Chapter IV

The Brauer Group

The Brauer group of a field K may be defined to be the group of similarity classes of central simple algebras over K or, equivalently, the cohomology group

$$H^2(\mathrm{Gal}(K_{\mathrm{sep}}/K), K_{\mathrm{sep}}^*) = H^2((\mathrm{Spec}\, K)_{\mathrm{et}}, \mathbb{G}_m).$$

Both of these definitions generalize to schemes, but they may give different groups. The first group, that of similarity classes of Azumaya algebras over X, is called the Brauer group $\mathrm{Br}(X)$ of X, and the second, $H^2(X_{\mathrm{et}}, \mathbb{G}_m)$, is called the cohomological Brauer group $\mathrm{Br}'(X)$ of X.[1] There is always an injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$, which is known to be surjective in many cases. It would be useful to know exactly when this map is surjective, that is, when every cohomology class in $H^2(X_{\mathrm{et}}, \mathbb{G}_m)$ is represented by an Azumaya algebra, for much the same reasons as it is useful to know that every cohomology class in $H^1(X, \mathbb{G}_m)$ is represented by an invertible sheaf.[2] From a geometric point of view, the Brauer group classifies the cohomology 2-classes that do not arise from an algebraic divisor, that is, it classifies the transcendental classes.

1 The Brauer Group of a Local Ring

Throughout this section, R will be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and A a not necessarily commutative algebra over R. We assume that A has an identity element and that the map $R \to A$, $r \mapsto r1$, identifies R with a subring of the center of A. Residue class maps modulo $\mathfrak{m}$ will be written $a \mapsto \bar{a}$. Ideal will mean two-sided ideal. We assume as known the theory of Brauer groups over fields (see, for example, Herstein [1, Chapter 4] or Blanchard [1]).

Let A^o denote the opposite algebra to A, that is, the algebra with the multiplication reversed. We say that A is an *Azumaya algebra* over R if it is free of finite rank as an R-module and if the map $A \otimes_R A^o \to \mathrm{End}_{R\text{-mdl}}(A)$ that sends $a \otimes a'$ to the endomorphism $(x \mapsto axa')$ is an isomorphism. (Compare Bourbaki [2, II, 5, Ex. 14].)

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[1] The terminology has changed since the book was written. Because $\mathrm{Br}(X)$ is always torsion for quasi-compact schemes, the cohomological Brauer group $\mathrm{Br}'(X)$ is more naturally defined to be the *torsion subgroup* of $H^2(X, \mathbb{G}_m)$.

[2] Gabber and de Jong have shown that $\mathrm{Br}(X) = \mathrm{Br}'(X)_{\mathrm{tors}}$ if X admits an ample invertible sheaf; see later.

PROPOSITION 1.1 *Let A be an Azumaya algebra over R. Then A has center R; moreover, for any ideal*[3] $\mathfrak{J}$ *of* A, $\mathfrak{J} = (\mathfrak{J} \cap R)A$, *and for any ideal* $\mathfrak{I}$ *of* R, $\mathfrak{I} = (\mathfrak{I}A) \cap R$. *Thus* $\mathfrak{J} \mapsto \mathfrak{J} \cap R$ *is a bijection from the ideals of A to those of R.*

PROOF. Let ϕ be an endomorphism of A as an R-module. As ϕ is multiplication by an element of $A \otimes A^o$, it follows that $\phi(ac) = \phi(a)c$ for every c in the center of A and that $\phi(\mathfrak{J}) \subset \mathfrak{J}$ for every ideal $\mathfrak{J}$ of A.

Let $a_1 = 1, a_2, \ldots, a_n$ be a basis for A as an R-module, and let $\chi_1, \ldots, \chi_n$ be R-linear endomorphisms of A such that $\chi_i(a_j) = \delta_{ij}$ (Kronecker delta).

Let c lie in the center of A, and write it as $c = \sum r_i a_i$, $r_i \in R$. Then

$$c = \chi_1(a_1)c = \chi_1(a_1 c) = \chi_1(\sum r_i a_i) = r_1 \in R.$$

Let $\mathfrak{J}$ be an ideal of A. Let $a \in \mathfrak{J}$, and write it as $a = \sum r_i a_i$, $r_i \in R$. Then $r_i = \chi_i(a) \in \mathfrak{J}$, and so $a \in (\mathfrak{J} \cap R)A$. Thus $\mathfrak{J} = (\mathfrak{J} \cap R)A$.

Finally, let $\mathfrak{I}$ be an ideal of R. Let $a \in \mathfrak{I}A$, and write it as $a = \sum r_j a_j$, $r_i \in \mathfrak{I}$. Now $a \in R$ if and only if $r_i = 0$ for $i > 1$, and so $\mathfrak{I}A \cap R = \mathfrak{I}$. □

In particular, if $R = k$ is a field, then an Azumaya algebra over k is a central simple algebra over k. The converse is also true.

PROPOSITION 1.2 *(a) If A is an Azumaya algebra over R and R' is a commutative local R-algebra, then $A \otimes_R R'$ is an Azumaya algebra over R' (we do not require the map $R \to R'$ to be local).*

(b) If A is free of finite rank as an R-module and $\bar{A} \overset{\text{def}}{=} A \otimes (R/\mathfrak{m})$ is an Azumaya algebra over $R/\mathfrak{m}$, then A is an Azumaya algebra over R.

PROOF. For any R-algebra A that is free and of finite rank as an R-module and any (commutative) R-algebra R', there is a commutative diagram:

$$\begin{array}{ccc} (A \otimes_R A^o) \otimes R' & \xrightarrow{\phi \otimes R'} & \mathrm{End}_{R\text{-mdl}}(A) \otimes R' \\ \downarrow \simeq & & \downarrow \simeq \\ (A \otimes R') \otimes (A \otimes R')^o & \xrightarrow{\phi'} & \mathrm{End}_{R'\text{-mdl}}(A \otimes R'). \end{array}$$

Thus, if ϕ is an isomorphism, then ϕ' is also. On the other hand, if A is free of finite rank and $\phi \otimes (R/\mathfrak{m})$ is an isomorphism, then ϕ is is surjective by Nakayama's lemma, and it is injective by the elementary Lemma 1.11 below. □

COROLLARY 1.3 *(a) If A and A' are Azumaya algebras over R, then $A \otimes_R A'$ is an Azumaya algebra over R.*

(b) The matrix ring $M_n(R)$ is an Azumaya algebra over R.

PROOF. Both statements follow from (1.2b) and the corresponding statements for fields. □

We now define the Brauer group $\mathrm{Br}(R)$ of R. Two Azumaya algebras A and A' over R are said to be *similar* if

$$A \otimes_R M_n(R) \approx A' \otimes_R M_n(R)$$

[3] PUP used a font that doesn't distinguish fraktur I from fraktur J.

for some n and n'. Similarity is an equivalence relation, and if A_1 is similar to A_1' and A_2 to A_2', then $A_1 \otimes_R A_2$ is similar to $A_1' \otimes_R A_2'$ (because $M_n(R) \otimes M_m(R) \approx M_{nm}(R)$). Write $[A]$ for the similarity class of A. Then the similarity classes form a group under the law of composition $[A][A'] = [A \otimes_R A']$. The identity element is $[R]$, and $[A^o]$ is the inverse of $[A]$. This group is called the *Brauer group* of R. Clearly $\mathrm{Br}(-)$ is a functor from local rings to Ab.

PROPOSITION 1.4 (SKOLEM-NOETHER) . *Let A be an Azumaya algebra over R. Every automorphism of A as an R-algebra is inner, that is, of the form $a \mapsto uau^{-1}$ with u a unit in A.*

PROOF. Let $\phi: A \to A$ be an automorphism of A. It is possible to make A into a left $A \otimes_R A^o$-module in two different ways, namely,

$$\begin{cases} (a_1 \otimes a_2^o)a = a_1 a a_2 \\ (a_1 \otimes a_2^o)a = \phi(a_1) a a_2 \end{cases}$$

We denote the resulting $A \otimes_R A^o$-modules by A and A' respectively. Both $\bar{A}' = A' \otimes_R R/\mathfrak{m}$ and $\bar{A}$ are simple left $\bar{A} \otimes_{\bar{R}} \bar{A}^o$-modules, and as $\bar{A} \otimes_{\bar{R}} \bar{A}^o$ is a central simple algebra over $\bar{R}$, there is an isomorphism $\bar{\psi}: \bar{A} \to \bar{A}'$ of $\bar{A} \otimes_{\bar{R}} \bar{A}^o$-modules.

