \downarrow{\scriptstyle \lambda} \\ & & \mathsf{G}. \end{array}$$

The functor λ is faithful on the subcategory $i(\mathsf{G}^0(S:G))$ of $\mathsf{G}(S:G)$. Let $X \in \mathrm{ob}\,\mathsf{G}_T$, and let pr_1 and pr_2 be the projections from $T \times S$ to T and S respectively. They are faithfully flat because the structure morphisms $T \to \mathrm{Spec}\,k$ and $S \to \mathrm{Spec}\,k$ are. By definition, there exists a faithfully flat map $f : T' \to T \times S$ and an isomorphism $f^* \mathrm{pr}_2^* Q \to f^* \mathrm{pr}_1^* X$. This means that X corresponds to a descent datum on $(\mathrm{pr}_2 \circ f)^*Q$. This comes from a descent datum on an object in $\mathsf{G}(S:G)_{T'}$, which because $\mathsf{G}(S:G)$ is a stack, comes from an object in $\mathsf{G}(S:G)_T$, that corresponds under λ to an object in the isomorphism class of X. One shows similarly that λ is faithful on the whole of $\mathsf{G}(S:G)$. Therefore λ is an equivalence of categories.

(b) Φ *is surjective on isomorphic classes of* 1*-morphisms (between any two objects).*

(c) Φ *is bijective on 2-morphisms (between any two* 1*-morphisms).* Let $f, g : G \to H$ be morphisms of k-groupoids acting on S. We can regard G and H as functors from k-algebras to sets, and then a 2-morphism $f \Rightarrow g$ is a natural transformation satisfying certain conditions (2.11). Let $F, G : \mathsf{G}(S:G) \to \mathsf{G}(S:H)$ be the corresponding 1-morphisms of gerbes. Then $\mathsf{G}(S:G)_0$ and $\mathsf{G}(S:H)_0$ are functors from k-algebas to sets. A 2-morphism $F \Rightarrow G$ is a natural transformation $F_0 \Rightarrow G_0$ satisfying certain conditions. The two sets of natural transformations are in bijective correspondence.

1.29 We now define a 2-functor $\Psi : \mathcal{G}erb^{\bullet}_S \to \mathcal{G}rpd_S$. Let G be an affine gerbe over Aff_k, and let $Q \in \mathrm{ob}(\mathsf{G}_S)$. We let $\Psi\mathsf{G} = \mathcal{A}ut_k(Q)$, the groupoid of k-automorphisms of Q (see 1.17).

Let (F, μ) be a morphism from (G, Q) to (H, R). Given an affine scheme $(a, b) : T \to S \times S$ and a $\varphi \in \mathcal{A}ut_k(Q)(T)$, we define $\widetilde{\varphi}$ by the following diagram

$$\begin{array}{ccc} a^*(FQ) & \xrightarrow{F\varphi} & b^*(FQ) \\ {\scriptstyle \simeq}\downarrow{\scriptstyle a^*\mu} & & {\scriptstyle \simeq}\downarrow{\scriptstyle b^*\mu} \\ a^*R & \xrightarrow{\bar{\varphi}} & b^*R \end{array}$$

Then $\varphi \mapsto \bar{\varphi}$ is a natural transformation from $\mathcal{A}ut_k(Q)$ to $\mathcal{A}ut_k(R)$. As it respects the identity, source, target, and composition, it corresponds to a morphism f of the corresponding groupoids. We let $\Psi(F,\mu) = f$.

THEOREM 1.30 *There is a biequivalence*

$$\mathcal{G}rpd_S \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{G}erbe_S^{\bullet}, \quad \mathrm{id}_{\mathcal{G}rpd_S} \to \Psi\circ\Phi, \quad \Phi\circ\Psi \to \mathrm{id}_{\mathcal{G}erbe_S^{\bullet}}.$$

PROOF We define $\Psi\circ\Phi \simeq \mathrm{id}$. Let G be an affine k-groupoid acting transitively on S, and let $Q = \mathrm{id}_S$ be the distinguished object of the affine gerbe $\Phi(G)$. For any affine k-scheme T,

$$\begin{aligned}\mathcal{A}ut_k(Q)(T) &= \{(b,a,\varphi) \mid b,a : T \to S,\ \varphi : a^*Q \xrightarrow{\simeq} b^*Q\}\\ &= \{(b,a,f) \mid b,a \in S(T),\ f \in G(T) \text{ with } s\circ f = a,\ t\circ f = b\}\\ &= \bigsqcup_{(b,a)\in(S\times S)(T)} \mathrm{Hom}_{S\times S}(T,G)\\ &= \mathrm{Hom}_k(T,G).\end{aligned}$$

Therefore, G represents the functor $\mathcal{A}ut_k(Q)$.

From a morphism $f : G \to H$ of k-groupoids acting transitively on S, we get a natural transformation

$$(a^*Q \xrightarrow{\varphi} b^*Q) \mapsto (a^*R \xrightarrow{f\circ\varphi} b^*R) : \mathcal{A}ut_k(Q) \to \mathcal{A}ut_k(R),$$

which equals that induced by f. Hence, $\Psi\circ\Phi \simeq \mathrm{id}$.

We define $\Phi\circ\Psi \cong \mathrm{id}$. Let $(\mathsf{G}, Q) \in \mathrm{ob}(\mathcal{G}erbe_S^{\bullet})$. The functor $\mathcal{A}ut_k(Q)$ is represented by an affine k-groupoid G acting transitively on S. From G we get a prestack $\mathsf{G}^0(S:G)$ with $\mathrm{ob}\,\mathsf{G}^0(S:G)_T = S(T)$ and $\mathrm{ar}\,\mathsf{G}^0(S:G)_T = G(T)$. We define a cartesian functor

$$\lambda^0 : \mathsf{G}^0(S:G) \to \mathsf{G}$$

by setting $\lambda^0_T : \mathsf{G}^0(S:G)_T \to \mathsf{G}_T$ equal to

$$\begin{cases} a \in S(T) \mapsto a^*Q \\ f \in G(T) \mapsto ((s\circ f)^*Q \xrightarrow{f} (t\circ f)^*Q). \end{cases}$$

By construction, λ^0 is faithful. We have a stack $\mathsf{G}(S:G)$, a faithful cartesian functor i, and a cartesian functor λ such that the following diagram commutes up to isomorphism,

$$\begin{array}{ccc} \mathsf{G}^0(S:G) & \xrightarrow{\ i\ } & \mathsf{G}(S:G) \\ & \searrow^{\lambda^0} & \downarrow{\scriptstyle\lambda} \\ & & \mathsf{G}. \end{array}$$

The functor λ is faithful on the subcategory $i(\mathsf{G}^0(S:G))$ of $\mathsf{G}(S:G)$. Let $X \in \mathrm{ob}\,\mathsf{G}_T$, and let pr_1 and pr_2 be the projections from $T\times S$ to T and S respectively. They are faithfully flat because the structure morphisms $T \to \mathrm{Spec}\,k$ and $S \to \mathrm{Spec}\,k$ are. By definition, there exists a faithfully flat map $f : T' \to T\times S$ and an isomorphism $f^*\,\mathrm{pr}_2^*\,Q \to f^*\,\mathrm{pr}_1^*\,X$. This means that X corresponds to a descent datum on $(\mathrm{pr}_2\circ f)^*Q$. This comes from a descent datum on an object in $\mathsf{G}(S:G)_{T'}$, which because $\mathsf{G}(S:G)$ is a stack, comes from

an object in $\mathsf{G}(S:G)_T$, that corresponds under λ to an object in the isomorphism class of X. One shows similarly that λ is faithful on the whole of $\mathsf{G}(S:G)$. Therefore λ is an equivalence of categories. It is easy to check that λ is natural in $\mathsf{G}(S:G)$. It follows that $\Phi\circ\Psi \sim \mathrm{id}$. □

Let B be a band over k. The cohomology set $H^2(k,B)$ is defined to be the set of B-equivalence classes of gerbes over Aff_k banded by B (see Appendix C, §6). We define the cohomology class of a k-groupoid G acting transitively on S to be the cohomology class of the associated gerbe $\mathsf{G}(S:G)$.

PROPOSITION 1.31 *Let G and G' be k-groupoids acting transitively on S. A morphism $\varphi: G^\Delta \to G'^\Delta$ of bands over k extends to a morphism of groupoid schemes if and only if it maps the cohomology class of G to that of G'.*

PROOF Almost by definition, φ extends to a morphism of gerbes if and only if it maps the cohomology class of G to that of G'. Now use the relation (1.28) between groupoids and gerbes. □

NOTES Lattermann 1989, 1.3.7, asserts that functors Φ and Ψ in 1.27 and 1.29 define an equivalence of the underlying 1-categories of $\mathscr{G}rpd_S$ and $\mathscr{G}erbe_S^\bullet$. This is incorrect – the objects of $\mathscr{G}erbe_S^\bullet$ are only equivalent (not isomorphic) to objects in the image of Φ.[1] The correct statement, as above, is that they define an equivalence of 2-categories. A similar remark applies to the assertion (Lattermann 1989, 4.2.13) that the 1-categories of Galois groupoids and pointed gerbes are equivalent and to the assertion (Langlands and Rapoport 1987, §4, p. 152) that the 1-categories of "Galoisgerbes" and pointed Giraud-gerbes are equivalent.

2 The gerbe of fibre functors

Fix a field k.

THEOREM 2.1 *Let T be an essentially small tannakian category over k.*

(a) *The fibre functors on T form an affine gerbe $\mathrm{FIB}(\mathsf{T})$ over Aff_k.*

(b) *The canonical functor $\mathsf{T} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))$ is an equivalence of tensor categories.*

Conversely, if G is an affine gerbe over Aff_k, then $\mathsf{Repf}(\mathsf{G})$ is a tannakian category and the canonical functor $\mathsf{G} \to \mathrm{FIB}(\mathsf{Repf}(\mathsf{G}))$ is an equivalence of gerbes.

The canonical functor $\gamma: \mathsf{T} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))$ sends an object X of T to the representation $\omega \rightsquigarrow \omega(X)$ of $\mathrm{FIB}(\mathsf{T})$.

FIRST PROOF OF THEOREM 2.1 (SANS GROUPOIDS)

(a) To prove that $\mathrm{FIB}(\mathsf{T})$ is a gerbe, it remains to prove that any two fibre functors are locally isomorphic for the fpqc topology. This we did in II, 8.7.

(b) Let K be a field containing k. It suffices to show that the functor γ becomes an equivalence after we have extended scalars to K. Choose K so that T has a fibre functor over K. Then $\mathsf{T}_{(K)}$ is a neutral tannakian category over K (12.20), and so the composite

$$\mathsf{T}_{(K)} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))_{(K)} \sim \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}_{(K)}))$$

is an equivalence (1.5). Thus, $\mathsf{T}_{(K)} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))_{(K)}$ is an equivalence, which implies that $\mathsf{T} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))$ is an equivalence.

For the converse, we defer to the next proof.

[1] Note that the gerbes in the image are split. While every gerbe is equivalent to a split gerbe, it need not be isomorphic to one.

SECOND PROOF OF THEOREM 2.1 (GROUPOIDS).

(a) To prove that FIB(T) is a gerbe, it remains to prove that any two fibre functors are locally isomorphic for the fpqc topology. This we did in IV, 1.4.

(b) Choose a fibre functor ω over an affine k-scheme $S \neq \emptyset$, and let G be the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$. Then $\mathsf{G}(S:G) \sim$FIB(T), and the composite of the equivalences

$$\mathsf{T} \xrightarrow[\text{IV, 1.1}]{\sim} \mathsf{Repf}(S:G) \xrightarrow[1.16]{\sim} \mathsf{Repf}(\mathsf{G}(S:G)) \approxeq \mathsf{Repf}(\text{FIB}(\mathsf{T}))$$

is the required equivalence.

Let G be an affine gerbe over Aff_k, and let G be the groupoid of k-automorphisms of some $Q \in \mathrm{ob}\,\mathsf{G}_S$, $S \neq \emptyset$ (see 1.17). Then the final statement for G follows from the similar statement for G (IV, 1.1).

Tannakian categories and categories of gerbes

THEOREM 2.2 *Let* T *be an essentially small tannakian category over* k.

(a) *The fibre functors on* T *form an affine gerbe* FIB(T) *over* Aff_k.

(b) *The canonical functor* $\mathsf{T} \to \mathsf{Repf}(\text{FIB}(\mathsf{T}))$ *is an equivalence of tensor categories.*

Conversely, if G *is an affine gerbe over* Aff_k, *then* $\mathsf{Repf}(\mathsf{G})$ *is a tannakian category and the canonical functor* $\mathsf{G} \to$FIB($\mathsf{Repf}(\mathsf{G})$) *is an equivalence of gerbes.*

PROOF Statement (a) follows from IV, 1.4.

In (b), the canonical functor $\mathsf{T} \to \mathsf{Repf}(\text{FIB}(\mathsf{T}))$ sends an object X of T to the representation $\omega \rightsquigarrow \omega(X)$ of FIB(T). Choose a fibre functor ω over an affine k-scheme $S \neq \emptyset$, and let G be the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$. Then $\mathsf{G}(S:G) \sim$FIB(T), and the composite of the equivalences

$$\mathsf{T} \xrightarrow[\text{IV, 1.1}]{\sim} \mathsf{Repf}(S:G) \xrightarrow[1.16]{\sim} \mathsf{Repf}(\mathsf{G}(S:G)) \approxeq \mathsf{Repf}(\text{FIB}(\mathsf{T}))$$

is the required equivalence.

Let G be an affine gerbe over Aff_k, and let G be the groupoid of k-automorphisms of some $Q \in \mathrm{ob}\,\mathsf{G}_S$, $S \neq \emptyset$ (see 1.17). Then the final statement for G follows from the similar statement for G (IV, 1.1). □

Let k be a field. We show that the 2-category of tannakian categories over k is equivalent to the 2-category of affine gerbes over Aff_k. In particular, there is a dictionary between tannakian categories over k and affine gerbes over Aff_k.

DEFINITION 2.3 The 2-category $\mathcal{G}erbe_k$ has

- objects the affine gerbes over Aff_k;
- 1-morphisms the cartesian functors of fibred categories;
- 2-morphisms the natural transformations between 1-morphisms.

In particular, for any affine gerbes G and H over Aff_k, we have a category $\mathsf{Hom}(\mathsf{G},\mathsf{H})$ whose objects are the cartesian functors from G to H and whose morphisms are the natural transformations between cartesian functors.

DEFINITION 2.4 The 2-category $\mathcal{T}ann_k$ of tannakian categories over k has

- objects the essentially small tannakian categories over k;

- 1-morphisms the exact k-linear $\otimes$-functors;
- 2-morphisms the monoidal natural transformations (I, 2.11).

In particular, for any tannakian categories C and D over k, we have a category $\mathsf{Hom}(\mathsf{C},\mathsf{D})$ whose objects are the exact k-linear $\otimes$-functors from C to D and whose morphisms are the monoidal natural transformations.

There are canonical 2-functors

$$\mathscr{G}erbe^{\mathrm{opp}} \to \mathscr{T}ann, \quad \begin{cases} \mathsf{G} \rightsquigarrow \mathsf{Repf}_k(\mathsf{G}) \\ (\mathsf{G} \xrightarrow{F} \mathsf{H}) \rightsquigarrow (X \mapsto X \circ F) \end{cases}$$

$$\mathscr{T}ann^{\mathrm{opp}} \to \mathscr{G}erbe, \quad \begin{cases} \mathsf{C} \rightsquigarrow \mathrm{FIB}(\mathsf{C}) \\ (\mathsf{C} \xrightarrow{F} \mathsf{D}) \rightsquigarrow (\omega \mapsto \omega \circ F)\,. \end{cases}$$

THEOREM 2.5 *The* 2*-functor*

$$\mathsf{C} \rightsquigarrow \mathrm{FIB}(\mathsf{C}) : \mathscr{T}ann_k^{\mathrm{opp}} \to \mathscr{G}erbe_k$$

is a biequivalence. Explicitly, for any essentially small tannakian category C *over* k*, the canonical functor*

$$\mathsf{C} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{C}))$$

is an equivalence of symmetric monoidal categories over k*, and for any affine gerbe* G *over* Aff_k*, the canonical functor*

$$\mathsf{G} \to \mathrm{FIB}(\mathsf{Repf}(\mathsf{G}))$$

is an equivalence of fibred categories over Aff_k*.*

PROOF The explicit statements were proved in Theorem 2.1, and they imply that FIB is an equivalence of 2-categories. □

Under the equivalence, neutral gerbes correspond to neutral tannakian categories.

COROLLARY 2.6 *For any tannakian categories* C *and* D *over* k*, the functor*

$$\mathsf{Hom}(\mathsf{C},\mathsf{D}) \to \mathsf{Hom}(\mathrm{FIB}(\mathsf{D}), \mathrm{FIB}(\mathsf{C}))$$

defined by FIB *is an equivalence of categories.*

PROOF This is an immediate consequence of the theorem. □

EXAMPLE 2.7 Let C be a neutral tannakian category over k. Theorem II, 8.1 shows that the choice of a fibre functor ω with values in k determines an equivalence of fibred categories $\mathrm{FIB}(\mathsf{C}) \to \mathrm{TORS}(G)$, where G represents $\mathscr{A}ut^{\otimes}(\omega)$. This shows that $\mathrm{FIB}(\mathsf{C})$ is an affine gerbe, and the commutative diagram of functors

$$\begin{array}{ccc} \mathsf{C} & \longrightarrow & \mathsf{Repf}(\mathrm{FIB}(\mathsf{C})) \\ {\scriptstyle\sim}\downarrow{\scriptstyle\omega} & & {\scriptstyle\sim}\downarrow \\ \mathsf{Repf}(G) & \xleftarrow{\sim} & \mathsf{Repf}(\mathrm{TORS}(G)) \end{array}$$

shows that $\mathsf{C} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{C}))$ is an equivalence of categories.

3 Ind-tannakian categories

This section summarizes Saavedra 1972, Chap. III.

Let R be a commutative ring. Ind-tannakian categories over R are defined in such a way that it is relatively easy to show that the 2-category of ind-tannakian categories over R is equivalent to the 2-category of tannakian gerbes over Aff_R (those locally bound by a flat affine group scheme). The important theorem, whose presentation op. cit. is flawed, is that, when R is a field, the category of ind-objects of a tannakian category is ind-tannakian.

3.1 Let R be a commutative ring, and let C be an R-linear abelian category with small inductive limits. When M is an R-module and X is an object of C, $M \otimes X$ denotes the object of C representing the functor

$$Y \rightsquigarrow \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathsf{C}}(X, Y)).$$

For any R-algebra R', we let $\mathsf{C}_{R'}$ denote the category of R'-modules in C. The functor

$$i_{R'/R} : \mathsf{C} \to \mathsf{C}_{R'}$$

sending X to $R' \otimes X$ is R-linear and commutes with inductive limits. A symmetric monoidal structure on C induces a similar structure on $\mathsf{C}_{R'}$, and $i_{R'/R}$ is symmetric monoidal. See I, 12.22, 12.23.

DEFINITION 3.2 (OP. CIT. 1.1.1) Let C be an abelian R-linear $\otimes$-category with small inductive limits such that $\otimes$ commutes with $\varinjlim$ in each variable. We say that C is ***ind-tannakian*** if there exists a faithfully flat R–algebra R' such that

(a) the functor $i_{R'/R} : \mathsf{C} \to \mathsf{C}_{R'}$ is faithful and exact;

(b) $C_{(R')}$ is $\otimes$-equivalent to a $\otimes$-category $\mathrm{Rep}(G')$, where G' is a flat affine R'–group.

The ind-tannakian categories over R form a 2-category in a natural way.

DEFINITION 3.3 (OP. CIT. II, 3.2.1) Let C be an ind-tannakian category over R and R' an R-algebra. A ***fibre functor on*** C with values in R' is an exact faithful R'-linear $\otimes$-functor

$$\omega : \mathsf{C}_{R'} \longrightarrow \mathsf{Mod}(R')$$

that commutes with inductive limits.

3.4 When we take the morphisms to be monoidal natural transformations, the R'-valued fibre functors on C form a category $\mathsf{Fib}(\mathsf{C}, R')$. As R' varies, these categories form a stack $\mathrm{FIB}(\mathsf{C})$ over Aff_R.

PROPOSITION 3.5 (OP. CIT. 1.1.6) *Let C be an ind-tannakian category over R and $\omega : \mathsf{C}_{R'} \to \mathsf{Mod}(R')$ a fibre functor with values in an R-algebra R'.*

(a) *The sheaf $\mathcal{A}ut^{\otimes}(\omega)$ on $\mathrm{Spec}(R')_{\mathrm{fpqc}}$ is representable by a flat affine group scheme G.*

(b) *The $\otimes$-functor $\mathsf{C}_{R'} \to \mathsf{Rep}(G)$ defined by ω is an equivalence of categories.*

PROOF First show that, if C is an ind-tannakian category over R and R' is an R-algebra, then $\mathsf{C}_{R'}$ is an ind-category over R' (op. cit. 1.1.4). Next prove the proposition for the category $\mathsf{Rep}(G)$ of representations of an affine group scheme over R (op. cit. 1.1.5). Now the proposition holds when ω is the fibre functor implicit in condition (b) of Definition 3.2. A descent argument now proves the proposition for a general fibre functor (op. cit. 1.1.2). □

3.6 Let G be an affine group scheme over R, and let ω be the forgetful functor on $\mathsf{Rep}(G)$. Then $G \simeq \mathcal{A}ut^{\otimes}(\omega)$, and, for any other R-valued fibre functor η, $\mathcal{H}om^{\otimes}(\omega,\eta)$ is a torsor under G over R. It follows that two fibre functors on $\mathsf{Rep}(G)$ are locally isomorphic for the fpqc topology on Aff_R.

3.7 Let C be an ind-tannakian category over R. It follows from 3.6 and condition (b) of Definition 3.2, that the stack of fibre functors $\textsc{Fib}(\mathsf{C})$ is a gerbe, and that it is locally bound by a flat affine group. A gerbe satisfying the last condition is said to be ***affine***.[2] The formation of the affine gerbe $\textsc{Fib}(\mathsf{C})$ is 2-functorial in C.

3.8 Let G be an affine gerbe over R. Denote by $\mathsf{Rep}(\mathsf{G})$ the category of cartesian functors

$$\mathsf{G} \longrightarrow \textsc{Mod}(R).$$

Then $\mathsf{Rep}(\mathsf{G})$ inherits a tensor product from that on $\textsc{Mod}(R)$ which makes it into an ind-tannakian category over R (op. cit. 2.2.3.1). For example, if G is a flat affine R-group then the torsors under G form an affine gerbe G, and there is an equivalence of ind-tannakian categories

$$\mathsf{Rep}(\mathsf{G}) \longrightarrow \mathsf{Rep}(G).$$

THEOREM 3.9 (OP. CIT. 2.3.2) *For an ind-tannakian category C over R and an affine gerbe B over Aff_R, there are canonical equivalences*

$$\mathsf{C} \longrightarrow \mathsf{Rep}(\textsc{Fib}(C)), \qquad \mathsf{G} \longrightarrow \textsc{Fib}(\mathsf{Rep}(\mathsf{G})).$$

Let $\mathcal{IT}ann(R)$ denote the 2-category of ind-tannakian categories over R and $\mathcal{G}erbe(R)$ the 2-category of affine gerbes over Aff_R. The 2-functor

$$\textsc{Fib} : \mathcal{IT}ann(R) \to \mathcal{G}erbe(R)$$

is an equivalence of 2-categories.

The case of a base field

Let k be a field. The key result is that, when C is a tannakian category, $\mathrm{Ind}\,\mathsf{C}$ is an ind-tannakian category. More precisely, there is the following statements.

THEOREM 3.10 (SAAVEDRA 1972, III, 3.2.2) *Let C_0 be an essentially small tensor category over a field k, and let $\mathsf{C} = \mathrm{Ind}\,\mathsf{C}_0$. Let ω_0 be a fibre functor on C_0 with values in a field k' containing k, and let ω' be the extension of ω_0 to a fibre functor on $\mathsf{C}_{(k')}$. The functor $\mathcal{A}ut^{\otimes}(\omega)$ of k'-algebras is represented by an affine group scheme G' over k', and ω defines an equivalence of k'-linear $\otimes$-categories $\mathsf{C}_{(k')} \to \mathsf{Rep}(G')$.*

The functor $i_{k'/k} : \mathsf{C} \to \mathsf{C}_{(k')}$ is exact and faithful, and so C is an ind-tannakian category.

THEOREM 3.11 (SAAVEDRA 1972, III, 3.2.3.1) *Let C_0 be an essentially small tannakian category over k. The stack $\textsc{Fib}(\mathsf{C}_0)$ is a gerbe over Aff_k for the fpqc topology, and the canonical functor*

$$\mathsf{C}_0 \to \mathsf{Rep}_0(\textsc{Fib}(\mathsf{C}_0))$$

is an equivalence of tannakian categories.

[2] Saavedra (1972), III, 2.2.1, calls this a tannakian gerbe.

If C_0 is a tannakian category over k, define, as in [(9), 2.2], the fibered category $\mathrm{FIB}_0(C_0)$ of fiber functors

$$C_0 \longrightarrow \mathrm{Loclib}(T),$$

where T is a variable k–scheme.

THEOREM 3.12 *If C_0 is a tannakian category over k, then the $\otimes$-category $C = \mathrm{Ind}(C_0)$ of ind–objects of C_0 is ind–tannakian over k. Moreover, there is an equivalence of tannakian gerbes*

$$\mathrm{FIB}(C) \simeq \mathrm{FIB}_0(C_0). \tag{2.2.1}$$

3.13 An important case is that in which the gerbe $\mathrm{FIB}_0(C_0)$ is ***algebraic***, i.e., is locally bound by a group of finite type. We also say in this case that the tannakian category C_0 is ***algebraic***. One can see that C_0 is algebraic if and only if it has a $\otimes$-generator. It follows that every tannakian category is the union of its algebraic tannakian subcategories, so its gerbe is pro-algebraic, i.e., a projective limit of algebraic tannakian gerbes. Conversely, if C is an ind–tannakian category over k and its gerbe $\mathrm{FIB}(C)$ is pro-algebraic, then C is equivalent to an ind-tannakian category

$$\mathrm{Ind}(C_0).$$

I do not know whether this condition is always satisfied; it is satisfied, at any rate, if the band of $\mathrm{FIB}(C)$ [see (4),chap. III] is representable by a group. The essential result for the proof of that which proceeds is the following.

THEOREM 3.14 *If C is an ind–tannakian category over k whose gerbe $\mathrm{FIB}(C)$ is algebraic, then C is a locally noetherian category [(3),II, 4], and the full subcategory of its noetherian objects is tannakian. Moreover, C has a fiber functor over a finite extension of k.*

The proof of this theorem uses the following result from nonabelian homological algebra.

THEOREM 3.15 *For a fibered category E over $\mathrm{Sch}_{/k}$, being an algebraic tannakian gerbe for the fpqc topology is equivalent to being one for the fppf topology.*

3.16 As pointed out by Deligne, in his definition of a tannakian category over k, Saavedra (ibid. p. 193) omits the condition $k \simeq \mathrm{End}(\mathbb{1})$. Without it the fibre functors are not locally isomorphic (I, 5.16). For example, let K be a finite Galois extension of k, and let $\mathsf{C} = \mathsf{Vecf}(K)$ with $\otimes = \otimes_K$. Then C is a k-linear abelian rigid $\otimes$-category with $K = \mathrm{End}(\mathbb{1}) \neq k$. For a K-vector space V and $\sigma \in \mathrm{Gal}(K/k)$, let $\omega_\sigma(V)$ denote V with K acting through σ (i.e., $\omega_\sigma(V) = V \otimes_{K,\sigma} K$). For any V in C and endomorphism of α of V, $\mathrm{Tr}(\omega_\sigma(\alpha) \mid \omega_\sigma(V)) = \sigma\,\mathrm{Tr}(\alpha|V)$. Since locally isomorphic fibre functors give the same trace (I, 5.14), it follows that ω_σ and $\omega_{\sigma'}$ are not locally isomorphic if $\sigma \neq \sigma'$.

3.17 Let k be a field and C_0 a k–linear abelian ACU $\otimes$-category possessing internal *Hom* objects. We say that C_0 is a tannakian category over k if there exists an extension k'/k and k'–linear ACU $\otimes$- functor

$$C_0 \longrightarrow \mathrm{Mod}(k')$$

that is faithful and exact and commutes with internal *Hom* objects.

4 Algebraic geometry in a tannakian category

In I, §10, we studied algebra geometry in a tensor category. We show in this section that the existence of a fibre functor, hence a gerbe of fibre functors, makes this easier. Throughout, k is a field.

Affine T*-schemes*

Recall (I, 9.13) that, for a tensor category T over k, we have the notion of an affine T-scheme (affine scheme in Ind T).

4.1 Let T be a tannakian category over k, and let ω_1 and ω_2 be fibre functors on T over an affine k-scheme S. Recall that the functor sending an affine S-scheme $u: S' \to S$ to the set of isomorphisms from $u^*\omega_1$ to $u^*\omega_2$ is represented by a scheme $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$, affine over S. For a fibre functor ω over S, we let $\mathcal{A}ut_S^{\otimes}(\omega)$ denote $\mathcal{I}som_S^{\otimes}(\omega, \omega)$. The main result of Chapter II says the following: if ω is a k-valued fibre functor on T, then ω induces an equivalence of symmetric monoidal categories

$$\mathsf{T} \to \mathsf{Repf}(\mathcal{A}ut_S^{\otimes}(\omega)).$$

The interpretation (I, 9.17) of T-schemes as equivariant affine k-schemes is then available. Unfortunately, this is scarcely convenient and depends on the choice of the fibre functor ω. Here we prefer to work with all fibre functors.

EXAMPLE 4.2 (DELIGNE 1989, 5.10) Let G be an affine group scheme over k, let X be a finite-dimensional representation of G, and let X also denote the corresponding vectorial group scheme $\mathrm{Spec}(\mathrm{Sym}^*(X^\vee))$. An extension

$$0 \to X \to E \to \mathbb{1} \to 0$$

of the trivial representation (k with the trivial action) by X determines an equivariant X-torsor, namely, the inverse image of $1 \in k$ in E. This construction is an equivalence of categories.

We deduce a similar statement for a tannakian category T over k: for any X in T, there is an equivalence from the category of extensions of $\mathbb{1}$ by X to that of X-torsors,

$$(\text{extensions of } \mathbb{1} \text{ by } X) \xrightarrow{\approx} (X\text{-torsors}).$$

We define a functor as follows. Let A be the vectorial T-scheme defined by the identity object. It is also the image (see I, 9.15) of the affine line $\mathrm{Spec}\, k[T]$ over k, and the point $T = 1$ defines a point $1: (\mathrm{pt}) \to A$. An extension of E of $\mathbb{1}$ by X defines a vectorial scheme E mapping onto A. The action by translation of E on itself induces an action of X on E stabilizing the fibre $P \overset{\text{def}}{=} E \times_A (\mathrm{pt})$ at 1 of $E \to A$. This fibre is the torsor sought.

This description is independent of the choice of a fibre functor, but once we choose a fibre functor, the interpretation (I, 9.15) shows that it is an equivalence.

Interpretation in terms of gerbes; constructions

4.3 Let T be a tannakian category over k. Recall that we have proved the following statements.

(a) The fibre functors form a gerbe $\mathrm{FIB}(\mathsf{T})$ over Aff_k for the fpqc topology: they form a stack (fibre functors given locally on S patch to a fibre functor on S); over some $S \neq \emptyset$, there exists a fibre functor; any two fibre functors over S become isomorphic on a scheme T faithfully flat and affine over S.

(b) Each object X of T defines a morphism of stacks $\omega \rightsquigarrow \omega(X)$

$$\{\text{fibre functors over } S \text{ (variable)}\} \to \{\text{vector bundles over } S\}.$$

This construction is an equivalence of T with the category $\mathsf{Repf}(\mathrm{FIB}\mathsf{T})$ of these functors. In other words, it is "the same" to give an object X of T or to give, for each fibre functor ω over a k-scheme S, a vector bundle over S functorial in ω, and compatible with base change $S' \to S$.

(c) By passage to ind-objects, a fibre functor ω on S defines a $\otimes$-functor, again denoted ω, from $\mathrm{Ind}\,\mathsf{T}$ to the category of quasi-coherent sheaves on S. Each object X of $\mathrm{Ind}\,\mathsf{T}$ defines a morphism of stacks

$$\{\text{fibre functors over } S \text{ (variable)}\} \to \{\text{quasi-coherent sheaves over } S\}.$$

This construction is an equivalence of $\mathrm{Ind}\,\mathsf{T}$ with the category of cartesian functors from the stack $\mathrm{FIB}(\mathsf{T})$ over Aff_k to the stack MOD),

(d) The passage from T to $\mathrm{FIB}(\mathsf{T})$ has an inverse. Let G be an affine gerbe. Let $\mathsf{Repf}(\mathsf{G})$ be the category of morphisms of stacks

$$\mathsf{G} \to (\text{vector bundles over } S \text{ variable}).$$

Then $\mathsf{Repf}(\mathsf{G})$ is a tannakian category, and

$$\mathsf{G} \xrightarrow{\cong} \mathrm{FIB}(\mathsf{Repf}(\mathsf{G})).$$

It follows from (c) that to give an affine T-scheme X (resp. an affine group T-scheme G, resp. a T-torsor under G) is the same as giving, for each fibre functor ω over an affine k-scheme S, an affine scheme X_ω (resp. an affine group scheme G_ω, resp. a torsor under G_ω) over S, natural in ω, and compatible with base changes $S' \to S$. For example, $X = \mathrm{Sp}(A)$ corresponds to the system $X_\omega \overset{\text{def}}{=} \mathrm{Spec}(\omega(A))$.

To construct a morphism $F : X \to Y$ between affine T-schemes, it suffices to construct, for every fibre functor ω, a morphism $X_\omega \to Y_\omega$ natural in ω. If ω is a fibre functor over S, it suffices, for every affine S-scheme T, to construct a map

$$X_\omega(T) \to Y_\omega(T),$$

natural in T. Here $X(T) = \mathrm{Hom}_S(T, X)$.

4.4 For (X_ω) as above, each X_ω/S automatically has the following property (concerning X/S).

(*) There exists an extension k' of k and a scheme $u : T \to S$ faithfully flat over S, such that the inverse image $u^*X = T \times_S X$ of X over T is the inverse image over T of a k'-scheme by a morphism of T to k'.

Indeed, there exists a fibre functor ω_0 over an extension k' of k and, because $\mathrm{FIB}(\mathsf{T})$ is a gerbe, ω and ω_0 become isomorphic over some T faithfully flat over $S \times \mathrm{Spec}(k')$, so the inverse images of X_ω and X_{ω_0} over T are isomorphic,

$$\begin{array}{ccccc} X_\omega & \longleftarrow & X_\omega \times_S T & \longrightarrow & X_{\omega_0} \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & T & \longrightarrow & \mathrm{Spec}(k'). \end{array}$$

In other words, locally for the fpqc topology, X comes from a scheme over an field extension of k. A similar statement holds for schemes equipped with additional data.

4.5 Suppose that we have a construction Ξ taking an affine scheme (possibly with additional data) over an affine k-scheme S to another affine scheme over S (possibly with additional data). If the construction applies to all schemes (with additional data) satisfying 4.4(*) and is compatible with base change, we can apply it to an affine T-scheme. Given such a T-scheme X, apply Ξ to the schemes X_ω to get a system $Y_\omega \overset{\text{def}}{=} \Xi(X_\omega)$, which arises from a T-scheme, denoted $\Xi(X)$. Rather than trying to make this more precise, we give some examples.

EXAMPLE 4.6 For an affine group scheme G over S, let $\Xi(G)$ be the nth term $C^n(G)$ of the lower central series

$$G = C^1(G) \supset C^2(G) \supset \cdots, \qquad C^{n+1} = [G, C^n(G)].$$

This construction is not compatible with base change for an arbitrary G/S, but it is for affine group schemes satisfying 4.4(*). Thus, for any affine group T-scheme G, we have an affine T-scheme $C^n(G)$ such that $\omega(C^n(G)) = C^n(\omega(G))$ for all fibre functors ω.

EXAMPLE 4.7 Let H be a normal subgroup scheme of G, and let $\Xi(G, H) = G/H$. Even when H is not normal, we may consider G/H provided it is affine. The same discussion as in 4.6 applies.

EXAMPLE 4.8 For affine group schemes over an affine k-scheme satisfying 4.4(*), the property of being unipotent is stable under base change. Thus, it makes sense to say that an affine group T-scheme is unipotent. Similarly, it is possible to define the unipotent radical $R_u G$ of an affine group T-scheme G, and even its semisimple quotient $G/R_u G$.

APPLICATION 4.9 Over an arbitrary base S, giving an extension $\mathcal{E}$ of $\mathcal{O}$ by a vector bundle $\mathcal{V}$ is equivalent to giving a torsor under the vectorial group scheme defined by $\mathcal{V}$. This construction is compatible with base change. It follows that in any tannakian category, giving an extension E of $\mathbb{1}$ by an object V is equivalent to giving a torsor under the T-vectorial scheme V. We have already proved this in 4.2 for a neutral T.

Relation between the two points of view

4.10 Here is the relation between the points of view (I, 9.17) and (4.3) in the case that $\mathsf{T} = \mathsf{Repf}(G)$. Let ω_0 be the forgetful fibre functor. For ω a fibre functor over S, $\mathcal{I}som^{\otimes}(\omega_0, \omega)$ is a G-torsor P over S. Conversely, a G-torsor P over S defines a fibre functor

$$\omega_P : V \rightsquigarrow V^P \overset{\text{def}}{=} V \wedge^G P \qquad (V \text{ twisted by } P)$$

over S.

If $P(S) \neq \emptyset$, then V^P is a vector bundle over S equipped, for each $p \in P(S)$, with an isomorphism

$$\rho(p) : V \otimes \mathcal{O}_S \xrightarrow{\simeq} V^P,$$

such that $\rho(pg) = \rho(p)\rho(g)$ for $g \in G(S)$. The case $P(S) = \emptyset$ can be treated by descent, and so we have an equivalence

$$\text{FIB}(\mathsf{Repf}(G)) \sim (G\text{-torsors over } S \text{ variable}).$$

If X is a T-scheme, identified by I, 9.17, to a G-equivariant affine scheme, then for every fibre functor ω_P, $\omega_P(X)$ is the twist X^P of X by P.

4.11 *For a torsor P and corresponding fibre functor ω_P, we have*

$$\mathcal{A}ut^{\otimes}(\omega_P) = \mathcal{A}ut(P) \simeq G^P$$

(twist of G for the inner action of G on itself).

PROOF By definition, $G^P = P \wedge^G G$, where $P \wedge^G G$ is the quotient of $P \times G$ by the action of G,

$$(p, g)t = (pt, t^{-1}xt), \quad p \in P(S), \quad g, t \in G(S).$$

Define an action of G^P on P by

$$[p, g](p) = pg.$$

When P is the trivial torsor, this gives an isomorphism $G^P \to \mathcal{A}ut(P)$, and the general case follows by descent. □

Tensor products of tannakian categories

4.12 From 4.3 we get a dictionary between tannakian categories over k and affine gerbes. We define the ***tensor product*** of two tannakian categories by

$$\text{FIB}(\mathsf{T}_1 \times \mathsf{T}_2) \cong \text{FIB}(\mathsf{T}_1) \times \text{FIB}(\mathsf{T}_2).$$

Giving an object X of $\mathsf{T}_1 \otimes \mathsf{T}_2$ is equivalent to giving, for ω_1 and ω_2 fibre functors over S of T_1 and T_2, a vector bundle X_{ω_1,ω_2} on S, the formation of X_{ω_1,ω_2} being functorial in ω_1 and ω_2 and compatible with base change.

We have a tensor product

$$\boxtimes : \mathsf{T}_1 \times \mathsf{T}_2 \to \mathsf{T}_1 \otimes \mathsf{T}_2,$$

such that, for fibre functors ω_1 and ω_2 on T_1 and T_2, there is a fibre functor on $\mathsf{T}_1 \otimes \mathsf{T}_2$ sending $X_1 \boxtimes X_2$ to $\omega_1(X_1) \otimes \omega_2(X_2)$. In Chapter II, §10, we showed that $\mathsf{T}_1 \otimes \mathsf{T}_2$ is the universal target of such a tensor product with suitable properties.

If T_1, T_2 are $\mathsf{Rep}(G_1)$, $\mathsf{Rep}(G_2)$, then $\mathsf{T}_1 \otimes \mathsf{T}_2 \cong \mathsf{Rep}(G_1 \times G_2)$.

The fundamental group of a tannakian category

Let T be a tannakian category over k. For each fibre functor ω over a k-scheme S, $\mathcal{A}ut^{\otimes}_S(\omega)$ is an affine group scheme over S (see 4.1). Its formation is compatible with base change, i.e., for any morphism $T \to S$,

$$\mathcal{A}ut^{\otimes}_T(\omega_T) \simeq \mathcal{A}ut^{\otimes}_S(\omega)_T.$$

By 4.3(c), the system of affine group schemes $\mathcal{A}ut^{\otimes}_S(\omega)$ arises from an affine group T-scheme.

DEFINITION 4.13 The ***fundamental group*** $\pi(\mathsf{T})$ of T is the affine group T-scheme such that

$$\omega(\pi(\mathsf{T})) \simeq \mathcal{A}ut^{\otimes}(\omega). \tag{89}$$

functorially in ω.

Let $X \in \mathrm{ob}\,\mathsf{T}$. For each fibre functor ω over S, $\omega(\pi(\mathsf{T})) = \mathcal{A}ut^{\otimes}(\omega)$ acts on $\omega(X)$. We deduce an action (9.16) of $\pi(\mathsf{T})$ on X, functorial in X and compatible with tensor products. By passage to ind-objects, these actions furnish an action of $\pi(\mathsf{T})$ on all ind-objects. We deduce an action of $\pi(\mathsf{T})$ on all affine T-schemes. The action of $\pi(\mathsf{T})$ on the T-scheme $\pi(\mathsf{T})$ is the action of $\pi(\mathsf{T})$ on itself by inner automorphisms. Indeed, for any fibre functor ω, the action by functoriality of $\mathrm{Aut}^{\otimes}(\omega)$ on itself is its action by inner automorphisms.

EXAMPLE 4.14 Let G be an affine group scheme over k, and let $\mathsf{T} = \mathsf{Rep}(G)$. After 4.11, the fundamental group $\pi(\mathsf{T})$, viewed as an equivariant affine group scheme, is G equipped with the inner action on itself. The action of $\pi(\mathsf{T})$ on a representation V of G is the given action of G. It is G-equivariant,

$$h(gv) = hgh^{-1} \cdot hv.$$

4.15 Let $u : \mathsf{T}_1 \to \mathsf{T}$ be an exact k-linear $\otimes$-functor between tannakian categories over k. For any fibre functor ω on T over a k-scheme, $\omega \circ u$ is a fibre functor on T_1 over S, and there is a canonical homomorphism

$$\mathcal{A}ut^{\otimes}(\omega) \to \mathcal{A}ut^{\otimes}(\omega \circ u) \tag{90}$$

On applying u to the group T_1-scheme $\pi(\mathsf{T}_1)$, we get a group T-scheme $u(\pi(\mathsf{T}))$ and (90) is a morphism, functorial in ω, of $\omega(\pi(\mathsf{T}))$ into $\omega \circ u(\pi(\mathsf{T})) = \omega(u(\pi(\mathsf{T}_1)))$. By 4.3, it defines a morphism of T-schemes

$$U : \pi(\mathsf{T}) \to u(\pi(\mathsf{T}_1)). \tag{91}$$

For any object X_1 of T_1, the action of $\pi(\mathsf{T}_1)$ on X_1 induces an action of $u(\pi(\mathsf{T}_1))$ on $u(X_1)$. The action of $\pi(\mathsf{T})$ on $u(\pi(\mathsf{T}_1))$ is the action by conjugation defined by U. It suffices to check this after applying a fibre functor.

THEOREM 4.16 *With the preceding notation, u induces an equivalence of T_1 with the category of objects of T equipped with an action of $u\pi(\mathsf{T}_1)$ extending the action of $\pi(\mathsf{T})$.*

PROOF Suppose that T is neutral, say, $\mathsf{T} = \mathsf{Rep}(G)$, and let ω be the forgetful functor. Let $G_1 = \mathcal{A}ut^{\otimes}(\omega\circ u)$. The morphisms (90) define

$$f : G \to G_1, \tag{92}$$

which can also be obtained from (91) by applying ω. Via the equivalences $\mathsf{T} \approxeq \mathsf{Rep}(G)$, $\mathsf{T}_2 \approxeq \mathsf{Rep}(G_1)$, the functor u is the restriction to G (by f) of the action of G_1, and 4.16 becomes to the following triviality: for a vector space V, to give an action of G_1 on V is equivalent to giving an action of G plus a G-equivariant action of G_1 factoring through the action of G.

For the general case, we refer the reader to Deligne 1990, 8.17. □

4.17 Let $u : \mathsf{T}_1 \to \mathsf{T}$ be an exact k-linear $\otimes$-functor between tannakian categories over k. If u is fully faithful and identifies T_1 to a full subcategory of T stable under subquotients, then the morphism $U : \pi(\mathsf{T}) \to u\pi(\mathsf{T}_1)$ is an epimorphism (= faithfully flat). Moreover, Proposition 4.16 shows that u identifies T_1 with the subcategory of T formed of the objects on which the action of $\pi(\mathsf{T})$ induces the trivial action of $H \overset{\text{def}}{=} \mathrm{Ker}(U : \pi(\mathsf{T}) \to u\pi(\mathsf{T}_1))$. See Saavedra 1972, II, 4.3.2(g).

EXAMPLE 4.18 Let $\mathsf{T}_1 = \mathsf{Vecf}(k)$ and consider the functor $V \rightsquigarrow V \otimes \mathbb{1} : \mathsf{T}_1 \to \mathsf{T}$. Then $\pi(\mathsf{T}_1) = \{e\}$ and the functor is an equivalence of $\mathsf{Vecf}(k)$ with the subcategory of objects of T on which $\pi(\mathsf{T})$ acts trivially (cf. 9.15).

Semisimplicity

In this subsection, T is a tannakian category over a field k of characteristic zero.

EXAMPLE 4.19 Suppose that T has a fibre functor ω_0 with values in k. The semisimple objects of the abelian category category of representations of the affine group scheme $\mathcal{A}ut^{\otimes}(\omega_0)$ are those on which the unipotent radical $R_u\mathcal{A}ut^{\otimes}(\omega_0)$ acts trivially. Therefore the subcategory $\mathsf{T}_1 \subset \mathsf{T}$ of semisimple objects is stable under tensor products, and is again a tannakian category over k. The morphism (91) corresponding to the inclusion is

$$\pi(\mathsf{T}) \to \pi(\mathsf{T})/R_u\pi(\mathsf{T}).$$

EXAMPLE 4.20 Let T be an object of dimension 1 of T, and let $T^{\otimes -n} = (T^\vee)^{\otimes n}$ for $n \in \mathbb{N}$. To give a representation ρ of $\mathbb{G}_m$ is the same as giving a graded vector space $V = \bigoplus V^j$, with $(\lambda)v^j = \lambda^j v^j$ for $v^j \in V^j$, and we define

$$u : \mathsf{Rep}(\mathbb{G}_m) \to \mathsf{T}$$

by $V \rightsquigarrow \bigoplus(V^j \otimes T^{\otimes j})$. From there, we get a morphism

$$\pi(\mathsf{T}) \to \mathbb{G}_m, \tag{93}$$

such that the action of $\pi(\mathsf{T})$ on T factors through $\mathbb{G}_m$, with λ acting as multiplication by λ. In (93), $\mathbb{G}_m$ is regarded as a group T-scheme as in I, 9.15. The morphism 93 is characterized by the action on the invertible object T.

If, for all $n > 0$, we have $\mathrm{Hom}(\mathbb{1}, T^{\otimes n}) = 0$, we can apply 4.17 to deduce that (93) is an epimorphism.

If the $T^{\otimes n}$ ($n \in \mathbb{Z}$) are the only simple objects of T, and no two are isomorphic, we can conclude from (4.19) and (4.20) (in characteristic 0), that (93) realizes $\pi(\mathsf{T})$ an extension of $\mathbb{G}_m$ by a unipotent group.

4.21 Again, let T be a tannakian category over a field k of characteristic 0 and, to simplify, suppose again that T is neutral. Let T^{ss} be the category of semisimple objects of T. The group T-scheme $R_u\pi(\mathsf{T})$ acts trivially on $(R_u\pi(\mathsf{T}))^{ab}$, which is a group T^{ss}-scheme. It is commutative and unipotent, and we can identify it with a pro-object in T^{ss}, for example, by writing it as a projective limit of vectorial group T-schemes.

PROPOSITION 4.22 *Let* T *be a neutral tannakian category over a field k of characteristic zero. For any semisimple object X of* T,

$$\mathrm{Ext}^1(\mathbb{1}, X) \xrightarrow{\simeq} \mathrm{Hom}((R_u\pi(\mathsf{T}))^{ab}, X). \tag{94}$$

PROOF We first explain the statement. In (94), on the left X is an object of T and on the right it is the corresponding vectorial T-scheme. We have

$$\mathrm{Hom}(R_u\pi(\mathsf{T}), X) \xrightarrow{\simeq} \mathrm{Hom}(R_u\pi(\mathsf{T}))^{ab}, X) \xrightarrow{\simeq} \mathrm{Hom}(\mathrm{Lie}(R_u\pi(\mathsf{T}))^{ab}, X).$$

If a group G acts on an extension E of A by B and acts trivially on A and B, then the maps $\rho(g) - 1 : E \to E$ factor through morphisms from A to B. The statements 4.3, 4.5 allow us to repeat this "in T".

If E is an extension of $\mathbb{1}$ by X, the action 4.13 of $R_u\pi(\mathsf{T}) \subset \pi(\mathsf{T})$ on E is trivial on $\mathbb{1}$ and X (see 4.19). It defines a morphism

$$R_u\pi(\mathsf{T}) \to \mathrm{Hom}(\mathbb{1}, X) = X.$$

This construction defines the arrow (94).

Injectivity of (94): if the class of an extension E has trivial image under (94), the action of $R_u\pi(\mathsf{T})$ on E is trivial: E is semisimple and the extension is trivial.

Surjectivity of (94): we may suppose that $\mathsf{T} = \mathsf{Repf}(G)$. Write G as a semi-direct product of a proreductive group scheme G^{ss} by R_uG (Levi decomposition; here we use characteristic 0). For (X, ρ) a representation of $G^{ss} = G/R_uG$ and a a G^{ss}-morphism of R_uG^{ab} into X, we define an extension E of the trivial representation by the representation X by making $u \cdot g$ ($g \in G^{ss}$, $u \in R_uG$) act on $\mathbb{1} \otimes X$ by $\left(\begin{smallmatrix} 1 & 0 \\ a(u) & \rho(g) \end{smallmatrix}\right)$. Its image by (94) is the morphism a. □

NOTATION 4.23 For V a vector space over k and X in T, $\mathcal{H}om(V, X)$ is the pro-object of T, projective limit of the $W^\vee \otimes X$ for W a subspace of finite dimension of V.

EXAMPLE 4.24 Let T be the category $\mathsf{Rep}(\mathbb{G}_m)$. Let $T(n)$ be the k-vector space on which $\lambda \in \mathbb{G}_m$ acts by multiplication by λ^n. For any pro-object X of T, if we put $V(n) = \mathrm{Hom}(X, T(n))$, then we have

$$X = \prod_n \mathcal{H}om(V(n), T(n)). \tag{95}$$

4.25 Let T be a neutral tannakian category over k of characteristic 0 and $T \in \mathrm{ob}\,\mathsf{T}$. We assume that T has dimension 1 and we put $T(n) = T^{\otimes n}$. We assume that the morphism 4.20 of $\pi(\mathsf{T})$ into $\mathbb{G}_m$ is an epimorphism with unipotent kernel, i.e., that the conditions of last paragraph of 4.20 are satisfied. Let $U = \mathrm{Ker}(\pi(\mathsf{T}) \to \mathbb{G}_m)$. Applying 4.22 and 4.23 and identifying U^{ab} with its Lie algebra, we find that

$$U^{ab} = \prod_n \mathcal{H}om(\mathrm{Ext}^1(\mathbb{1}, T(n)), T(n)). \tag{96}$$

The fundamental groupoid

4.26 To two fibre functors ω_1, ω_2 of T over S, we attach the affine scheme $\mathrm{Isom}^{\otimes}_S(\omega_2, \omega_1)$ over S. This construction is compatible with change of base. By 4.3 and 4.12, it defines a $\mathsf{T} \otimes \mathsf{T}$-scheme $G(\mathsf{T})$ such that

$$(\omega_1 \otimes \omega_2)(G(\mathsf{T})) = \mathcal{I}som^{\otimes}_S(\omega_2, \omega_1).$$

It is the ***fundamental groupoid*** of T.

For any mapping between finite sets $\varphi : I \to J$, we define $T(\varphi) : \mathsf{T}^{\otimes I} \to \mathsf{T}^{\otimes J}$ by

$$T(\varphi)(\boxtimes X_i) = \boxtimes_j \Big(\bigotimes_{\varphi(i)=j} X_i \Big),$$

where the tensor product over the $i \in \varphi^{-1}(j)$ is taken in T, and is $\mathbb{1}$ if $\varphi^{-1}(j) = \emptyset$.

Put $j_{a,b} = T(\varphi)$ for

$$\varphi : \{1,2\} \to \{1,2,3\}, \quad 1 \mapsto a, \ 2 \mapsto b.$$

Composition of isomorphisms defines

$$j_{1,2}(G(\mathsf{T})) \times j_{2,3}(G(\mathsf{T})) \to j_{1,3}(G(\mathsf{T})) \tag{97}$$

in $\mathsf{T} \otimes \mathsf{T} \otimes \mathsf{T}$. For $\varphi : \{1,2\} \to \{1\}$, $T(\varphi)$ is

$$T : \mathsf{T} \otimes \mathsf{T} \to \mathsf{T}, X \boxtimes_j Y \mapsto X \otimes_{\mathsf{T}} Y.$$

We have

$$T(G(\mathsf{T})) = \pi(\mathsf{T}). \tag{98}$$

For any fibre functor ω over S, $(\mathrm{pr}_1^* \omega, \mathrm{pr}_2^* \omega)$ defines a fibre functor $\omega \boxtimes \omega$ on $\mathsf{T} \otimes \mathsf{T}$ over $S \times S$. The image of $G(\mathsf{T})$ by $\omega \boxtimes \omega$ is the groupoid $\mathcal{A}ut^{\otimes}_k(\omega) \overset{\text{def}}{=} \mathcal{I}som_{S\times S}(\mathrm{pr}_2^* \omega, \mathrm{pr}_2^* \omega)$ over S, and the groupoid structure is deduced from (97).

4.27 In Chapter II, we gave the following description of the algebra Λ in $\mathrm{Ind}(\mathsf{T} \otimes \mathsf{T})$ whose spectrum is $G(\mathsf{T})$. As as an ind-object, it is the target of the universal morphism

$$X^\vee \otimes_k X \to \Lambda \qquad (X \text{ in } \mathsf{T}) \tag{99}$$

making, for all $f : X \to Y$, the following diagram commutative

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{f^t \otimes 1} & X^\vee \otimes X \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow \\ Y^\vee \otimes Y & \longrightarrow & \Lambda \end{array} \tag{100}$$

For any fibre functor ω over S, the groupoid $\mathcal{A}ut^{\otimes}_k(\omega) \overset{\text{def}}{=} \mathcal{I}som^{\otimes}_{S\times S}(\mathrm{pr}_2^* \omega, \mathrm{pr}_1^* \omega)$ is therefore the spectrum of $(\omega \boxtimes \omega)(\Lambda)$, that is, the quasi-coherent sheaf of algebras L on $S \times S$ which, as a quasi-coherent sheaf, is the universal target of morphisms

$$\mathrm{pr}_1^* \omega(X)^\vee \otimes \mathrm{pr}_2^* \omega(X) \to L \qquad (X \text{ in } \mathsf{T})$$

satisfying a compatibility similar to (100) for all $f : X \to Y$.

NOTES This section closely follows the original source, Deligne 1989, 5.9–6.14.

5 Morphisms from one tannakian category to a second

5.1 An exact k-linear $\otimes$-functor $u : \mathsf{T}_1 \to \mathsf{T}_2$ of tannakian categories defines a homomorphism $\pi(u) : \pi(\mathsf{T}_2) \to u(\pi(\mathsf{T}_1))$ (4.15). Moreover:

(a) u induces an equivalence of T_1 with the category of objects of T equipped with an action of $u\pi(\mathsf{T}_1)$ extending the action of $\pi(\mathsf{T})$ (see 4.16);

(b) $\pi(u)$ is flat and surjective if and only if u is fully faithful and every subobject of $u(X)$, for X in T_1, is isomorphic to the image of a subobject of X (cf. 8.2);

(c) $\pi(u)$ is a closed immersion if and only if every object of T_2 is a subquotient of an object in the image of u (cf. 8.2).

5.2 Let T be a tannakian category over k. Recall that, by definition, an affine group T-scheme G is a (commutative) Hopf algebra A in $\operatorname{Ind}\mathsf{T}$. We define a representation of G to be a right comodule over A, i.e., an object V of T together with a morphism $\rho : V \to V \otimes A$ such that certain diagrams commute (II, 1.13). With the obvious notion of morphism, we obtain a category $\mathsf{Rep}_{\mathsf{T}}(G)$ of representations of G (in T). For example, if $\mathsf{T} = \mathsf{Vecf}(k)$, then G is an affine group scheme over k in the usual sense, and $\mathsf{Rep}_{\mathsf{T}}(G) = \mathsf{Repf}(G)$. Note that in this last case $\mathsf{Rep}_{\mathsf{T}}(G)$ contains T as the subcategory of objects on which G acts trivially, and that there is a forgetful functor to T with the property that the composite

$$\mathsf{T} \longrightarrow \mathsf{Rep}_{\mathsf{T}}(G) \xrightarrow{\text{forget}} \mathsf{T}$$

is the identity functor. Similarly, in the general case, $\mathsf{Rep}_{\mathsf{T}}(G)$ contains T as the subcategory of objects on which G acts trivially, and that there is a forgetful functor to T with the same property.

5.3 Let $u : \mathsf{T}_1 \to \mathsf{T}_2$ be an exact k-linear $\otimes$-functor of tannakian categories over k. When $\mathsf{T}_2 = \mathsf{Vecf}(k)$, u induces an equivalence of T_1 with $\mathsf{Rep}_{\mathsf{T}_2}(G)$, where $G = \mathcal{A}ut^{\otimes}(u)$. In the general case, we can ask whether there is an affine group T_2-scheme G such that u induces an equivalence $\mathsf{T}_1 \xrightarrow{\approx} \mathsf{Rep}_{\mathsf{T}_2}(G)$. A quasi-inverse to such an equivalence will restrict to a functor $s : \mathsf{T}_2 \to \mathsf{T}_1$ such that $u \circ s \approx \mathrm{id}_{\mathsf{T}_2}$. Thus, a necessary condition is the existence of such an section s. The condition is also sufficient.

THEOREM 5.4 *Let $u : \mathsf{T}_1 \to \mathsf{T}_2$ be an exact k-linear $\otimes$-functor of tannakian categories over k. Suppose that there exists an exact k-linear $\otimes$-functor $s : \mathsf{T}_2 \to \mathsf{T}_1$ and an isomorphism $\alpha : u \circ s \to \mathrm{id}_{\mathsf{T}_2}$. Then there is an affine group T_2-group G such that u factors into*

$$\mathsf{T}_1 \xrightarrow[\sim]{} \mathsf{Rep}_{\mathsf{T}_2}(G) \xrightarrow[\text{forget}]{} \mathsf{T}_2, \qquad \mathsf{T}_1 \xrightarrow{\ u\ } \mathsf{T}_2.$$

PROOF Let $U : \pi(\mathsf{T}_2) \to u(\pi(\mathsf{T}_1))$ and $S : \pi(\mathsf{T}_1) \to s(\pi(\mathsf{T}_2))$ be the morphisms defined by u and s. The morphism $\pi(\mathsf{T}_2) \to (u \circ s)(\pi(\mathsf{T}_2))$ defined by $u \circ s$ is equal $u(S) \circ U$, and it is an isomorphism because the composite

$$\pi(\mathsf{T}_2) \xrightarrow{u(S)\circ U} (u \circ s)(\pi(\mathsf{T}_2)) \xrightarrow[\approx]{\alpha} \pi(\mathsf{T}_2)$$

is the identity map. So, if we define the group T_2-scheme G to be the kernel of $u(S)$, then

$$u(\pi(\mathsf{T}_1)) \simeq G \rtimes \pi(\mathsf{T}_2)$$

and the action of $\pi(\mathsf{T}_2)$ on G by conjugation is the restriction of its action by conjugation on $u(\pi(\mathsf{T}_1))$, which is the natural action. So this action is also the natural action. According to Theorem 4.16, u induces an equivalence of T_1 with the category of objects of T_2 equipped with an action ρ of $G \rtimes \pi(\mathsf{T}_2)$ whose restriction to $\pi(\mathsf{T}_2)$ is the natural action.

An arbitrary action ρ of $G \rtimes \pi(\mathsf{T}_2)$ on an object of T_2 is given by actions ρ_1 and ρ_2 of G and $\pi(\mathsf{T}_2)$ with the property that

$$\rho_2(p)\rho_1(g)\rho_2(p)^{-1} = \rho_1({}^p g), \quad \text{all } g \in G(R),\ p \in \pi(\mathsf{T}_2),\ R \text{ a } k\text{-algebra}.$$

But in the case that ρ_2 is the natural action, this condition is always satisfied, because the action of $\pi(\mathsf{T}_2)$ on G by conjugation is equal to the natural action and ρ_1 respects the natural action. This shows that u induces an equivalence of T_2 with $\mathsf{Rep}_{\mathsf{T}_2}(G)$ as required. □

Theorem 5.4 is Corollary 5.3 of Jafari and Einollahzadeh 2018. The group G depends on the choice of s (ibid., 5.5).

EXAMPLE 5.5 For a smooth algebraic variety over $\mathbb{C}$, let $E(X)$ denote the category of admissable variations of mixed Hodge structures on X. It is a tannakian category over $\mathbb{Q}$. Let $x \in X$. We have an exact $\mathbb{Q}$-linear tensor fibre functor $F : E(X) \to E(x)$. The functor sending an object of $E(x)$ to the corresponding constant local system is a section for F, and so there exists an affine group $E(x)$-scheme $\pi_1^E(X, x)$ such that F defines an equivalence of symmetric monoidal categories

$$E(X) \approxeq \mathsf{Rep}_{E(x)}(\pi_1^E(X, x)).$$

See Arapura 2010 for more details and more examples.

5.6 Let C and D be tannakian categories over k, and let $\alpha : H \to G$ be a morphism from the band of D to that of C. Then the morphisms $\mathsf{C} \to \mathsf{D}$ banded by α form a gerbe banded by the centralizer of α. See Giraud 1971, IV, 2.3.2.

5.7 For a group G, a right G-object X, and a left G-object Y, $X \wedge^G Y$ denotes the contracted product of X and Y, i.e., the quotient of $X \times Y$ by the diagonal action of G, $(x, y)g = (xg, g^{-1}y)$. When $G \to H$ is a homomorphism of groups, $X \wedge^G H$ is the H-object obtained from X by extension of the structure group. In this last case, if X is a G-torsor, then $X \wedge^G H$ is also an H-torsor. See Giraud 1971, III 1.3, 1.4.

5.8 Let T be a tannakian category over k, and assume that the fundamental group π of T is commutative. A torsor P under π in T defines a $\otimes$-equivalencee $\mathsf{T} \to \mathsf{T}, X \rightsquigarrow P \wedge^\pi X$, bound by the identity map on $\mathrm{Bd}(\mathsf{T})$, and every such equivalence arises in this way from a torsor under π (cf. Saavedra 1972, III 2.3). For any k-algebra R and R-valued fibre functor ω on T, $\omega(P)$ is an R-torsor under $\omega(\pi)$ and $\omega(P \wedge^\pi X) \simeq \omega(P) \wedge^{\omega(\pi)} \omega(X)$.

6 Quotients of tannakian categories

Given a tannakian category T and a tannakian subcategory S, we ask whether there exists a quotient of T by S, by which we mean an exact $\otimes$-functor $q : \mathsf{T} \to \mathsf{Q}$ from T to a tannakian category Q such that

(a) the objects of T that become trivial in Q (i.e., isomorphic to a direct sum of copies of $\mathbb{1}$ in Q) are precisely those in S, and

(b) every object of Q is a subquotient of an object in the image of q.

When T is the category $\mathsf{Repf}(G)$ of finite-dimensional representations of an affine group scheme G the answer is obvious: there exists a unique normal subgroup H of G such that the objects of S are the representations on which H acts trivially, and there exists a canonical functor q satisfying (a) and (b), namely, the restriction functor $\mathsf{Repf}(G) \to \mathsf{Repf}(H)$ corresponding to the inclusion $H \hookrightarrow G$. By contrast, in the general case, there need not exist a quotient, and when there does there will usually not be a canonical one. In fact, we prove that there exists a q satisfying (a) and (b) if and only if S is neutral, in which case the q are classified by the k-valued fibre functors on S. Here $k \overset{\text{def}}{=} \mathrm{End}(\mathbb{1})$ is assumed to be a field.

