Motivic complexes over finite fields and the ring of correspondences at the generic point

James S. Milne   Niranjan Ramachandran

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Abstract

Already in the 1960s Grothendieck understood that one could obtain an almost entirely satisfactory theory of motives over a finite field when one assumes the full Tate conjecture. In this note we prove a similar result for motivic complexes. In particular Beilinson's $\mathbb{Q}$-algebra of "correspondences at the generic point" is then defined for all connected varieties. We compute this for all smooth projective varieties (hence also for varieties birational to such a variety).

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Introduction

Forty years after Grothendieck predicted that the standard cohomology functors factor through a tannakian category of pure motives, we still do not know how to construct such a category. However, when the field is finite and one assumes the full Tate conjecture, there is an almost entirely satisfactory theory of pure motives. Deligne tells us that this was known to Grothendieck, but it was re-discovered by Langlands and Rapoport (1987), who used it to state a conjecture, more precise than earlier attempts by Langlands, on the structure of the points modulo a prime on a Shimura variety. For a detailed account, see Milne 1994.

It is generally hoped that the standard cohomology functors to triangulated categories will factor through a triangulated category of motivic complexes with t-structure whose heart is (defined to be) the category of mixed motives (see, for example, Deligne 1994, §3). We show that, over a finite field, a triangulated category of motivic complexes exists with the expected properties if and only if the Tate conjecture holds and homological equivalence coincides with rational equivalence with $\mathbb{Q}$-coefficients (see Theorems 4.2 and 5.3 for more precise statements). Moreover, then a category of effective motivic complexes exists with the properties (A,B,C) of Beilinson 2002, and so there is a well-defined semisimple $\mathbb{Q}$-algebra of "correspondences at the generic point" attached to every variety over a finite field. We compute this $\mathbb{Q}$-algebra for smooth projective varieties (hence also for varieties birational to such a variety). As this requires the generalized Tate conjecture (in the sense of Grothendieck 1968, §10), we begin by giving an elementary proof that this follows from the usual Tate conjecture.

Notations

A variety is a geometrically-reduced separated scheme of finite type over a field. For a variety X over a perfect field k of characteristic $p \neq 0$ and algebraic closure $\bar{k}$, we set

$$H_l^i(X) = H_{\text{et}}^i(X_{\bar{k}}, \mathbb{Q}_l), \quad \text{if } l \neq p, \text{ and}$$
$$H_p^i(X) = H_{\text{crys}}^i(X/W) \otimes \mathbb{Q}, \quad W = W(k).$$

We use (r) to denote a Tate twist, and we write $\mathrm{hom}(l)$ for the equivalence relation on the space $Z^*(X)$ of algebraic cycles defined by H_l. Similarly, we write num and rat for numerical and rational equivalence. For an adequate equivalence relation $\sim$, $Z_\sim^i(X) = Z^i(X)/\sim$ and $Z_\sim^i(X)_\mathbb{Q} = Z_\sim^i(X) \otimes \mathbb{Q}$. For example, $Z_{\text{rat}}^i(X)$ is the Chow group $CH^i(X)$.

By a functor between additive categories, we mean an additive functor. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ of triangulated categories together with an isomorphism of functors $F \circ T \approx T' \circ F$ is said to be ***triangulated*** (formerly, exact; Verdier 1977, p4) if it takes distinguished triangles to distinguished triangles.

A triangulated category with t-structure (Gelfand and Manin 1996, IV 4.2, p278) will be referred to simply as a t-category. All t-structures will be assumed to be bounded (i.e., $\bigcup_{n\geq 0} \mathcal{D}^{\leq n} = \mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}^{\geq -n}$) and nondegenerate (i.e., $\bigcap_{n\geq 0} \mathcal{D}^{\leq -n} = 0 = \bigcap_{n\geq 0} \mathcal{D}^{\geq n}$).

The symbol $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$, and the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted $\mathbb{Q}^{\text{al}}$. Reductive groups are not required to be connected. Isomorphisms are denoted $\approx$ and canonical isomorphisms $\simeq$.

1 The generalized Tate conjecture

In this section, k is the subfield $\mathbb{F}_q$ of $\mathbb{F}$, and $l \neq p$.

1.1 By the ***full Tate conjecture*** for a smooth complete variety X over k and an $r \geq 0$, we mean the statement that the order of the pole of the zeta function $Z(X,t)$ at $t = q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r on X. If the Tate conjecture holds for X and r, then, for all $l \neq p$,

$T^r(X,l)$: the cycle class map $Z^r(X) \otimes \mathbb{Q}_l \to H_l^{2r}(X)(r)^{\mathrm{Gal}(\mathbb{F}/k)}$ is surjective, and

$E^r(X,l)$: the quotient map $Z_{\mathrm{hom}(l)}^r(X)_\mathbb{Q} \to Z_{\mathrm{num}}^r(X)_\mathbb{Q}$ is injective (i.e., $\mathrm{hom}(l)$ and num coincide with $\mathbb{Q}$-coefficients).

Conversely, if $T^r(X,l)$ and $E^r(X,l)$ hold for a single l, then the full Tate conjecture holds for X and r (Tate 1994, §2).[1] The statement $T^r(X,l)$ is the ***Tate conjecture for*** X, r, and l.[2]

Statement of the generalized Tate conjecture

Define a ***Tate structure*** to be a finite-dimensional $\mathbb{Q}_l$-vector space with a linear (Frobenius) map ϖ whose characteristic polynomial lies in $\mathbb{Q}[T]$ and whose eigenvalues are Weil q-numbers, i.e., algebraic numbers α such that, for some integer m (called the weight of α), $|\rho(\alpha)| = q^{m/2}$ for every homomorphism $\rho\colon \mathbb{Q}[\alpha] \to \mathbb{C}$, and, for some integer n, $q^n\alpha$ is an algebraic integer. When the eigenvalues are all of weight m (resp. algebraic integers, resp. semisimple), we say that V is of ***weight*** m (resp. ***effective***, resp. ***semisimple***). For example, for any smooth complete variety X over k, $H^i_l(X)$ is an effective Tate structure of weight $i/2$ (Deligne 1980), which is semisimple if X is an abelian variety (Weil 1948b, no. 70) or if the full Tate conjecture holds for $X \times X$ (Milne 1986b, 8.6).

Let X be a smooth complete variety over k. For each r, let $F^r_a H^i_l(X) \subset H^i_l(X)$ denote the subspace of classes with support in codimension at least r, i.e.,

$$F^r_a H^i_l(X) = \bigcup\nolimits_U \operatorname{Ker}(H^i_l(X) \to H^i_l(U))$$

where U runs over the open subvarieties of X such that $X \smallsetminus U$ has codimension $\geq r$.

EXAMPLE 1.2 If Z is a *smooth* closed subvariety of X of codimension r, then there is an exact Gysin sequence

$$\cdots \to H^{i-2r}_l(Z)(-r) \to H^i_l(X) \to H^i_l(U) \to \cdots, \quad U = X \smallsetminus Z,$$

(e.g., Milne 1980, VI 5.4), and so the kernel of $H^i_l(X) \to H^i_l(U)$ is an effective Tate structure of weight i whose twist by $\mathbb{Q}_l(r)$ is still effective.

CONJECTURE 1.3 *(Generalized Tate conjecture; cf. Grothendieck 1968, 10.3.). For a smooth complete variety X over k, every Tate substructure $V \subset H^i_l(X)$ such that $V(r)$ is still effective is contained in $F^r_a H^i_l(X)$.*

REMARK 1.4 Let X be a smooth complete variety over k. For any i and r, the set of eigenvalues α of ϖ_X on $H^i_l(X)$ such that α/q^r is an algebraic integer is stable under Galois conjugation. It follows that there exists a Tate substructure $F^r_b H^i_l(X)$ of $H^i_l(X)$ whose eigenvalues are exactly these α. It is the largest Tate substructure of $H^i_l(X)$ whose twist by $\mathbb{Q}_l(r)$ is still effective, and so the generalized Tate conjecture 1.3 is the statement:

$$F^r_b H^i_l(X) \subset F^r_a H^i_l(X).$$

EXAMPLE 1.5 Let Z' be a closed irreducible subvariety of $X_{\mathbb{F}}$ of codimension r. Then

$$H^{2r}_{Z'}(X_{\mathbb{F}}, \mathbb{Q}_l(r)) \to H^{2r}(X_{\mathbb{F}}, \mathbb{Q}_l(r)) \to H^{2r}(X_{\mathbb{F}} \smallsetminus Z', \mathbb{Q}_l(r))$$

[1] Therefore, if $T^r(X,l)$ holds for an abelian variety X over k, an integer r, and a set l of prime numbers of density one, then the full Tate conjecture holds for X and r (apply Clozel 1999).