Next we show that A is projective as an $A \otimes A^o$-module. As $A \otimes A^o \simeq \mathrm{End}_{R\text{-mdl}}(A)$, it suffices to show that A is projective as an $\mathrm{End}(A)$-module. As A is a free R-module, there exists a homomorphism $g: A \to R$ of R-modules such that $g(r) = r$ for all $r \in R$. The surjection $\mathrm{End}_{R\text{-mdl}}(A) \to A$, $f \mapsto f(1)$, has an $\mathrm{End}_{R\text{-mdl}}(A)$-module section, namely, $a \mapsto (a' \mapsto g(a')a)$, which shows that A is projective.

Now the map $A \to \bar{A} \xrightarrow{\bar{\psi}} \bar{A}'$ lifts to a homomorphism of $A \otimes A^o$-modules, $\psi: A \to A'$. The surjectivity of $\bar{\psi}$ implies that $\psi(A) + \mathfrak{m}A' = A'$, and Nakayama's lemma applied to A' as an R-module shows that ψ is surjective. Let $u = \psi(1)$; then for an $a \in A$, $\psi(a) = \psi(a1) = \phi(a)u$ and $\psi(a) = \psi(1a) = ua$. Hence $\phi(a)u = ua$ for all $a \in A$, and it remains for us to check that u is a unit. But if $a_0 \in A$ is such that $\psi(a_0) = 1$, then $1 = \psi(a_0) = \phi(a_0)u$, and $\phi(a_0) = u^{-1}$. □

COROLLARY 1.5 *The automorphism group of $M_n(R)$ (as an R-algebra) is $\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/R^*$.*

PROOF. By definition, $\mathrm{GL}_n(R)$ is the group of units of $M_n(R)$, and the inner automorphism defined by $U \in \mathrm{GL}_n(R)$ is the identity map if and only if U is in the center R of $M_n(R)$. □

PROPOSITION 1.6 *If R is Henselian, then the canonical map $\mathrm{Br}(R) \to \mathrm{Br}(\bar{R})$, where $\bar{R} = R/\mathfrak{m}$, is injective.*

PROOF. Let A be an Azumaya algebra over R such that there exists an isomorphism $\bar{A} \to M_n(\bar{R})$, and let $\epsilon \in \bar{A}$ map to the matrix with 1 in the (1,1)-position and 0 elsewhere. Note that ϵ is idempotent, $\epsilon^2 = \epsilon$. Choose an $a \in A$ such that $\bar{a} = \epsilon$. Then $R[a]$ is a finite commutative R-subalgebra of A, and (I, 4.2) and (I, 4.3) imply that ϵ lifts to an idempotent e in $R[a]$. As $A = Ae \oplus A(1-e)$, the R-module Ae is a finitely generated and free, and so it remains to show that the map $\phi: A \to \mathrm{End}_R(Ae)$ sending a to left multiplication by a, is an isomorphism. The kernel of ϕ is an ideal in A whose intersection with R is zero (as Ae is free over R); thus ϕ is injective by (1.1). The same argument shows that $\bar{\phi}$ is injective, and as $\bar{A}$ and $\mathrm{End}_{\bar{R}}(\bar{A}\epsilon)$ have the same dimension, $\bar{\phi}$ is an isomorphism; now Nakayama's lemma shows that ϕ is surjective. □

COROLLARY 1.7 *The Brauer group of a strictly local ring is zero.*

PROOF. The Brauer group of a separably closed field is zero. □

COROLLARY 1.8 *If A is an Azumaya algebra over a Henselian local ring R, then there exists a finite étale faithfully flat homomorphism $R \to R'$ such that $A \otimes_R R' \approx M_n(R')$.*

PROOF. This follows from (1.6), (I, 4.4), and the fact that it is true for R a field. □

REMARK 1.9 We shall prove later (2.13) that the map $\mathrm{Br}(R) \to \mathrm{Br}(\bar{R})$ is an isomorphism for local Henselian rings R.

We say that an R-algebra R' *splits* an Azumaya R-algebra A if $A \otimes_R R' \approx M_n(R')$ for some n. According to (1.7), every A is split by a faithfully flat R-algebra.

THEOREM 1.10 *Let A be an Azumaya R-algebra of rank n^2.*

(a) *Let $a \in A$. Let R' be a faithfully flat R-algebra that splits A, and let $\phi\colon A \otimes R' \to M_n(R')$ be an isomorphism. Then the characteristic polynomial $c_a(T)$ of the matrix $\phi(a \otimes 1)$ belongs to $R[T]$, is independent of R', and $c_a(a) = 0$.*

(b) *There exists a commutative étale subalgebra R' of A of rank n that is a direct summand of A as an R-module;.*

(c) *Every maximal étale subalgebra of A splits it.*

PROOF. (a) We first remark that if ϕ_1, ϕ_2 are isomorphisms $A \otimes_R R' \rightrightarrows M_n(R')$, then $\phi_1(a \otimes 1)$ and $\phi_2(a \otimes 1)$ have the same characteristic polynomials. Indeed, for a maximal ideal $\mathfrak{m}$ of R', (1.4) shows that there is a $u \in \mathrm{GL}_n(R'_{\mathfrak{m}})$ such that $\phi_2(a \otimes 1) = u\phi_1(a \otimes 1)u^{-1}$ in $M_n(R'_{\mathfrak{m}})$, and so the characteristic polynomials have the same image in $R'_{\mathfrak{m}}[T]$ for all $\mathfrak{m}$; this proves the remark.

Let $c_a(T)$ be the characteristic polynomial of $\phi(a \otimes 1)$. The remark applied to $R' \otimes R'$ shows that the images of $c_a(T)$ under the two maps $R'[T] \rightrightarrows R' \otimes_R R'[T]$ agree. As

$$R[T] \to R'[T] \rightrightarrows R'[T] \otimes_{R[T]} R'[T] = R' \otimes_R R'[T]$$

is exact (I, 2.18), $c_a(T) \in R[T]$.

The independence assertion follows easily from the special case shown in the above remark. The final statement follows from the fact that $c_a(\phi(a \otimes 1)) = 0$ in $M_n(R')$.

(b) Choose an $a \in A$ such that $\bar{R}[\bar{a}]$ is a maximal étale subalgebra of $\bar{A}$. Then $\bar{R}[\bar{a}]$ has rank n over $\bar{R}$. Let $R' = R[T]/(c_a(T))$; it is an étale R-algebra of rank n, and there is a canonical map $R' \to A$, $T \mapsto a$. As $R' \otimes_R \bar{R} \xrightarrow{\approx} \bar{R}[\bar{a}] \hookrightarrow \bar{A}$ is injective, a standard result (Lemma 1.11) below) shows that $R' \to A$ is injective and R' is a direct summand of A.

(c) Regard A as a right R'-module. The map

$$a_0 \otimes r' \mapsto (a \mapsto a_0 a r')\colon A \otimes R' \to \mathrm{End}_{R'\text{-mdl}}(A)$$

is well-known to be an isomorphism modulo $\mathfrak{m}$ and hence is an isomorphism. □

The polynomial $c_a(T)$ in (a) of the theorem is called the *Cayley-Hamilton polynomial* of a. A subalgebra as in (b) of the theorem is called a *maximal étale subalgebra.*

LEMMA 1.11 *Let M and N be finitely generated R-modules with N free. If $\phi\colon M \to N$ is an R-linear map such that $\bar{\phi} \overset{\text{def}}{=} \phi \otimes_R \bar{R}\colon \bar{M} \to \bar{N}$ is injective, then ϕ has a section; in particular, it is injective. If $\bar{\phi}$ is an isomorphism, then so also is ϕ.*

PROOF. Let $\phi': N \to M$ be such that $\bar{\phi}'\bar{\phi} = \mathrm{id}_{\bar{M}}$, and let $\psi = \phi'\phi$. According to Nakayama's lemma, $\psi: M \to M$ is surjective. If M is regarded as an $R[T]$-module by means of ψ, then Atiyah-Macdonald [1, 2.5] shows that there exists a polynomial $f(T)$ such that $(1 - f(\psi)\psi)M = 0$, that is, such that $f(\psi)\psi = \mathrm{id}_M$. Now $f(\psi)\phi': N \to M$ has the property that $(f(\psi)\phi')\phi = \mathrm{id}_M$. The second statement is a consequence of the first and Nakayama's lemma. □

2 The Brauer Group of a Scheme

Let X be a locally Noetherian scheme. An $\mathcal{O}_X$-algebra A is called an *Azumaya algebra* over X if it is coherent as an $\mathcal{O}_X$-module and if, for all closed points x of X, A_x is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$. The conditions imply that A is locally free and of finite rank as an $\mathcal{O}_X$-module (I, 2.9), and that, for *every* point x of X, A_x is an Azumaya algebra over $\mathcal{O}_{X,x}$ (1.2a).