From a different perspective, one can ask the following question: given a subgroup H of the fundamental group $\pi(\mathsf{T})$ of T, does there exist an exact $\otimes$-functor $q : \mathsf{T} \to \mathsf{Q}$ such that the resulting homomorphism $\pi(\mathsf{Q}) \to q(\pi(\mathsf{T}))$ maps $\pi(\mathsf{Q})$ isomorphically onto $q(H)$? Again, there exists such a q if and only if the subcategory T^H of T, whose objects are those on which H acts trivially, is neutral, in which case the functors q correspond to the k-valued fibre functors on T^H.

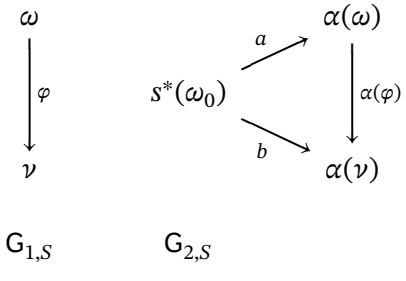
The two questions are related by the "tannakian correspondence" between tannakian subcategories of T and subgroups of $\pi(\mathsf{T})$ (see 6.6).

Preliminaries

We fix a field k and consider only tannakian categories over k.

Gerbes

6.1 Let $\alpha : \mathsf{G}_1 \to \mathsf{G}_2$ be a morphism of gerbes over Aff_k, and let ω_0 be an object of $\mathsf{G}_{2,k}$. Define $(\omega_0 \downarrow \mathsf{G}_1)$ to be the fibred category over Aff_k whose fibre over $s : S \to \mathrm{Spec}\, k$ has as objects the pairs (ω, a) consisting of an object ω of $\mathrm{ob}(\mathsf{G}_{1,S})$ and an isomorphism $a : s^*\omega_0 \to \alpha(\omega)$ in $\mathsf{G}_{2,S}$; the morphisms $(\omega, a) \to (\nu, b)$ are the isomorphisms $\varphi : \omega \to \nu$ in $\mathsf{G}_{1,S}$ giving rise to a commutative triangle,



If the morphism of bands defined by α is epi, then $(\omega_0 \downarrow \mathsf{G}_1)$ is a gerbe, and the sequence of bands

$$1 \to \mathrm{Bd}(\omega_0 \downarrow \mathsf{G}_1) \to \mathrm{Bd}(\mathsf{G}_1) \to \mathrm{Bd}(\mathsf{G}_2) \to 1 \tag{101}$$

is exact (Giraud 1971, IV 2.5.5(i)).

6.2 Recall (5) that a gerbe is said to be affine if its band is locally defined by an affine group scheme. It is clear from the exact sequence (101) that if G_1 and G_2 are affine, then so also is $(\omega_0 \downarrow \mathsf{G}_1)$.

6.3 Recall (IV, 1.4) that the fibre functors on a tannakian category T form a gerbe $\mathrm{FIB}(\mathsf{T})$ over Aff_k. Each object X of T defines a representation $\omega \rightsquigarrow \omega(X)$ of $\mathrm{FIB}(\mathsf{T})$, and in this way we get an equivalence $\mathsf{T} \to \mathsf{Rep}(\mathrm{FIB}(\mathsf{T}))$ of symmetric monoidal categories (2.1). Every affine gerbe arises in this way from a tannakian category (2.1).

Fundamental groups

6.4 Recall (4.13) that the fundamental group $\pi(\mathsf{T})$ of a tannakian category T is an affine group scheme in $\mathrm{Ind}\,\mathsf{T}$ such that

$$\omega(\pi(\mathsf{T})) \simeq \mathcal{A}ut^{\otimes}(\omega)$$

functorially in the fibre functor ω on T. The group $\pi(\mathsf{T})$ acts on each object X of T, and ω transforms this action into the natural action of $\mathcal{A}ut^{\otimes}(\omega)$ on $\omega(X)$. The various realizations $\omega(\pi(\mathsf{T}))$ of $\pi(\mathsf{T})$ determine the band of T.

6.5 A ***subgroup*** of $\pi(\mathsf{T})$ is a subgroup object of the fundamental group object of T. Subgroups of $\pi(\mathsf{T})$ are automatically normal. For example, the fundamental group π of the category $\mathsf{Repf}(G)$ of representations of the affine group scheme $G = \mathrm{Spec}(A)$ is A regarded as an object of $\mathrm{Ind}(\mathsf{Repf}(G))$. The action of G on A is that defined by inner automorphisms. A subgroup of π is a quotient $A \to B$ of A (as a bi-algebra) such that the action of G on A defines an action of G on B. Such quotients correspond to normal subgroups of G.

6.6 For a subgroup $H \subset \pi(\mathsf{T})$, we let T^H denote the full subcategory of T whose objects are those on which H acts trivially. It is a tannakian subcategory of T and $\pi(\mathsf{T}^H) = \pi(\mathsf{T})/H$. It follows from 4.16, that every tannakian subcategory $i : \mathsf{T}_1 \to \mathsf{T}$ of T is of the form T^H with $H = \mathrm{Ker}(\pi(i) : \pi(\mathsf{T}) \to i(\pi(\mathsf{T}_1)))$. In this way, we get a one-to-one correspondence between the subgroups of $\pi(\mathsf{T})$ and the tannakian subcategories of T.

For example, the objects of $\mathsf{T}^{\pi(\mathsf{T})}$ are exactly the trivial objects of T, and there exists a unique (up to a unique isomorphism) fibre functor $\gamma^{\mathsf{T}} : \mathsf{T}^{\pi(\mathsf{T})} \to \mathsf{Vecf}(k)$, namely, $\gamma^{\mathsf{T}}(X) = \mathrm{Hom}(\mathbb{1}, X)$.

6.7 For a subgroup H of $\pi(\mathsf{T})$ and an object X of T, we let X^H denote the largest subobject of X on which the action of H is trivial. Thus $X = X^H$ if and only if X is in T^H.

6.8 When H is contained in the centre of $\pi(\mathsf{T})$, it is an affine group scheme in $\mathsf{T}^{\pi(\mathsf{T})}$, and so γ^{T} identifies it with an affine group scheme over k in the usual sense. For example, γ^{T} identifies the centre of $\pi(\mathsf{T})$ with $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{T}})$ (cf. 9.2).

Quotients

For any exact $\otimes$-functor $q : \mathsf{T} \to \mathsf{T}'$, the full subcategory T^q of T whose objects become trivial in T' is a tannakian subcategory of T (obviously).

We say that an exact $\otimes$-functor $q : \mathsf{T} \to \mathsf{Q}$ of tannakian categories is a ***quotient functor*** if every object of Q is a subquotient of an object in the image of q; equivalently,

if the homomorphism $\pi(q)\colon \pi(\mathsf{Q}) \to q(\pi\mathsf{T})$ is a closed immersion (see 5.1(c)). If, in addition, the homomorphism $\pi(q)$ is normal (i.e., its image is a normal subgroup of $q(\mathsf{T})$), then we say that q is ***normal***.

EXAMPLE 6.9 Consider the exact $\otimes$-functor $\omega^f : \mathsf{Repf}(G) \to \mathsf{Repf}(H)$ defined by a homomorphism $f : H \to G$ of affine group schemes. The objects of $\mathsf{Repf}(G)^{\omega^f}$ are those on which H (equivalently, the intersubsection of the normal subgroups of G containing $f(H)$) acts trivially. The functor ω^f is a quotient functor if and only if f is a closed immersion, in which case it is normal if and only if $f(H)$ is normal in G.

PROPOSITION 6.10 *An exact $\otimes$-functor $q : \mathsf{T} \to \mathsf{Q}$ of tannakian categories is a normal quotient functor if and only if there exists a subgroup H of $\pi(\mathsf{T})$ such that $\pi(q)$ induces an isomorphism $\pi(\mathsf{Q}) \to q(H)$.*

PROOF $\Leftarrow$: Let H be a subgroup of $\pi(\mathsf{T})$ satisfying the hypothesis. Because q is exact, $q(H) \to q(\pi\mathsf{T})$ is a closed immersion. Therefore $\pi(q)$ is a closed immersion, and its image is the normal subgroup $q(H)$ of $q(\pi\mathsf{T})$.

$\Rightarrow$: Because q is a quotient functor, $\pi(q)$ is a closed immersion. Let H be the kernel of the homomorphism $\pi(\mathsf{T}) \to \pi(\mathsf{T}^q)$ defined by the inclusion $\mathsf{T}^q \hookrightarrow \mathsf{T}$. The image of $\pi(q)$ is contained in $q(H)$, and equals it if and only if q is normal. To see this, let $G = q\pi(\mathsf{T})$, and identify T with the category of objects of Q with an action of G compatible with that of $\pi(\mathsf{Q}) \subset G$. Then q becomes the forgetful functor, and $\mathsf{T}^q = \mathsf{T}^{\pi(\mathsf{Q})}$. Thus, $q(H)$ is the subgroup of G acting trivially on those objects on which $\pi(\mathsf{Q})$ acts trivially. It follows that $\pi(\mathsf{Q}) \subset q(H)$, with equality if and only if $\pi(\mathsf{Q})$ is normal in G. □

In the situation of the proposition, we sometimes call Q ***a quotient of*** T ***by*** H.

Let $q : \mathsf{T} \to \mathsf{Q}$ be an exact $\otimes$-functor of tannakian categories. By definition, q maps T^q into $\mathsf{Q}^{\pi(\mathsf{Q})}$, and so we acquire a k-valued fibre functor $\omega^q \overset{\text{def}}{=} \gamma^{\mathsf{Q}} \circ (q|\mathsf{T}^q)$ on T^q:

$$\begin{array}{ccccc} \mathsf{T}^q & \xrightarrow[q|\mathsf{T}^q]{} & \mathsf{Q}^{\pi(\mathsf{Q})} & \xrightarrow{\gamma^{\mathsf{Q}}} & \mathsf{Vecf}_k \\ \downarrow & & \downarrow & & \\ \mathsf{T} & \xrightarrow{q} & \mathsf{Q}. & & \end{array} \qquad \omega^q(X) = \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, qX).$$

In particular, T^q is neutral. A fibre functor ω on Q, defines a fibre functor $\omega \circ q$ on T, and the (unique) isomorphism $\gamma^{\mathsf{Q}} \to \omega|\mathsf{Q}^{\pi(\mathsf{Q})}$ defines an isomorphism $a(\omega)\colon \omega^q \to (\omega\circ q)|\mathsf{T}^q$.

PROPOSITION 6.11 *Let $q : \mathsf{T} \to \mathsf{Q}$ be a normal quotient, and let H be the subgroup of $\pi(\mathsf{T})$ such that $\pi(\mathsf{Q}) \simeq q(H)$.*

(a) *For X, Y in T, $\mathcal{H}om(X,Y)^H$ is an object of T^q, and there is a canonical functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{Q}}(qX, qY) \simeq \omega^q(\mathcal{H}om(X,Y)^H).$$

(b) *The map $\omega \mapsto (\omega\circ q, a(\omega))$ defines an equivalence of gerbes*

$$r(q)\colon \mathrm{FIB}(\mathsf{Q}) \to (\omega^q \downarrow \mathrm{FIB}(\mathsf{T})).$$

PROOF (a) We have,

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Q}}(qX, qY) &\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, \mathcal{H}om(qX, qY)^{\pi(\mathsf{Q})}) && \text{(10), p. 18}\\
&\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, (q\mathcal{H}om(X, Y))^{q(H)}) && \text{(I, 4.20)}\\
&\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, q(\mathcal{H}om(X, Y)^{H}))\\
&\simeq \omega^q(\mathcal{H}om(X, Y)^{H}) && \text{(definition of } \omega^q\text{).}
\end{aligned}$$

(b) The functor $\mathrm{FIB}(\mathsf{T}) \to \mathrm{FIB}(\mathsf{T}^H)$ gives rise to an exact sequence

$$1 \to \mathrm{Bd}(\omega_Q \downarrow \mathrm{FIB}(\mathsf{T})) \to \mathrm{Bd}(\mathsf{T}) \to \mathrm{Bd}(\mathsf{T}^H) \to 0$$

(see 6.1). On the other hand, we saw in the proof of (6.10) that $H = \mathrm{Ker}(\pi(\mathsf{T}) \to \pi(\mathsf{T}^H))$. On comparing these statements, we see that the morphism $r(q)$ of gerbes is bound by an isomorphism of bands, which implies that it is an equivalence of gerbes (1.19). □

PROPOSITION 6.12 *Let (Q, q) be a normal quotient of T. An exact $\otimes$-functor $q' : \mathsf{T} \to \mathsf{T}'$ factors through q if and only $\mathsf{T}^{q'} \supset \mathsf{T}^q$ and $\omega^q \approx \omega^{q'}|\mathsf{T}^q$.*

PROOF The conditions are obviously necessary. For the sufficiency, choose an isomorphism $b : \omega^q \to \omega^{q'}|\mathsf{T}^q$. A fibre functor ω on T' then defines a fibre functor $\omega \circ q'$ on T and an isomorphism $a(\omega)|\mathsf{T}^q \circ b : \omega^q \to (\omega \circ q')|\mathsf{T}^q$. In this way we get a homomorphism

$$\mathrm{FIB}(\mathsf{T}') \to (\omega^q \downarrow \mathrm{FIB}(\mathsf{T})) \simeq \mathrm{FIB}(\mathsf{Q})$$

and we can apply (6.3) to get a functor $\mathsf{Q} \to \mathsf{T}'$ with the correct properties. □

THEOREM 6.13 *Let T be a tannakian category over k, and let ω_0 be a k-valued fibre functor on T^H for some subgroup $H \subset \pi(\mathsf{T})$. There exists a quotient (Q, q) of T by H such that $\omega^q \simeq \omega_0$.*

PROOF The gerbe $(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$ is affine (see 6.2). From the morphism of gerbes

$$(\omega, a) \mapsto \omega : (\omega_0 \downarrow \mathrm{FIB}(\mathsf{T})) \to \mathrm{FIB}(\mathsf{T}),$$

we obtain a morphism of tannakian categories

$$\mathsf{Rep}(\mathrm{FIB}(\mathsf{T})) \to \mathsf{Rep}(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$$

(see 6.3). We define Q to be $\mathsf{Rep}(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$ and we define q to be the composite of the above morphism with the equivalence (see 6.3)

$$\mathsf{T} \to \mathsf{Rep}(\mathrm{FIB}(\mathsf{T})).$$

Since a gerbe and its tannakian category of representations have the same band, an argument as in the proof of Proposition 6.11 shows that $\pi(q)$ maps $\pi(\mathsf{Q})$ isomorphically onto $q(H)$. A direct calculation shows that ω^q is canonically isomorphic to ω_0. □

We sometimes write T/ω for the quotient of T defined by a k-valued fibre functor ω on a subcategory of T.

EXAMPLE 6.14 Let $(\mathsf{T}, w, \mathbb{T})$ be a Tate triple (see Chapter V below), and let S be the full subcategory of T of objects isomorphic to a direct sum of integer tensor powers of the Tate object $\mathbb{T}$. Define ω_0 to be the fibre functor on S,

$$X \rightsquigarrow \varinjlim_n \operatorname{Hom}\Big(\bigoplus_{-n\le r\le n} \mathbb{1}(r), X\Big).$$

Then the quotient tannakian category T/ω_0 is that described more explicitly in V, 11.9, below.

REMARK 6.15 Let $q : \mathsf{T} \to \mathsf{Q}$ be a normal quotient functor. Then T can be recovered from Q, the homomorphism $\pi(\mathsf{Q}) \to q(\pi(\mathsf{T}))$, and the actions of $q(\pi(\mathsf{T}))$ on the objects of Q (apply 5.1(a)).

REMARK 6.16 A fixed k-valued fibre functor on a tannakian category T determines a Galois correspondence between the subgroups of $\pi(\mathsf{T})$ and the equivalence classes of quotient functors $\mathsf{T} \to \mathsf{Q}$.

EXERCISE 6.17 Use (5.7, 5.8) to express the correspondence between fibre functors on tannakian subcategories of T and normal quotients of T in the language of 2-categories.

ASIDE 6.18 Let G be the fundamental group $\pi(\mathsf{T})$ of a tannakian category T, and let H be a subgroup of G. We use the same letter to denote an affine group scheme in T and the band it defines. Then, under certain hypotheses, for example, if all the groups are commutative, there will be an exact sequence

$$\cdots \to H^1(k,G) \to H^1(k,G/H) \to H^2(k,H) \to H^2(k,G) \to H^2(k,G/H).$$

The category T defines a class $c(\mathsf{T})$ in $H^2(k,G)$, namely, the G-equivalence class of the gerbe of fibre functors on T, and the image of $c(\mathsf{T})$ in $H^2(k,G/H)$ is the class of T^H. Any quotient of T by H defines a class in $H^2(k,H)$ mapping to $c(\mathsf{T})$ in $H^2(k,G)$. Thus, the exact sequence suggests that a quotient of T by H will exist if and only if the cohomology class of T^H is neutral, i.e., if and only if T^H is neutral as a tannakian category, in which case the quotients are classified by the elements of $H^1(k,G/H)$ (modulo $H^1(k,G)$). When T is neutral and we fix a k-valued fibre functor on it, then the elements of $H^1(k,G/H)$ classify the k-valued fibre functors on T^H. Thus, the cohomology theory suggests the above results, and in the next subsubsection we prove that a little more of this heuristic picture is correct.

The cohomology class of the quotient

For an affine group scheme G over a field k, $H^r(k,G)$ denotes the cohomology group computed with respect to the flat topology. When G is not commutative, this is defined only for $r = 0, 1, 2$ (Giraud 1971).

PROPOSITION 6.19 *Let (Q, q) be a quotient of T by a subgroup H of the centre of $\pi(\mathsf{T})$. Suppose that T is neutral, with k-valued fibre functor ω. Let $G = \mathcal{A}ut^{\otimes}(\omega)$, and let $\wp(\omega^q)$ be the $G/\omega(H)$-torsor $\mathcal{H}om(\omega|\mathsf{T}^H, \omega^q)$. Under the connecting homomorphism*

$$H^1(k,G/H) \to H^2(k,H)$$

the class of $\wp(\omega^q)$ in $H^1(k,G/H)$ maps to the class of Q in $H^2(k,H)$.

PROOF Note that $H = \operatorname{Bd}(\mathsf{Q})$, and so the statement makes sense. According to Giraud 1971, IV, 4.2.2, the connecting homomorphism sends the class of $\wp(\omega^q)$ to the class of the gerbe of liftings of $\wp(\omega^q)$, which can be identified with $(\omega^q \downarrow \text{FIB}(\mathsf{T}))$. Now Proposition 6.11 shows that the H-equivalence class of $(\omega^q \downarrow \text{FIB}(\mathsf{T}))$ equals that of $\text{FIB}(\mathsf{Q})$ which (by definition) is the cohomology class of Q. □

Semisimple normal quotients

Everything can be made more explicit when the categories are semisimple. Throughout this subsection, k has characteristic zero.

PROPOSITION 6.20 *Normal quotients of semisimple tannakian categories are semisimple.*

PROOF A tannakian category is semisimple if and only if the identity component of its fundamental group is pro-reductive (cf. 6.21), and the identity component of a normal subgroup of a reductive group is reductive (because its unipotent radical is a characteristic subgroup). □

Let T be a semisimple tannakian category over k, and let ω_0 be a k-valued fibre functor on a tannakian subcategory S of T. We can construct an explicit quotient T/ω_0 as follows. First, let $(\mathsf{T}/\omega_0)'$ be the category with one object $\bar{X}$ for each object X of T, and with

$$\mathrm{Hom}_{(\mathsf{T}/\omega_0)'}(\bar{X},\bar{Y}) = \omega_0(\mathcal{H}om(\bar{X},\bar{Y})^H),$$

where H is the subgroup of $\pi(\mathsf{T})$ defining S. There is a unique structure of a k-linear symmetric monoidal category on $(\mathsf{T}/\omega_0)'$ for which $q\colon \mathsf{T} \to (\mathsf{T}/\omega_0)'$ is a $\otimes$-functor. With this structure, $(\mathsf{T}/\omega_0)'$ is rigid, and we define T/ω_0 to be its pseudo-abelian hull. Thus, T/ω_0 has

$$\begin{aligned} \text{objects:}\quad & \text{pairs } (\bar{X},e) \text{ with } X \in \mathrm{ob}(\mathsf{T}) \text{ and } e \text{ an idempotent in } \mathrm{End}(\bar{X}),\\ \text{morphisms:}\quad & \mathrm{Hom}_{\mathsf{T}/\omega_0}((\bar{X},e),(\bar{Y},f)) = f\circ \mathrm{Hom}_{(\mathsf{T}/\omega_0)'}(\bar{X},\bar{Y})\circ e. \end{aligned}$$

Then $(\mathsf{T}/\omega_0, q)$ is a quotient of T by H, and $\omega^q \simeq \omega_0$.

Let ω be a fibre functor on T, and let a be an isomorphism $\omega_0 \to \omega|\mathsf{T}^H$. The pair (ω, a) defines a fibre functor ω_a on T/ω_0 whose action on objects is determined by

$$\omega_a(\bar{X}) = \omega(X)$$

and whose action on morphisms is determined by

$$\begin{array}{ccccc} \mathrm{Hom}(\bar{X},\bar{Y} & & \xrightarrow{\quad\omega_a\quad} & & \mathrm{Hom}(\omega_a(\bar{X}),\omega_a(\bar{Y})) \\ \| \text{def} & & & & \uparrow \\ \omega_0(\mathcal{H}om(X,Y)^H) & \xrightarrow{\ a\ } & \omega(\mathcal{H}om(X,Y)^H) & \xrightarrow{\ \simeq\ } & \mathcal{H}om(\omega(X),\omega(Y))^{\omega(H)} \end{array}$$

The map $(\omega, a) \mapsto \omega_a$ defines an equivalence $(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T})) \to \mathrm{FIB}(\mathsf{T}/\omega_0)$.

Let $H_1 \subset H_0 \subset \pi(\mathsf{T})$, and let ω_0 and ω_1 be k-valued fibre functors on T^{H_0} and T^{H_1} respectively. A morphism $\alpha\colon \omega_0 \to \omega_1|\mathsf{T}^{H_0}$ defines an exact $\otimes$-functor $\mathsf{T}/\omega_0 \to \mathsf{T}/\omega_1$ whose action on objects is determined by

$$\bar{X}\ (\text{in } \mathsf{T}^{H_0}) \ \mapsto \bar{X}\ (\text{in } \mathsf{T}^{H_1}),$$

and whose action on morphisms is determined by

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathsf{T}/\omega_0}(\bar{X},\bar{Y}) & & \longrightarrow & & \mathrm{Hom}_{\mathsf{T}/\omega_1}(\bar{X},\bar{Y}) \\ \| \text{def} & & & & \| \text{def} \\ \omega_0(\mathcal{H}om_{\mathsf{T}}(X,Y)^{H_0}) & \xrightarrow{\ \alpha\ } & \omega_1(\mathcal{H}om_{\mathsf{T}}(X,Y)^{H_0}) & \hookrightarrow & \omega_1(\mathcal{H}om_{\mathsf{T}}(X,Y)^{H_1})) \end{array}$$

When $H_1 = H_0$, this is an isomorphism (!) of symmetric monoidal categories $\mathsf{T}/\omega_0 \to \mathsf{T}/\omega_1$.

Let (Q_1, q_1) and (Q_2, q_2) be quotients of T by H. For simplicity, assume that $\pi \overset{\text{def}}{=} \pi(\mathsf{T})$ is commutative. Then $\mathcal{H}om(\omega^{q_1}, \omega^{q_2})$ is π/H-torsor, and we assume that it lifts to a π-torsor P in T, so $P \wedge^{\pi} (\pi/H) = \mathcal{H}om(\omega^{q_1}, \omega^{q_2})$. Then

$$\mathsf{T} \xrightarrow{X \mapsto P \wedge^{\pi} X} \mathsf{T} \xrightarrow{q_2} \mathsf{Q}_2$$

realizes Q_2 as a quotient of T by H, and the corresponding fibre functor on T^H is $P \wedge^{\pi} \omega^{q_2} \simeq \omega^{q_1}$. Therefore, there exists a commutative diagram of exact $\otimes$-functors

$$\begin{array}{ccc} \mathsf{T} & \xrightarrow{X \mapsto P \wedge^{\pi} X} & \mathsf{T} \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} \\ \mathsf{Q}_1 & \longrightarrow & \mathsf{Q}_2, \end{array}$$

which depends on the choice of P lifting $\mathcal{H}om(\omega^{q_1}, \omega^{q_2})$ in an obvious way.

EXERCISE 6.21 Re-express the theory of quotients in terms of (Galois) groupoids.

NOTES This section adapted from Milne 2007a.

Chapter VI

Polarizations

Consider an abelian variety A over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. For any prime number $\ell \neq p$, we have a finite-dimensional $\mathbb{Q}_\ell$-vector space $V_\ell A$, and, for any polarization of A, we have a Weil pairing $\varphi : V_\ell A \times V_\ell A \to \mathbb{Q}_\ell$. As $\mathbb{Q}_\ell$ is not a subfield of $\mathbb{R}$, it makes no sense to ask if φ is positive definite. However, φ defines an involution of the finite-dimensional $\mathbb{Q}$-subalgebra $\mathbb{Q}\operatorname{End}(A)$ of $\operatorname{End}_{\mathbb{Q}_\ell}(V_\ell A)$, and Weil proved that this involution is positive. Indeed, this is the key to his proof of the Riemann hypothesis for abelian varieties over finite fields. Grothendieck extended Weil's ideas to tannakian categories.

From another perspective, the category of motives (however defined) over a field is a tannakian category over $\mathbb{Q}$ equipped with, for each prime number l (including p), a canonical realization functor to a tannakian category over $\mathbb{Q}_l$. A polarization on the category plays the role of the missing realization at the infinite place.

For most of this chapter, C is a tannakian category over $\mathbb{R}$ and C' is its extension to $\mathbb{C}$, so $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$ (I, 6.12). Complex conjugation on $\mathbb{C}$ is denoted by ι or by $z \mapsto \bar{z}$. Sesquilinear forms are linear in the first variable.

1 Preliminaries

TERMINOLOGY 1.1 An additive map $f : V \to W$ of $\mathbb{C}$-vector spaces is ***semilinear*** if $f(zv) = \bar{z}f(v)$ for $z \in \mathbb{C}$ and $v \in V$. An additive functor $F : \mathsf{C}_1 \to \mathsf{C}_2$ of $\mathbb{C}$-linear categories is ***semilinear*** if $F(z_X) = \bar{z}_{FX}$, where z_X denotes the action of $z \in \mathbb{C}$ on X. A morphism of $\mathbb{C}$-schemes $\alpha : T \to S$ is ***semilinear*** if $f \mapsto f \circ \alpha : \Gamma(S, \mathcal{O}_S) \to \Gamma(T, \mathcal{O}_T)$ is semilinear as a map of $\mathbb{C}$-vector spaces.

Semisimplicity

1.2 Let A be a ring (not necessarily commutative). An ideal in A is ***nil*** if its elements are all nilpotent. Finitely generated nil ideals are nilpotent.

The ***Jacobson radical*** $R(A)$ of A is the intersection of the maximal left ideals in A. Equivalently it is the intersection of the annihilators of simple A-modules. It is a two-sided ideal in A. Every left (or right) nil ideal is contained in $R(A)$. For any ideal $\mathfrak{a}$ of A contained in $R(A)$, $R(A/\mathfrak{a}) = R(A)/\mathfrak{a}$. The radical of an artinian ring A is nilpotent, and it is the largest nilpotent two-sided ideal in A. A ring is said to be ***semisimple*** if its Jacobson radical is zero.

A finite product of matrix algebras over division algebras is semisimple and, conversely, every artinian semisimple ring is of the form (theorem of Wedderburn and Artin).

Recall I, 6.16, that an abelian category is said to be semisimple if every object is a finite direct sum of simple objects.

LEMMA 1.3 *Let $\alpha: M \to N$ be a nonzero morphism in an additive category, and let $A = \mathrm{End}(M \oplus N)$. If A is semisimple, then there exists a $\beta: N \to M$ such that $\beta\circ\alpha \neq 0$.*

PROOF Otherwise the nonzero left ideal $A\left(\begin{smallmatrix}0&0\\ \alpha&0\end{smallmatrix}\right)$ is contained $\left\{\left(\begin{smallmatrix}0&0\\ *&0\end{smallmatrix}\right)\right\}$, and hence is nil, contradicting the semisimplicity of A. □

PROPOSITION 1.4 *Let* A *be a pseudo-abelian category*[1] *whose objects are artinian. Then* A *is abelian semisimple if and only if the ring* $\mathrm{End}(X)$ *is semisimple for all X in* A.

PROOF If A is abelian semisimple, then every object X is a finite direct sum $X = \bigoplus_i m_i S_i$ of its isotypic subobjects $m_i S_i$: this means that each object S_i is simple, and S_i is not isomorphic to S_j if $i \neq j$. Because S_i is simple, $\mathrm{End}(S_i)$ is a division algebra, and because $\mathrm{End}(X) \simeq \prod_i M_{m_i}(\mathrm{End}(S_i))$, it is semisimple.

Conversely, suppose that the endomorphism rings are semisimple, and let N be a nonzero object of A. If N is not simple, then it properly contains a nonzero subobject S, which we may suppose to be minimal, hence simple. Let α be the inclusion $S \hookrightarrow N$. As $\mathrm{End}(S \oplus N)$ is semisimple, there exists a $\beta: N \to S$ such that $\beta\circ\alpha \neq 0$. As S is simple, $\beta\circ\alpha$ is an isomorphism, and we may modify β so that $\beta\circ\alpha = \mathrm{id}_S$. Now $\alpha\circ\beta$ is an idempotent endomorphism of N, which decomposes it into a direct sum $N = S \oplus N'$. If N' is not simple, then we continue. Because N is artinian, the process ends with N a finite direct sum of simple objects. As this holds for all objects, it follows easily that A is abelian. □

We next prove a criterion for an algebra to be semisimple. Let F be a field.

DEFINITION 1.5 An ***involution*** of an F-algebra A is an F-linear map $*: A \to A$ such that

$$1^* = 1, \quad (ab)^* = b^*a^*, \quad a^{**} = a, \quad \text{for all } a, b \in A.$$

PROPOSITION 1.6 *Let A be an F-algebra and $u \mapsto u'$ an involution of A. If there exists an F-linear (trace) map $\sigma: A \to F$ such that $\sigma(uv) = \sigma(vu)$ for all $u, v \in A$ and $\sigma(u'u) \neq 0$ when $u \neq 0$, then A is semisimple.*

PROOF Let u be a nonzero element of the radical of A. Then $v \overset{\text{def}}{=} u'u$ again lies in the radical of A and it is nonzero because $\sigma(u'u) \neq 0$. Now

$$\begin{aligned} v \neq 0,\ v' = v &\Rightarrow v^2 = v'v \neq 0,\\ v^2 \neq 0,\ v' = v &\Rightarrow v^4 = (v^2)'(v^2) \neq 0,\\ &\cdots \end{aligned}$$

contradicting the nilpotence of v. □

NOTES Proposition 1.4 was extracted from Harada 1970. Some finiteness condition is needed in the proposition: in the category of vector spaces modulo finite-dimensional vector spaces, every monomorphism splits, but there are no simple objects.

[1] In particular, additive.

Positive involutions

Let F be a subfield of $\mathbb{R}$, and let A be a finite-dimensional F-algebra (not necessarily commutative).

1.7 An ***involution*** of A is said to be ***positive*** if $\mathrm{Tr}_{A/F}(a^*a) > 0$ for all nonzero $a \in A$.

Note that an involution on A over F is positive if it becomes positive on $A \otimes \mathbb{R}$ (because it suffices to check the condition on the elements of a basis for A over F).

PROPOSITION 1.8 *If A admits a positive involution, then it is semisimple.*

PROOF The trace map $\mathrm{Tr}_{A/F} : A \to F$ satisfies the hypotheses of Proposition 1.6. □

Let $*$ be an involution on A, and let V be an A-module. An F-bilinear form $\psi : V \times V \to F$ is said to be ***balanced*** if

$$\psi(au, v) = \psi(u, a^*v) \text{ for all } a \in A \text{ and } u, v \in V.$$

A ***hermitian form*** on V is a balanced symmetric F-bilinear form. For example, if $F = \mathbb{R}$, $A = \mathbb{C}$, and $*$ is complex conjugation, such a form can be written uniquely as $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \circ \phi$ with $\phi : V \times V \to \mathbb{C}$ a hermitian form in the usual sense. A hermitian form ψ is positive definite if $\psi(v, v) > 0$ for all nonzero $v \in V$.

PROPOSITION 1.9 *Assume that A is semisimple. The following conditions on an involution $*$ of A are equivalent:*

(a) *some faithful A-module admits a positive-definite hermitian form;*

(b) *every A-module admits a positive-definite hermitian form;*

(c) *the involution $*$ is positive.*

PROOF We may suppose that $F = \mathbb{R}$.

(a)⇒(b). Let V be a faithful A-module. Every simple A-module occurs as a direct summand of V, and so every A-module occurs as a direct summand of a direct sum of copies of V. Hence, if V carries a positive-definite hermitian form, then so does every A-module.

(b)⇒(c). Let V be a A-module with a positive-definite hermitian form $(\ |\ )$, and choose an orthonormal F-basis $e_1, \dots, e_n$ for V. Then

$$\mathrm{Tr}_F(a^*a|V) = \textstyle\sum_i (e_i|a^*ae_i) = \sum_i (ae_i|ae_i),$$

which is > 0 unless a acts as the zero map on V. On applying this remark with $V = A$, we obtain (c).

(c)⇒(a). Condition (c) says that the hermitian form $(a, b) \mapsto \mathrm{Tr}_{A/F}(a^*b)$ on the (faithful) A-module A is positive definite. □

An element of a finite-dimensional semisimple $\mathbb{R}$-algebra A is said to be ***totally positive*** if the roots of its characteristic polynomial P_α are all > 0. This condition is equivalent to α being invertible in A and a square in $\mathbb{R}[\alpha]$.

Real algebraic groups

1.10 Let G be an algebraic group over $\mathbb{R}$. Recall (II, 9.7) that G is said to be compact if $G(\mathbb{R})$ is compact and every connected component of G has a real point. Then $G(\mathbb{R})$ is Zariski dense in G and the functor

$$\mathsf{Repf}_{\mathbb{R}}(G) \to \mathsf{Repf}_{\mathbb{R}}(G(\mathbb{R}))$$

sending a representation of G to the corresponding continuous representation of $G(\mathbb{R})$ is an equivalence of categories.

1.11 Let G be an algebraic group over $\mathbb{R}$. If G is compact, then so also is any algebraic subgroup of G (Deligne 1972, 2.5).

1.12 Let G be an algebraic group over $\mathbb{R}$. If G is compact, then every finite-dimensional real representation of $G \to \mathrm{GL}(V)$ carries a G-invariant positive-definite symmetric bilinear form. Conversely, if one faithful finite-dimensional real representation of G carries such a form, then G is compact. Indeed, G is then an algebraic subgroup of an orthogonal group, which is compact, so we can apply 1.11.

1.13 We can restate 1.12 for complex representations of G. If G is compact, then every finite-dimensional complex representation of G carries a G-invariant positive-definite hermitian form.[2] Conversely, if some faithful finite-dimensional complex representation of G carries a G-invariant positive-definite hermitian form, then G is compact. Indeed, G is then an algebraic subgroup of a unitary group (which is a compact algebraic group over $\mathbb{R}$).

Cartan involutions and compact real forms

1.14 Let G be an algebraic group over $\mathbb{R}$, and let $g \mapsto \bar{g}$ denote complex conjugation on $G(\mathbb{C})$. Let θ be an involution of G (as an algebraic group over $\mathbb{R}$). There is a unique real form $G^{(\theta)}$ of G such that complex conjugation on $G^{(\theta)}(\mathbb{C})$ is $g \mapsto \theta(\bar{g})$. An involution is said to be ***Cartan*** if $G^{(\theta)}$ is compact (in the sense of 1.10). The compact real forms of G correspond exactly to the Cartan involutions.

1.15 Let G be an algebraic group over $\mathbb{R}$. There exists a Cartan involution of G if and only if G° is reductive, in which case, any two are conjugate by an element of $G^\circ(\mathbb{R})$.

1.16 Let $G = \mathrm{GL}_V$ with V a finite-dimensional real vector space. The choice of a basis for V determines a transpose operator $M \mapsto M^\intercal$, and $M \mapsto (M^\intercal)^{-1}$ is obviously a Cartan involution. Statement 1.15 implies that all Cartan involutions of G arise in this way.

1.17 Let G be a connected algebraic group over $\mathbb{R}$ and $G \to \mathrm{GL}_V$ a faithful representation of G. Then G is reductive if and only if G is stable under $g \mapsto g^\intercal$ for some choice of a basis for V, in which case $g \mapsto (g^\intercal)^{-1}$ is a Cartan involution of G; all Cartan involutions of G arise in this way from the choice of a basis for V (Satake 1980, I, 4.4).

[2] A sesquilinear form φ is said to be G-invariant if $\varphi(gu, \bar{g}v) = \varphi(u, v)$, all $g \in G(\mathbb{C})$, $u, v \in V$, i.e., if φ is G-invariant when viewed as a map $V \otimes \bar{V} \to \mathbb{C}$.

Maximal compact subgroups

1.18 Let G be an algebraic group over $\mathbb{C}$.

(a) Any two maximal compact subgroups of $G(\mathbb{C})$ are conjugate (Hochschild 1965, XV, 3.1).

(b) If G° is reductive, then the maximal compact subgroups K of $G(\mathbb{C})$ are the subgroups $G_0(\mathbb{R})$ with G_0 a compact real form of G (Springer 1979).

Cartan involutions and C-polarizations

1.19 Let G be an algebraic group over $\mathbb{R}$, and let C be an element of $G(\mathbb{R})$ whose square is central (so that $\mathrm{ad}(C)$ is an involution). A ***C-polarization*** on a real representation V of G is a G-invariant bilinear form φ such that the form φ_C,

$$(u, v) \mapsto \varphi(u, Cv),$$

is symmetric and positive definite.

Proposition 1.20 *Let G and C be as above. If $\mathrm{ad}(C)$ is a Cartan involution of G, then every finite-dimensional real representation of G carries a C-polarization; conversely, if one faithful finite-dimensional real representation of G carries a C-polarization, then $\mathrm{ad}(C)$ is a Cartan involution.*

Proof We first remark that an $\mathbb{R}$-bilinear form φ on a real vector space V extends to a sesquilinear form φ' on $V(\mathbb{C})$, namely,

$$\varphi' : V(\mathbb{C}) \times V(\mathbb{C}) \to \mathbb{C}, \quad \text{where } \varphi'(u, v) = \varphi_{\mathbb{C}}(u, \bar{v}).$$

Moreover, φ' is hermitian (resp. hermitian and positive definite) if and only if φ is symmetric (resp. symmetric and positive definite).

Let $\rho : G \to \mathrm{GL}(V)$ be a real representation of G. For any G-invariant bilinear form φ on V, $\varphi_{\mathbb{C}}$ is $G(\mathbb{C})$-invariant, and so

$$\varphi'(gu, \bar{g}v) = \varphi'(u, v), \quad \text{all } g \in G(\mathbb{C}), \quad u, v \in V(\mathbb{C}). \tag{102}$$

On replacing v with Cv in this equality, we find that

$$\varphi'(gu, C(C^{-1}\bar{g}C)v) = \varphi'(u, Cv), \quad \text{all } g \in G(\mathbb{C}), \quad u, v \in V(\mathbb{C}). \tag{103}$$

This can be rewritten as

$$\varphi'_C(gu, ((\mathrm{ad}\, C)\bar{g})v) = \varphi'_C(u, v),$$

where $\varphi'_C = (\varphi_C)'$. This last equation says that φ'_C is invariant under $G^{(\mathrm{ad}\, C)}$.

If ρ is faithful and φ is a C-polarization, then φ'_C is a positive-definite hermitian form, and so $G^{(\mathrm{ad}\, C)}(\mathbb{R})$ is compact (1.13). Thus, $\mathrm{ad}\, C$ is a Cartan involution.

Conversely, if $G^{(\mathrm{ad}\, C)}(\mathbb{R})$ is compact, then every real representation $G \to \mathrm{GL}(V)$ carries a $G^{(\mathrm{ad}\, C)}(\mathbb{R})$-invariant positive-definite symmetric bilinear form φ (1.12). Similar calculations to the above show that $\varphi_{C^{-1}}$ is a C-polarization on V. □

1.21 Let G be an algebraic group over $\mathbb{Q}$, and let C be an element of $G(\mathbb{R})$ whose square is central. A C-***polarization*** on a $\mathbb{Q}$-representation V of G is a G-invariant bilinear form $\varphi : V \times V \to \mathbb{Q}$ such that $\varphi_{\mathbb{R}}$ is a C-polarization on $V_{\mathbb{R}}$. In order to show that a $\mathbb{Q}$-representation V of G is polarizable, it suffices to check that $V_{\mathbb{R}}$ is polarizable. We prove this when C^2 acts as $+1$ or -1 on V, which are the only cases we shall need. Let $P(\mathbb{Q})$ (resp. $P(\mathbb{R})$) denote the space of G-invariant bilinear forms on V (resp. on $V_{\mathbb{R}}$) that are symmetric when C^2 acts as $+1$ or skew-symmetric when it acts as -1. Then $P(\mathbb{R}) = P(\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R})$. The C-polarizations of $V_{\mathbb{R}}$ form an open subset of $P(\mathbb{R})$, whose intersection with $P(\mathbb{Q})$ consists of the C-polarizations of V.

2 Tensor categories over $\mathbb{R}$

2.1 Let C be a tensor category over $\mathbb{R}$, and let $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Recall (I, 6.12) that an object of C' is an object X of C together with an $\mathbb{R}$-linear action $\alpha : \mathbb{C} \to \mathrm{End}(X)$ of $\mathbb{C}$. For such an object X, we let $\bar{X}$ denote the same object but with the complex conjugate action. In this way, we get a semilinear $\otimes$-functor $X \rightsquigarrow \bar{X} : \mathsf{C}' \to \mathsf{C}'$, and a canonical $\otimes$-isomorphism $\mu_X : X \to \bar{\bar{X}}$ such that

$$\mu_{\bar{X}} = \overline{\mu_X}. \tag{104}$$

The category C can be recovered from the triple $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ as the collection of pairs (X, a) with X an object of C' and $a : X \to \bar{X}$ an isomorphism such that $\bar{a} \circ a = \mu_X$ (i.e., a descent datum on X). Every triple $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ satisfying (104) arises in this way from a tensor category over $\mathbb{R}$. Recall (IV, 10.1) that if C is tannakian, then it admits a $\mathbb{C}$-valued fibre functor.

2.2 More formally, a $\mathbb{C}/\mathbb{R}$-descent datum on a tensor category C' over $\mathbb{C}$ is a semilinear $\otimes$-functor $J : \mathsf{C}' \to \mathsf{C}'$ together with a natural isomorphism $\mu_X : X \to J^2X$ such that $\mu_{JX} = J(\mu_X)$. If C is a tensor category over $\mathbb{R}$, then $\mathsf{C}_{(\mathbb{C})}$ is equipped with a $\mathbb{C}/\mathbb{R}$-descent datum,

$$J(X, \alpha) = (X, \alpha \circ \iota)$$
$$\mu_X : (X, \alpha) \to J^2(X, \alpha) \quad \text{is the identity map.}$$

Conversely, a tensor category C' over $\mathbb{C}$ equipped with a $\mathbb{C}/\mathbb{R}$-descent datum gives rise to a tensor category $(\mathsf{C}')_0$ over $\mathbb{R}$ whose objects are the pairs (X, a) such that X is an object of C' and $a : X \to J(X)$ is an isomorphism such that $J(a) \circ a = \mu_X$.

These operations are inverse in the sense that there are canonical equivalences of tensor categories

$$\mathsf{C} \to (\mathsf{C}_{(k')})_0, \quad (\mathsf{C}'_0)_{(k')} \to \mathsf{C}'.$$

Therefore, to give a tensor category over $\mathbb{R}$ amounts to giving a tensor category over $\mathbb{C}$ together with a $\mathbb{C}/\mathbb{R}$ descent datum (cf. I, 6.21).

2.3 Let C be a tensor category over $\mathbb{R}$, and assume that there is a $\mathbb{C}$-valued fibre functor ω on C. Then $\mathcal{G}(\omega) \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$ is $\mathbb{C}/\mathbb{R}$-Galois groupoid,

$$1 \to G(\mathbb{C}) \to \mathcal{G}(\omega) \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1,$$

with $G = \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$. The functor $\mathsf{C} \to \mathsf{Repf}(\mathcal{G}(\omega))$ defined by ω is an equivalence of tensor categories over k (III, 3.6). Choose an $s \in \mathcal{G}(\omega)$ mapping to $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, let $\sigma : G(\mathbb{C}) \to G(\mathbb{C})$ be conjugation by s, and let $c = s^2 \in G(\mathbb{C})$. Then σ is semilinear and

$$\sigma^2 = \mathrm{ad}(c), \quad \sigma(c) = c. \tag{105}$$

If s is replaced by sm, then (σ, c) is replaced by $(\sigma \circ \mathrm{ad}(m), \sigma(m)cm)$.

Conversely, let G be an affine group scheme over $\mathbb{C}$. A pair (σ, c) satisfying (105) with σ semilinear provides a descent datum on $\mathsf{Repf}(G)$: given a representation of G on V, define a representation of G on $\bar{V}$ by the rule $\overline{gv} = \sigma(g)\bar{v}$, and define μ_V to be the map $cv \mapsto \bar{v} : V \xrightarrow{\simeq} \bar{\bar{V}}$.

The operations in the last two paragraphs are inverse.

In summary, we have a diagram,

$$\begin{array}{ccc} \mathsf{C}\ (\text{over } \mathbb{R}) & \xleftarrow{\quad 2.1 \quad} & (\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X) \\ \Big\updownarrow \text{III,3.6} & & \Big\updownarrow 2.3 \\ \mathcal{G} = \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega) & \xleftarrow{\text{III,4.1}} & (G = \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega), \sigma, c). \end{array}$$

To pass from the top row to the bottom row, choose a fibre functor ω over $\mathbb{C}$.

3 Bilinear and sesquilinear forms

We review some definitions and formulas concerning bilinear and sesquilinear forms on a tensor category. When the category is tannakian, the formulas can be proved by applying a fibre functor. Proving them for tensor categories may require drawing diagrams. See Saavedra 1972, V, 2.1, 2.2, for the details.

Bilinear forms in tensor categories.

3.1 Let C be a tensor category over a field k (for example $\mathbb{R}$), and let T be an invertible object of C. A ***bilinear form with values in*** T is a morphism

$$\phi : X \otimes X \to T.$$

Let $\phi^\sim$ and ${}^\sim\phi$ be the morphisms $X \to X^\vee \otimes T$ corresponding to ϕ under the canonical isomorphisms p. 18

$$\begin{array}{rl} & \mathrm{Hom}(X \otimes X, T) \simeq \mathrm{Hom}(X, \mathcal{H}om(X, T)) \simeq \mathrm{Hom}(X, X^\vee \otimes T) \\ \mathrm{Hom}(X \otimes X, T) & \overset{\gamma_{X,X}}{\simeq} \mathrm{Hom}(X \otimes X, T) \simeq \mathrm{Hom}(X, \mathcal{H}om(X, T)) \simeq \mathrm{Hom}(X, X^\vee \otimes T), \end{array}$$

so

$$\begin{cases} \phi^\sim(x)(y) = \phi(x, y) \\ {}^\sim\phi(x)(y) = \phi(y, x). \end{cases}$$

The form ϕ is said to be ***nondegenerate*** if $\phi^\sim$ (equivalently ${}^\sim\phi$) is an isomorphism. The ***parity*** of a nondegenerate bilinear form ϕ is the unique morphism $\varepsilon_\phi : X \to X$ such that

$$\begin{cases} \phi^\sim = {}^\sim\phi \circ \varepsilon_\phi \\ \phi(x, y) = \phi(y, \varepsilon_\phi x). \end{cases}$$

Then

$$\phi(\varepsilon_\phi x, \varepsilon_\phi y) = \phi(x, y).$$

The ***transpose*** (or ***adjoint***) u^ϕ of $u \in \mathrm{End}(X)$ relative to ϕ is determined by

$$\begin{cases} \phi\circ(u \otimes \mathrm{id}_X) = \phi\circ(\mathrm{id}_X \otimes u^\phi) \\ \phi(ux, y) = \phi(x, u^\phi y). \end{cases}$$

There are the formulas[3]

$$(uv)^\phi = v^\phi u^\phi, \quad (\mathrm{id}_X)^\phi = \mathrm{id}_X, \quad (u^\phi)^\phi = \varepsilon_\phi u \varepsilon_\phi^{-1}, \quad (\varepsilon_\phi)^\phi = \varepsilon_\phi^{-1},$$

so $u \mapsto u^\phi$ is a bijective k-linear antihomomorphism $\mathrm{End}(X) \to \mathrm{End}(X)$.

3.2 If ϕ is a nondegenerate bilinear form on X, then any other nondegenerate bilinear form can be written

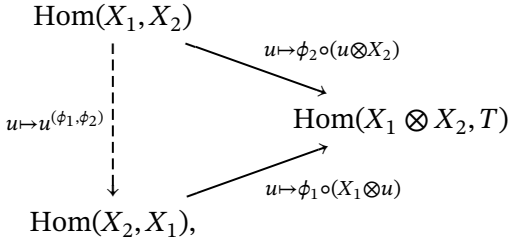
$$\begin{cases} \phi_\alpha = \phi\circ(\alpha \otimes \mathrm{id}) \\ \phi_\alpha(x, y) = \phi(\alpha x, y) \end{cases}$$

for a uniquely determined automorphism α of X. There are formulas

$$u^{\phi_\alpha} = (\alpha u \alpha^{-1})^\phi, \quad \varepsilon_{\phi_\alpha} = (\alpha^\phi)^{-1} \varepsilon_\phi \alpha.$$

Therefore, when ε_ϕ is in the centre of $\mathrm{End}(X)$, ϕ_α has the same parity as ϕ if and only if $\alpha^\phi = \alpha$.

3.3 Let $\phi_1 : X_1 \otimes X_1 \to T$ and $\phi_2 : X_2 \otimes X_2 \to T$ be nondegenerate bilinear forms. Then the sloping arrows in the following diagram are isomorphisms, and we define the vertical arrow to make the diagram commute,



so

$$\phi_2(ux_1, x_2) = \phi_1(x_1, u^{(\phi_1,\phi_2)} x_2), \quad u : X_1 \to X_2, \quad x_1 \in X_1,\ x_2 \in X_2.$$

3.4 Let (U, u) be a unit object, and let $\phi_1 : X_1 \otimes X_1 \to U$ and $\phi_2 : X_2 \otimes X_2 \to U$ be bilinear forms. Define the bilinear pairing $\phi_1 \otimes \phi_2$ on $X_1 \otimes X_2$ to be the composite

$$(X_1 \otimes X_2) \otimes (X_1 \otimes X_2) \simeq (X_1 \otimes X_1) \otimes (X_2 \otimes X_2) \xrightarrow{\phi_1 \otimes \phi_2} U \otimes U \xrightarrow{u} U.$$

[3] For example,

$$\begin{aligned} \phi(uvx, y) &= \phi(vx, u^\phi y) = \phi(x, v^\phi u^\phi y), = \phi(x, (uv)^\phi y) \\ \phi(ux, y) &= \phi(x, u^\phi y) = \phi(u^\phi y, \epsilon x) = \phi(y, u^{\phi\phi}\epsilon x) = \phi(u^{\phi\phi}\epsilon x, \epsilon y) = \phi(\epsilon^{-1} u^{\phi\phi} \epsilon x, y) \\ \phi(\epsilon x, y) &= \phi(x, \epsilon^\phi y), = \phi(x, \epsilon^{-1} y). \end{aligned}$$

3.5 Let $\phi_1 : X_1 \otimes X_1 \to T$ and $\phi_2 : X_2 \otimes X_2 \to T$ be bilinear forms. Define the bilinear form $\phi = \phi_1 \oplus \phi_2$ on $X = X_1 \oplus X_2$ to be the composite

$$X \otimes X \to X_1 \otimes X_1 \oplus X_2 \otimes X_2 \xrightarrow{\phi_1+\phi_2} T.$$

The pairing ϕ is nondegenerate if and only if ϕ_1 and ϕ_2 are nondegenerate. Assume this, and let $u \in \mathrm{End}(V)$. If $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$, then $u^\phi = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, where $v_{ji} = (u_{ij})^{(\phi_j,\phi_i)}$.

3.6 Let $\phi : X \otimes Y \to \mathbb{1}$ be a nondegenerate bilinear pairing, and let ϕ' denote the pairing $\phi \circ \gamma_{X,Y} : Y \otimes X \to \mathbb{1}$. Under the canonical isomorphisms

$$\mathrm{Hom}(Y \otimes X, \mathbb{1}) \simeq \mathrm{Hom}(Y, \mathcal{H}om(X, \mathbb{1})) \simeq \mathrm{Hom}(Y, X^\vee)$$

ϕ' corresponds to an isomorphism $Y \to X^\vee$. We make this more explicit. Consider the pairings

$$(\phi \otimes \phi) : (X \otimes Y) \otimes (X \otimes Y) \to \mathbb{1}$$
$$e : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1},$$

and let $\phi^* = \phi^{(\phi \otimes \phi, e)}$. Then Y is dual to X with evaluation morphism $\phi' : Y \otimes X \to \mathbb{1}$ and coevaluation morphism $\phi^* : \mathbb{1} \to X \otimes Y$.

Sesquilinear forms on vector spaces

3.7 A ***sesquilinear*** form on a complex vector space V is a biadditive mapping

$$\phi : V \times V \to \mathbb{C}$$

such that

$$\phi(ax, by) = a\bar{b}\phi(x, y) \text{ for } x, y \in V, a, b \in \mathbb{C}.$$

The form is ***nondegenerate*** if the mapping

$$x \mapsto \phi(x, -) : V \to \bar{V}^\vee$$

is an isomorphism. Then, the ***transpose*** u^ϕ of an endomorphism u of V relative to ϕ is the unique endomorphism u^ϕ such that

$$\phi(ux, y) = \phi(x, \overline{u^\phi} y), \quad \text{all } x, y \in V.$$

We transfer these definitions to a tensor category. Note that we can regard ϕ as a $\mathbb{C}$-bilinear map $V \times \bar{V} \to \mathbb{C}$, and hence as a $\mathbb{C}$-linear map $V \otimes_{\mathbb{C}} \bar{V} \to \mathbb{C}$.

Sesquilinear forms in tensor categories

Let C be tensor category over $\mathbb{R}$ and $(\mathsf{C}', X \mapsto \bar{X}, \mu_X)$ the associated triple (2.1).

3.8 Let $(\mathbb{1}, e)$, $e : \mathbb{1} \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$, be a unit object for C'. Then $(\overline{\mathbb{1}}, \bar{e})$ is again a unit object, and the unique isomorphism of unit objects $a : (\mathbb{1}, e) \to (\overline{\mathbb{1}}, \bar{e})$ is a descent datum. We use it to identify $\overline{\mathbb{1}}$ with $\mathbb{1}$.

3.9 A ***sesquilinear form*** on an object X of C' is a morphism

$$\phi : X \otimes \bar{X} \to \mathbb{1}.$$

On applying $-$, we obtain a morphism $\bar{X} \otimes \bar{\bar{X}} \to \overline{\mathbb{1}}$, which can be identified (using μ_X) with a morphism

$$\bar{\phi} : \bar{X} \otimes X \to \mathbb{1}.$$

Let $\phi^\sim$ and ${}^\sim\phi$ be the morphisms $X \to \bar{X}^\vee$ corresponding to ϕ and $\bar{\phi}$ under the canonical isomorphisms

$$\begin{aligned} \operatorname{Hom}(X \otimes \bar{X}, \mathbb{1}) &\simeq \operatorname{Hom}(X, \mathcal{H}om(\bar{X}, \mathbb{1})) \simeq \operatorname{Hom}(X, \bar{X}^\vee) \\ \operatorname{Hom}(\bar{X} \otimes X, \mathbb{1}) &\simeq \operatorname{Hom}(X \otimes \bar{X}, \mathbb{1}) \simeq \operatorname{Hom}(X, \mathcal{H}om(\bar{X}, \mathbb{1})) \simeq \operatorname{Hom}(X, \bar{X}^\vee), \end{aligned}$$

so

$$\begin{cases} \phi^\sim(x)(y) = \phi(x \otimes y) \\ {}^\sim\phi(x)(y) = \bar{\phi}(y \otimes x) \end{cases} \tag{106}$$

The form ϕ is said to be ***nondegenerate*** if $\phi^\sim$ (equivalently ${}^\sim\phi$) is an isomorphism. The ***parity*** of a nondegenerate sesquilinear form ϕ is the unique morphism $\varepsilon_\phi : X \to X$ such that

$$\begin{cases} \phi^\sim = {}^\sim\phi \circ \varepsilon_\phi; \\ \phi(x, y) = \bar{\phi}(y, \varepsilon_\phi x). \end{cases} \tag{107}$$

Note that

$$\begin{cases} \phi \circ (\varepsilon_\phi \otimes \bar{\varepsilon}_\phi) = \phi; \\ \phi(\varepsilon_\phi x, \bar{\varepsilon}_\phi y) = \phi(x, y) \end{cases} \tag{108}$$

The ***transpose*** u^ϕ of $u \in \operatorname{End}(X)$ relative to ϕ is determined by

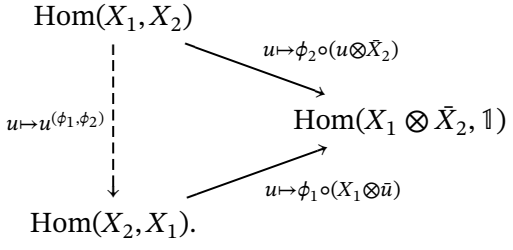
$$\begin{cases} \phi \circ (u \otimes \mathrm{id}_{\bar{X}}) = \phi \circ (\mathrm{id}_X \otimes \overline{u^\phi}); \\ \phi(ux, y) = \phi(x, \overline{u^\phi} y). \end{cases} \tag{109}$$

There are formulas

$$(uv)^\phi = v^\phi u^\phi, \quad (\mathrm{id}_X)^\phi = \mathrm{id}_X, \quad (u^\phi)^\phi = \varepsilon_\phi u \varepsilon_\phi^{-1}, \quad (\varepsilon_\phi)^\phi = \varepsilon_\phi^{-1} \tag{110}$$

and so $u \mapsto u^\phi$ is a semilinear bijective antihomomorphism $\operatorname{End}(X) \to \operatorname{End}(X)$.

3.10 Let $\phi_1 : X_1 \otimes \bar{X}_1 \to \mathbb{1}$ and $\phi_2 : X_2 \otimes \bar{X}_2 \to \mathbb{1}$ be nondegenerate sesquilinear forms. The sloping arrows in the following diagram are isomorphisms, and we define the vertical arrow to make the diagram commute,



3.11 Let ϕ and ψ be sesquilinear forms on X and Y. Define the sesquilinear form $\phi \otimes \psi$ on $X \otimes Y$ to be the composite

$$X \otimes Y \otimes \bar{X} \otimes \bar{Y} \xrightarrow{X \otimes \gamma_{Y,\bar{X}} \otimes Y} X \otimes \bar{X} \otimes Y \otimes \bar{Y} \xrightarrow{\phi \otimes \psi} \mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1}.$$

3.12 If ϕ is a nondegenerate sesquilinear form on X, then any other nondegenerate sesquilinear form can be written

$$\phi_\alpha = \phi\circ(\alpha\otimes \mathrm{id}), \quad \phi_\alpha(x,y) = \phi(\alpha x, y) = \phi(x, \overline{\alpha^\phi} y) \tag{111}$$

for a uniquely determined automorphism α of X. There are the formulas

$$u^{\phi_\alpha} = (\alpha u \alpha^{-1})^\phi, \quad \varepsilon_{\phi_\alpha} = (\alpha^\phi)^{-1}\varepsilon_\phi \alpha. \tag{112}$$

Therefore, when ε_ϕ is in the centre of $\mathrm{End}(X)$, ϕ_α has the same parity as ϕ if and only if $\alpha^\phi = \alpha$.

Variant

3.13 In an alternative notation, a sesquilinear form is a morphism $\phi : \bar{X}\otimes X \to \mathbb{1}$. With this convention, $(\bar{X},\phi)$ is a dual of V if ϕ is nondegenerate. Under the canonical isomorphisms

$$\mathrm{Hom}(\bar{X}\otimes X, \mathbb{1}) \simeq \mathrm{Hom}(\bar{X}, \mathcal{H}om(X,\mathbb{1})) \simeq \mathrm{Hom}(\bar{X}, X^\vee),$$

ϕ corresponds to an isomorphism $\bar{X}\to X^\vee$.

4 Weil forms

Let C be a tensor category over $\mathbb{R}$ and $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ its extension to $\mathbb{C}$ (as in 2.1). Let X be an object of C'. Then $\mathrm{End}(X)$ is a finite-dimensional $\mathbb{C}$-algebra.

DEFINITION 4.1 A nondegenerate sesquilinear form $\phi : X\otimes\bar{X}\to\mathbb{1}$ is a ***Weil form*** if

(a) its parity ε_ϕ lies in the centre of $\mathrm{End}(X)$ and

(b) $\mathrm{Tr}_X(u^\phi\circ u) > 0$ for all nonzero u in $\mathrm{End}(X)$.

PROPOSITION 4.2 *Let ϕ be a Weil form on X.*

(a) *The map $u\mapsto u^\phi$ is an involution of* $\mathrm{End}(X)$ *inducing complex conjugation on $\mathbb{C} = \mathbb{C}\cdot\mathrm{id}_X$, and $(u,v)\mapsto \mathrm{Tr}(u^\phi\circ v|X)$ is a positive-definite hermitian form on* $\mathrm{End}(X)$.

(b) $\mathrm{End}(X)$ *is a semisimple $\mathbb{C}$-algebra.*

(c) *Any commutative sub-$\mathbb{R}$-algebra A of* $\mathrm{End}(X)$ *composed of symmetric elements (i.e., elements such that $u^\phi = u$) is a product of copies of $\mathbb{R}$.*

PROOF (a) Condition (a) says that $u\mapsto u^\phi$ is an involution (see (110)) and condition (b) says that the hermitian form is positive definite.

(b) The map $u\mapsto u^\phi$ is a positive involution, and so we can apply 1.8.

(c) As A admits a positive involution, it is semisimple, and therefore a product of fields. Moreover, for any $u\in A$, $\mathrm{Tr}_X(u^2) = \mathrm{Tr}_X(u^\phi u) > 0$. If $\mathbb{C}$ occurs as a factor of A, then $\mathrm{Tr}_X|\mathbb{C}$ is a multiple of the identity map, which contradicts $\mathrm{Tr}_X(u^2) > 0$. □

DEFINITION 4.3 Two Weil forms, ϕ on X and ψ on Y, are said to be ***compatible*** if the sesquilinear form $\phi\oplus\psi$ on $X\oplus Y$ is again a Weil form.

REMARK 4.4 Let ϕ and ψ be Weil forms on X and Y respectively, and consider

$$\mathrm{End}(X \oplus Y) = \mathrm{End}(X) \oplus \mathrm{Hom}(X,Y) \oplus \mathrm{Hom}(Y,X) \oplus \mathrm{End}(Y).$$

The transpose involution on $\mathrm{End}(X \oplus Y)$ defined by $\phi \oplus \psi$ is $u \mapsto u^{\phi}$ on $\mathrm{End}(X)$, $u \mapsto u^{\psi}$ on $\mathrm{End}(Y)$ and defines semilinear isomorphisms

$$\begin{aligned} u &\mapsto u^{(\phi,\psi)} : \ \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X) \\ u &\mapsto u^{(\psi,\phi)} : \ \mathrm{Hom}(Y,X) \to \mathrm{Hom}(X,Y). \end{aligned}$$

$\psi(ux,y) = \phi(x,\overline{u^*}y)$. The pairing $\phi \oplus \psi$ is a Weil form (so ϕ and ψ are compatible) if and only if

$$\begin{aligned} &\mathrm{Tr}(u^{(\psi,\phi)} \circ u | X) > 0 \quad \text{all nonzero } u \in \mathrm{Hom}(X,Y) \\ &\mathrm{Tr}(u^{(\phi,\psi)} \circ u | Y) > 0 \quad \text{all nonzero } u \in \mathrm{Hom}(Y,X). \end{aligned}$$

In particular, if $\mathrm{Hom}(X,Y) = 0 = \mathrm{Hom}(Y,X)$, then ϕ and ψ are automatically compatible.

PROPOSITION 4.5 *Let ϕ be a Weil form on X, and let $\phi_\alpha = \phi \circ (\alpha \otimes \mathrm{id}_X)$ for some $\alpha \in \mathrm{Aut}(X)$.*

(a) *The sesquilinear form ϕ_α has the same parity as ϕ if and only if α is symmetric, i.e., $\alpha^{\phi} = \alpha$.*

(b) *Assume that α is symmetric. Then ϕ_α is a Weil form if and only if α is a square in $\mathbb{R}[\alpha] \subset \mathrm{End}(X)$.*

(c) *If ϕ_α is a Weil form with the same parity as ϕ, then ϕ_α is compatible with ϕ.*

(d) *For any Weil form ϕ on X, the map $\alpha \mapsto \phi_\alpha$ defines a one-to-one correspondence between the set of totally positive symmetric endomorphisms of X and the set of Weil forms on X that have the same parity as ϕ and are compatible with ϕ.*

PROOF (a) According to (112), the parity of ϕ_α is $(\alpha^{\phi})^{-1}\epsilon_\phi\alpha$. As ϵ_ϕ is in the centre of $\mathrm{End}(X)$, this equals ϵ_ϕ if and only if $\alpha^{\phi} = \alpha$.