[2] More precisely, it is Conjecture 1 of Tate 1965. Statement $E^r(X,l)$ is a $\mathbb{Q}$-version of the "conjectural statement" (**a**′) of Tate 1965. Our notation follows that of tate1994.

is exact, and $H^{2r}_{Z'}(X_{\mathbb{F}}, \mathbb{Q}_l(r)) \simeq \mathbb{Q}_l$; moreover, the image of 1 under the first map is the cohomology class of Z' (cf. Milne 1980, p269). For any open $U \subset X$, the kernel of

$$H_l^{2r}(X)(r) \to H_l^{2r}(U)(r)$$

is spanned by the cohomology classes of the irreducible components of $(X \smallsetminus U)_{\mathbb{F}}$, and some power of ϖ_X acts as 1 on it. On the other hand, $F_b^r H_l^{2r}(X)(r)$ is the subspace of $H_l^{2r}(X)(r)$ on which the eigenvalues of ϖ are roots of 1. Thus, the generalized Tate conjecture with $i = 2r$ states that this subspace is spanned by the classes of algebraic cycles of codimension r on $X_{\mathbb{F}}$. This is the Tate conjecture stated over $\mathbb{F}$ rather than $\mathbb{F}_q$.

The Tate conjecture implies the generalized Tate conjecture

Recall that, for a proper map $\pi\colon Y \to X$ of smooth varieties over an algebraically closed field, the Gysin map

$$\pi_*\colon H^i(Y, \mathbb{Q}_l) \to H^{i-2c}(X, \mathbb{Q}_l(-c)), \quad c = \dim Y - \dim X,$$

is defined to be the Poincaré dual of

$$\pi^*\colon H_c^{2d-i}(X, \mathbb{Q}_l(d)) \to H_c^{2d-i}(Y, \mathbb{Q}_l(d)), \quad d = \dim Y$$

(Milne 1980, VI 11.6). We shall need to know that these maps are compatible with restriction to open subvarieties.

LEMMA 1.6 *Let $\pi\colon Y \to X$ be a proper map of smooth complete varieties over an algebraically closed field, and let $j\colon U \hookrightarrow X$ an open immersion. Then the commutative diagram at left gives rise to the commutative diagram at right:*

$$\begin{array}{ccc} Y & \xleftarrow{j'} & \pi^{-1}U \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xleftarrow{j} & U \end{array} \qquad \begin{array}{ccc} H^i(Y, \mathbb{Q}_l) & \xrightarrow{j'^*} & H^i(\pi^{-1}U, \mathbb{Q}_l) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi'_*} \\ H^{i-2c}(X, \mathbb{Q}_l(-c)) & \xrightarrow{j^*} & H^{i-2c}(U, \mathbb{Q}_l(-c)) \end{array}$$

PROOF. Exercise for the reader. □

PROPOSITION 1.7 *Every effective semisimple Tate structure is a Tate substructure of $H_l^*(A)$ for some abelian variety A over $\mathbb{F}_q$.*

PROOF. We may assume that the Tate structure V is simple (i.e., irreducible). Then V has weight m for some $m \geq 0$, and the characteristic polynomial $P(T)$ of ϖ is a monic irreducible polynomial with coefficients in $\mathbb{Z}$ whose roots all have real absolute value $q^{m/2}$. According to Honda's theorem (Honda 1968; Tate 1968), $P(T)$ is the characteristic polynomial of an abelian variety A over $\mathbb{F}_{q^m}$. Let B be the abelian variety over $\mathbb{F}_q$ obtained from A by restriction of the base field. The eigenvalues of the Frobenius map on $H_l^1(B)$ are the m^{th}-roots of the eigenvalues of the Frobenius map on $H_l^1(A)$, and it follows that V is a Tate substructure of $H_l^m(B)$. □

LEMMA 1.8 *Let z be an algebraic cycle of codimension $\dim T + r$ on the product $T \times X$ of two smooth complete varieties over k (i.e., z is an algebraic correspondence of degree r from T to X). Assume that the push-forward of z on X is nonzero. Then the image of the map*

$$z_*\colon H_l^{i-2r}(T)(-r) \to H_l^i(X)$$

defined by z is contained in $F_a^r H_l^i(X)$.

PROOF. Let p, q denote the projection maps $T \times X \rightrightarrows T, X$, and let $[z]$ denote the cohomology class of z in $H_l^{2d_T+2r}(T \times X)(d_T + r)$, $d_T = \dim T$. Then

$$z_*(a) \overset{\text{def}}{=} q_*([z] \cup p^*(a)), \quad a \in H_l^{i-2r}(T)(-r).$$

As the push-forward $q_*(z)$ of z is nonzero, its support Z has codimension r.[3] Let $U = X \smallsetminus Z$. Then z has support in $T \times Z$, and so $[z]$ maps to zero in $H_l^{2d_T+2r}(T \times U)(d_T + r)$ (cf. 1.5). According to (1.6), the diagram

$$\begin{array}{ccc} H_l^{i+2d_T}(T \times X)(d_T) & \longrightarrow & H_l^{i+2d_T}(T \times U)(d_T) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\ H_l^i(X) & \longrightarrow & H_l^i(U) \end{array}$$

commutes, which shows that $z_*(a)$ maps to zero in $H_l^i(U)$, and therefore lies in $F_a^r H_l^i(X)$. □

LEMMA 1.9 *Let X be a smooth complete variety over k and let $i, r \in \mathbb{N}$. If there exists a smooth complete variety T such that*

- *$H_l^{i-2r}(T)$ is a semisimple Tate structure,*
- *the Tate conjecture $T^{\dim(T)+r}(T \times X, l)$ holds, and*
- *$F_b^r H_l^i(X)(r)$ is isomorphic to a Tate substructure of $H_l^{i-2r}(T)$*

then $F_b^r H_l^i(X) \subset F_a^r H_l^i(X)$.

PROOF. Let $d = \dim(T)$ and let V be a Tate substructure of $H_l^{i-2r}(T)$ for which there exists an isomorphism $f\colon V(-r) \to F_b^r H_l^i(X)$. Then

$$\begin{aligned} H_l^{2d+2r}(T \times X)(d + r) &\supset H_l^{2d+2r-i}(T)(d + r) \otimes H_l^i(X) \\ &\simeq \mathrm{Hom}(H_l^{i-2r}(T)(-r), H_l^i(X)) \\ &\supset \mathrm{Hom}(V(-r), F_b^r H_l^i(X)) \ni f. \end{aligned}$$

(The last inclusion depends on the choice of stable complement for V in $H_l^{i-2r}(T)$.) As f is fixed by $\mathrm{Gal}(\mathbb{F}/k)$, it can be approximated by the cohomology class of an algebraic correspondence z of degree r from T to X. Moreover, z can be chosen so that z_* is injective on V. Obviously z_* maps $H_l^{i-2r}(T)(-r)$ into $F_b^r H_l^i(X)$, and so

$$F_b^r H_l^i(X) \subset z_* V(-r) \overset{1.8}{\subset} F_a^r H_l^i(X).$$

□

[3]Recall that the push-forward $q_*(z)$ of an irreducible z is defined to be zero if $\dim(q(z)) < \dim z$.

THEOREM 1.10 *Let X be a smooth complete variety over k. If the Tate conjecture holds for all varieties of the form $A \times X$ with A an abelian variety (and some l), then the generalized Tate conjecture holds for X (and the same l).*

PROOF. As we noted above, $H_l^*(A)$ is a semisimple, and so this follows from (1.7) and 1.9). □

COROLLARY 1.11 *If the Tate conjecture holds for all abelian varieties over k (or for all smooth complete varieties over k) and some l, then the generalized Tate conjecture holds for the same class and that l.*

REMARK 1.12 As others have noted (Kahn 2002, Theorem 2; André 2004, 8.2), when one assumes the full Tate conjecture, the generalized Tate conjecture follows directly from from the description of the simple motives in terms of Weil numbers (see Milne 1994, Proposition 2.6).

Complements

1.13 Let X be a smooth projective variety over k, and let $V = F_b^r H_l^i(X)$. We know that $V(-r) \subset H_l^{i-2r}(A)$ for some abelian variety A over k (see 1.7). If $\dim A = d > i - 2r$, then, according to the Lefschetz hypersurface-section theorem, for any smooth hypersurface section Y of A (which exists by Gabber 2001), $V(-r) \subset H_l^{i-2r}(Y)$. Continuing in this fashion, we get that $V(-r) \subset H_l^{i-2r}(T)$ for some smooth projective T of dimension $i-2r$. Therefore, under the assumption of the Tate conjecture, there exists a smooth projective variety T of dimension at most $i-2r$ over k and an algebraic correspondence z from T to X of degree r such that $z_* H_l^{i-2r}(T)(r) = F_b^r H_l^i(X)$.

1.14 Deligne (1974b, 8.2.8) proves the following:

> Let X be a smooth complete variety over $\mathbb{C}$, and let Z be a closed subvariety of X of codimension r. For any desingularization $\tilde{Z} \to Z$ of Z, the sequence
>
> $$H^{i-2r}(\tilde{Z}, \mathbb{Q}(-r)) \to H^i(X, \mathbb{Q}) \to H^i(U, \mathbb{Q}), \quad U = X \smallsetminus Z,$$
>
> is exact.