PROPOSITION 2.1 *Let A be an $\mathcal{O}_X$-algebra that is of finite-type as an $\mathcal{O}_X$-module. The following conditions on A are equivalent:*

(a) *A is an Azumaya algebra over X;*

(b) *A is locally free as an $\mathcal{O}_X$-module and $A(x) \stackrel{\mathrm{def}}{=} A_x \otimes \kappa(x)$ is a central simple algebra over $\kappa(x)$ for all x in X;*

(c) *A is locally free as an $\mathcal{O}_X$ -module and the canonical homomorphism $A \otimes_{\mathcal{O}_X} A^\circ \to \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mdl}}(A)$ is an isomorphism;*

(d) *there is a covering $(U_i \to X)$ for the étale topology on X such that for each i, there exists an r_i, for which $A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \approx M_{r_i}(\mathcal{O}_{U_i})$;*

(e) *there is a covering $(U_i \to X)$ for the flat topology on X such that for each i, there exists an r_i, for which $A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \approx M_{r_i}(\mathcal{O}_{U_i})$.*

PROOF. (a)⇔(b). This follows from (1.2b).

(a)⇔(c). As A is locally free, $(A \otimes A^o)_x = A_x \otimes A^o_x$ and $(\mathrm{End}_{\mathcal{O}_X}(A))_x = \mathrm{End}_{\mathcal{O}_{X,x}}(A_x)$. Thus this equivalence follows from the definitions.

(a)⇒(d). Let $\bar{x}$ be a geometric point of X. Corollary 1.7 shows that $A \otimes \mathcal{O}_{X,\bar{x}} \approx M_r(\mathcal{O}_{X,\bar{x}})$, and it follows that there exists an étale morphism $U \to X$ whose image contains x and is such that $A \otimes_{\mathcal{O}_X} \mathcal{O}_U \approx M_r(\mathcal{O}_U)$.

(d)⇒(e). Trivial.

(e)⇒(b). Let $U = \coprod U_i$. As U is faithfully flat over X, $A \otimes \mathcal{O}_U$ being flat as an $\mathcal{O}_U$-module implies that A is flat and hence locally free, as an $\mathcal{O}_X$-module. Also (e) implies that $A(x) \otimes_{\kappa(x)} k' \approx M_r(k')$ for some extension field k' of $\kappa(x)$, and this implies that $A(x)$ is a central simple algebra over $\kappa(x)$. □

REMARK 2.2 (a) Let $X = \operatorname{Spec} R$ be affine. An Azumaya algebra over X corresponds to an R-algebra A. The conditions in (2.1c) say exactly that A is projective and finitely generated as an R-module and that the canonical map $A \otimes_R A^o \to \mathrm{End}_{R\text{-mdl}}(A)$ is an isomorphism (note that $(A \otimes_R A^o)^\sim \approx \tilde{A} \otimes_{\mathcal{O}_X} \tilde{A}^o$ and $(\mathrm{End}_R(A))^\sim \approx \underline{\mathrm{End}}_{\mathcal{O}_X\text{-mdl}}(\tilde{A})$, because $\tilde{A}$ is coherent). Thus the notion of an Azumaya algebra over X corresponds exactly to that of a central separable algebra over R in the sense of Auslander-Goldman [1].

(b) Condition (2.1d) holds in a stronger form: there is a Zariski covering (U_i) of X and finite surjective étale maps $U_i' \to U_i$ such that, for all i, $A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i'} \approx M_{r_i}(\mathcal{O}_{U_i})$. This follows from (1.10b,c).

We now define the Brauer group of X. Two Azumaya algebras A and A' over X are said to be *similar* if there exist locally free $\mathcal{O}_X$-modules E and E', of finite rank over $\mathcal{O}_X$, such that

$$A \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(E) \approx A' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(E').$$

Similarity is an equivalence relation, because $\underline{\mathrm{End}}(E) \otimes \underline{\mathrm{End}}(E') \approx \underline{\mathrm{End}}(E \otimes E')$. Clearly the tensor product of two Azumaya algebras is an Azumaya algebra (use (1.3a)), and this operation is compatible with the similarity relation. The set of similarity classes of Azumaya algebras on X becomes a group under the operation $[A][A'] = [A \otimes A']$: the identity element is $[\mathcal{O}_X]$ and $[A]^{-1} = [A^o]$. This is the *Brauer group* $\mathrm{Br}(X)$ of X. Clearly $\mathrm{Br}(-)$ is a functor from schemes to abelian groups.

In relating $\mathrm{Br}(X)$ to the cohomology group $H^2(X_{\mathrm{et}}, \mathbb{G}_m)$, we shall need the following generalization of the Skolem-Noether theorem.

PROPOSITION 2.3 *Let A be an Azumaya algebra on X. Every automorphism ϕ of A is locally, for the Zariski topology on X, an inner automorphism, that is, there is a covering of X by open sets U_i such that $\phi|U_i$ is of the form $a \mapsto uau^{-1}$ for some $u \in \Gamma(U_i, A)^*$.*

PROOF. Let $x \in X$. From the Skolem-Noether theorem for local rings (1.4), we see that there exists a neighborhood U of x and a $u \in \Gamma(U, A)^*$ such that $\phi_x(a) = u_x^{-1} a u_x$ for all $a \in A_x$. Now the maps $\phi\colon A|U \to A|U$ and $(a \mapsto u^{-1}au)$: $A|U \to A|U$ agree on some neighborhood $V \subset U$ of x. □

Let GL_n be the functor $\mathsf{Sch} \to \mathsf{Gp}$ such that

$$\mathrm{GL}_n(S) = \mathrm{GL}_n(\Gamma(S, \mathcal{O}_S)) = M_n(\Gamma(S, \mathcal{O}_S))^*$$

for all schemes S. Then GL_n is representable by

$$\mathrm{Spec}\left(\frac{\mathbb{Z}[T_{11}, \ldots, T_{nn}, T]}{(T\det(T_{ij}) - 1)}\right)$$

and so defines a sheaf on X for the flat, or any coarser, topology (II, 1.7).

Let PGL_n be the functor $\mathsf{Sch} \to \mathsf{Gp}$ such that $\mathrm{PGL}_n(S) = \mathrm{Aut}(M_n(\mathcal{O}_S))$ (automorphisms of $M_n(\mathcal{O}_S)$ as a sheaf of $\mathcal{O}_S$-algebras). Then PGL_n is also representable and so defines a sheaf on X for the flat (or coarser) topology. Indeed, any automorphism of $M_n(\mathcal{O}_S)$ as an $\mathcal{O}_S$-algebra may also be regarded as an endomorphism of $M_n(\mathcal{O}_S)$ as an $\mathcal{O}_S$-module. Thus PGL_n is a subfunctor of M_{n^2}. The condition that an $\mathcal{O}_S$-linear endomorphism be an automorphism of algebras is described by polynomials, and hence PGL_n is represented by a closed subscheme of $M_{n^2} = \mathrm{Spec}\,\mathbb{Z}[T_{11}, \ldots, T_{n^2n^2}]$. In fact, PGL_n is represented by $\mathrm{Spec}\,S$ where S is the subring of elements of degree zero in $\mathbb{Z}[T_{11}, \ldots, T_{nn}, \det(T_{ij})^{-1}]$.

The next result is an immediate consequence of the Skolem-Noether theorem.

COROLLARY 2.4 *The sequence*

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

is exact as a sequence of sheaves on $X_{\mathrm{Zar}}, X_{\mathrm{et}}$, or X_{fl}.

THEOREM 2.5 *There is a canonical injective homomorphism* $\mathrm{Br}(X) \to H^2(X_{\mathrm{et}}, \mathbb{G}_m)$.

We give one proof based on Čech cohomology, which assumes that X satisfies the hypotheses of (III, 2.17), and sketch a second, based on Giraud's nonabelian cohomology, which is general.