(b) As $\alpha = \alpha^{\phi}$, (112) and (110) show that $u^{\phi_\alpha} = \alpha^{-1} \cdot u^{\phi} \cdot \alpha$. Thus, ϕ_α is a Weil form if and only if

$$\mathrm{Tr}_X(u \cdot \alpha^{-1} \cdot u^{\phi} \cdot \alpha) > 0, \text{ all } u \neq 0,\ u \in \mathrm{End}(X).$$

If $\alpha = \beta^2$ with $\beta \in \mathbb{R}[\alpha]$, then

$$\begin{aligned} \mathrm{Tr}_X(u\alpha^{-1}u^{\phi}\alpha) &= \mathrm{Tr}_X((u\beta^{-1})\beta^{-1}u^{\phi}\alpha^{-1}) \\ &= \mathrm{Tr}_X(\beta^{-1}u^{\phi}\alpha^{-1}(u\beta^{-1})) \qquad (\mathrm{Tr}_X(vw) = \mathrm{Tr}_X(wv)) \\ &= \mathrm{Tr}_X((\beta u \beta^{-1})^{\phi}(\beta^{-1}u\beta)) > 0 \end{aligned}$$

for $u \neq 0$. Conversely, if ϕ_α is a Weil form, then $\mathrm{Tr}_X(u^2\alpha) > 0$ for all $u \neq 0$ in $\mathbb{R}[\alpha]$, which implies that α is a square in $\mathbb{R}[\alpha]$ (because $\mathbb{R}[\alpha]$ is a product of copies of $\mathbb{R}$ (4.2(c))).

(c) Let u be a nonzero endomorphism of X. Then $u' = u^{\phi_\alpha}$, and so ϕ and ϕ_α are compatible if and only if $\mathrm{Tr}_X(u \cdot u^{\phi_\alpha}) > 0$ for all $u \neq 0$, but this is implied by ϕ_α's being a Weil form.

(d) According to (111), every nondegenerate sesquilinear form on X is of the form ϕ_α for a unique automorphism α of X. Thus, the proposition is an immediate consequence of the preceding statements. □

The relation of compatibility on the set of Weil forms on X is obviously reflexive and symmetric, and the next corollary implies that it is also transitive on any set of Weil forms on X having a fixed parity.

COROLLARY 4.6 *Let ϕ and ϕ' be compatible Weil forms on X with the same parity, and let ψ be a Weil form on Y. If ϕ is compatible with ψ, then so also is ϕ'.*

PROOF This follows easily from writing $\phi' = \phi_\alpha$. □

EXAMPLE 4.7 Let X be a simple object in C', so that $\mathrm{End}(X) = \mathbb{C}$, and let ε be an endomorphism of X central in $\mathrm{End}(X)$. If $\bar{X}$ is isomorphic to $X^\vee$, so that there exists a nondegenerate sesquilinear form on X, then (111) shows that the sesquilinear forms on X are parametrized by $\mathbb{C}$; moreover, (112) shows that if there is a nonzero such form with parity ε, then the set of sesquilinear forms on X with parity ε is parametrized by $\mathbb{R}$; finally, (4.5) shows that if there is a Weil form with parity ε, then the set of such forms falls into two compatibility classes, each parametrized by $\mathbb{R}_{>0}$.

Variant

4.8 Let X_0 be an object in C and let ϕ_0 be a nondegenerate bilinear form $\phi_0 : X_0 \otimes X_0 \to \mathbb{1}$. The form ϕ_0 is said to be a ***Weil form*** on X_0 if

(a) its parity ε_{ϕ_0} is in the centre of $\mathrm{End}(X_0)$ and

(b) $\mathrm{Tr}_{X_0}(u \circ u^{\phi_0}) > 0$ for all nonzero $u \in \mathrm{End}(X_0)$.

Two Weil forms ϕ_0 and ψ_0 are said to be ***compatible*** if $\phi_0 \oplus \psi_0$ is also a Weil form.

Let X_0 correspond to the pair (X, a) with $X \in \mathrm{ob}(\mathsf{C}')$. Then ϕ_0 defines a bilinear form ϕ on X, and

$$\psi \overset{\text{def}}{=} (X \otimes \bar{X} \xrightarrow{1 \otimes a^{-1}} X \otimes X \xrightarrow{\phi} \mathbb{1})$$

is a nondegenerate sesquilinear form on X. If ϕ_0 is a Weil form, then ψ is a Weil form on X that is compatible with its conjugate $\bar{\psi}$, and every such ψ arises from a ϕ_0; moreover, $\varepsilon_\psi = \varepsilon_{\phi_0}$.

5 Polarizations

Let C be a tensor category over $\mathbb{R}$ and $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ its extension to $\mathbb{C}$ (as in 2.1).

DEFINITION 5.1 Let $\varepsilon \in \mathrm{Aut}^\otimes(\mathrm{id}_\mathsf{C})$ and, for each $X \in \mathrm{ob}(\mathsf{C}')$, let $\Pi(X)$ be an equivalence class (for the relation of compatibility) of Weil forms on X with parity ε_X. Then Π is a ***polarization*** on C if

(a) for all X, $\bar{\phi} \in \Pi(X)$ whenever $\phi \in \Pi(\bar{X})$, and

(b) for all X and Y, $\phi \oplus \psi \in \Pi(X \oplus Y)$ and $\phi \otimes \psi \in \Pi(X \otimes Y)$ whenever $\phi \in \Pi(X)$ and $\psi \in \Pi(Y)$.

We call ε the ***parity*** of Π and say that ϕ is ***positive*** for Π if $\phi \in \Pi(X)$. Thus, the conditions require that $\bar{\phi}$, $\phi \oplus \psi$, and $\phi \otimes \psi$ are positive for Π whenever ϕ and ψ are.

PROPOSITION 5.2 *Let C be a tensor category over $\mathbb{R}$. If C is polarizable, then it is tannakian. Indeed, C' is a neutral tannakian category.*

PROOF Let X be a nonzero object of C', and let $\phi \in \Pi(X)$. The morphism $\phi \circ \gamma_{\bar{X},X} : \bar{X} \otimes X \to \mathbb{1}$ realizes $\bar{X}$ as the dual of X with coevaluation map $\mathbb{1} \to X \otimes \bar{X}$ the adjoint ϕ^* of ϕ with respect to the sesquilinear form $\phi \otimes \bar{\phi}$ on $X \otimes \bar{X}$ and the obvious sesquilinear form on $\mathbb{1}$ (cf. 3.6). Therefore

$$\dim X = \mathrm{Tr}_{\mathbb{1}}(\phi \circ \phi^*) > 0.$$

Recall (I, 10.2) that

$$\dim(\textstyle\bigwedge^n X) = \dfrac{(\dim X)(\dim X - 1)\cdots(\dim X - n + 1)}{n \cdot n - 1 \cdot \cdots \cdot 1}.$$

If $\dim X \notin \mathbb{N}$, then $\dim(\bigwedge^n X) < 0$ for some integer $n > 0$ – contradiction. Hence $\dim X \in \mathbb{N}$. As this holds for all X in C, Theorem 10.1 of Chapter I shows that C is tannakian. Finally, Theorem 10.1 of Chapter IV shows that C admits a $\mathbb{C}$-valued fibre functor. □

PROPOSITION 5.3 *Let Π be a polarization on C.*

(a) *The categories C and C' are semisimple.*

(b) *If $\phi \in \Pi(X)$ and $Y \subset X$, then $X = Y \oplus Y^\perp$ and the restriction ϕ_Y of ϕ to Y is in $\Pi(Y)$.*

PROOF (a) Let X be an object of $\mathsf{C}'$and let $u : Y \hookrightarrow X$ be a nonzero simple subobject of X. Choose $\phi \in \Pi(Y)$ and $\psi \in \Pi(X)$. Consider

$$v = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} : X \oplus Y \to X \oplus Y, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} uy \\ 0 \end{pmatrix}$$

and let $u' : X \to Y$ be such that

$$v^{\psi \oplus \phi} = \begin{pmatrix} 0 & 0 \\ u' & 0 \end{pmatrix}.$$

Then $\mathrm{Tr}_Y(u'u) = \mathrm{Tr}_{Y \oplus X}(v^{\psi \oplus \phi} \circ v) > 0$, and so $u'u$ is an automorphism w of Y. The map $p = w^{-1} \circ u'$ projects X onto Y, which shows that Y is a direct summand of X. We have shown that X is semisimple. Thus, C' is semisimple, and the same argument, using the bilinear forms (4.8) shows that C is semisimple.[4]

(b) Let $Y' = Y \cap Y^\perp$, where $Y^\perp$ is the largest subobject of X such that ϕ is zero on $Y \otimes \bar{Y}^\perp$, and let $p : X \to X$ be the projection of X onto Y' (by which we mean that $p(X) \subset Y'$ and $p|Y' = \mathrm{id}_{Y'}$). As ϕ is zero on $Y' \otimes \overline{Y'}$,

$$0 = \phi \circ (p \otimes \bar{p}) = \phi \circ (\mathrm{id} \otimes \overline{p^\phi p}),$$

and so $p^\phi p = 0$. Therefore, $\mathrm{Tr}_X(p^\phi p) = 0$, and so p, and Y', are zero. Thus $X = Y \oplus Y^\perp$ and $\phi = \phi_Y \oplus \phi_Y^\perp$. Let $\phi_1 \in \Pi(Y)$ and $\phi_2 \in \Pi(Y^\perp)$. Then $\phi_1 \oplus \phi_2$ is compatible with ϕ, and this implies that ϕ_1 is compatible with ϕ_Y. □

Let C be a tannakian category over $\mathbb{R}$, and let Z be the centre of the band attached to C. Thus, Z is a commutative affine group scheme over $\mathbb{R}$ such that

$$Z(\mathbb{C}) \simeq \mathrm{Centre}(\mathrm{Aut}^\otimes(\omega))$$

for any $\mathbb{C}$-valued fibre functor ω on C'. Moreover, $Z(\mathbb{R}) = \mathrm{Aut}^\otimes(\mathrm{id}_{\mathsf{C}})$. We usually regard the parity of a polarization as an element of $Z(\mathbb{R})$.

[4] Alternatively, use that $\mathrm{End}(X)$ is semisimple for all X; see VI, 1.4.

EXAMPLE 5.4 Suppose that C is defined by a triple (G,σ,c), as in (2.1), so that $\mathsf{C}' = \mathsf{Repf}_{\mathbb{C}}(G)$. A sesquilinear form $\phi : X \otimes \bar{X} \to \mathbb{1}$ defines a sesquilinear form ϕ' on X in the usual, vector space, sense by the formula

$$\phi'(x,y) = \phi(x \otimes \bar{y}), \quad x,y \in X. \tag{113}$$

The conditions that ϕ be a G-morphism and have parity $\varepsilon \in Z(\mathbb{R})$ become respectively

$$\begin{array}{rcll} \phi'(x,y) & = & \phi'(gx,\sigma^{-1}(g)y), & g \in G(\mathbb{C}), \\ \overline{\phi'(y,x)} & = & \phi'(x,\varepsilon c^{-1}y). & \end{array} \tag{114}$$

When G acts trivially on X, the last equation becomes

$$\overline{\phi'(y,x)} = \phi'(x,y),$$

and so ϕ' is a hermitian form in the usual sense on X. When X is one-dimensional, ϕ' is positive definite (for otherwise $\phi \otimes \phi \notin \Pi(X)$). Now (5.3(b)) shows that the same is true for any X on which G acts trivially, and (4.5) shows that $\{\phi' \mid \phi \in \Pi(X)\}$ is the complete set of positive-definite hermitian forms on X. In particular, $\mathsf{Vecf}_{\mathbb{R}}$ has a unique polarization.

REMARK 5.5 Let Π be a polarization on C with parity ε. Then Π defines, for each simple object X of C', an orientation of the real line of sesquilinear forms on X with parity ε (see 4.7), and Π is obviously determined by this family of orientations.

Choose for each simple object X_i a $\phi_i \in \Pi(X_i)$. Then

$$\Pi(X_i) = \{r\phi_i \mid r \in \mathbb{R}_{>0}\}.$$

Assume that C' has a $\mathbb{C}$-valued fibre functor. If X is isotypic of type X_i, then $\omega(X) = W \otimes \omega(X_i)$ for some finite-dimensional vector space W on which $\mathcal{A}ut^{\otimes}(\omega)$ acts trivially, and

$$\{\omega(\phi)' \mid \phi \in \Pi(X)\} = \{\psi \otimes \omega(\phi_i)' \mid \psi \text{ hermitian } \psi > 0\}.$$

If $X = \bigoplus X^{(i)}$, where the $X^{(i)}$ are the isotypic components of X, then

$$\Pi(X) = \bigoplus \Pi(X^{(i)}).$$

POLARIZATIONS ON TANNAKIAN CATEGORIES OVER SUBFIELDS OF $\mathbb{R}$

Let C be a tannakian category over a subfield F of $\mathbb{R}$.

5.6 The ***parity*** ε_φ of a nondegenerate bilinear form $\varphi : X \otimes X \to \mathbb{1}$ on an object X of C is the unique endomorphism of X satisfying

$$\varphi \circ (X \otimes \varepsilon_\varphi) = \varphi \circ \gamma_{X,X}, \quad [\varphi(x,\varepsilon_\varphi y) = \varphi(y,x)]$$

where γ is the commutativity constraint. The transpose u^φ of an endomorphism u of X with respect to φ is defined by

$$\varphi \circ (u \otimes X) = \varphi \circ (X \otimes u^\varphi), \quad [\varphi(ux,y) = \varphi(x,u^\varphi y)].$$

5.7 A nondegenerate bilinear form $\varphi : X \otimes X \to \mathbb{1}$ is a ***Weil form*** if its parity ε_φ lies in the centre of $\mathrm{End}(X)$ and $\mathrm{Tr}_X(uu^\varphi) > 0$ for all nonzero $u \in \mathrm{End}(X)$. Weil forms φ_1 on X_1 and φ_2 on X_2 are ***compatible*** if $\varphi_1 \oplus \varphi_2$ is a Weil form on $X_1 \oplus X_2$.

5.8 Suppose $F = \mathbb{R}$, and let $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. An object X_0 of C corresponds to a pair (X, a) with X in $\operatorname{ob}\mathsf{C}'$. A bilinear form ϕ_0 on X_0 defines a bilinear form ϕ on X, and hence a sesquilinear form

$$\psi : X \otimes \bar{X} \xrightarrow{X\otimes a^{-1}} X \otimes X \xrightarrow{\phi} \mathbb{1}$$

on X. If ϕ_0 is a Weil form, then ψ is a Weil form such that ψ is compatible with $\bar{\psi}$, and every such ψ arises in this way.

5.9 If ε is an endomorphism of X, central in $\operatorname{End}(X)$, then compatibility is an equivalence relation on Weil forms on X with parity ε. Let $w_\varepsilon(X)$ denote the quotient set. If $X_{\mathbb{R}}$ is the object of $\mathsf{C}_{(\mathbb{R})}$ defined by X, then, using that F is dense in $\mathbb{R}$, one obtains a canonical bijection

$$w_\varepsilon(X) \simeq w_\varepsilon(X_{\mathbb{R}}).$$

5.10 Let $\varepsilon \in \operatorname{Aut}^{\otimes}(\mathrm{id}_{\mathsf{C}})$, and, for each $X \in \operatorname{ob}(\mathsf{C})$, let $\pi(X)$ be a compatibility class of Weil forms on X with parity ε. Then π is a ***polarization*** on C if $\phi \oplus \psi \in \pi(X \oplus Y)$ and $\phi \otimes \psi \in \pi(X \otimes Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$. When $F = \mathbb{R}$, the polarizations π on C in this sense are in canonical one-to-one correspondence with the polarizations Π in the sense of 5.1.

For example, there is a polarization π on $\mathsf{C} \overset{\text{def}}{=} \mathsf{Vecf}(\mathbb{R})$ such that, for each finite-dimensional $\mathbb{R}$-vector space V, $\pi(V)$ consists of the positive-definite bilinear forms on V. For each V in $\mathsf{C}' \overset{\text{def}}{=} \mathsf{Vecf}(\mathbb{C})$, $\Pi(V)$ consists of the positive-definite hermitian forms on V (regarded as morphisms $V \otimes \bar{V} \to \mathbb{C}$).

5.11 For $\varepsilon \in \operatorname{Aut}^{\otimes}(\mathrm{id}_{\mathsf{C}})$, let $\operatorname{Pol}_\varepsilon(\mathsf{C})$ denote the set of polarizations on C with parity ε. There is a canonical one-to-one correspondence

$$\operatorname{Pol}_\varepsilon(\mathsf{C}) \simeq \operatorname{Pol}_\varepsilon(\mathsf{C}_{(\mathbb{R})}).$$

To some extent, this justifies studying polarizations only for $F = \mathbb{R}$.

See Saavedra 1972, V, 2.3, 2.4.

6 Description of the polarizations

Let C be an algebraic tannakian category over $\mathbb{R}$ and ω a $\mathbb{C}$-valued fibre functor on C. Then $\mathcal{G} \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$ is a Galois groupoid

$$1 \to G(\mathbb{C}) \to \mathcal{G} \to \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

with G an algebraic group over $\mathbb{C}$. Choose an $s \in \mathcal{G}$ mapping to $\iota \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})$, and let $\sigma : G(\mathbb{C}) \to G(\mathbb{C})$ be conjugation by s. Then σ is semilinear and

$$\sigma^2 = \operatorname{ad}(c), \quad \sigma(c) = c, \tag{115}$$

where $c = s^2 \in G(\mathbb{C})$. See I, 4.1.

Let K be a maximal compact subgroup of $G(\mathbb{C})$. As all maximal compact subgroups of $G(\mathbb{C})$ are conjugate (1.18), there exists an $m \in G(\mathbb{C})$ such that $\sigma(K) = m^{-1}Km$. After replacing s with ms, hence σ with $\sigma \circ \operatorname{ad}(m)$, we have $\sigma(K) = K$. Subject to this constraint, (σ, c) is determined up to modification by an element m in the normalizer of K.

Assume that C is polarizable. Then C′ is semisimple (5.3(a)), and so the identity component of G is reductive (II, 6.21). It follows that K is a compact real form of G (1.18). Let σ_K denote the semilinear automorphism of G sending an element of $G(\mathbb{C})$ to its conjugate relative to the real structure on G defined by K. Then σ_K determines K. The normalizer of K is $K \cdot Z(\mathbb{C})$, and so $c \in K \cdot Z(\mathbb{C})$.

Now fix a polarization Π on C, and let ε be its parity.

Let X be a simple representation of G, and let ψ be a positive-definite K-invariant hermitian form on X. For any $\phi \in \Pi(X)$, the associated sesquilinear form $\phi'(x,y) \overset{\text{def}}{=} \phi(x \otimes \bar{y})$ can be expressed

$$\phi(x \otimes \bar{y}) = \psi(x, \beta y)$$

for some $\beta \in \mathrm{Aut}(X)$. The conditions (114) expressing that ϕ is G-invariant and has parity ε can be re-written as

$$\begin{aligned} \beta \cdot g_X &= \sigma(g)_X \cdot \beta \qquad \text{all } g \in K(\mathbb{R}) \\ \beta^* &= \beta \cdot \varepsilon_X \cdot c_X^{-1} \end{aligned} \tag{116}$$

where β^* is the adjoint of β relative to ψ,

$$\psi(\beta x, y) = \psi(x, \beta^* y).$$

As $K(\mathbb{R})$ is Zariski dense in $K(\mathbb{C})$, X is also simple as a representation of $K(\mathbb{R})$, and so the set $c(X,\Pi)$ of such β is parametrized by $\mathbb{R}_{>0}$ (see 4.7).

An arbitrary finite-dimensional representation X of G can be written

$$X = \bigoplus\nolimits_i W_i \otimes X_i,$$

where the sum is over the distinct simple representations X_i of G and G acts trivially on each W_i. Let ψ_i' and ψ_i be K-invariant positive-definite hermitian forms on W_i and X_i respectively, and let $\psi = \bigoplus \psi_i' \otimes \psi_i$. Then for any $\phi \in \Pi(X)$,

$$\phi'(x,y) = \psi(x, \beta y), \quad \beta \in \mathrm{Aut}(X),$$

where $\beta = \bigoplus \beta_i' \otimes \beta_i$ with $\beta_i \in c(X_i,\Pi)$ and β_i' positive-definite and hermitian relative to ψ_i'. We again let $c(X,\Pi)$ denote the set of β arising from some ϕ in $\Pi(X)$. The condition (5.1(b)) that

$$\Pi(X_1) \otimes \Pi(X_2) \subset \Pi(X_1 \otimes X_2)$$

becomes

$$c(X_1,\Pi) \otimes c(X_2,\Pi) \subset c(X_1 \otimes X_2, \Pi).$$

LEMMA 6.1 *There exists a $b \in K$ with the following properties:*

(a) $b_X \in c(X,\Pi)$ *for all simple X;*

(b) $\sigma = \sigma_K \circ \mathrm{ad}(b)$, *where σ_K denotes complex conjugation on $G(\mathbb{C})$ relative to K;*

(c) $\varepsilon^{-1} c = \sigma b \cdot b = b^2$.

PROOF Let $a = \varepsilon c^{-1} \in G(\mathbb{C})$.

When X is simple, the first equality in (116) applied twice shows that

$$\beta^2 \cdot g \cdot x = \sigma^2(g) \cdot \beta^2 \cdot x = c \cdot g \cdot c^{-1} \cdot \beta^2 \cdot x$$

for $\beta \in c(X,\Pi)$, $g \in K$, and $x \in X$. Therefore

$$(c^{-1}\beta^2) g x = g (c^{-1}\beta^2) x,$$

and so $c^{-1}\beta^2$ acts as a scalar on X. Hence $a\beta^2 = \varepsilon c^{-1}\beta^2$ also acts as a scalar. Moreover, $\beta^2 a = \beta\beta^*$ (by the second equation in (116)) and so

$$\mathrm{Tr}_X(a\beta^2) = \mathrm{Tr}_X(\beta^2 a) > 0.$$

We conclude that $a_X\beta^2 \in \mathbb{R}_{>0}$. It follows that there is a unique $\beta \in c(X,\Pi)$ such that

$$\begin{cases} a_X = \beta^{-2}, \\ \beta g_X = \sigma(g)_X\beta, & \text{all } g \in K, \\ \beta^* = \beta^{-1} & \text{(i.e., } \beta \text{ is unitary).} \end{cases}$$

For an arbitrary X, we let $X = \bigoplus W_i \otimes X_i$ as before, and set $\beta = \bigoplus_i \mathrm{id}_{W_i} \otimes \beta_i$, where β_i is the canonical element of $c(X_i,\Pi)$ just defined. We have

$$\begin{cases} a_X = \beta^{-2} \\ \beta g_X = \sigma(g)_X\beta, & \text{all } g \in K, \\ \beta \in c(X,\Pi). \end{cases}$$

Moreover, these conditions characterize β: if $\beta' \in c(X,\Pi)$ has the same properties, then $\beta' = \bigoplus_i \gamma_i \otimes \beta_i$ (this expresses that $\beta' g_X = \sigma(g)_X\beta'$, $g \in K$) with $\gamma_i^2 = 1$ (as $\beta'^2 = a_X^{-1}$) and γ_i positive definite and hermitian. Hence $\gamma_i = 1$.

These conditions are compatible with tensor products and maps of representations, and so the canonical β are compatible with tensor products and maps of representations. They therefore define an element $b \in G(\mathbb{C})$ (II, 2.1). As b is unitary on all irreducible representations, it lies in K. The equations $\beta^2 = a_X^{-1}$ show that $b^2 = a^{-1} = \varepsilon^{-1}c$. Finally, $\beta g_X = \sigma(g)_X\beta$ implies that $\sigma(g) = \mathrm{ad}(b(g))$ for all $g \in K$; therefore $\sigma \circ \mathrm{ad}(b)^{-1}$ fixes K, and as it has order 2, it must equal σ_K. □

THEOREM 6.2 *Let* C *be a tannakian category over* $\mathbb{R}$*, let* ω *be a* $\mathbb{C}$*-valued fibre functor on* C*, and let* Π *be a polarization on* C *with parity* ε*. For any compact real form* K *of* $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$*, the pair* (σ_K, ε) *satisfies (115), and the equivalence* $\mathsf{C}' \to \mathsf{Repf}_{\mathbb{C}}(G)$ *defined by* ω *carries the descent datum on* C′ *defined by* C *into that on* $\mathsf{Repf}_{\mathbb{C}}(G)$ *defined by* (σ_K, ε)*:*

$$\omega(\bar{X}) = \overline{\omega(X)}, \quad \omega(\mu_X) = \mu_{\omega(X)}.$$

For any simple X *in* C′*,*

$$\{\omega(\phi)' \mid \phi \in \Pi(X)\}$$

is the set of K*-invariant positive-definite hermitian forms on* $\omega(X)$*.*

PROOF Let (C,ω) correspond to the triple (G,σ_1,c_1) (see **??**(a)), and let $b \in K$ be the element constructed in the lemma. Then $\sigma_1 = \sigma_K \circ \mathrm{ad}(b)$ and $c = \varepsilon \cdot \sigma b \cdot b = \sigma b \cdot \varepsilon \cdot b$. Therefore, (σ_K,ε) has the same property as (σ_1,c_1) (see **??**(b)), which proves the first assertion. The second assertion follows from the fact that $b \in c(\omega(X),\Pi)$ for any simple X. □

In other words, (σ_K,ε) satisfies (115), and the Galois groupoid of (C,ω) is that defined by (G,σ_K,ε) (cf. III, 4.1).

7 Classification of polarized tannakian categories

THEOREM 7.1 *(a) An algebraic tannakian category* C *over* $\mathbb{R}$ *is polarizable if and only if its band is defined by a compact real algebraic group* K.

(b) For any compact real algebraic group K *and* $\varepsilon \in Z(\mathbb{R})$, *where* Z *is the centre of* K, *there exists a tannakian category* C *over* $\mathbb{R}$ *whose gerbe is banded by the band* $\mathrm{Bd}(K)$ *of* K *and a polarization* Π *on* C *with parity* ε.

(c) Let (C_1, Π_1) *and* (C_2, Π_2) *be polarized algebraic tannakian categories over* $\mathbb{R}$ *with isomorphic bands* B_1 *and* B_2. *If there exists an isomorphism* $B_2 \to B_1$ *sending* $\varepsilon(\Pi_1)$ *to* $\varepsilon(\Pi_2)$ *(as elements of* $Z(B_i)(\mathbb{R})$*), then there is a* $\otimes$*-equivalence* $\mathsf{C}_1 \to \mathsf{C}_2$ *respecting the polarizations and the actions of* B_1 *and* B_2 *(i.e., such that* $\mathrm{FIB}(\mathsf{C}_2) \to \mathrm{FIB}(\mathsf{C}_1)$ *is a banded by* $B_2 \to B_1$*), and this equivalence is unique up to isomorphism.*

PROOF We have already seen (5.3) that if C is polarizable, then C' is semisimple, and so, for any fibre functor ω with values in $\mathbb{C}$, the identity component of $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$ is reductive, and so has a compact real form K. This proves the necessity in (a). Statement (b) is proved in the first lemma below, and the sufficiency in (a) follows from (b) and the second lemma below. Statement (c) follows from Theorem 6.2. □

LEMMA 7.2 *Let* K *and* ε *be as in (b) of the theorem, and let* $G = K_{\mathbb{C}}$. *For* $g \in G(\mathbb{C})$, *let* $\sigma(g) = \sigma'(\bar{g})$, *where* σ' *is the Cartan involution corresponding to* K *(notation as in 1.14). The pair* (σ, ε) *then satisfies (105) and the tannakian category* C *defined by* (G, σ, ε) *has a polarization with parity* ε.

PROOF Since $\sigma^2 = \mathrm{id}$ and σ fixes all elements of K, it is obvious that (σ, ε) satisfies (105). There exists a polarization Π on C such that, for all simple X, $\{\phi' \mid \phi \in \Pi(X)\}$ is the set of positive-definite K-invariant hermitian forms on X.[5] This polarization has parity ε. □

Let C be an algebraic tannakian category over $\mathbb{R}$, and let $(\mathsf{C}', X \mapsto \bar{X}, \mu)$ be the corresponding triple, as in 2.1. Let Z be the centre of the band B of C. For any $z \in Z(\mathbb{R})$, the triple $(\mathsf{C}', X \mapsto \bar{X}, \mu \circ z)$ defines a new tannakian category ${}^z\mathsf{C}$ over $\mathbb{R}$.

LEMMA 7.3 *Every tannakian category over* $\mathbb{R}$ *whose gerbe is banded by* B *is of the form* ${}^z\mathsf{C}$ *for some* $z \in Z(\mathbb{R})$. *There is a* $\otimes$*-equivalence* ${}^z\mathsf{C} \to {}^{z'}\mathsf{C}$ *respecting the action of* B *if and only if* $z'z^{-1} \in Z(\mathbb{R})^2$.

PROOF Let ω be a $\mathbb{C}$-valued fibre functor on C, and let (G, σ, c) be the corresponding triple, as in **??**. We may suppose that the second category C_1 corresponds to a triple (G, σ_1, c_1). Let γ and γ_1 be the functors $V \mapsto \bar{V}$ defined by (σ, c) and (σ_1, c_1) respectively. Then $\gamma_1^{-1} \circ \gamma$ defines a $\otimes$-automorphism of ω, and so corresponds to an element $m \in G(\mathbb{C})$. We have $\sigma = \sigma_1 \circ \mathrm{ad}(m)$, and so we can modify (σ_1, c_1) in order to get $\sigma_1 = \sigma$. Let μ and μ_1 be the natural isomorphisms $V \to \bar{\bar{V}}$ defined by (σ, c) and (σ, c_1) respectively. Then $\mu_1^{-1} \circ \mu$ defines a symmetric monoidal automorphism of id_{C}, and so $\mu_1^{-1} \circ \mu = z^{-1}$, $z \in Z(\mathbb{R})$. We have $\mu_1 = \mu \circ z$.

The second part of the lemma is obvious. □

REMARK 7.4 In Saavedra 1972, V, 1, there is a table of tannakian categories whose bands are simple, from which it is possible to read off those that are polarizable (loc. cit. V, 2.8.3).

[5] In the notation of 6.1, $b = 1$.

Cohomological interpretation

Let C be a tannakian category with band B. Assume that B is defined by a compact real algebraic group K, and let Z denote the centre of B.

7.5 As Z is an algebraic subgroup of a compact real algebraic group, it is also compact (1.11). It is easy to compute its cohomology. One finds that

$$H^1(\mathbb{R}, Z) = {}_2Z(\mathbb{R}) \overset{\text{def}}{=} \operatorname{Ker}(2:\ Z(\mathbb{R}) \to Z(\mathbb{R}))$$
$$H^2(\mathbb{R}, Z) = Z(\mathbb{R})/Z(\mathbb{R})^2.$$

See Serre 1964, III, 4.5.

7.6 The general theory (Saavedra 1972, III 2.3.4.2, p. 184) shows that there is an isomorphism $H^1(\mathbb{R}, Z) \to \operatorname{Aut}_B(\mathsf{C})$, which can be described explicitly as the map sending $z \in {}_2Z(\mathbb{R})$ to the automorphism w_z

$$\begin{cases} (X, a_X) & \mapsto & (X, a_X z_X) \\ f & \mapsto & f. \end{cases}$$

7.7 The tannakian categories banded by B are classified up to B-equivalence by $H^2(\mathbb{R}, B)$, and $H^2(\mathbb{R}, B)$, if nonempty, is an $H^2(\mathbb{R}, Z)$-torsor. The action of $H^2(\mathbb{R}, Z) = Z(\mathbb{R})/Z(\mathbb{R})^2$ on the set of B-equivalence classes is made explicit in (7.3).

7.8 Let $\operatorname{Pol}(\mathsf{C})$ denote the set of polarizations on C. For $\Pi \in \operatorname{Pol}(\mathsf{C})$ and $z \in Z(\mathbb{R})$ we define $z\Pi$ to be the polarization such that

$$\phi(x, y) \in z\Pi(X) \iff \phi(x, zy) \in \Pi(X).$$

It has parity $\varepsilon(z\Pi) = z^2\varepsilon(\Pi)$. The pairing

$$(z, \Pi) \mapsto z\Pi:\ Z(\mathbb{R}) \times \operatorname{Pol}(\mathsf{C}) \to \operatorname{Pol}(\mathsf{C})$$

makes $\operatorname{Pol}(\mathsf{C})$ into a $Z(\mathbb{R})$-torsor.

7.9 Let $\Pi \in \operatorname{Pol}(\mathsf{C})$ have parity $\varepsilon = \varepsilon(\Pi)$, and let $\varepsilon' \in Z(\mathbb{R})$. There is a polarization on C with parity ε' if and only if $\varepsilon' = \varepsilon z^2$ for some $z \in Z(\mathbb{R})$.

8 Neutral polarized categories

The results in the last section can be made more explicit when the tannakian category is neutral.

Let G be an algebraic group over $\mathbb{R}$, and let $C \in G(\mathbb{R})$. A G-invariant sesquilinear form $\psi:\ V \times V \to \mathbb{C}$ on $V \in \operatorname{ob}(\mathsf{Repf}_{\mathbb{C}}(G))$ is said to be a C-***polarization*** if

$$\psi^C(x, y) \overset{\text{def}}{=} \psi(x, Cy)$$

is a positive-definite hermitian form on V. If every object of $\mathsf{Repf}_{\mathbb{C}}(G)$ has a C-polarization, then C is called a ***Hodge element***.

As usual, we let Z denote the centre of G.

PROPOSITION 8.1 *Assume that $G(\mathbb{R})$ contains a Hodge element C.*

(a) *There is a polarization* Π_C *on* $\mathsf{Repf}_{\mathbb{R}}(G)$ *for which the positive forms are exactly the* C*-polarizations. It has parity* C^2.

(b) *For any* $g \in G(\mathbb{R})$ *and* $z \in Z(\mathbb{R})$, $C' = zgCg^{-1}$ *is also a Hodge element and* $\Pi_{C'} = z\Pi_C$.

(c) *Every polarization on* $\mathsf{Repf}_{\mathbb{R}}(G)$ *is of the form* $\Pi_{C'}$ *for some Hodge element* C'.

PROOF Let ψ be a C-polarization on $V \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{C}}(G))$; then

$$\psi(x,y) = \psi(Cx,Cy)$$

because ψ is G-invariant, and

$$\psi(Cx,Cy) = \psi^C(Cx,y) = \overline{\psi^C(y,Cx)} = \overline{\psi(y,C^2x)}.$$

This shows that ψ has parity C^2. For any V and $g \in G(\mathbb{R})$,

$$\begin{aligned}\overline{\psi(y,C^2x)} &= \psi(x,y)\\ &= \psi(gx,gy)\\ &= \overline{\psi(gy,C^2gx)}\\ &= \overline{\psi(y,g^{-1}C^2gx)}.\end{aligned}$$

This shows that $C^2 \in Z(\mathbb{R})$. For any $u \in \mathrm{End}(V)$, $u^{\psi} = u^{\psi^C}$, and so $\mathrm{Tr}(uu^{\psi}) > 0$ if $u \neq 0$. This shows that ψ is a Weil form with parity C^2. Statement (a) is now easy to check. Statement (b) is straightforward to prove, and statement (c) follows from it and (7.3). □

PROPOSITION 8.2 *The following conditions on* G *are equivalent:*

(a) *there exists a Hodge element in* $G(\mathbb{R})$*;*

(b) *the category* $\mathsf{Repf}_{\mathbb{R}}(G)$ *is polarizable;*

(c) G *is an inner form of a compact real algebraic group* K.

PROOF (a)⇒(b). This is proved in (8.1).

(b)⇒(c). To say that G is an inner form of K is the same as to say that G and K define the same band; this implication therefore follows from (7.1(a)).

(c)⇒(a). Let Z be the centre of K (and therefore also of G) and let $K^{\mathrm{ad}} = K/Z$. That G is an inner form of K means that its cohomology class is in the image of

$$H^1(\mathbb{R},K^{\mathrm{ad}}) \to H^1(\mathbb{R},\mathcal{A}ut(K)).$$

More explicitly, this means that there is an isomorphism $\gamma : K_{\mathbb{C}} \to G_{\mathbb{C}}$ such that

$$\bar{\gamma} = \gamma \circ c, \quad \text{some } c \in K^{\mathrm{ad}}(\mathbb{C}).$$

According to Serre 1964, III, Thm 6, $H^1(\mathbb{R},K^{\mathrm{ad}}) \simeq H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}),K^{\mathrm{ad}}(\mathbb{R}))$, which is equal to the set of conjugacy classes in $K^{\mathrm{ad}}(\mathbb{R})$ consisting of elements of order 2. Thus, we can assume that $c \in K(\mathbb{R})$ and $c^2 = 1$. Consider the cohomology sequence

$$K(\mathbb{R}) \;\to\; K^{\mathrm{ad}}(\mathbb{R}) \;\to\; H^1(\mathbb{R},Z) \;\to\; H^1(\mathbb{R},K).$$

The last map is injective, and so $K(\mathbb{R}) \to K^{\mathrm{ad}}(\mathbb{R})$ is surjective. Thus, $c = \mathrm{ad}(C')$ for some $C' \in K(\mathbb{R})$ whose square is in $Z(\mathbb{R})$. Let $C = \gamma(C')$; then $\bar{C} = \bar{\gamma}(C') = \gamma(C') = C$ and $\bar{\gamma}^{-1} \circ \mathrm{ad}(C) = \gamma^{-1}$. This shows that $C \in G(\mathbb{R})$ and that K is the form of G defined by C; the next lemma completes the proof. □

LEMMA 8.3 *An element $C \in G(\mathbb{R})$ such that $C^2 \in Z(\mathbb{R})$ is a Hodge element if and only if the real form K of G defined by C is a compact real group.*

PROOF Identify $K_{\mathbb{C}}$ with $G_{\mathbb{C}}$ and let $\bar{g}$ and g^* be the complex conjugates of $g \in G(\mathbb{C})$ relative to the real forms K and G. Then

$$g^* = \mathrm{ad}(C^{-1})(\bar{g}) = C^{-1}\bar{g}C.$$

Let ψ be a sesquilinear form on $V \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{C}}(G))$. Then ψ is G-invariant if and only if

$$\psi(gx, \bar{g}y) = \psi(x, y), \quad g \in G(\mathbb{C}).$$

On the other hand, ψ^C is K-invariant if and only if

$$\psi^C(gx, g^*y) = \psi^C(x, y), \quad g \in G(\mathbb{C}).$$

These conditions are equivalent. Therefore, V has a C-polarization if and only if V has a K-invariant positive-definite hermitian form. Thus, C is a Hodge element if and only if, for every complex representation V of K, the image of K in $\mathrm{Aut}(V)$ is contained in the unitary group of a positive-definite hermitian form; this last condition is implied by K being compact and implies that K is contained in a compact real group, and so is compact (1.11). □

REMARK 8.4 (a) The centralizer of a Hodge element C of G is a maximal compact subgroup of G, and is the only maximal compact subgroup of G containing C; in particular, if G is compact, then C is a Hodge element if and only if it is in the centre of G (Saavedra 1972, V, 2.7.3.5).

(b) If C and C' are Hodge elements of G, then there exists a $g \in G(\mathbb{R})$ and a unique $z \in Z(\mathbb{R})$ such that $C' = zgCg^{-1}$ (Saavedra 1972, V, 2.7.4). As $\Pi_{C'} = z\Pi_C$, this shows that $\Pi_{C'} = \Pi_C$ if and only if C and C' are conjugate in $G(\mathbb{R})$.

VARIANT 8.5 It is possible to state the above results in terms of bilinear forms. A G-invariant bilinear form $\phi : V_0 \times V_0 \to \mathbb{R}$ on $V_0 \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{R}}(G))$ is a ***C-polarization*** if

$$\phi^C(x, y) \overset{\text{def}}{=} \phi(x, Cy)$$

is a positive-definite symmetric form on V_0. If every object of $\mathsf{Repf}_{\mathbb{R}}(G)$ admits a C-polarization, then C is called a ***Hodge element***. If $G(\mathbb{R})$ contains a Hodge element C, then there is a polarization Π_C on $\mathsf{Repf}_{\mathbb{R}}(G)$ (in the sense of 5.10) for which the positive forms are exactly the C-polarizations. Every polarization on $\mathsf{Repf}_{\mathbb{R}}(G)$ is of the form $\Pi_{C'}$ for some Hodge element C'.

9 Symmetric polarizations

A polarization is said to be ***symmetric*** if its parity is 1.

Let K be a compact algebraic group over $\mathbb{R}$. As 1 is a Hodge element (8.3), $\mathsf{Repf}_{\mathbb{R}}(K)$ has a symmetric polarization Π such that, for any $X_0 \in \mathsf{Repf}_{\mathbb{R}}(K)$,

$$\Pi(X_0) = \{K\text{-invariant positive-definite symmetric bilinear forms on } X_0\},$$

and $\mathsf{Rep}_{\mathbb{C}}(K)$ has a symmetric polarization Π such that, for any $X \in \mathsf{Repf}_{\mathbb{C}}(K)$,

$$\Pi(X) = \{K\text{-invariant positive-definite hermitian forms on } X\}.$$

See 8.1(a) and 8.5.

THEOREM 9.1 *Let* C *be an algebraic tannakian category over* $\mathbb{R}$*, and let* Π *be a symmetric polarization on* C*. Then* C *has a unique (up to isomorphism) fibre functor* ω *with values in* $\mathbb{R}$ *transforming positive bilinear forms for* Π *into positive-definite symmetric bilinear forms. Moreover,* ω *defines a* $\otimes$*-equivalence* $C \to \mathsf{Repf}_{\mathbb{R}}(K)$*, where* $K \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$ *is a compact real algebraic group.*

PROOF Let ω_1 be a fibre functor with values in $\mathbb{C}$, and let $G = \mathcal{A}ut^{\otimes}(\omega_1)$. Because C is polarizable, G has a compact real form K. According to (6.2), $\omega_1' : C' \to \mathsf{Repf}_{\mathbb{C}}(G)$ carries the descent datum on C' defined by C into that on $\mathsf{Repf}_{\mathbb{C}}(G)$ defined by $(\sigma_K, 1)$. It therefore defines a $\otimes$-equivalence $\omega : C \to \mathsf{Repf}_{\mathbb{R}}(K)$ transforming Π into the polarization on $\mathsf{Repf}_{\mathbb{R}}(K)$ defined by the Hodge element 1. The rest of the proof is now obvious. □

REMARK 9.2 Suppose that C has a polarization Π. Then it follows from 7.9 that C has a symmetric polarization if and only if $\varepsilon(\Pi)$ is a square in $Z(\mathbb{R})$.

10 Polarizations with parity ε of order 2

10.1 For $u = \pm 1$, define a ***real u-space*** to be a finite-dimensional complex vector space V together with a semilinear automorphism σ such that $\sigma^2 = u$. A bilinear form ϕ on a real u-space is ***u-symmetric*** if $\phi(x, y) = u\phi(y, x)$. Thus, a 1-symmetric form is a symmetric form, and a -1-symmetric form is a skew-symmetric form. A u-symmetric form on a real u-space is ***positive-definite*** if $\phi(x, \sigma x) > 0$ for all $x \neq 0$.

10.2 Let V_0 be the category (I, 8.5) whose objects are the pairs (V, a) with $V = V^0 \oplus V^1$ a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{C}$ and $a : V \to V$ a semilinear automorphism such that $a^2x = (-1)^{\deg(x)}x$. It is a tannakian category over $\mathbb{R}$ with a $\mathbb{C}$-valued fibre functor $\omega_0 : (V, a) \rightsquigarrow V$. Note that $\mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega_0) = \mu_2 = \{1, \varepsilon\}$.

There is a polarization $\Pi = \Pi^{\mathsf{V}_0}$ on V_0 such that, for each homogeneous V, $\Pi(V, a)$ consists of the $(-1)^{\deg V}$-symmetric positive-definite forms on V.

THEOREM 10.3 *Let* C *be an algebraic tannakian category over* $\mathbb{R}$*, and let* Π *be a polarization on* C *with parity* ε*, where* $\varepsilon^2 = 1$*,* $\varepsilon \neq 1$*. There exists a unique (up to isomorphism) exact* $\mathbb{R}$*-linear* $\otimes$*-functor* $\omega : C \to \mathsf{V}_0$ *such that*

(a) ω *carries the gradation on* C *defined by* ε *into the gradation on* V_0*, i.e.,* $\omega(\varepsilon)$ *acts as* $(-1)^m$ *on* $\omega(V)^m$*;*

(b) ω *carries* Π *into* Π^{V_0}*, i.e.,* $\phi \in \Pi(X)$ *if and only if* $\omega(\phi) \in \Pi^{\mathsf{V}_0}(\omega(X))$*.*

PROOF Note that V_0 is defined by the triple $(\mu_2, \sigma_0, \varepsilon_0)$, where σ_0 is the unique semilinear automorphism of μ_2 and ε_0 is the unique element of $\mu_2(\mathbb{R})$ of order 2. We can assume (by **??**) that C corresponds to a triple (G, σ, ε). Let G_0 be the subgroup of G generated by ε; then $(G_0, \sigma|G_0, \varepsilon) \approx (\mu_2, \sigma_0, \varepsilon_0)$, and so the inclusion $(G_0, \sigma|G_0, \varepsilon) \hookrightarrow (G, \sigma, \varepsilon)$ induces a functor $C \to \mathsf{V}_0$ having the required properties.

Let ω and ω' be two functors $C \to \mathsf{V}_0$ satisfying (a) and (b). There exists an isomorphism $\lambda : \omega \to \omega'$ from ω to ω' viewed as $\mathbb{C}$-valued fibre functors (II, 8.3). As $\lambda_X : \omega(X) \to \omega'(X)$ commutes with action of ε, it preserves the gradations; as λ commutes with $\omega(\phi)$, all $\phi \in \Pi(X)$, it also commutes with σ; it follows that λ is an isomorphism from ω to ω' as functors to V_0. □

10.4 We restate the theorem in terms of Galois groupoids. By definition, $\mathsf{V}_0 = \mathsf{Repf}(\mathcal{G}_0)$, where $\mathcal{G}_0$ is the (unique) nonsplit $\mathbb{C}/\mathbb{R}$-Galois groupoid

$$1 \to \mu_2 \to \mathcal{G}_0 \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

with kernel μ_2. Let C be an algebraic tannakian category over $\mathbb{R}$ with a polarization Π. Choose a $\mathbb{C}$-valued fibre functor ω for C, and let $\mathcal{G} = \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$ regarded as a $\mathbb{C}/\mathbb{R}$-Galois groupoid,

$$1 \to G \to \mathcal{G} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1.$$

If the parity ε of Π is such that $\varepsilon^2 = 1$, $\varepsilon \neq 1$, then there is a unique isomorphism from μ_2 to the subgroup of G generated by ε, and the theorem says that the homomorphism $\mu_2 \hookrightarrow G$ extends uniquely (up to isomorphism) to a homomorphism $\mathcal{G}_0 \to \mathcal{G}$ of $\mathbb{C}/\mathbb{R}$-groupoids,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\ & & \downarrow & & \vdots\downarrow & & \| & & \\ 1 & \longrightarrow & G & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1. \end{array}$$

From this, we get exact $\otimes$-functors

$$\mathsf{C} \xrightarrow[\approx]{\omega} \mathsf{Repf}(\mathcal{G}) \longrightarrow \mathsf{Repf}(\mathcal{G}_0) \overset{\text{def}}{=} \mathsf{V}_0.$$

The exact $\otimes$-functor $\mathsf{C} \to \mathsf{V}_0$ sends Π to the canonical polarization on V_0, and is uniquely determined up to isomorphism by this property.

We have an exact $\mathbb{R}$-linear $\otimes$-functor $\omega : \mathsf{C} \to \mathsf{V}_0$. Suppose that we also have an exact $\mathbb{R}$-linear $\otimes$-functor $s : \mathsf{V}_0 \to \mathsf{C}$ and an isomorphism $\omega \circ s \to \mathrm{id}_{\mathsf{V}_0}$. Then there exists an affine group scheme G in V_0 such that ω factors into

$$\mathsf{C} \xrightarrow[\approx]{} \mathsf{Rep}_{\mathsf{V}_0}(G) \xrightarrow[\text{forget}]{} \mathsf{V}_0, \qquad \mathsf{C} \xrightarrow{\ \omega\ } \mathsf{V}_0.$$

11 Tate triples

Definition and examples

11.1 A ***Tate triple*** T over k is a triple (C, w, T) consisting of

- a tannakian category C over k,
- a $\mathbb{Z}$-gradation $w : \mathbb{G}_m \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ on C (called the ***weight gradation***),
- an invertible object T (called the ***Tate object***) of weight -2.

For any $X \in \mathrm{ob}(\mathsf{C})$ and $n \in \mathbb{Z}$, we set $X(n) = X \otimes T^{\otimes n}$.

A ***fibre functor*** on T with values in a k-algebra R is a fibre functor $\omega : \mathsf{C} \to \mathsf{Mod}(R)$ on C together with the structure

$$u : \omega(T) \otimes \omega(T) \simeq \omega(T^{\otimes 2}) \to \omega(T)$$

of a unit object on $\omega(T)$. There then exists a unique isomorphism $a : \omega(T) \to R$ such that

$$\begin{array}{ccc} \omega(T) \otimes \omega(T) & \xrightarrow{\ u\ } & \omega(T) \\ \downarrow{\scriptstyle a\otimes a} & & \downarrow{\scriptstyle a} \\ R \otimes R & \xrightarrow{\text{multiplication}} & R \end{array}$$

commutes.

If T has a fibre functor with values in k, then T is said to be ***neutral***.

A ***morphism*** of Tate triples $(\mathsf{C}_1, w_1, T_1) \to (\mathsf{C}_2, w_2, T_2)$ is an exact k-linear $\otimes$-functor $\eta : \mathsf{C}_1 \to \mathsf{C}_2$, preserving the gradations, together with an isomorphism $\eta(T_1) \to T_2$.

EXAMPLE 11.2 The triple $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ in which

- $\mathsf{Hod}_{\mathbb{R}}$ is the category of real Hodge structures (II, 9.14),
- w is the weight gradation on $\mathsf{Hod}_{\mathbb{R}}$, and
- $\mathbb{R}(1)$ is the unique real Hodge structure with weight -2 and underlying vector space $2\pi i\mathbb{R}$,

is a neutral Tate triple over $\mathbb{R}$.

EXAMPLE 11.3 (a) The category of finite-dimensional $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$ with the Tate object

$$\mathbb{Q}_B(1) \overset{\text{def}}{=} 2\pi i\mathbb{Q}$$

is a neutral Tate triple T_B over $\mathbb{Q}$.

(b) For l a prime number, the category of finite-dimensional $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}_l$ with the Tate object

$$\mathbb{Q}_l(1) \overset{\text{def}}{=} (\varprojlim_r \mu_{l^r}) \otimes_{\mathbb{Z}} \mathbb{Q}_l, \quad \mu_{l^r} = \{\zeta \in \mathbb{Q}^{\text{al}} \mid \zeta^{l^r} = 1\},$$

is a neutral Tate triple T_l over $\mathbb{Q}_l$.

(c) The category of $\mathbb{Z}$-graded vector spaces over a field k with the Tate object

$$k_{\text{dR}}(1) = k$$

is a neutral Tate triple T_{dR} over k.

See Deligne 1982, §1, for the significance of these examples (they are the natural targets of the Betti, l-adic étale, and de Rham cohomologies).

EXAMPLE 11.4 Let V be the category whose objects are pairs (V, a) with V a $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces and a semilinear automorphism a such that $a^2v = (-1)^n v$ if $v \in V^n$. With the obvious tensor structure, V becomes a tannakian category over $\mathbb{R}$, and $\omega : (V, a) \mapsto V$ is a fibre functor with values in $\mathbb{C}$. We have $\mathbb{G}_m = \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$, and V corresponds (as in **??**(a)) to the triple $(\mathbb{G}_m, g \mapsto \bar{g}, -1)$.

Let $w : \mathbb{G}_m \to \mathbb{G}_m$ be the identity map, and let $T = (\mathbb{C}, z \mapsto \bar{z})$, where $\mathbb{C}$ is viewed as a homogeneous vector space of weight -2. Then (V, w, T) is a non-neutral Tate triple over $\mathbb{R}$.

Neutral Tate triples

Giving a Tate triple and a fibre functor over k is essentially the same as giving an affine group scheme over k and additional data, as we now explain.

11.5 Consider a triple (G, w, t), where

- G is an algebraic group over a field k,
- $w : \mathbb{G}_m \to G$ is a central homomorphism,
- $t : G \to \mathbb{G}_m$ is a homomorphism such that $t \circ w = -2$, i.e., such that $t(w(s)) = s^{-2}$, $s \in \mathbb{G}_m(k)$.

Let T be the representation of G on k such that g acts as multiplication by $t(g)$. Then $(\mathsf{Repf}(G), w, T)$ is a Tate triple over k with the forgetful functor as a k-valued fibre functor.

11.6 Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over k, and let ω be a k-valued fibre functor on T. Let $G = \mathcal{A}ut_k^{\otimes}(\omega)$. Then w is a homomorphism $\mathbb{G}_m \to Z(G) \subset G$, and the action of G on T defines a homomorphism $t : G \to \mathbb{G}_m$ such that $t \circ w = -2$. The equivalence $\mathsf{C} \to \mathsf{Repf}(G)$ of tannakian categories over k defined by ω extends to an equivalence of Tate triples.

Thus, to give a Tate triple over k and a k-valued fibre functor is essentially the same as giving a triple (G, w, t) with $t \circ w = -2$.

In general, a Tate triple T determines a band B, a homomorphism $w : \mathbb{G}_m \to Z$ into the centre Z of B, and a homomorphism $t : B \to \mathbb{G}_m$ such that $t \circ w = -2$. We say that T is ***banded*** by (B, w, t).

The quotient of a Tate triple by its Tate object

Let (C, w, T) be a Tate triple. On setting $T = \mathbb{1}$, we obtain a quotient tannakian category C_0 equipped with a $\mathbb{Z}/2\mathbb{Z}$-gradation defined by an element $\varepsilon \in \mathcal{A}ut(\mathrm{id}_{\mathsf{C}_0})$, $\varepsilon^2 = 1$ (V, 6.14). It is possible to recover (C, w, T) from $(\mathsf{C}_0, \varepsilon)$. We first consider the neutral case.

11.7 Let (G, w, t) be a triple with $t \circ w = -2$, and let $\mathsf{C} = \mathsf{Repf}(G)$ – it has the structure of a Tate triple (11.5). Let $G_0 = \mathrm{Ker}(t : G \to \mathbb{G}_m)$, and let $\mathsf{C}_0 = \mathsf{Repf}(G_0)$. The restriction of w to a homomorphism $\varepsilon : \mu_2 \to G_0$ defines a $\mathbb{Z}/2\mathbb{Z}$-gradation on C_0. The $\otimes$-functor $Q : \mathsf{C} \to \mathsf{C}_0$ defined by the inclusion $G_0 \hookrightarrow G$ has the following properties:

(a) if X is homogeneous of weight n, then $Q(X)$ is homogeneous of weight $n \pmod 2$;

(b) $Q(T) = \mathbb{1}$;

(c) if X and Y are homogeneous of the same weight, then

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(Q(X), Q(Y));$$

(d) if X and Y are homogeneous with weights m and n respectively and $Q(X) \approx Q(Y)$, then $m - n$ is an even integer $2k$ and $X(k) \approx Y$;

(e) Q is essentially surjective.

The first four of these statements are obvious. For the last, note that

$$G = (G_0 \times \mathbb{G}_m)/\mu_2,$$

and so we only have to show that every representation of μ_2 extends to a representation of $\mathbb{G}_m$, but this is obvious.

REMARK 11.8 (a) The identity component of G_0 is reductive if and only if the identity component of G is reductive. If G_0 is connected, then so also is G, but the converse is false (e.g., $G_0 = \mu_2$, $G = \mathbb{G}_m$).

(b) It is possible to reconstruct (C, w, T) from $(\mathsf{C}_0, \varepsilon)$: an object of C is an object of C_0 together with a $\mathbb{Z}$-gradation compatible with its $\mathbb{Z}/2\mathbb{Z}$-gradation (see 6.15).

(c) The following pushout diagram makes it clear how to reconstruct (G, w, t) from (G_0, ε):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \xrightarrow{-2} & \mathbb{G}_m & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle w} & & \| & & \\ 1 & \longrightarrow & G_0 & \longrightarrow & G & \xrightarrow{t} & \mathbb{G}_m & \longrightarrow & 1. \end{array}$$

Note that to give a homomorphism $\varepsilon : \mu_2 \to G$ of algebraic groups is the same as giving an element $\varepsilon \in G(k)$ such that $\varepsilon^2 = 1$.

PROPOSITION 11.9 *Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over k with C algebraic. There exists a tannakian category C_0 over k, an element ε in $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}_0})$ with $\varepsilon^2 = 1$, and a functor $Q : \mathsf{C} \to \mathsf{C}_0$ having the properties 11.7.*

PROOF Indeed, the quotient of C by the tannakian subcategory generated by T has these properties (V, 6.14). We make the construction explicit. For any fibre functor ω on C with values in a k-algebra R, $\mathcal{I}som(R, \omega(T))$, viewed as a sheaf on $\operatorname{Spec} R$, is a torsor for $\mathbb{G}_m$. This association gives rise to a morphism of gerbes

$$\text{FIB}(\mathsf{C}) \xrightarrow{t} \text{TORS}(\mathbb{G}_m),$$

and we define G_0 to be the gerbe of liftings of the canonical section of $\text{TORS}(\mathbb{G}_m)$, i.e., G_0 is the gerbe of pairs (ω, ξ), where ω is a fibre functor on C and ξ is an isomorphism $t(\omega) \to \mathbb{G}_m$ (Giraud 1971, IV, 3.2.1). The category $\mathsf{C}_0 \overset{\text{def}}{=} \mathsf{Repf}(\mathsf{G}_0)$ of representations of G_0 is tannakian (V, 2.1). If $Z = \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ and $Z_0 = \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}_0})$, then the homomorphism

$$\alpha \mapsto \alpha_T : Z \to \mathcal{A}ut(T) = \mathbb{G}_m,$$

determined by t has kernel Z_0, and the composite $t \circ w = -2$. We let $\varepsilon = w(-1) \in Z_0$.

There is an obvious (restriction) functor $Q : \mathsf{C} \to \mathsf{C}_0$. In showing that Q has the properties 11.7, we can make a finite field extension $k \to k'$. We can therefore assume that T is neutral, but this case is covered by (11.6) and (11.7). □

EXAMPLE 11.10 If (V, w, T) is the Tate triple defined in 11.4, then $(\mathsf{V}_0, \varepsilon)$ is the pair defined in 10.2.

REMARK 11.11 The functor $\omega \rightsquigarrow \omega|\mathsf{C}_0$ defines an equivalence from the gerbe of fibre functors on the Tate triple T to the gerbe of fibre functors on C_0.

As in the neutral case, T can be reconstructed from $(\mathsf{C}_0, \varepsilon)$, but there is a stronger result.

THEOREM 11.12 *The functor $\mathsf{T} \rightsquigarrow (\mathsf{C}_0, \varepsilon)$ is an equivalence from the 2-category of Tate triples to the 2-category of $\mathbb{Z}/2\mathbb{Z}$-graded tannakian categories.*

PROOF See Saavedra V, 3.1.4. □

Tate triples over $\mathbb{R}$

11.13 Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over $\mathbb{R}$, and let ω be a fibre functor on T with values in $\mathbb{C}$. On combining **??** with 11.6 we find that (T, ω) corresponds to a quintuple (G, σ, c, w, t) in which

(a) G is an algebraic group over $\mathbb{C}$;

(b) (σ, c) satisfies (105), i.e., $\sigma^2 = \mathrm{ad}(c)$, $\sigma(c) = c$;

(c) $w : \mathbb{G}_m \to G$ is a central homomorphism; that the gradation is defined over $\mathbb{R}$ means that w is defined over $\mathbb{R}$, i.e., $\sigma(w(g)) = w(\bar{g})$;

(d) $t : G \to \mathbb{G}_m$ is such that $t \circ w = -2$; that T is defined over $\mathbb{R}$ means that $t(\sigma(g)) = \overline{t(g)}$ and there exists an $a \in \mathbb{G}_m(\mathbb{C})$ such that $t(c) = \sigma(a) \cdot a$.

Let $G_0 = \mathrm{Ker}(t)$, and let $m \in G(\mathbb{C})$ be such that $t(m) = a^{-1}$. After replacing (σ, c) with $(\sigma \circ \mathrm{ad}(m), \sigma(m) \cdot c \cdot m)$ we find that the new c is in G_0. The pair $(\mathsf{C}_0, \omega|\mathsf{C}_0)$ corresponds to $(G_0, \sigma|G_0, c)$.

In terms of Galois groupoids, we have a diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) & \longrightarrow & \mathbb{G}_m(\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle w} & & \downarrow & & \| & & \\
1 & \longrightarrow & G(\mathbb{C}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle t} & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) & \longrightarrow & \mathbb{G}_m(\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1.
\end{array}$$

12 Polarizations on Tate triples

In this section, $\mathsf{T} = (\mathsf{C}, w, T)$ is a Tate triple over $\mathbb{R}$ with C algebraic. We use the earlier notation; in particular $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Let U be an invertible object of C' that is defined over $\mathbb{R}$, i.e., U is endowed with an identification $U \simeq \bar{U}$. Then in the definitions and results of §4 concerning sesquilinear forms and Weil forms, it is possible to replace $\mathbb{1}$ with U.

DEFINITION 12.1 Suppose that for each $X \in \mathrm{ob}(\mathsf{C}')$ homogeneous of weight n, some $n \in \mathbb{Z}$, we have an equivalence class $\Pi(X)$ of Weil forms $X \otimes \bar{X} \to \mathbb{1}(-n)$ of parity $(-1)^n$. Then Π is a ***graded polarization*** on T if

(a) for all X, $\bar{\phi} \in \Pi(X)$ whenever $\phi \in \Pi(\bar{X})$;

(b) for all X and Y homogeneous of the same weight, $\phi \oplus \psi \in \Pi(X \oplus Y)$ whenever $\phi \in \Pi(X)$ and $\psi \in \Pi(Y)$;

(c) for all homogeneous X and Y, $\phi \otimes \psi \in \Pi(X \otimes Y)$ whenever $\phi \in \Pi(X)$ and $\psi \in \Pi(Y)$;

(d) the map $T \otimes \bar{T} \to T^{\otimes 2} = \mathbb{1}(2)$, defined by $T \simeq \bar{T}$, is in $\Pi(T)$.

PROPOSITION 12.2 *Let $(\mathsf{C}_0, \varepsilon)$ be the quotient of T by its Tate object (11.9). There is a canonical bijection*

$$Q : \mathrm{Pol}(\mathsf{T}) \to \mathrm{Pol}_\varepsilon(\mathsf{C}_0)$$

from the set of graded polarizations on T to the set of polarizations on C_0 of parity ε.

PROOF For any $X \in \mathrm{ob}(\mathsf{C}')$ that is homogeneous of degree n, 11.7(b) and 11.7(c) give an isomorphism

$$Q : \mathrm{Hom}(X \otimes \bar{X}, \mathbb{1}(-n)) \to \mathrm{Hom}(Q(X) \otimes \overline{Q(X)}, \mathbb{1}).$$

We define $Q\Pi$ to be the polarization such that, for any homogeneous X,

$$Q\Pi(QX) = \{Q\phi \mid \phi \in \Pi(X)\}.$$

It is clear that the map $\Pi \mapsto Q\Pi$ is a bijection. □

COROLLARY 12.3 *The Tate triple T is polarizable if and only if C_0 has a polarization Π with parity $\varepsilon(\Pi) \equiv \varepsilon \pmod{Z_0(\mathbb{R})^2}$.*

PROOF Apply 7.9. □

COROLLARY 12.4 *For each $z \in {}_2Z_0(\mathbb{R})$ and polarization Π on T, there is a polarization $z\Pi$ on T defined by the condition*

$$\phi(x,y) \in z\Pi(X) \iff \phi(x,zy) \in \Pi(X).$$

If Π exists, then

$$(z,\Pi) \mapsto z\Pi : {}_2Z_0(\mathbb{R}) \times \mathrm{Pol}(\mathsf{T}) \to \mathrm{Pol}(\mathsf{T})$$

makes $\mathrm{Pol}(\mathsf{T})$ *into a torsor for* ${}_2Z_0(\mathbb{R})$.

PROOF Apply 7.8. □

THEOREM 12.5 *Let Π be a polarization on T, and let ω be a fibre functor on C' with values in $\mathbb{C}$. Let (G,w,t) correspond to $(\mathsf{T}_{(\mathbb{C})},\omega)$ (11.5, 11.6). For any real form K of (G,w,t) such that $K_0 \overset{\text{def}}{=} \mathrm{Ker}(t)$ is compact, the pair (σ_K,ε), where $\varepsilon = w(-1)$, satisfies (105), and ω defines an equivalence between T and the Tate triple defined by $(G,\sigma_K,\varepsilon,w,t)$. For any simple X in C',*

$$\{\omega(\phi)' \mid \phi \in \Pi(X)\}$$

is the set of K_0-invariant positive-definite hermitian forms on $\omega(X)$.