A similar argument[4] proves the following l-adic analogue:

[4]For any proper surjective morphism $f: Y \to Z$ from a smooth projective variety Y, we can find a smooth projective simplicial scheme $Y_\bullet$ with $Y_0 = Y$ that is a proper hypercovering of Z. The corresponding spectral sequence (l-adic analogue of the spectral sequence Deligne 1974b, 8.1.19.1) degenerates at E_2 with $\mathbb{Q}_l$-coefficients because of weight considerations, and gives an exact sequence

$$0 \to \frac{H_l^i(Z)}{W_{i-1} H_l^i(Z)} \to H_l^i(Y_0) \xrightarrow{\delta_0 - \delta_1} H_l^i(Y_1).$$

This implies that the image of $H_l^i(Z)$ in $H_l^i(Y)$ is the (largest) quotient of pure weight i of $H_l^i(Z)$. This implies the l-adic analogue of Deligne 1974b, 8.2.7, (the proof there works as the Gr_*^W functor is exact) and of ibid. 8.2.8.

> Let X be a smooth complete variety over a perfect field k, and let Z be a closed subvariety of X of codimension r. For any smooth alteration $\tilde{Z} \to Z$ of Z, the sequence
>
> $$H_l^{i-2r}(\tilde{Z})(-r) \to H_l^i(X) \to H_l^i(U), \quad U = X \smallsetminus Z,$$
>
> is exact.

Since de Jong (1996, 3.1) shows that smooth alterations always exist, this implies that

$$F_a^r H_l^i(X) \subset F_b^r H_l^i(X).$$

The generalized Tate conjecture then states that

$$F_a^r H_l^i(X) = F_b^r H_l^i(X).$$

1.15 The above statements hold *mutatis mutandis* for p. For a smooth complete variety X, $H_p^i(X)$ is an F-isocrystal, i.e., a finite-dimensional vector space over $B(\mathbb{F}_q) \stackrel{\text{def}}{=} W(\mathbb{F}_q) \otimes \mathbb{Q}$ equipped with a σ-linear bijection $F\colon H_p^i(X) \to H_p^i(X)$. The full Tate conjecture for X and r is equivalent to

$T^r(X, p)$: the cycle class map $Z^r(X) \otimes \mathbb{Q}_p \to H_p^{2r}(X)(r)^{F=1}$ is surjective (***Tate conjecture for*** p), and

$E^r(X, p)$: the quotient map $Z^r_{\mathrm{hom}(p)}(X)_{\mathbb{Q}} \to Z^r_{\mathrm{num}}(X)_{\mathbb{Q}}$ is injective.

Define

$$F_a^r H_p^i(X) = \bigcup\nolimits_Z \mathrm{Im}(H_p^{i-2r}(\tilde{Z})(-r) \to H_p^i(X))$$

where Z runs over the closed subvarieties of X such that Z is of codimension at least r and $\tilde{Z}$ is a smooth alteration of Z. If the Tate conjecture holds for smooth complete varieties over k and p, then

$$F_a^r H_p^i(X) = F_b^r H_p^i(X)$$

where $F_b^r H_p^i(X) = H_p^i(X)_{[r,\infty)}$, the sub-isocrystal of $H_p^i(X)$ with slopes at least r. The proofs are similar to those in the case $l \neq p$ — we omit the details.

1.16 Similar arguments show that the generalized Tate conjecture over number fields follows from the Tate conjecture and an effective version of the Fontaine-Mazur conjecture (Fontaine and Mazur 1995, Conjecture 1, p44) that specifies which representations arise from effective motives.

NOTES It was known to Grothendieck that the generalized Hodge conjecture follows from the usual Hodge conjecture and the following weak analogue of (1.7),

> Let V be a simple Hodge substructure of the cohomology of a smooth complex projective variety; if its Tate twist $V(r)$ is still effective (i.e., has only nonnegative Hodge numbers), then $V(r)$ occurs in the cohomology of a smooth complex projective variety.

presumably by more-or-less the above argument. See Grothendieck 1969, top of p301 (also Abdulali 1997, §2 and Schoen 1989, §0).

2 The category of pure motives

In this section $k = \mathbb{F}_q$.

For any adequate equivalence relation $\sim$, Grothendieck's construction gives a rigid pseudo-abelian tensor $\mathbb{Q}$-category $\mathcal{M}_\sim(k)$ of pure motives (Saavedra Rivano 1972, VI 4.1.3.5, p359) and a map h from the smooth projective varieties over k to $\mathcal{M}_\sim(k)$ which is natural for algebraic correspondences modulo $\sim$. Because rational equivalence is the finest adequate equivalence relation, h factors through a tensor functor $\mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{M}_\sim(k)$. Conversely, a tensor functor from $\mathcal{M}_{\mathrm{rat}}(k)$ to an additive tensor category with $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$ defines an adequate equivalence relation (cf. Jannsen 2000, 1.7). When $\sim$ is numerical equivalence, $\mathcal{M}_\sim(k)$ is a semisimple (Jannsen 1992).

For a smooth projective variety X over k, there are well-defined polynomials $P_{X,i}(T) \in \mathbb{Q}[T]$ such that $P_{X,i}(T) = \det(1 - \varpi_X T \mid H_l^i(X))$ for all l; moreover, $P_{X,i}$ has reciprocal roots of absolute value $q^{\frac{i}{2}}$ (Deligne 1974a). The $P_{X,i}(T)$ are relatively prime, and so there exist $P^i(T) \in \mathbb{Q}[T]$, well-defined up to a multiple of $\prod_i P_{X,i}(T)$, such that

$$P^i(T) \equiv \begin{cases} 1 & \mod P_{X,i}(T) \\ 0 & \mod P_{X,j}(T) \text{ for } j \neq i. \end{cases} \tag{1}$$

Because $\prod_i P_{X,i}(\varpi_X)$ acts as zero on $H_l^*(X)$, the graph p^i of $P^i(\varpi_X)$ is a well-defined element of $Z_{\mathrm{hom}(l)}(X \times X)_{\mathbb{Q}}$ (or $Z_{\mathrm{num}}(X \times X)_{\mathbb{Q}}$), and $\{p^0, \ldots, p^{2d}\}$ is a complete set of orthogonal idempotents. Let $hX = \bigoplus_i h^i X$ be the corresponding decomposition. When we use this decomposition to modify the commutativity constraint in $\mathcal{M}_{\mathrm{num}}(k)$, the rank of each object of $\mathcal{M}_{\mathrm{num}}(k)$ becomes a nonnegative integer, and so $\mathcal{M}_{\mathrm{num}}(k)$ is a tannakian category (Deligne 1990, 7.1).

The category $\mathcal{M}_{\mathrm{num}}(k)$ has a canonical (Frobenius) element $\varpi \in \mathrm{Aut}^{\otimes}(\mathrm{id}_{\mathcal{M}_{\mathrm{num}}(k)})$ and a canonical (weight) $\mathbb{Z}$-gradation. An object M of $\mathcal{M}_{\mathrm{num}}(k)$ is of pure weight m if and only if its Frobenius element ϖ_M has eigenvalues of absolute value $q^{m/2}$.

Recall (Deligne 1989, §6) that the fundamental group $\pi(\mathcal{T})$ of a Tannakian category is an affine group scheme in $\mathrm{Ind}\,\mathcal{T}$ that acts on each object of $\mathcal{T}$ in such a way that these actions define an isomorphism

$$\omega(\pi(\mathcal{T})) \simeq \underline{\mathrm{Aut}}^{\otimes}(\omega)$$

for each fibre functor ω. Any subgroup of the centre of $\pi(\mathcal{T})$ lies in $\mathrm{Ind}\,\mathcal{T}^0$ where $\mathcal{T}^0$ is the full subcategory of trivial objects (those isomorphic to a multiple of $\mathbb{1}$). Since $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, -)$ defines an equivalence of $\mathcal{T}^0$ with the finite-dimensional vector spaces over the ground field, such a subgroup can be identified with an affine group scheme in the usual sense. For example, the centre of $\pi(\mathcal{T})$ is $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathcal{T}})$ (cf. Saavedra Rivano 1972, II 3.3.3.2).

Recall (e.g., Milne 1994, §2) that the Weil-number group P is the affine group scheme of multiplicative type over $\mathbb{Q}$ whose character group consists of the Weil q-numbers in $\mathbb{Q}^{\mathrm{al}}$. Define the Frobenius element ϖ_{univ} in $P(\mathbb{Q})$ to be that corresponding to $\alpha \mapsto \alpha$ under the bijection

$$P(\mathbb{Q}) \simeq \mathrm{Hom}(X^*(P), \mathbb{Q}^{\mathrm{al}})^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})}.$$

Note that, for any smooth projective variety X over $\mathbb{F}_q$, the roots of $P_{X,i}(T)$ in $\mathbb{Q}^{\mathrm{al}}$ are Weil q-integers of weight i (i.e., Weil q-numbers of weight i that are algebraic integers).