STEP 1 *The set of isomorphism classes of Azumaya algebras of rank n^2 over X is equal to $\check{H}^1(X_{\text{et}}, \mathrm{PGL}_n)$.*

PROOF. Since, by definition, an Azumaya algebra is a twisted-form of $M_n(\mathcal{O}_X)$ for the étale topology and $\underline{\mathrm{Aut}}(M_n(O_X)) = \mathrm{PGL}_n$, this is a special case of the theory discussed at the end of (III, 4). The only problem is in seeing that every 1-cocycle does arise from an Azumaya algebra. A section of PGL_n may be regarded as an automorphism of M_n as an $\mathcal{O}_X$-module. Thus a 1-cocycle for PGL_n defines a 1-cocycle for GL_{n^2}, which we know (III, 4) arises from a locally free $\mathcal{O}_X$-module of rank n^2. The fact that the 1-cocycle came from a 1-cocycle for PGL_n means that this locally free O_X-module automatically has the structure of an Azumaya algebra. □

STEP 2 *The set of isomorphism classes of locally free modules of rank n over X is equal to $\check{H}^1(X_{\text{et}}, \mathrm{GL}_n)$; the map $\check{H}^1(X_{\text{et}}, \mathrm{GL}_n) \to \check{H}^1(X_{\text{et}}, \mathrm{PGL}_n)$ defined by the surjection $\mathrm{GL}_n \to \mathrm{PGL}_n$ sends an $\mathcal{O}_X$-module E to $\underline{\mathrm{End}}_{\mathcal{O}_X}(E)$.*

PROOF. For the first statement see (III, 4). For the second, let E be an $\mathcal{O}_X$-module and $\mathcal{U} = (U_i)$ a Zariski covering of X for which there exist isomorphisms $\phi_i : \mathcal{O}_U^n \to E|U_i$. Then E corresponds to the 1-cocycle $(\phi_i^{-1}\phi_j)$. Let $A = \underline{\mathrm{End}}(E)$. There are isomorphisms

$$\psi_i : M_n(\mathcal{O}_{U_i}) \to A|U_i, \quad \psi_i(a) = \phi_i a \phi_i^{-1}.$$

Thus A corresponds to the 1-cocycle $(\psi_i^{-1}\psi_j) = (\alpha_{ij})$, where $\alpha_{ij}(a) = \phi_i^{-1}\phi_j a \phi_j^{-1}\phi_i$ for $a \in \Gamma(U_{ij}, M_n)$. This is the image of $(\phi_i^{-1}\phi_j)$ because $\mathrm{GL}_n \to \mathrm{PGL}_n$ maps u to the automorphism $(a \mapsto uau^{-1})$ of M_n. □

STEP 3 *Assume that X satisfies the hypotheses of (III, 2.17). There is an exact sequence of pointed sets,*

$$\to \check{H}^1(X_{\text{et}}, \mathbb{G}_m) \to \check{H}^1(X_{\text{et}}, \mathrm{GL}_n) \to \check{H}^1(X_{\text{et}}, \mathrm{PGL}_n) \xrightarrow{d} \check{H}^2(X_{\text{et}}, \mathbb{G}_m);$$

moreover, the maps d are compatible for varying n, and $d(c(A \otimes A')) = dc(A) \cdot dc(A')$ where $c(A)$ denotes the class in $\check{H}^1(X, \mathrm{PGL}_n)$ of an Azumaya algebra A.

PROOF. The map d is defined as follows: let $\gamma \in \check{H}^1(X_{\text{et}}, \mathrm{PGL}_n)$ be represented by a cocycle (c_{ij}) for the covering (U_i); after refining (U_i), we may assume (III, 2.19) that each c_{ij} is the image of an element $c'_{ij} \in \Gamma(U_{ij}, \mathrm{GL}_n)$; then $d(\gamma)$ is the class of the 2-cocycle (a_{ijk}) where

$$a_{ijk} = c'_{jk}(c'_{ik})^{-1}c'_{ij} \in \Gamma(U_{ijk}, \mathbb{G}_m).$$

The verification of the exactness and the other statements is routine and hence is omitted. (Compare Giraud [2, IV.3.5].) □

STEP 4 *Definition of* $\mathrm{Br}(X) \hookrightarrow \check{H}^2(X, \mathbb{G}_m)$.

PROOF. Let A be an Azumaya algebra over X. If X is connected, then A has constant rank on X and so defines an element $c(A) \in \check{H}^1(X_{\mathrm{et}}, \mathrm{PGL}_n)$ for some n. The element $dc(A) \in \check{H}^2(X_{\mathrm{et}}, \mathbb{G}_m)$ depends, according to Step 3, only on the similarity class of A. Thus we have an injection $\mathrm{Br}(X) \to \check{H}^2(X_{\mathrm{et}}, \mathbb{G}_m)$, which, according to Step 3 again, is a homomorphism. If X is not connected, then as it is quasi-compact, both $\mathrm{Br}(X)$ and $\check{H}^2(X, \mathbb{G}_m)$ break up into products following the splitting of X into a disjoint sum of its connected components. Thus it suffices to define the map on each component. This completes the first proof. □

We next sketch the general proof. Let $\phi\colon \mathsf{F} \to \mathsf{C}$ be a functor; for any object U of C write $\mathsf{F}(U)$ for the category whose objects are those u in F such that $\phi(u) = U$ and whose morphism are those f such that $\phi(f) = \mathrm{id}_U$. Let $f\colon v \to u$ be a morphism in F, and let $\phi(f\colon v \to u) = (g\colon V \to U)$; we say that f is *Cartesian* or that v is the *inverse image* $g^*(u)$ of u with respect to g, if for every v' in $\mathsf{F}(V)$, the map

$$f' \mapsto ff'\colon \mathrm{Hom}_{\mathsf{F}(V)}(v', v) \to \mathrm{Hom}_g(v', u) \stackrel{\mathrm{def}}{=} \{h \in \mathrm{Hom}(v', u) \mid \phi(h) = g\}$$

is an isomorphism:

$$\begin{tikzcd} v' \arrow[d, dashed, "f'"'] \arrow[dr, "h"] & \\ v \arrow[r, "f"'] & u \end{tikzcd}$$

$$V \xrightarrow{g} U.$$

We say $\phi\colon \mathsf{F} \to \mathsf{C}$ is a *fibered category* if inverse images always exist and if the composite of two Cartesian morphisms is Cartesian. Then g^* can be made into a functor $\mathsf{F}(U) \to \mathsf{F}(V)$, and $(g_1 g_2)^*$ is canonically isomorphic to $g_2^* g_1^*$.

Now consider a fibered category $\phi\colon \mathsf{F} \to \mathsf{C}/X$ where $(\mathsf{C}/X)_E$ is a site. Let $(U_i \xrightarrow{g_i} U)$ be a covering in $(\mathsf{C}/X)_E$. Every $u \in \mathsf{F}(U)$ gives rise to a family (u_i), $u_i = g_i^* u \in \mathsf{F}(U_i)$, and the inverse images of u_i and u_j on $U_{ij} = U_i \times_X U_j$ are isomorphic; moreover, the isomorphisms satisfy the cocycle condition on U_{ijk}, that is, there is a descent datum on the family (u_i). If conversely, every family (u_i), $u_i \in \mathsf{F}(U_i)$, with a descent datum arises from a $u \in \mathsf{F}(U)$ and if, moreover, for any $u_1, u_2 \in \mathsf{F}(U)$, the functor

$$(V \xrightarrow{g} U) \rightsquigarrow \mathrm{Hom}_{\mathsf{F}(V)}(g^* u_1, g^* u_2)$$

is a sheaf on $(\mathsf{C}/U)_E$, then ϕ is a *stack* (champ in French), (Giraud [2, II, 1.2.1]). A stack is a *gerbe* if:

(a) each $\mathsf{F}(U)$ is a groupoid; that is, all morphisms in $\mathsf{F}(U)$ are isomorphisms;

(b) there exists a covering (U_i) of X such that each $\mathsf{F}(U_i)$ is nonempty;

(c) for every U in C/X, any two objects of $\mathsf{F}(U)$ are locally isomorphic, that is, their inverse images on some covering of U are isomorphic.

A gerbe is said to be *trivial* if $\mathsf{F}(X)$ is nonempty. A gerbe F is *bound* by a sheaf of groups G on $(\mathsf{C}/X)_E$ if for any U in C/X and any u in $\mathsf{F}(U)$ there are functorial isomorphisms $G(U) \xrightarrow{\simeq} \mathrm{Aut}_{\mathsf{F}(U)}(u)$. Giraud defines $H^2(X_E, G)$ to be the set of gerbes bound by G, modulo G-equivalence (Giraud [2, IV.3.1.1]). To avoid confusion when G is abelian, we shall denote this set by $H^2_g(X_E, G)$. To prove the theorem we must show:

(i) there is a canonical isomorphism $H^2_g(X_E, G) \to H^2(X_E, G)$ when G is abelian;

(ii) there is a canonical injective homomorphism $\mathrm{Br}(X) \to H_g^2(X_{\mathrm{et}}, \mathbb{G}_m)$.

It is easy to describe the map in (ii). Associate with an Azumaya algebra A on X the category F_A over X_{et} such that an object of $\mathsf{F}_A(U)$ is a pair (E,α) where E is a locally free $\mathcal{O}_U$-module of finite rank and α is an isomorphism $\mathrm{End}(E) \to A \otimes \mathcal{O}_U$; a morphism $(E,\alpha) \to (E',\alpha')$ is an isomorphism $E \to E'$ such that the obvious diagram commutes. Descent theory shows that F_A is a stack, and (2.1d) shows that it is a gerbe. The map $\mathbb{G}_m(U) \to \mathrm{Aut}_U(E,\alpha)$ that sends an element of $\Gamma(U,\mathcal{O}_U^*)$ to multiplication on E by that element is an isomorphism. Thus F_A is bound by $\mathbb{G}_m$ and so defines an element of $H_g^2(X_{\mathrm{et}}, \mathbb{G}_m)$. Clearly the element is trivial if and only if $[A] = 0$ in $\mathrm{Br}(X)$.