PROOF Apply 6.2. □

REMARK 12.6 Theorem 7.1 implies the following: a triple (B,w,t), where B is an affine algebraic band over $\mathbb{R}$ and $t\circ w = -2$, bounds a polarizable Tate triple if and only if $B_0 \overset{\text{def}}{=} \mathrm{Ker}(t : B \to \mathbb{G}_m)$ is the band defined by a compact real algebraic group; when this condition holds, the polarizable Tate triple banded by (B,w,t) is unique up to a $\otimes$-equivalence preserving the action of B and the polarization, and the equivalence is unique up to isomorphism. The Tate triple is neutral if and only if $\varepsilon \overset{\text{def}}{=} w(-1) \in Z_0(\mathbb{R})^2$.

The neutral case

Let (G,w,t) be a triple as in (11.5) defined over $\mathbb{R}$, and let $G_0 = \mathrm{Ker}(t)$ and $\varepsilon = w(-1)$. A Hodge element $C \in G_0(\mathbb{R})$ is said to be a ***Hodge element*** for (G,w,t) if $C^2 = \varepsilon$. A G-invariant sesquilinear form $\psi : V \times V \to \mathbb{1}(-n)$ on a homogeneous complex representation V of G of degree n is said to be a ***C-polarization*** if

$$\psi^C(x,y) \overset{\text{def}}{=} \psi(x,Cy)$$

is a positive-definite hermitian form on V. When C is a Hodge element for (G,w,t) there is a polarization Π_C on the Tate triple defined by (G,w,t) for which the positive forms are exactly the C-polarizations.

PROPOSITION 12.7 *Every polarization on the Tate triple defined by (G,w,t) is of the form Π_C for some Hodge element C.*

PROOF Apply 8.1 and 8.2. □

Variant

12.8 Let $(\mathsf{C}, w, \mathbb{T})$ be a Tate triple. In particular, C is a rigid tensor category, and so each object X has a dual $(X^\vee, \mathrm{ev}_X)$; moreover,

$$\underline{\mathrm{End}}(X)^\vee \cong (X^\vee \otimes X)^\vee \cong X \otimes X^\vee \cong \underline{\mathrm{End}}(X^\vee). \tag{117}$$

Let X be an object of weight n in C. A nondegenerate $(-1)^n$-symmetric bilinear form $\psi : X \otimes X \to \mathbb{1}(-n)$ on X defines an isomorphism $X \to X^\vee(-n)$, and hence an isomorphism

$$\underline{\mathrm{End}}(X) \to \underline{\mathrm{End}}(X^\vee(-n)) \cong \underline{\mathrm{End}}(X^\vee).$$

This, together with the pairing

$$\mathrm{ev} : \underline{\mathrm{End}}(X)^\vee \otimes \underline{\mathrm{End}}(X) \to \mathbb{1}$$

and the isomorphism (117), gives a symmetric bilinear form

$$T^\psi : \underline{\mathrm{End}}(X) \otimes \underline{\mathrm{End}}(X) \to \mathbb{1}.$$

On $\mathrm{End}(X) \subset \underline{\mathrm{End}}(X)$, T^ψ is the form $(u, v) \mapsto \mathrm{Tr}_X(u^\psi \cdot v)$, and so to say that ψ is a Weil form amounts to saying that the form induced by T^ψ on $\mathrm{End}(X)$ is positive-definite.

Fibre functors on polarized Tate triples

PROPOSITION 12.9 *Let T be a Tate triple over $\mathbb{R}$ and Π a polarization on T. If $w(-1) = 1$, then there is a unique (up to isomorphism) fibre functor ω on T with values in $\mathbb{R}$ transforming the positive bilinear forms for Π into positive-definite symmetric bilinear forms.*

PROOF With the notation of 12.2, $Q\Pi$ is a symmetric polarization on C_0, and so we can apply 9.1. □

In the next proposition, V is the Tate triple defined in 11.4. There is a unique polarization Π^V on V such that, if (V, a) is homogenous of weight n, then

$$\Pi^\mathsf{V}(V, a) = \{(-1)^n\text{-symmetric positive-definite forms on } V\}.$$

THEOREM 12.10 *Let T be a Tate triple over $\mathbb{R}$ and Π a polarization on T. If $w(-1) \neq 1$, then there exists a unique (up to isomorphism) exact $\mathbb{R}$-linear $\otimes$-functor $\omega : \mathsf{C} \to \mathsf{V}$ such that*

(a) *ω is a morphism of Tate triples, and*

(b) *ω carries Π into Π^V.*

PROOF The polarization $Q\Pi$ on C_0 has parity $\varepsilon = w(-1) \neq 1$, and so we can apply 10.3.

In more detail, the functor

$$\mathsf{V} \overset{\text{def}}{=} \mathsf{Repf}(\mathcal{G}) \xrightarrow{Q} \mathsf{V}_0 \overset{\text{def}}{=} \mathsf{Repf}(\mathcal{G}_0)$$

is defined by a morphism of $\mathbb{C}/\mathbb{R}$ Galois groupoids,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1. \end{array}$$

The top row is the unique nontrivial extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\{\pm 1\}$, and the second row is obtained from the first by pushout.

Let $\mathsf{T} = (\mathsf{C}, w, T)$ and Π be as in the statement. Let ω be a $\mathbb{C}$-valued fibre functor on C and let $\mathcal{G}(\omega)$ be the $\mathbb{C}/\mathbb{R}$-Galois groupoid $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\mathbb{C})$. Its kernel is $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$, and w is a homomorphism $\mathbb{G}_m \to Z(G) \subset G$. As in the proof of 10.3, the map $\mu_2(\omega) \to G(\mathbb{C})$ sending -1 to $w(-1)$ extends to a homomorphism of groupoids,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & G(\mathbb{C}) & \longrightarrow & \mathcal{G}(\omega) & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1. \end{array}$$

The homomorphism $w : \mathbb{G}_m \to G$ extends the inclusion $\mu_2 \hookrightarrow G$, and so the homomorphism $\mathcal{G}_0 \to \mathcal{G}(\omega)$ extends to the pushout,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m(\mathbb{C}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle w} & & \downarrow & & \| & & \\ 1 & \longrightarrow & G(\mathbb{C}) & \longrightarrow & \mathcal{G}(\omega) & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1. \end{array}$$

The exact k-linear $\otimes$-functor

$$\mathsf{C} \xrightarrow[\sim]{\omega} \mathsf{Repf}(\mathcal{G}(\omega)) \longrightarrow \mathsf{Repf}(\mathcal{G}) \overset{\text{def}}{=} \mathsf{V}.$$

satisfies (a) and (b).

If ω and ω' are two functors $\mathsf{C} \to \mathsf{V}$ satisfying (a) and (b), then exists an isomorphism $\lambda : \omega \to \omega'$ from ω to ω' viewed as $\mathbb{C}$-valued fibre functors (II, 8.3), and the conditions (a) and (b) imply that λ is an isomorphism from ω to ω' as functors to V. □

EXAMPLE 12.11 Let T be the Tate triple $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ defined in 11.2. A ***polarization*** on a real Hodge structure V of weight n is a bilinear form $\phi : V \times V \to \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi i)^n \phi(x, Cy)$, where C denotes the element $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, is positive-definite and symmetric. These polarizations are the positive (bilinear) forms for a polarization Π on the Tate triple T. The functor $\omega : \mathsf{Hod}_{\mathbb{R}} \to \mathsf{V}$ provided by the last theorem is $V \mapsto (V \otimes \mathbb{C}, v \mapsto C\bar{v})$. (Note that $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ is not quite the Tate triple associated, as in (11.5), with $(\mathbb{S}, w, t)$ because we have chosen a different Tate object; this difference explains the occurrence of $(2\pi i)^n$ in the above formula; Π is essentially the polarization defined by the canonical Hodge element C.)

13 Polarizations on quotient categories

Let V denote the category of $\mathbb{Z}$-graded complex vector spaces endowed with a semilinear automorphism a such that $a^2 v = (-1)^n v$ if $v \in V^n$. It has a natural structure of a Tate triple (11.4) and a canonical polarization Π^{V} (p. 270).

Recall that a morphism $F : (\mathsf{C}_1, w_1, T_1) \to (\mathsf{C}_2, w_2, T_2)$ of Tate triples is an exact $\otimes$-functor $F : \mathsf{C}_1 \to \mathsf{C}_2$ preserving the gradations together with an isomorphism $F(T_1) \simeq T_2$. We say that such a morphism is ***compatible*** with graded polarizations Π_1 and Π_2 on T_1 and T_2 (denoted $F : \Pi_1 \mapsto \Pi_2$) if

$$\psi \in \Pi_1(X) \Rightarrow F\psi \in \Pi_2(FX),$$

in which case, for any X homogeneous of weight n, $\Pi_1(X)$ consists of the sesquilinear forms $\psi : X \otimes \bar{X} \to \mathbb{1}(-n)$ such that $F\psi \in \Pi_2(FX)$. In particular, given F and Π_2, there exists at most one graded polarization Π_1 on T_1 such that $F : \Pi_1 \mapsto \Pi_2$.

Let $\mathsf{T} = (\mathsf{C}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Given a graded polarization Π on T, there exists a morphism of Tate triples $\xi_\Pi : \mathsf{T} \to \mathsf{V}$, unique up to isomorphism, such that $\xi_\Pi : \Pi \mapsto \Pi^{\mathsf{V}}$ (Theorem 12.10). Let ω_Π be the composite

$$\mathsf{C}^{w(\mathbb{G}_m)} \xrightarrow{\xi_\Pi} \mathsf{V}^{w(\mathbb{G}_m)} \xrightarrow{\gamma^{\mathsf{V}}} \mathsf{Vecf}(\mathbb{R}),$$

where $\mathsf{C}^{w(\mathbb{G}_m)}$ is the full subcategory of objects on which $w(\mathbb{G}_m)$ acts trivially (see V, 6.6). Then ω_Π is a fibre functor on $\mathsf{T}^{w(\mathbb{G}_m)}$.

A criterion for the existence of a polarization

PROPOSITION 13.1 *Let $\mathsf{T} = (\mathsf{C}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$, and let $\xi : \mathsf{T} \to \mathsf{V}$ be a morphism of Tate triples. There exists a graded polarization Π on T (necessarily unique) such that $\xi : \Pi \mapsto \Pi^{\mathsf{V}}$ if and only if the real algebraic group $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi | \mathsf{C}^{w(\mathbb{G}_m)})$ is compact.*

PROOF Let $G = \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi | \mathsf{C}^{w(\mathbb{G}_m)})$. Assume Π exists. The restriction of Π to $\mathsf{C}^{w(\mathbb{G}_m)}$ is a symmetric polarization, which the fibre functor $\gamma^{\mathsf{V}} \circ \xi$ maps to the canonical polarization on $\mathsf{Vecf}_{\mathbb{R}}$. This implies that G is compact (1.12).

For the converse, let X be an object of weight n in $\mathsf{C}_{(\mathbb{C})}$. A sesquilinear form $\psi : \xi(X) \otimes \overline{\xi(X)} \to \mathbb{1}(-n)$ arises from a sesquilinear form on X if and only if it is fixed by G. Because G is compact, there exists a $\psi \in \Pi^{\mathsf{V}}(\xi(X))$ fixed by G (1.12), and we define $\Pi(X)$ to consist of all sesquilinear forms ϕ on X such that $\xi(\phi) \in \Pi^{\mathsf{V}}(\xi(X))$. It is now straightforward to check that $X \mapsto \Pi(X)$ is a polarization on T. □

COROLLARY 13.2 *Let $F : (\mathsf{C}_1, w_1, \mathbb{T}_1) \to (\mathsf{C}_2, w_2, \mathbb{T}_2)$ be a morphism of Tate triples, and let Π_2 be a graded polarization on $(\mathsf{C}_2, w_2, \mathbb{T}_2)$. There exists a graded polarization Π_1 on $(\mathsf{C}_1, w_1, \mathbb{T}_1)$ such that $F : \Pi_1 \mapsto \Pi_2$ if and only if the real algebraic group*

$$\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi_{\Pi_2} \circ F | \mathsf{T}_1^{w(\mathbb{G}_m)})$$

is compact.

Polarizations on quotients

The next proposition gives a criterion for a polarization on a Tate triple to pass to a quotient Tate triple.

PROPOSITION 13.3 *Let $\mathsf{T} = (\mathsf{C}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Let (Q, q) be a quotient of T by $H \subset \pi(\mathsf{T})$, and let ω^q be the corresponding fibre functor on C^H (V, 6.11). Assume $H \supset w(\mathbb{G}_m)$, so that Q inherits a Tate triple structure from that on C, and that Q is semisimple. Given a graded polarization Π on T, there exists a graded polarization Π' on Q such that $q : \Pi \mapsto \Pi'$ if and only if $\omega^q \approx \omega_\Pi | \mathsf{C}^H$.*

PROOF $\Rightarrow$: Let Π' be such a polarization on Q, and consider the functors

$$\mathsf{C} \xrightarrow{q} \mathsf{Q} \xrightarrow{\xi_{\Pi'}} \mathsf{V}, \quad \xi_{\Pi'} : \Pi' \mapsto \Pi^{\mathsf{V}}.$$

Both $\xi_{\Pi'}\circ q$ and ξ_Π are compatible with Π and $\Pi^\vee$ and with the Tate triple structures on C and V, and so $\xi_{\Pi'}\circ q \approx \xi_\Pi$ (Theorem 12.10). On restricting everything to $\mathsf{C}^{w(\mathbb{G}_m)}$ and composing with γ^V, we get an isomorphism $\omega_{\Pi'}\circ(q|\mathsf{C}^{w(\mathbb{G}_m)}) \approx \omega_\Pi$. Now restrict this to C^H, and note that

$$\left(\omega_{\Pi'}\circ(q|\mathsf{C}^{w(\mathbb{G}_m)})\right)|\mathsf{C}^H = (\omega_{\Pi'}|\mathsf{Q}^{\pi(\mathsf{Q})})\circ(q|\mathsf{C}^H) \simeq \omega^q$$

because $\omega_{\Pi'}|\mathsf{Q}^{\pi(\mathsf{Q})} \simeq \gamma^Q$.

$\Leftarrow$: The choice of an isomorphism $\omega^q \to \omega_\Pi|\mathsf{C}^H$ determines an exact $\otimes$-functor

$$\mathsf{C}/\omega^q \to \mathsf{C}/\omega_\Pi.$$

As the quotients C/ω^q and C/ω_Π are $\otimes$-equivalent respectively to Q and V, this shows that there is an exact $\otimes$-functor $\xi:\ \mathsf{Q}\to\mathsf{V}$ such that $\xi\circ q \approx \xi_\Pi$. Evidently $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^\vee\circ\xi|\mathsf{Q}^{w(\mathbb{G}_m)})$ is isomorphic to a subgroup of $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^\vee\circ\xi_\Pi|\mathsf{C}^{w(\mathbb{G}_m)})$. Since the latter is compact, so also is the former (1.11). Hence ξ defines a graded polarization Π' on Q (Proposition 13.1), and clearly $q:\ \Pi\mapsto\Pi'$. □

Polarizations on semisimple quotients

LEMMA 13.4 *Let $F:\ \mathsf{C}\to\mathsf{Q}$ be a morphism of Tate triples, and assume that every object of Q is a direct summand of an object in the image of F. Let Π_1 be a polarization on C. There exists a polarization Π_2 on Q such that $F:\ \Pi_1\mapsto\Pi_2$ if and only if, for all X in C and all $\psi\in\Pi_1(X)$, $F\psi$ is a Weil form on FX.*

PROOF $\Rightarrow$: This follows directly from the definitions.

$\Leftarrow$: For Y a direct summand of FX, define $\Pi_2(Y)$ to be the compatibility class of $(F\psi)|Y$ for some $\psi\in\Pi_1(X)$. It is straightforward to verify that the sets $\Pi_2(Y)$ are well-defined and form a polarization on Q. □

Recall that an exact tensor functor $q:\ \mathsf{C}\to\mathsf{Q}$ of Tannakian categories defines a morphism $\pi(q):\ \pi(\mathsf{Q})\to q(\pi(\mathsf{C}))$ (Deligne 1990, 8.15.2).

Let $q:\ \mathsf{C}\to\mathsf{Q}$ be an exact tensor functor, and let H be a closed subgroup of $\pi(\mathsf{C})$. Recall that (Q,q) is a quotient of C by H if $\pi(q)$ is an isomorphism of $\pi(\mathsf{Q})$ onto $q(H)$.

When (Q,q) is a quotient of C by $H\subset\pi(\mathsf{C})$, every object in Q is a subquotient of an object in the image of q. Moreover, q maps C^H into $\mathsf{Q}^{\pi(\mathsf{Q})}$, and so, for $X\in\mathsf{C}^H$, we can identify qX with the vector space $\mathrm{Hom}(\mathbb{1},qX)$. With this identification, there is a functorial isomorphism

$$\mathrm{Hom}_{\mathsf{Q}}(qX,qY)\cong q(\underline{\mathrm{Hom}}(X,Y)^H),\ X,Y\in\mathrm{ob}(\mathsf{C}).$$

PROPOSITION 13.5 *Let $(\mathsf{C},w,\mathbb{T})$ be a Tate triple over $\mathbb{R}$. Let (Q,q) be a quotient of C by $H\subset\pi(\mathsf{C})$, and let Π_1 be a polarization on C. Suppose that $H\supset w(\mathbb{G}_m)$, so that Q inherits a Tate triple structure from that on C, and that Q is semisimple. Assume:*

(*) for all X in C^H and all $\psi\in\Pi_1(X)$, $q\psi$ is a positive-definite form on the vector space qX.

Then there exists a polarization on Π_2 on Q such that $q:\ \Pi_1\mapsto\Pi_2$.

PROOF Because Q is semisimple, every object of Q is a direct summand of an object in the image of q. We shall check the condition in Lemma 13.4.

Let $\psi \in \Pi_1(X)$. Then T^ψ is positive for Π_1, and hence so also is its restriction $T^\psi|$ to $\underline{\mathrm{End}}(X)^H$. Therefore, (*) implies that $q(T^\psi|)$ is a positive-definite form on the vector space $q(\underline{\mathrm{End}}(X)^H)$. But $q(\underline{\mathrm{End}}(X)^H) \cong \mathrm{End}_{\mathsf{Q}}(qX)$ and $q(T^\psi|) \cong T^{q\psi}$, and so $q\psi$ is a Weil form, as required. □

REMARK 13.6 Instead of (*), it suffices to assume that there exists a single X in C^H such that $\pi(\mathsf{C})/H$ acts faithfully on X and a single $\psi \in \Pi(X)$ such that $q\psi$ is a positive-definite form on qX.

14 The Doplicher–Roberts theorem

We explain the relation of the DR theorems (Doplicher and Roberts 1989b, §7) to the results of this chapter. One difference is that the physicists consider categories in which the objects are equipped with a positive-definite hermitian form (for example, the category of finite-dimensional Hilbert spaces) whereas we prefer to categories in which the objects are equipped with a distinguished class of positive-definite hermitian forms (for example, the category $\mathsf{Vecf}(\mathbb{R})$ equipped with its canonical polarization).

Statement of the DR theorems

We state the DR theorems, following Müger 2007.

14.1 A ***positive*** $*$***-operation*** on a $\mathbb{C}$-linear category C is a family of semilinear maps

$$* : \ \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X), \quad X,Y \in \mathrm{ob}\,\mathsf{C},$$

such that $(\mathrm{id}_X)^* = \mathrm{id}_X$ all X, $(s^*)^* = s$ all s, and $s^* \circ s = 0 \ \Rightarrow \ s = 0$. A $*$***-category*** is a $\mathbb{C}$-linear category equipped with a positive $*$-operation. A morphism $v : X \to Y$ in a $*$-category is an ***isometry*** if $v^* \circ v = \mathrm{id}_X$ and is ***unitary*** if, in addition, $v \circ v^* = \mathrm{id}_Y$.

14.2 A ***symmetric monoidal*** $*$***-category*** is a $\mathbb{C}$-linear symmetric monoidal category equipped with a positive $*$-operation for which the constraints are unitary and $(s \otimes t)^* = s^* \otimes t^*$ for all s, t.

14.3 Let X be an object of a symmetric monoidal $*$-category C. An object $\bar{X}$ of C is a ***conjugate object*** if there exist morphisms

$$r : \mathbb{1} \to \bar{X} \otimes X$$
$$\bar{r} : \mathbb{1} \to X \otimes \bar{X}$$

such that

$$\left(X \simeq \mathbb{1} \otimes X \xrightarrow{\bar{r} \otimes X} X \otimes \bar{X} \otimes X \xrightarrow{X \otimes r^*} X \otimes \mathbb{1} \simeq X\right) = \mathrm{id}_X$$
$$\left(\bar{X} \simeq \mathbb{1} \otimes \bar{X} \xrightarrow{r \otimes \bar{X}} \bar{X} \otimes X \otimes \bar{X} \xrightarrow{\bar{X} \otimes \bar{r}^*} \bar{X} \otimes \mathbb{1} \simeq \bar{X}\right) = \mathrm{id}_{\bar{X}}.$$

Such pair $(r, \bar{r})$ is said to be ***standard*** if

$$(r^* \circ \mathrm{id}_{\bar{X}}) \otimes (s \circ r) = (\bar{r}^* \circ s) \otimes (\mathrm{id}_{\bar{X}} \circ \bar{r}), \quad \text{all } s \in \mathrm{End}(X).$$

A standard pair always exists.

DEFINITION 14.4 An ***STC**** is a symmetric monoidal $*$-category with direct sums and subobjects such that

- every $\mathbb{C}$-vector space $\mathrm{Hom}(X, Y)$ is finite dimensional,
- every nonzero object has a conjugate, and
- $\mathrm{End}(\mathbb{1}) \simeq \mathbb{C}$.

There is also the notion of a TC* (drop the symmetry).

EXAMPLE 14.5 The category $\mathsf{Hilbf}_{\mathbb{C}}$ of finite-dimensional Hilbert spaces is an STC*. A standard conjugate of H is $(\bar{H}, r, \bar{r})$, where $\bar{H}$ is the Hilbert space dual of H and

$$r = \sum\nolimits_i f_i \otimes e_i$$
$$\bar{r} = \sum\nolimits_i e_i \otimes f_i$$

for any basis (e_i) of H and its dual (f_i).

In the same way, the category $\mathsf{Repf}(G)^{\mathrm{unitary}}$ of finite-dimensional unitary representations of a compact group G is an STC*.

14.6 Let C be an STC*, and let $X \in \mathrm{ob}\,\mathsf{C}$. For any standard conjugage $(\bar{X}, r, \bar{r})$ of X, let $\Theta(X)$ denote the composite

$$X \simeq \mathbb{1} \otimes X \xrightarrow{r \otimes X} \bar{X} \otimes X \otimes X \xrightarrow{\bar{X} \otimes \gamma_{X,X}} \bar{X} \otimes X \otimes X \xrightarrow{r^* \otimes X} \mathbb{1} \otimes X \simeq X.$$

Then $\Theta(X)$ is independent of the choice of the standard conjugate, $\Theta(X)^2 = \mathrm{id}_X$ all X, and $\Theta : \mathrm{id}_{\mathsf{C}} \to \mathrm{id}_{\mathsf{C}}$ is a monoidal natural transformation (Müger 2007, 343).

For example, $\Theta(X) = \mathrm{id}_X$ for all $X \in \mathsf{Hilbf}$, hence also for all $X \in \mathsf{Repf}^{\mathrm{u}}(G)$

The following is the bosonic version of the Doplicher–Roberts theorem.

THEOREM 14.7 *Let* C *be an STC* such that* $\Theta(X) = \mathrm{id}_X$ *for all* X*. Then there is a compact group* G*, unique up to isomorphism, such that there exists an equivalence* $F : \mathsf{C} \to \mathsf{Repf}^{\mathrm{u}}(G)$ *of symmetric monoidal* $*$*-categories.*

DEFINITION 14.8 Let G be a compact group, and ε an element of order 2 in the centre of G, and let $\mathsf{Repf}^{u}(G, \varepsilon)$ equal $\mathsf{Repf}(G)^{\mathrm{u}}$ as a monoidal $*$-category. The element ϵ defines a $\mathbb{Z}/2\mathbb{Z}$-gradation on $\mathsf{Repf}(G)$, and we use this to change the commutativity constraint,

$$\gamma_\epsilon(x \otimes y) = (-1)^{\deg(x)\deg(y)}(y \otimes x).$$

When $(G, \varepsilon) = (\mathbb{Z}/2\mathbb{Z}, 1)$, we let $\mathsf{sHilb} = \mathsf{Repf}^{u}(G, \varepsilon)$, and call it the category of ***super Hilbert spaces***.

The following is the fermionic version of the Doplicher–Roberts theorem.

THEOREM 14.9 *Let* C *be an STC* such that* $\Theta \neq 1$*. Then there exists a compact group* G *and an element* ϵ *of order* 2 *in the centre of* G*, the pair being unique up to isomorphism, such that there exists an equivalence* $F : \mathsf{C} \to \mathsf{Repf}^{\mathrm{u}}(G, \epsilon)$ *of symmetric monoidal* $*$*-categories.*

The composite of F with the forgetful functor to sHilb is $*$-preserving faithful symmetric monoidal functor.

ASIDE 14.10 Doplicher and Roberts conceived their theorem as an intrinsic characterization of the superselection sectors of a quantum field theory. The theorem is usually stated for C^*-monoidal categories, so, in particular for categories whose Hom sets are Banach algebras. However, the Banach algebra norms are not needed for the DR theorems. More precisely, every C^*-monoidal category with direct sums, subobjects, conjugates, and an irreducible unit (in particular, every C^*-monoidal category satisfying the hypotheses of the DR theorem) is a TC*. Therefore the DR theorem stated above implies the original theorem. Conversely, for any TC*, there are unique norms on the Hom spaces making the category a C^*-monoidal category, so the two statements are essentially the same (Longo and Roberts; Müger 2007, 346).

NOTES This subsection has been extracted from Müger 2007, where the reader can find proofs of the Doplicher–Roberts theorems, independent of the material in this book.

Tannakian interpretation

14.11 Let C' be an STC*. By definition C' is a $\mathbb{C}$-linear category with finite direct sums, so, in particular, it is an additive category. The existence of subobjects (**??**) implies that it is pseudo-abelian, and the existence of the positive $*$-operation implies that the endomorphism algebras are semisimple. Therefore, C' is a semisimple category (1.4). In particular, it is abelian, and so C' is $\mathbb{C}$-linear abelian symmetric monoidal category. Let $X \in \mathrm{ob}\,\mathsf{C}'$, and let $(\bar{X}, r_X, \bar{r}_X)$ be a standard conjugate of X (14.3). Then $(\bar{X}, r^*)$ is a dual of X, and so C' is a tensor category over $\mathbb{C}$.

14.12 For each $X \in \mathrm{ob}\,\mathsf{C}'$, choose a standard conjugate $(\bar{X}, r_X, \bar{r}_X)$ for each X. Then $X \rightsquigarrow X'$ is a semilinear symmetric monoidal functor $F : \mathsf{C}' \to \mathsf{C}'$, and there is a natural isomorphism $\mu_X : X \to JJ(X)$ such that $J(\mu_X)\circ\mu_X = \mathrm{id}_{JX}$ .Therefore, (C', J, μ) defines a tensor category C over $\mathbb{R}$.

14.13 There is a polarization Π on C' such that, for all X, Y in C', there are positive Weil forms ϕ and ψ on X and Y respectively for which the maps

$$*: \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$$
$$*: \mathrm{Hom}(Y,X) \to \mathrm{Hom}(X,Y)$$

provided by the STC* structure are $u \mapsto u^{(\phi,\psi)}$ and $u \mapsto u^{(\psi,\phi)}$ respectively (cf. 4.4). In particular, the category C' is tannakian (5.2.

We now have the following variants of the DR theorems.

THEOREM 14.14 *Let C' be a finitely generated STC* such that $\Theta(X) = \mathrm{id}_X$ for all X. Then there is a unique (up to isomorphism) fibre functor $\mathsf{C} \to \mathsf{Vecf}(\mathbb{R})$ transforming positive bilinear forms Π into postive-definite symmetric bilinear forms. Moreover, ω defines a $\otimes$-equivalence $\mathsf{C} \to \mathsf{Repf}_{\mathbb{R}}(G)$, where $K \stackrel{\mathrm{def}}{=} \mathcal{A}ut^{\otimes}(\omega)$ is a compact real algebraic group.*

PROOF Apply Theorem 9.1. □

NOTES This section has been extracted from Milne 2002. The chapter largely follows Saavedra 1972, Chapt. V, and Deligne and Milne 1982, §§4,5.

Chapter VII

Applications of tannakian categories to motives

As noted in the introduction, Grothendieck introduced tannakian categories to provide a framework for the study of motives. The theory of motives has become a very large topic. Here we describe only a small fragment. In particular, we discuss only pure motives.[1]

After reviewing Grothendieck's construction of the category of motives, we describe his conjectural motivic paradise. We then construct a lesser paradise, based on abelian varieties, with all the properties of Grothendieck's paradise except that its cycles may be only almost-algebraic.

1 Motives

> *One recognizes the Master's touch in the idea that the problem is not to define what a motive is: the problem is to define the category of motives and to identify the structures it carries.*[2]
>
> P. Deligne (1998)

In this section, we describe Grothendieck's original construction of the category of motives corresponding to an adequate equivalence relation.

Throughout, we fix a field k. All algebraic varieties are smooth and projective over k. We let $\mathsf{V}(k)$ denote the category of smooth projective varieties over k, and $\mathsf{V}'(k)$ the category of connected smooth projective varieties over k.[3]

Note that $\mathsf{V}(k)$ becomes a symmetric monoidal category with

- $X \otimes Y = X \times Y$,
- the associativity constraint

$$X \times (Y \times Z) \to (X \times Y) \times Z, \quad (x,(y,z)) \mapsto ((x,y),z),$$

- the commutativity constraint

$$X \times Y \to Y \times X, \quad (x,y) \mapsto (y,x).$$

[1] For a more general introduction to motives, see André 2004.

[2] On reconnaît la patte du Maître dans l'idée que le problème n'est pas de définir ce qu'est un motif: le problème est de définir la catégorie des motifs, et de dégager les structures qu'elle porte.

[3] Recall that an algebraic variety over k is a geometrically reduced separated scheme of finite type over k. By a map of algebraic varieties, we mean a morphism over k, sometimes called a regular map.

Algebraic cycles

1.1 Let X be an algebraic variety of dimension n. The ***group of algebraic cycles*** $Z(X)$ on X is the free abelian group generated by the closed integral subschemes Y of X. It is graded by codimension,

$$Z(X) = \bigoplus\nolimits_r Z^r(X), \quad 0 \le r \le n.$$

If $X = \bigsqcup X_i$ is the decomposition of X into its connected components, then

$$Z(X) \simeq \bigoplus\nolimits_i Z(X_i). \tag{118}$$

The closed integral subschemes of X are in canonical one-to-one correspondence with the points of X: to a closed integral subscheme, attach its generic point; to a point of X, attach its closure. We sometimes regard the points of X as forming a basis for $Z(X)$. Then $\dim(x) \stackrel{\text{def}}{=} \dim \overline{\{x\}}$.

1.2 When Y is a closed irreducible subscheme (not necessarily reduced) of X, the local ring $\mathcal{O}_{Y,\eta}$ at the generic point of Y is artinian, and the class of Y in $Z(X)$ is defined to be

$$[Y] = \text{length}(\mathcal{O}_{Y,\eta})Y_{\text{red}}.$$

1.3 When X is connected of dimension n, the ***degree map*** $\langle\ \rangle : Z(X) \to \mathbb{Z}$ is defined by

$$\langle x\rangle = \begin{cases} [k(x) : k] & \text{if } \operatorname{codim}(x) = n \quad \text{(i.e., } x \text{ is a closed point)}, \\ 0 & \text{if } \operatorname{codim}(x) < n. \end{cases}$$

1.4 When $f : X \to Y$ is a map of algebraic varieties, we define $f_* : Z(X) \to Z(Y)$ to be the $\mathbb{Z}$-linear map such that, for a basis element x of $Z(X)$,

$$f_*(x) = \begin{cases} [k(x) : k(f(x)] \cdot f(x) & \text{if } \dim((f(x)) = \dim(x), \\ 0 & \text{if } \dim(f(x)) < \dim(x). \end{cases}$$

1.5 When Y and Z are closed integral subschemes of X and W is an irreducible component of $Y \cap Z$ such that

$$\operatorname{codim}(W) = \operatorname{codim}(Y) + \operatorname{codim}(Z), \tag{119}$$

we define

$$i(Y, Z; W) = \sum (-1)^i \text{length } \operatorname{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b}),$$

where $R = \mathcal{O}_{W,\eta}$ and $\mathfrak{a}$ and $\mathfrak{b}$ are the ideals of Y and Z in R. We say that Y and Z ***intersect properly*** if (119) holds for all irreducible components W of $Y \cap Z$, and we then set

$$Y \cdot Z = \sum i(Y, Z; W)W.$$

In this way, we obtain a partially defined ***intersection product***

$$a, b \mapsto a \cdot b : Z(X) \times Z(X) \dashrightarrow Z(X).$$

If a and b are homogeneous of degrees i and j, then $a \cdot b$ is homogeneous of degree $i + j$.

1.6 Let $f : X \to Y$ be a map of algebraic varieties. By the ***graph*** of f, we mean either the closed immersion

$$x \mapsto (x, f(x)) : X \to X \times Y$$

or its image Γ_f. If Z is a closed integral subscheme of Y, we let

$$f^*(Z) = p_{X*}(X \times Z \cdot \Gamma_f)$$

when this is defined. In this way, we get a partially defined $\mathbb{Z}$-linear map

$$f^* : Z(Y) \dashrightarrow Z(X).$$

When f is flat, we can extend f^* to the whole of $Z(Y)$ by setting

$$f^*(Z) = [f^{-1}(Z)],$$

where $f^{-1}(Z)$ is the closed subscheme $X \times_Y Z$ of X.

1.7 Let $f : X \to Y$ be a map of algebraic varieties. The operations f_* and f^* are related by the ***projection formula***

$$f_*(f^*(b) \cdot a) = b \cdot f_*(a), \quad a \in Z(X),\ b \in Z(Y), \tag{120}$$

whenever both sides are defined.

1.8 An ***adequate equivalence relation*** is a family of equivalence relations $\sim_X$ on the graded groups $Z(X)$ that is respected by both f^* and f_* and by intersections, and that satisfies the following condition: for all $a, b \in Z(X)$, there exist $a' \sim_X a$ and $b' \sim_X b$ such that $a' \cdot b'$ is defined.

Let $\sim$ be an admissible equivalence relation. There is a unique multiplication on $Z_\sim(X) \overset{\text{def}}{=} Z(X)/\sim$ such that

$$\begin{array}{ccccc} Z(X) & \times & Z(X) & \overset{\cdot}{\dashrightarrow} & Z(X) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ Z_\sim(X) & \times & Z_\sim(X) & \longrightarrow & Z_\sim(X) \end{array}$$

commutes. It makes $Z_\sim(X)$ into a graded ring.

Let $f : X \to Y$ be a map of algebraic varieties. Then $f^* : Z_\sim(Y) \to Z_\sim(X)$ is a homomorphism of graded rings and $f_* : Z_\sim(X) \to Z_\sim(Y)$ is a homomorphism of abelian groups. The two operations are related by the projection formula (120).

If $X \xrightarrow{f} X' \xrightarrow{f'} X''$ are maps of algebraic varieties, then

$$f'_* \circ f_* = (f' \circ f)_*, \quad f^* \circ f'^* = (f' \circ f)^*.$$

Thus, $Z_\sim$ is a contravariant functor of graded rings and a covariant functor of abelian groups, with the two structures being related by the projection formula (120).

1.9 Two algebraic cycles Z and Z' on X are said to be ***rationally equivalent*** (denoted $Z \sim_{\text{rat}} Z'$) if one can be transformed into the other by a series of rational deformations.[4]

[4] In more detail, $Z \sim_{\text{rat}} 0$ if there is a cycle W on $X \times \mathbb{P}^1$ and two points $a, b \in \mathbb{P}^1(k)$ such that $W(a) \overset{\text{def}}{=} \text{pr}_2^*(a)$ and $W(b) \overset{\text{def}}{=} \text{pr}_2^*(b)$ are defined and $Z = W(a) - W(b)$.

They are ***algebraically equivalent*** (denoted $Z \sim_{\text{alg}} Z'$) if one can be transformed into the other by algebraic deformations. They are ***homologically equivalent relative to some Weil cohomology theory*** H (denoted $Z \sim_H Z'$) if they have the same cohomology class for H, and they are ***homology equivalent*** (denoted $Z \sim_{\text{hom}} Z'$) if they are homologically equivalent relative to every Weil cohomology satisfying the weak Lefschetz theorem (see below for this terminology). They are ***numerically equivalent*** (denoted $Z \sim_{\text{num}} Z'$) if their intersection numbers with any subvariety of complementary dimension coincide. We have

$$\sim_{\text{rat}} \succ \sim_{\text{alg}} \succ \sim_{\text{hom}} \succ \sim_{\text{num}} .$$

Rational equivalence is the finest adequate equivalence relation and numerical equivalence is the coarsest.

1.10 The above are the classical adequate equivalence relations. Grothendieck's standard conjectures (see below) predict that, for all the standard cohomology theories, $\sim_H = \sim_{\text{num}}$, and it was sometimes speculated that $\sim_{\text{alg}} = \sim_{\text{num}}$, but, in fact, $\sim_{\text{alg}} \neq \sim_{\text{hom}}$.

Voevodsky introduced the notion of smash-nilpotence and conjectured that it coincides with numerical equivalence. An algebraic cycle Z on a smooth projective variety X (with $\mathbb{Q}$-coefficients) is ***smash-nilpotent*** if there exists an integer $n > 0$ such that $Z^n \overset{\text{def}}{=} Z \times \cdots \times Z$ is rationally equivalent to zero on X^n. We have

$$\sim_{\text{alg}} \succ \sim_{\text{nil}} \succ \sim_{\text{hom}} \succ \sim_{\text{num}} .$$

NOTES For more details, see Saavedra 1972, VI, Appendice, A 0, or André 2004, 3.2.

Motives

We fix an admissible equivalence relation $\sim$ and write $C(X)$ for $C_\sim(X) \overset{\text{def}}{=} Z_\sim(X) \otimes \mathbb{Q}$.

1.11 For algebraic varieties X and Y with X connected, we let

$$C^r(X,Y) = C^{\dim X + r}(X \times Y)$$

(***correspondences of degree*** r from X to Y). When $X = \bigsqcup_{i\in I} X_i$ is the decomposition of X into its connected components, we let

$$C^r(X,Y) = \bigoplus\nolimits_{i\in I} C^r(X_i, Y).$$

For algebraic varieties X, Y, Z, there is a bilinear pairing

$$f, g \mapsto g \circ f : C^r(X,Y) \times C^s(Y,Z) \to C^{r+s}(X,Z)$$

with

$$g \circ f \overset{\text{def}}{=} (p_{XZ})_*(p_{XY}^* f \cdot p_{YZ}^* g).$$

Here, the p are the projection maps from $X \times Y \times Z$,

$$\begin{array}{ccccc} X \times Y & \xleftarrow{p_{XY}} & X \times Y \times Z & \xrightarrow{p_{YZ}} & Y \times Z \\ & & \downarrow{\scriptstyle p_{XZ}} & & \\ & & X \times Z. & & \end{array}$$

These pairing are associative, and so we get a category $\mathsf{CV}(k)$ of ***correspondences***, which has one object hX for each variety over k, and whose Homs are defined by

$$\mathrm{Hom}(hX, hY) = C^0(X, Y) = C^{\dim X}(X \times Y).$$

Let $f : X \to Y$ be a map algebraic varieties. The transpose of the graph of f is an element of $C^0(Y, X)$, and $X \rightsquigarrow hX$ is a contravariant functor.

1.12 The category $\mathsf{CV}(k)$ is $\mathbb{Q}$-linear with direct sums,

$$hX \oplus hY = h(X \times Y).$$

There is a structure of a $\mathbb{Q}$-linear $\otimes$-category on $\mathsf{CV}(k)$ for which

- $hX \otimes hY = h(X \times Y)$,
- the associativity constraint is induced by $X \times (Y \times Z) \to (X \times Y) \times Z$,
- the commutativity constraint is induced by $X \times Y \to Y \times X$,
- $h(\text{point})$ is a unit object.

1.13 The category $\mathsf{CV}(k)$ is not pseudo-abelian (much less abelian). Recall that a category is pseudo-abelian if it is additive and if, for every idempotent endomorphism e of an object M, there is a decomposition

$$M = M_1 \oplus M_2$$

with $e|M_1 = \mathrm{id}_{M_1}$ and $e|M_2 = 0$. Then M_1 is the image eM of M. To construct the category $\mathsf{M}^+(k)$ of ***effective motives***, we enlarge $\mathsf{CV}(k)$ by adding images of idempotents. More precisely, we define $\mathsf{M}^+(k)$ to be the category with one object $h(X, p)$ for each algebraic variety X and idempotent p in the ring $\mathrm{End}(hX) \stackrel{\text{def}}{=} C^{\dim X}(X \times X)$, and with

$$\mathrm{Hom}(h(X, p), h(Y, q)) = q \circ \mathrm{Hom}(hX, hY) \circ p = q \circ C^0(X, Y) \circ p.$$

When we identify hX with $h(X, \Delta)$, $\mathsf{CV}(k)$ becomes a subcategory of $\mathsf{M}^+(k)$, and $h(X, p)$ becomes the image of $p : hX \to hX$. The rule

$$h(X, p) \otimes h(Y, q) = (hX \otimes hY, p \otimes q) \stackrel{\text{def}}{=} h(X \times Y, p \times q)$$

makes $\mathsf{M}^+(k)$ into a $\mathbb{Q}$-linear pseudo-abelian symmetric monoidal category.

1.14 The category $\mathsf{M}^+(k)$ is not rigid. In $\mathsf{M}^+(k)$, the motive $h\mathbb{P}^1$ decomposes as $h\mathbb{P}^1 = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1$, and it turns out that to obtain duals for all objects, it suffices to "invert" $h^2\mathbb{P}^1$. This is most conveniently done by defining the category $\mathsf{M}(k)$ of motives to have one object $h(X, p, m)$ for each pair (X, p) as before and integer m, and whose Homs are defined by

$$\begin{aligned}\mathrm{Hom}((X, p, m), (Y, q, n)) &= q \circ C^{m-n}(X, Y) \circ p \\ &= q \circ C^{\dim(X)+m-n}(X \times Y) \circ p.\end{aligned}$$

The tensor product on $\mathsf{M}^+(k)$ extends to $\mathsf{M}(k)$,

$$h(X, p, m) \otimes h(Y, q, n) = h(X \times Y, p \times q, m + n).$$

When X is connected of dimension n, there is a canonical decomposition

$$hX = h^0X \oplus h'X \oplus h^nX$$

(Saavedra 1972, VI, 4.1.2). For example,

$$h\mathbb{P}^1 = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1.$$

We define the ***Lefschetz motive*** to be $h^2\mathbb{P}^1$, and note that in passing from $\mathsf{M}^+(k)$ to $\mathsf{M}(k)$, we have inverted $h^2\mathbb{P}^1$ to get the ***Tate motive*** T. For X connected of dimension n, $h^0X \simeq \mathbb{1}$ and $h^nX \simeq \left(h^2\mathbb{P}^1\right)^{\otimes n}$.

When we identify $h(X,p)$ with $h(X,p,0)$, $\mathsf{M}^+(k)$ becomes a subcategory of $\mathsf{M}(k)$. We set $M(n) = M \otimes T^{\otimes n}$, so $(h^nX)(n) \simeq \mathbb{1}$ when X is connected of dimension n.

THEOREM 1.15 *The category of motives* $\mathsf{M}(k)$ *is a rigid pseudo-abelian symmetric monoidal category such that* $\mathrm{End}(\mathbb{1}) \simeq \mathbb{Q}$.

PROOF The tensor structure on $\mathsf{M}(k)$ is symmetric monoidal – this uses that the permutation (123) acts as the identity on $\left(h^2\mathbb{P}^1\right)^{\otimes 3}$. It is pseudo-abelian by construction and $\mathrm{End}(\mathbb{1}) \simeq \mathbb{Q}$, so it remains to show that it is rigid.

Let X be a connected algebraic variety over k. Let

$$h(X)^\vee = h(X)(n)$$

and define

$$\mathrm{ev}_X : h(X)^\vee \otimes h(X) \to \mathbb{1}$$

to be the composite of the morphisms

$$h(X)^\vee \otimes h(X) \simeq h(X \times X)(n) \xrightarrow{h(\Delta)} h(X)(n) \to h^n(X)(n) \simeq \mathbb{1}.$$

There exists a coevaluation map $\delta : \mathbb{1} \to h(X) \otimes h(X)^\vee$ satisfying (19), p. 21. This construction extends to $\mathsf{M}(k)$ because of the universal properties of the functors $\mathsf{V}'(k) \to \mathsf{V}(k)$ (for direct sums), $\mathsf{CV}(k) \to \mathsf{M}^+(k)$, and $\mathsf{M}^+(k) \to \mathsf{M}(k)$ (Saavedra 1972, VI, 4.1.3.5). □

Alas, as we shall see, it is not an abelian category except when $\sim{=}$ num, in which case it is semisimple.

An ***ideal*** $\mathcal{I}$ in an F-linear category T is a family of F-subspaces $\mathcal{I}(A,B) \subset \mathrm{Hom}(A,B)$, $A, B \in \mathrm{ob}\,\mathsf{T}$, stable under left and right composition by morphisms in T. The quotient category $\mathsf{T}/\mathcal{I}$ has the same objects as T but with

$$\mathrm{Hom}_{\mathsf{T}/\mathcal{I}}(A,B) = \mathrm{Hom}_{\mathsf{T}}(A,B)/\mathcal{I}(A,B).$$

When T is a symmetric monoidal category, we say that $\mathcal{I}$ is a ***tensor ideal*** if it stable under tensor products with morphisms of the form $\mathrm{id}_C \otimes f$ and $f \otimes \mathrm{id}_C$, $C \in \mathrm{ob}\,\mathsf{T}$. It is then stable under tensor products with any morphism, and $\mathsf{T}/\mathcal{I}$ acquires a tensor structure from T.

PROPOSITION 1.16 *Every tensor ideal* $\mathcal{I}$ *in* $\mathsf{M}_{\mathrm{rat}}(k)$ *is of the form* $\mathcal{I}_\sim$ *for some adequate equivalence relation* $\sim$. *The category* $\mathsf{M}_\sim(k)$ *is then the pseudo-abelian envelope of* $\mathsf{M}_{\mathrm{rat}}(k)/\mathcal{I}_\sim$.

PROOF See André 2004, 4.4.1.1. □

NOTATION 1.17 Let F be a field of characteristic zero, and replace $\mathbb{Q}$ with F in the above; for example, set $C_\sim(X) = Z_\sim(X) \otimes_{\mathbb{Z}} F$. We then get an F-linear category of motives $\mathsf{M}_\sim(k)_F$.

The following notation is common,

$$\begin{aligned} \mathrm{CH}_{\mathbb{Q}}(X) &= C_{\mathrm{rat}}(X) \\ \mathsf{CHM}(k) &= \mathsf{M}_{\mathrm{rat}}(k) && \text{(Chow motives)} \\ \mathsf{NM}(k) &= \mathsf{M}_{\mathrm{num}}(k) && \text{(numerical motives).} \end{aligned}$$

QUESTION 1.18 The category $M_H(k)$ of motives over k relative to a Weil cohomology theory H is a pseudo-abelian tensor category over $\mathbb{Q}$. If $M_H(k)$ is semisimple at $\mathbb{1}$ (i.e., every morphism $X \to \mathbb{1}$ splits) and the Lefschetz standard conjecture holds for H, does $M_H(k)$ admit an abelian envelope (in the sense of I, 12.44)?

NOTES For more details, see Scholl 1994.

Weil cohomology theories

DEFINITION

1.19 We fix a field k and a field Q of characteristic zero. A contravariant functor $X \rightsquigarrow H^*(X)$ from the category of smooth projective varieties over k to the category of finite-dimensional, graded, anti-commutative Q-algebras is said to be a ***Weil cohomology theory*** if it carries disjoint unions to direct sums and admits a Poincaré duality, a Künneth formula, and a cycle map.

Poincaré duality Let X be a connected smooth projective variety over k of dimension d.

(a) The vector spaces $H^s(X)$ are zero except for $0 \le s \le 2d$, and $H^{2d}(X)$ has dimension 1.

(b) Let $Q(-1) = H^2(\mathbb{P}^1)$. For any Q-vector space V and integer m, let $V(m) = V \otimes_Q Q(-1)^{\otimes -m}$ or $V \otimes_Q Q(-1)^{\vee \otimes m}$ according as m is positive or negative. Then, for each X, there is given a natural isomorphism $\eta_X : H^{2d}(X)(d) \to Q$.

(c) The pairings
$$H^r(X) \times H^{2d-r}(X)(d) \to H^{2d}(X)(d) \simeq Q$$
induced by the product structure on $H^*(X)$ are non-degenerate.

Let $\phi : X \to Y$ be a morphism of smooth projective varieties over k, and let $\phi^* = H^*(\phi) : H^*(Y) \to H^*(X)$. Because the pairing in (c) is nondegenerate, there is a unique linear map
$$\phi_* : H^*(X) \to H^{*+2c}(Y)(c), \quad c = \dim Y - \dim X$$
such that the projection formula
$$\eta_Y(\phi_*(x) \cup y) = \eta_X(x \cup \phi^* y)$$
holds for all $x \in H^{2\dim X - 2s}(X)(\dim X - s)$, $y \in H^{2s}(Y)(s)$.

Künneth formula Let $p, q : X \times Y \to X, Y$ be the projection maps. Then the map
$$x \otimes y \mapsto p^*x \cup q^*y : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$
is an isomorphism of graded k-algebras.

Cycle map There are given homomorphisms

$$\mathrm{cl}_X : \mathrm{CH}^r_{\mathbb{Q}}(X) \to H^{2r}(X)(r)$$

satisfying the following conditions:

(a) (functoriality) For any regular map $\phi : X \to Y$,

$$\phi^* \circ \mathrm{cl}_Y = \mathrm{cl}_X \circ \phi^*, \quad \phi_* \circ \mathrm{cl}_X = \mathrm{cl}_Y \circ \phi_*.$$

(b) (multiplicativity) For any X, Y

$$\mathrm{cl}_{X\times Y}(Z \times W) = \mathrm{cl}_X(Z) \otimes \mathrm{cl}_Y(W).$$

(c) (non-triviality) If P is a point, so $\mathrm{CH}^*_{\mathbb{Q}}(P) = \mathbb{Q}$ and $H^*(P) = Q$, then cl_P is the natural inclusion map $\mathbb{Q} \hookrightarrow Q$.

In the functoriality statement, the ϕ^* and ϕ_* on the right of the equality signs refer to the standard operations on the $\mathbb{Q}$-algebras of algebraic cycles modulo rational equivalence (see 4.3).

The field Q is called the ***coefficient field*** for the Weil cohomology theory. Note that if $X \rightsquigarrow H^*(X)$ is a Weil cohomology theory with coefficient field Q, and $Q' \supset Q$, then $X \rightsquigarrow H^*(X) \otimes_Q Q'$ is a Weil cohomology theory with coefficient field Q'.

PROPOSITION 1.20 *Let $i : Z \hookrightarrow X$ be a smooth closed subvariety of X. Then $\mathrm{cl}_X(Z) = i_*(1_Z)$, where 1_Z is the identity element of the algebra $H^*(Z)$.*

PROOF Let $P = \operatorname{Spec} k$ and let $\phi : Z \to P$ be the structure map. Then

$$1_Z = \phi^*(1_P) = \phi^*(\mathrm{cl}_P(P)) = \mathrm{cl}_Z(\phi^* P) = \mathrm{cl}_Z(Z).$$

Therefore

$$i_*(1_Z) = i_*(\mathrm{cl}_Z(Z)) = \mathrm{cl}_X(i_*(Z)) = \mathrm{cl}_X(Z).$$

□

An element of $H^{2r}(X)(r)$ is said to be ***algebraic*** if it is in the image of $\mathrm{cl}_X : \mathrm{CH}^r_{\mathbb{Q}}(X) \to H^{2r}(X)(r)$.

COHOMOLOGICAL CORRESPONDENCES

Let H be a Weil cohomology theory. We define

$$\langle\ \rangle : H^*(X) \to Q$$

to be zero on $H^i(X)$ for $i < 2\dim(X)$ and η_X on $H^{2\dim(X)}(X)$.

1.21 From the Künneth formula and Poincaré duality, we obtain canonical isomorphisms

$$H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y) \simeq H_*(X) \otimes H^*(Y) \simeq \mathrm{Hom}(H^*(X), H^*(Y)).$$

Specifically, ($n = \dim(X)$)

$$\begin{aligned} H^{2n+i}(X \times Y) &\simeq \bigoplus\nolimits_j H^{2n-j}(X) \otimes H^{j+i}(Y) \quad \text{(Künneth formula)} \\ &\simeq \bigoplus\nolimits_j H^j(X)^\vee \otimes H^{j+i}(Y) \quad \text{(Poincaré duality)} \\ &\simeq \bigoplus\nolimits_j \mathrm{Hom}(H^j(X), H^{j+i}(Y)) \quad \text{(linear algebra).} \end{aligned}$$

Let $u = \sum_j a^{2n-j} \otimes b^{j+i}$ and $c \in H^{j'}(X)$; then

$$u(c) = \sum_j \langle c \cdot a^{2n-j} \rangle b^{j+i} = \langle c \cdot a^{2n-j'} \rangle b^{j'+i}.$$

More generally, an element $u = a \otimes b \in H^*(X \times Y)$ corresponds to the Q-linear map

$$u : H^*(X) \to H^*(Y), \quad u(c) = \langle c \cdot a \rangle b; \tag{121}$$

An arbitrary element $u \in H^*(X \times Y)$ corresponds to the composite

$$H^*(X) \xrightarrow{p_X^*} H^*(X \times Y) \xrightarrow{v \mapsto v \cdot u} H^*(X \times Y) \xrightarrow{p_{Y*}} H^*(Y),$$

i.e.,

$$u(a) = p_{Y*}(p_X^*(a) \cdot u), \quad a \in H^*(X). \tag{122}$$

1.22 Composition of correspondences is defined as composition of the linear maps. Explicitly, for $u \in H^*(X \times Y)$ and $v \in H^*(Y \times Z)$,

$$\begin{aligned} v \circ u &= (p_{X\times Z})_*(u \otimes 1_Z \cdot 1_X \otimes v) \\ &= (p_{X\times Z})_*(p_{X\times Y}^*(u) \cdot p_{Y\times Z}^*(v)). \end{aligned}$$

The self-correspondences of X form an associative anti-commutative graded Q-algebra under composition,

$$H^*(X \times X) \simeq \mathrm{End}(H^*(X)).$$

1.23 All of this is compatible with the cycle class maps. A map $u : H^*(X) \to H^*(Y)$ is said to be ***algebraic*** if there exists an algebraic class γ such that

$$\begin{array}{ccc} H^*(X \times Y) & \xrightarrow{\cup\gamma} & H^*(X \times Y) \\ {\scriptstyle \mathrm{pr}_X^*}\uparrow & & \downarrow{\scriptstyle \mathrm{pr}_{Y*}} \\ H^*(X) & \xrightarrow{u} & H^*(Y) \end{array}$$

commutes.

1.24 Let $f : X \to Y$ be a morphism. Then f_* is the map defined by Γ_f (as an element of $H^*(X \times Y)$) and f^* is the map defined by Γ_f^t. [5]

1.25 Let $a \in H^*(X)$, and let $u : H^*(X) \to H^*(X)$ be right multiplication $x \mapsto xa$ by a. Then u, as an element of $H^*(X \times X)$, is $\delta_* a$ where $\delta : X \to X \times X$ is the diagonal map. In particular, u is equal to its own transpose u^t.

Indeed, if $q : X \times X \to X$ is the projection onto the second factor, then

$$(\delta_*(a))^* x = q_*((x \otimes 1) \cdot (\delta_*(a))) = (q_* \circ \delta_*)(\delta^*(x \otimes 1) \cdot a) = x \cdot a.$$

[5] In many accounts, this is reversed because Γ_f and Γ_f^t are reversed.

The Lefschetz trace formula

We fix a Weil cohomology theory H. Let X and Y be varieties of dimension n and m. For $u \in H^*(X \times X)$ of degree 0, we let $\mathrm{Tr}_i(u)$ denote the trace of u acting on $H^i(X)$.

PROPOSITION 1.26 *Let $v \in H^*(X \times Y)$, and $w \in H^*(Y \times X)$ be correspondences of degrees d, $-d$ respectively. Then*

$$\langle v \cdot w^t \rangle = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}_i(w \circ v).$$

PROOF We may regard v and w as elements of $H^*(X) \otimes H^*(Y)$ and $H^*(Y) \otimes H^*(X)$ respectively, and we may suppose that

$$\begin{aligned} v &\in H^{2n-i}(X) \otimes H^j(Y) \simeq H_i(X) \otimes H^j(Y) \simeq \mathrm{Hom}(H^i(X), H^j(Y)) \\ w &\in H^{2m-j}(Y) \otimes H^i(X) \simeq H_j(Y) \otimes H^i(X) \simeq \mathrm{Hom}(H^j(Y), H^i(X). \end{aligned}$$

where $j = i + d$. Choose a basis $e_1, e_2, \ldots$ for $H^i(X)$, and let $f_1, f_2, \ldots$ be the dual basis for $H^{2n-i}(X)$, so $\langle f_\ell \cdot e_k \rangle = \delta_{\ell k}$ and $\langle e_\ell \cdot f_k \rangle = (-1)^i \delta_{\ell k}$ (the algebras are anti-commutative). Write

$$\begin{aligned} v &= \textstyle\sum_\ell f_\ell \otimes y_\ell \in H^{2n-i}(X) \otimes H^j(Y) \\ w &= \textstyle\sum_k x_k \otimes e_k \in H^{2m-j}(Y) \otimes H^i(X), \end{aligned}$$

so

$$w^t = \textstyle\sum_k (-1)^{ij} e_k \otimes x_k.$$

Then

$$\begin{aligned} \langle v \cdot w^t \rangle &= \textstyle\sum_{k,\ell} (-1)^{ij} \langle f_\ell \otimes y_\ell \cdot e_k \otimes x_k \rangle \\ &= \textstyle\sum_{k,\ell} \langle f_\ell \cdot e_k \rangle \langle y_\ell \cdot x_k \rangle \\ &= \textstyle\sum_\ell \langle y_\ell \cdot x_\ell \rangle \end{aligned}$$

On the other hand

$$\begin{aligned} v(e_k) &= \Big(\textstyle\sum_\ell f_\ell \otimes y_\ell\Big)(e_k) = \textstyle\sum_\ell \langle e_k \cdot f_\ell \rangle y_\ell = (-1)^i y_k \\ (w \circ v)(e_k) &= (-1)^i w(y_k) = (-1)^i (\textstyle\sum_\ell x_\ell \otimes e_\ell)(y_k) = (-1)^i \langle y_k \cdot x_k \rangle e_k + \cdots, \end{aligned}$$

so

$$\mathrm{Tr}_i(w \circ v) = (-1)^i \textstyle\sum_k \langle y_k \cdot x_k \rangle = (-1)^i \langle v \cdot w^t \rangle.$$

□

PROPOSITION 1.27 *Let $u \in H^*(X \times X)$ be a correspondence of degree zero.*

(a) *(Trace formula)*

$$\mathrm{Tr}_i(u) = (-1)^i \langle u \cdot \pi^{2n-i} \rangle.$$

(b) *(Lefschetz fixed-point formula)*

$$\langle u \cdot \Delta \rangle = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}_i(u).$$

PROOF Both statements are special cases of 1.26. □

Let X_0 be a smooth projective variety over $\mathbb{F}_q$. For each m, we let N_m be the number of points on X_0 with coordinates in $\mathbb{F}_{q^m}$, and we define the zeta function of X_0 to be

$$Z(X_0,t) = \exp\Big(\sum_{m\geq 1} N_m \frac{t^m}{m}\Big) = 1 + \sum_{m\geq 1} N_m \frac{t^m}{m} + \frac{1}{2!}\Big(\sum_{m\geq 1} N_m \frac{t^m}{m}\Big)^2 + \cdots$$

It is a formal power series with coefficients in $\mathbb{Q}$, i.e., $Z(X_0,t) \in \mathbb{Q}[[t]]$. Note that

$$\frac{d}{dt}\log Z(X_0,t) = \sum_{m\geq 1} N_m t^{m-1},$$

so $\frac{d}{dt}\log Z(X_0,t)$ is essentially the generating function for the sequence $N_1, N_2, N_3, \ldots$.

Let A_0 be an affine $\mathbb{F}_q$-algebra. Then $a \mapsto a^q$ is a homomorphism $f_0 : A_0 \to A_0$ of $\mathbb{F}_q$-algebras. By extension of scalars, we get a homomorphism $f : A \to A$ of $\mathbb{F}$-algebras, where $A = A_0 \otimes_{\mathbb{F}_q} \mathbb{F}$. The corresponding morphism $F : \operatorname{Spec} A \to \operatorname{Spec} A$ is called the ***Frobenius map***. For a variety X_0 over $\mathbb{F}_q$, the ***Frobenius map*** $F : X \to X$ is the unique morphism such that, for every open affine $U_0 \subset X_0$, $F(U) \subset U$ and $F|U$ is the Frobenius map on U. The Frobenius map acts on any subvariety of $\mathbb{P}^n$ as $(t_0 : \ldots : t_n) \mapsto (t_0^q : \ldots : t_n^q)$.

COROLLARY 1.28 *Let X be a smooth projective geometrically irreducible variety of dimension n over $\mathbb{F}_q$, and let H be a Weil cohomology theory. Then*

$$Z(X/\mathbb{F}_1, T) = \prod_{i=0}^{2n} P^i(X/\mathbb{F}_q, T)^{(-1)^{i+1}},$$

where $P^i(X/\mathbb{F}_q, T) = \det(1 - TF \mid H^i(X)$.

PROOF An element a of $\mathbb{F}$ lies in $\mathbb{F}_q$ if and only if $a^q = a$, and so it is clear from the description of F in terms of the coordinates of points that the $X^F = X(\mathbb{F}_q)$. At a fixed point P of F, the graph of F intersects the diagonal transversally, and so $(\Gamma_F \cdot \Delta)_P = 1$. Now apply Proposition 1.27 to the graph of F. □

ASIDE 1.29 Recall (4.11) that, for any adequate equivalence relation, $M_\sim(k)$ is a rigid pseudo-abelian $\otimes$-category such that $\operatorname{End}(\mathbb{1}) \simeq \mathbb{Q}$. Kimura and O'Sullivan conjecture that every object X of $M_\sim(k)$ decomposes into a sum $X = X^+ \oplus X^-$ with $\bigwedge^n X^+ = 0 = \operatorname{Sym}^n X^-$. The conjecture implies that the radical of $\operatorname{End}(X)$ is nilpotent for all objects X (see André 2004, 12.1).

WEIL COHOMOLOGIES AND $\otimes$-FUNCTORS

PROPOSITION 1.30 *Every Weil cohomology theory H with coefficients in a field Q of characteristic 0 extends uniquely to $\otimes$-functor*

$$\omega_H : \mathsf{CHM}(k) \to \mathbb{Z}\text{-}\mathsf{Vecf}(Q).$$

PROOF This follows from the universality of $\mathsf{CHM}(k)$. □

ASIDE 1.31 Let $M \in \mathsf{CHM}(k)$ and $i \in \mathbb{N}$. Then the dimension of the Q-vector space $H^i(M)$ is independent of the standard Weil cohomology H^*. When k has characteristic zero, this follows from the comparison isomorphisms. For nonzero characteristic, we refer the reader to André 2004, 4.2.5.2.

The Lefschetz theorems

1.32 A Weil cohomology theory is said to satisfy the ***weak Lefschetz theorem*** if for every X, there exists an integer $d_0(X)$ such that if $f : Y \hookrightarrow X$ is a smooth hypersurface section of X of degree $d \geq d_0(X)$, then $f^* : H^i(X) \to H^i(Y)$ is an isomorphism for $i \leq \dim X - 2$ and an injection for $i = n - 1$. The standard Weil cohomology theories (see below) satisfy the weak Lefschetz theorem.

1.33 Let X be an absolutely irreducible smooth projective variety of dimension n over k. Let H^* be a Weil cohomology theory. Let H be a hyperplane section, let $\xi = \mathrm{cl}_X(H) \in H^2(X)(1)$, and let L the operator of degree 2,

$$a \mapsto a \cdot \xi : H^*(X) \to H^{*+2}(X).$$

We call L the ***Lefschetz operator*** defined by H. The Weil cohomology theory is said to satisfy the ***strong Lefschetz theorem*** if, for all $i \leq n = \dim X$, the map

$$L^{n-i} : H^i(X) \to H^{2n-i}(X)(n-i)$$

is an isomorphism.

1.34 The strong Lefschetz theorem has been proved for the standard Weil cohomology theories (see below). When k has characteristic zero, it suffices (by the comparison theorems) to prove it for $k = \mathbb{C}$ and Betti cohomology, where it was proved by transcendental means by Hodge. In arbitrary characteristics, it suffices (by specialization) to prove it for k the algebraic closure of a finite field, and then it suffices to prove it for étale cohomology (see 1.37). In this case, it was proved by Deligne (1980).

Weil cohomologies over finite fields

Following Katz and Messing 1974, we explain some consequences of Deligne's proof of the Weil conjectures.