LEMMA 2.1 *The group of Weil q-numbers is generated by the Weil q-numbers of abelian varieties over k.*

PROOF. Let α be a Weil q-number. After multiplying α by a power of q, we may suppose that it is a Weil q-integer, of weight m say. Then $\alpha^{1/m}$ is a Weil q-integer of weight 1, and hence arises from an abelian variety by Honda's theorem. □

PROPOSITION 2.2 *The affine subgroup scheme of $\pi(\mathcal{M}_{num}(k))$ generated by ϖ_{univ} is canonically isomorphic to P. It equals $\pi(\mathcal{M}_{num}(k))$ if and only if the full Tate conjecture holds over k.*

PROOF. Let $Z = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id})$ be the centre of $\pi(\mathcal{M}_{\mathrm{num}}(k))$. Because $\mathcal{M}_{\mathrm{num}}(k)$ is semisimple, $\pi(\mathcal{M}_{\mathrm{num}}(k))$ is pro-reductive (cf. Deligne and Milne 1982, 2.23). Therefore Z is of multiplicative type, which implies that the closed subgroup scheme $\langle\varpi_{\mathrm{univ}}\rangle$ generated by ϖ_{univ} is also of multiplicative type. The homomorphism $\chi \mapsto \chi(\varpi_{\mathrm{univ}})\colon X^*(\langle\varpi_{\mathrm{univ}}\rangle) \to \mathbb{Q}^{\mathrm{al}\times}$ is injective, and its image consists of the Weil q-numbers that occur as roots of the characteristic polynomial of ϖ_M for some M in $\mathcal{M}_{\mathrm{num}}(k)$. According to Lemma 2.1, this consists of all Weil q-numbers, and so $X^*(\langle\varpi_{\mathrm{univ}}\rangle) \simeq X^*(P)$. Hence $\langle\varpi_{\mathrm{univ}}\rangle \simeq P$.

If the full Tate conjecture holds, then, for any fibre functor ω over $\mathbb{Q}^{\mathrm{al}}$ and smooth projective variety X, the $\mathbb{Q}^{\mathrm{al}}$-span of the algebraic cycles in $\omega(h^{2i}(X)(i))$ consists of the tensors fixed by ϖ_{univ}. Therefore, the inclusion $\langle\varpi_{\mathrm{univ}}\rangle \hookrightarrow \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an isomorphism, i.e., $\omega(P) \hookrightarrow \omega(\pi(\mathcal{M}_{\mathrm{num}}(k))$ is an isomorphism, which implies that $P \hookrightarrow \pi(\mathcal{M}_{\mathrm{num}}(k))$ is an isomorphism. The converse can be proved by the same argument as in the proof of Milne 1999, Proposition 7.4. □

If num and $\mathrm{hom}(l)$ coincide with $\mathbb{Q}$-coefficients, then H_l defines a fibre functor ω_l on $\mathcal{M}_{\mathrm{num}}(k)$. Without any assumptions, it is known that there exists a polarizable semisimple tannakian category with fundamental group P and with fibre functors ω_l for all l. Moreover, any two such systems are equivalent (Langlands and Rapoport 1987; Milne 2003, §6). However, it has not been shown that there exists a natural functor from $\mathcal{M}_{\mathrm{rat}}(k)$ to such category. In fact, we have the following:

PROPOSITION 2.3 *If there exists a full tensor functor r preserving Frobenius elements from $\mathcal{M}_{rat}(k)$ to a tannakian category $\mathcal{M}$ with fundamental group P, then the full Tate conjecture holds over k, and r defines an equivalence of tensor categories $\mathcal{M}_{num}(k) \to \mathcal{M}$.*

PROOF. Such a functor r defines an adequate equivalence relation $\sim$ (see above) such that r factors into

$$\mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{M}_{\sim}(k) \xrightarrow{\bar{r}} \mathcal{M}$$

with $\bar{r}$ a fully faithful tensor functor. Because P is a pro-reductive, $\mathcal{M}$ is semisimple (cf. Deligne and Milne 1982, 2.23). It follows that $\mathcal{M}_{\sim}(k)$ is semisimple (apply the criterion in Jannsen 1992, Lemma 2), and so $\sim$ is numerical equivalence (ibid. Theorem 1). The simple objects of $\mathcal{M}$ are classified by the orbits of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acting on $X^*(P)$, i.e., by the conjugacy classes of Weil q-numbers, and so Lemma 2.1 shows that $\mathcal{M}$ is generated as a tensor category by the images of abelian varieties. Therefore, $\bar{r}$ is a tensor equivalence, and so defines an isomorphism of P with $\pi(\mathcal{M}_{\mathrm{num}}(k))$. We can now apply Proposition 2.2. □

When we drop the condition that r is full, we obtain a conjecture that is much weaker than the full Tate conjecture, but which has many of the same consequences.

CONJECTURE 2.4 *There exists a $\mathbb{Q}$-linear tannakian category $\mathcal{M}$ with fundamental group P and a $\mathbb{Q}$-linear tensor functor $r\colon \mathcal{M}_{rat}(k) \to \mathcal{M}$ preserving Frobenius elements such that, for each prime l (including p), there is a fibre functor ω_l on $\mathcal{M}$ whose composite with r is H_l.*

2.5 There are canonical homomorphisms

$$\mathbb{G}_m \xrightarrow{w} P \xrightarrow{t} \mathbb{G}_m$$

such that $t(w(x)) = x^{-2}$ and $X^*(w)(\varpi) = \mathrm{weight}(\varpi)$. Therefore, on any $\mathbb{Q}$-linear Tannakian category with fundamental group P, there is a natural weight gradation for which an object M is of pure weight m if and only if its Frobenius element ϖ_M has eigenvalues of absolute value $q^{m/2}$, and there is a Tate object of weight -2. Assume that there exists an r as in the conjecture, and set $h_r^i(X) = (rh(X))^i$ for a smooth projective variety X. Then, $\omega_l(h_r^i(X)) = H_l^i(X)$ and the natural map

$$\mathrm{Hom}_{\mathcal{M}}(\mathbb{1}, h_r^{2i}(X)(i)) \otimes \mathbb{Q}_l \longrightarrow \mathrm{Hom}_{\mathbb{Q}_l\text{-linear}}(\mathbb{Q}_l, H_l^{2i}(X)(i)) = H_l^{2i}(X)(i) \qquad (2)$$

is injective (Deligne 1990, 2.13) with image the $\mathbb{Q}_l$-space of Tate classes in $H_l^{2i}(X)(i)$ (i.e., the subspace conjectured by Tate to be the $\mathbb{Q}_l$-span of the algebraic classes). In other words, (2) is a $\mathbb{Q}$-structure on the space of l-adic Tate classes, and so the elements of

$$T^i(X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{M}}(\mathbb{1}, h_r^{2i}(X)(i))$$

deserve to be called rational Tate classes. In essence, Conjecture 2.4 says that there is a "good" theory of rational Tate classes on smooth projective varieties over k.

2.6 One may even hope that the r in Conjecture 2.4 factors through $\mathcal{M}_{\mathrm{num}}(k)$. The theory of quotients of tannakian categories (Milne 2005) gives the following description of the exact $\mathbb{Q}$-linear tensor functors $r\colon \mathcal{M}_{\mathrm{num}}(k) \to \mathcal{M}$ identifying $\pi(\mathcal{M})$ with the subgroup P of $\pi(\mathcal{M}_{\mathrm{num}}(k))$.

Let $\mathcal{M}_{\mathrm{num}}(k)^{\langle\varpi_{\mathrm{univ}}\rangle}$ be the full subcategory of $\mathcal{M}_{\mathrm{num}}(k)$ of objects with trivial Frobenius element. Assume $\mathcal{M}_{\mathrm{num}}(k)^{\langle\varpi_{\mathrm{univ}}\rangle}$ is neutral, and choose a $\mathbb{Q}$-valued functor ω_0 on $\mathcal{M}(k)^{\langle\varpi_{\mathrm{univ}}\rangle}$. Let $(\mathcal{M}_{\mathrm{num}}(k)/\omega_0)'$ be the category with one object $\bar{X}$ for each object X of $\mathcal{M}_{\mathrm{num}}(k)$, and with

$$\mathrm{Hom}_{(\mathcal{M}_{\mathrm{num}}(k)/\omega_0)'}(\bar{X}, \bar{Y}) = \omega_0(\underline{\mathrm{Hom}}(\bar{X}, \bar{Y})^{\langle\varpi_{\mathrm{univ}}\rangle}).$$

There is a unique tensor structure on $(\mathcal{M}_{\mathrm{num}}(k)/\omega_0)'$ for which

$$q\colon \mathcal{M}_{\mathrm{num}}(k) \to (\mathcal{M}_{\mathrm{num}}(k)/\omega_0)'$$

is a tensor functor. With this structure, $(\mathcal{M}_{\mathrm{num}}(k)/\omega_0)'$ is rigid, and we define $\mathcal{M}_{\mathrm{num}}(k)/\omega_0$ to be its pseudo-abelian hull. Then $\mathcal{M}_{\mathrm{num}}(k)/\omega_0$ is a tannakian category with fundamental group P, and every "quotient" of $\mathcal{M}_{\mathrm{num}}(k)$ with fundamental group P arises in this way. In particular, such a quotient exists if and only if $\mathcal{M}_{\mathrm{num}}(k)^{\langle\varpi_{\mathrm{univ}}\rangle}$ is neutral (which will be so, for example, if its fundamental group satisfies a Hasse principle for H^1).