For (i) it suffices to check:

(i$'$) for every exact sequence of abelian sheaves

$$0 \to G' \xrightarrow{u} G \xrightarrow{v} G'' \to 0$$

there is an exact sequence of abelian groups

$$\cdots \to H^1(X,G'') \to H^1(X,G') \to H_g^2(X,G') \to H_g^2(X,G) \to H_g^2(X,G');$$

(i$''$) if G is an injective abelian sheaf, then $H_g^2(X,G) = 0$.

Let $u\colon G' \to G$ be a homomorphism of abelian sheaves, and let F' and F be gerbes bound by G' and G respectively. Then $\mathsf{HOM}_u(\mathsf{F}',\mathsf{F})$ is defined to be the fibered category over C/X whose fiber over U is all functors $\mathsf{F}'|U \to \mathsf{F}|U$ that preserve the fiberings, preserve Cartesian arrows, and that commute with the actions of G' on F' and G on F (Giraud [2, IV.2.3.2]). It is in fact a gerbe. The map $H_g^2(u)\colon H_g^2(X,G') \to H_g^2(X,G)$ sends F' to $\mathsf{HOM}_u(\mathsf{F}',\mathsf{F}_0)$ where F_0 is the trivial gerbe, that is, $\mathsf{F}_0(U) =$ set of torsors for $G|U$.

The group structure on $H_g^2(X,G)$ (G abelian) is defined by the maps

$$H_g^2(X,G) \times H_g^2(X,G) \to H_g^2(X,G \times G) \to H_g^2(X,G)$$

induced by $G \rightrightarrows G \times G \xrightarrow{+} G$.

The map $H^1(X,G'') \to H^2(X,G)$ is defined as follows: let P be a G''-torsor, and let F be the gerbe such that $\mathsf{F}(U)$ consists of all pairs (Q,α) with Q a G-torsor on U and α an isomorphism of G -torsors $v_*(Q) \to P$. As F is bound by G' we may define its class in $H_g^2(X,G')$ to be the image of P. By definition, F is trivial if and only if P arises from a G-torsor.

The proof of (i$'$) now only involves straightforward, but tedious, checking (Giraud [2, IV.3.4]).

For (i$''$), let F be a gerbe for the injective sheaf G. There is a covering (U_i) of X such that $\mathsf{F}|U_i$ is trivial for all i. Consider $G \to \pi_*\pi^*G$ where $\pi\colon \coprod U_i \to X$. As F maps to zero in $H_g^2(X,\pi_*\pi^*G)$ and as there is a section $G \rightleftarrows \pi_*\pi^*G$, F is zero in $H_g^2(X,G)$ (Giraud [2, IV, 3.4.3]).

COROLLARY 2.6 *Let X be a regular integral scheme and let $K = R(X)$. The canonical map* $\mathrm{Br}(X) \to \mathrm{Br}(K)$ *is injective.*

PROOF. This follows from the theorem, using the fact that the map $H^2(X,\mathbb{G}_m) \to H^2(K,\mathbb{G}_m)$ is injective (III, 2.22). □

PROPOSITION 2.7 *The image of $\check{H}^1(X,\mathrm{PGL}_n)$ in $H^2(X,\mathbb{G}_m)$ is killed by n. Thus, if X has only finitely many connected components,* $\mathrm{Br}(X)$ *is torsion.*

PROOF. We use the flat topology (and, by implication, Giraud [2, IV, 3.4.5]). Consider the diagram,

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r, "n"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{SL}_n \arrow[r] \arrow[d] & \mathrm{GL}_n \arrow[r, "\det"] \arrow[d] & \mathbb{G}_m \arrow[r] & 1 \\
 & \mathrm{PGL}_n \arrow[r, equal] \arrow[d] & \mathrm{PGL}_n \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

where SL_n is defined to be the kernel of the determinant mapping. The top row is exact (II, 2.18), which implies that det is surjective, and the middle row is exact. A diagram chase now shows that the first column is exact.

On comparing the flat cohomology sequences of the first two columns, one finds that the map $H^1(X, \mathrm{PGL}_n) \to H^2(X, \mathbb{G}_m)$ factors through $H^2(X, \mu_n)$, which is killed by n. (A much more complicated proof, but which avoids referring to Giraud [2], can be found in Knus and Ojanguren [1, IV.6.1].) □

REMARK 2.8 By making use of an example of Mumford, Grothendieck [4, II, 1.11b] has shown that there exists a normal (but singular) surface X over the complex numbers such that $H^2(X_{\mathrm{et}}, \mathbb{G}_m)$ is not torsion (and $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is not injective). Thus, the map $\mathrm{Br}(X) \to H^2(X, \mathbb{G}_m)$ cannot be surjective in this case. However, it is natural to ask the following question.

QUESTION 2.9 Is $\mathrm{Br}(X) \to H^2(X, \mathbb{G}_m)_{\mathrm{tors}}$ surjective for quasi-compact X?

When X is a topological space and $\mathcal{O}_X$ is the sheaf of continuous functions into $\mathbb{C}$, it is possible to mimic the above definitions and so define the Brauer group of X. In the case that X is a finite CW-complex it is known that the map $\mathrm{Br}(X) \to H^2(X, \mathcal{O}_X^*)_{\mathrm{tors}}$ is surjective (see Grothendieck [4, I, 1]). We shall prove below that the same is true for certain classes of schemes. Apparently there is no example known where the map is not surjective. Positive answers to (2.9) are valuable because they enable cohomological techniques to be applied to the Brauer group and, from the point of view of the cohomology, they allow a cohomology class to be explicitly represented by an Azumaya algebra.

We write $\mathrm{Br}'(X)$ for $H^2(X, \mathbb{G}_m)$ and call it the *cohomological Brauer group*.[4] In view of (2.6) and the fact that the Brauer group of a field is torsion, one may also ask:

QUESTION 2.9′ If X is a regular scheme with only finitely many connected components, then is $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ surjective?

REMARK 2.10 For any scheme X, let $\underline{\mathrm{Br}}$ (respectively, $\underline{\mathrm{Br}}'$) be the sheaf on X_{Zar} associated with the presheaf $U \rightsquigarrow \mathrm{Br}(U)$ (respectively, $U \rightsquigarrow \mathrm{Br}'(U)$). If $f\colon X_{\mathrm{et}} \to X_{\mathrm{Zar}}$ is the obvious

[4] As noted earlier, the cohomological Brauer group $\mathrm{Br}'(X)$ is now defined to be the torsion subgroup of $H^2(X, \mathbb{G}_m)$.

morphism, then $R^2 f_*\mathbb{G}_m = \underline{\mathrm{Br}}'$. Since $R^1 f_*\mathbb{G}_m = 0$ (III, 4.9), the Leray spectral sequence gives the first row of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^2(X_{\mathrm{Zar}},\mathcal{O}_X^*) \arrow[r] & \mathrm{Br}'(X) \arrow[r] & \Gamma(X,\underline{\mathrm{Br}}') \arrow[r] & H^3(X_{\mathrm{Zar}},O_X^*) \\
0 \arrow[r] & \mathrm{Br}(X_{\mathrm{Zar}}) \arrow[r] \arrow[u] & \mathrm{Br}(X) \arrow[r] \arrow[u] & \Gamma(X,\underline{\mathrm{Br}}). \arrow[u] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Here $\mathrm{Br}(X_{\mathrm{Zar}})$ is the subgroup of $\mathrm{Br}(X)$ generated by Azumaya algebras that are split by a Zariski covering of X. The first vertical map is induced by the other two. We leave it as an exercise to the reader to give an explicit description of ϕ in terms of Azumaya algebras. The question (2.9) can now be broken into three questions.

(a) (Local question) Is $\underline{\mathrm{Br}}(X) \to \underline{\mathrm{Br}}'(X)$ an isomorphism? Equivalently, is $\mathrm{Br}(\mathcal{O}_{X,x}) \to \mathrm{Br}'(\mathcal{O}_{X,x})$ an isomorphism for all $x \in X$?

(b) (Global question). Does every element of $\Gamma(X,\underline{\mathrm{Br}})$ that maps to zero in $H^3(X_{\mathrm{Zar}},\mathcal{O}_X^*)$ arise from an element in $\mathrm{Br}(X)$?

(c) (Singular question). Is $\mathrm{Br}(X_{\mathrm{Zar}}) \to H^2(X_{\mathrm{Zar}},\mathcal{O}_X^*)$ an isomorphism? (If X is regular, $H^2(X_{\mathrm{Zar}},\mathcal{O}_X^*) = 0 = \mathrm{Br}(X_{\mathrm{Zar}})$ and $U \rightsquigarrow \mathrm{Br}'(U)$ is a sheaf.)