1.35 Let X be a smooth projective geometrically irreducible variety of dimension n over $\mathbb{F}_q$. Let ℓ be a prime number $\neq \mathrm{char}(\mathbb{F}_q)$, and let F denote the Frobenius element relative to $\mathbb{F}_q$ acting on $H^i_\ell(X) \overset{\text{def}}{=} H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$. For a polynomial $g(T) = \prod_i(1 - \alpha_i T)$ and $r \geq 1$, we let $g(T)^{(r)} = \prod_i(1 - \alpha_i^r T)$. Deligne (1980) proved the following statements.

(a) For every integer $i \geq 0$, the polynomial

$$P^i_\ell(X/\mathbb{F}_q, T) \overset{\text{def}}{=} \det(1 - TF \mid H^i_\ell(X))$$

lies in $\mathbb{Z}[T]$, and its reciprocal roots have complex absolute value $q^{i/2}$.

(b) For every integer $d \geq 2$ and Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$ of hypersurface sections of degree d of X, the polynomial $P^{n-1}_\ell(X/\mathbb{F}_q, T)$ is the least common multiple of the complex polynomials $f(T)$ with the property that, whenever $t \in \mathbb{F}_{q^r}$ is such that X_t is smooth, $f(T)^{(r)}$ divides $P^{n-1}_\ell(X_t/\mathbb{F}_{q^r}, T)$.

(c) (strong Lefschetz) For all hyperplane sections $\xi \in H^2_\ell(X)$, the map

$$L^i : H^{n-i}_\ell(X) \to H^{n+i}_\ell(X)$$

is an isomorphism.

THEOREM 1.36 *Let H be a Weil cohomology theory satisfying the weak Lefschetz theorem, and let $X/\mathbb{F}_q$ be as above. Then, for all $i \geq 0$,*

$$\det(1 - TF \mid H^i(X)) = \det(1 - TF \mid H^i_\ell(X)),$$

i.e., $P^i(X/\mathbb{F}_q, T) = P^i_\ell(X/\mathbb{F}_q, T)$. In particular, $\det(1 - TF \mid H^i(X))$ is independent of H and

$$\dim_Q H^i(X) = \dim_{\mathbb{Q}_\ell} H^i_\ell(X).$$

PROOF Katz and Messing 1974, Theorem 1. □

COROLLARY 1.37 *The statements (a), (b), and (c) of 1.35 hold with H^i_ℓ and P^i_ℓ replaced by H^i and P^i.*

PROOF Ibid., Corollary 1. □

THE STANDARD WEIL COHOMOLOGY THEORIES

Let X be a (smooth projective) algebraic variety over a field k.

	k	Q	$H^s(X)$
Betti cohomology	$k \subset \mathbb{C}$	$\mathbb{Q}$	$H^s(X(\mathbb{C}), \mathbb{Q})$
étale cohomology	arbitrary	$\mathbb{Q}_\ell$, $\ell \neq \text{char}(k)$	$H^s_{\text{et}}(X \otimes_k k^{\text{al}}, \mathbb{Q}_\ell)$
de Rham cohomology	char $= 0$	k	$\mathbb{H}^s_{\text{Zar}}(X, \Omega^\bullet_{X/\Omega})$
crystalline cohomology	char $\neq 0$	ff(W))	$H^s_{\text{crys}}(X/W) \otimes_W k$

Let $Q(1) = H^2(\mathbb{P}^1)^\vee$. For example, for the Betti cohomology theory $\mathbb{Q}(1) = 2\pi i\mathbb{Q}$, and for the étale cohomology theory $\mathbb{Q}_\ell(1) = (\varprojlim \mu_{\ell^n}(\Omega)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. In every case, $Q(1)$ is a one-dimensional vector space over k.

The Betti cohomology group $H^r_B(X)(m) \overset{\text{def}}{=} H^r(X(\mathbb{C}), \mathbb{Q}(m))$ has a canonical structure of polarizable rational Hodge structure.

The de Rham cohomology groups $H^r_{\text{dR}}(X)(m) \overset{\text{def}}{=} \mathbb{H}^r(X_{\text{Zar}}, \Omega^\bullet_{X/k})(m)$ are finite-dimensional k-vector spaces. For any homomorphism $\sigma : k \to k'$ of fields, there is a canonical base change isomorphism

$$k' \otimes_k H^r_{\text{dR}}(X)(m) \xrightarrow{\sigma} H^r_{\text{dR}}(\sigma X)(m). \tag{123}$$

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_{\text{dR}}(X)(m). \tag{124}$$

Let k be an algebraically closed field. For each prime number $\ell \neq \text{char}(k)$, the étale cohomology groups $H^r_\ell(X)(m) \overset{\text{def}}{=} H^r_\ell(X_{\text{et}}, \mathbb{Q}_\ell(m))$ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces. For any homomorphism $\sigma : k \to k'$ of algebraically closed fields, there is a canonical base change isomorphism

$$H^r_\ell(X)(m) \xrightarrow{\sigma} H^r_\ell(\sigma X)(m), \quad \sigma X \overset{\text{def}}{=} X \otimes_{k,\sigma} k'. \tag{125}$$

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_\ell(X)(m). \tag{126}$$

Here $H^r_B(X)$ denotes the Betti cohomology group $H^r(X^{\text{an}}, \mathbb{Q})$.

2 Grothendieck's motivic paradise

Ten things, conjectured only, none inspiring conviction, but which illuminate and complement each other, and seem to contribute to a single, still mysterious, harmony, acquiring in this way the force of a vision. Even if all ten were eventually revealed to be false, the work that led to this provisional vision would not have been in vain.[6]

Grothendieck, quoted in Serre 1994a.

Grothendieck gave a series of lectures on motives at the IHES. One part was about the standard conjectures…Eventually, it was Kleiman who wrote down the notes in Dix Exposés…

Illusie, NAMS, 2010, p. 1110.

Motives for numerical equivalence

Throughout this subsection, H is a Weil cohomology theory with coefficient field Q, and $A_H^r(X)$ denotes the $\mathbb{Q}$-subspace of $H^{2r}(X)(r)$ spanned by the algebraic classes. We let $H^{\text{even}}(X) = \bigoplus_{i\geq 0} H^{2i}(X)$ and $H^{\text{odd}}(X) = \bigoplus_{i\geq 0} H^{2i+1}(X)$.

Let X be a smooth projective variety over a field k. For an admissible equivalence relation $\sim$, we let $A^i_\sim(X)$ denote the $\mathbb{Q}$-vector space $Z^i_\sim(X)\otimes\mathbb{Q}$.

2.1 The space $A^r_{\text{num}}(X)$ is finite-dimensional over $\mathbb{Q}$. Indeed, if $f_1,\dots,f_s \in A^{d-r}_{\text{hom}}(X)$ span the subspace $Q\cdot A_H^{d-r}(X)$ of $H^{2d-2r}(X)(d-r)$, then the map

$$x \mapsto (x\cdot f_1,\dots,x\cdot f_s)\colon\ A_H^r(X)\to\mathbb{Q}^s$$

has image $A^{d-r}_{\text{num}}(X)$.

2.2 Let $A_H^r(X,Q) = Q\cdot A_H$. Define $A^r_{\text{num}}(X,Q)$ to be the quotient of $A_H^r(X,Q)$ by the left kernel of the pairing

$$A_H^r(X,Q)\times A_H^{d-r}(X,Q)\to A_H^d(X,Q)\simeq Q$$

induced by cup product. Then $A_H^r(X)\to A^r_{\text{num}}(X,Q)$ factors through $A^r_{\text{num}}(X)$,

$$\begin{array}{ccc} A_H^r(X) & \longrightarrow & A^r_{\text{num}}(X) \\ \downarrow & & \vdots\downarrow \\ A_H^r(X,Q) & \longrightarrow & A^r_{\text{num}}(X,Q), \end{array}$$

and I claim that

$$Q\otimes A^r_{\text{num}}(X)\to A^r_{\text{num}}(X,Q)$$

[6] Dix choses soupçonnées seulement, dont aucune n'entraîne conviction, mais qui mutuellement s'éclairent et se complètent et semblent concourir à une même harmonie encore mystérieuse, acquièrent dans cette harmonie force de vision. Alors même que toutes les dix finiraient par se révéler fausses, le travail qui a abouti à cette vision provisoire n'a pas été fait en vain.

is an isomorphism. As $A^r_{\text{num}}(X,Q)$ is spanned by the image of $A^r_H(X)$, the map is obviously surjective. Let $e_1,\dots,e_m$ be a $\mathbb{Q}$-basis for $A^r_{\text{num}}(X)$, and let $f_1,\dots,f_m$ be the dual basis in $A^{d-r}_{\text{num}}(X)$. If $\sum_{i=1}^m a_i\otimes e_i$, $a_i\in Q$, is zero in $A^r_{\text{num}}(X,Q)$, then $a_j=(\sum a_ie_i)\cdot f_j=0$ for all j.

THEOREM 2.3 (JANNSEN 1992) *For any smooth projective variety X over a field k, the $\mathbb{Q}$-algebra $A^*_{\text{num}}(X\times X)$ is semisimple.*

PROOF Let $B=A^*_{\text{num}}(X\times X)$. Recall (2.1) that B has finite dimension over $\mathbb{Q}$, and (4.6) that multiplication in B is composition $\circ$ of correspondences. By definition of numerical equivalence, the pairing

$$f,g\mapsto\langle f\cdot g\rangle:\ B\times B\to\mathbb{Q}$$

is nondegenerate. Let f be an element of the Jacobson radical $R(B)$ of B. We have to show that $\langle f\cdot g\rangle=0$ for all $g\in B$.

Let H be a Weil cohomology theory with coefficient field Q, and let $A=A^*_H(X\times X,Q)$. Then A is a finite-dimensional Q-algebra, and there is a surjective homomorphism

$$A\overset{\text{def}}{=}A^d_H(X\times X,Q)\to A^d_{\text{num}}(X\times X,Q)\simeq Q\otimes B$$

(2.2). This maps the radical of A onto that of $Q\otimes B$ (see VI, 1.2). Therefore, there exists an $f'\in R(A)$ mapping to $1\otimes f$. For all $g\in A$,

$$\langle f'\cdot g^t\rangle=\sum\nolimits_i(-1)^i\operatorname{Tr}(f'\circ g\mid H^i(X)) \tag{127}$$

by Lefschetz trace formula 1.26. As $f'\circ g$ lies in $R(A)$, it is nilpotent (see VI, 1.2), and so (127) shows that $\langle f'\cdot g^t\rangle=0$. □

COROLLARY 2.4 *The category $\mathsf{M}_\sim(k)$ of motives over k is semisimple if and only if $\sim$ is numerical equivalence.*

PROOF The sufficiency follows from the theorem and VI, 1.4. For the necessity, let $\mathcal{N}$ be the ideal in $\mathsf{M}_\sim(k)$ corresponding to numerical equivalence. If $\alpha:\ M\to N$ is nonzero, then there exists a $\beta:\ N\to M$ such $\beta\circ\alpha\neq0$ (by VI, 1.3) and so $\alpha\notin\mathcal{N}(M,N)$. Hence $\mathcal{N}=0$. □

ASIDE 2.5 In fact, $\mathsf{M}_\sim(k)$ is abelian if and only if $\sim$ is numerical equivalence (see André 1996).

2.6 Let X be a smooth projective variety over a field k. We say that X satisfies the ***sign conjecture*** for H if there exists an algebraic cycle e on $X\times X$ such that $eH^*(X)=H^{\text{even}}(X)$. Smooth projective varieties over a finite field satisfy the sign conjecture for the standard Weil cohomology theories, as do abelian varieties over any field.

THEOREM 2.7 *Assume the sign conjecture. The kernel of the $\mathbb{Q}$-algebra homomorphism*

$$A\overset{\text{def}}{=}A^*_H(X\times X)\overset{S}{\longrightarrow}A^*_{\text{num}}(X\times X)$$

is the radical of A, and it is a nilpotent ideal.

PROOF As $A^*_{\text{num}}(X\times X)$ is semisimple, $R(A)\subset\operatorname{Ker}(S)$. For the converse, it suffices to show that $\operatorname{Ker}(S)$ is a nil ideal (see VI, 1.2). Let $f\in\operatorname{Ker}(S)$ – we want to show that f is nilpotent. Clearly, we may suppose that f is homogeneous. If $\deg(f)\neq0$, then it is

obviously nilpotent, and so we may suppose that $\deg(f) = 0$. By assumption, $f \cdot e^t = 0$ for all j (here is where we use that e is algebraic), and so

$$0 = \langle f \cdot e^t \rangle = \sum_i (-1)^i \operatorname{Tr}(f \circ e | H^i(X)) = \operatorname{Tr}(f | H^{\text{even}}(X)).$$

Therefore $\operatorname{Tr}(f|H^{\text{even}}(X)) = 0$ and, similarly, $\operatorname{Tr}(f|H^{\text{odd}}(X)) = 0$. Since this is true for all powers of f, we see that the image f_H of f in $H^{2d}(X \times X)(d)$ is nilpotent.

Let QA_H be the Q-subspace spanned by the image of A in $H^*(X \times X)(d)$. This is a finite-dimensional Q-algebra, and the Q-span of $\{f_H \mid f \in \operatorname{Ker}(S)\}$ is a nil ideal in QA_H, and so it is contained in $R(QA_H)$ (see VI, 1.2). Now $R(QA_H)^r = 0$ for some r. As $\operatorname{Ker}(S)$ maps into $R(QA_H)$, it follows that $\operatorname{Ker}(S)^r$ maps into $R(QA_H)^r = 0$. As $A \to QA_H$ is injective, this shows that $\operatorname{Ker}(S)^r = 0$. □

When the sign conjecture holds, we can modify the commutativity constraint in $\mathsf{M}_\sim(k)$ so that, for any $\alpha \in \operatorname{End}(M)$ and Weil cohomology theory H, we have

$$\operatorname{Tr}(\alpha|M) = \operatorname{Tr}(H^{\text{even}}) + \operatorname{Tr}(H^{\text{odd}})$$

instead of $\operatorname{Tr}(H^{\text{even}}) - \operatorname{Tr}(H^{\text{odd}})$ (see later, 2.29).

THEOREM 2.8 *Let $\mathsf{M}_{\text{num}}(k)$ denote the category of numerical motives over k generated by the algebraic varieties over k satisfying the sign conjecture. With the modified commutativity constraint, $\mathsf{M}_{\text{num}}(k)$ is a semisimple tannakian category over $\mathbb{Q}$.*

PROOF From 1.12 and 2.4, we know that $\mathsf{M}_{\text{num}}(k)$ is a semisimple abelian $\otimes$-category over $\mathbb{Q}$. With the modified commutativity constraint, $\dim(M)$ is an integer ≥ 0 for all M, and so we can apply Theorem 10.1 of Chapter I. □

Note that the characteristic polynomial of an endomorphism of an object of $\mathsf{M}_{\text{num}}(k)$ is well defined, and equals its characteristic polynomial under any Weil cohomology theory.

The standard conjecture of Lefschez type

For a smooth projective variety X over k and a Weil cohomology theory H, we let $A^i_H(X)$ denote the $\mathbb{Q}$-subspace of $H^{2i}(X)(i)$ generated by the classes of algebraic cycles. When X has dimension n, we let $\mathcal{A}^*(X)$ denote the graded ring with $\mathcal{A}^i(X) = A^{n+i}_H(X \times X)$ (self-correspondences of degree i).

THE KÜNNETH STANDARD CONJECTURE

Let X be a smooth projective variety over k and H a Weil cohomology theory. The ***Künneth projector*** π^i_X is the projection of $H^*(X)$ onto $H^i(X)$,

$$H^*(X) \to H^i(X) \to H^*(X).$$

CONJECTURE 2.9 ($C(X)$) *The Künneth projectors π^i_X are algebraic.*

CONJECTURE 2.10 ($C^+(X)$) *The even Künneth projector $\pi^+_X \overset{\text{def}}{=} \sum_i \pi^{2i}_X$ is algebraic.*

These are called the ***Künneth standard conjecture*** and the ***sign conjecture*** respectively.

Consequences of $C(X)$

2.11 If X satisfies $C(X)$, then $A_H(X\times X)$ is stable under the Künneth decomposition, and

$$\mathcal{A}^i(X) = \sum_{j=0}^{n-i} \pi_{j+i} A(X\times X)\pi_j.$$

Theorem 2.12 *Let u be the endomorphism of $H^i(X)$ defined by an integral algebraic cycle on $X\times X$ (i.e., an element of $Z^n(X\times X)$). If π^{2n-i} is algebraic, then the characteristic polynomial $P(T) \overset{\text{def}}{=} \det(1-uT \mid H^i(X))$ has coefficients in $\mathbb{Z}$; moreover, these coefficients are given by universal polynomials in the rational numbers*

$$\langle u^m\cdot\pi^{2n-i}\rangle, \quad m = 1,\dots,\dim H^i(X).$$

Proof The Newton identities (see Wikipedia) express the coefficients of $P(t)$ as polynomials with rational coefficients in the power sums

$$S_m = a_1^m + a_2^m + \cdots$$

of the eigenvalues a_i of u on $H^i(X)$. By the trace formula (1.27(a)),

$$S_m = \operatorname{Tr}(u^m|H^i(X)) = (-1)^i\langle u^m\cdot\pi^{2n-i}\rangle \in \mathbb{Q}. \qquad \square$$

Case of a finite base field

Theorem 2.13 *Let X be a smooth projective variety over $\mathbb{F}_q$ of dimension n. There are unique projectors π_X^i in $A_H^n(X\times X)$ such that $H(\pi_X^i)$ projects $H^*(X)$ onto $H^i(X)$ for every Weil cohomology theory satisfying the weak Lefschetz theorem. Moreover, the π_X^i are $\mathbb{Q}$-linear combinations of the graphs of the Frobenius endomorphism and its iterates.*

Proof According to Theorem 1.36, there are polynomials $P_i(T)\in\mathbb{Q}[T]$ such that $P_i(T) = \det(T-F \mid H^i(X))$ for every Weil cohomology theory H satisfying the weak Lefschetz theorem. These polynomials are relatively prime because their roots have different values, and so there are polynomials $P^i(T)\in\mathbb{Q}[T]$ such that

$$P^i(T) \equiv \begin{cases} 1 \bmod P_i(T) \\ 0 \bmod P_j(T) \text{ for } j\neq i. \end{cases}$$

For any H, $P^i(F)$ acts on $H^i(X)$ as 1 and on $H^j(X)$, $j\neq i$, as 0. We can take π_X^i to be the graph of $P^i(F)$. The P^i are uniquely determined up to a polynomial Q such that $Q(F)$ acts trivially on all H satisfying the weak Lefschetz theorem. $\square$

Corollary 2.14 *Let X be a smooth projective geometrically irreducible variety over $\mathbb{F}_q$ of dimension n, and let H be a Weil cohomology theory satisfying the weak Lefschetz theorem. For any integrally algebraic cycle Z on $X\times X$ of codimension n, the characteristic polynomial of the induced endomorphism of $H^i(X)$ lies in $\mathbb{Z}[T]$ and is independent of H.*

Proof Apply Theorem 2.12. $\square$

Summary 2.15 Conjecture C is known over finite fields for any Weil cohomology theory satisfying the weak Lefschetz theorem (2.13), for example, the standard Weil cohomology theories. It follows that the Künneth projectors are almost-algebraic (see **??** for this terminology). In characteristic zero, if Conjecture $C(X)$ holds for one standard Weil cohomology theory, then it holds for all (by the comparison theorems).

Consequences of the strong Lefschetz theorem

In this subsection, H is a Weil cohomology theory satisfying the strong Lefschetz theorem.

2.16 For $j \le n-1$, we have a diagram

$$H^i(X) \xrightarrow{L^j} H^{i+2j}(X) \xrightarrow{L^{n-i-j}} H^{2n-i}(X), \qquad H^i(X) \xrightarrow{L^{n-i}}_{\simeq} H^{2n-i}(X),$$

and so $L^j : H^i(X) \to H^{i+2j}(X)$ is injective and $L^{n-i-j} : H^{i+2j}(X) \to H^{2n-i}(X)$ is surjective. Therefore,

$$\begin{aligned} 1 = \beta_0 \le \beta_2 \le \cdots \le \beta_{2i} \qquad & \text{for } 2i \le n \\ \beta_1 \le \beta_3 \le \cdots \le \beta_{2j+1} \qquad & \text{for } 2j+1 \le n. \end{aligned}$$

Lemma 2.17 *Consider homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of vector spaces. If $\beta\circ\alpha$ is an isomorphism, then $B = \alpha A \oplus \operatorname{Ker}(\beta)$.*

Proof Let $b \in \alpha A \cap \operatorname{Ker}(\beta)$. Then $b = \alpha(a)$ for some $a \in A$ and $0 = \beta(b) = \beta\alpha(a)$, so $a = 0$. Therefore $b = 0$. Let $b \in B$. Then there exists an $a \in A$ such that $\beta(b) = \beta\alpha(a)$. Now $b = \alpha(a) + (b - \alpha(a))$ and $b - \alpha(a) \in \operatorname{Ker}(\beta)$. □

For $i \le n$, define

$$P^i(X) = \{a \in H^i(X) \mid L^{n-i+1}(a) = 0\}.$$

The elements of $P^i(X)$ are said to be ***primitive***. On applying the lemma to

$$H^{i-2}(X) \xrightarrow{L} H^i(X) \xrightarrow{L^{n-i+1}} H^{2n-i+2}(X), \qquad H^{i-2}(X) \xrightarrow[\simeq]{L^{n-i+2}} H^{2n-i+2}(X),$$

we find that

$$H^i(X) \simeq P^i(X) \oplus LH^{i-2}(X).$$

On repeating the argument with $H^{i-2}(X)$ …we obtain a decomposition

$$H^i(X) = P^i(X) \oplus LP^{i-2}(X) \oplus L^2P^{i-4}(X) \oplus \cdots.$$

More precisely, if $i \le n$, then

$$H^i(X) = \bigoplus_{j\ge 0} L^j P^{i-2j}(X).$$

In other words, every $a \in H^i(X)$ can be written uniquely in the form

$$a = a_0 + La_1 + L^2a_2 + \cdots = \sum_{j\ge 0} L^j a_j \qquad \left(a_j \in P^{i-2j}(X)\right).$$

If $i \ge n$, then

$$H^i(X) = L^{i-n}P^{2n-i} + L^{i-n+1}P^{2n-i-2} + \cdots = \bigoplus_{j\ge i-n} L^j P^{i-2j}(X).$$

In other words, every $a \in H^i(X)$ can be written uniquely in the form

$$a = L^{i-n}a_{i-n} + L^{i-n+1}a_{i-n+1} + \cdots = \sum_{j \geq \max(0,i-n)} L^j a_j \qquad \big(a_j \in P^{i-2j}(X)\big).$$

There is an operator

$$\begin{cases} \Lambda : H^i(X) \to H^{i-2}(X), & \Lambda x = \sum_{j \geq 1,\, i-n} L^{j-1}x_j, \end{cases}$$

where $x = \sum L^j x_j$ is the primitive decomposition of $x \in H^i(X)$. For $0 \leq i \leq n$, Λ is determined on H^i by the diagram

$$\begin{array}{ccc} H^i(X) & \xrightarrow[\simeq]{L^{n-i}} & H^{2n-i}(X) \\ \big\downarrow \Lambda & & \big\downarrow L \\ H^{i-2}(X) & \xrightarrow[\simeq]{L^{n-i+2}} & H^{2n-i+2}(X), \end{array}$$

and on H^{2n-i+2} by the diagram

$$\begin{array}{ccc} H^i(X) & \xrightarrow[\simeq]{L^{n-i}} & H^{2n-i}(X) \\ L \big\uparrow & & \Lambda \big\uparrow \\ H^{i-2}(X) & \xrightarrow[\simeq]{L^{n-i+2}} & H^{2n-i+2}(X), \end{array}$$

Clearly, Λ is surjective on $H^i(X)$ and injective on H^{2n-i+2}.

Similarly, there are operators (here $x = \sum L^j x_j \in H^i(X)$)

$$\begin{cases} {}^c\Lambda : H^i(X) \to H^{i-2}(X), & {}^c\Lambda x = \sum_{j \geq 1,\, i-n} j(n-i+j+1)L^{j-1}x_j \\ * : H^i(X) \to H^{2n-i}(X), & * x = \sum_{j \geq \max(0,i-n)} (-1)^{(i-2j)(i-2j+1)/2} L^{n-i+j}x_j \\ p^j : H^i(X) \to P^j(X), & p^j x = \delta_{ij} x_{\max(0,i-n)} \text{ for } j = 0, \ldots 2n. \end{cases}$$

PROPOSITION 2.18 (a) *For $i \leq n$, $\Lambda^{n-i} : H^{2n-i}(X) \to H^i(X)$ is inverse to L^{n-i} and ${}^c\Lambda^{n-1} : L^{n-i}P^i(X) \to P^i(X)$ is inverse to a multiple of L^{n-i}.*

(b) *For all i, $*^2 = \mathrm{id}$ and $\Lambda = * L *$.*

(c) *For $j = 0, \ldots n$, the operator p^j is a projector onto $P^j(X)$; for $j = n, \ldots, 2n$, $p^j = p^{2n-j}\Lambda^{n-j}$.*

(d) *Λ, ${}^c\Lambda$, $*$, $\pi^0, \ldots, \pi^{2n}, p^0, \ldots, p^{n-1} \in \mathbb{Z}[L, p^n, \ldots, p^{2n}]$ (ring of noncommutative polynomials).*

PROOF Straightforward from the definitions (Kleiman 1968, 1.4.3) □

PROPOSITION 2.19 *We have*

$$\mathbb{Q}[L, \Lambda] = \mathbb{Q}[L, {}^c\Lambda] = \mathbb{Q}[L, *] = \mathbb{Q}[L, p^n, \ldots, p^{2n}],$$

and this $\mathbb{Q}$-algebra contains $p^0, \ldots, p^{n-1}$ and $\pi^0, \ldots, \pi^{2n}$.

PROOF The $\mathbb{Q}$-algebra generated by L and $p^n, \dots, p^{2n}$ contains Λ, ${}^c\Lambda$, $*$, $\pi^0, \cdots, \pi^{2n}$, and $p^0, \dots, p^{n-1}$ by 2.18(d). As it contains L and $*$, it also contains $\Lambda = * L *$. Finally, as it contains L and Λ (resp. ${}^c\Lambda$), it contains $p^n, \dots, p^{2n}$ by 2.18(a) and the next lemma. □

LEMMA 2.20 *For $i \le n$, let $\theta^i : H^*(X) \to H^*(X)$ be a map of degree $-2(n-1)$ that induces the map $L^{n-i}P^i(X) \to P^i(X)$ inverse to L^{n-i}. Then p^{2n-i} is given by a universal noncommutative polynomial with integer coefficients in L and $\theta^0, \dots, \theta^i$.*

PROOF The statement follows by induction on i from the following easily verified formulas:

$$\varphi_i = \sum_{j=1}^{2n-i} \pi^j = \mathrm{id} - \sum_{\ell \notin [i,2n-i]} \sum_{j \ge \max(0,\ell-n)} L^j p^{2n-\ell+2j} L^{n-i+j},$$

$$p^{2n-i} = \varphi_i \theta^i = \Big(\mathrm{id} - \sum_{j \ge 1+n-i} L^j p^{i+2j} L^{i-n+j}\Big)\varphi_i, \text{ or maybe}$$

$$p^{2n-i} = \varphi_i \theta^i \Big(\mathrm{id} - \sum_{j \ge 1+n-i} L^j p^{i+2j} L^{i-n+j}\Big)\varphi_i.$$ □

PROPOSITION 2.21 *(a) ${}^c\Lambda$ is the unique operator of degree -2 such that*

$$[{}^c\Lambda, L] = \sum_{i=0}^{2n} (n-i)\pi^i. \tag{128}$$

(b) Let X, Y, and $X \times Y$ satisfy the strong Lefschetz theorem, and polarize $X \times Y$ with the Segre immersion $L_{X\times Y} = L_X \otimes \mathrm{id} + \mathrm{id} \otimes L_Y$. Then

$${}^c\Lambda_{X\times Y} = {}^c\Lambda_X \otimes \mathrm{id} + \mathrm{id} \otimes {}^c\Lambda_Y.$$

PROOF (a) It follows easily from the definition that ${}^c\Lambda$ satisfies (128). On the other hand, any operator λ satisfying (128) is easily seen by induction to satisfy

$$[\lambda, L^j] = L^{j-1} \sum_{\ell=0}^{j-1} \sum_{i=0}^{2n} (n-i)\pi^{i-2\ell}. \tag{129}$$

Assume that λ has degree -2, and let $a \in P^i(X)$. Then $L^{n-i+2}\lambda a = \lambda L^{n-i+2}a - rL^{n-i+1}a = 0$, where r is the integer given by (129); hence $\lambda a = 0$. Then, for any $j \ge 1$,

$$\lambda L^j a = [\lambda, L^j]a + L^j \lambda a = {}^c\Lambda L^j a.$$

Thus, $\lambda = {}^c\Lambda$.

(b) This follows formally from (a). □

STATEMENT OF THE STANDARD CONJECTURE OF LEFSCHETZ TYPE

Fix a Weil cohomology theory H with coefficient field Q, and write $A_H^i(X)$ for the image of $\mathrm{cl}_X : \mathrm{CH}^i(X) \to H^{2i}(X)(i)$.

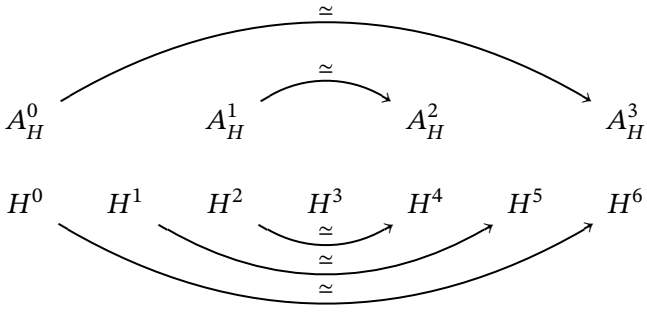
2.22 Let X be a smooth projective variety of dimension n and L a Lefschetz operator. The main variants of the standard conjecture of Lefschetz type are the following:

$A(X, L)$: The map $L^{n-2i} : A_H^i(X) \to A_H^{n-i}(X)$ is an isomorphism for all $i \le n/2$.

$B(X,L)$: The operator Λ is algebraic.

As we shall see, the two conjectures are essentially equivalent. The Conjecture $B(X)$ implies conjecture $C(X)$, which can be regarded as a weak form of the Lefschetz standard conjecture.

2.23 Assume $A(X,L)$. When $n=3$, we get the following picture



In the general case, we define

$$A^i_{\text{prim}}(X) = A^i_H(X) \cap P^{2i}(X) = \{a \in A^i_H(X) \mid L^{n-2i+1}a = 0\}.$$

Applying Lemma 2.17, we get a decomposition

$$A^i_H(X) = A^i_{\text{prim}}(X) \oplus LA^i_{\text{prim}}(X) \oplus \cdots$$

In other words, every $a \in A^i_H(X)$ can be written uniquely in the form

$$a = a_0 + La_1 + \cdots = \sum L^j a_j \quad (a_j \in A^{i-j}_H(X)).$$

Variants and relations between the conjectures

Let H be a Weil cohomology theory satisfying the strong Lefschetz theorem.

Theorem 2.24 *Let X be a smooth projective variety and L a Lefschetz operator. Then the following statements are equivalent:*

$B(X,L)$: *The operator Λ is algebraic.*

$\nu(X)$: *For each $i \le n$, there exists an algebraic correspondence θ^i inducing an isomorphism $H^{2n-i}(X) \to H^i(X)$.*

$*\,(X,L)$: *The operator $*$ is algebraic.*

As the statement $\nu(X)$ does not involve L, we see that if any one of the remaining statements holds for one L, then they all hold for all L.

Proof See Kleiman 1994, 4-1. □

Corollary 2.25 *Conjecture $A(X,L)$ holds for all X and L if and only if $B(X)$ holds for all X, in which case $C(X)$ holds for all X.*

Proof See Kleiman 1994, 4-1, 4-2. □

Application to motives

Let H be a Weil cohomology theory satisfying the Lefschetz standard conjectures and the Lefschetz theorems. We shall construct a natural $\otimes$-gradation on the category $\mathsf{M}_H(k)$. Recall that this means that each motive M is equipped with a canonical decomposition

$$M = \bigoplus_{n\in\mathbb{Z}} M^n$$

compatible in an obvious sense with $\otimes$. The construction requires several steps.

2.26 Let X be a connected smooth projective k-scheme of dimension n. We have a decomposition in the ring $C^n_H(X\times X)$ of endomorphisms in $\mathsf{CV}^0(k)$

$$\mathrm{id}_X = \sum_{i\in\mathbb{Z}} \pi^i_X, \tag{130}$$

where $\pi^i_X = 0$ for $i \notin [0, 2n]$. On the other hand, if Y has dimension m, we have

$$\pi^k_{X\times Y} = \sum_{i+j=k} \pi^i_X \otimes \pi^j_Y. \tag{131}$$

2.27 The π^i_X are orthogonal projectors, as one sees immediately, and the formula (130) defines a decomposition of the motive $h(X)$ in $\mathsf{M}^+(k)$

$$h(X) = \bigoplus_{i\in\mathbb{Z}} h^i(X), \quad h^i(X) = (X, \pi^i_X). \tag{132}$$

The decomposition (130) is functorial on the category $\mathsf{CV}^0(k)$, which means the following: if $f \in C^n(X\times Y)$, i.e., $f : X \to Y$ is in $\mathsf{CV}^0(k)$, then

$$\pi^i_Y \circ f = f \circ \pi^i_X,$$

which is only another way of saying that $H(f) : H(X) \to H(Y)$ is homogeneous of degree 0. In fact, the commutant of π^i_X $(0 \le i \le 2n)$ in the ring $C(X\times X)$ of algebraic correspondences is exactly $C^n(X\times X)$.

From the above and (131), we deduce a $\otimes$-gradation of $\mathsf{M}^+(k)$, characterized by the equality

$$h(X)^i = h^i(X).$$

We note that the motive $h^2\mathbb{P}^1$ is homogeneous of degree 2.

2.28 The $\otimes$-gradation of $\mathsf{M}^+_H(k)$ extends directly to a $\otimes$-gradation of $\mathsf{M}_H(k)$. We have

$$(M, n)^i = (M^{i+2n}, n), \quad i \in \mathbb{Z}. \tag{133}$$

2.29 We use this gradation to modify the commutativity constraint of $\mathsf{M}_H(k)$, as follows. If $\dot\alpha : M\otimes N \simeq N\otimes M$ is the original constraint, then

$$\dot\alpha = \bigoplus \dot\alpha^{p,q}, \quad \text{where } \dot\alpha^{p,q} : M^p \otimes M^q \simeq N^q \otimes M^p.$$

We define the new constraint α by

$$\begin{cases} \alpha = \bigoplus \dot\alpha^{p,q} \\ \alpha^{p,q} = (-1)^{pq}\dot\alpha^{p,q}. \end{cases} \tag{134}$$

If H also satisfies the Hodge standard conjecture (2.30), then the π^i_X are independent of H (satisfying the weak Lefschetz theorem) – see 2.39 below. As the above construction depend only on the π^i_X, it also is independent of H.

The standard conjecture of Hodge type

As before, all varieties are projective and smooth over k. We fix a Weil cohomology theory H satisfying the strong Lefschetz theorem and let $A^i(X) = C^i_H(X)$.

STATEMENTS

2.30 For $2i \le n = \dim(X)$, let

$$A^i_H(X)_{\text{prim}} = \{a \in A^i(X) \mid L^{n-2i+1}a = 0\}$$

The ***standard conjecture of Hodge type*** says the following:
$I(X,L)$**:** For all $i \le n/2$, the bilinear form

$$a, b \mapsto (-1)^i\langle L^{n-2i}a \cdot b\rangle : \ A^i_H(X)_{\text{prim}} \times A^i_H(X)_{\text{prim}} \to \mathbb{Q}$$

is positive definite.

PROPOSITION 2.31 *Let X be a smooth projective variety of dimension n, and let p be such that $2p \le n$. Assume $A(X,L)$. Then the following statements are equivalent:*

(a) *$I^q(X,L)$ holds for all $q \le p$;*

(b) *the quadratic form*

$$a, b \mapsto \langle a \cdot * b\rangle : \ A^p(X) \times A^p(X) \to \mathbb{Q}$$

is positive definite (hence the canonical pairing $A^p(X) \times A^{n-p}(X) \to \mathbb{Q}$ is nondegenerate).

PROOF Recall that $A(X,L)$ gives a decomposition

$$A^p(X) = A^p_H(X)_{\text{prim}} \oplus LA^{p-1}_H(X)_{\text{prim}} \oplus \cdots \oplus A^{p-i}_H(X)_{\text{prim}} \oplus \cdots.$$

Let

$$a = \sum_i L^i a_i \quad \text{with } a_i \in A^{p-i}_H(X)_{\text{prim}},$$
$$b = \sum_j L^j b_j \quad \text{with } b_j \in A^{p-j}_H(X)_{\text{prim}}.$$

Note that

$$* b \overset{\text{def}}{=} \textstyle\sum_j (-1)^{(2p-2j)(2p-2j+1)/2} L^{n-2p+j} b_j = \sum_j (-1)^{(p-j)} L^{n-2p+j} b_j,$$

Therefore,

$$a * b = \textstyle\sum_{i,j} (-1)^{p-j} L^{n-2p+i+j} a_i \cdot b_j.$$

If $i \ne j$, then

$$L^{n-2p}L^i a_i \cdot L^j b_j = 0; \tag{135}$$

for example, if $i < j$, then it equals $L^{n-(2p-2i)+j-i}a_i \cdot b_j$, which is zero because $L^{n-(2p-2i)+1}a_i = 0$.

Thus

$$\langle a * b\rangle = \sum_{i \ge 0} (-1)^{p-i}\langle L^{n-(2p-2i)}a_i \cdot b_i\rangle,$$

from which the statement is obvious. □

ASIDE 2.32 For Weil cohomology theories satisfying the weak Lefschetz theorem, to prove the Hodge standard conjecture for all varieties, it suffices to prove it in middle dimension (Kleiman 1968, 3.7).

2.33 There is another conjecture, which is sometimes considered part of the standard conjectures and sometimes part of the Tate conjectures.

$D(X)$: Homological equivalence coincides with numerical equivalence,

$$\sim_H = \sim_{\text{num}}.$$

PROPOSITION 2.34 *Conjecture $D(X)$ implies $A(X,L)$, and the converse holds in the presence of $I(X,L)$.*

PROOF Assume $D(X)$. Then the $\mathbb{Q}$-vector spaces $A_H^i(X)$ and $A_H^{n-i}(X)$ are finite-dimensional (2.1) and the pairing $A_H^i(X) \times A_H^{n-i}(X) \to \mathbb{Q}$ nondegenerate, and so the two spaces have the same dimension. The map

$$L^{n-2i} : A_H^i(X) \to A_H^{n-i}(X)$$

is injective because of the strong Lefschetz theorem, hence is bijective. Thus $A(X,L)$ holds.

Conversely, we saw in 2.31 that $A(X,L)$ and $I(X,L)$ imply that the pairing $A_H^i(X) \times A_H^{n-i}(X) \to \mathbb{Q}$ is nondegenerate, i.e., that $D(X)$ holds. □

In characteristic zero, all the standard conjectures follow from Conjecture D. Indeed, the Hodge standard conjecture is known, and so $A(X,L)$ is known for all X and L, which implies that $B(X)$ holds for all X (2.25).

THE HODGE STANDARD CONJECTURE AND POSITIVITY

Let X, Y be polarized varieties. The bilinear forms

$$(x,x') \mapsto \text{Tr}_X(x \cdot * x') : H(X) \otimes H(X) \to Q$$
$$(y,y') \mapsto \text{Tr}_Y(y \cdot * y') : H(Y) \otimes H(Y) \to Q$$

are nondegenerate (by Poincaré duality and the fact that $*$ is an isomorphism (2.18(b)).

THEOREM 2.35 *Let X and Y satisfy $B(X)$ and $B(Y)$. Let $u : H^*(X) \to H^*(Y)$ be a correspondence, and let $u' : H^*(Y) \to H^*(X)$ be its transpose with respect to the above pairings, so $u' = *_X \circ u^t \circ *_Y$. If u is algebraic, then u' is algebraic, and*

$$\text{Tr}(u \circ u') = \text{Tr}(u' \circ u) \in \mathbb{Q};$$

if, moreover, $I(X \times Y, L_X \otimes 1 + 1 \otimes L_Y)$ holds, then

$$u \neq 0 \Rightarrow \text{Tr}(u' \circ u) > 0.$$

PROOF Recall (2.24) that $B(X)$ and $B(Y)$ imply that $*_X$ and $*_Y$ are algebraic, and so u' is algebraic if u is. Now $\text{Tr}(u' \circ u) \in \mathbb{Q}$ because of the trace formula 1.27(a).

We prove the second statement. From the strong Lefschetz theorem, we obtain decompositions

$$H^i(X) = P^i(X) \oplus \cdots \oplus L^j P^{i-2j}(X) \oplus \cdots$$
$$H^r(Y) = P^r(Y) \oplus \cdots \oplus L^s P^{r-2s}(Y) \oplus \cdots.$$

$B(X)$ implies that the projection operators

$$q_X^{ij} : H^*(X) \to L^j P^{i-2j}(X),$$

where $j = \max(0, i-n)$ and $n = \dim(X)$ are algebraic (2.18(c), 2.24). Moreover,

$$(q_Y^{rs} \circ u \circ q_X^{ij})' = q_X^{ij} \circ u' \circ q_Y^{rs}$$

i.e.,

$$\begin{aligned}\Big(H^*(X) \xrightarrow{q_X^{ij}} L^j P^{i-2j}(X) \subset H^*(X) \xrightarrow{u} H^*(Y) \xrightarrow{q_Y^{rs}} L^s P^{r-2s}(Y) \subset H^*(Y)\Big)' \\ = H^*(Y) \xrightarrow{q_Y^{rs}} L^s P^{r-2s}(Y) \subset H^*(Y) \xrightarrow{u'} H^*(X) \xrightarrow{q_X^{ij}} L^j P^{i-2j}(X) \subset H^*(X).\end{aligned}$$

by the orthogonality of primitive elements (135). Therefore

$$\begin{aligned}\operatorname{Tr}(u' \circ u) &= \sum \operatorname{Tr}\left((q_Y^{r_1 s_1} \circ u \circ q_X^{i_1 j_1})' \circ (q_Y^{rs} \circ u \circ q_X^{ij})\right) \\ &= \sum \operatorname{Tr}\left((q_Y^{rs} \circ u \circ q_X^{ij})' \circ (q_Y^{rs} \circ u \circ q_X^{ij})\right),\end{aligned}$$

and so we may assume that $u = q_Y^{rs} \circ u \circ q_X^{ij}$.

Let $v = \Lambda_Y^s \circ u \circ \Lambda_X^{n-i+j}$. Then $v' = L_X^{n-i+j} \circ u' \circ L_Y^s$; so $\operatorname{Tr}(v' \circ v) = \operatorname{Tr}(u' \circ u)$. Replacing u with v, $i-2j$ with i, and $r-2s$ with j, we may assume that $u \in P^i(X) \otimes P^j(Y)$.

By the Lefschetz trace formula 1.26 we now have

$$\operatorname{Tr}(u' \circ u) = (-1)^i \langle u \cdot *_X \circ u \circ *_Y \rangle; \tag{136}$$

moreover,

$$*_X \circ u \circ *_Y = (*_X \otimes *_Y)u. \tag{137}$$

Furthermore, it is easily seen that, if

$$L_{X\times Y} = L_X \otimes 1 + 1 \otimes L_Y,$$

then $u \in P^{i+j}(X \times Y)$ and

$$\begin{aligned}\binom{n-i+m-j}{n-i}((*_X \otimes *_Y)u) &= (-1)^{i(i+1)/2}(-1)^{j(j+1)/2} L_{X\times Y}^{n-i+m-j} u \\ &= (-1)^{ij} *_{X\times Y} u.\end{aligned}$$

Since u is algebraic, $i+j$ is even and $(-1)^{ij} = (-1)^i$. Therefore, $I(X \times Y, L_{X\times Y})$ implies that

$$(-1)^i \langle u \cdot *_X \circ u \circ *_Y \rangle > 0$$

when $u \neq 0$, which completes the proof.

To recap:

$$\begin{aligned}\operatorname{Tr}(u' \circ u) &= (-1)^i \langle u \cdot *_X \circ u \circ *_Y \rangle \quad \text{by (??)} \\ &= (-1)^i \langle u \cdot (*_X \otimes *_Y)u \rangle \quad \text{by (??)} \\ &= c(-1)^i \langle u \cdot *_{X\times Y} u \rangle, c > 0\end{aligned}$$

Indeed, $I(X \times Y, L_{X\times Y})$ says that

$$(-1)^{(i+j)/2} \langle u \cdot *_{X\times Y} u \rangle > 0.$$

□

COROLLARY 2.36 *Assume that X satisfies $B(X)$, and that $X \times X$ satisfies $I(X \times X, L \otimes 1 + 1 \otimes L)$. Then $C(X \times X)$ holds, and the $\mathbb{Q}$-algebra $\mathcal{A}^* \overset{\text{def}}{=} C_H^{\dim X + *}(X \times X)$ of algebraic correspondences is semisimple. In fact, every subalgebra of $\mathcal{A}^*(X)$ that is closed under the involution $u \mapsto u' \overset{\text{def}}{=} * \circ u^t \circ *$ is semisimple.*

PROOF Indeed, the involution $u \mapsto u'$ on $\mathcal{A}^*$ is positive, and so this follows from VI, 1.6. □

COROLLARY 2.37 *Let X and Y be varieties satisfying the strong Lefschetz theorem and $B(X)$, $B(Y)$. Let*

$$u : H^i(X) \to H^j(Y)$$

be an algebraic correspondence.

(a) *If $I(X \times Y, L_X \otimes 1 + 1 \otimes L_Y)$ holds and u is injective, then u has an algebraic left inverse $v : H^j(Y) \to H^i(X)$. Consequently, if $a \in H^i(X)$ is such that $u(a)$ is algebraic, then a is algebraic.*

(b) *If $I(X \times Y, L_X \otimes 1 + 1 \otimes L_Y)$ holds and u is surjective, then u has an algebraic right inverse $v : H^j(Y) \to H^i(X)$. Consequently, if $b \in H^j(Y)$ is algebraic, then there exists an algebraic $a \in H^i(X)$ such that $b = u(a)$.*

PROOF (a) Let $y = u' \circ u$ and $x = u \circ u'$. Then $x' = x$; hence, s is semisimple by 3.12, and so $\operatorname{Ker}(x) = \operatorname{Ker}(x^2) = \operatorname{Ker}(u \circ y \circ u')$. Since u is injective, u' is surjective; it follows that $y : H^i(X) \to H^i(X)$ is injective, and is an automorphism. Hence, by 2.12 and the Cayley-Hamilton theorem, y^{-1} is algebraic. Therefore, $v = y^{-1}u'$ is a left inverse u, and it is algebraic. The proof of (b) is similar. □

LEMMA 2.38 *Let $E = \bigoplus_{\nu=-n}^{n} E^\nu$ be a graded noncommutative ring (with 1). There is at most one complete set of orthogonal idempotents $\{\pi^0, \dots, \pi^{2n}\}$ in E satisfying the following conditions:*

(a) *$E^\nu = \bigoplus_i \pi^{i+\nu} E \pi^i$, and*

(b) *for $i = 0, \dots, n$, there exist $v^i \in E^{2n-2i}$ and $w^i \in E^{-(2n-2i)}$ such that*

$$(w^i v^i - 1)\pi^i = 0 = (v^i w^i - 1)\pi^{2n-i}.$$

PROOF The condition on the π^i means that

$$\pi^i \pi^j = \begin{cases} \pi^i & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

and $\pi^0 + \cdots + \pi^{2n} = 1$. By (a), $u \in E^0$ if and only if $u = \sum \pi^i u \pi^i$, hence if and only if $\pi^i u = \pi^i u \pi^i = u \pi^i$ for all i. In particular, the π^i lie in the centre $Z(E^0)$ of E^0.

We prove the uniqueness by induction on $i \le n$. Suppose that $\pi^0, \dots, \pi^{i-1}$ and $\pi^{2n-i+1}, \dots, \pi^{2n}$ are uniquely determined by the conditions. Let

$$\varphi^i = 1 - \sum_{\alpha \notin [i, 2n-i]} \pi^\alpha = \sum_{\alpha \in [i, 2n-i]} \pi^\alpha.$$

By (a), $\varphi^i E^{2n-2i} \varphi^i = E^{2n-2i} \pi^i$. If $E^{2n-2i} \pi^i u = 0$, then $w^i v^i \pi^i = 0$, so, by (b), $\pi^i u = 0$. Therefore, the right annihilator of π^i is uniquely determined. However, $\pi^i \in Z(E^0)$, and an idempotent in a commutative ring is uniquely determined by its annihilator. Similarly, π^{2n-1} is uniquely determined. □

THEOREM 2.39 *Assume that H satisfies the weak Lefschetz theorem. Then the following two conditions are equivalent:*

(i) the standard conjectures hold, i.e., $B(X)$ and $I(X,L)$ hold for all varieties X over k;

(ii) for all varieties X over k and all integers p such that $2p \le n = \dim(X)$, $D(X)$ holds and the quadratic form

$$a, b \mapsto (-1)^p \langle L^{n-2p} a \cdot b \rangle$$

is positive definite on the set of $a \in A^p(X) = C^p_{\mathrm{num}}(X)$ such that $L^{n-2p+1}a = 0$.

Moreover, if these conditions hold for several Weil cohomology theories satisfying the weak Lefschetz theorem, then

(a) *the operators Λ, ${}^c\Lambda$, $*$, $p^0, \dots, p^{2n}, \pi^0, \dots, \pi^{2n}$ are the classes of algebraic cycles that do not depend on the theory.*

(b) *The Betti numbers $b_i = \dim H^i(X)$ do not depend on the theory.*

(c) *The characteristic polynomial of an endomorphism induced by a rationally (resp. integrally) algebraic cycle has rational (resp. integer) coefficients that do not depend on the theory.*

(d) *If the map $H^i(X) \to H^j(Y)$ induced by an algebraic cycle is bijective (resp. injective, resp. surjective) in one theory, then it bijective (resp. injective, resp. surjective) in every theory. In fact, the inverse (resp. one left inverse, resp. one right inverse) may be induced by an algebraic cycle that does not depend on the theory.*

PROOF The equivalence of (i) and (ii) results immediately from 2.34 (and 2.25). If these conditions hold, then $\pi^0, \dots, \pi^{2n}$ are the classes of algebraic cycles by **??**. By 2.38 applied to the ring of algebraic correspondences, these cycles are uniquely determined modulo homological or, what is the same, numerical equivalence. By 2.24, ${}^c\Lambda$ is the class of an algebraic cycle, which, therefore is uniquely determined modulo numerical equivalence by (128). Finally, $p^n, \dots, p^{2n}$ (resp. $\Lambda, *, p^0, \dots, p^{n-1}$) are given by universal (noncommutative) polynomials with rational coefficients in L and ${}^c\Lambda$ by 2.18 and 2.20 (resp. in L and $p^n, \dots, p^{2n}$ by 2.18). Thus (a) holds.

By (a), the π^i are intrinsically determined. Therefore, (b) results from the formula $b_i = (-1)^i \langle \Delta \cdot \pi^{2n-i} \rangle$, and (c) results from the proof of 2.12. Further, a correspondence $u : H^*(X) \to H^*(Y)$ induces a map $u' : H^i(X) \to H^j(Y)$ if and only if $\pi^\ell_Y u \pi^i_X = 0$ for $\ell \ne j$, and u' is injective (resp. surjective, resp. bijective) if and only if there exists a correspondence $v : H^*(Y) \to H^*(X)$ such that $vu\pi^i_X = \pi^i_X$ (resp. ...); hence, (d) results from 2.36. □

APPLICATIONS TO MOTIVES

Recall that decompositions of rings, $R = \bigoplus_i R_i$, correspond to decompositions, $1 = \sum_i e_i$, of 1 into a sum of orthogonal central idempotents; then $R_i = e_i R e_i = R e_i$. Let R be a semisimple algebra.

Let H be a Weil cohomology theory satisfying the strong Lefschetz theorem and Conjecture C, and let $\mathcal{A}(X) = C_H^{\dim(X)}(X \times X)$ be the ring of algebraic correspondences of degree 0. For idempotents e in $\mathcal{A}(X)$ and f in $A(Y)$, we let

$$\mathrm{Hom}((X,e),(Y,f)) = f \circ C_H^{\dim(X)}(X \times Y) \circ e.$$

We write $h^i(X)$ for the motive (X, π^i); thus

$$\mathrm{End}(h^i(X)) = \pi^i \circ \mathcal{A}^0(X) \circ \pi^i.$$

Let X be a smooth projective variety. Assume $B(X)$, and fix a Lefschetz operator L. Then $*$ is a morphism of motives (2.24)

$$*: h^i(X) \to h^{2n-i}(X)(n-i),$$

and we define

$$\phi_L: h^i(X) \otimes h^i(X) \to T(-i)$$

to be the composite of

$$h^i(X) \otimes h^i(X) \xrightarrow{\mathrm{id}\otimes *_L} h^i(X) \otimes h^{2n-i}(X)(n-i) \to h^{2n}(X)(n-i) \xrightarrow{\mathrm{Tr}} T(-i).$$

Polarizations

Proposition 2.40 *Assume $B(X)$, and let L be a Lefschetz operator. The map $u \mapsto u' = *\circ u^t \circ *$ is an involution on* $\mathrm{End}(h^i(X)) \overset{\mathrm{def}}{=} \pi^i \circ C_H^{\dim(X)}(X\times X)\circ\pi^i$. *It is positive if and only if $I(X\times X, L\otimes 1 + 1\otimes L)$ holds.*

Proof Only have to prove necessity. Let $u = u_1 \otimes u_2 \in P^i(X)\otimes P^j(X)$. Then $\mathrm{Tr}(u'\circ u) = (-1)^i(u\cdot *_X u *_X)$. Now

$$*_X u *_X = (*_X \otimes *_X)(u) = \pm *_{X\times X} u.$$

Now $I(X\times X, L\otimes 1 + 1\otimes L)$ says that

$$(-1)^i\langle u\cdot *_{X\times X} u\rangle > 0.$$

So

$$(-1)^i\langle u\cdot *_{X\times X} u\rangle = \mathrm{Tr}(u'\circ u).$$

□

The Weil forms one gets in this way are all compatible.

Polarizations and the Hodge standard conjecture

Throughout this subsection, k is an algebraically closed field. By a Weil cohomology theory, we mean a Weil cohomology theory satisfying the strong Lefschetz theorem.

Recall that a Weil cohomology theory satisfies Conjecture D if homological equivalence coincides with numerical equivalence on algebraic cycles with $\mathbb{Q}$-coefficients.

Let H be a Weil cohomology theory. For a connected variety X of dimension n, define $P^r(X)$ to be the subspace of $A^r_H(X)$ on which L^{n-2r+1} is zero. Let θ^r be the bilinear form

$$(x,y) \mapsto (-1)^r\langle L^{n-2r}x\cdot y\rangle: P^r(X)\times P^r(X) \to \mathbb{Q}, \quad r \le n/2.$$

As originally stated (Grothendieck 1969), the Hodge standard conjecture asserts that these pairings are positive-definite when H is ℓ-adic étale cohomology. Kleiman (1968, §3) states the conjecture for any Weil cohomology theory. Note that if the Hodge standard conjecture holds for one Weil cohomology theory satisfying Conjecture D, then it holds for all.

For a Weil cohomology theory H, π^r denotes the projection onto H^r and Λ, ${}^c\Lambda$, $*$, p^r denote the maps defined above (p. 295).

Proposition 2.41 *For all Weil cohomology theories H satisfying Conjecture D, the operators Λ, ${}^c\Lambda$, $*$, p^r, and π^r are defined by algebraic cycles that, modulo numerical equivalence, depend only on L (not H).*

PROOF See 2.18. □

When there exists a Weil cohomology theory satisfying Conjecture D, we define $\mathsf{Mot}(k)$ to be the category of motives with the commutativity constraint modified using the π^r given by (2.41). It is semisimple (2.3, 2.4), hence Tannakian (I, 10.1), and it has a natural structure of a Tate triple. Let X be connected of dimension n, and let Z be a smooth hyperplane section of X. Then $l \overset{\text{def}}{=} \Delta_X(Z) \in A^{n+1}_{\text{num}}(X \times X)$ is a morphism

$$l : h(X) \to h(X)(1)$$

of degree 2. Define

$$\varphi^r : h^r(X) \otimes h^r(X) \to \mathbb{1}(-r)$$

to be the composite

$$h^r(X) \otimes h^r(X) \overset{\text{id} \otimes *}{\to} h^r(X) \otimes h^{2n-r}(X)(n-r) \to h^{2n}(X)(n-r) \cong \mathbb{1}(-r)$$

($*$ as in 2.41). Let $p^r(X)$ be the largest subobject of

$$\operatorname{Ker}(l^{n-2r+1} : h^{2r}(X)(r) \to h^{2n-2r+2}(X)(n-r+1))$$

on which $\pi \overset{\text{def}}{=} \pi(\mathsf{Mot}(k))$ acts trivially. For any Weil cohomology theory H satisfying Conjecture D,

$$\gamma(p^r(X)) \cong P^r(X)$$

where γ is the $\otimes$-equivalence $\operatorname{Hom}(\mathbb{1}, -)$ from $\mathsf{Mot}(k)^\pi$ to finite-dimensional $\mathbb{Q}$-vector spaces, and there is a pairing

$$\vartheta^r : p^r(X) \otimes p^r(X) \to \mathbb{1},$$

also fixed by π, such that $\gamma(\vartheta^r) = \theta^r$.

THEOREM 2.42 *Let H be a Weil cohomology theory satisfying Conjecture D. The following statements are equivalent:*

(a) *H satisfies the Hodge standard conjecture;*

(b) *there exists a polarization Π on $\mathsf{Mot}(k)$ for which the forms φ^r are positive;*

(c) *there exists a polarization Π on $\mathsf{Mot}(k)$ for which the forms ϑ^r are positive.*

PROOF (a)⇒(b): Let X be a connected smooth projective k-scheme of dimension n equipped with a "polarization" $\lambda \in C^1(X)$. The polarization of X defines morphisms of motives

$$* : h^i(X) \to h^{2n-i}(X)(n-i).$$

We let φ_λ denote the bilinear form

$$\varphi_\lambda : h^i(X) \otimes h^i(X) \to T^{\otimes -i}$$

equal to the composite

$$h^i(X) \otimes h^i(X) \overset{\text{id} \otimes *}{\longrightarrow} h^i(X) \otimes h^{2n-i}(X)(n-i) \to h^{2n}(X)(n-i) \overset{\text{Tr}}{\longrightarrow} T^{\otimes -i}.$$ □

It is $(-1)^i$-symmetric, and it follows from 2.35 that the forms φ_λ are Weil forms compatible in pairs. Moreover, if X, Y are connected smooth projective k-schemes equipped with

polarizations λ_X, λ_Y, the form $\varphi_X \otimes \varphi_Y$ on $h^i(X) \otimes h^j(Y)$ with values in $T^{\otimes -i-j}$ is deduced (up to a positive factor, see 2.35) from the form $\varphi_{X\times Y}$ on $h^{i+j}(X \times Y)$ when one polarizes $X \times Y$ by $\lambda_X \otimes 1_Y + 1_X \otimes \lambda_Y$.

(b)⇒(c): The restriction of $\varphi^{2r} \otimes \mathrm{id}_{\mathbb{1}(2r)}$ to the subobject $p^r(V)$ of $h^{2r}(V)(r)$ is the form ϑ^r.

(c)⇒(a): Let Π be a polarization on $\mathsf{Mot}(k)$ for which the forms ϑ^r are positive. The restriction of Π to $\mathsf{Mot}(k)^{w(\mathbb{G}_m)}$ is a symmetric polarization, and so there exists an $\mathbb{R}$-valued fibre functor ω on $\mathsf{Mot}(k)^{w(\mathbb{G}_m)}$ carrying Π-positive forms to positive-definite symmetric forms (VI, 9.1). The restriction of ω to $\mathsf{Mot}(k)^{\pi}_{(\mathbb{R})}$ is (uniquely) isomorphic to γ, and so $\gamma(\vartheta^r)$ is positive-definite.

We deduce from this discussion that the set of Weil forms of the preceding type define a polarization on the Tate triple of motives.

SUMMARY 2.43 Assume that the standard conjectures hold for smooth projective varieties over k and all the standard Weil cohomology theories. Then $\mathsf{Mot}(k)$ is a tannakian category over $\mathbb{Q}$ with

(a) a canonical fibre functor for each standard Weil cohomology theory, and

(b) a unique polarization such that the geometric Weil forms are positive.

To say more, for example, to describe fundamental group of the category, we need to know more.

The Hodge conjecture

CONJECTURE 2.44 (HODGE$^r(X)$) *Let X be a smooth projective variety over $\mathbb{C}$, and let $r \in \mathbb{N}$. The $\mathbb{Q}$-subspace of $H^{2r}_B(X)(r)$ generated by the classes of algebraic cycles is*

$$\mathcal{B}^r(X) \overset{\text{def}}{=} H^{2r}_B(X)(r) \cap H^{r,r}(X).$$

Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and L a Lefschetz operator. As Betti cohomology satisfies the strong Lefschetz theorem, the map

$$L^{n-2r} : H^{2r}_B(X)(r) \to H^{2n-2r}_B(X)(r)$$

is an isomorphism, and it sends $\mathcal{B}^r(X)$ isomorphically onto $\mathcal{B}^{n-r}(X)$. Therefore, if the Hodge conjecture holds for X, then so does the Lefschetz standard conjecture $A(X, L)$. It follows that, if the Hodge conjecture holds for all smooth projective varieties over $\mathbb{C}$, then so do the all the standard conjectures (for Betti cohomology).

Recall that $\mathsf{Hdg}_{\mathbb{Q}}$ is the category of polarizable rational Hodge structures.

THEOREM 2.45 *Let $\mathsf{Mot}(\mathbb{C})$ be the category of motives over $\mathbb{C}$ for Betti homological equivalence. The Hodge conjecture holds for all smooth projective varieties over $\mathbb{C}$ if and only if the Betti fibre functor*

$$\omega_B : \mathsf{Mot}(\mathbb{C}) \to \mathsf{Hdg}_{\mathbb{Q}}$$

is fully faithful.

PROOF Restatement of the conjecture. □

There is no description, even conjectural, for the essential image. As Mumford has pointed out, a Baire category argument using Griffiths transversality shows that "most" polarized rational Hodge structures are not motivic.

Assuming the Hodge conjecture, we obtain the following tannakian subcategories

$$\mathsf{CM}(\mathbb{C}) \subset \mathsf{AMot}(\mathbb{C}) \subset \mathsf{Mot}(\mathbb{C}) \subset \mathsf{Hdg}_{\mathbb{Q}}$$

of the category of polarizable rational Hodge structures. Here $\mathsf{AMot}(\mathbb{C})$ (resp. $\mathsf{CM}(\mathbb{C})$ is the category of numerical motives based on abelian varieties over $\mathbb{C}$ (resp. CM abelian varieties over $\mathbb{C}$). For each category, we have an affine group scheme $G = \mathcal{A}ut^{\otimes}(\omega_B)$ and a homomorphism $h: \mathbb{S} \to G$, and we have surjective homomorphisms

$$G_{\mathsf{Hdg}} \twoheadrightarrow G_{\mathsf{Mot}} \twoheadrightarrow G_{\mathsf{AMot}} \twoheadrightarrow G_{\mathsf{CM}}$$

As we shall explain later, G_{CM} is the well-known Serre group S. Both G_{Hdg} and G_{AMot} have S as their largest commutative quotient, and, in each case, the algebraic quotients of their derived groups have been classified (II, 9.15; Milne 1994b, 1.27). On the other hand, little is known about the derived group of G_{Mot} (see Serre 1994a for some conjectures).

The Tate conjecture

We consider the conjecture only for a finite base field.

CONJECTURE 2.46 (TATE$^r(X,\ell)$) *Let X be a smooth projective variety over a finite field k, and let $r \in \mathbb{N}$. For all $\ell \neq \mathrm{char}(k)$, the $\mathbb{Q}_\ell$-subspace of $H^{2r}_{\mathrm{et}}(X_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))$ generated by the classes of algebraic cycles is*

$$\mathcal{T}^r_\ell(X) \stackrel{\mathrm{def}}{=} H^{2r}_{\mathrm{et}}(X_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(k^{\mathrm{al}}/k)}.$$

CONJECTURE 2.47 (TATE$^r(X,p)$) *Let X be a smooth projective variety over a finite field k, and let $H^r_{\mathrm{crys}}(X/B)$ be the crystalline cohomology group with coefficients in the field of fractions of the Witt vectors $W(k)$. The $\mathbb{Q}_p$-subspace of $H^r_{\mathrm{crys}}(X/B)$ generated by the classes of algebraic cycles is*

$$\mathcal{T}^r_p(X) \stackrel{\mathrm{def}}{=} H^r_{\mathrm{crys}}(X/B)^{F=1}.$$

CONJECTURE 2.48 (FULL TATE) *Let X be a smooth projective variety X over $\mathbb{F}_{m71}$. The pole of the zeta function of X at $t = q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r.*

THEOREM 2.49 (FOLKLORE) *Let X be smooth projective variety over a finite field, and let $r \in \mathbb{N}$. The following statements are equivalent:*

(a) *Tate$^r(X,l)$ and $D^r(X,l)$ hold for one prime number l ($l = p$ allowed);*

(b) *Tate$^r(X,l)$ and $D^r(X,l)$ hold for all prime number l (including p);*

(c) *The full Tate conjecture holds for X and r.*

2.50 Let $q \in p^{\mathbb{N}}$. An algebraic number is a Weil q-number of weight m if

(a) for every homomorphism $\rho: \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $\rho(\pi) \cdot \overline{\rho(\pi)} = q^m$, and

(b) for some $n > 0$, $q^m\pi$ is an algebraic integer.