2.7 For a smooth projective variety X over k, let $H_f^*(X)$ be the restricted topological product of the cohomology algebras $H_l^*(X)$ (see Milne and Ramachandran 2004, §2). Call a class α in $H_f^*(X)$ strongly motivated if $\eta^r \cup \alpha$ is algebraic for some ample divisor class η on X and $r \geq 0$, and motivated if it is of the form $p_{X*}(\beta \cup \gamma)$ with β and γ algebraic and strongly motivated classes respectively on $X \times Y$ for some smooth projective variety Y. Then the motivated classes on X form a graded $\mathbb{Q}$-subalgebra of $H_f^{2*}(X)(*)$, and, for every regular map γ, the maps γ^* and γ_* send motivated classes to motivated classes; moreover, the motivated classes form the smallest collection containing the algebraic classes and satisfying these conditions and the Lefschetz standard conjecture (André 1996, §2). Assume André's conjecture:

> for all motivated classes α, β of complementary dimension on a smooth projective variety over k, $\langle \alpha \cup \beta \rangle \in \mathbb{Q}$ (inside $\mathbb{A}_f$).

and define $\mathcal{M}'(k)$ using motivated classes modulo numerical equivalence instead of algebraic classes. There is a canonical $\mathbb{Q}$-linear tensor functor $\mathcal{M}_{\text{rat}}(k) \to \mathcal{M}'(k)$, and one hopes that the fundamental group of $\mathcal{M}'(k)$ is P. The arguments of Milne 1999 (with "algebraic class" replaced by "motivated class") show that this is true of the subcategory generated by abelian varieties,[5] and so P is a direct factor of $\pi(\mathcal{M}'(k))$.

3 The category of motives

The next observation goes back to Grothendieck.

PROPOSITION 3.1 *Let $\mathcal{MM}(\mathbb{F}_q)$ be a pseudo-abelian category containing $\mathcal{M}_{num}(\mathbb{F}_q)$ as a full subcategory. Assume*

(a) *each object M of $\mathcal{MM}(\mathbb{F}_q)$ has a (weight) filtration*

$$\cdots \subset W_{i-1}M \subset W_iM \subset \cdots$$

such that $W_iM/W_{i-1}M$ is a pure motive of weight i;

(b) *the Frobenius element extends to $\mathcal{MM}(\mathbb{F}_q)$ and preserves the weight filtrations.*

Then the inclusion $\mathcal{M}_{num}(\mathbb{F}_q) \to \mathcal{MM}(\mathbb{F}_q)$ is an equivalence of tensor categories.

PROOF. For X in $\mathcal{MM}(\mathbb{F}_q)$, let $P_i(T)$ be the characteristic polynomial of $\varpi_{W_iM/W_{i-1}M}$, and define $P^i(T)$ to satisfy (1). Let $p^i = P^i(\varpi_M)$. Then the p^i form a complete set of orthogonal idempotents in $\operatorname{End}(M)$ which decompose M into a direct sum isomorphic to $\bigoplus_i W_iM/W_{i-1}M$. □

4 Triangulated motivic categories

By a ***triangulated motivic category*** over a field k, we mean a triangulated rigid tensor category[6] $\mathcal{D}$ together with a covariant functor

$$R\colon \mathcal{M}_{\text{rat}}(k) \to \mathcal{D}$$

[5]This uses the Abdulali-André theorem (André 1996, 0.6.2) that all Hodge classes on complex abelian varieties are motivated, and André's theorem that motivated classes specialize to motivated classes (André 2006).

[6]Not being able to find a definition in the literature, we suggest one. A ***triangulated rigid tensor category*** is a category $\mathcal{C}$ with a rigid tensor structure (in the sense of Deligne and Milne 1982) and a triangulated structure (in the sense of Verdier 1977) satisfying the following compatibility conditions:

– for each object A of $\mathcal{C}$, the functor $C \mapsto A \otimes C\colon \mathcal{C} \to \mathcal{C}$ is triangulated;

– the functor $C \mapsto C^\vee\colon \mathcal{C}^{\text{opp}} \to \mathcal{C}$ is triangulated.

and isomorphisms for all smooth projective varieties X and all $i, j \in \mathbb{Z}$

$$K_{2j-i}(X)^{(j)} \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(hX)(j)[i]) \tag{3}$$

that are natural for the maps defined by algebraic correspondences and reduce to the identity map when X is a point and $i = j = 0$ (see Jannsen 2000, §7, p257, which omits the final condition). Here $K_i(X)^{(j)}$ is the subspace of $K_i(X) \otimes \mathbb{Q}$ on which each Adams operator ψ^m acts as m^j. According to ibid., p257, over any field k that admits resolution of singularities, triangulated motivic categories have been constructed (independently) by Hanamura (1995, 1999, 2004), Levine (1998), and Voevodsky (2000). When $k = \mathbb{F}_q$, ψ^q acts as ϖ_X (Hiller 1981, §5; Soulé 1985, 8.1), and so[7] $K_i(X)^{(j)}$ is the subspace on which ϖ_X acts as q^j.

Let $\mathcal{D} = \mathcal{D}(k)$ be a triangulated motivic category. As we noted in the introduction, for the "true" triangulated motivic category, there should be a t-structure on $\mathcal{D}(k)$ whose heart $\mathcal{MM}(k) \overset{\text{def}}{=} \mathcal{D}(k)^{\heartsuit}$ is the category of mixed motives. As Jannsen (2000, §7, p257) explains, there should be the following compatibilities between R and the t-structure:

(a) for each standard Weil cohomology, the composite

$$\begin{array}{ccccc} \mathcal{M}_{\mathrm{rat}}(k) & \xrightarrow{R} & \mathcal{D} & \xrightarrow{\bigoplus_i H^i} & \mathcal{MM}(k) \\ & & K & \mapsto & \bigoplus_i H^i(K) \end{array}$$

factors through $\mathcal{M}_{\mathrm{hom}}(k)$, and defines a fully faithful functor $\bar{R}\colon \mathcal{M}_{\mathrm{hom}}(k) \to \mathcal{MM}(k)$ (here $H^i(K) = \tau_{\leq 0}\tau_{\geq 0}(K[-i])$);

(b) for each smooth projective variety X, $\bigoplus_i H^i(R(hX))$ is the weight gradation of hX.

When k is finite, condition (b) says that $H^i(R(hX)) = \bar{R}(h^i(X))$.

Evidently, there should also be the following compatibilities between the tensor structures and the t-structure:

(c) the subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ are tensor subcategories of $\mathcal{D}$, and $M \mapsto M^{\vee}$ interchanges $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$, and

(d) $R\colon \mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{MM}(k)$ is a tensor functor.

Note that (c) implies that $\mathcal{MM}(k)$ is a rigid tensor subcategory of $\mathcal{D}$.

DEFINITION 4.1 A t-structure on a triangulated motivic category is said to be ***admissible*** if it satisfies the conditions (a,b,c,d).

THEOREM 4.2 *Let k be a finite field. If there exists a triangulated motivic category $\mathcal{D}$ over k and a admissible t-structure on $\mathcal{D}$ such that*

- *the heart of $\mathcal{D}$ is a tannakian category $\mathcal{M}$ with fundamental group P, and*
- *the functor $\mathcal{M}_{rat}(k) \to \mathcal{M}$ in (a) above preserves Frobenius elements,*

then

(a) *the full Tate conjecture holds for all smooth projective varieties over k;*

(b) *for each l, the functor $R_l\colon \mathcal{M}_{hom(l)}(k) \to \mathcal{M}$ defined by R is an equivalence of abelian categories;*

(c) *rational equivalence equals numerical equivalence ($\mathbb{Q}$-coefficients);*

(d) *for all M, N in $\mathcal{M}(k)$ and $i \neq 0$, $\mathrm{Hom}_{\mathcal{D}}(M, N[i]) = 0$.*

---

[7]Because the m^i-eigenspace of ψ^m is independent of m (Seiler 1988, Theorem 1).

PROOF. Proposition 2.3 shows that the full Tate conjecture holds and that R_l is essentially surjective (hence an equivalence). Moreover, it allows us to identify $\mathcal{M}$ with $\mathcal{M}_{\mathrm{num}}(k)$.