PROPOSITION 2.11 *Let $X = \operatorname{Spec} R$, where R is a local ring, and let $\gamma \in \mathrm{Br}'(X)$. The following are equivalent:*

(a) $\gamma \in \mathrm{Br}(X)$*;*

(b) *there exists a finite étale surjective map $Y \to X$ such that γ maps to zero in* $\mathrm{Br}'(Y)$*;*

(c) *there exists a finite flat surjective map $Y \to X$ such that γ maps to zero in* $\mathrm{Br}'(Y)$*;*

PROOF. Proof. (a)⇒(b). This follows from (1.10).

(b)⇒(c). This is trivial.

(c)⇒(a). Let $Y = \operatorname{Spec} S$. The spectral sequence (III, 2.7) for the covering $Y \to X$ gives an exact sequence,

$$\check{H}^0(Y/X,\underline{\mathrm{Pic}}) \to \check{H}^2(Y/X,\mathbb{G}_m) \to \mathrm{Br}'(Y/X) \to \check{H}^1(Y/X,\underline{\mathrm{Pic}}) \to \check{H}^3(Y/X,\mathbb{G}_m)$$

where we have written $\mathrm{Br}'(Y/X)$ for $\ker(\mathrm{Br}'(X) \to \mathrm{Br}'(Y))$. As S and $S \otimes S$ are semilocal rings, their Picard groups are zero, and the above sequence reduces to an isomorphism $\check{H}^2(Y/X,\mathbb{G}_m) \approx \mathrm{Br}'(Y/X)$. Let $u \in (S \otimes S \otimes S)^*$ represent the class in $\check{H}^2(Y/X,\mathbb{G}_m)$ corresponding to γ in $\mathrm{Br}'(Y/X)$, and let E be S regarded as a free R-module of rank n. Multiplication by u defines an $S \otimes S$-linear automorphism $\phi\colon S \otimes S \otimes E \to S \otimes S \otimes E$ that may be regarded as an element of $\mathrm{GL}_n(S \otimes S)$. Writing ϕ_i for ϕ tensored with id in the i^{th} place, we find that

$$\phi_1\phi_2^{-1}\phi_3 : S \otimes S \otimes S \otimes E \to S \otimes S \otimes S \otimes E$$

is $\phi_4 =$ multiplication by $u \otimes 1$ (because $\phi_1\phi_2^{-1}\phi_3\phi_4^{-1}$ is the coboundary of the 2-cocycle u). As

$$u \otimes 1 \in (S \otimes S \otimes S)^* = \text{ center } (\mathrm{Aut}_{S\otimes S\otimes S}(S \otimes S \otimes S \otimes E)),$$

the image of ϕ in $\mathrm{PGL}_n(S \otimes S)$ is a 1-cocycle and so defines an element $\gamma' \in \check{H}^1(X,\mathrm{PGL}_n)$. The image of γ' under the boundary map $\check{H}^1(X,\mathrm{PGL}_n) \to \check{H}^2(X,\mathbb{G}_m)$ is obviously γ. Thus any Azumaya algebra with class γ' will represent γ. □

COROLLARY 2.12 *If X is the spectrum of a Henselian local ring, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

PROOF. Every $\gamma \in \mathrm{Br}'(X)$ satisfies (2.11b), according to (III, 2.11) and (I, 4.2). □

COROLLARY 2.13 *If R is a Henselian local ring, then* $\mathrm{Br}(R) \simeq \mathrm{Br}(R/\mathfrak{m})$.

PROOF. This follows from (2.12) and (III, 3.lla). □

REMARK 2.14 (a) A direct proof of (b)⇔(a) of the proposition may be given as follows: the map $Y \to X$ may be assumed to be Galois, with Galois group G; the Hochschild-Serre spectral sequence for Y/X shows that γ corresponds to an element of $H^2(G, \Gamma(Y, \mathcal{O}_Y)^*)$; such an element defines a crossed product algebra over X in the same way as for Galois extensions of fields.

(b) The algebra representing γ in (2.11c) can be described by descent theory. Regard $E = S \otimes S$ as a free S-module by letting S act on the first factor and define an $S \otimes S$-linear isomorphism $\phi: S \otimes E \to E \otimes S$ by setting

$$\phi(x \otimes y \otimes z) = \sum a_i x \otimes c_i z \otimes b_i y$$

where

$$u = \sum a_i \otimes b_i \otimes c_i.$$

One checks easily that $\phi_2^{-1}\phi_3\phi_1$ is multiplication by $u_1 u_2^{-1} u_3 = u_4$, and so

$$g \mapsto \phi^{-1} g \phi \colon \mathrm{End}_{S\otimes S}\ (E \otimes S) \to \mathrm{End}_{S \otimes S}(S \otimes E)$$

is a descent datum on $\mathrm{End}_S(E)$. The corresponding Azumaya algebra A over R, such that $A \otimes_R S \approx \mathrm{End}_S(E)$, represents γ.

(c) Let X be an arbitrary quasi-compact scheme, and let $Y \to X$ be a finite faithfully flat map. The proof of (c)⇒(a) above can be extended to show that in this case also an element of $\mathrm{Br}'(X)$ that maps to zero in $\mathrm{Br}'(Y)$ arises from an element of $\mathrm{Br}(X)$. See Hoobler [1, Pptn 3.1] or prove it in the affine case by comparing the sequence of Chase-Rosenberg [1, Thm 7.6] with the exact sequence in the proof of (2.11).

PROPOSITION 2.15 *If X is a smooth variety over a field k, then* $\underline{\mathrm{Br}}(X) = \underline{\mathrm{Br}}'(X)$, *that is, every element γ of* $\mathrm{Br}'(X)$ *arises locally from an Azumaya algebra.*

PROOF. Let $\gamma \in \mathrm{Br}'(U)$, where U is a Zariski open subset of X. For any point x of U we must show that there exists a Zariski open neighborhood V of x in U and a finite flat surjective map $V' \to V$ such that γ becomes zero in $\mathrm{Br}'(V')$. Clearly we may suppose that k is algebraically closed. If $n\gamma = 0$, then the Kummer sequence shows that γ arises from an element $\gamma' \in H^2(U, \mu_n)$. If the characteristic p of k does not divide n, then $\mu_n \approx \mathbb{Z}/n\mathbb{Z}$, and the existence of Artin neighborhoods implies the existence of a V' on which γ' becomes zero (III, 3.16). Now suppose $p = n$. Then (compare III, 4.14) there is an exact sequence

$$H^0(U, \Omega^1_{\mathrm{cl}}) \xrightarrow{C-1} H^0(U, \Omega^1) \to H^2(U, \mu_p) \to H^1(U, \Omega^1_{\mathrm{cl}}).$$

Since $H^1(U, \Omega^1_{\mathrm{cl}})$ may be computed using the Zariski topology (Ω^1_{cl} is a coherent sheaf over $\mathcal{O}_{X^{(p)}}$), the image of γ' in $H^1(U, \Omega^1_{\mathrm{cl}})$ becomes zero on some Zariski neighborhood of x. Thus we may assume that γ' arises from some element γ'' of $H^0(U, \Omega^1)$. But then (see the proof of (III, 4.14)) there is a Zariski neighborhood V of x and a finite Artin-Schreier covering $V' \to V$ such that $\gamma''|V'$ is in the image of $H^0(V', \Omega^1_{\mathrm{cl}}) \xrightarrow{C-1} H^0(V', \Omega^1)$. □

THEOREM 2.16 *Let X be a quasi-compact scheme, and let $\gamma \in \mathrm{Br}'(X)$. There exists an open subset U of X with codim$(X-U) > 1$ such that $\gamma|U$ is represented by an Azumaya algebra on U. Moreover, if X is regular, then U may be chosen so that* $\mathrm{codim}(X-U) > 2$, *and if X is a smooth affine variety, then U may be taken to be X.*

We first (partially) answer the local question.

LEMMA 2.17 *Let R be a local ring of dimension ≤ 1. Then* $\mathrm{Br}(R) \xrightarrow{\simeq} \mathrm{Br}'(R)$.