The set of Weil q-numbers in $\mathbb{Q}^{\mathrm{al}}$ is denoted by $W(q)$. It is a subgroup of the multiplicative group of $\mathbb{Q}^{\mathrm{al}}$, stable under the action of $\Gamma \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. We let $P(q)$ denote the affine group scheme of multiplicative type with

$$X^*(P(q)) = W(q).$$

There is a universal element $\pi_{\mathrm{univ}} \in P(q)(\mathbb{Q})$ such that $\chi(\pi_{\mathrm{univ}}) = \pi$ if χ is the character of $P(q)$ corresponding to π — it is the element corresponding to the inclusion map under

$$P(q)(\mathbb{Q}) \simeq \mathrm{Hom}(W(q), \mathbb{Q}^{\mathrm{al}\times})^{\mathrm{Gal}\,\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})}.$$

2.51 If $n|n'$, then $\pi \mapsto \pi^{n'/n}$ is a homomorphism $W(p^m) \to W(p^{n'})$. Define

$$W(p^\infty) = \varinjlim W(p^m).$$

There is an action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $W(p^\infty)$, and the protorus P over $\mathbb{Q}$ such that

$$X^*(P) = W(p^\infty)$$

is called the Weil-number protorus.

THEOREM 2.52 *Assume that the standard conjectures hold for smooth projective varieties over $\mathbb{F}_q$ (resp. $\mathbb{F}$) and all the standard Weil cohomologies. Then* $\mathsf{Mot}(\mathbb{F}_q)$ *(resp.* $\mathsf{Mot}(\mathbb{F})$*) is a tannakian category over $\mathbb{Q}$ such that*

(a) *each standard Weil cohomology defines a fibre functor;*

(b) *there is a unique polarization such that the geometric Weil forms are positive;*

(c) *the fundamental group is $P(q)$ (resp. P).*

PROOF See Milne 1994a. □

Initially, the rest of this chapter was intended to describe Deligne's theory of motives for absolute Hodge classes in characteristic zero, and the extension of this to motives for rational Tate classes in characteristic p, which was conditional on the rationality conjecture. The proof of this last conjecture made the theory unconditional, and the proof of the Hodge conjecture for CM abelian varieties made it almost-algebraic.

3 Motives for absolute Hodge classes

> *For me, [the Hodge conjecture] is part of the story of motives, and it is not crucial whether it is true or false. If it is true, that's very good, and it solves a large part of the problem of constructing motives in a reasonable way. If one can find another purely algebraic notion of cycles for which the analogue of the Hodge conjecture holds, this will serve the same purpose, and I would be as happy as if the Hodge conjecture were proved. For me it is motives, not Hodge, that is crucial.*
>
> Deligne, May 2013.[7]

[7] Interview on the award of the Abel prize, Eur. Math. Soc. Newsl. No. 89 (2013), 15–23.

Let X be a smooth complete algebraic variety over an algebraically closed field k of characteristic zero. We let $H^r_{k\times\mathbb{A}_f}(X)(m)$ denote the product of $H^r_{\mathrm{dR}}(X)(m)$ with the restricted product of the topological spaces $H^r_\ell(X)(m)$ relative to their subspaces $H^r(X_{\mathrm{et}},\mathbb{Z}_\ell)(m)$. This is a finitely generated free module over the ring $k\times\mathbb{A}_f$. For any homomorphism $\sigma: k\to k'$ of algebraically closed fields, there is a base change homomorphism

$$H^r_{k\times\mathbb{A}_f}(X)(m)\xrightarrow{\sigma} H^r_{k'\times\mathbb{A}_f}(\sigma X)(m). \tag{138}$$

When $k=\mathbb{C}$, the maps (124) and (126) give a comparison isomorphism

$$(\mathbb{C}\times\mathbb{A}_f)\otimes_{\mathbb{Q}} H^r_B(X)(m)\to H^r_{\mathbb{C}\times\mathbb{A}_f}(X)(m). \tag{139}$$

Absolute Hodge classes

Let X be a smooth complete algebraic variety over $\mathbb{C}$. The cohomology group $H^{2r}_B(X)(r)$ has a Hodge structure of weight 0. An element of type $(0,0)$ in it is called a ***Hodge class of codimension*** r on X.[8] We wish to extend this notion to all base fields of characteristic zero. Of course, given a variety X over a field k, we can choose a homomorphism $\sigma: k\to\mathbb{C}$ and define a Hodge class on X to be a Hodge class on σX, but this notion depends on the choice of the embedding. Deligne's idea for avoiding this problem is to use all embeddings (Deligne 1979, 0.7).

Let X be a smooth complete algebraic variety over an algebraically closed field k of characteristic zero, and let σ be a homomorphism $k\to\mathbb{C}$. An element γ of $H^{2r}_{k\times\mathbb{A}_f}(X)(r)$ is a σ-***Hodge class of codimension*** r if $\sigma\gamma$ lies in the subspace $H^{2r}_B(\sigma X)(r)\cap H^{0,0}$ of $H^{2r}_{\mathbb{C}\times\mathbb{A}_f}(\sigma X)(r)$. When k has finite transcendence degree over $\mathbb{Q}$, an element γ of $H^{2r}_{k\times\mathbb{A}}(X)(r)$ is an ***absolute Hodge class*** if it is σ-Hodge for all homomorphisms $\sigma: k\to\mathbb{C}$,

$$\begin{array}{ccc} H^{2r}_B(\sigma X)(r)\cap H^{0,0} & \xhookrightarrow{(139)} & H^{2r}_{\mathbb{C}\times\mathbb{A}_f}(\sigma X)(r) \\ \uparrow & & \uparrow{\scriptstyle (138)}\,\sigma \\ AH^r(X) & \hookrightarrow & H^{2r}_{k\times\mathbb{A}_f}(X)(r). \end{array}$$

The absolute Hodge classes of codimension r on X form a $\mathbb{Q}$-subspace $AH^r(X)$ of $H^{2r}_{k\times\mathbb{A}_f}(X)(r)$.

We list the basic properties of absolute Hodge classes.

3.1 The inclusion $AH^r(X)\subset H^{2r}_{k\times\mathbb{A}_f}(X)(r)$ induces an injective map

$$\big(k\times\mathbb{A}_f\big)\otimes_{\mathbb{Q}} AH^r(X)\to H^{2r}_{k\times\mathbb{A}_f}(X)(r).$$

In particular $AH^r(X)$ is a finite dimensional $\mathbb{Q}$-vector space. This follows from (139) because $AH^r(X)$ is isomorphic to a $\mathbb{Q}$-subspace of $H^{2r}_B(\sigma X)(r)$ (each σ).

3.2 For any homomorphism $\sigma: k\to k'$ of algebraically closed fields of finite transcendence degree over $\mathbb{Q}$, the map (138) induces an isomorphism $AH^r(X)\to AH^r(\sigma X)$ (Deligne 1982, 2.9a).

[8] As $H^{2r}_B(X)(r)\simeq H^{2r}_B(X)\otimes\mathbb{Q}(r)$, this is essentially the same as an element of $H^{2r}_B(X)$ of type (r,r).

This allows us to define $AH^r(X)$ for a smooth complete variety over an arbitrary algebraically closed field k of characteristic zero: choose a model X_0 of X over an algebraically closed subfield k_0 of k of finite transcendence degree over $\mathbb{Q}$, and define $AH^r(X)$ to be the image of $AH^r(X_0)$ under the map $H^{2r}_{k_0\times\mathbb{A}_f}(X_0)(r) \to H^{2r}_{k\times\mathbb{A}_f}(X)(r)$. With this definition, (3.2) holds for all homomorphisms of algebraically closed fields k of characteristic zero. Moreover, if k admits an embedding in $\mathbb{C}$, then a cohomology class is absolutely Hodge if and only if it is σ-Hodge for one (hence every) such embedding.

3.3 The cohomology class of an algebraic cycle on X is absolutely Hodge; thus, the algebraic cohomology classes of codimension r on X form a $\mathbb{Q}$-subspace $A^r(X)$ of $AH^r(X)$ (Deligne 1982, 2.1a).

3.4 The Künneth components of the diagonal are absolute Hodge classes (ibid., 2.1b).

3.5 Let X_0 be a model of X over a subfield k_0 of k such that k is algebraic over k_0; then $\mathrm{Gal}(k/k_0)$ acts on $AH^r(X)$ through a finite discrete quotient (ibid. 2.9b).

3.6 Let

$$AH^*(X) = \bigoplus\nolimits_{r\geq 0} AH^r(X);$$

then $AH^*(X)$ is a $\mathbb{Q}$-subalgebra of $\bigoplus H^{2r}_{k\times\mathbb{A}_f}(X)(r)$. For any regular map $\alpha: Y \to X$ of complete smooth varieties, the maps α_* and α^* send absolute Hodge classes to absolute Hodge classes. This follows easily from the definitions.

3.7 (Deligne 1982, 2.12, 2.14) Let S be a smooth connected algebraic variety over $\mathbb{C}$, and let $\pi: X \to S$ be a smooth proper morphism. Let $\gamma \in \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$, and let γ_s be the image of γ in $H^{2r}_B(X_s)(r)$ $(s \in S(\mathbb{C}))$. If γ_s is a Hodge (resp. absolutely Hodge) class for one $s \in S(\mathbb{C})$, then it is a Hodge (resp. absolutely Hodge class) for every $s \in S(\mathbb{C})$.

Conjecture 3.8 (Deligne 1979a, 0.10) *Every σ-Hodge class on a smooth complete variety over an algebraically closed field of characteristic zero is absolutely Hodge.*

In other words,

$$\sigma\text{-Hodge for one } \sigma \implies \sigma\text{-Hodge for all } \sigma, \text{ i.e., absolutely Hodge.}$$

Theorem 3.9 (Deligne 1982, 2.11) *Conjecture 3.8 is true for abelian varieties.*

To prove the theorem, it suffices to show that every Hodge class on an abelian variety over $\mathbb{C}$ is absolutely Hodge.[9]

It is convenient to restate the main result of Deligne 1982 in the following more abstract form.

Theorem 3.10 (Deligne) *Suppose that for each abelian variety A over $\mathbb{C}$ we have a $\mathbb{Q}$-subspace $C^r(A)$ of the Hodge classes of codimension r on A. Assume:*

(a) *$C^r(A)$ contains all algebraic classes of codimension r on A;*

(b) *pull-back by a homomorphism $\alpha: A \to B$ of abelian varieties maps $C^r(B)$ into $C^r(A)$;*

[9] Let A be an abelian variety over k, and suppose that γ is σ_0-Hodge for some homomorphism $\sigma_0: k \to \mathbb{C}$. We have to show that it is σ-Hodge for every $\sigma: k \to \mathbb{C}$. But, using the Zorn's lemma, one can show that there exists a homomorphism $\sigma': \mathbb{C} \to \mathbb{C}$ such that $\sigma = \sigma'\circ\sigma_0$. Now γ is σ-Hodge if and only if $\sigma_0\gamma$ is σ'-Hodge.

(c) *let $\pi : \mathcal{A} \to S$ be an abelian scheme over a connected smooth complex algebraic variety S, and let $t \in \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$; if t_s lies in $C^r(A_s)$ for one $s \in S(\mathbb{C})$, then it lies in $C^r(A_s)$ for all s.*

Then $C^r(A)$ contains all the Hodge classes of codimension r on A.

PROOF This is essentially what is proved in Deligne 1982. For a sketch of the proof, with some simplifications, see Milne 2013, Theorem 9.11. □

To obtain Theorem 3.9, take $C^r(A)$ to be $AH^r(A)$.

COROLLARY 3.11 *If hypothesis (c) of the theorem holds for algebraic classes on abelian varieties, then the Hodge conjecture holds for abelian varieties. (In other words, for abelian varieties, the variational Hodge conjecture implies the Hodge conjecture.)*

PROOF Immediate consequence of the theorem, because the algebraic classes satisfy (a) and (b). □

ASIDE 3.12 Let $X_{\mathbb{C}}$ be a smooth complete algebraic variety over $\mathbb{C}$. Then $X_{\mathbb{C}}$ has a model X_0 over a subfield k_0 of $\mathbb{C}$ finitely generated over $\mathbb{Q}$. Let k be the algebraic closure of k_0 in $\mathbb{C}$, and let $X = X_{0k}$. For a prime number ℓ, let

$$\mathcal{T}^r_\ell(X) = \bigcup\nolimits_U H^{2r}_\ell(X)(r)^U \qquad \text{(space of Tate classes)}$$

where U runs over the open subgroups of $\mathrm{Gal}(k/k_0)$ — as the notation suggests, $\mathcal{T}^r_\ell(X)$ depends only on X/k. The Tate conjecture (Tate 1965, Conjecture 1) says that the $\mathbb{Q}_\ell$-vector space $\mathcal{T}^r_\ell(X)$ is spanned by algebraic classes. Statement 3.5 implies that $AH^r(X)$ projects into $\mathcal{T}^r_\ell(X)$, and (3.1) implies that the map $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} AH^r(X) \to \mathcal{T}^r_\ell(X)$ is injective. Therefore the Tate conjecture implies that $A^r(X) = AH^r(X)$, and so the Tate conjecture for X and one ℓ implies that all absolute Hodge classes on $X_{\mathbb{C}}$ are algebraic. Thus, in the presence of Conjecture 3.8, the Tate conjecture implies the Hodge conjecture. In particular, Theorem 3.9 shows that, for an abelian variety, the Tate conjecture implies the Hodge conjecture.

Motives for absolute Hodge classes

We fix a base field k of characteristic zero; "variety" will mean "smooth projective variety over k".

3.13 For varieties X and Y with X connected, we let

$$C^r(X, Y) = AH^{\dim X + r}(X \times Y).$$

Then, as in 4.6, 4.8, we get a $\mathbb{Q}$-linear $\otimes$-category of correspondences with objects hX and morphisms

$$\mathrm{Hom}(hX, hY) = C^0(X, Y) = AH^{\dim X}(X \times Y).$$

3.14 The category of correspondences is additive, but not abelian, and so, as in 4.9 we enlarge it by adding the images of idempotents. In this way, we get a category of effective motives, which has one object $h(X, e)$ for each variety X and idempotent e in the ring $\mathrm{End}(hX) = AH^{\dim X}(X \times X)$, and whose Homs are defined by

$$\mathrm{Hom}(h(X, e), h(Y, f)) = f \circ AH^{\dim X}(X \times Y) \circ e.$$

The category of effective motives is abelian, but objects need not have duals, so, as in 4.10, we invert $h^2\mathbb{P}^1$. In this way, we get a category of motives which has one object $h(X,e,m)$ for each pair (X,e) as before and integer m, and whose Homs are defined by

$$\mathrm{Hom}(h(X,e,m),h(Y,f,n)) = f\circ AH^{n-m}(X\times Y)\circ e.$$

Finally, we use that the Künneth components of the diagonal are absolutely Hodge (3.4) to modify the commutativity constraint (see 2.29) to obtain the category motives $\mathsf{Mot}(k)$ for absolute Hodge classes.

3.15 The category of motives for absolute Hodge classes $\mathsf{Mot}(k)$ is a $\mathbb{Q}$-linear semisimple tannakian category over $\mathbb{Q}$. The standard cohomology functors factor through $\mathsf{Mot}(k)$. For example, define

$$\omega_\ell(h(X,e,m)) = e\left(\bigoplus\nolimits_i H^i_\ell(X)(m)\right)$$

(image of e acting on $\bigoplus_i H^i_\ell(X)(m)$). Then ω_ℓ is an exact faithful functor $\mathsf{Mot}(k)\to\mathsf{Vec}_{\mathbb{Q}_\ell}$ commuting with tensor products. Similarly, de Rham cohomology defines an exact tensor functor $\omega_{\mathrm{dR}}:\mathsf{Mot}(k)\to\mathsf{Vec}_k$, and, when $k=\mathbb{C}$, Betti cohomology defines an exact tensor functor $\mathsf{Mot}(k)\to\mathsf{Vec}_{\mathbb{Q}}$. The functors ω_ℓ, ω_{dR}, and ω_B are called the ℓ-adic, de Rham, and Betti fibre functors, and they send a motive to its ℓ-adic, de Rham, or Betti ***realization***. See Deligne and Milne 1982, §6.

The Betti fibre functor on $\mathsf{Mot}(\mathbb{C})$ takes values in $\mathsf{Hdg}_{\mathbb{Q}}$, and is faithful (almost by definition). Deligne's conjecture 3.8 is equivalent to saying that it is full.

Abelian motives

DEFINITION 3.16 A motive is ***abelian*** if it lies in the tannakian subcategory $\mathsf{AMot}(k)$ of $\mathsf{Mot}(k)$ generated by the motives of abelian varieties.

The Tate motive, being isomorphic to $\bigwedge^2 h_1E$ for any elliptic curve E, is an abelian motive. It is known that $h(X)$ is an abelian motive if X is a curve, a unirational variety of dimension ≤ 3, a Fermat hypersurface, or a $K3$ surface.

Deligne's theorem 3.9 implies that $\omega_B:\mathsf{AMot}(\mathbb{C})\to\mathsf{Hdg}_{\mathbb{Q}}$ is fully faithful.

The category of motives of CM-type

First we review the theory of abelian varieties of CM-type.

ABELIAN VARIETIES OF CM-TYPE (CM ABELIAN VARIETIES)

Let E be a CM-field. A ***CM-type*** on E is a locally constant map $\varphi:\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{al}})\to\mathbb{Z}$ such that $\varphi(\tau)\ge 0$ and $\varphi(\tau)+\varphi(\iota\circ\tau)=1$ for all τ. A CM-type on E is said to be ***primitive*** if it is not the extension $\tau\mapsto\varphi_0(\tau|E_0)$ of a CM-type φ_0 on a proper subfield E_0. Every CM-type φ on E is the extension of a unique primitive CM-type.

A simple abelian variety A over C is said to be of ***CM-type*** if $\mathrm{End}^0(A)$ is a field (necessarily CM) of degree $2\dim A$ over $\mathbb{Q}$, and an arbitrary abelian variety over C is said to be of ***CM-type*** if all its simple isogeny factors are of CM-type.

Let A be a simple abelian variety over C of CM-type, and let $E=\mathrm{End}^0(A)$. Let i be the inclusion $\mathbb{Q}^{\mathrm{al}}\hookrightarrow C$. For $\tau\in\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{al}})$, define $\varphi(\tau)$ to be 1 or 0 according as $i\circ\tau$ does, or does not, occur in the representation of E on the tangent space to A at 0.

PROPOSITION 3.17 *With the above notations, φ is a primitive CM-type on E, and the map $A \mapsto (E,\varphi)$ defines a bijection from the set of isogeny classes of simple abelian varieties over C of CM-type to the set of isomorphism classes of pairs (E,φ) consisting of a CM-field of finite degree over $\mathbb{Q}$ and a primitive CM-type on the field.*

PROOF Suppose first that $C = \mathbb{C}$. Let φ be a CM-type on a CM-field E, and let $\Sigma = \{\tau \mid \varphi(\tau) = 1\}$. Define A_φ to be the abelian variety over $\mathbb{C}$ such that $A_\varphi(\mathbb{C}) = \mathbb{C}^\Sigma/\Sigma(\mathcal{O}_E)$ where $\mathbb{C}^\Sigma = \mathrm{Hom}(\Sigma,\mathbb{C})$ and $\mathcal{O}_E$, the ring of integers in E, is embedded in $\mathbb{C}^\Sigma$ by $a \mapsto (\sigma a)_{\sigma\in\Sigma}$. Then $(E,\varphi) \mapsto A_\varphi$ provides an inverse to the map $A \mapsto (E,\varphi)$.

To extend the result to fields other than $\mathbb{C}$, use the following observation: let $C \hookrightarrow C'$ be an inclusion of algebraically closed fields of characteristic zero, and let (A,i) be an abelian variety of CM-type (E,φ) over C'; then any specialization of (A,i) to C is again of CM-type (E,φ), and hence becomes isogenous to (A,i) over C'. □

Let φ be a CM-type on a CM-field E. For each $\tau : E \to \mathbb{Q}^{\mathrm{al}}$ and $\sigma \in \Gamma$, define

$$\psi_\tau(\sigma) = \varphi(\sigma^{-1}\circ\tau).$$

Then ψ_τ depends only on the restriction of σ to $\mathbb{Q}^{\mathrm{cm}}$, and ψ_τ, when regarded as a map $\mathrm{Hom}(\mathbb{Q}^{\mathrm{cm}},\mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$, is a CM-type on $\mathbb{Q}^{\mathrm{cm}}$. Moreover, for any $\rho \in \Gamma$, $\psi_{\rho\circ\tau}(\sigma) = \psi_\tau(\rho^{-1}\circ\sigma) = (\rho\psi_\tau)(\sigma)$, and so, as τ runs over the embeddings $E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, ψ_τ runs over a Γ-orbit of CM-types on $\mathbb{Q}^{\mathrm{cm}}$.

PROPOSITION 3.18 *The map $(E,\varphi) \mapsto \{\psi_\tau\}$ defines a bijection from the set of isomorphism classes of pairs (E,φ) consisting of a CM-field of finite degree over $\mathbb{Q}$ and a primitive CM-type on the field to the set of Γ-orbits of CM-types on $\mathbb{Q}^{\mathrm{cm}}$.*

PROOF We construct an inverse. For a CM-type ψ on $\mathbb{Q}^{\mathrm{cm}}$, define Γ_ψ to be the stabilizer of ψ in Γ and E_ψ to be the fixed field of Γ_ψ. Let $\tau_0 : E_\psi \hookrightarrow \mathbb{Q}^{\mathrm{al}}$ be the given embedding. Then any embedding $\tau : E_\psi \to \mathbb{Q}^{\mathrm{al}}$ can be written $\tau = \sigma\circ\tau_0$ with $\sigma \in \Gamma$, and we define

$$\varphi_\psi(\tau) = \psi(\sigma^{-1}).$$

Then φ_ψ is a CM-type on E_ψ, and the map $\psi \mapsto (E_\psi,\varphi_\psi)$ gives the required inverse. □

The ***reflex field*** K of (E,φ) is defined to be the fixed field of the stabilizer of φ in Γ. Thus $\sigma \in \Gamma$ fixes K if and only if $\varphi(\sigma^{-1}\circ\tau) = \varphi(\tau)$ for all $\tau : E \to \mathbb{Q}^{\mathrm{al}}$. For any $\tau : E \hookrightarrow C$, ψ_τ is the extension to $\mathbb{Q}^{\mathrm{cm}}$ of a primitive CM-type on K.

The ***reflex field*** of a simple abelian variety over C of CM-type is defined to be the reflex field of its associated CM-type.

PROPOSITION 3.19 *Let K be a CM-subfield of C. There is a natural one-to-one correspondence between the set of isogeny classes of simple abelian varieties over C of CM-type whose reflex field is contained in K and the set of Γ-orbits of CM-types on K.*

PROOF When $K = \mathbb{Q}^{\mathrm{cm}}$, this is an immediate consequence of the preceding two propositions. The remark following the definition of the reflex field of a CM-type allows one to extend it to an arbitrary CM-subfield of C. □

REMARK 3.20 Let E be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$, and let φ be a CM-type on E. Let K be the reflex field of (E,φ), and let $\psi = \psi_{\tau_0}$ where τ_0 is the given inclusion of E into $\mathbb{Q}^{\mathrm{al}}$. Then $\psi(\sigma|K) = \varphi(\sigma^{-1}|E)$ for any $\sigma \in \Gamma$, and (K,ψ) is the reflex of (E,φ) in the classical sense (Shimura 1971, p126).

MOTIVES OF CM TYPE

A motive over $\mathbb{C}$ is of ***CM-type*** if its Hodge realization is of CM-type, and a rational Hodge structure is of CM-type if it is polarizable and its Mumford–Tate group if commutative (hence a torus). We first describe the tannakian category of polarizable rational Hodge structure is of CM-type

A number field E is a ***CM field*** if it is a totally imaginary quadratic extension of a totally real field. Let $\mathbb{Q}^{\mathrm{cm}}$ be the union of the CM-subfields of $\mathbb{Q}^{\mathrm{al}}$. Then $\mathbb{Q}^{\mathrm{cm}}$ is the largest Galois extension of $\mathbb{Q}$ in $\mathbb{Q}^{\mathrm{al}}$ such that complex conjugation is in the centre of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q})$.

LEMMA 3.21 *Let T be a torus over $\mathbb{Q}$ and μ a cocharacter of T over $\mathbb{Q}^{\mathrm{al}}$. The following conditions on μ are equivalent:*

(a) *the weight of μ is defined over $\mathbb{Q}$ and μ is defined over a CM field;*

(b) *for all $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $(\sigma - 1)(\iota + 1)\mu = 0 = (\iota + 1)(\sigma - 1)\mu$.*

PROOF The first equality in (b) says that the weight of μ is defined over $\mathbb{Q}$, and then the second says that $\sigma\iota\mu = \iota\sigma\mu$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, i.e., that μ is defined over $\mathbb{Q}^{\mathrm{cm}}$. □

Condition (b) of the lemma is called the ***Serre condition***. When T is split by a CM field, the Serre condition simply says that the weight of μ is defined over $\mathbb{Q}$. When (V, h) is simple, this means that $\mathrm{End}(V, h)$ is either a CM-field or $\mathbb{Q}$. We have the following criterion.

THEOREM 3.22 *Let T be a torus over $\mathbb{Q}$ and μ a cocharacter of T over $\mathbb{Q}^{\mathrm{al}}$. Then (T, μ) is a Mumford–Tate group if and only if*

(a) *the weight $-\mu - \iota\mu$ of μ is defined over $\mathbb{Q}$,*

(b) *T is split by a CM field, and*

(c) *μ generates T.*

PROOF Let (T, μ) be a pair such that μ generates T. Then

$$\begin{aligned} \text{(a)} + \text{(b)} &\iff \text{(a) of Lemma 3.21} \\ 3.24 + 3.25 &\iff \text{(b) of Lemma 3.21,} \end{aligned}$$

and so the theorem restates Propositions 3.24 and 3.25 for the case of a torus. □

Let E be a CM subfield of $\mathbb{Q}^{\mathrm{al}}$. Then $(\mathbb{G}_m)_{E/\mathbb{Q}}$ is a torus with character group $\mathbb{Z}^{\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{al}})}$, and we define S^E to be the quotient of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ such that

$$X^*(S^E) = \{\lambda \in \mathbb{Z}^{\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{al}})} \mid \lambda(\sigma) + \lambda(\iota\circ\sigma) = \text{constant}, \ \ \sigma \in \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})\}.$$

Define μ^E to be the cocharacter of S^E such that

$$\langle \lambda, \mu^E \rangle = \lambda(\sigma_0), \quad \text{all } \lambda \in X^*(S^E),$$

where σ_0 is the given embedding of E into $\mathbb{Q}^{\mathrm{al}}$. If $E \subset E' \subset \mathbb{Q}^{\mathrm{al}}$, then the norm map defines a homomorphism $S^{E'} \to S^E$ carrying $\mu^{E'}$ to μ^E. We set

$$(S, \mu_{\mathrm{can}}) = \varprojlim (S^E, \mu^E).$$

The pair (S, μ_{can}) is called the ***Serre group***. Note that $X^*(S)$ is the set of all locally constant functions $\lambda : \text{Gal}(\mathbb{Q}^{\text{cm}}/\mathbb{Q}) \to \mathbb{Z}$ such that

$$\lambda(\sigma) + \lambda(\iota\sigma\sigma) = -m$$

for some integer m (called the ***weight*** of λ).

THEOREM 3.23 *(a) The cocharacter μ^E of S^E satisfies the Serre condition.*

(b) Let T be a torus over $\mathbb{Q}$ and μ a cocharacter satisfying the Serre condition. Then there is a unique homomorphism $\rho_\mu : S \to T$ such that $(\rho_\mu)_{\mathbb{Q}} \circ \mu_{can} = \mu$.

(c) We have

$$(S, \mu_{can}) \simeq \varprojlim (T, \mu), \tag{140}$$

where the limit runs over the pairs (T, μ) such that μ satisfies the Serre condition and generates T.

PROOF (a) The weight of μ is defined over $\mathbb{Q}$ and T is split by a CM field. (b) For $\chi \in X^*(T)$ and $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$, define

$$f_\chi(\sigma) = \langle \sigma^{-1}\chi, \mu \rangle.$$

Then

$$\chi \mapsto f_\chi : X^*(T) \to X^*(S) \tag{141}$$

is a $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$-equivariant homomorphism, and so corresponds to a homomorphism

$$\rho : S \to T.$$

For σ_0 the given inclusion of E into $\mathbb{Q}^{\text{al}}$, $f_\chi(\sigma_0) = \langle \mu, \chi \rangle$, i.e., $\langle \mu_{\text{can}}, f_\chi \rangle = \langle \mu, \chi \rangle$, which shows that $(\rho_\mu)_{\mathbb{Q}} \circ \mu_{\text{can}} = \mu$. (c) For every (T, μ), we have a surjective homomorphism $\rho_\mu : (S, \mu_{\text{can}}) \to (T, \mu)$. The corresponding homomorphism on characters (141) is injective and $X^*(S)$ is the union of their images, and so $(S, \mu_{\text{can}}) \simeq \varprojlim (T, \mu)$. □

COROLLARY 3.24 *The polarizable rational Hodge structures of CM-type form a tannakian category $\mathsf{Hdg}^{\text{CM}}_{\mathbb{Q}}$, and S is the pro-algebraic group attached to the forgetful fibre functor.*

PROOF For any rational Hodge structures X and Y, $\text{MT}(X \oplus Y) \subset \text{MT}(X) \times \text{MT}(Y)$, and so $X \oplus Y$ is of CM type if X and Y are. The category of polarizable rational Hodge structures of CM-type is the directed union of the categories $\langle X \rangle^{\otimes}$ with X of CM-type, and is therefore tannakian. Correspondingly, the pro-algebraic group attached to the forgetful fibre functor is $\varprojlim (\text{MT}(X), \mu_X)$, which, according to 4.1 and (140), is equal to (S, μ_{can}). □

The functor sending a rational Hodge structure (V, h) to the real Hodge structure $(V \otimes \mathbb{R}, h)$ defines a homomorphism $h_{\text{can}} : \mathbb{S} \to S$; its associated cocharacter is μ_{can}.

PROPOSITION 3.25 *Every Hodge structure of CM-type is the Betti realization of an abelian motive.*

PROOF We have to show that $\mathsf{Rep}_{\mathbb{Q}}(S)$ is generated by the representations of S on $\{H_B(A)\}$. For this, it suffices to show that $X^*(S)$ is generated by the set of characters arising from abelian varieties of CM-type over $\mathbb{C}$. Let $L \subset \mathbb{Q}^{\text{cm}}$ be Galois over $\mathbb{Q}$. A CM-type Φ for L is a function $\Phi : \text{Hom}(L, \mathbb{C}) \to \{0, 1\}$ such that $\Phi + \iota\Phi = \text{id}$. An abelian variety A over $\mathbb{C}$ together with a homomorphism $L \to \text{End}(A) \otimes \mathbb{Q}$ is said to be of CM-type (L, Φ) if $H_B(A)$

is a one-dimensional vector space over L and the representation of L on the tangent space to A at 0 is equivalent to $\sum \Phi(\varphi)\varphi$. An abelian variety of CM-type (L,Φ) always exists, and for such a variety A, Φ, when regarded as a character of S, occurs in the representation of S on $H_B(A)\otimes\mathbb{Q}^{\mathrm{al}}$. Thus it suffices to show that, for any CM-field L Galois over $\mathbb{Q}$, $X^*(S^L)$ is generated by CM-types. Choose a set of representatives $R=\{\varphi_1,\dots,\varphi_g\}$ for $\mathrm{Hom}(L,\mathbb{C})/\{1,\iota\}$, and let Φ_j be the CM-type with support $\{\varphi_1,\dots,\varphi_{j-1},\iota\varphi_j,\varphi_{j+1},\dots,\varphi_g\}$. For any $\lambda\in X^*(S)$, $\lambda-\sum_{i=1}^g\lambda(\iota\varphi_i)\Phi_i$ takes the value 0 on any element of ιR, and hence is a multiple of the CM-type Φ having support R. □

COROLLARY 3.26 *The Betti fibre functor defines an equivalence of from the category of abelian motives of CM-type to the category of Hodge structures of CM-type.*

PROOF The functor is fully faithful on all abelian motives, and the proposition shows that it becomes essentially surjective when we restrict to CM motives and CM Hodge structures. □

4 Motives for Lefschetz classes

The category of Lefschetz motives

In this section we define the category of Lefschetz motives.

PRELIMINARIES

Let $\sim$ be an adequate equivalence relation on algebraic cycles, and let $\mathcal{C}_\sim(X)\overset{\text{def}}{=}\oplus_r\mathcal{C}^r_\sim(X)$ be the graded $\mathbb{Q}$-algebra of algebraic cycles on X modulo $\sim$. We define $\mathcal{D}_\sim(X)$ to be the $\mathbb{Q}$-subalgebra of $\mathcal{C}_\sim(X)$ generated by $\mathcal{C}^1_\sim(X)$, i.e., by the divisor classes. The elements of $\mathcal{D}_\sim(X)$ are called the ***Lefschetz classes*** on X for the relation $\sim$. We list some properties of Lefschetz classes.

4.1 *For any regular map $\phi: X\to Y$, ϕ^* maps Lefschetz classes on Y to Lefschetz classes on X (for any adequate equivalence relation).*

Because ϕ^* is a homomorphism of graded $\mathbb{Q}$-algebras.

4.2 *For any n and any adequate equivalence relation,*

$$\mathcal{D}_\sim(\mathbb{P}^m)=\mathbb{Q}[t]/(t^{n+1}),$$

where t denotes the class of any hyperplane in $\mathbb{P}^m$, and for any X,

$$\mathcal{D}_\sim(X\times\mathbb{P}^m)\cong\mathcal{D}_\sim(X)\otimes\mathcal{D}_\sim(\mathbb{P}^m).$$

This follows from the similar statement with $\mathcal{D}$ replaced by $\mathcal{C}$.

Now let $\mathsf{V}(k)$ be the class of algebraic varieties over k whose connected components are products of projective spaces and varieties admitting the structure of an abelian variety.

4.3 *For any variety X in $\mathsf{V}(k)$, the diagonal $\Delta_X\subset X\times X$ is a Lefschetz class (for any adequate equivalence relation).*

It suffices to prove this for the finest adequate equivalence relation, namely, rational equivalence. For an abelian variety, there is an explicit expression of Δ_X as a Lefschetz class in Scholl 1994, 5.9 (see also Milne 1999a, 5.10). To extend the statement to a product abelian varieties and projective spaces, use (4.2).

Note that (4.1) and (4.3) imply that the graph Γ_ϕ of any regular map $\phi: X \to Y$ of varieties in $\mathsf{V}(k)$ is Lefschetz, because

$$\Gamma_\phi = (\phi \times \mathrm{id}_Y)^*(\Delta_Y).$$

4.4 *For any regular map $\phi: X \to Y$ of varieties in $\mathsf{V}(k)$, ϕ_* maps $\mathcal{D}_{\mathrm{num}}(X)$ into $\mathcal{D}_{\mathrm{num}}(Y)$.*

See Milne 1999a, 5.5.

Let X and Y be varieties in $\mathsf{V}(k)$, and let $X = \coprod X_i$ be the decomposition of X into its equidimensional components. Then

$$\mathcal{D}_\sim(X \times Y) = \bigoplus\nolimits_i \mathcal{D}_\sim(X_i \times Y),$$

and we set

$$\mathrm{LCorr}^m(X,Y) = \bigoplus\nolimits_i \mathcal{D}_{\mathrm{num}}^{\dim X_i + m}(X_i \times Y).$$

The map

$$\alpha, \beta \mapsto \beta \circ \alpha \stackrel{\mathrm{def}}{=} p_{XZ*}(p_{XY}^*\alpha \cdot p_{YZ}^*\beta)$$

is a pairing

$$\mathrm{LCorr}^m(X,Y) \times \mathrm{LCorr}^m(Y,Z) \to \mathrm{LCorr}^{m+n}(X,Z).$$

Define $\mathsf{LCV}^0(k)$ to be the category whose objects are symbols hX, one for each $X \in \mathcal{V}(k)$, and whose morphisms are

$$\mathrm{Hom}(hX, hY) = \mathrm{LCorr}^0(X,Y).$$

The transpose of the graph of a regular map $\phi: X \to Y$ defines an element $h\phi \stackrel{\mathrm{def}}{=} [{}^t\Gamma_\phi] \in \mathrm{LCorr}^0(Y,X)$, and h is a contravariant functor $\mathsf{V}(k) \to \mathsf{LCV}^0(k)$.

4.5 *For an abelian variety A of dimension g, there are unique elements $p_i \in \mathrm{LCorr}^0(A,A)$ such that*

(a) $[\Delta_A] = p_0 + \cdots + p_g$

(b) $p_i \circ p_j = 0$ *if* $i \neq j$, *and* $p_i \circ p_i = p_i$;

(c) *for any integer n, $h(n_A) \circ p_i = n^i[\Delta_A] \circ p_i$, where n_A is the endomorphism of A "multiplication by n".*

This is proved in Scholl 1994, 5.2.

Now let $X \rightsquigarrow H^*(X)$ be a Weil cohomology theory, and write $H^{2*}(X)(*) = \bigoplus^r H^{2r}(X)(r)$. By assumption, there is given a homomorphism of graded $\mathbb{Q}$-algebras $\mathrm{cl}: \mathrm{CH}^*(X) \to \bigoplus H^{2*}(X)(*)$.

4.6 *For a Lefschetz class x on a variety $X \in \mathsf{V}(k)$, the following are equivalent:*

(a) $\mathrm{cl}(x) \cdot y = 0$ *for all cohomology classes y;*

(b) $x \cdot y = 0$ *for all algebraic classes y;*

(c) $x \cdot y = 0$ *for all Lefschetz classes y.*

Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c), but (c) $\Rightarrow$ (a) is proved in Milne 1999a, 5.2.

In particular, $\mathcal{D}_{\mathrm{hom}}$ is independent of the cohomology theory and equals $\mathcal{D}_{\mathrm{num}}$. From now on, I drop the subscript. Thus "$\mathcal{D}(X)$" means "$\mathcal{D}_{\mathrm{num}}(X)$", and "Lefschetz class on X" means "element of $\mathcal{D}(X)$".

The category of Lefschetz motives

The category $\mathsf{LMot}(k)$ of Lefschetz motives is defined as follows. An object is a symbol $h(X,e,m)$ where X is a variety in $\mathsf{V}(k)$, e is an idempotent in $\mathrm{LCorr}^0(X,X)$, and $m \in \mathbb{Z}$. If $h(X,e,m)$ and $h(Y,f,n)$ are two motives, then

$$\mathrm{Hom}(h(X,e,m),h(Y,f,n)) = \{f\circ\alpha\circ e \mid \alpha \in \mathrm{LCorr}^{n-m}(X,Y)\}.$$

The composite of two morphisms of motives is their composite as correspondences.

Exactly as in the usual case (Scholl 1994, §1), one shows that $\mathsf{LMot}(k)$ is a $\mathbb{Q}$-linear pseudo-abelian rigid tensor category, with

$$\begin{aligned} h(X,e,m)\oplus h(Y,f,m) &= h(X\coprod Y, e\oplus f, m);\\ h(X,e,m)\otimes h(Y,f,n) &= h(X\times Y, e\otimes f, m+n);\\ h(X,e,m)^\vee &= h(X,{}^te,d-m) \text{ if } X \text{ has pure dimension } d. \end{aligned}$$

Moreover,

$$h(\mathbb{P}^m,\mathrm{id},0) = 1\oplus\mathbb{L}\oplus\cdots\oplus\mathbb{L}^{\otimes n}$$

where $\mathbb{L} \overset{\text{def}}{=} (\mathrm{Spec}\,k,\mathrm{id},-1)$. The proofs of these facts use (4.1–4.5).

Note that $hX \leftrightarrow h(X,\mathrm{id},0)$ identifies $\mathsf{LCV}^0(k)$ with a full subcategory of $\mathsf{LMot}(k)$. Moreover, every motive is a direct sum of motives of the form $h(A,e,m)$ with A an abelian variety.

From (4.5), we find that $\mathsf{LMot}(k)$ has a canonical $\mathbb{Z}$-grading for which $h(A,p_i,m)$ has weight $i-2m$. This can be used to modify the commutativity constraint as in 2.27 to obtain the "true" category of Lefschetz motives. The method of Jannsen 1992 shows that (4.6) implies that $\mathsf{LMot}(k)$ is a semisimple abelian category. Finally, Deligne 1990, 7.1, implies that $\mathsf{LMot}(k)$ is Tannakian. In summary:

THEOREM 4.7 *The category* $\mathsf{LMot}(k)$ *is a semisimple* $\mathbb{Q}$*-linear Tannakian category endowed with a canonical* $\mathbb{Z}$*-gradation* w *and a canonical (Tate) object* $\mathbb{T} = (\mathrm{Spec}\,k,\mathrm{id},1)$.

The fundamental group of $\mathsf{LMot}(k)$

We now assume k to be algebraically closed, and we fix a Weil cohomology theory $X \rightsquigarrow H^*(X)$ with coefficient field Q. There is a unique fibre functor $\omega_H : \mathsf{LMot}(k) \to \mathsf{Vec}_Q$ such that $\omega_H(h(A)) = H^*(A)$ for all abelian varieties A.

Let $H_1(A)$ be the linear dual of $H^1(A)$, and let $C(A)$ be the centralizer of $\mathrm{End}^0(A)$ in $\mathrm{End}(H_1(A))$. A polarization $\lambda : A \to A^\vee$ of A determines an involution

$$\alpha \mapsto \alpha^\dagger \overset{\text{def}}{=} H_1(\lambda)^{-1}\circ H_1(\alpha^\vee)\circ H_1(\lambda)$$

of $\mathrm{End}(H_1(A))$ whose restriction to $C(A)$ is independent of the choice of λ. The ***Lefschetz group*** $L(A)$ of A is the algebraic subgroup of $\mathrm{GL}(H_1(A))\times\mathbb{G}_m$ such that

$$L(A)(R) = \{(\gamma,c)\in (C(A)\otimes R)^\times\times R^\times \mid \gamma^\dagger\gamma = c\}$$

for all Q-algebras R (Milne 1999a, 4.3, 4.4). It is reductive (not necessarily connected), and $(\gamma,c)\mapsto c$ is a homomorphism $l(A): L(A)\to\mathbb{G}_m$ rational over Q.

Let $h_1(A) = h^1(A)^\vee$, and let $\langle A\rangle^\otimes$ be the Tannakian subcategory of $\mathsf{LMot}(k)$ generated by $h_1(A)$ and $\mathbb{T}$. Because $h(A^r)\simeq\bigwedge h^1(A^r)$ and $h_1(A^r)\simeq h_1(A)\oplus\cdots\oplus h_1(A)$, $\langle A\rangle^\otimes$ contains $h(A^r)$ for all r. Let $\boldsymbol{\pi}(A)$ be the fundamental group of the Tannakian category $\langle A\rangle^\otimes$.

PROPOSITION 4.8 *There is a canonical isomorphism* $\omega_H(\pi(A)) \simeq L(A)$.

PROOF Almost by definition,

$$\omega_H(\boldsymbol{\pi}(A)) \simeq \mathcal{A}ut^{\otimes}(\omega_H|\langle A\rangle^{\otimes}).$$

Therefore the action of $\boldsymbol{\pi}(A)$ on $h_1(A)$ and $\mathbb{T}$ identifies $\omega_H(\boldsymbol{\pi}(A))$ with the subgroup of $\mathrm{GL}(H_1(A))\times\mathbb{G}_m$ fixing $\omega_H(\phi)$ for all morphisms ϕ of objects in $\langle A\rangle^{\otimes}$. On the other hand, $L(A)$ is the largest subgroup $\mathrm{GL}(H_1(A))\times\mathbb{G}_m$ fixing all Lefschetz classes on A^r for all r (Milne 1999a, 4.3). These two groups are equal. □

Let $(L,l) = \prod_B(L(B),l(B))$, where B runs over a set of representatives for the simple isogeny classes of abelian varieties over k.

COROLLARY 4.9 *Let* $\boldsymbol{\pi}$ *be the fundamental group of* $\mathsf{LMot}(k)$. *Then* $\omega_H(\boldsymbol{\pi})$ *is canonically isomorphic to* L.

PROOF For any abelian variety A, $(L(A),l(A)) \simeq \prod_B(L(B),l(B))$ where B runs over a set of representatives for the simple isogeny factors of A (Milne 1999a, 4.7). Therefore the corollary follows from the proposition by passing to the limit over A. □

REMARK 4.10 Let A be an abelian variety over k. For each Weil cohomology theory H we have a Lefschetz group $L(A)_H$, which is an algebraic group over the field of coefficients of H. Proposition 1.8 shows each $L(A)_H$ is a realization of $\boldsymbol{\pi}(A)$, which should therefore be considered as the archetype for all the Lefschetz groups of A. Unfortunately, $\boldsymbol{\pi}(A)$ is only an algebraic group in a Tannakian category, and hence is a somewhat mysterious object. There are two situations in which $\boldsymbol{\pi}(A)$ can be identified with an algebraic group over $\mathbb{Q}$ in the usual sense. The first is when $k = \mathbb{C}$. Here there is a canonical Weil cohomology theory with coefficients in $\mathbb{Q}$, namely, the Betti cohomology, and so we can identify $\boldsymbol{\pi}(A)$ with the Betti Lefschetz group of A. The second is when A has "many endomorphisms", which we now explain.

For any k-linear Tannakian category T, the category Vec_k of finite-dimensional vector spaces over k can be identified with the full subcategory of T of objects on which $\pi(\mathsf{T})$ acts trivially. If $\pi(\mathsf{T}) = \mathrm{Sp}(R)$ is commutative, then the action of $\pi(\mathsf{T})$ on R is trivial, and so $\pi(\mathsf{T})$ is an affine group scheme in the Tannakian category $\mathsf{Vec}_k \subset \mathsf{T}$, i.e., it is an affine group scheme over k in the usual sense (cf. Milne 1994, 2.37, p428).

A semisimple algebra R of finite degree over $\mathbb{Q}$ is a product of simple algebras, say, $R = R_1\times\cdots\times R_m$, and the centre K_i of each R_i is a field. The reduced degree $[R:\mathbb{Q}]_{\mathrm{red}}$ of R over $\mathbb{Q}$ is $\sum[R_i:K_i]^{1/2}\cdot[K_i:\mathbb{Q}]$. For an abelian variety A, $[\mathrm{End}^0(A):\mathbb{Q}]_{\mathrm{red}} \le 2\dim A$, and when equality holds we say that A has is of CM-type.

Let A be a simple abelian variety of CM-type, and let $C_0(A)$ be the centre of $\mathrm{End}^0(A)$. A Rosati involution on $\mathrm{End}^0(A)$ defines an involution on $C_0(A)$, which is independent of the choice of the Rosati involution. For any Weil cohomology theory H with coefficient field Q, the canonical map

$$C_0(A)\otimes_{\mathbb{Q}} Q \to C(A)$$

is an isomorphism—this follows easily from the definition of A being of CM-type and the fact that $H_1(A)$ is a free $C_0(A)\otimes_{\mathbb{Q}} Q$-module (Milne 1999a, 2.1). Therefore,

$$L(A) \simeq L_0(A)_{/Q}$$

where $L_0(A)$ is the algebraic group over $\mathbb{Q}$ such that

$$L_0(A)(R) = \{(\gamma,c)\in C_0(A)^{\times}\times R^{\times} \mid \gamma^{\dagger}\gamma = c\}$$

for all $\mathbb{Q}$-algebras R. This shows that $\pi(A)$ is commutative (because its realizations are), and hence can be regarded as an algebraic group in the usual sense; moreover, the action of $L_0(A)$ on $h_1(A)$ identifies $L_0(A)$ with $\pi(A)$.

Below we consider two categories of Lefschetz motives generated by abelian varieties with many endomorphisms. The remark shows that their fundamental groups are affine group schemes of multiplicative type in the usual sense. In each case, there is a classification of the isogeny classes and a description of the endomorphism algebra of each isogeny class, which allow us to compute the fundamental groups explicitly.

NOTES The original source of the above theory is Milne 1999a, Milne 1999b. For a more recent exposition, see Kahn 2026.

Lefschetz Motives of CM-Type

The theory of abelian varieties of CM-type provides a classification of the simple isogeny classes of such varieties, which allows us to compute the fundamental group of the category of Lefschetz motives generated by abelian varieties of CM-type.

Throughout this section, C is an algebraically closed field of characteristic zero and ι is an involution of C restricting to complex conjugation on every CM-subfield. We let $\mathbb{Q}^{\mathrm{al}}$ denote the algebraic closure of $\mathbb{Q}$ in C. Recall that $\mathbb{Q}^{\mathrm{cm}} \subset \mathbb{Q}^{\mathrm{al}}$ and that $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

The fundamental group of the category of Lefschetz motives of CM-type

Fix a CM-field $K \subset \mathbb{Q}^{\mathrm{al}}$ and define $\mathsf{LCM}^K(C)$ to be the Tannakian subcategory of $\mathsf{LMot}(C)$ generated by the motives of simple abelian varieties over C of CM-type with reflex field contained in K. When $K = \mathbb{Q}^{\mathrm{cm}}$, we omit the superscript. We fix a Weil cohomology theory $X \rightsquigarrow H^*(X)$ with coefficient field $\mathbb{Q}$, and write ω_H for the corresponding fibre functor on $\mathsf{LMot}(C)$ or its Tannakian subcategories.

For a Γ-orbit Ψ of CM-types on $\mathbb{Q}^{\mathrm{cm}}$, define T^Ψ to be the torus over $\mathbb{Q}$ with character group

$$X^*(T^\Psi) = \frac{\{f : \Psi \to \mathbb{Z}\}}{\{f \mid f = \iota f \text{ and } \sum_\psi f(\psi) = 0\}}.$$

The element $\psi + \iota\psi$ of $X^*(T^\Psi)$ is independent of the choice of $\psi \in \Psi$ and is fixed by Γ. It therefore defines a homomorphism $t^\Psi : T^\Psi \to \mathbb{G}_m$ rational over $\mathbb{Q}$. Let A^Ψ be a simple abelian variety corresponding (as in Proposition 3.19) to the orbit Ψ. Although A^Ψ is defined only up to isogeny, its Lefschetz group $L(A^\Psi)$ with respect to $X \mapsto H^*(X)$ is well-defined up to a unique isomorphism.

PROPOSITION 4.11 *For any* Ψ, $(L(A^\Psi), l(A^\Psi)) = (T^\Psi, t^\Psi)$.

PROOF Choose a $\psi \in \Psi$. Let (E_ψ, φ_ψ) be as in the proof of Proposition 3.18, and let A^Ψ be the abelian variety A_{φ_ψ} defined in the proof of Proposition 3.17. Then $L(A^\Psi)$ is the subtorus of $(\mathbb{G}_m)_{E_\psi/\mathbb{Q}}$ such that

$$L(A^\Psi)(\mathbb{Q}) = \{\alpha \in E_\psi^\times \mid \alpha \cdot \iota\alpha \in \mathbb{Q}^\times\}$$

and its canonical character $l(A^\Psi)$ sends α to $\alpha \cdot \iota\alpha$. Therefore, $X^*(L(A^\Psi))$ is the quotient of $\mathbb{Z}^{\mathrm{Hom}(E_\psi, \mathbb{Q}^{\mathrm{al}})}$ by the subgroup of f such that $f(\tau) = f(\iota\tau)$ for all $\tau : E_\psi \to \mathbb{Q}^{\mathrm{al}}$ and

$\sum f(\tau) = 0$, and $l(A^\Psi)$ is represented by $\iota + 1$. By definition, $\mathrm{Hom}(E_\psi, \mathbb{Q}^{\mathrm{al}}) = \Gamma/\Gamma_\psi$ where Γ_ψ is the group fixing ψ, and $\sigma \mapsto \sigma\psi$ is a bijection from Γ/Γ_ψ onto Ψ. This map identifies $X^*(L(A^\Psi))$ with $X^*(T^\Psi)$ and $l(A^\Psi)$ with t^Ψ. □

The fundamental group of $\mathsf{LCM}^K(C)$ is commutative, and so can be identified with an affine group scheme over $\mathbb{Q}$ in the usual sense.

THEOREM 4.12 *For any CM-subfield K of $\mathbb{Q}^{\mathrm{al}}$, the fundamental group (T^K, t^K) of $\mathsf{LCM}^K(C)$ is $\prod_\Psi(T^\Psi, t^\Psi)$, where the product is over the set of Γ-orbits of CM-types on K.*

PROOF For any abelian variety A over C of CM-type, the fundamental group of $\langle A\rangle^\otimes$ is equal to the Lefschetz group of A, which is $\prod_B(L(B), l(B))$ where B runs over a set of representatives for the simple isogeny factors of A. When $[K:\mathbb{Q}] < \infty$, $\mathsf{LCM}^K(C) = \langle\prod_\Psi A^\Psi\rangle^\otimes$ where Ψ runs through the Γ-orbits of CM-types on K, and so

$$(T^K, t^K) = (L(\textstyle\prod A^\Psi), l(\prod A^\Psi)) = \prod(L(A^\Psi), l(A^\Psi)) = \prod(T^\Psi, t^\Psi).$$

The case when $[K:\mathbb{Q}]$ is infinite follows by passing to the limit over the CM-subfields of K finite over $\mathbb{Q}$. □

The functor from Lefschetz motives of CM-type to Hodge motives of CM-type

Let $\mathbb{Q}^{\mathrm{al}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. Fix a CM-field $K \subset \mathbb{C}$. Since a Lefschetz class is a Hodge class, there is a tensor functor $\mathsf{LCM}^K(\mathbb{C}) \to \mathsf{CM}^K(\mathbb{C})$ sending $h(A, e, m)$ to $h(A, e, m)$ (e now regarded as a Hodge class). We describe the homomorphism $S^K \to T^K$ of fundamental groups that it defines. For any Γ-orbit Ψ of CM-types on K, the map

$$f \mapsto \sum_{\psi\in\Psi} f(\psi)\psi : \mathbb{Z}^\Psi \to X^*(S^K)$$

factors through $X^*(T^\Psi)$, and hence defines a homomorphism $\gamma^\Psi : S^K \to T^\Psi$. Its composite with t^Ψ is s^K, and so the t^Ψ define a homomorphism $\gamma^K : (S^K, s^K) \to \prod_\Psi(T^\Psi, t^\Psi)$.

PROPOSITION 4.13 *The homomorphism $(S^K, s^K) \to (T^K, t^K)$ defined by the tensor functor $\mathsf{LCM}(\mathbb{C}) \to \mathsf{CM}(\mathbb{C})$ is γ^K.*

PROOF As we noted above, if A is a simple abelian variety corresponding to the Γ-orbit Ψ of CM-types on K, then the representation of S^K on $H_1(A)$ has the elements of Ψ as its characters. The shows that the homomorphism $(S^K, s^K) \to (T^\Psi, t^\Psi)$ defined by the tensor functor $\langle A\rangle^\otimes \to \mathsf{CM}^K(\mathbb{C})$ is γ^Ψ. Therefore, the two homomorphisms $(S^K, s^K) \to (T^K, t^K)$ agree when composed with the projections $(T^K, t^K) \to (T^\Psi, t^\Psi)$, which implies that they are equal. □

REMARK 4.14 The homomorphism $\gamma^K : S^K \to T^K$ is injective. Indeed, its kernel is killed by every CM-type on K, but these generate $X^*(S^K)$.

REMARK 4.15 Let $k \hookrightarrow k'$ be an inclusion of algebraically closed fields of characteristic zero. If (A, i) is an abelian variety of CM-type (E, φ) over k', then any specialization of (A, i) to k is again of CM type (E, φ), and hence becomes isogenous to (A, i) over k'. The functor extension of scalars sending an abelian variety over k to an abelian variety over k' is fully faithful. The last remark shows that it is also essentially surjective on CM abelian varieties. This extends the results of this section from $\mathbb{C}$ to any algebraically closed field of characteristic zero.

Lefschetz motives over $\mathbb{F}$

Weil numbers and abelian varieties

Let p be a prime number. An element π of a field algebraic over $\mathbb{Q}$ is said to be a ***Weil*** p^m-number of weight $-m$ if

(a) for all embeddings $\rho: \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $\rho(\pi)\cdot\iota\rho(\pi) = (p^m)^m$;

(b) for some N, $p^N\pi$ is an algebraic integer.

Condition (a) implies that $\pi \mapsto p^{nm}/\pi$ defines an involution (possibly trivial) ι' of $\mathbb{Q}[\pi]$ such that $\rho\circ\iota' = \iota\circ\rho$ for all embeddings $\rho: \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$. Therefore $\mathbb{Q}[\pi]$ is either a CM-field or is totally real.

Let $W(p^m)$ be the group of Weil p^m-numbers in $\mathbb{Q}^{\mathrm{al}}$. If $n|n'$, then $\pi \mapsto \pi^{\frac{n'}{n}}$ maps $W(p^m)$ into $W(p^{n'})$, and we define $W(p^\infty) = \varinjlim_{m70} W(p^m)$. Thus an element of $W(p^\infty)$ is represented by an element of $W(p^m)$ for some n, and elements $\pi \in W(p^m)$ and $\pi'^{n'})$ represent the same element of $W(p^\infty)$ if and only $\pi^{n'}$ and π'^n differ by a root of unity. We let $[\pi]$ denote the element of $W(p^\infty)$ represented by π.

There is a natural action of Γ on $W(p^\infty)$, and the ***Weil-number torus*** P is defined to be the pro-torus over $\mathbb{Q}$ with $X^*(P) = W(p^\infty)$.

Let $W_{1,+}(p^m)$ be the subset of $W(p^m$ consisting of those π that are of weight -1 and are algebraic integers, and let $W_{1,+}(p^\infty) = \varinjlim W_{1,+}(p^m)$.

Let A be a simple abelian variety over $\mathbb{F}$. Choose a model A_0 of A over a finite field $\mathbb{F}_{p^m}$ such that all endomorphisms of A are rational over $\mathbb{F}_{p^m}$, and let π be the Frobenius endomorphism of $A_0/\mathbb{F}_{p^m}$. According to Tate 1966, Theorem 2, $\mathbb{Q}[\pi]$ is the centre $Z(A)$ of $\mathrm{End}^0(A)$. For any embedding $\rho: Z(A) \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, $\rho(\pi) \in W^K_{1,+}(p^m)$. The class $[\rho(\pi)]$ of $\rho(\pi)$ in $W^K_{1,+}(p^\infty)$ is independent of the choice of the model A_0, and as ρ runs over the embeddings of $Z(A)$ into $\mathbb{Q}^{\mathrm{al}}$, $[\rho(\pi)]$ runs over a Γ-orbit in $W^K_{1,+}(p^\infty)$.

Proposition 4.16 *The map* $A \mapsto \{[\rho(\pi)] \mid \rho \in \mathrm{Hom}(Z(A), \mathbb{Q}^{\mathrm{al}})\}$ *defines a bijection from the set of isogeny classes of simple abelian varieties over* $\mathbb{F}$ *to* $\Gamma\backslash W_{1,+}(p^\infty)$.

Proof The injectivity follows from Tate 1966, Theorem 1, and the surjectivity from Honda 1968. □

The fundamental group of LMot($\mathbb{F}$)

For a Γ-orbit Π in $W^K_{1,+}(p^\infty)$, define L^Π to be the torus over $\mathbb{Q}$ with character group

$$X^*(L^\Pi) = \frac{\{f: \Pi \to \mathbb{Z}\}}{\{f \mid f = \iota f \text{ and } \sum_{\pi\in\Pi} f(\pi) = 0\}}.$$

The element $\pi + \iota\pi$ of $X^*(L^\Pi)$ is independent of the choice of $\pi \in \Pi$ and is fixed by Γ. It therefore defines a homomorphism $l^\Pi: L^\Pi \to \mathbb{G}_m$ rational over $\mathbb{Q}$. Let A^Π be a simple abelian variety over $\mathbb{F}$ corresponding (as in Proposition 4.16) to the orbit Π. Although A^Π is defined only up to isogeny, its Lefschetz group $L(A^\Pi)$ is well-defined up to a unique isomorphism.

Lemma 4.17 *For any* Γ*-orbit* Π *of Weil numbers of weight* -1, $(L(A^\Pi), l(A^\Pi)) = (L^\Pi, l^\Pi)$.

PROOF The Lefschetz group $L(A^\Pi)$ of A^Π is the subtorus of $(\mathbb{G}_m)_{Z(A^\Pi)/\mathbb{Q}}$ such that

$$L(A^\Pi)(\mathbb{Q}) = \{\alpha \in Z(A^\Pi) \mid \alpha \cdot \iota\alpha \in \mathbb{Q}^\times\},$$

and $l(A^\Pi)$ sends α to $\alpha \cdot \iota\alpha$ (cf. 1.10). Choose a model for A^Π over a finite field whose Frobenius endomorphism π_0 generates $Z(A^\Pi)$ as a $\mathbb{Q}$-algebra. Then the bijection $\rho \mapsto [\rho(\pi_0)] : \operatorname{Hom}(Z(A^\Pi), \mathbb{Q}^{\mathrm{al}}) \to \Pi$ induces an isomorphism $X^*(L^\Pi) \to X^*(L(A^\Pi))$, which is independent of the choice of the model and maps l^Π to $l(A^\Pi)$. □

Because the Lefschetz group of an abelian variety over $\mathbb{F}$ is commutative, so also is the fundamental group of $\mathsf{LMot}(\mathbb{F})$, which therefore may be identified with an affine group scheme over $\mathbb{Q}$ in the usual sense.

THEOREM 4.18 *The fundamental group of* $\mathsf{LMot}(\mathbb{F})$ *is* $\displaystyle\prod_{\Pi \in \Gamma\backslash W_{1,+}(p^\infty)} (L^\Pi, l^\Pi)$.

PROOF Combine Lemma 4.17 with Proposition 4.16 and 4.8. □

THE FUNDAMENTAL GROUP OF $\mathsf{LMot}^K(\mathbb{F})$

For a Weil p^m-number π in a field K finite over $\mathbb{Q}$ and a prime w of K lying over p, define

$$f_\pi(w) = \frac{\operatorname{ord}_w(\pi)}{\operatorname{ord}_w(p^m)}[K_w : \mathbb{Q}_p].$$

Now let K be a CM-subfield of $\mathbb{C}$, finite and Galois over $\mathbb{Q}$. Define $W^K(p^m)$ to be the set of Weil p^m-numbers π in K such that $f_\pi(w) \in \mathbb{Z}$ for all $w|p$, and set $W^K(p^\infty) = \varinjlim W^K(p^m)$. It is a Γ-submodule of $W(p^\infty)$, and we define P^K to be the corresponding quotient of P. Let Y be the set of primes of K lying over p. The number $f_\pi(w)$ depends only on the class $[\pi]$ of π in $W^K(p^\infty)$, and so $[\pi] \mapsto f_\pi$ is a homomorphism from $W^K(p^\infty)$ to the set of functions $f : Y \to \mathbb{Z}$. This homomorphism is obviously injective, and the functions in the image have the property that $f(w) + f(\iota w)$ is an integer independent of $w \in Y$ and divisible by $[K_w : \mathbb{Q}_p]$. Later we shall see that every f with this property is in the image.

Define $W^K_{1,+}(p^m)$ and $W^K_{1,+}(p^\infty)$ similarly. Then $[\pi] \mapsto f_\pi$ defines an isomorphism

$$W^K_{1,+}(p^\infty) \xrightarrow{\sim} \{f : Y \to \mathbb{Z} \mid f(w) + f(\iota w) = [K_w : \mathbb{Q}_p], \quad f(w) \geq 0\}.$$

Let A be a simple abelian variety over $\mathbb{F}$. According to Tate 1968/69, Théorème 1, the invariant of $\operatorname{End}^0(A)$ at a prime v of its centre $Z(A)$ is given by

$$\operatorname{inv}_v(\operatorname{End}^0(A)) = \frac{\operatorname{ord}_v(\pi)}{\operatorname{ord}_v(p^m)}[Z(A)_v : \mathbb{Q}_p] \quad (= f_\pi(v))$$

where $\pi \in Z(A)$ is the Frobenius endomorphism of a model $A_0/\mathbb{F}_{p^m}$ of A with the property that $\operatorname{End}^0(A_0) = \operatorname{End}(A)$. Therefore, for any embedding $\rho : Z(A) \hookrightarrow K$ and $w \in Y$,

$$\operatorname{inv}_w(\operatorname{End}^0(A) \otimes_{Z(A),\rho} K) = f_{\rho(\pi)}(w).$$

Consequently, $f_{\rho(\pi)}(w)$ is an integer for all $w|p$ if and only if K splits $\operatorname{End}^0(A)$. Therefore, under the bijection in Proposition 4.16, $\Gamma\backslash W^K_{1,+}(p^\infty)$ corresponds to the set of isogeny classes of A's having the following property:

(*) for all $\rho: Z(A) \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, $\rho(Z(A)) \subset K$ and $\mathrm{End}^0(A) \otimes_{Z(A),\rho} K$ is a matrix algebra over K.