We next prove (c). When $i = 2j$, the isomorphism (3) becomes

$$K_0(X)^{(j)} \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(hX)(j)[2j]). \tag{4}$$

As we noted above, ϖ_X acts on $K_0(X)^{(j)}$ as q^j. The Tate conjecture implies the Lefschetz standard conjecture, and so, for any smooth projective variety X, there exists an isomorphism

$$R(hX)(j)[2j] \approx \bigoplus\nolimits_s h^s(X)(j)[2j-s] \tag{5}$$

(Deligne 1968, Van den Bergh 2004). The characteristic polynomial $P_{X,s}$ of ϖ_X on $h^s X$ has roots of absolute value $q^{s/2}$, and $P_{X,s}(\varpi_X)$ acts as zero on $h^s(X)$ and hence on $\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[2j-s])$. But we know from (4) that it acts as $P_{X,s}(q^j)$. Therefore, $\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[2j-s]) = 0$ unless $s = 2j$, and so (4) becomes

$$K_0(X)^{(j)} \simeq \mathrm{Hom}_{\mathcal{M}_{\mathrm{num}}(k)}(\mathbb{1}, h^{2j}(X)(j)).$$

Under Grothendieck's isomorphism $K_0(X)_{\mathbb{Q}} \simeq CH^*(X)_{\mathbb{Q}}$, the factors $K_0(X)^{(j)}$ and $CH^j(X)_{\mathbb{Q}}$ correspond (this is obvious over a finite field), and (by definition)

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{num}}(k)}(\mathbb{1}, h^{2j}(X)(j)) = Z^j_{\mathrm{num}}(X)_{\mathbb{Q}}.$$

Moreover, our conditions imply that the isomorphism

$$CH^j(X)_{\mathbb{Q}} \simeq Z^j_{\mathrm{num}}(X)_{\mathbb{Q}} \tag{6}$$

obtained by combining these isomorphisms is the canonical one.[8] Hence, we have proved (c), and we have shown that

$$\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(hX)(j)[i]) = 0 \tag{7}$$

when $i = 2j \neq 0$.

Finally, we prove (d). Because $\mathcal{M}$ is a rigid subcategory of $\mathcal{D}$, for M, N in $\mathcal{M}$ there exists an object $\underline{\mathrm{Hom}}(M,N)$ in $\mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{D}}(T \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{D}}(T, \underline{\mathrm{Hom}}(M,N))$ for all T in $\mathcal{D}$. In particular,

$$\mathrm{Hom}_{\mathcal{D}}(M, N[i]) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, \underline{\mathrm{Hom}}(M,N)[i]).$$

[8]Let p and q be the projection maps

$$\mathrm{pt} \xleftarrow{p} X \times \mathrm{pt} \xrightarrow{q} X.$$

Let $\gamma \in CH^j(X)$, and let f be the map $CH^*(\mathrm{pt}) \to CH^{*+j}(X)$ defined by the correspondence $q^*(\gamma)$. Then

$$f(1_{\mathrm{pt}}) \overset{\mathrm{def}}{=} q_*(q^*(\gamma) \cup p^*(1_{\mathrm{pt}})) = \gamma \cup q_* p^*(1_{\mathrm{pt}}) = \gamma \cup 1_X = \gamma.$$

As (6) is functorial for correspondences, and the bottom row in

$$\begin{array}{ccc} CH^j(X) & \longrightarrow & Z^j_{\mathrm{num}}(X) \\ \uparrow f & & \uparrow f \\ CH^0(\mathrm{pt}) & \longrightarrow & Z^0_{\mathrm{num}}(\mathrm{pt}) \end{array}$$

sends 1 to 1 (by assumption), it follows that the top row sends γ to γ.

Therefore, because every object of $\mathcal{M}$ is a direct summand of $R(hX)(j)$ for some smooth projective variety X and integer j, it suffices to prove (d) with $M = \mathbb{1}$ and $N = R(hX)(j)$. We know it when $i = 2j$ (see (7)), and so, to complete the proof of (d), it remains to prove (7) when $i \neq 2j$. Because of (3), it suffices to show that (a) and (c) imply that $K_i(X)_{\mathbb{Q}} = 0$ whenever $i \neq 0$. This is done in Geisser 1998, 3.3. We recall the proof. The functors $K_i(X) \otimes \mathbb{Q}$ factor through $\mathcal{M}_{\mathrm{rat}}(k)$ (Soulé 1984), and hence (because of (c)) through $\mathcal{M}_{\mathrm{num}}(k)$. Therefore, it suffices to prove that $K_i(M) \otimes \mathbb{Q} = 0$ $(i \neq 0)$ for M a simple motive in $\mathcal{M}_{\mathrm{num}}(k)$. If $M = \mathbb{L}^j$, then $K_i(\mathbb{L}^j)$ is a direct factor of $K_i(\mathbb{P}^j)$, which is torsion (Quillen 1973). If $M \neq \mathbb{L}^j$, then $P_M(T)$ does not have q^j as a root (Milne 1994, 2.6). As $P_M(\varpi_X)$ acts as the nonzero rational number $P_M(q^j)$ on $K_i(M)^{(j)}$, and also as zero, the group $K_i(M)^{(j)}$ must be zero. □

COROLLARY 4.3 *Let $\mathcal{D}$ be as in the theorem, and let $\mathcal{M}$ be its heart. If the inclusion $\mathcal{M} \to \mathcal{D}$ extends to a functor $\mathcal{D}^b(\mathcal{M}) \to \mathcal{D}$ (e.g., if $\mathcal{D}$ is endowed with a filtered triangulated category; see 4.6a below), then that functor is an equivalence.*

PROOF. It suffices to show that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(M, N[i]) \to \mathrm{Hom}_{\mathcal{D}}(M, N[i])$ is an isomorphism for all M, N in $\mathcal{M}$ and all i (see 4.6b below). For $i = 0$ this is automatic, and for $i \neq 0$, both groups are zero (recall that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(M, N[i]) \simeq \mathrm{Ext}^i_{\mathcal{M}}(M, N)$, and that $\mathcal{M}$ is semisimple). □

REMARK 4.4 The existence of a admissible t-structure on a triangulated motivic category $\mathcal{D}$ implies the existence of a Bloch-Beilinson filtration on the Chow groups of smooth projective varieties for which

$$Gr^s(CH^j(X)) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^{2j-s}(X)(j)[s]) \tag{8}$$

(Jannsen 2000, p258, 4.3). For a finite field, the existence of a Bloch-Beilinson filtration implies that rational equivalence equals numerical equivalence ($\mathbb{Q}$-coefficients) (ibid., 4.17).

REMARK 4.5 Beilinson has conjectured that, for a smooth projective variety X,

$$Gr^s(CH^j(X)) = \mathrm{Ext}^s_{\mathcal{MM}_{\mathrm{num}}(k)}(\mathbb{1}, h^{2j-s}(X)(j)).$$

This is compatible with (8) only if $\mathcal{D} = \mathcal{D}^b(\mathcal{MM}_{\mathrm{num}}(k))$ (see the next remark).

REMARK 4.6 (a) Let $\mathcal{D}$ be a t-category with heart $\mathcal{C}$. Then $D^b(\mathcal{C})$ is also a t-category with heart $\mathcal{C}$, but in general there is no obvious relation between $D^b(\mathcal{C})$ and $\mathcal{D}$ (cf. Gelfand and Manin 1996, IV 4.13, p285). In particular, there will be no obvious functor $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ extending the inclusion of $\mathcal{C}$ into $\mathcal{D}$ unless $\mathcal{D}$ is endowed with an additional structure. Beilinson (1987) defines the notion of a filtered triangulated category, and states[9] that such a category over a t-category $\mathcal{D}$ gives rise to a well-defined t-exact functor $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ inducing the identity functor on $\mathcal{C}$ (ibid. A.6). The usual triangulated categories are endowed with filtered triangulated categories over them (ibid. A.2; Beilinson et al. 1982, 3.1).

(b) Let $\mathcal{D}$ be a t-category with heart $\mathcal{C}$. A t-exact functor $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ inducing the identity functor on $\mathcal{C}$ need not be an equivalence even when $\mathcal{C}$ is semisimple (Deligne 1994, 3.1). We need the following well-known criterion:

[9]Without proof; cf. the discussion Beilinson et al. 1982, 3.1, which, however, states that (at that time) the situation had not been axiomatised.

> Let $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ be a t-exact functor inducing the identity functor on $\mathcal{C}$; then r is an equivalence of t-categories if and only if the maps $\mathrm{Hom}_{D^b(\mathcal{C})}(M, N[i]) \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{C}}}(M, N[i])$ it defines are isomorphisms for all M, N in $\mathcal{C}$ and all i.

For M, N in $\mathcal{C}$, let $\mathrm{Ext}^i_{\mathcal{C}}(M, N)$ denote the Yoneda Ext-group, and for M, N in the heart of $\mathcal{D}$, let

$$\mathrm{Ext}^i_{\mathcal{D}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N[i]).$$

Since $\mathrm{Ext}^i_{\mathcal{C}}(M, N) \simeq \mathrm{Hom}_{D^b(\mathcal{C})}(M, N[i])$ (Verdier 1996, III.3.2.12), the criterion states that $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ is an equivalence of t-categories if and only if the maps $\mathrm{Ext}^i_{\mathcal{C}}(M, N) \to \mathrm{Ext}^i_{\mathcal{D}}(M, N)$ it defines are isomorphisms for all M, N, and i.

5 The motivic t-category

Throughout this section, $k = \mathbb{F}_q$.