PROOF. If R is of dimension 0, then it is a local Artin ring and hence Henselian. Thus the result follows from (2.12). Now suppose that R has dimension 1. There is an exact sequence of sheaves on Spec R_{et} (compare II, 3.9):

$$0 \to \mathbb{G}_m \to g_*\mathbb{G}_{m,K} \to \mathrm{Div} \to 0$$

where K is the total ring of fractions of R and Div is the sheaf of Cartier divisors. Since Div has support on the closed point Spec k of R, there is an exact sequence,

$$H^1(k, \mathrm{Div}) \to \mathrm{Br}'(R) \to \mathrm{Br}(K).$$

Every element of $\mathrm{Br}(K)$ or $H^1(k, \mathrm{Div})$ is killed by a finite flat extension of K or k, and such extensions are induced by a finite flat extension of R. (For example, if L has basis $x_1 = 1, x_2, \ldots, x_n$ over K, and

$$x_i x_j = \sum c_{ijk} x_k$$

with $c_{ijk} \in K, dc_{ijk} \in R$, then $L = S \otimes_R K$ where $S \subset L$ has basis 1, $dx_2, \ldots, dx_n$ over R.) Thus the lemma follows from (2.11). □

Now let X be as in the theorem and let $x_1, \ldots, x_r$ be the generic points of the irreducible components of X. Since the local rings at the x_i have dimension 0, the lemma shows that there exist open neighborhoods $U_1, \ldots, U_r$ (which we may take to be disjoint) of $x_1, \ldots, x_r$ and an Azumaya algebra A_i on each U_i that represents $\gamma|U_i$. Now $U = U_1 \cup \cdots \cup U_r$ is an open dense subset of X, and we have an Azumaya algebra A on it representing $\gamma|U$.

Let $z \notin U$ with $\dim(\mathcal{O}_{X,z}) = 1$. According to (2.17), there exists an Azumaya algebra A_z over $\mathcal{O}_{X.z}$ representing $\gamma|\mathrm{Spec}(\mathcal{O}_{X,z})$. Let K be the total ring of fractions of $\mathcal{O}_{X,z}$. Then $A_z \otimes K$ is similar to $A \otimes_{\mathcal{O}_U} K$, and so after replacing them by matrix algebras over them if necessary, we may assume they are equal. Thus A extends over z. After a finite number of such extensions, we have $\mathrm{codim}(X-U) > 1$.

Before considering a regular scheme X, we recall some definitions and facts about orders (Reiner [1]).

Let R be an integral domain with field of fractions K, and let A_K be a finite K-algebra (not necessarily commutative). An R-*order* in A_K is a sub-R-algebra of A_K that contains a basis for A_K as a K-vector space and that is finitely generated as an R-module. If A is an R-order, then $A_{\mathfrak{m}}$ is an $R_{\mathfrak{m}}$-order for all maximal ideals of R and conversely. Thus, given an integral scheme X such that $R(X) = K$, we may define an $\mathcal{O}_X$-order in A_K to be a coherent $\mathcal{O}_X$-algebra A that, locally for the Zariski topology on X, is an order.

If R is integrally closed and A_K is an Azumaya algebra over K, then every subalgebra of A_K that contains a basis for A_K and is integral over R is an order. To show this, we only have to show that A is finitely generated as an R-module, but the reduced trace on A_K defines a nondegenerate bilinear form $A_K \times A_K \to K$, and so the usual proof in the commutative case (Atiyah-Macdonald [1, 5.17]) carries over. It follows that any sequence of orders $A_1 \subset A_2 \subset \cdots$ terminates because $A = \bigcup A_i$ is finitely generated. Thus there exist maximal orders. Clearly this result extends to orders over quasi-compact normal schemes.

Let X be a connected quasi-compact regular scheme, let $\gamma \in \mathrm{Br}'(X)$, and let A_K be the Azumaya algebra over $K = R(X)$ that represents the image of γ in $\mathrm{Br}'(K)$. Choose a maximal $\mathcal{O}_X$-order A in A_K. Clearly the set U of points $x \in X$ such that A_x is an Azumaya algebra over $O_{X,x}$ is open and dense in X, and the injectivity of

$$\mathrm{Br}(U) \hookrightarrow \mathrm{Br}'(U) \hookrightarrow \mathrm{Br}'(K)$$

implies that $A|U$ represents $\gamma|U$. It remains to show that codim $(X - U) > 2$ (after possibly replacing[5] A by a matrix algebra over it).

(It should be noted that A being a maximal order in A_K implies that $M_n(A)$ is a maximal order in $M_n(A_K)$. In proving this, we may assume that X is affine, $X = \mathrm{Spec}\, R$. Let B be an order such that $M_n(A) \subset B \subset M_n(A_K)$, and let B_0 be the set of elements of A_K that occur as an entry in some matrix in B. From the matrix identity

$$e_{1k}(b_{ij})e_{l1} = \begin{pmatrix} b_{kl} & 0 \\ 0 & 0 \end{pmatrix},$$

it follows that B_0 is a ring. Since it is generated as an R-module by the entries in any finite set of generators for B and it contains A, it is an order. Thus $B_0 = A$ and $M_n(A) = B$.)

Suppose that there is a point x in $X \setminus U$ such that $R = \mathcal{O}_{X,x}$ has dimension one. Then A_x is a maximal R-order in A_K, and, from the lemma, we know that there exists another R-order A' in A_K that is an Azumaya algebra representing $\gamma|\,\mathrm{Spec}\, R$ (after possibly replacing A by a matrix algebra over it). The set $\mathfrak{I} = \{a \in A_K \mid aA_x \subset A'\}$ is a left ideal in A'. Since left ideals in Azumaya algebras over fields are principal (Herstein [1, 1.4.2]), Nakayama's lemma shows that $\mathfrak{I}$ is principal, $\mathfrak{I} = A'u$. One sees easily that $\mathfrak{I} \cap K \neq 0$, which shows that u is a unit in A_K. Since $\mathfrak{I}$ is a right A_x-module, $uA_x \subset \mathfrak{I} = A'u$, and so $A_x \subset u^{-1}A'u$. The maximality of A_x implies that $A_x = u^{-1}A'u$, and so A_x is an Azumaya algebra. Thus we may assume that $x \in U$.

Write $A^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(A, \mathcal{O}_X) \subset A_K^\vee = \mathrm{Hom}_K(A_K, K)$. We may identify A_K with $A_K^{\vee\vee}$, and the maximality of A shows that the natural injection $A \hookrightarrow A^{\vee\vee}$ is an isomorphism.

Let x be a point of X such that $R = \mathcal{O}_{X,x}$ has dimension two, and consider a sequence

$$0 \to M \to F_0 \to A_x^\vee \to 0$$

with F_0 a free and finitely generated R-module. On applying $\mathrm{Hom}_R(-, R)$ to this, we get an exact sequence,

$$0 \to A_x \to F_0^\vee \to N \to 0$$

with N a submodule of $M^\vee$. As R is an integral domain, $M^\vee$ and N are torsion free, and so $N \hookrightarrow N \otimes_R K$. Thus N may be embedded in a free, finitely generated, R-submodule F_1 of $N \otimes_R K$. As R has homological dimension two, the existence of a sequence

$$0 \to A_x \to F_0^\vee \to F_1 \to F_1/N \to 0$$

[5]From Tate 12/4/92: "It was not at all clear to me, especially for schemes as opposed to rings, that a maximal order is locally maximal. And I think the comment on line -7 (after possibly replacing...) is misleading, since (I hope) if $M_n(A)$ is Azumaya, then A is also, and so is maximal."

shows that A_x is free.

The canonical map $A_x \otimes A_x^o \to \mathrm{End}_R(A_x)$ is injective because it is a submap of $A_K \otimes A_K^o \to \mathrm{End}_K(A_K)$; let M be its cokernel. As $A_x \otimes A_x^o$ and $\mathrm{End}_R(A_x)$ are both free, M has homological dimension ≤ 1. Thus the maximal ideal of R is not an associated prime of M (see Serre [9, p. IV-36]). But $M \otimes R_{\mathfrak{p}} = 0$ for every prime $\mathfrak{p}$ of height ≤ 1 since we know that for such $\mathfrak{p}$, $A \otimes R_{\mathfrak{p}}$ is an Azumaya algebra. Thus no prime is associated with M, $M = 0$, A_x is an Azumaya algebra, and $x \in U$.