For a fixed CM-field $K \subset \mathbb{Q}^{\mathrm{al}}$ of finite degree and Galois over $\mathbb{Q}$, let $\mathsf{LMot}^K(\mathbb{F})$ be the full subcategory of $\mathsf{LMot}(\mathbb{F})$ whose objects are direct sums of motives of the form $h(A,p,m)$ with A satisfying the condition (*). It is a Tannakian subcategory of $\mathsf{LMot}(\mathbb{F})$, with fundamental group

$$\prod_{\Pi \in \Gamma\backslash W^K_{1,+}(p^\infty)} (L^\Pi, t^\Pi).$$

THE MAP $\beta^K: P^K \to L^K$

The element $p \in K$ is a Weil p-number of weight -2. Its class $[p]$ in $W^K(p^\infty)$ is fixed under the action of Γ, and so defines homomorphism $p^K: P^K \to \mathbb{G}_m$ rational over $\mathbb{Q}$.

Let Π be a Γ-orbit in $W^K_{1,+}(p^\infty)$. The map

$$f \mapsto \prod_{\pi \in \Pi} \pi^{f(\pi)}: \mathbb{Z}^\Pi \to W^K(p^\infty)$$

factors through $X^*(L^\Pi)$, and hence defines a homomorphism

$$\beta^\Pi: P^K \to L^\Pi.$$

This map sends p^K to l^Π, and hence the family $(\beta^\Pi)_{\Pi \in W_{1,+}(p^\infty)}$ defines a homomorphism

$$\beta^K: (P^K, p^K) \to (L^K, l^K),$$

which is injective because it corresponds to a surjective map on the character groups. On passing to the inverse limit over all $K \subset \mathbb{Q}^{\mathrm{cm}}$ finite and Galois over $\mathbb{Q}$, we obtain an injective homomorphism

$$\beta: (P,p) \to (L,l).$$

The Reduction Functor on Lefschetz Motives of CM-Type

Because an abelian variety of CM-type has potential good reduction, for each prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ there is a "reduction" functor from the category of Lefschetz motives of CM-type over $\mathbb{Q}^{\mathrm{al}}$ to the category of Lefschetz motives over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Using the theorem of Shimura and Taniyama, we shall compute the map of the fundamental groups it defines.

Throughout this section $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We fix a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ lying over p, and denote its residue field by $\mathbb{F}$.

THE MAP $P \to S$

We review the construction of the map $P \to S$ that is conjecturally associated with the reduction of motives of CM-type.

Fix a CM-subfield K of $\mathbb{Q}^{\mathrm{al}}$ of finite degree and Galois over $\mathbb{Q}$ and a prime w_0 of K lying over p. Recall that $X^*(S^K)$ consists of the homomorphisms $g: \mathrm{Hom}(K, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$ such that $g + \iota g$ is constant, and that the weight of g is $-g - \iota g$.

For $g \in X^*(S^K)$ and $a \in K$, define

$$g(a) = \prod_{\tau:\, K \to \mathbb{Q}^{\mathrm{al}}} (\tau g)^{g(\tau)} \in \mathbb{Q}^{\mathrm{al}}.$$

Then $g(a)\cdot \iota g(a) = \mathrm{Nm}_{K/\mathbb{Q}}\, a^{-\mathrm{wt}(g)}$. If a lies in the real subfield F of K, then $g(a) = \mathrm{Nm}_{F/\mathbb{Q}}(a)^{-\mathrm{wt}(g)}$. Because the group of units of F has finite index in the group of units of K, this shows that g maps units in K to roots of unity.

Let ϖ generate the ideal $\mathfrak{P}_{w_0}^h$, where h is the order of the prime ideal $\mathfrak{P}_{w_0}$ corresponding to w_0 in the class group of K. According to the above remarks, $g(\varpi)$ is independent of the choice of ϖ up to a root of unity, and it is a Weil $p^{f(\frac{w_0}{p})h}$-number of weight $\mathrm{wt}(g)$. Moreover, for any prime w of K lying over p,

$$\mathrm{ord}_w(g(\varpi)) = h\sum_{\tau,\tau w_0=w} g(\tau).$$

Therefore, with the notation of Section 4,

$$f_{g(\varpi)}(w) = \sum_{\tau w_0=w} g(\tau) \in \mathbb{Z}, \tag{142}$$

and so $g(\varpi)\in W^K(p^{f(\frac{w_0}{p})})$. The class it represents in $W^K(p^\infty)$ is independent of the choice of ϖ, and so we have a homomorphism

$$g\mapsto [g(\varpi)]: X^*(S^K)\to W^K(p^\infty).$$

We sometimes denote this map as $g\mapsto \pi(g)$. It commutes with the action of Γ, and so defines a homomorphism

$$\alpha^K: P^K\to S^K.$$

LEMMA 4.19 *The maps*

$$X^*(S^K)\xrightarrow{g\mapsto[g(\varpi)]} W^K(p^\infty)\xrightarrow{[\pi]\mapsto f_\pi}\{f: Y\to\mathbb{Z}\mid f+\iota f\in[K_{w_0}:\mathbb{Q}_p]\mathbb{Z}\}$$

are surjective.

PROOF We know that the second map is injective, and so it suffices to prove that the composite map is surjective. But it sends $g\in X^*(S^K)$ to the map $f: Y\to\mathbb{Z}$ such that $f(w)=\sum_{\tau w=w_0} g(\tau)$. Choose a section s to the map $\tau\mapsto\tau w_0: \mathrm{Gal}(K/\mathbb{Q})\to Y$ such that $s(\iota w)=\iota s(w)$, and define g so that $g(\tau)$ is $f(\tau w_0)$ or 0 according as τ is in the image of s or not. Then $g\mapsto f$. □

REMARK 4.20 (a) The lemma shows that the homomorphism $\alpha^K: P^K\to S^K$ is injective. On passing to the limit over all K, we obtain an injective homomorphism $\alpha: P\to S$.

(b) The lemma shows that

$$[\pi]\mapsto f_\pi: X^*(P^K)\to\{f: Y\to\mathbb{Z}\mid f+\iota f\in[K_{w_0}:\mathbb{Q}_p]\mathbb{Z}\}$$

is an isomorphism.

(c) The homomorphism $\alpha^K: P^K\to S^K$ sends p^K to s^K.

The reduction of abelian varieties

Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type, and let A' be a model of A over a subfield L of $\mathbb{Q}^{\mathrm{al}}$ finite over $\mathbb{Q}$. After possibly replacing L by a larger field, A' will have good reduction at w_0 (Serre and Tate 1968, Theorem 6). Let A'_0 be the reduction of A'. Then $A_0 \overset{\text{def}}{=} A'_0 \times_{\operatorname{Spec} k(w_0)} \mathbb{F}$ is independent of all choices (up to a well-defined isomorphism) and $A \mapsto A_0$ is a functor $\mathsf{Isab}^{\mathrm{cm}}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Isab}(\mathbb{F})$.

Now assume A to be simple, and let $E = \mathrm{End}^0(A)$. It is a CM-field, and the action of E on $\mathrm{Tgt}_0(A)$ defines a CM-type $\varphi : \operatorname{Hom}(E, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$.

The centre $Z(A_0)$ of $\mathrm{End}^0(A_0)$ is a subfield of E. Let $\pi \in E$ be the Frobenius endomorphism of some model of A_0 over a finite subfield, say $\mathbb{F}_{p^m}$, of $\mathbb{F}$. Any two such π's represent the same class in $W(p^\infty)$. For any $\rho : E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, let $\rho^{-1}w_0$ be the valuation on E such that $|c|_{\rho^{-1}w_0} = |\rho c|_{w_0}$. According to the Theorem of Shimura and Taniyama (Tate 1968/69, Lemme 5), for any prime $w|p$ of E,

$$f_\pi(w) \overset{\text{def}}{=} \frac{\operatorname{ord}_w(\pi)}{\operatorname{ord}_w(p^m)}[E_w : \mathbb{Q}_p] = \sum_{\rho, \rho^{-1}w_0 = w} \varphi(\rho).$$

Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$, and large enough to contain all conjugates of E (and hence also the reflex field of (E, φ)). The choice of an embedding $\rho_0 : E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$ determines a Weil q-integer $\rho(\pi)$ of weight -1 in K and a CM-type ψ_{ρ_0} on K (see Section 2). From the inclusion $K \subset \mathbb{Q}^{\mathrm{al}}$, K acquires a valuation $w_0|p$, and we choose an $\varpi \in K$ such that $(\varpi) = \mathfrak{P}_{w_0}^h$. Then $\psi_{\rho_0}(\varpi) \overset{\text{def}}{=} \prod_{\tau : K \to \mathbb{Q}^{\mathrm{al}}} (\tau\varpi)^{\psi_{\rho_0}(\tau)}$ is a Weil $p^{f(\frac{w_0}{p})h}$-integer of weight -1 in K.

PROPOSITION 4.21 *The Weil numbers $\rho_0(\pi)$ and $\psi_{\rho_0}(\varpi)$ represent the same element of $W^K(p^\infty)$.*

PROOF Because of the injectivity of the map $[\pi] \mapsto f_\pi$, it suffices to show that $f_{\rho_0(\pi)} = f_{\psi_{\rho_0}(\varpi)}$. From the Theorem of Shimura and Taniyama, we find that

$$f_{\rho_0(\pi)}(w) = [K_w : (\rho_0 E)_v] f_{\rho_0(\pi)}(v) \overset{\text{S–T}}{=} [K_w : (\rho_0 E)_v] \sum_{\sigma^{-1}w_0 = v} \varphi(\sigma \circ \rho_0)$$

where v is the restriction of w to ρE and the sum is over the embeddings $\sigma : \rho E \to K$.

On the other hand, we know (142) that

$$f_{\psi_{\rho_0}(\varpi)} = \sum_{\tau w_0 = w} \psi_{\rho_0}(\tau)$$

where τ runs over the elements of $\mathrm{Gal}(K/\mathbb{Q})$. As $\psi_{\rho_0}(\tau) \overset{\text{def}}{=} \varphi(\tau^{-1} \circ \rho_0)$, the two sums are equal. □

The reduction functor

The functor $A \mapsto A_0$ extends to a functor

$$R : \mathsf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{LMot}(\mathbb{F}), \quad h(A, e, m) \mapsto h(A_0, e_0, m).$$

Above, we defined a surjective homomorphism $f \mapsto \pi(f) : X^*(S^K) \to W^K(p^\infty)$ which sends CM-types on K to Weil integers of weight -1. Since the map is Γ-equivariant, to

each Γ-orbit Ψ of CM-types it attaches a Γ-orbit $\Pi(\Psi)$ of Weil integers of weight -1 and a surjective Γ-equivariant homomorphism $\Psi \to \Pi(\Psi)$. This last map induces a surjective homomorphism

$$\sum_{\psi\in\Psi} f(\psi)\psi \mapsto \sum_{\psi\in\Psi} f(\psi)\pi(\psi) : X^*(T^\Psi) \to X^*(L^\Pi)$$

sending t^Ψ to l^Π, and hence an injective homomorphism

$$\alpha'^\Psi : (L^{\Pi(\Psi)}, l^{\Pi(\Psi)}) \to (T^\Psi, t^\Psi).$$

On combining these maps for all Ψ, we obtain a injective homomorphism $\alpha'^K : (L^K, l^K) \to (T^K, t^K)$.

THEOREM 4.22 *The homomorphism $(L^K, l^K) \to (T^K, t^K)$ of fundamental groups defined by the reduction functor $\mathsf{LCM}^K(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{LMot}^K(\mathbb{F})$ is α'.*

PROOF It suffices to check this on $\langle A\rangle^\otimes$ for A a simple abelian variety of CM-type, but here it follows from Proposition 4.21. □

5 The Hodge conjecture for CM abelian varieties

THEOREM 5.1 (OPENAI, 2026) *The Hodge conjecture holds for CM abelian varieties.*

We sketch the proof.

5.2 *It suffices to prove the conjecture in codimension 2 (Hazama 2000, 2003).*

Fix a CM-field K finite and Galois over $\mathbb{Q}$ with Galois group G. To give a CM abelian variety over $\mathbb{C}$ split by K, i.e., such that $\mathrm{End}^0(A)\otimes_{\mathbb{Q}} K$ is a product of matrix algebras over K, amounts to giving a finite G-set S and a subset S^+ of S such that $S = S^+ \sqcup \iota S^+$. For example, if A has complex multiplication by a CM-algebra E, then $S = \mathrm{Hom}(E, K)$ and S^+ is the CM type. If A corresponds to (S, S^+), then it is possible to read off $H^*(A, \mathbb{Q})$ with its Hodge structure from (S, S^+), and even $H^*(A^n, \mathbb{Q})$ because A^m corresponds to the disjoint union of n copies of (S, S^+).

Let $[K : \mathbb{Q}] = 2m$. Let $H_1, \dots, H_m$ be the coordinate hyperplanes in $\mathbb{R}^m$, and take S to be the set of connected regions of $\mathbb{R}^m \smallsetminus \bigcup H_i$, and S^+ the subset on which $x_1 > 0$. For a suitable action of G on $\mathbb{R}^m$, hence on (S, S^+), there is a CM abelian variety $A(S, S^+)$ over $\mathbb{C}$ and Hazama proves:

(a) for every $n \geq 0$, the $\mathbb{Q}$-algebra $\mathcal{B}^*(A^n)$ of Hodge classes is generated by the classes of degree ≤ 2;

(b) every CM abelian variety split by K is isogenous to an isogeny factor of A^n for some n.

Thus, if the Hodge conjecture holds in codimension ≤ 2 for abelian varieties split by K, then it holds in all all codimensions for for $A(S, S^+)^n$ by (a), and hence for every CM abelian variety split by K by (b).

5.3 *It suffices to prove the conjecture in dimension 2.*

We know the standard conjecture of Lefschetz type for abelian varieties: the map on algebraic cycles

$$L^{n-4}: \mathcal{A}^2(A) \xrightarrow{\simeq} \mathcal{A}^{n-2}(A)$$

is an isomorphism. Here L is the Lefschetz operator (cup-product with the class of an ample divisor).

5.4 *Proof of the conjecture in dimension* 2.

Thus, it comes down to embedding surfaces into A. Let S be a surface. Suppose the Hodge structure $H^1(A,\mathbb{Q})$ occurs in $H^1(S,\mathbb{Q})$. Then A is an isogeny factor of the albanese of S, and so you get a map $f: S \to A$.

For S we take Picard surfaces. These are quotients of the 2-dimensional complex ball

$$\mathbb{B}^2 = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$$

by a discrete group attached to a three-dimensional hermitian space

$$\psi: V \times V \to E$$

over a CM-field E such that ψ has signature $(2,1)$ at one embedding of the totally real subfield of E and signature $(3,0)$ at all other embeddings.

From the Liu 2014, 2021, we find that the Hodge structure $H^1(A,\mathbb{Q})$ occurs in $H^1(S,\mathbb{Q})$ for S a Picard surface.

Now comes the hard part. We have to show that $f(S)$ is not numerically equivalent to zero. For this, we have to prove that $Z \cdot f(S) \neq 0$ for some Hodge class Z on A of codimension 2. Here we refer the reader to van Hoften and Sempliner 2026, which in addition to the fifty-page OpenAI manuscript "The rational Hodge conjecture for CM abelian varieties", draws on the manuscript Bergeron et al. 2016, which proves the Hodge conjecture in small degrees for some arithmetic locally symmetric varieties..

6 The Tate conjecture for abelian varieties over finite fields.

THEOREM 6.1 (Milne 1999b) *The full Tate conjecture (2.49) holds for abelian varieties over finite fields.*

The proof will occupy the rest of this section, but first we give some consequences.

6.2 Recall (2.48) that, for a smooth projective variety X over $\mathbb{F}_q$, the full Tate conjecture says that

$$Z(X,t) \sim c(X)\cdot(1-q^r t)^{-\rho_r}$$

where $c(X) \neq 0$ and ρ_r is the is the rank of the group of numerical equivalence classes of algebraic cycles of codimension r. Let $H^i(X_{Wet},\mathbb{Z}(r))$ denote the Weil-étale cohomology of complex of sheaves defined by Bloch's higher Chow groups, let $e: H^{2r}(X_{Wet},\mathbb{Z}(r)) \to H^{2r+1}(X_{Wet},\mathbb{Z}(r))$ denote cup-product with the canonical element of $H^1(k_{Wet},\mathbb{Z})$, and let

$$\chi(X_{Wet},\mathbb{Z}(r)) = \prod_{i\neq 2r,2r+1} [H^i(X_{Wet},\mathbb{Z}(r))]^{(-1)^i}\,\frac{[\mathrm{Ker}(e)]}{[\mathrm{Coker}(e)].}$$

The conjecture of Lichtenbaum and Milne asserts that

$$Z(X,t) = \pm\chi(X_{Wet}, \mathbb{Z}(r)) \cdot q^{\chi(A,\mathcal{O}_A,r)} \cdot (1-q^r t)^{-\rho_r},$$

where $\chi(X,\mathcal{O}_X,r) = \sum_{i\le r,j}(-1)^{i+j}(r-i)\dim H^j(X,\Omega^i_{X/k})$. This is now a theorem when A is an abelian variety. See Milne 2015, Theorem 11 and Remark 12.

6.3 Let X be a smooth projective variety over a field k. Recall (Tate 1994, p. 76, an ***almost-algebraic class*** of codimension n on X is an element $\gamma \in H^{2n}_{\mathbb{A}}(X)(n)$ such that there exists a cartesian square

$$\begin{array}{ccc} \widetilde{X} & \longleftarrow & X \\ \downarrow f & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

and a global section $\widetilde{\gamma}$ of $R^{2n}f_*\mathbb{A}(n)$ satisfying the following conditions

- S is a connected normal scheme of finite type over Spec $\mathbb{Z}$;
- $f : X \to S$ is a smooth projective scheme over S;
- the fibre of $\widetilde{\gamma}$ over Spec(k) is γ, and the specialization of $\widetilde{\gamma}$ at s is algebraic for all closed points s in a dense open subset U of S.

Note that the residue field $\kappa(s)$ at a closed point of S is finite.

It follows from Theorem 6.1 that all absolute Hodge classes on abelian varieties are almost-algebraic.

A serendipitous equality

Recall that we have defined maps α, α', β, and γ.

THEOREM 6.4 *The diagram*

$$\begin{array}{ccc} S & \xhookrightarrow{\gamma} & T \\ \uparrow\alpha & & \uparrow\alpha' \\ P & \xhookrightarrow{\beta} & L \end{array}$$

commutes and

$$P = L \cap S \quad \textit{(intersection inside } T\textit{)}.$$

More precisely, let $K \subset \mathbb{Q}^{\mathrm{al}}$ be a CM field, finite and Galois over $\mathbb{Q}$. We prove that

$$P^K = L^K \cap S^K \quad (\text{intersection inside } T^K).$$

Let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$ and let $D \subset \Gamma$ be the decomposition group at the given p-adic prime on K. We write P, S, L, T for P^K, S^K, L^K, T^K. Recall that we have diagrams

$$\begin{array}{ccc} T & \longleftarrow & S \\ \uparrow & & \uparrow \\ L & \longleftarrow & P \end{array} \qquad \begin{array}{ccc} X^*(T) & \xrightarrow{j} & X^*(S) \\ \downarrow r & & \downarrow s \\ X^*(L) & \xrightarrow{i} & X^*(P) \end{array}$$

and have to show that the induced map $S/P \to T/L$ is injective, or, equivalently, that the map $j:\ \mathrm{Ker}(r) \to \mathrm{Ker}(s)$ is surjective.

Recall that ι denotes complex conjugation (on K say), so $\langle\iota\rangle$ is the subgroup $\{1,\iota\}$ of Γ. For a positive integer d and a finite set Y with an action of $\langle i\rangle$, set

$$\begin{aligned}\mathbb{Z}[Y] &= \text{free abelian group on } Y = \{\text{maps } f: Y \to \mathbb{Z} \text{ of sums } \textstyle\sum f(y)y\}\\ \mathbb{Z}[Y]^d &= \{f \in \mathbb{Z}[Y] \mid \text{there exists a } c \in \mathbb{Z} \text{ such that } f + \iota f = d\cdot c\}\\ \mathbb{Z}[Y]_0 &= \mathbb{Z}[Y]/\{f \mid f = \iota f \text{ and } \textstyle\sum f(y) = 0\}.\end{aligned}$$

We sometimes regard the elements of $\mathbb{Z}[Y]$ as being maps $f: Y \to \mathbb{Z}$ or sums $\sum f(y)y$. With this notation, the right-hand of the above diagrams can be identified with

$$\begin{array}{ccc}\mathbb{Z}[\mathcal{S}]_0 & \xrightarrow{\ j\ } & \mathbb{Z}[\Gamma]^1\\ \downarrow r & & \downarrow s\\ \mathbb{Z}[\mathcal{P}]_0 & \xrightarrow{\ i\ } & \mathbb{Z}[\Gamma/D]^d,\end{array}$$

where $\mathcal{S}$ is the set of CM types on K and $\mathcal{P}$ is the set of functions $\pi: \Gamma/D \to \{0,1,\dots,d\}$, $d = (D:1)$ such that $\pi + \iota\pi = d$. The horizontal maps take a formal sum to a sum of functions. The map r send φ to $r(\varphi)$, where $r(\varphi)(\tau D) = \sum_{\sigma\in D}\varphi(\tau\sigma)$, and s sends f to $s(f)$, where $s(f)(\tau D) = \sum_{\sigma\in D} f(\tau\sigma)$.

We first consider the case $\iota \in D$. Choose a set B of coset representatives of $\langle\iota\rangle$ in D and a set A of coset representatives for D in Γ. Then, as a map of $\langle\iota\rangle$-sets, $\Gamma \to \Gamma/D$ can be identified with the projection map

$$A\times B\times\{0,1\} \to A\times\{0,1\}.$$

Here ι acts only on $\{0,1\}$. Thus, the diagram becomes

$$\begin{array}{ccc}\mathbb{Z}[\mathcal{S}]_0 & \xrightarrow{\ j\ } & \mathbb{Z}[A\times B\times\{0,1\}]^1\\ \downarrow r & & \downarrow s\\ \mathbb{Z}[\mathcal{P}]_0 & \xrightarrow{\ i\ } & \mathbb{Z}[A\times\{0,1\}]^d,\end{array}$$

With this notation, a CM type on K is a function $\varphi: A\times B\times\{0,1\} \to \{0,1\}$ such that

$$\varphi(a,b,0) + \varphi(a,b,1) = 1 \quad \text{all } a,b,$$

and an element of $\mathcal{P}$ is a function $\pi: A\times\{0,1\} \to \{0,\dots,d\}$ such that

$$\pi(a,0) + \pi(a,1) = d, \quad \text{all } a.$$

For each a,b, let $\varphi_{a,b}$ be the CM-type such that

$$\varphi_{a,b}(x,y,0) = 1 \iff (x,y) = (a,b),$$

and let φ' be the CM-type such that

$$\varphi(x,y,z) = z.$$

Then φ' and the $\varphi_{a,b}$ generate $\mathbb{Z}[A\times B\times\{0,1\}]^1$: if f is a function on $A\times B\times\{0,1\}$ such that $f + \iota f = c \in \mathbb{Z}$, then

$$f = \sum_{a,b} f(a,b,0)\varphi_{a,b} + (c - \sum_{a,b} f(a,b,0))\varphi'.$$

Let $\pi_a = r(\varphi)_{a,b}$ and $\pi' = r(\varphi')$, so π_a is the element of $\mathcal{P}$ such that $\pi_a(x,0) = 1$ for $x = a$ and 0 otherwise, and $\pi'(x,z) = d \cdot z$. Clearly, the π_a and π' are linearly independent.

We now prove this case of the theorem. Let $f \in \mathbb{Z}[A \times B \times \{0,1\}]^1$, and write it

$$f = \sum_{a,b} n(a,b) \cdot \varphi_{a,b} + n\varphi',$$

so

$$s(f) = \sum_a (\sum_b n(a,b)\pi_a + n\pi'.$$

If $s(f) = 0$, then, because of the linear independence of the π_a and π',

$$\sum\nolimits_{b\in B} n(a,b) = 0, \quad \text{all } a \in A, \text{ and}$$
$$n = 0.$$

The first equation implies that $\sum_{b\in B} n(a,b)\varphi_{a,b}$ is in the kernel of r, which completes the proof.

The proof of the remaining case, $\iota \notin D$, is similar.

COROLLARY 6.5 *The groups S^K/P^K acts faithfully on $X = \mathcal{E}nd(h_1A)^P$.*

PROOF I begin with an elementary remark. Let $T \supset L$ be tori with T acting on a finite dimensional vector space V. Let $\chi_1, \dots, \chi_{m70}$ be the characters of T occurring in V. Then T acts faithfully on V if and only if $\chi_1, \dots, \chi_{m70}$ span $X^*(T)$ as a $\mathbb{Z}$-module — assume this. The characters of T occurring in $\mathrm{End}(V)$ are $\{\chi_i - \chi_j\}$, and the set of those occurring in $\mathrm{End}(V)^L$ is

$$\{\chi_i - \chi_j \mid \chi_i|L = \chi_j|L\}. \tag{*}$$

On the other hand,

$$X^*(T/L) = \{\sum a_i\chi_i \;\Big|\; \sum a_i\chi_i|L = 0\}. \tag{**}$$

Thus, T/L will act faithfully on $\mathrm{End}(V)^L$ if the set (*) spans the $\mathbb{Z}$-module (**).

I now prove the statement. With the notations of Milne 1999b, Section 6 (especially p22,23 of the pdf file on my site), T^Ψ acts on a realization of h_1A^Ψ through the characters $\psi_0, \dots, \psi_{n-1}, \iota\psi_0, \dots, \iota\psi_{n-1}$, where the ψ_i have been numbered so that $\pi(\psi_0) = \cdots = \pi(\psi_{d-1}) = \pi_0$, $\pi(\psi_d) = \cdots = \pi(\psi_{2d-1}) = \pi_1$, etc.. Now $\sum a_i \cdot \psi_i|L^\Psi = \sum a_i \cdot \pi(\psi_i)$, which is zero if and only if $\sum_{i=0}^{d-1} a_i = 0$, $\sum_{i=d}^{2d-1} a_i = 0$,…; but then

$$\sum a_i\psi_i = \sum_{i=0}^{d-1} a_i(\psi_i - \psi_0) + \cdots,$$

which (by the remark) shows that T^Ψ/L^Π acts faithfully on $\mathcal{E}nd(h_1A^\Psi)^{L^\Pi}$. It is even easier to show that $T^{\bar\Psi}/L^{\bar\Pi}$ acts faithfully on $\mathcal{E}nd(h_1A^{\bar\Psi})^{L^{\bar\Pi}}$ and deduce that $T^{A^\Psi\times A^{\bar\Psi}}/L^{A^\Pi\times A^{\bar\Pi}}$ acts faithfully on $\mathcal{E}nd(h_1(A^\Psi \times A^{\bar\Psi}))^{L^{A^\Pi\times A^{\bar\Pi}}}$. As

$$P^K/L^K \hookrightarrow T^{A^\Psi\times A^{\bar\Psi}}/L^{A^\Pi\times A^{\bar\Pi}}$$

(cf. ibid. Lemma 6.9), this implies the statement. □

Proof of Theorem 6.1

Let $M \subset T$ be the subgroup scheme fixing the algebraic classes. Then $M \subset L$ because Lefschetz classes are algebraic, and $M \subset S$ because (5.1) the absolute Hodge classes on CM abelian varieties are algebraic. It follows from Theorem 6.4 that $M = P$. This implies the Tate conjecture because M and P act on $H^{2r}(A, \mathbb{Q}_\ell(r))$ for any abelian variety A over $\mathbb{F}$; the classes fixed by M are the algebraic classes, and the classes fixed by P are the Tate classes.

ASIDE 6.6 I call the equality $P = L \cap S$ serendipitous because, a priori, there is no reason to expect it to be true, but it allows us to determine the group scheme M, hence prove the Tate conjecture, knowing only that it is contained in both L and S.

NOTES Previously, the Tate conjecture was known divisors on abelian varieties over finite fields (Tate 1966) and for abelian fourfolds over finite fields (Broe 2026).

7 Grothendieck's standard conjectures for abelian varieties

The standard conjecture of Lefschetz type

THEOREM 7.1 (KLEIMAN, LIEBERMAN) *The standard conjecture of Lefschetz type holds for abelian varieties.*

For the original proof, see Kleiman 1968, pp. 373–378. We prove the following more precise statement.

THEOREM 7.2 (MILNE 1999B) *The cohomology classes on abelian varieties, predicted to be algebraic by the Lefschetz standard conjecture, are, in fact, Lefschetz.*

PROOF Let A be an abelian variety over an algebraically closed field k, and let H be a Weil cohomology theory satisfying the strong Lefschetz theorem. Let $L: H^*(X) \to H^{*+1}(X)$ be the operator defined by a hyperplane section. Because H satisfies the strong Lefschetz theorem, any $x \in H^s(X)$ can be written uniquely in the form $x = \sum_{i \geq 0, s-d} L^i x_i$ with x_i a primitive element of $H^{s-2i}(A)$. The operator L is $L(A \times A)$-equivariant, and so the map $x \mapsto x_i$ is $L(A \times A)$-equivariant, hence Lefschetz. It follows that the operator $x \mapsto \Lambda x = \sum L^{i-1} x_i$ is also Lefschetz, and similarly for the operators ${}^c\Lambda$ and $*$. □

REMARK 7.3 The Lefschetz standard conjecture B is known for a product of varieties if it is known for the factors, and it is known for any hyperplane section of a variety for which it is known. Using this, one can enlarge the collection of varieties for which it is known. See Kleiman 1994, 4-3.

The standard conjecture of Hodge type

THEOREM 7.4 (MILNE 2002) *The standard conjecture of Hodge type holds for abelian varieties.*

The proof uses both Theorem 5.1 and Theorem 6.1, but otherwise is a fairly straightforward application of the theory of polarized tannakian categories.

By an obvious specialization argument it suffices to prove the conjecture for abelian varieties over $\mathbb{F}$. According to (2.42), it suffices to prove that there exists a graded polarization on the Tate triple $\mathsf{Mot}(\mathbb{F})$ for which the geometric Weil forms are positive. We prove a more precise statement. We prove a more precise statement.

Let $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ be the category of numerical motives based on CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ and $\mathsf{Mot}(\mathbb{F})$ the category of numerical motives based on abelian varieties over $\mathbb{Q}^{\mathrm{al}}$. There is a reduction functor $R:\ \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F})$.

THEOREM 7.5 *There exists a graded polarization* Π *on* $\mathsf{Mot}(\mathbb{F})$ *such that*

(a) *all geometric Weil forms are positive for* Π*;*

(b) *the reduction functor* $R:\ \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F})$ *is compatible with* Π^{CM} *and* Π.

Moreover, Π *is uniquely determined by the each of the conditions (a) and (b).*

PROOF I claim that to prove the theorem it suffices to show:

(*) there exists a polarization Π on $\mathsf{Mot}(\mathbb{F})$ such that $R:\ \Pi^{\mathrm{CM}} \mapsto \Pi$.

The geometric Weil forms are positive for Π^{CM} and every polarized abelian variety A over $\mathbb{F}$ lifts (up to isogeny) to a polarized abelian variety of CM-type over $\mathbb{Q}^{\mathrm{al}}$ (Zink 1983, 2.7), and so if $R:\ \Pi^{\mathrm{CM}} \mapsto \Pi$, then every geometric Weil form is positive for Π. In particular, the geometric Weil forms on a supersingular elliptic curve are positive, and so $\Pi = \Pi^{\mathrm{Mot}}$. This proves the claim.

We now prove (*). Fix a CM-field $K \subset \mathbb{Q}^{\mathrm{al}}$ such that

(a) K is finite and Galois over $\mathbb{Q}$, and

(b) K properly contains an imaginary quadratic field in which p splits.

It suffices to prove (*) for

$$R^K:\ \mathsf{Mot}^K(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}^K(\mathbb{F}).$$

Let A be the product of a set of representatives for the simple isogeny classes of abelian varieties split by K, and let $X = \mathcal{E}nd(h_1A)^P$. Then S^K/P^K acts faithfully on X (6.5) and it follows that T^K/L^K acts faithfully on $\mathcal{E}nd(h_1A)^{L^K}$ because the canonical map

$$S^K/P^K \to T^K/L^K$$

is injective (6.4).

Let ϕ be the geometric Weil form on h_1A defined by an ample divisor D, and let $\psi = T^\phi|X$. Then $\psi \in \Pi^{\mathrm{CM}}(X)$, and it suffices to show that $R^K(\psi)$ is positive-definite (13.5, 13.6). But $R^K(X) = \mathrm{End}(A_{\mathbb{F}})_{\mathbb{Q}}$ and $R^K(\phi)$ is the trace pairing $u, v \mapsto \mathrm{Tr}(u \cdot v^\dagger)$ of the Rosati involution defined by $D_{\mathbb{F}}$, which is positive-definite (Weil 1948, Théorème 38).□

THEOREM 7.6 *Let* k *be an algebraically closed field. For all* $\ell \neq \mathrm{char}(k)$*,*

(a) *numerical equivalence coincides with* ℓ*-adic étale homological equivalence on abelian varieties over* k*, and*

(b) *the Hodge standard conjecture holds for all abelian varieties over* k *and the* ℓ*-adic étale cohomology theory.*

PROOF (a) for $k = \mathbb{F}$. See Theorem 6.1.

(b) for $k = \mathbb{F}$. For abelian varieties over $\mathbb{Q}^{\mathrm{al}}$, the Betti cohomology theory is satisfies Conjecture D (Lieberman 1968) and the Hodge standard conjecture holds, and so (2.42) there is a polarization Π on $\mathsf{Mot}(\mathbb{Q}^{\mathrm{al}}()$ for which the forms

$$\varphi^r:\ h^r(A) \otimes h^r(A) \to \mathbb{1}(-r)$$

are positive. Clearly, Π is the canonical polarization Π^{CM} in §2. Let Z be the hyperplane section of A used in the definition of φ^r. Because $R:\ \Pi^{\mathrm{CM}} \mapsto \Pi^{\mathrm{Mot}}$ (Theorem 7.5), the form

$$\varphi^r:\ h^r(A_{\mathbb{F}}) \otimes h^r(A_{\mathbb{F}}) \to \mathbb{1}(-r)$$

defined by the reduction $Z_{\mathbb{F}}$ of Z on $A_{\mathbb{F}}$ is positive for Π^{Mot}. Every polarized abelian variety A over $\mathbb{F}$ lifts (up to isogeny) to a polarized abelian variety of CM-type over $\mathbb{Q}^{\mathrm{al}}$ (Zink 1983, 2.7), and so

$$(A_{\mathbb{F}}, Z_{\mathbb{F}} \text{ modulo numerical equivalence})$$

is arbitrary. Proposition 2.42 now gives (b).

(a,b) for arbitrary k. For an abelian variety A of dimension n over k, consider the commutative diagram:

$$\begin{array}{ccccc} H^{2r}(A,\mathbb{Q}_\ell(r)) & \times & H^{2n-2r}(A,\mathbb{Q}_\ell(n-r)) & \xrightarrow{\cup} & \mathbb{Q}_\ell \\ \uparrow{\scriptstyle \mathrm{cl}} & & \uparrow{\scriptstyle L^{n-2r}\circ \mathrm{cl}} & & \uparrow \\ P^r(A) & \times & P^r(A) & \xrightarrow{\theta} & \mathbb{Q} \end{array}$$

There is a similar diagram for a smooth specialization $A_{\mathbb{F}}$ of A to an abelian variety over $\mathbb{F}$. The specialization maps on the cohomology groups are bijective and hence they are injective on the P's. Since the pairings are compatible, this implies the Hodge standard conjecture for A and ℓ-adic étale cohomology. Because the Lefschetz standard conjecture is known for abelian varieties, this in turn implies that numerical equivalence coincides with ℓ-adic homological equivalence for A (Kleiman 1994, 5-4). □

COROLLARY 7.7 *If the Hodge conjecture holds for complex abelian varieties of CM-type, then, for any algebraically closed field k,* $\mathsf{Mot}(k)$ *has a polarization (necessarily unique) for which the forms ϑ^r and φ^r are positive.*

PROOF For $\ell \neq \mathrm{char}(k)$, Theorem 7.6(a) shows that ℓ-adic étale cohomology is good. Now apply (7.6b) and (2.42). □

REMARK 7.8 Let H be a Weil cohomology theory on over an algebraically closed field k. Because the Lefschetz standard conjecture is known for abelian varieties, if H does not satisfy Conjecture D, then the Hodge standard conjecture fails for H. Thus, the Hodge standard conjecture holds for H if and only if H satisfies Conjecture D.

REMARK 7.9 Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$ satisfying conditions (a) and (b) of the proof of (7.5). The preceding arguments can be modified to show that, if the Hodge conjecture holds for all complex abelian varieties with reflex field contained in K, then the conclusions of Theorem 7.6 hold for all abelian varieties over $\mathbb{F}$ whose endomorphism algebra is split by K. In fact, condition (b) is not necessary for this statement.

REMARK 7.10 Most of the preceding arguments hold with "algebraic cycle" replaced by "Lefschetz cycle". Let A be an abelian variety over k. Recall that, for any Weil cohomology theory, if a Lefschetz class a on A is not homologically equivalent to zero, then there exists a Lefschetz class b on A of complementary dimension such that $\langle a \cdot b\rangle \neq 0$. Thus, homological equivalence on Lefschetz classses is independent of the Weil cohomology theory, and coincides with numerical equivalence.

Let $D^r(A)$ be the $\mathbb{Q}$-space of Lefschetz classes modulo numerical equivalence on A of codimension r, and let $DP^r(A)$ be the $\mathbb{Q}$-subspace on which L^{n-2r+1} is zero. The categories $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$ and $\mathsf{LMot}(\mathbb{F})$ of Lefschetz motives have canonical polarizations, and the reduction functor $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{LMot}(\mathbb{F})$ maps one to the other. The same argument as in the proof of Theorem 7.6 shows that the bilinear forms

$$(x, y) \mapsto (-1)^r \langle L^{n-2r} x \cdot y \rangle : \ DP^r(A) \times DP^r(A) \to \mathbb{Q}$$

are positive-definite for $r \leq n/2$ and A an abelian variety over $\mathbb{F}$. In other words, the Lefschetz analogue of the Hodge standard conjecture holds for abelian varieties over $\mathbb{F}$. A specialization argument (as in the proof of 7.6) extends the statement to arbitrary k.

ASIDE 7.11 Grothendieck (1969) stated: "Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry." Grothendieck once said that he hope to see the standard conjectures proved in his lifetime. In this he was disappointed. At least we now know that his standard conjectures are true for all abelian varieties.

NOTES Previously, the standard conjecture of Hodge type was known for abelian fourfolds (Ancona 2021).

8 Specialization of motives

The category of motives over a finite field

8.1 The category $\mathsf{Mot}(\mathbb{F}_q)$ of abelian motives is semisimple. The Frobenius element π_M of a simple motive M is a Weil q-number of weight m. The isomorphism classes of simple motives are classified by the orbits of $\Gamma \overset{\text{def}}{=} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acting on $W(q)$,

$$\Sigma(\mathsf{Mot}(\mathbb{F}_q)) \simeq \Gamma \backslash W(q).$$

Let M be a simple motive over $\mathbb{F}_q$. Then $E \overset{\text{def}}{=} \mathrm{End}(M)$ is a simple $\mathbb{Q}$-algbra with centre π_M, and its invariant at a prime v of $\mathbb{Q}[\pi_M]$ is given by

$$\mathrm{inv}_v(E) = \begin{cases} 1/2 \text{ if } v \text{ is real and } M \text{ has odd weight,} \\ \dfrac{\mathrm{ord}_v(\pi_X)}{\mathrm{ord}_v(q)} \cdot [\mathbb{Q}[\pi_X]_v : \mathbb{Q}_p] \text{ if } v|p, \\ 0 \text{ otherwise.} \end{cases}$$

Specialization of motives

8.2 Let $\mathbb{Q}^{\mathrm{al}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let w be a p-adic prime. There are compatible commutative diagrams,

$$\begin{array}{ccc} G & & \mathsf{Mot}'(\mathbb{Q}^{\mathrm{al}}) \\ \uparrow \ \nwarrow & & \downarrow R \ \searrow^{\xi_l} \\ P \longleftarrow P_l & & \mathsf{Mot}(\mathbb{F}) \xrightarrow{\eta_l} R_l(\mathbb{F}) \end{array} \qquad l = 2, \dots, p, \dots, \infty,$$

where

- $\mathsf{Mot}'(\mathbb{Q}^{\mathrm{al}})$ is the category of abelian motives over $\mathbb{Q}^{\mathrm{al}}$ with visibly good reduction at w (i.e., of the form $h(A,e,m)$, where A has good reduction at w);
- $R_l(\mathbb{F})$ is the realization category at l (a tannakian category over $\mathbb{Q}_l$);
- ξ_l is the exact tensor functor defined by étale cohomology ($l \neq p,\infty$), crystalline cohomology ($l = p$), or the canonical polarization on $\mathsf{Mot}^w(\mathbb{Q}^{\mathrm{al}})$ ($l = \infty$);
- $\mathsf{Mot}(\mathbb{F})$ is the category of abelian motives over $\mathbb{F}$;.
- G is the affine group scheme over $\mathbb{Q}$ attached to $\mathsf{Mot}'(\mathbb{Q}^{\mathrm{al}})$ and the Betti fibre functor;
- the arrows $P_l \to G$ and $P_l \to P$ are the morphisms of bands induced by ξ_l and η_l;
- $P \to G$ is the (unique) morphism of bands making the diagrams at left commute for all l;
- $R:\ \mathsf{Mot}'(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F})$ is a $\mathbb{Q}$-linear exact tensor functor, banded by the morphism $P \to G$, and making the diagrams at right commute.

Applications

There are many heuristic arguments deriving a statement from a sufficiently good theory of motives. Many of these heuristic arguments are now proofs.

8.3 Let A be an abelian variety over a number field k, and let G be the Mumford-Tate group of $A_{\mathbb{C}}$. It follows from Deligne's theorem on absolute Hodge classes that, after making a finite extension of k, there are well-defined Frobenius conjugacy classes in $G(\mathbb{Q}_\ell)$, ℓ a prime. It is an immediate consequence of the above theory that these are rational, i.e., arise from a single conjugacy class in $G(\mathbb{Q})$. See Serre 1994b, 12.6.

8.4 We leave it as a challenge to the reader to prove the conjecture of Langlands and Rapoport (1987) following the heuristic argument they use to derive its statement.

Appendix A

Categories and 2-Categories

We review some basic definitions and statements. Throughout, 2-category means strict 2-category and bicategory means weak 2-category.

Equivalence of categories

A.1 Let C and D be categories. An ***equivalence*** of C and D is a system

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}, \qquad \eta : \mathrm{id}_{\mathsf{C}} \to GF, \quad \epsilon : FG \to \mathrm{id}_{\mathsf{D}}, \tag{143}$$

where F and G are functors and η and ϵ are invertible natural transformations. We call a functor $F : \mathsf{C} \to \mathsf{D}$ an ***equivalence*** if it can be extended to such a system. A functor F is an equivalence if and only if it is fully faithful and ***essentially surjective*** (i.e., every object of D is isomorphic to an object in the image of F).

In practice, these conditions may be too strong. For example, we shall need to consider functors $F : \mathsf{C} \to \mathsf{D}$ such that the objects of D are only "equivalent" to objects in the image. To be able talk about such functors, we need morphisms of morphisms. In other words, we need 2-categories.[1] But first, we review adjoint functors.

Review of adjoint functors

Let C and D be categories.

A.2 An ***adjunction*** is a triple (F, G, ϕ) consisting of a pair of functors

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \tag{144}$$

and a family of bijections

$$\phi_{X,Y} : \mathrm{Hom}_{\mathsf{D}}(FX, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, GY),$$

[1] If a functor of 1-categories is an isomorphism of categories, then it is surjective on objects; if an equivalence, then it is surjective on isomorphism classes of objects; if part of a biequivalence of 2-categories, then it is surjective on equivalence classes of objects.

natural in $X \in \operatorname{ob}\mathsf{C}$ and $Y \in \operatorname{ob}\mathsf{D}$. We call (F, G) an ***adjoint pair***, with F the ***left adjoint*** of G and G the ***right adjoint*** of F . There are natural transformations

$$\eta : \mathrm{id}_{\mathsf{C}} \to GF, \quad \eta_X = \phi(\mathrm{id}_{FX}) : X \to GFX$$
$$\epsilon : FG \to \mathrm{id}_{\mathsf{D}}, \quad \epsilon_Y = \phi^{-1}(\mathrm{id}_{GY}) : FGY \to Y,$$

called the ***unit*** and ***counit*** of the adjunction, satisfying the ***triangle identities***,

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle \mathrm{id}_F} \searrow & \downarrow {\scriptstyle \epsilon F} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & {\scriptstyle \mathrm{id}_G} \searrow & \downarrow {\scriptstyle G\epsilon} \\ & & G \end{array} \tag{145}$$

See A.9 for the notation $F\eta$ and ϵF.

A.3 Let F and G be functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$, and let $\eta : \mathrm{id}_{\mathsf{C}} \to GF$ and $\epsilon : FG \to \mathrm{id}_{\mathsf{D}}$ be natural transformations satisfying the triangle identities. Then the map

$$\phi_{X,Y} : \mathrm{Hom}_{\mathsf{D}}(FX, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, GY), \quad f \mapsto Gf \circ \eta_X,$$

is natural in X and Y, and has inverse

$$\psi_{X,Y} : \mathrm{Hom}_{\mathsf{C}}(X, GY) \to \mathrm{Hom}_{\mathsf{D}}(FX, Y), \quad f \mapsto \epsilon_X \circ Ff.$$

In particular, $\phi_{X,Y}$ is a bijection, and so the triple (F, G, ϕ) is an adjunction with unit η and counit ϵ.

A.4 Let F and G be functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$, and let $\epsilon : FG \to \mathrm{id}_{\mathsf{D}}$ be a natural transformation. There exists a natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \to GF$ such that the triangle identities hold if and only if the map

$$\psi_{X,Y} : \mathrm{Hom}_{\mathsf{C}}(X, GY) \to \mathrm{Hom}_{\mathsf{D}}(FX, Y), \quad f \mapsto \epsilon_X \circ Ff,$$

is bijective for all X in C and Y in D, in which case (F, G, ψ^{-1}) is an adjunction with η and ϵ as its unit and counit. The natural transformation η is unique if it exists.

A.5 An equivalence (F, G, η, ϵ) is an ***adjoint equivalence*** if it satisfies the triangle identities (A.2). Any equivalence can be made into an adjoint equivalence by replacing either one of η or ϵ. Indeed, the composite

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

is an isomorphism, but it need not be the identity. Modifying either ϵ or η to absorb the relevant inverse automorphism gives an adjoint equivalence (Riehl 2016, 4.4.5).

A.6 When the categories are abelian, every adjunction preserves the abelian group structure on the Hom-sets. Let C and D be abelian categories with small inductive limits, and assume that small filtered inductive limits in C are exact and that C admits a generator. Then an additive functor $F : \mathsf{C} \to \mathsf{D}$ has a right adjoint if and only if it preserves small inductive limits (special adjoint functor theorem). An additive functor between abelian categories preserves small inductive limits if and only if is right exact and preserves small direct sums.

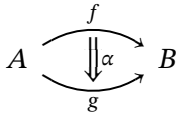
ToDo 1 Check t241 and applications.

2-categories

A 2-category is a category enriched over the category of small categories equipped with its cartesian monoidal structure (I, 1.9). When we forget the arrows in the Hom-categories, we get a category in the usual sense. Thus, a 2-category can be viewed as a category in the usual sense enriched with morphisms between morphisms.

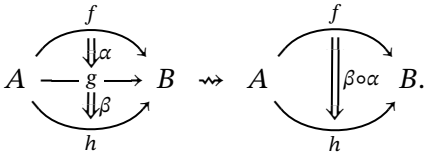
DEFINITION A.7 A 2-***category*** has objects, 1-morphisms, and 2-morphisms. The objects and 1-morphisms form a category in the usual sense.

For each pair A, B of objects, there is a small category $\mathsf{Hom}(A,B)$ having the 1-morphisms $f : A \to B$ as objects and the 2-morphisms[2]

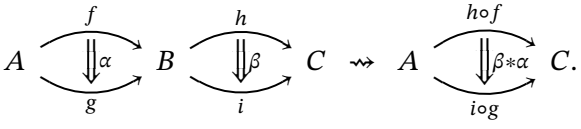


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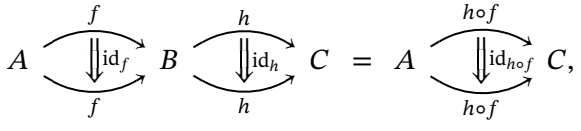
as morphisms (from f to g). Composition of morphisms in the category $\mathsf{Hom}(A,B)$ is called ***vertical composition***,



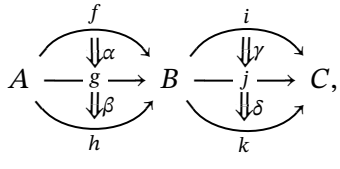
For each triple A, B, C of objects, there is a ***horizontal composition***



Finally, horizontal composition is functorial with respect to vertical composition. This means that the horizontal composite of identity 2-morphisms is an identity 2-morphism,



and the system satisfies the middle four interchange law: given a diagram,



the 2-morphism $i \circ f \Rightarrow k \circ h$ obtained by first composing vertically and then horizontally equals that obtained by first composing horizontally and then vertically,

$$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha).$$

See Riehl and Verity 2022, B.1.1.

[2]We sometimes use the symbol $\Rightarrow$ to denote 2-morphisms.

A.8 When A and B are objects of a 2-category $\mathscr{C}$, we usually write $\mathscr{C}(A,B)$ for the 1-category $\mathsf{Hom}_{\mathscr{C}}(A,B)$. From horizontal composition we get

(a) a bifunctor

$$c_{ABC} : \mathscr{C}(A,B) \times \mathscr{C}(B,C) \to \mathscr{C}(A,C),$$

which is associative in the sense that

$$\begin{array}{ccc} \mathscr{C}(A,B)\times\mathscr{C}(B,C)\times\mathscr{C}(C,D) & \xrightarrow{1\times c_{BCD}} & \mathscr{C}(A,B)\times\mathscr{C}(B,D) \\ \downarrow{\scriptstyle c_{ABC}\times 1} & & \downarrow{\scriptstyle c_{ABD}} \\ \mathscr{C}(A,C)\times\mathscr{C}(C,D) & \xrightarrow{c_{ACD}} & \mathscr{C}(A,D) \end{array}$$

commutes for all objects A, B, C, D, and

(b) functors $u_A : \mathbb{1} \to \mathcal{C}(A,A)$, where $\mathbb{1}$ is the category with one object and one morphism, such that

$$\begin{array}{ccccc} \mathbb{1}\times\mathscr{C}(A,B) & \xleftarrow{\cong} & \mathscr{C}(A,B) & \xrightarrow{\cong} & \mathscr{C}(A,B)\times\mathbb{1} \\ \downarrow{\scriptstyle u_A\times 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1\times u_B} \\ \mathscr{C}(A,A)\times\mathscr{C}(A,B) & \xrightarrow{c_{AAB}} & \mathscr{C}(A,B) & \xleftarrow{c_{ABB}} & \mathscr{C}(A,B)\times\mathscr{C}(B,B) \end{array}$$

commutes for all objects A, B.

Conversely, to give a 2-category amounts to giving a class of objects, small categories $\mathscr{C}(A,B)$ for each pair of objects A and B, identity functors u_A for each object A, and composition functors for each triple of objects A, B, and C satisfying (a) (associativity axiom) and (b) (unit axiom) (Borceux 1994a, 7.1.1).

When $\mathscr{C}$ is a 2-category, we write $\mathscr{C}_0$ for the collection of objects, $\mathscr{C}_1$ for the collection of 1-morphisms, and $\mathscr{C}_2$ for the collection of 2-morphisms. A 2-category $\mathscr{C}$ is ***small*** if $\mathscr{C}_0$ is a set.

To distinguish them, we sometimes call categories in the usual sense 1-categories.

From a 2-category, we get a 1-category by forgetting the 2-morphisms. Conversely, a 1-category can be turned into a 2-category by regarding each Hom-set as a discrete category (i.e., the only morphisms are the identity morphisms).

EXAMPLE A.9 The 2-category $\mathscr{C}at$ has objects the small categories, 1-morphisms the functors between categories, and 2-morphisms the natural transformations between functors.

Let F, G, and H be functors $\mathsf{A} \to \mathsf{B}$. The vertical composite of natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ is the natural transformation $\beta \circ \alpha : F \Rightarrow H$ such that

$$(\beta\circ\alpha)_a = \beta_a \circ \alpha_a : F(a) \to H(a)$$

for all objects a of A.

Let F, G be functors $\mathsf{A} \to \mathsf{B}$ and H, I functors $\mathsf{B} \to \mathsf{C}$. The horizontal composite of natural transformations $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow I$,

$$\mathsf{A} \;\overset{F}{\underset{G}{\Downarrow \alpha}}\; \mathsf{B} \;\overset{H}{\underset{I}{\Downarrow \beta}}\; \mathsf{C},$$

is the natural transformation $\beta * \alpha : H \circ F \to I \circ G$ such that $(\beta * \alpha)_a$ is the diagonal map in the commutative square

$$\begin{array}{ccc} (H\circ F)(a) & \xrightarrow{\beta_{F(c)}} & (I\circ F)(a) \\ \downarrow{\scriptstyle H(\alpha_a)} & & \downarrow{\scriptstyle I(\alpha_a)} \\ (H\circ G)(a) & \xrightarrow{\beta_{G(c)}} & (I\circ G)(a) \end{array}$$

for all objects a of A. For example, from

$$\mathsf{A} \xrightarrow{F} \mathsf{B} \underset{I}{\overset{H}{\Downarrow \beta}} \mathsf{C},$$

we get a natural transformation $\beta F : HF \to IF$, namely, $\beta * \mathrm{id}_F$, and from

$$A \underset{G}{\overset{F}{\Downarrow \alpha}} B \xrightarrow{H} C$$

we get a natural transformation $H\alpha : H \circ F \to H \circ G$, namely, $\mathrm{id}_H * \alpha$. With this notation,

$$\beta * \alpha = (\beta G) \circ (H\alpha) = (I\alpha) \circ (\beta F).$$

See Riehl 2016, 1.7.

EXAMPLE A.10 The category of topological spaces and continuous maps can be turned into a 2-category by taking the 2-morphisms $f \Rightarrow g$, where f and g are continuous maps $A \to B$, to be the homotopies.

EXAMPLE A.11 The category of groups and homomorphisms can be turned into a 2-category by taking the 2-morphisms $f \Rightarrow g$, where f and g are homomorphisms $G \to H$, to be the inner automorphisms α of H such that $g = \alpha \circ f$.

We can now make the following definition.

DEFINITION A.12 Let A and B be objects in a 2-category. An ***equivalence*** between A and B is a system

$$A \underset{g}{\overset{f}{\rightleftarrows}} B, \qquad \eta : \mathrm{id}_A \Rightarrow g \circ f, \qquad \epsilon : f \circ g \Rightarrow \mathrm{id}_B,$$

where f and g are 1-morphisms and η and ϵ are 2-isomorphisms (invertible 2-morphisms). We call a 1-morphism $f : A \to B$ an ***equivalence*** if it can be extended to such a system, and we say that A and B are ***equivalent***, denoted $A \approxeq B$, when there exists such an f.

A.13 An equivalence (f, g, η, ϵ) is an ***internal*** or ***adjoint equivalence*** if it satisfies the triangle identities (A.2). As in A.5, any equivalence can be made into an adjoint equivalence by changing at most one of η or ϵ (Johnson and Yau 2021, 6.2.4).

A.14 If $A \approxeq A'$ and $B \approxeq B'$, then $\mathsf{Hom}(A, B) \approxeq \mathsf{Hom}(A', B')$. More precisely, from equivalences between A and A' and between B and B', we get an equivalence between $\mathsf{Hom}(A, B)$ and $\mathsf{Hom}(A', B')$.

ASIDE A.15 As a rough rule of thumb, in a 1-category, objects can be considered to be the "same" if they are isomorphic, and parallel morphisms if they are equal. In a 2-category, objects can be considered to be the "same" if they are equivalent, parallel 1-morphisms if they are isomorphic, and parallel 2-morphisms if they are equal.

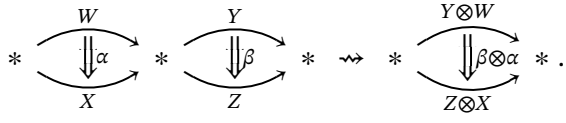
NOTES Because we require the Hom categories to be small, what we call a 2-category is called a locally small 2-category in Johnson and Yau 2021, 2.3.9. Our definition of a small 2-category agrees with their definition.

Bicategories

The definition of a ***bicategory*** is the same as that of a 2-category except that the associativity and unit axioms hold only up to coherent invertible 2-morphisms. More precisely, to give a bicategory $\mathscr{C}$ amounts to giving a class of objects, small categories $\mathscr{C}(A,B)$ for each pair of objects A and B, identity arrows for each object, and composition functors such that the diagrams (a) and (b) of A.8 commute up to natural isomorphisms, which are then part of the data of a bicategory, and which themselves satisfy certain coherence conditions. See Borceux 1994a, 7.7.

Every bicategory is biequivalent to a 2-category (Johnson and Yau 2021, 8.4.1), but bicategories also arise naturally. In this work, we only use 2-categories, but we need to consider biequivalences in addition to 2-equivalences.

EXAMPLE A.16 Every monoidal category $(\mathsf{C}, \otimes)$ can be thought of as a bicategory with one object $*$. The 1-morphisms are the objects of C and the 2-morphisms are the morphisms of C. Composition is given by tensor product,



Note that $X \otimes (Y \otimes Z)$ is not equal to $(X \otimes Y) \otimes Z$, but there is a natural isomorphism $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ satisfying a coherence condition (the pentagon axiom).

2-functors and 2-equivalence

A 2-functor between 2-categories assigns objects to objects, 1-morphisms to 1-morphisms, and 2-morphisms to 2-morphisms, preserving the 2-category structure strictly.[3]

DEFINITION A.17 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-***functor*** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and a family of functors

$$F_{A,B} : \mathscr{C}(A,B) \to \mathscr{D}(F_0A, F_0B),$$

indexed by the pairs of objects A, B in $\mathscr{C}$, satisfying the following conditions,

(a) F is a functor between the underlying 1-categories of $\mathscr{C}$ and $\mathscr{D}$,

(b) F preserves horizontal compositions of 2-morphisms.

[3] What we call 2-functor, 2-(natural transformation), etc. are often called strict 2-functor, strict 2-(natural transformation), etc.

The conditions (a) and (b) require that the diagrams

$$\begin{array}{ccc} \mathscr{C}(A,B)\times\mathscr{C}(B,C) & \xrightarrow{c} & \mathscr{C}(A,C) \\ \downarrow{\scriptstyle F_{A,B}\times F_{B,C}} & & \downarrow{\scriptstyle F_{A,C}} \\ \mathscr{D}(FA,FB)\times\mathscr{D}(FB,FC) & \xrightarrow{c} & \mathscr{D}(FA,FC) \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{u_A} & \mathscr{C}(A,A) \\ & {\scriptstyle u_{FA}}\searrow & \downarrow{\scriptstyle F_{A,A}} \\ & & \mathscr{D}(FA,FA) \end{array} \tag{147}$$

commute for all objects A, B, C in $\mathscr{C}$.

See Johnson and Yau 2021, 4.1.8

DEFINITION A.18 Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be a parallel pair of 2-functors of 2-categories. A 2-***(natural transformation)*** $\alpha : F \to G$ is a family of 1-morphisms

$$\alpha_A : FA \to GA$$

indexed by the objects A of $\mathscr{C}$ such that

(a) for each 1-morphism $f : A \to B$ in $\mathscr{C}$ the diagram

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\alpha_B} & GB \end{array}$$

commutes in $\mathscr{D}$, and

(b) for each 2-morphism $\theta : f \Rightarrow g$ in $\mathscr{C}(A,B)$, the diagram

$$\begin{array}{ccc} (Gf)\circ\alpha_A & = & \alpha_B\circ(Ff) \\ \downarrow{\scriptstyle (G\theta)*\mathrm{id}_{\alpha_A}} & & \downarrow{\scriptstyle \mathrm{id}_{\alpha_A}*(G\theta)} \\ (Gg)\circ\alpha_A & = & \alpha_B\circ(Fg) \end{array}$$

commutes in $\mathscr{D}(FA, GB)$.

Ibid., 4.2.11.

DEFINITION A.19 A 2-(natural transformation) $\alpha : F \to G$ is a 2-***(natural isomorphism)*** if there exists a 2-(natural transformation) $\beta : G \to F$ such that $\beta\alpha = \mathrm{id}_F$ and $\alpha\beta = \mathrm{id}_G$.

DEFINITION A.20 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-***equivalence*** of $\mathscr{C}$ and $\mathscr{D}$ is a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D} \ , \quad \eta : \mathrm{id}_{\mathscr{C}} \to GF, \quad \epsilon : FG \to \mathrm{id}_{\mathscr{D}},$$

where F and G are 2-functors and η and ϵ are 2-(natural isomorphisms). We call a 2-functor $F : \mathscr{C} \to \mathscr{D}$ a 2-***equivalence*** if it can be extended to such a system.

A 2-equivalence between 2-categories is precisely the same as a strict $\mathscr{C}at$-enriched equivalence. Ibid., 1.3.11, 6.2.12.

THEOREM A.21 (WHITEHEAD THEOREM FOR 2-EQUIVALENCES) *Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-functor $F : \mathscr{C} \to \mathscr{D}$ is a 2-equivalence if and only if*

(a) *the underlying functor on the* 1*-categories is an equivalence, and*

(b) *F is fully faithful on 2-morphisms.*

PROOF See Johnson and Yau 2021, Theorem 7.5.8 and Definition 7.0.1. □

Condition (b) in the theorem means that, for all objects A, B in $\mathscr{C}$, the functor

$$F_{A,B} : \mathscr{C}(A,B) \to \mathscr{D}(F_0A, F_0B)$$

is fully faithful.

A.22 The conditions in the theorem are equivalent to

(a) F_0 is surjective on isomorphism classes of objects, and

(b) for all objects A, B in $\mathscr{C}$, the functor

$$F_{A,B} : \mathscr{C}(A,B) \to \mathscr{D}(F_0A, F_0B)$$

is an isomorphism of 1-categories.

As the condition (b) suggests, a 2-equivalence is the analogue for 2-categories of an isomorphism of 1-categories. As we explain in the next subsection, the analogue of the more useful notion of an equivalence of 1-categories is that of a biequivalence.

Pseudofunctors and biequivalences

We need to relax some conditions in the last subsection. The notion of a pseudo widget is obtained from that of a widget by allowing certain equalities to be isomorphisms.

DEFINITION A.23 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A ***pseudofunctor*** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and a family of functors

$$F_{A,B} : \mathscr{C}(A,B) \to \mathscr{D}(F_0A, F_0B)$$

indexed by the pairs of objects A, B in $\mathscr{C}$ such that the diagrams (147) commute up to given natural isomorphisms. In detail, there are given natural isomorphisms

$$\gamma_{A,B,C} : c\circ(F_{A,B} \times F_{B,C}) \Rightarrow F_{A,C}\circ c$$
$$\delta_A : u_{FA} \Rightarrow F_{A,A}\circ u_A$$

for all objects A, B, C of $\mathscr{C}$, satisfying the following conditions:

(a) (associativity) for every triple of 1-morphisms in $\mathscr{C}$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

the diagram

$$\begin{array}{ccc} Fh\circ Fg\circ Ff & \xrightarrow{\mathrm{id}_{Fh}*\gamma_{f,g}} & Fh\circ F(g\circ f) \\ \downarrow{\scriptstyle \gamma_{g,h}*\mathrm{id}_{Ff}} & & \downarrow{\scriptstyle \gamma_{g\circ f,h}} \\ F(h\circ g)\circ Ff & \xrightarrow{\gamma_{f,h\circ g}} & F(h\circ g\circ f) \end{array}$$

commutes;

(b) (unity) for every 1-morphism $f : A \to B$ in $\mathscr{C}$,

$$\gamma_{\mathrm{id}_A,f} \circ (\mathrm{id}_{Ff} * \delta_A) = \mathrm{id}_{Ff} = \gamma_{f,\mathrm{id}_B} \circ (\delta_A * \mathrm{id}_{Ff}).$$

See Borceux 1994a, 7.5.1; Johnson and Yau 2021, 4.1.2).

A.24 Pseudo-functors preserve internal (adjoint) equivalences: if (f, g, η, ϵ) is an internal equivalence and F is a pseudofunctor, then Ff and Fg are adjoint members of an internal equivalence (Johnson and Yau 2021, 6.2.3).

DEFINITION A.25 Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be two pseudofunctors of 2-categories. A ***pseudo natural transformation*** $\alpha : F \Rightarrow G$ is a family of 1-morphisms

$$\alpha_A : FA \to GA$$

indexed by the objects A of $\mathscr{C}$ and a family of invertible 2-morphisms

$$\alpha_f : G(f) \circ \alpha_A \;\Rightarrow\; \alpha_B \circ F(f)$$

indexed by the 1-morphisms f of $\mathscr{C}$. The families are required to satisfy certain coherence conditions, ibid., 4.2.1. An invertible pseudo natural transformation is called a ***pseudo natural isomorphism***.

A pseudo natural transformation $F \Rightarrow G$ is invertible if and only if each component $F(A) \to G(A)$ is an invertible 1-morphism in $\mathscr{D}$ (ibid., 6.2.16).