If we want the category of motives to have the Weil-number group P as its fundamental group, then Corollary 4.3 shows that $\mathcal{D}^b(\mathcal{M}_{\mathrm{num}}(k))$ is essentially the only candidate for a triangulated motivic category, and that it will have a admissible t-structure only if the Tate conjecture holds over k and rational equivalence equals numerical equivalence ($\mathbb{Q}$-coefficients). In this section, we prove that, when we assume these two conjectures, $\mathcal{D}^b(\mathcal{M}_{\mathrm{num}}(k))$ does have the hoped for properties.

PROPOSITION 5.1 *Let $\mathcal{D} = D^b(\mathcal{M}_{num}(k))$. Then $\mathcal{D}$ is a triangulated rigid tensor category with t-structure, and there exists a tensor functor*

$$R\colon \mathcal{M}_{rat}(k) \to \mathcal{D},$$

unique up to a unique isomorphism, such that $H^i(RX) = h^i(X)[-i]$ for all i.

PROOF. Let $C^b(\mathcal{M}_{\mathrm{num}}(k))$ be the category of bounded complexes of objects in $\mathcal{M}_{\mathrm{num}}(k)$, and let $C^b_0(\mathcal{M}_{\mathrm{num}}(k))$ be the full subcategory of bounded complexes whose differentials are zero. Because $\mathcal{M}_{\mathrm{num}}(k)$ is semisimple, the functor $D^b(\mathcal{M}(k)) \to C^b_0(\mathcal{M}_{\mathrm{num}}(k))$ sending A to

$$\bigoplus_r H^r(A)[-r] = \cdots \to H^{r-1}(A) \xrightarrow{0} H^r(A) \to \cdots$$

is an equivalence of categories which is quasi-inverse to the inclusion functor (Gelfand and Manin 1996, III 2.4, p146). Since $C^b_0(\mathcal{M}_{\mathrm{num}}(k))$ is a direct sum of copies of $\mathcal{M}_{\mathrm{num}}(k)$, and $\mathcal{M}_{\mathrm{num}}(k)$ is tannakian, it follows that $\mathcal{D}$ is a rigid tensor category. Define R to be

$$X \mapsto (\cdots \to h^{r-1}(X) \xrightarrow{0} h^r(X) \to \cdots). \tag{9}$$

The uniqueness is obvious. □

REMARK 5.2 Deligne (1968, 1.11, 1.13)[10] proves the following:

> Let $\mathcal{A}$ be an abelian category, and suppose that an object C of $D^b(\mathcal{A})$ admits endomorphisms $p_i\colon C \to C$ such that $H^j(p_i) = \delta_{ij}$ and the p_i are orthogonal idempotents; then there is a unique isomorphism $C \simeq \bigoplus_i H^i(C)[-i]$ inducing the identity map on cohomology and such that p_i is the i^{th} projection map.

[10] This also applies to t-categories. To check this, one only has to check that the spectral sequence in Deligne's proof exists for t-categories (for which there exist references).

Let R' be a tensor functor $\mathcal{M}_{\text{rat}}(k) \to \mathcal{D}^b(\mathcal{M}_{\text{num}}(k))$ and let R be as in (9). Then Deligne's result shows that, for any smooth projective variety X over k, there is a unique isomorphism $R'(X) \simeq R(X)$ inducing the identity on cohomology and such that $P^i(\varpi_X)$ is the projection from $R'(X)$ onto $h^i(X)[-i]$. Here P^i is as in (1).

THEOREM 5.3 *Assume that the Tate conjecture holds over k and that numerical equivalence coincides with rational equivalence (with $\mathbb{Q}$-coefficients).*

(a) *$D^b(\mathcal{M}_{num}(k))$ has a natural structure of a triangulated motivic category.*
(b) *The standard t-structure on $D^b(\mathcal{M}_{num}(k))$ is admissible (in the sense of §4), and it is the unique t-structure on $D^b(\mathcal{M}_{num}(k))$ with heart $\mathcal{M}_{num}(k)$.*
(c) *The functor $X \mapsto RX$ sending a smooth projective variety over k to its motivic complex (see 5.1) has a unique extension to all varieties over k.*
(d) *For each l (including p) there is a t-exact functor R_l from $D^b(\mathcal{M}_{num}(k))$ to a t-category $\mathcal{D}_l$ such that $X \mapsto R_l(RX)$ is the functor giving rise to the absolute l-adic cohomology.*

In the remainder of this section, we explain these statements in more detail and prove them.

Statement (a). We have to construct isomorphisms (3). In computing the right hand side of (3), we can replace $\mathcal{D}^b(\mathcal{M}_{\text{num}}(k))$ with the equivalent category $C_0^b(\mathcal{M}_{\text{num}}(k)) \simeq \bigoplus_r \mathcal{M}_{\text{num}}(k)[r]$. Therefore,

$$\operatorname{Hom}_{\mathcal{D}}(\mathbb{1}, R(X)(j)[i]) = \bigoplus\nolimits_s \operatorname{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[i-s]),$$

and

$$\operatorname{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[i-s]) = \operatorname{Ext}^{i-s}_{\mathcal{M}_{\text{num}}(k)}(\mathbb{1}, h^s(X)(j)).$$

This group is zero for $i \neq s$ because $\mathcal{M}_{\text{num}}(k)$ is semisimple, and it is zero for $i = s$, $s \neq 2j$, because $\mathbb{1}$ and $h^s(X)(j)$ will then have different weights. It is immediate from the definition of $\mathcal{M}_{\text{num}}(k)$, that

$$\operatorname{Hom}(\mathbb{1}, h^{2j}(X)(j)) \simeq Z^j_{\text{num}}(X)_{\mathbb{Q}}.$$

On the other hand, $K_i(X)_{\mathbb{Q}} = 0$ for $i \neq 0$ (see the proof 4.2), and $K_0(X)^{(j)} \simeq CH^j(X)_{\mathbb{Q}}$. Therefore, we can define (3) to be the natural map

$$CH^j(X)_{\mathbb{Q}} \to Z^j_{\text{num}}(X)_{\mathbb{Q}}$$

when $i = 2j$ and zero otherwise.

Statement (b). By hypothesis, rational, l-homological, and numerical equivalence coincide ($\mathbb{Q}$-coefficients), and so the standard t-structure is obviously admissible. It is the unique t-structure with heart $\mathcal{M}_{\text{num}}(k)$ because the heart determines the t-structure (Beilinson et al. 1982, 1.2, 1.3).

Statement (c). We only sketch the argument, leaving the details as an exercise for the reader. The key point is that de Jong 1996, Theorem 3.1, allows one to define a simplicial resolution

$$V \xleftarrow{f} U_\bullet \xrightarrow{j} X_\bullet$$

of any variety V over k in which j is a simplicial strict compactification and f is a proper hypercovering of V by a split simplicial smooth variety (cf. Berthelot 1997, 6.3). One first extends R to the category of strict compactifications, and then to the simplicial objects in the category of strict compactifications. Then one defines $RV = R(U_\bullet \to X_\bullet)$ for any simplicial resolution $V \xleftarrow{f} U_\bullet \xrightarrow{j} X_\bullet$ of V. One verifies that RV is independent of the choice of the simplicial resolution (up to a well-defined isomorphism), and the map $V \mapsto RV$ is contravariant for morphisms of varieties.

Statement (d), $l \neq p$**.** For $l \neq p$, let $D^b_c(k, \mathbb{Z}_l)$ be the category defined in Deligne 1980, 1.1.2. It is a t-category whose heart is the category $\mathcal{R}(k, \mathbb{Z}_l)$ of finitely generated $\mathbb{Z}_l$-modules endowed with a continuous action of $\mathrm{Gal}(\mathbb{F}/k)$. Each variety V over k defines an object $R\Gamma V$ in $D^b_c(k, \mathbb{Z}_l)$ such that $H^i(R\Gamma V) \simeq H^i_{\mathrm{et}}(V, \mathbb{Z}_l)$ (as objects of $\mathcal{R}(k, \mathbb{Z}_l)$). It is known that $D^b_c(k, \mathbb{Z}_l) \simeq D^b(\mathcal{R}(k, \mathbb{Z}_l))$. Now quotient out by the torsion objects to obtain an equivalence $D^b_c(k, \mathbb{Q}_l) \simeq D^b(\mathcal{R}(k, \mathbb{Q}_l))$ of $\mathbb{Q}_l$-linear categories. We define

$$R_l\colon D^b(\mathcal{M}_{\mathrm{num}}(k)) \to D^b(\mathcal{R}(k, \mathbb{Q}_l)) \simeq D^b_c(k, \mathbb{Q}_l)$$

to be the derived functor of the fibre functor $\mathcal{M}_{\mathrm{num}}(k) \to \mathcal{R}(k, \mathbb{Q}_l)$. Applying Deligne 1968, 1.11, 1.13 (cf. 5.2), we see that, for each smooth projective variety V over k, there is a unique isomorphism $R_l(V) \simeq \bigoplus_i H^i_l(V)[-i]$ inducing the identity map on cohomology and such that $P^i(\varpi_V)$ is the i^{th} projection map. Here P^i is the polynomial in (1). This shows that

$$R\Gamma(V) \simeq R_l(RV) \tag{10}$$

when V is projective and smooth. For an arbitrary V, we choose a simplicial resolution $V \xleftarrow{f} U_\bullet \xrightarrow{j} X_\bullet$ of V. Because (10) holds for smooth projective varieties,

$$R\Gamma(U_\bullet \xrightarrow{j} X_\bullet) \simeq R_l(R(U_\bullet \xrightarrow{j} X_\bullet)).$$

Moreover,

$$R(U_\bullet \xrightarrow{j} X_\bullet) \simeq R(V) \quad \text{(definition of } R(V))$$

$$R\Gamma(U_\bullet \xrightarrow{j} X_\bullet) \simeq R\Gamma(V) \quad \text{(Saint-Donat 1973, 4.3.2; also Huber 1995, 1.1.3),}$$

and so (10) holds for all varieties.