Finally we come to the case that X is a smooth affine variety, say $X = \operatorname{Spec} R$. Let $\mathfrak{I}$ be the set of $f \in R$ such that $\gamma | \operatorname{Spec} R_f$ is represented by an Azumaya algebra. If we can show that $\mathfrak{I}$ is an ideal, then (2.15) will show that it equals R, and the theorem will be proved. Obviously $fg \in \mathfrak{I}$ if $f \in \mathfrak{I}$, $g \in R$. Let f and $g \in \mathfrak{I}$; it remains to show that $f + g \in \mathfrak{I}$. By definition, there are Azumaya algebras A and B over R_f and R_g respectively representing $\gamma | \operatorname{Spec} R_f$ and $\gamma | \operatorname{Spec} R_g$. As A_g and B_f are similar on $\operatorname{Spec} R_{fg}$ there exist finitely generated, locally free, R_{fg}-modules E and F such that $A_g \otimes \mathrm{End}(E) \approx B_f \otimes \mathrm{End}(F)$. The maps on the Grothendieck groups

$$K_0(R_f) \to K_0(R_{fg}), \quad K_0(R_g) \to K_0(R_{fg})$$

are surjective; this is obvious for K_0 (finitely generated modules) but as R is regular, this group agrees with K_0 (finitely generated projective modules) according to Bass [1, IX, 2.1]. Thus the classes of E and F in $K_0(R_{fg})$ extend to R_f and R_g. Since we may choose E and F to have arbitrarily high rank, the stability theorems (Bass [1, IX, 4.1]) show that E and F themselves extend to locally free, finitely generated, R_f and R_g-modules E' and F'. After replacing A and B by $A \otimes \mathrm{End}(E')$ and $B \otimes \mathrm{End}(F')$ respectively, we find that $A_g \approx B_f$, that is, A and B patch and γ is represented by an Azumaya algebra on $\operatorname{Spec} R_f \cup \operatorname{Spec} R_g$. As this contains $\operatorname{Spec} R_{f+g}$, the proof is complete.

REMARK 2.18 (a) If X is an affine regular scheme, then arguments similar to the above show that $\mathrm{Br}(X) \xrightarrow{\approx} H^2(X_{\mathrm{etf}}, \mathbb{G}_m)$ where (etf) is the class of all composites of finite étale morphisms and open immersions.

(b) The proof of the second case of (2.17) is essentially that used by Auslander and Goldman to prove that for any regular connected scheme X of dimension 2, $\mathrm{Br}(X)$ is the subgroup $\bigcap_{x \in X_1} \mathrm{Br}(\mathcal{O}_{X,x})$ of $\mathrm{Br}(R(X))$. Hoobler, to whom the proof of the third case of (2.17) is due, shows that (2.17) implies the same result for X a smooth affine variety. The result may be regarded as a noncommutative analogue of the purity of branch locus (I, 3.7): if the class of an Azumaya algebra in the Brauer group ramifies then it does so on a set of pure codimension one.[6]

(c) The proof of the third case of (2.17) shows that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ if X is a smooth variety with a covering $X = X_1 \cup X_2$ where X_1, X_2 and $X_1 \cap X_2$ are open affine. This holds, for example, if X is a smooth projective curve over a smooth affine variety, a fact that has been exploited by Artin to generalize his result (Grothendieck [4, III, 3.1]).

(d) Ofer Gabber and R. Hoobler have announced proofs that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for every affine scheme X.[7]

[6] The orginal states this for Azumaya algebras instead of their classes in the Brauer group. But there exists a vector bundle V on $\mathbb{A}^3 \smallsetminus \{(0,0,0)\}$ that doesn't extend, and then $\mathrm{End}(V)$ is an Azumaya algebra on the same variety that doesn't extend (Tate email).

[7] See

Gabber, Ofer, Some theorems on Azumaya algebras, Ph.D. Thesis, Harvard Univ., Cambridge, Mass., 1978

Finally, we mention a question that, because of its relation to Tate's conjecture, is the most interesting concerning the Brauer group.

QUESTION 2.19 (ARTIN) If X is proper over $\operatorname{Spec}\mathbb{Z}$, is $\operatorname{Br}(X)$ finite?

For X of dimension 1, class field theory shows that the answer is yes, but already the case of a surface over a finite field presents serious problems.[8]

EXERCISE 2.20 (a) Show that $\operatorname{Br}(R) = 0$ for any ring R in which every element r satisfies an equation $r^n = r$, where n is an integer ≥ 2 depending on r. (Hint: show that R is a union of finite rings.)

(b) Show that for every perfect field k, $\operatorname{Br}(k) \approx \operatorname{Br}(k[T])$. (Hint: use (III, 2.21) for $k_{\text{sep}}[T]/k[T]$, and note that $\operatorname{Br}(k_{\text{sep}}[T]) \subset \operatorname{Br}(k_{\text{sep}}(T)) = 0$.)

(c) Show that if R is a Henselian discrete valuation ring, then $\operatorname{Br}(K) = \operatorname{Br}(\hat{K})$, where K and $\hat{K}$ are the fields of fractions of R and its completion, except possibly for the p-components when $\operatorname{char}(K) = p \neq 0$. (Hint: K and $\hat{K}$ have the same Galois groups (Serre [7, II]), so one only needs to examine $L^* \hookrightarrow \hat{L}^*$ for L a finite extension of K; the quotient $\hat{L}^*/L^*$ is uniquely divisible by every integer prime to p.)

(d) Show that for any field k, there is an exact sequence

$$0 \to \operatorname{Br}(k[T]) \to \operatorname{Br}(\tilde{K}) \to X(G(k_s/k)) \to 0$$

where $\tilde{K}$ is the field of fractions of $(k[T]_{(T)})^{\mathrm{h}}$ and the first map is induced by $T \mapsto T^{-1}$. (Hint: note that $\tilde{K}$ is the Henselization of $k(\mathbb{P}^1)$ at the point at infinity; use (III, 2.23b), (III, 1.25), and (III, 1.28) to prove it in the case that $k = k_{\text{sep}}$; use (III, 2.21) to prove it in general.) (Compare Yuan [1].)

(e) Let R be the ring of integers in a quadratic number field $\mathbb{Q}[\sqrt{d}]$; use elementary number theory (in particular, no class field theory) to show that $\operatorname{Br}(R/\mathbb{Z}) = 0$. (Hint: note that

$$\operatorname{Br}(R/\mathbb{Z}) = \ker(\operatorname{Br}(\mathbb{Q}[\sqrt{d}]/\mathbb{Q}) \to \prod\nolimits_p \operatorname{Br}(\mathbb{Q}_p[\sqrt{d}]/\mathbb{Q}_p)$$

where $\mathbb{Q}_p = p$-adic numbers; by the periodicity of the cohomology of $G = \operatorname{Gal}(\mathbb{Q}[\sqrt{d}]/\mathbb{Q})$ the exercise is equivalent to showing that $a \in \mathbb{Q}$ is a norm from $\mathbb{Q}[\sqrt{d}]$ if it is a norm from $\mathbb{Q}_p[\sqrt{d}]$, all p; now use the proof of case $n = 3$ of Serre [10, IV. Thm. 8]. This proof is due to Legendre.)

(f) Let X be a smooth variety over a field of characteristic p. Show that every p-torsion element in $\operatorname{Br}'(X)$ becomes zero in $\operatorname{Br}'(Y)$ for some finite flat morphism $Y \to X$. Deduce that such an element is in $\operatorname{Br}(X)$ (2.14c). (Hint: use the Frobenius map.)

(published in: The Brauer group (Sem., Les Plans-sur-Bex, 1980), pp. 129–209, Lecture Notes in Math., 844, Springer, Berlin-New York, 1981).

Hoobler, Raymond T., A cohomological interpretation of Brauer groups of rings. Pacific J. Math. 86 (1980), no. 1, 89–92.

Hoobler, Raymond T. When is $\operatorname{Br}(X) = \operatorname{Br}'(X)$? In: Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), pp. 231–244, Lecture Notes in Math., 917, Springer, Berlin-New York, 1982.

In the above papers, it is shown that $\operatorname{Br}(X) = \operatorname{Br}'(X)_{\text{tors}}$ if X is a union of two affine schemes whose intersection is also affine. More recently, Gabber and de Jong have shown that $\operatorname{Br}(X) = \operatorname{Br}'(X)_{\text{tors}}$ for every scheme X admitting an ample invertible sheaf. See:

A.J. de Jong, A result of Gabber (n.d.). Available on the author's website.

[8] Indeed, there has been little progress on this question since the book was written. The author showed that $\operatorname{Br}(X)$ is finite if some l-primary component is finite, and this last assertion has been proved recently for all $K3$ surfaces.

(g) Let A be an Azumaya algebra over a scheme X. Show that there exists a scheme Y_A over X such that, for any X-scheme U, $A|U$ is trivial if and only if Y_A has a point in U. (Hint: use (III, 4.24).) (Compare Roquette [1].)

Comments on the Literature

Azumaya algebras were first studied over local rings by Azumaya [1], over arbitrary rings by Auslander and Goldman [1], and over schemes by Grothendieck [4]. Grothendieck was the first to give a satisfactory cohomological description of the Brauer group. We have largely followed these three sources. There is now a large literature on the Brauer group (see DeMeyer and Ingraham [1], Orzech and Small [1], Knus-Ojanguren, [1], and their bibliographies) much of which, unfortunately ignores the powerful methods introduced by Grothendieck. Yu. Manin has used the Brauer group to study the arithmetic and the geometry of cubic surfaces [1].[9]

[9] Since this was written, there has been much work done on the "Brauer-Manin" obstruction to the Hasse principle.