Pseudo natural transformations are sometimes called strong transformations.

DEFINITION A.26 Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be pseudofunctors of 2-categories, and let $\alpha, \beta : F \Rightarrow G$ be pseudo natural transformations. A ***modification*** $\Gamma : \alpha \rightsquigarrow \beta$ is a family of 2-morphisms

$$\Gamma_A : \alpha_A \Rightarrow \beta_A,$$

indexed by the objects of $\mathscr{C}$, such that the diagram

$$\begin{array}{ccc} (Gf)\alpha_A & \xrightarrow{Gf * \Gamma_A} & (Gf)\beta_A \\ \downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle \beta_f} \\ \alpha_B(Ff) & \xrightarrow{\Gamma_B * Ff} & \beta_B(Ff) \end{array}$$

commutes in $\mathscr{D}$ (ibid., 4.4.1, 4.4.3).

DEFINITION A.27 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories with $\mathscr{C}$ small. There exists a 2-category $\mathscr{PsFun}(\mathscr{C}, \mathscr{D})$ with

- objects the pseudofunctors $\mathscr{C} \to \mathscr{D}$,
- 1-morphisms the pseudo natural transformations between such pseudofunctors,
- 2-morphisms the modifications between such pseudo natural transformations.

Ibid., 4.4.13, where the category is denoted $\mathsf{Bicat}^{\mathrm{ps}}(\mathscr{C}, \mathscr{D})$.

DEFINITION A.28 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A ***biequivalence*** of $\mathscr{C}$ and $\mathscr{D}$ is a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}. \qquad \mathrm{id}_{\mathscr{C}} \to GF, \quad FG \to \mathrm{id}_{\mathscr{D}},$$

where F and G are pseudofunctors and $\mathrm{id}_{\mathscr{C}} \to GF$ and $FG \to \mathrm{id}_{\mathscr{D}}$ are pseudo natural transformations that are equivalences as 1-morphisms in $\mathscr{PsFun}(\mathscr{C}, \mathscr{C})$ and $\mathscr{PsFun}(\mathscr{D}, \mathscr{D})$ respectively (A.12). We call a pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ a ***biequivalence*** if it can be extended to such a system (ibid., 6.2.8).

A biequivalence is sometimes called an ***equivalence of*** 2***-categories***. It is the correct analogue of an equivalence of 1-categories.

A.29 When $\mathscr{C}$ and $\mathscr{D}$ are small, the internal equivalences in $\mathscr{PS}\mathscr{F}un(\mathscr{C},\mathscr{C})$ are pseudo natural isomorphisms (ibid., 6.2.7). In this case, we can define a biequivalence of $\mathscr{C}$ and $\mathscr{D}$ to be a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}. \qquad \mathrm{id}_{\mathscr{C}} \to GF, \quad FG \to \mathrm{id}_{\mathscr{D}}$$

with $\mathrm{id}_{\mathscr{C}} \to GF$ and $FG \to \mathrm{id}_{\mathscr{D}}$ pseudo natural isomorphisms.

THEOREM A.30 (WHITEHEAD THEOREM FOR BIEQUIVALENCES) *Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ is a biequivalence if and only if*

(a) *F_0 is surjective on equivalence classes of objects, and*

(b) *for all objects A, B of $\mathscr{C}$, the component functor*

$$F_{A,B} : \mathscr{C}(A,B) \to \mathscr{D}(F_0A, F_0B)$$

is an equivalence of 1*-categories.*

PROOF See Johnson and Yau 2021, Theorem 7.4.1 and Definition 7.0.1. □

A.31 The conditions (a) and (b) in the theorem are equivalent to the following conditions:

(a) F is surjective on equivalence classes of objects,

(b) F is surjective on isomorphism classes of 1-morphisms (between any two objects),

(c) F is bijective on 2-morphisms (between any two 1-morphisms).

REMARK A.32 (a) When the F in the theorem is a 2-functor, there need not exist a 2-functor G satisfying the conditions in A.28, only a pseudofunctor (ibid., 6.2.11).

(b) When F is a biequivalence, the underlying functor on 1-categories need not be full, faithful, or essentially surjective.

(c) Let $F : \mathscr{C} \to \mathscr{D}$ be a 2-functor of 2-categories. If F is a 2-equivalence, then F_0 is surjective on isomorphism classes of objects; if it is a biequivalence, it only required to be surjective on equivalence classes of objects.

TODO 2 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories whose objects are categories and whose morphisms are functors. It seems to me that to give a biequivalence $\mathscr{C} \to \mathscr{D}$ it should suffice to give 2-functors F, G and natural equivalences of categories $\mathsf{C} \to GF(\mathsf{C})$ and $FG(\mathsf{D}) \to \mathsf{D}$.

NOTES The Whitehead theorems are folklore. They were named by Johnson and Yau 2021 in analogy with the Whitehead theorem in homotopy. They are also proved in Gabber and Ramero 2018, 2.4.30.

Appendix B

Ind categories

Basic definitions

B.1 A set with a partial order $\leq$ is ***directed*** if, for every pair of elements a, b, there exists an element c such that $a, b \leq c$. A category is ***filtered*** if, for every pair i, j of objects, there exists an object k and morphisms $i \to k$, $j \to k$, and for every parallel pair of morphisms $u, v : i \rightrightarrows j$ there exists a morphism $w : j \to k$ such that $w \circ u = w \circ v$. A directed set can be viewed as a filtered category in an obvious way.

B.2 Let C be an essentially small category. An ind-object in C is a functor $\alpha \rightsquigarrow X_\alpha : \mathsf{A} \to \mathsf{C}$, where A a small filtered category. Given two ind-objects $\alpha \rightsquigarrow X_\alpha$ and $\beta \rightsquigarrow Y_\beta$, we set

$$\mathrm{Hom}((X_\alpha), (Y_\beta)) = \varprojlim_\alpha \varinjlim_\beta \mathrm{Hom}(X_\alpha, Y_\beta). \tag{148}$$

In this way, we obtain the category $\mathrm{Ind}\,\mathsf{C}$ of ind-objects of C. The same category is obtained when A is required to be a directed set (Adámek and Rosický 1994, 1.5). The functor ι_C sending an object of C to a constant inductive system in C is fully faithful. The category $\mathrm{Ind}\,\mathsf{C}$ is essentially small (Kashiwara and Schapira 2006, 6.1.2).

B.3 The functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim h_{X_\alpha}$$

is an equivalence of $\mathrm{Ind}\,\mathsf{C}$ with the category of functors $\mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$ that are small filtered inductive limits of representable functors. This allows us to identify $\text{“}\varinjlim\text{”} X_\alpha$ with its limit in $\mathsf{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Set})$, and $\mathrm{Ind}\,\mathsf{C}$ with the full subcategory of $\mathsf{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Set})$ of limits of small filtered inductive systems of representable functors.

When C has finite inductive limits, a functor $\mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$ lies in $\mathrm{Ind}\,\mathsf{C}$ if and only if it is left exact and satisfies a certain smallness condition, so

$$\mathsf{C} \approx \mathsf{Lex}_k(\mathsf{C}_0^{\mathrm{opp}}, \mathsf{Set}),$$

(Kashiwara and Schapira 2006, 6.1.7).

B.4 A functor $F : \mathsf{C} \to \mathsf{D}$ has a unique extension to a functor of ind-objects,

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{F} & \mathsf{D} \\ \downarrow{\scriptstyle \iota_\mathsf{C}} & & \downarrow{\scriptstyle \iota_\mathsf{D}} \\ \mathrm{Ind}\,\mathsf{C} & \xrightarrow{\mathrm{Ind}\,F} & \mathrm{Ind}\,\mathsf{D}, \end{array}$$

commuting with small filtered inductive limits. Moreover,

(a) this construction respects composition,

$$\mathrm{Ind}(G\circ F) = \mathrm{Ind}(G)\circ \mathrm{Ind}(F);$$

(b) if F is faithful (resp. fully faithful), then so also in $\mathrm{Ind}\, F$.

See Kashiwara and Schapira 2006, 6.1.9, 6.1.10, 6.1.11.

B.5 When C has finite inductive limits, $\mathrm{Ind}\,\mathsf{C}$ has small inductive limits and the functor $i_{\mathsf{C}} : \mathsf{C} \to \mathrm{Ind}\,\mathsf{C}$ is right exact, i.e., commutes with finite inductive limits. If, in addition, C admits finite projective limits, then small filtered inductive limits are exact in $\mathrm{Ind}\,\mathsf{C}$.
See Kashiwara and Schapira 2006, 6.1.6, 6.1.18, 6.1.19.

An object X of a category C with filtered inductive limits is ***finitely presentable*** if $\mathrm{Hom}(X, -)$ commutes with small filtered inductive limits, i.e.,

$$\mathrm{Hom}(X, \varinjlim_{\alpha} Y_\alpha) \simeq \varinjlim_{\alpha} \mathrm{Hom}(X, Y_\alpha)$$

for every functor $\alpha \mathrel{\rightsquigarrow} Y_\alpha : \mathsf{A} \to \mathsf{C}$ with A a small filtered category.

PROPOSITION B.6 *Let C be an additive category with finite inductive limits. If C is idempotent complete, then the essential image of ι_{C} consists of the finite presented objects of* $\mathrm{Ind}\,\mathsf{C}$. *Hence, ι_{C} defines an equivalence*

$$\mathsf{C} \xrightarrow{\approx} (\mathrm{Ind}\,\mathsf{C})^{\mathrm{fp}}.$$

PROOF Almost by definition, the objects of C are finitely presentable when regarded as objects of $\mathrm{Ind}\,\mathsf{C}$. For the converse, let X be a finitely presentable object of $\mathrm{Ind}\,\mathsf{C}$, and write $X = \varinjlim_{\alpha} \iota X_\alpha$ with X_α in C. As X is finitely presentable,

$$\mathrm{Hom}(X, X) = \mathrm{Hom}(X, \varinjlim_{\alpha} \iota X_\alpha) \simeq \varinjlim_{\alpha} \mathrm{Hom}(X, \iota X_\alpha).$$

In particular, id_X factors through some ιX_α,

$$\mathrm{id}_X = (X \xrightarrow{f} \iota X_\alpha \xrightarrow{g} \varinjlim X_\alpha = X).$$

Let $e = f\circ g : \iota X_\alpha \to \iota X_\alpha$. Then e is an idempotent endomorphism of X_α in $\mathrm{Ind}\,\mathsf{C}$, hence in C because ι_{C} is faithful. As C is idempotent complete, e splits in C: there exist morphisms $p : X_\alpha \to Y$ and $i : Y \to X_\alpha$ in C such that $p\circ i = \mathrm{id}_Y$ and $i\circ p = e$. The factorization of id_X through X_α identifies X_α with Y, and so X lies in the essential image of ι_{C}. □

Ind categories of abelian categories

B.7 Recall that an essentially small abelian category is a ***Grothendieck category*** if small inductive limits exist, small filtered inductive limits are exact, and there exists a generator.

(a) Grothendieck categories admit small projective limits.

(b) If C is Grothendieck, then every functor $\mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$ commuting with small projective limits is representable.

(c) If C is Grothendieck, then every functor $\mathsf{C} \to \mathsf{C}'$ commuting with small inductive limits admits a right adjoint.

(d) Every Grothendieck category admits enough injectives.

See Kashiwara and Schapira 2006, 8.3.27, 9.6.2.

B.8 Let C be an essentially small abelian category. Then $\operatorname{Ind}\mathsf{C}$ is an abelian category and the fully faithful functor $\iota_{\mathsf{C}} : \mathsf{C} \to \operatorname{Ind}\mathsf{C}$ is exact. In particular, C is an abelian subcategory of $\operatorname{Ind}\mathsf{C}$ (i.e., closed under taking kernels and cokernels). Small inductive limits exist in $\operatorname{Ind}\mathsf{C}$, and small filtered inductive limits are exact. See Kashiwara and Schapira 2006, 8.6.5.

B.9 Let C be an essentially small abelian category. If S is any set of objects of C such that every object of C is isomorphic to one in S, then S generates $\operatorname{Ind}\mathsf{C}$. To see this, note that every object X of $\operatorname{Ind}\mathsf{C}$ can be written $X = \varinjlim_{i\in I} X_i$ with the X_i in S and, almost by definition, $\varinjlim_{i\in I} X_i$ is a quotient of $\bigoplus_{i\in I} X_i$. In particular $\operatorname{Ind}\mathsf{C}$ has a set of generators, and is therefore a Grothendieck category.

Recall that a ***subobject*** of an object X is an equivalence class of monomorphisms with target X. A category is ***well-powered*** if the subobjects of any object form a set, in which case they form a partially ordered set. For example, Grothendieck categories are well-powered. An object of such a category is ***artinian*** (resp. ***noetherian***) if the set of its subobjects satisfies the descending chain condition (resp. ascending chain condition).

PROPOSITION B.10 *Let C be an essentially small abelian category whose objects are noetherian.*

(a) *Every object X of $\operatorname{Ind}\mathsf{C}$ is of the form "$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is filtered inductive system in C whose transition morphisms are monomorphisms.*

(b) *The category C is stable under subquotients in $\operatorname{Ind}\mathsf{C}$, i.e., if X is in C and Z is a subquotient of X in $\operatorname{Ind}\mathsf{C}$, then Z is in C.*

(c) *Let $X =$"$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is an inductive system as in (a). Each X_α is a subobject of X, and the set of X_α is cofinal in the collection of all subobjects of X.*

(d) *An object of $\operatorname{Ind}\mathsf{C}$ is finitely presentable if and only if it is noetherian.*

PROOF (a). Let $X =$"$\varinjlim$"X_α. We may suppose that the indexing category is a directed set (B.2). For each α, the kernels of the morphisms $X_\alpha \to X_\beta$, $\beta > \alpha$, form an increasing system of subobjects of X_α, which is stationary because X_α is noetherian. Let $K_\alpha = \operatorname{Ker}(X_\alpha \to X_\beta)$ for all β sufficiently large. The canonical morphism

$$\text{“}\varinjlim\text{”}X_\alpha \to \text{“}\varinjlim\text{”}(X_\alpha/K_\alpha)$$

is an isomorphism, and the inductive system $(X_\alpha/K_\alpha)_\alpha$ satisfies (a).

(b) Because C is an abelian subcategory of $\operatorname{Ind}\mathsf{C}$, quotients in C agree with quotients in $\operatorname{Ind}\mathsf{C}$. Therefore, C is stable under subquotients if it is stable under subobjects. Let $Y \in \operatorname{ob}\mathsf{C}$, and let X be a subobject of Y in $\operatorname{Ind}\mathsf{C}$. Write $X =$"$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is an inductive system as in (a). For each α, the inductive limit of the monomorphisms $X_\alpha \to X_\beta$, $\beta > \alpha$, is a monomorphism $X_\alpha \to X$ (exactness of inductive limits in $\operatorname{Ind}\mathsf{C}$; see B.8). Thus the X_α are subobjects of Y. As Y is noetherian, they form a stationary system, and so $X = X_\alpha \in \operatorname{ob}\mathsf{C}$ for α sufficiently large.

(c) As in the proof of (b), the morphisms $X_\alpha \to X$ are monic, and so the X_α are subobjects of X. For any other subobject Z of X in C, we have

$$\mathrm{Hom}(Z,X) = \varinjlim_\alpha \mathrm{Hom}(Z,X_\alpha),$$

and so $Z \to X$ factors through X_α for some α.

(d) Every finitely presentable object of $\mathrm{Ind}\,\mathsf{C}$ is in the essential image of $\iota_{\mathsf{C}} : \mathsf{C} \to \mathrm{Ind}\,\mathsf{C}$ (see B.6), hence is noetherian.

For the converse, let N be a noetherian object of $\mathrm{Ind}\,\mathsf{C}$ and $(X_\alpha)_\alpha$ a filtered inductive sytem in $\mathrm{Ind}\,\mathsf{C}$. We have to show that the canonical map

$$\varinjlim_\alpha \mathrm{Hom}(N,X_\alpha) \to \mathrm{Hom}(N, \varinjlim_\alpha X_\alpha) \qquad (*)$$

is bijective. Let $X = \varinjlim_\alpha X_\alpha$.

For the surjectivity, let f be a morphism $N \to X$, and consider the diagram

$$\begin{array}{ccc} N & \xrightarrow{\ f\ } & X \\ \uparrow & & \uparrow{\scriptstyle i_\alpha} \\ N \times_X X_\alpha & \xrightarrow{\ f_\alpha\ } & X_\alpha. \end{array}$$

Because filtered inductive limits are exact in $\mathrm{Ind}\,\mathsf{C}$, $f = \varinjlim_\alpha f_\alpha$. Clearly,

$$\mathrm{Im}(f) \subset \bigcup_\alpha \mathrm{Im}(i_\alpha \circ f_\alpha) \quad \text{(filtered union of subobjects)},$$

and so $\mathrm{Im}(f) \subset \mathrm{Im}(i_\alpha \circ f_\alpha)$ for some α (because N, hence $\mathrm{Im}(f)$, is noetherian). For such an α, f factors through X_α, and so arises from an element of $\varinjlim \mathrm{Hom}(N,X_\alpha)$. Hence (*) is surjective.

For the injectivity, let $(f_\alpha)_\alpha$ be an inductive system of morphisms $N \to X_\alpha$ such that $\varinjlim_\alpha f_\alpha = 0$. Because $\varinjlim_\alpha f_\alpha = 0$, the filtered union $\bigcup_\alpha \mathrm{Ker}(f_\alpha) = N$, and because N is noetherian, $\mathrm{Ker}(f_\alpha) = N$ for some α. This means that $(f_\alpha)_\alpha = 0$. Hence (*) is injective. □

PROPOSITION B.11 *Let* D *be an abelian category and* C *a full subcategory stable under finite sums and subquotients in* D. *Assume that* C *is essentially small and that* D *is well-powered. If*

(a) *all objects of* C *are noetherian,*

(b) *small filtered inductive limits exist in* D *and are exact, and*

(c) *every object of* D *is a small filtered inductive limit of objects of* C,

then the functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim X_\alpha : \ \mathrm{Ind}\,\mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an object X *of* D *to the inductive system of its subobjects in* C.

Loosely, we can say that D is the category of ind-objects of C.

PROOF As in the proof of B.10(a),(c), we can deduce from (c) that each X in D is a small filtered inductive limit of subobjects X_α in C. Moreover, the X_α are cofinal in the collection of all subobjects of X in C. To see this, let $Z \in \operatorname{ob}\mathsf{C}$ be a subobject of X; on passing to the limit in

$$0 \to Z \cap X_\alpha \to X_\alpha \to X/Z,$$

we find that the $Z \cap X_\alpha$ have inductive limit Z; as Z is noetherian, they form a stationary system, and so this means that Z is contained in X_α for all sufficiently large α. We have shown that X is the inductive limit of its subobjects in C, and so the composite

$$\mathsf{D} \to \operatorname{Ind}\mathsf{C} \to \mathsf{D}$$

is naturally isomorphic to the identity functor. Conversely, every X in $\operatorname{Ind}\mathsf{C}$ is a "$\varinjlim$"X_α as in B.10(a) and the X_α are cofinal in the set of subobjects in C of $\varinjlim X_\alpha$; therefore the composite

$$\operatorname{Ind}\mathsf{C} \to \mathsf{D} \to \operatorname{Ind}\mathsf{C}$$

is naturally isomorphic to the identity functor. □

We list some examples where Proposition B.11 implies that D is equivalent to $\operatorname{Ind}\mathsf{C}$. Throughout k is a field.

EXAMPLE B.12 Let R be a noetherian ring. Take C to be the category $\mathsf{Modf}(R)$ of finitely generated R-modules and D to be the category $\mathsf{Mod}(R)$ of all R-modules. The conditions of B.11 are obviously satisfied. More generally, we can replace R with a noetherian scheme.

EXAMPLE B.13 Let L be a coalgebra over a noetherian ring R. If L is flat over R, then the category $\mathsf{coMod}(L)$ of right L-comodules is abelian. Moreover, every L-comodule is the filtered union of the L-subcomodules finitely generated over R (Serre 1968, Cor. 2). Therefore, we can take D to be $\mathsf{coMod}(L)$ and C to be the subcategory of those finitely generated over R.

EXAMPLE B.14 Let G be an affine group scheme over k. Take C to be the category $\mathsf{Repf}(G)$ of representations of G on finite-dimensional k-vector spaces, and D to be the category $\mathsf{Rep}(G)$ of representations of G on arbitrary k-vector spaces. Condition (c) of B.11 says that every representation is the inductive limit of its finite-dimensional subrepresentations (II, 1.17).

EXAMPLE B.15 Take C to be an abelian category whose objects are noetherian, and D to be the category of additive left exact functors from $\mathsf{C}^{\mathrm{opp}}$ to the category of abelian groups.

DEFINITION B.16 An abelian category is ***locally noetherian*** if small inductive limits exist, filtered inductive limits are exact, and there exists a family of noetherian generators with small index set.

THEOREM B.17 (GABRIEL 1962) *Let* C *be an essentially small abelian category whose objects are noetherian. There exists a locally noetherian abelian category* D *such that* C *is equivalent to the category of noetherian objects in* D*. Moreover, the conditions determine* D *up to an equivalence preserving* C*.*

PROOF Indeed, $\operatorname{Ind}\mathsf{C}$ has these properties. □

Extension of scalars

B.18 Let D be a k-linear abelian category with arbitrary small direct sums. For X in D and V a k-vector space, the functor $T \rightsquigarrow \mathrm{Hom}(V, \mathrm{Hom}(X,T))$ is representable, and we let $V \otimes X$ denote the object in D representing it, so

$$\mathrm{Hom}_{\mathsf{D}}(V \otimes X, T) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_{\mathsf{D}}(X,T)).$$

If $(e_i)_{i\in I}$ is a basis for V (possibly infinite), then $V \otimes X$ is a direct sum of copies of X indexed by I.

B.19 Let k be a field and k' an extension of k. Suppose that we are given a pair of k-linear abelian categories $\mathsf{C} \subset \mathsf{D}$ satisfying the conditions of Proposition B.11, a pair of k'-linear abelian categories $\mathsf{C}' \subset \mathsf{D}'$ satisfying the same conditions, and an adjoint pair of k-linear functors

$$\mathsf{D} \underset{\text{restriction}}{\overset{\text{extension}}{\rightleftarrows}} \mathsf{D}'$$

(extension of scalars, restriction of scalars). We have a diagram

$$\begin{array}{ccc} \mathsf{C} & \longrightarrow & \mathsf{C}' \\ \big\downarrow & & \big\downarrow \\ \mathsf{D} & \underset{\text{restriction}}{\overset{\text{extension}}{\rightleftarrows}} & \mathsf{D}' \end{array}$$

The functor restriction of scalars induces a functor

$$\mathsf{D}' \to \mathsf{D}_{k'}, \tag{149}$$

where $\mathsf{D}_{k'}$ is the category of objects of D equipped with k-linear k'-module structure. Such a structure can be interpreted as a morphism $k' \otimes_k X \to X$ such that certain diagrams commute. Let $X \in \mathrm{ob}\,\mathsf{D}_{k'}$. We say that $Y \subset X$ generates X as a k'-module if the composite

$$k' \otimes_k Y \to k' \otimes_k X \to X$$

is an epimorphism, i.e., any k'-submodule of X containing Y equals X. We let $\mathsf{C}_{(k')}$ denote the category of objects of $\mathsf{D} \cong \mathrm{Ind}\,\mathsf{C}$ equipped with a k-linear k'-module structure and generated as a k'-module by a subobject in C.

PROPOSITION B.20 *In the above situation, suppose*

(a) *that the functor (149) is an equivalence;*

(b) *the extension of scalars* $\mathsf{D} \to \mathsf{D}'$ *sends* C *into* C'.

Then the functor restriction of scalars induces an equivalence $\mathsf{C}' \cong \mathsf{C}_{(k')}$.

PROOF Identify D with $\mathrm{Ind}\,\mathsf{C}$, and, by (149), D' with the category of Ind-objects of C equipped with a k'-module structure. With these identification, the restriction of scalars functor becomes the forgetful functor. Its left adjoint (extension of scalars) becomes $X \rightsquigarrow k' \otimes_k X$. It remains to determine C'.

If $X \in \mathrm{ob}\,\mathsf{D}'$ is generated as a k'-module by a D-subobject $Y \subset X$ with $Y \in \mathrm{ob}\,\mathsf{C}$, then it is a quotient of $k' \otimes_k Y$ and is therefore in C'. Conversely, if $X \in \mathrm{ob}\,\mathsf{C}'$, we write it in D as a filtered union of its subobjects X_α in C. Since X is noetherian in C', the system of submodule images of the $k' \otimes_k X_\alpha$ in X is stationary. Therefore X is a quotient of $k' \otimes_k X_\alpha$ for α sufficiently large, and so lies in $\mathsf{C}_{(k')}$. □

EXAMPLE B.21 We list some examples where Proposition B.20 implies that $\mathsf{C}' \to \mathsf{C}_{(k')}$.

(a) Let X be a scheme of finite type over k and X' the scheme over k' deduced from X by extension of scalars. Take C and C' to be the categories of coherent sheaves on X and X' and D and D' the categories of quasi-coherent sheaves. In this case, "restriction of scalars" is direct image by the morphism $X' \to X$.

(b) Let G be an algebraic group over k and G' the group scheme over k' deduced from G by extension of scalars. Take $\mathsf{C} = \mathsf{Repf}(G)$ and $\mathsf{C}' = \mathsf{Repf}(G')$, and let D and D' be as in B.14.

Pro categories

B.22 On reversing the arrows, we get the notion of a pro-object. A pro-object in a category C is a small filtered projective system $\alpha \rightsquigarrow X_\alpha$, and

$$\mathrm{Hom}((X_\alpha),(Y_\beta)) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}(X_\alpha, Y_\beta).$$

If C is abelian, so also is $\mathrm{Pro}\,\mathsf{C}$, and the canonical functor $\mathsf{C} \to \mathrm{Pro}\,\mathsf{C}$ is exact. The opposites (duals) of Propositions B.10 and B.11 hold for abelian categories C whose objects are artinian.

EXAMPLE B.23 Let $\mathsf{C} = \mathsf{Vecf}(k)$, the category of finite-dimensional vector spaces over k. Then $\mathrm{Ind}\,\mathsf{C}$ is the category of all vector spaces over k and $\mathrm{Pro}\,\mathsf{C}$ is the category of linearly compact vector spaces over k. Duality is an antiequivalence betuween $\mathrm{Ind}\,\mathsf{C}$ and $\mathrm{Pro}\,\mathsf{C}$.

EXAMPLE B.24 Let k be a field. Take C to be the category of algebraic group schemes over k, and D to be the category of all affine group schemes over k. The functor

$$\text{“}\varprojlim\text{”} G_\alpha \rightsquigarrow \varprojlim G_\alpha : \ \mathrm{Pro}\,C \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an affine group scheme G to the projective system of its algebraic quotients (the system $(G/N)_N$, where runs over the normal subgroup schemes of G such that G/N is algebraic). Every affine group scheme over k is the limit of a filtered projective system of algebraic group schemes over k whose transition morphisms are faithfully flat.

If $G = \varprojlim G_\alpha$, we view its Lie algebra

$$\mathrm{Lie}(G) = \varprojlim \mathrm{Lie}(G_\alpha)$$

as a pro Lie algebra. If $G = \mathrm{Spec}(A)$ and I is ideal of $f \in A$ zero at the origin, the linearly compact vector space underlying $\mathrm{Lie}\,G$ is the dual of I/I^2.

NOTES The exposition in this section is largely based on Deligne 1989 §4.

Appendix C

Torsors, gerbes, and nonabelian cohomology

We review some definitions from Giraud 1971. Throughout, we work with affine schemes. This allows us to sheafify (basically bounded) presheaves for the fpqc topology without having to pass to a larger universe.

1 Fibred categories

Definition

Let A be a small category and $\phi : \mathsf{F} \to \mathsf{A}$ a functor. For an object S of A, the ***fibre*** of F at S is the category F_S whose objects are the X in F such that $\phi(X) = S$ and whose arrows are the f in F such that $\phi(f) = \mathrm{id}_S$. For an arrow $a : \phi(Y) \to \phi(X)$, we let $\mathrm{Hom}_a(Y,X)$ denote the set of $f : Y \to X$ such that $\phi(f) = a$; also $\mathrm{Hom}_T(Y,X) = \mathrm{Hom}_{\mathrm{id}_T}(Y,X)$.

Let $a : T \to S$ be an arrow in A and let $f \in \mathrm{Hom}_a(Y,X)$. We say f is ***cartesian***, and that (Y,f) is an ***inverse image*** X relative to a, written $Y = a^*X$, if, for every $Z \in \mathrm{ob}\,\mathsf{F}_T$ and $g \in \mathrm{Hom}_a(Z,X)$, there exists a unique $h \in \mathrm{Hom}_T(Z,Y)$ such that $f \circ h = g$:

$$\begin{array}{ccc} Z & & \\ \vdots_{h} \searrow^{g} & & \\ Y & \xrightarrow{f} & X \\ \\ T & \xrightarrow{a} & S. \end{array}$$

In other words, if composition with f is an isomorphism $\mathrm{Hom}_{\mathrm{id}_T}(Z, a^*X) \simeq \mathrm{Hom}_a(Z,X)$, natural in $Z \in \mathrm{ob}\,\mathsf{F}_T$. When it exists, the inverse image is uniquely determined up to a unique isomorphism.

The functor $\phi : \mathsf{F} \to \mathsf{A}$ is a ***fibred category*** [1] if

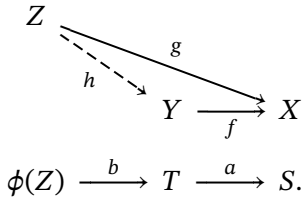
(a) (existence of inverse images) for every arrow $a : T \to S$ in A and $X \in \mathrm{ob}(\mathsf{F}_S)$, an inverse image a^*X exists, and

(b) (transitivity of inverse images) given arrows $U \xrightarrow{b} T \xrightarrow{a} S$ in A and $X \in \mathrm{ob}\,\mathsf{F}_S$, $b^*(a^*X)$ is an inverse image of X relative to $a \circ b$.

[1] Literally, a category furnished with fibres. Also called a (Grothendieck) ***fibration*** over A.

When we choose an inverse image a^*X of each X in F_S, a^* becomes a functor $F_S \to F_T$.

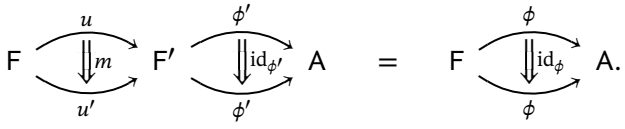
In fibred categories, cartesian arrows have a more general universal property. In such a category, let $f : Y \to X$ be a cartesian morphism over $a : T \to S$. For any morphism $g : Z \to X$ such that $\phi(g)$ factors as $b \circ a$, there exists a unique morphism $h : Z \to Y$ such that $\phi(h) = b$ and $f \circ h = g$,



(Borceux 1994b, 8.1.6, 8.1.7). Some authors use this as their definition of cartesian. In a fibred category, the two notions are equivalent.

Cartesian functors

Let $\phi : \mathsf{F} \to \mathsf{A}$ and $\phi' : \mathsf{F}' \to \mathsf{A}$ be fibred categories over A. A functor $u : \mathsf{F} \to \mathsf{F}'$ such that $\phi' \circ u = \phi$ is ***cartesian*** if it preserves cartesian morphisms, When u, u' are cartesian functors $\mathsf{F} \to \mathsf{F}'$, a ***cartesian natural transformation*** is a natural transformation $m : u \to u'$ such that $\mathrm{id}_{\phi'} * m = \mathrm{id}_{\phi}$,



We write $\mathsf{Cart}(\mathsf{F}, \mathsf{F}')$ for the category whose objects are the cartesian functors $\mathsf{F} \to \mathsf{F}'$ and whose morphisms are the cartesian natural transformations.

It is obvious that the fibred categories over base category A, together with the cartesian functors and cartesian natural transformations, form a 2-category $\mathcal{F}ib(\mathsf{A})$.

A ***cleavage*** of a fibred category $\phi : \mathsf{F} \to \mathsf{A}$ is the choice of a cartesian lifting $Y \to X$ for every arrow $a \to b$ in A and X with $\phi(X) = a$. A cleavage is a ***splitting*** if it contains the identity morphisms and is closed under composition, i.e., such that the objects of F and the distinguished cartesian arrows form a subcategory of F. A ***split*** fibred category is one equipped with a splitting. In general, a fibred category does not admit a splitting, but it is always equivalent to one that does (Borceux 1994b, 8.3.4; Vistoli 2005, 3.14, 3.45).

Let $\phi : \mathsf{F} \to \mathsf{A}$ be a fibred category. When we choose a cleavage, every morphism $a : T \to S$ in A defines a functor $a^* : \mathsf{F}_S \to \mathsf{F}_T$, and by universality, there are functorial isomorphisms

$$\mathrm{id}_S \to \mathrm{id}_S^*, \quad b^* \circ a^* \to (a \circ b)^*$$

for all objects S of A and composable morphisms a, b in A. In this way, we get a pseudofunctor $\mathsf{A}^{\mathrm{opp}} \to \mathcal{C}at$. Every such pseudofunctor arises from a fibred category, and every fibred category over A is isomorphic to one defined by a pseudofunctor. More precisely, there are a canonical 2-equivalences of 2-categories

$$\mathcal{F}ib(\mathsf{A}) \overset{\text{forget}}{\longleftarrow} \mathcal{F}ib^{\mathrm{cl}}(\mathsf{A}) \longrightarrow \mathcal{P}s\mathcal{F}un(\mathsf{A}^{\mathrm{opp}}, \mathcal{C}at)$$

under which split fibred categories correspond to 2-functors (Grothendieck's construction). Here $\mathcal{F}ib^{\mathrm{cl}}(\mathsf{A})$ has objects the fibred categories equipped with a cleavage. See Borceux 1994b, 8.3.1, or Johnson and Yau 2021, 9.1.9, 10.1.11, 10.6.16.

Thus, the theory of fibred categories sometimes allows us to deal with pseudofunctors without having to keep track of the natural isomorphims that measure the failure of certain diagrams to commute.

NOTES Fibred categories were introduced by Grothendieck in SGA 1 and developed in Giraud 1971. Our definitions are those of Giraud. For more recent accounts, see Borceux 1994b, Vistoli 2005, and Johnson and Yau 2021.

2 Sheaves for the fpqc topology

Let $S = \mathrm{Spec}(R)$ be an affine scheme, and let Aff_S denote the category of affine schemes over S. The ***fpqc topology*** on Aff_S is that for which the coverings are finite surjective families of flat morphisms $U_i \to U$ of affine S-schemes. A ***presheaf of sets***[2] on S is a contravariant functor $F : \mathsf{Aff}_S \to \mathsf{Set}$. A ***sheaf of sets*** on S is a presheaf satisfying the sheaf condition: for all coverings $(U_i \to U)_{i\in I}$, the sequence

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I\times I} F(U_i \times U_j)$$

is exact, i.e., the first arrow is the equalizer of the parallel pair. More concretely, a sheaf of sets for the fpqc topology on Aff_S is a functor $F : \mathrm{Alg}_R \to \mathsf{Set}$ such that

(a) $F(R_1 \times R_2) = F(R_1) \times F(R_2)$, and

(b) for any faithfully flat homomorphism $R \to R'$, the arrow $F(R) \to F(R')$ is the equalizer of the parallel pair of arrows $F(R') \rightrightarrows F(R' \otimes_R R')$ defined by $a \mapsto a \otimes 1, 1 \otimes a$.

For any S-scheme X, the functor $T \rightsquigarrow h_X(T) \overset{\text{def}}{=} \mathrm{Hom}_S(T, X)$ is a sheaf. Indeed, (a) is obvious, and (b) follows from the exactness of $R \to R' \rightrightarrows R' \otimes_R R'$ (Waterhouse 1979, 13.1).

Let G be a sheaf of groups. A ***sheaf torsor*** under G over S is a sheaf P of sets together with a right action of G such that

(a) for all maps $U \to S$ such that $P(U)$ is nonempty, $G(U)$ acts simply transitively on $P(U)$;

(b) for some covering $(U_i \to S)_i$, every set $P(U_i)$ is nonempty.

A ***morphism*** of G-torsors is a morphism of sheaves of sets with an action of G.

3 Stacks (Champs)

Let S be an affine scheme, and let $\phi : \mathsf{F} \to \mathsf{Aff}_S$ be a fibred category over Aff_S. In the following, we implicitly choose a cleavage.

[2] In order to be sure that associated sheaves exist, we should consider only basically bounded presheaves; see Waterhouse 1975.

Descent data

Let $a : V \to U$ be a faithfully flat morphism of affine S-schemes, and let $F \in \mathrm{ob}(\mathsf{F}_U)$. A ***descent datum*** on F relative to a is an isomorphism

$$u : \ \mathrm{pr}_1^*(F) \to \mathrm{pr}_2^*(F)$$

over $V \times_U V$ satisfying the "cocycle" condition

$$\mathrm{pr}_{31}^*(u) = \mathrm{pr}_{32}^*(u) \circ \mathrm{pr}_{21}^*(u)$$

over $V \times_U V \times_U V$,

$$\mathrm{pr}_1^* F \xrightarrow[\mathrm{pr}_{21}^*(u)]{} \mathrm{pr}_2^* F \xrightarrow[\mathrm{pr}_{32}^*(u)]{} \mathrm{pr}_3^* F, \qquad \mathrm{pr}_{31}^*(u) : \mathrm{pr}_1^* F \to \mathrm{pr}_3^* F.$$

Here pr_i is the projection onto the ith factor and pr_{ji} is the projection

$$V \times_U V \times_U V \to V \times_U V$$

onto the (i, j)th factor. With the obvious notion of morphism, the pairs (F, u) form a category $\mathsf{Desc}(V/U)$.

There is a functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ sending an object F of F_U to (a^*F, u) with u the canonical isomorphism

$$\mathrm{pr}_1^*(a^*F) \simeq (a \circ \mathrm{pr}_1)^* F = (a \circ \mathrm{pr}_2)^* F \simeq \mathrm{pr}_2^*(a^*F).$$

Definition

A ***stack*** is a fibred category $\phi : \mathsf{F} \to \mathsf{Aff}_S$ such that, for all faithfully flat morphisms $a : V \to U$ in Aff_S, the functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ is an equivalence of categories.

Explicitly, this means the following:

(a) for an affine S-scheme U and objects F, G in F_U, the functor sending $a : V \to U$ to $\mathrm{Hom}_V(a^*F, a^*G)$ is a sheaf of sets on U (for the fpqc topology);

(b) for every faithfully flat morphism $V \to U$ of affine S-schemes, descent is effective (that is, every descent datum for V/U is isomorphic to the descent datum defined by an object of F_U).

In other words, a fibred category is a stack if both morphisms and objects, given locally for the fpqc topology, patch uniquely to global objects.

The stacks over Aff_S form a 2-category with 1-morphisms the cartesian functors, with the 2-morphisms the cartesian natural transformations, and with

$$\mathsf{Hom}(\mathsf{F}, \mathsf{F}') = \mathsf{Cart}(\mathsf{F}, \mathsf{F}').$$

Examples

Let S be an affine scheme.

- There is a fibred category $\phi : \mathrm{MOD} \to \mathsf{Aff}_S$ such that MOD_U is the category of $\Gamma(U, \mathcal{O}_U)$-modules. Descent theory shows that this is a stack: if $R \to R'$ is faithfully flat, then $R' \otimes_R -$ is an equivalence from R-modules to R'-modules equipped with a descent datum (Waterhouse 1979, 17.2).

- There is a fibred category $\phi:\ \textsc{Proj}\to\ \mathsf{Aff}_S$ such that $\textsc{Proj}_U$ is the category of finitely generated projective $\Gamma(U,\mathcal{O}_U)$-modules. Descent theory shows that this is a stack: if $R\to R'$ is faithfully flat, then an R-module M is finitely generated and projective if and only if $R'\otimes_R M$ is.
- There is a fibred category $\phi:\ \textsc{Aff}\to\ \mathsf{Aff}_S$ such that $\textsc{Aff}_U=\mathsf{Aff}_U$, i.e., the fibre over U is the category of affine U-schemes. Descent theory shows that this is a stack (Waterhouse 1979, 17.3).
- There is a fibred category $\phi:\ \textsc{Shgr}\to\ \mathsf{Aff}_S$ such that SHGR_U is the category of sheaves of groups on U (for the fpqc) topology. This is a split stack, the inverse image of a sheaf being its restriction.
- Let G be a sheaf of groups on S (for the fpqc topology). There is stack $\phi:\ \textsc{Tors}(G)\to \mathsf{Aff}_S$ such that the fibre over U is the category of right torsors under $G|U$ over U.

Prestacks and the associated stack

A fibred category is a ***prestack*** if it satisfies the condition (a) to be a stack, i.e., for all $F,G\in \mathrm{ob}\,\mathsf{F}_U$, the functor

$$\left(V\xrightarrow{a}U\right)\rightsquigarrow \mathrm{Hom}(a^*F,a^*G)$$

is a sheaf on U for the fpqc topology.

Let $\phi:\ \mathsf{F}\to\mathsf{Aff}_S$ be a prestack. The ***associated stack*** $\phi:\ \mathsf{F}'\to\mathsf{Aff}_S$ of F contains it as a full subcategory and is characterized by having the property that every object of F' is locally in F. For any stack H over Aff_S, the inclusion functor $i:\ \mathsf{F}\to\mathsf{F}'$ induces an equivalence of categories

$$\mathsf{Hom}(\mathsf{F}',\mathsf{H})\xrightarrow{\cong}\mathsf{Hom}(\mathsf{F},\mathsf{H}),\tag{150}$$

compatible with base change. To construct F', we have to add an object over U for each faithfully flat morphism $V\to U$ and object over V equipped with a descent datum. We can do this by defining $\mathrm{ob}\,\mathsf{F}'_U=\varinjlim \mathsf{Desc}(V/U)$, where $V\to U$ runs over a suitably large collection faithfully flat morphisms. See Giraud 1971, II, §2.

For example, let G be a sheaf of groups on S. There is a fibred category $\mathsf{G}\to\mathsf{Aff}_S$ such that the fibre over U is $G|U$ regarded as a category with one object. This is a prestack, but not a stack. The functor $\mathsf{G}\to\textsc{Tors}$ sending the unique object of G_U to the trivial $G|U$-torsor realizes $\textsc{Tors}(G)$ as the associated stack of G: G is a full subcategory of $\textsc{Tors}(G)$ and every object of $\textsc{Tors}(G)$ is locally in G.

4 Gerbes

Let S be an affine scheme. A ***gerbe*** over S is a stack $\mathsf{G}\to\mathsf{Aff}_S$ such that,

(a) for all U, the category G_U is a groupoid (all morphisms are isomorphisms);

(b) there exists a faithfully flat morphism $U\to S$ such that G_U is nonempty;

(c) any two objects of a fibre G_U are locally isomorphic (i.e., their inverse images under some faithfully flat morphism $V\to U$ of affine S-schemes are isomorphic).

A ***morphism of gerbes*** over S is a morphism of stacks whose domain and codomain are gerbes, and similarly for a morphism of morphisms of gerbes. Thus the gerbes over Aff_k form a 2-category such that

$$\mathsf{Hom}(\mathsf{F},\mathsf{F}')=\mathsf{Cart}(\mathsf{F},\mathsf{F}').$$

A gerbe $\mathsf{G}\to \mathsf{Aff}_S$ is ***neutral*** if G_S is nonempty.

Example: torsors

The stack $\textsc{Tors}(G) \to \mathsf{Aff}_S$ defined by a sheaf G of groups on S is a gerbe. It is neutral because $\textsc{Tors}(G)_S$ contains the trivial torsor (G acting on itself on the right).

Conversely, let G be a neutral gerbe, and choose a $Q \in \mathrm{ob}(\mathsf{G}_S)$. Then $G \overset{\text{def}}{=} \mathcal{A}ut(Q)$ is a sheaf of groups on S, and, for any $a: U \to S$ and $P \in \mathrm{ob}(\mathsf{G}_U)$, the sheaf $\mathcal{I}som(a^*Q, a^*P)$ is a torsor under G over U. The functor

$$P \rightsquigarrow \mathcal{I}som_U(a^*Q, a^*P) : \mathsf{G} \to \textsc{Tors}(G)$$

is an isomorphism of gerbes.

5 Bands (Liens)

In this section, S is an affine scheme equipped with the fpqc topology.

C.1 Roughly speaking, we pass from groups to bands by systematically ignore inner automorphisms. For example, when two objects X and Y are isomorphic, the choice of an isomorphism $X \to Y$ determines an isomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(Y)$, well-defined up to an inner automorphism. In other words, $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ as bands. We make this precise in the category of sheaves of groups on S.

C.2 Let LI_U denote the category whose objects are sheaves of groups on U and whose morphisms $G \to H$ are the sections of the quotient sheaf

$$H\backslash\mathcal{H}om(G,H)/G,$$

where G and H act on $\mathcal{H}om(G,H)$ by inner automorphisms. Note that, as $f\circ \mathrm{inn}_g = \mathrm{inn}_{f(g)}\circ f$,

$$H\backslash\mathcal{H}om(G,H) \simeq H\backslash\mathcal{H}om(G,H)/G.$$

On varying U, we get a fibred category $\mathrm{LI} \to \mathsf{Aff}_S$. It is, in fact, a prestack, and the associated stack

$$\mathrm{LIEN} \to \mathsf{Aff}_S$$

is called the ***stack of bands***. Thus LI is a full subcategory of LIEN, and every object of LIEN is locally in LI. A ***band over*** S is an object of LIEN_S.

C.3 There are obvious morphisms of fibred categories

$$\textsc{Shgr} \to \mathrm{LI} \to \mathrm{LIEN}$$

over Aff_S. We write $\mathrm{Bd}(H)$ for the band defined by a sheaf of groups H. For sheaves of groups G, H on Aff_S, the functor Bd induces an isomorphism of sheaves of homomorphisms (Giraud 1971, IV, 1.1.7)

$$H\backslash\mathcal{H}om(G,H) \to \mathcal{H}om(\mathrm{Bd}(G), \mathrm{Bd}(H)).$$

Let

$$\mathcal{I}sex(G,H) \overset{\text{def}}{=} H\backslash\mathcal{I}som(G,H) = \mathcal{I}som(\mathrm{Bd}(G), \mathrm{Bd}(H))$$
$$\mathrm{Isex}(G,H) \overset{\text{def}}{=} \Gamma(S, \mathcal{I}sex(G,H)) = \mathrm{Isom}(\mathrm{Bd}(G), \mathrm{Bd}(H)).$$

C.4 Every band B over S is locally the band of a sheaf of groups. This means it comes by descent from a triple (U, G, u), where U is faithfully flat over S, G is a sheaf of groups on U, and $u \in \mathrm{Isex}(\mathrm{pr}_1^* G, \mathrm{pr}_2^* G)$ is such that

$$\mathrm{pr}_{31}^*(u) = \mathrm{pr}_{32}^*(u) \circ \mathrm{pr}_{21}^*(u).$$

If V is also a faithfully flat affine S-scheme, and $a : V \to U$ is an S-morphism, then (U, G, u) and $(V, a^*(G), (a \times a)^*(u))$ define the same band. If B_1 and B_2 are the bands defined by (U, G_1, u_1) and (U, G_2, u_2), then an element $\psi \in \mathrm{Isex}(G_1, G_2)$ such that $\mathrm{pr}_2^*(\psi) \circ u_1 = u_2 \circ \mathrm{pr}_1^*(\psi)$ defines an isomorphism $B_1 \to B_2$.

C.5 Recall that every sheaf of groups G on S defines a band $\mathrm{Bd}(G)$ over S, and that

$$\mathrm{Isom}(\mathrm{Bd}(G), \mathrm{Bd}(H)) = \mathrm{Isex}(G, H).$$

Thus, $\mathrm{Bd}(G)$ and $\mathrm{Bd}(H)$ are isomorphic if and only if H is an inner form of G, i.e., H becomes isomorphic to G on some faithfully flat S-scheme T, and the class of H in $H^1(S, \mathcal{A}ut(G))$ comes from $H^1(S, \mathrm{Inn}(G))$, where $\mathrm{inn}(G)$ is the image of the homomorphism of sheaves

$$\mathrm{inn} : \mathrm{Inn}(G) \to \mathrm{Aut}(G), \quad \mathrm{inn}(g)(x) = gxg^{-1}.$$

C.6 The fibred category LI contains the stack of sheaves of commutative groups as a full subcategory. Thus, we have an equivalence from the stack of sheaves of commutative groups to the stack of commutative bands.

C.7 The ***centre*** $Z(B)$ of the band B defined by (U, G, u) is defined by $(U, Z, u|\,\mathrm{pr}_1^* Z)$, where Z is the centre of G. The restriction $u|\,\mathrm{pr}_1^* Z$ lifts to an element $u_1 \in \mathrm{Isom}(\mathrm{pr}_1^* Z, \mathrm{pr}_2^* Z)$, and one checks immediately that $\mathrm{pr}_{31}^*(u_1) = \mathrm{pr}_{32}^*(u_1) \circ \mathrm{pr}_{21}^*(u_1)$. Thus $(U, Z, u|\,\mathrm{pr}_1^* Z)$ arises from a sheaf of groups on S, which we identify with $Z(B)$.

Affine and algebraic bands

C.8 A band is said to be ***affine*** if it is represented by a triple (U, G, u) with G an affine group scheme over U. When G is algebraic, the band is said to be ***algebraic***

C.9 Let k^{al} be an algebraic closure of k. Later, we shall see that every affine algebraic band over k is defined by a pair (G, u), where G is an algebraic group over k^{al} and u is an isomorphism $\mathrm{pr}_1 G \to \mathrm{pr}_2 G$.

The band of a gerbe

Let G be a gerbe on Aff_S. By definition, there exists an object $Q \in \mathsf{G}_U$ for some U faithfully flat over S. Let $G = \mathcal{A}ut(Q)$; it is a sheaf of groups on U. Again, by definition, $\mathrm{pr}_1^* Q$ and $\mathrm{pr}_2^* Q$ are locally isomorphic on $U \times_S U$, and the locally-defined isomorphisms determine an element $u \in \mathrm{Isex}(\mathrm{pr}_1^* G, \mathrm{pr}_2^* G)$. The triple (U, G, u) defines a band B which is uniquely determined up to a unique isomorphism. This is the band of the gerbe G.

A gerbe is said to be ***affine*** (resp. ***algebraic***) if it is banded by an affine (resp. algebraic) band.

Bands over a perfect field

Now let $S = \operatorname{Spec} k$, where k is a perfect field, and let G be a smooth affine group scheme over k^{al}. Recall that, if k has characteristic zero, then all algebraic group schemes are smooth.

C.10 *To give G the structure of a band over k is the same as giving an isomorphism $u\colon \operatorname{pr}_1 G \to \operatorname{pr}_2 G$ satisfying the cocycle condition modulo inner automorphisms.*

This is esentially the definition. Applying Galois descent (III, §5) to C.10, we obtain the following statement.

C.11 *To give G the structure of a band over k is the same as giving a family $(\phi_\sigma)_{\sigma\in\mathrm{Gal}(k^{\mathrm{al}}/k)}$ of isomorphisms $\phi_\sigma : \sigma G \to G$ satisfying the following two conditions:*

(a) *(cocycle) for all $\sigma,\tau \in \mathrm{Gal}(k^{\mathrm{al}}/k)$,*

$$\phi_\sigma \circ (\sigma\phi_\tau) = \phi_{\sigma\tau} \quad \textit{modulo inner automorphisms}$$

(b) *(continuity) (ϕ_σ) is split by some model of G over a subfield K of k^{al} finite over k.*

DEFINITION C.12 (SPRINGER 1966, 1.12) Let G be an algebraic group over k^{al}. A k^{al}/k-***kernel*** in G is a homomorphism

$$\kappa\colon \mathrm{Gal}(k^{\mathrm{al}}/k) \to \mathrm{Out}(G(k^{\mathrm{al}})) \stackrel{\mathrm{def}}{=} \frac{\mathrm{Aut}(G(k^{\mathrm{al}}))}{\mathrm{Inn}(G(k^{\mathrm{al}}))}$$

such that

(a) every automorphism $\widetilde{\kappa}(\sigma)$ of $G(k^{\mathrm{al}})$ lifting $\kappa(\sigma)$ is σ-linear (III, 6.1),

(b) for some finite extension $K \subset k^{\mathrm{al}}$ of k, the restriction of κ to $\mathrm{Gal}(\bar{k}/K)$ is defined by a model of G over k.

PROPOSITION C.13 *To give G the structure of a band over k is the same as giving it the structure of a k^{al}/k-kernel in G.*

PROOF Let κ be a k^{al}/k-kernel. For each σ, choose a lifting $\widetilde{\kappa}(\sigma)$ of $\kappa(\sigma)$. The condition (a) says that

$$\sigma P \mapsto \widetilde{\kappa}(\sigma)(P)\colon (\sigma G)(k^{\mathrm{al}}) \to G(k^{\mathrm{al}})$$

is defined by an isomorphism of algebraic varieties $\sigma G \to G$. The family (ϕ_σ) satisfies the cocycle and continuity conditions. Conversely, a family $(\phi_\sigma)_{\sigma\in\Gamma}$ satisfying these conditions defines a k^{al}/k-kernel by

$$\kappa(\sigma)(P) = \phi_\sigma(\sigma P).$$

□

C.14 Let G and H be algebraic group schemes over k. To give a morphism $\mathrm{Bd}(G) \to \mathrm{Bd}(H)$ of bands over k is the same as giving an $H(k^{\mathrm{al}})$-conjugacy class of homomorphisms $G_{k^{\mathrm{al}}} \to H_{k^{\mathrm{al}}}$ stable under the action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$.

TODO 3 Perhaps rethink the definition of kernel, aax.

6 Cohomology

Preliminaries

C.15 Let S be a site. For a sheaf F of abelian groups on S, let $\mathcal{H}^i(F)$ denote the presheaf $U \rightsquigarrow H^i(U,F)$. The Čech and derived cohomologies F are related by a spectral sequence

$$\check{H}^i(S,\mathcal{H}^j(F)) \Rightarrow H^{i+j}(S,F).$$

Moreover, $\check{H}^0(S,\mathcal{H}^j(F)) = 0$ for all $j > 0$, and so there is an exact sequence

$$0 \to \check{H}^2(S,F) \to H^2(S,F) \to \check{H}^1(S,\mathcal{H}^1(F)) \to \check{H}^3(S,F). \tag{151}$$

Note that the equality $\check{H}^0(S,\mathcal{H}^j(F)) = 0$, $j > 0$, says that for all U and $s \in H^j(U,F)$, $j > 0$, there exists a covering $(U_i \to U)$ such that s maps to zero in $H^j(U_i,F)$ for all i. See Milne 1980, III, 2.7, 2.9, 2.10, 2.11.

The fpqc cohomology versus the fppf cohomology

Let S be an affine scheme. Recall that the fpqc topology on Aff_S is that for which the coverings are finite surjective families of flat morphisms of affine S-schemes. For the ***fppf topology*** on Aff_S, the morphisms are required to be flat of finite presentation. The identity map on S is a continuous map $f : S_{\mathrm{fpqc}} \to S_{\mathrm{fppf}}$ of sites, and so, for any sheaf F of abelian groups on S_{fpqc} we have a canonical homomorphisms

$$H^i(S_{\mathrm{fppf}}, f_*F) \to H^i(S_{\mathrm{fpqc}}, F), \quad i \geq 0.$$

We now let $S = \operatorname{Spec} k$, k a field.

For a presheaf of abelian groups F on Aff_k and a k-algebra R, define $H^i(R/k,F)$ to be the ith cohomology group of the complex

$$F(R) \to F(R^{\otimes 2}) \to \cdots \to F(R^{\otimes i}) \to \cdots.$$

Then, by definition, the Čech groups

$$\check{H}^i(k_{\mathrm{fpqc}},F) = \varinjlim H^i(R/k,F) \quad \text{(limit over all } k\text{-algebras)}$$
$$\check{H}^i(k_{\mathrm{fppf}},F) = \varinjlim H^i(R/k,F) \quad \text{(limit over finitely generated } k\text{-algebras).}$$

PROPOSITION C.16 *Let F be a presheaf of abelian groups on Aff_k. If F transforms filtered projective limits of affine k-schemes into inductive limits, the canonical maps*

$$\check{H}^i(k_{\mathrm{fppf}},F) \to \check{H}^i(k_{\mathrm{fpqc}},F)$$

are isomorphisms for all i.

PROOF For any k-algebra R,

$$R^{\otimes i} = \varinjlim_{R'} R'^{\otimes i},$$

where the limit is over all finitely generated subalgebras R' of R, and so similarly

$$H^i(R/k,F) = \varinjlim_{R' \subset F,\ \text{finitely generated}} H^i(R'/k,F).$$

□

PROPOSITION C.17 *Let U be an affine scheme and G an affine group scheme flat of finite presentation over U. Then the canonical map*

$$H^1(U_{\mathrm{fppf}}, G) \to H^1(U_{\mathrm{fpqc}}, G)$$

is a bijection.

PROOF The sets classify the isomorphisms classes of torsors under G for the two topologies over U, and the functor

$$\mathsf{Tors}(U_{\mathrm{fppf}}, G) \to \mathsf{Tors}(U_{\mathrm{fpqc}}, G)$$

is an equivalence (even an isomorphism) of categories. Indeed, under the hypotheses, the torsors are representable by affine schemes flat and of finite presentation over U. The functor is obviously fully faithful, and every fpqc torsor T under G is also an fppf torsor because it has a point in an affine scheme flat and of finite presentation over U, namely, in T itself. □

PROPOSITION C.18 *Let Z be a commutative algebraic group scheme over k. The canonical map*

$$H^2(k_{\mathrm{fppf}}, Z) \to H^2(k_{\mathrm{fpqc}}, Z)$$

is an isomorphism.

PROOF For the fppf and fpqc topologies, the exact sequence (151) becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{H}^2(k_{\mathrm{fppf}}, Z) & \longrightarrow & H^2(k_{\mathrm{fppf}}, Z) & \longrightarrow & \check{H}^1(k_{\mathrm{fppf}}, \mathcal{H}^1(k_{\mathrm{fppf}}, Z)) & \longrightarrow & \check{H}^3(k_{\mathrm{fppf}}, Z) \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\
0 & \longrightarrow & \check{H}^2(k_{\mathrm{fpqc}}, Z) & \longrightarrow & H^2(k_{\mathrm{fpqc}}, Z) & \longrightarrow & \check{H}^1(k_{\mathrm{fpqc}}, \mathcal{H}^1(k_{\mathrm{fpqc}}, Z)) & \longrightarrow & \check{H}^3(k_{\mathrm{fpqc}}, Z).
\end{array}$$

The maps a and d are isomorphisms by C.16. The canonical map $\mathcal{H}^1(k_{\mathrm{fppf}}, Z) \to \mathcal{H}^1(k_{\mathrm{fpqc}}, Z)$ is an isomorphism by C.17, and the two functors transform filtered projective limits of affine schemes to inductive limits, so c is an isomorphism by C.16. Now the five-lemma shows that b is an isomorphism. □

ASIDE C.19 Flat algebras need not be filtered inductive limits of finitely presented flat algebras. For example, let p be a prime number, let $A = \mathbb{F}_p[X_1, ..., X_n]$, and let B be the integral closure of A in an algebraic closure of the field of fractions of A. Then B is flat over A by a theorem of Hochster and Huneke, but it is not a filtered direct limit of finitely presented flat A-algebras if $n \geq 3$ (Bhargav Bhatt).

NOTES This subsection is based on Saavedra 1972, III, 3.1.

The fppf cohomology versus the étale cohomology

THEOREM C.20 *Let G be a smooth affine group scheme over a scheme S. The canonical map*

$$H^i(S_{\mathrm{et}}, G) \to H^i(S_{\mathrm{fppf}}, G)$$

is an isomorphism for $i \leq 1$, and for all i if G is commutative.

PROOF Grothendieck 1968, 11.7, or Milne 1980, III, 3.9. □

Application to bands and gerbes

Let $S = \operatorname{Spec} k$, k a field. If a presheaf on S (i.e., a contravariant functor fom Aff_S to groups) is a sheaf for the fpqc topology, then it is also a sheaf for the fppf topology (obviously). This extends to a commutative diagram of fibred categories over Aff_S,

$$\begin{array}{ccc} \mathrm{SHGR}_{\mathrm{fppf}} & \longrightarrow & \mathrm{SHGR}_{\mathrm{fpqc}} \\ \downarrow & & \downarrow \\ \mathrm{LIEN}_{\mathrm{fppf}} & \longrightarrow & \mathrm{LIEN}_{\mathrm{fpqc}}. \end{array}$$

We write $B \rightsquigarrow B^\sharp$ for the functor from bands on S_{fpqc} to bands on S_{fppf}.

Let $\mathsf{L}_{\mathrm{fpqc}}$ (resp. $\mathsf{L}_{\mathrm{fppf}}$) denote the category whose objects are the affine algebraic bands on S_{fpqc} (resp. S_{fppf}) and whose morphisms are the isomorphisms between bands.

THEOREM C.21 *The functor $B \rightsquigarrow B^\sharp : \mathsf{L}_{\mathrm{fpqc}} \to \mathsf{L}_{\mathrm{fppf}}$ is an equivalence of categories.*

PROOF Saavedra 1972, III, 3.1.4.4, p. 191. □

THEOREM C.22 *Let G be a fibred category over Aff_S. Then G is an affine algebraic gerbe for the fpqc topology if and only if it is for the fppf topology.*

PROOF Saavedra 1972, III, 3.1.5, p. 191. □

COROLLARY C.23 *Let B be an affine algebraic band over k. Then the map*

$$H^2(S_{\mathrm{fppf}}, B) \to H^2(S_{\mathrm{fpqc}}, B)$$

is bijective, and induces a bijection

$$H^2(S_{\mathrm{fppf}}, B)' \to H^2(S_{\mathrm{fpqc}}, B)'.$$

COROLLARY C.24 *Let G be an affine algebraic gerbe over Aff_k. There exists a finite extension k' of k such that the fibre of G over $\operatorname{Spec} k'$ is nonempty.*

PROOF We may suppose that G is an affine algebraic gerbe for the fppf topology. Then $G(R) \neq \emptyset$ for some finitely generated k-algebra R. For any maximal ideal $\mathfrak{m}$ in R, $G(R/\mathfrak{m}) \neq \emptyset$ and $R/\mathfrak{m}$ is a finite field extension of k (Zariski's lemma). □

Alternative approach

C.25 If k is algebraically closed, then every algebraic band over k is the band of an algebraic group over k. To see this, let B be such a band. For some affine k-scheme U, B defines an element of $\check{H}^1(U/k, B^{\mathrm{ad}})$. There is an exact sequence of pointed sets

$$H^1(k, B) \to H^1(k, B^{\mathrm{ad}}) \to H^2(k, Z(B))$$

(fpqc cohomology groups). Now $H^2(k, Z(B))$ is equal to the fppf cohomology group (C.18), hence it is zero because k is algebraically closed. Thus, the class of B in $\check{H}^1(U/k, B^{\mathrm{ad}})$ lifts to a class in $\check{H}^1(U/k, B)$, which defines an algebraic group scheme over k.

C.26 Let B be a band on Aff_S, and let G and H be gerbes banded by B. Every morphism $m : \mathsf{G} \to \mathsf{H}$ banded by id_B is an equivalence (IV, 1.19). We say either that m is a B-morphism or a B-equivalence, since the two are the same. The cohomology set $H^2(S, B)$ is defined to be the set of B-equivalence classes of B-gerbes. If Z is the centre of B, then $H^2(S, Z)$ is equal to the cohomology group of Z in the usual sense of the fpqc topology on S, and either $H^2(S, B)$ is empty or $H^2(S, Z)$ acts simply transitively on it (Giraud 1971, IV, 3.3.3).

PROPOSITION C.27 *Let* G *be an affine algebraic gerbe over* Aff_k. *There exists a finite extension* k' *of* k *such that the fibre of* G *over* $\operatorname{Spec} k'$ *is nonempty.*

PROOF By assumption, the band B of G is defined by a triple (U, G, u) with G a group scheme of finite presentation over U. Let $U = \operatorname{Spec} R$. The k-algebra R can be replaced by a finitely generated subalgebra, and then by a quotient modulo a maximal ideal, and so we may suppose that $U = \operatorname{Spec} k'$, where k' is a finite field extension of k. We shall show that the gerbes G and $\textsc{Tors}(G)$ become B-equivalent over some finite field extension of k'. The statement preceding the proposition shows that we have to prove that an element of $H^2(U_{\mathrm{fpqc}}, Z)$, where Z is the centre of B, is killed by a finite field extension of k'. But this assertion is obvious for $H^2(U_{\mathrm{fppf}}, Z)$ (C.15), and we can apply C.18. □

REMARK C.28 The same argument (using C.20) shows that for a gerbe over Aff_k with smooth affine band, there exists a finite separable extension k' of k such that the fibre over $\operatorname{Spec} k'$ is nonempty. Deduce that a tannakian category over k with prosmooth band has a fibre functor over k^{sep}.

THEOREM C.29 *Let* G *be an affine gerbe over* Aff_k. *For any algebraically closed field* K *containing* k, *the fibre of* G *over* $\operatorname{Spec} K$ *is nonempty.*

PROOF We may suppose that k is algebraically closed, and then have to show that G is neutral. According to V, 2.5, G is equivalent (as a fibred category over Aff_k) to the gerbe of fibre functors on a tannakian category over k. Now the statement follows from IV, Theorem 10.1. □

EXERCISE C.30 Find a direct proof of Theorem C.29 (possibly using 7.9).

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