Statement (d), $l = p$**.** Let R be the Raynaud ring, and $D(R)$ the derived category of the category of graded R-modules (Illusie 1983, 2.1). For a smooth projective variety X over k, let $W\Omega^\bullet_X$ be the de Rham-Witt complex on X, and let $R\Gamma(W\Omega^\bullet_X)$ be its image under the derived functor of $\Gamma = \Gamma(X, -)$. Then $R\Gamma(W\Omega^\bullet_X)$ lies in the full subcategory $D^b_c(R)$ of $D(R)$ consisting of bounded R-complexes whose cohomology modules are coherent (Illusie and Raynaud 1983, II 2.2), and $H^i(R\Gamma(W\Omega^\bullet_X) \simeq H^i_{\mathrm{crys}}(X/W)$. When we endow $D^b_c(R)$ with Ekedahl's t-structure (Illusie 1983, 2.4.8) and quotient out by torsion objects, we obtain a $\mathbb{Q}_p$-linear t-category $D^b_c(R)_\mathbb{Q}$ whose heart is the category $\mathcal{R}(k, \mathbb{Q}_p)$ of F-isocrystals over k. It is known that $D^b_c(R)_\mathbb{Q} \simeq D^b_\mathcal{R}(B_\sigma[F])$ (derived category of bounded complexes of $B_\sigma[F]$-modules whose cohomology groups are F-isocrystals over k; recall $B = W \otimes \mathbb{Q}$ and that $B_\sigma[F]$ is the twisted polynomial ring). Define

$$R_p\colon D^b(\mathcal{M}_{\mathrm{num}}(k)) \to D^b(\mathcal{R}(k, \mathbb{Q}_p)) \to D^b_\mathcal{R}(B_\sigma[F]) \simeq D^b_c(R)_\mathbb{Q}$$

to be the composite of the derived functor of the fibre functor $\mathcal{M}_{\mathrm{num}}(k) \to \mathcal{R}(k, \mathbb{Q}_p)$ with the natural functors. The proof can now be completed as in the case $l \neq p$ except that the reference to Saint-Donat 1973 must be replaced by a reference to Tsuzuki 2003.

REMARK 5.4 Statement (c) and (d) of the theorem are very strong. Consider, for example, a closed subvariety Z of codimension r in a smooth projective variety X and a smooth alteration $\tilde{Z} \to Z$. Then the theorem says that there is an exact sequence

$$h^{i-2r}(\tilde{Z})(r) \to h^i(X) \to h^i(U), \quad U = X \smallsetminus Z,$$

whose l-adic realization is the sequence in (1.14) for $l \neq p$.

Application.

5.5 Using (c) and (d), we can extend the definition of $\mathbb{Q}_p$ cohomology (Milne 1986a, p309) from smooth projective varieties to all varieties, namely, for any variety X over k, define

$$H^i(X, \mathbb{Q}_p(r)) = \mathrm{Hom}_{\mathcal{D}(k;\mathbb{Q}_p)}(\mathbb{1}, R_p(RX)(r)[i]).$$

The main theorem of Milne and Ramachandran 2005 shows that this agrees with the original definition when X is smooth and projective.

6 The $\mathbb{Q}$-algebra of correspondences at the generic point

In this section, $k = \mathbb{F}_q$ and we assume that the Tate conjecture holds over k and that numerical equivalence equals rational equivalence ($\mathbb{Q}$-coefficients). We allow $l = p$.

Effective motives

Let $\mathcal{M}^{\mathrm{eff}}(k)$ be the category of effective motives given by Grothendieck's construction using algebraic classes modulo numerical equivalence as correspondences. It is an abelian nonrigid tensor category, and we let $\mathcal{D}^{\mathrm{eff}}(k) = D^b(\mathcal{M}^{\mathrm{eff}}(k))$. Much of Theorem 5.3 continues to hold. In particular, attached to a smooth projective variety X and an open subvariety U, there is a well-defined restriction map $h^i(X) \to h^i(U)$ whose l-adic realization is $H^i_l(X) \to H^i_l(U)$ (cf. 5.4, 5.5). We define

$$F^r_a h^i(X) = \bigcup\nolimits_U \mathrm{Ker}(h^i(X) \to h^i(U))$$

where U runs over the open subvarieties of X such that $X \smallsetminus U$ is of codimension at least r.

PROPOSITION 6.1 *For all l (including $l = p$)*

$$R_l(F^r_a h^i(X)) = F^r_b H^i_l(X).$$

PROOF. The functor R_l is exact, and so

$$R_l(F^r_a h^i(X)) = F^r_a H^i_l(X).$$

Therefore, the statement follows from the generalized Tate conjecture (1.10, 1.14, 1.16). □

COROLLARY 6.2 *For any smooth projective variety X over k, $F^r_a h^i(X)$ is the largest effective submotive of $h^i(X)$ of the form $M(-r)$ for some effective motive M.*

PROOF. Obvious. □

Definition of the $\mathbb{Q}$-algebra of correspondences at the generic point

In this subsection, we translate some definitions and results of Beilinson 2002 into our context. Let X be a connected algebraic variety of dimension n over a finite field k, and let η be its generic point. Define

$$CH^n(\eta \times \eta) = \varinjlim CH^n(U \times U),$$

where U runs over the open subvarieties of X. Following Beilinson 2002, 1.4, we define

$$A(X) = CH^n(\eta \times \eta) \otimes \mathbb{Q}.$$

Composition of correspondences makes $A(X)$ into an associative $\mathbb{Q}$-algebra, called the ***$\mathbb{Q}$-algebra of correspondences at the generic point.***

Denote by $\bar{h}^n(X)$ the image of the canonical map $h^n(X) \to h^n(\eta)$ (ind object of $\mathcal{M}^{\text{eff}}(k)$).

THEOREM 6.3 *For any connected smooth projective varieties X, X' of dimension n over k, the map*

$$CH^n(\eta' \times \eta) \otimes \mathbb{Q} \to \operatorname{Hom}(\bar{h}^n(\eta), \bar{h}^n(\eta'))$$

is an isomorphism.

PROOF. Beilinson's proof (2002, 4.9) applies in our context. □

COROLLARY 6.4 *For any connected smooth projective variety X of dimension n over k, there is a canonical isomorphism of $\mathbb{Q}$-algebras*

$$A(X) \simeq \operatorname{End}(\bar{h}^n(X)).$$

PROOF. It is only necessary to observe that composition of correspondences corresponds to composition of endomorphisms (Beilinson 2002, 4.10). □

COROLLARY 6.5 *The $\mathbb{Q}$-algebra $A(X)$ is finite-dimensional and semisimple.*

PROOF. Immediate from (6.4) because $\mathcal{M}^{\text{eff}}(k)$ is a semisimple category over $\mathbb{Q}$ with finite-dimensional Homs. □

Calculation of the $\mathbb{Q}$-algebra of correspondences at the generic point

PROPOSITION 6.6 *For a connected curve X over k,*

$$A(X) \simeq \operatorname{End}(J) \otimes \mathbb{Q}$$

where J is the Jacobian of a smooth complete model of X.

PROOF. As X is geometrically reduced, its smooth locus X' can be embedded in a smooth projective curve Y, and $X' \hookrightarrow Y$ is uniquely determined up to a unique isomorphism. As

$$A(X) \simeq A(X') \simeq A(Y)$$

we may as well assume that X itself is smooth and projective. For any nonempty open U, the map $h^1(X) \to h^1(U)$ is injective because $H^1_l(X) \to H^1_l(U)$ is injective, and so $\bar{h}^1(X) = h^1(X)$. Therefore, $A(X) \simeq \mathrm{End}(h^1(X))$, and it follows from the isomorphism

$$CH^1(X \times X) \simeq CH^1(X) \oplus CH^1(X) \oplus \mathrm{End}(J)$$

(Weil 1948a), that

$$\mathrm{End}(h^1(X)) \simeq \mathrm{End}(J) \otimes \mathbb{Q}. \qquad \square$$

For a connected smooth projective variety X of dimension n over k, define[11]

$$\bar{H}^n_l(X) = H^n_l(X)/F^0_b H^n_l(X).$$

For $l \neq p$, the quotient map $H^n_l(X) \to \bar{H}^n_l(X)$ defines an isomorphism of $\bar{H}^n_l(X)$ with the Tate substructure of $H^n_l(X)$ whose Frobenius eigenvalues α are such that a/q is not an algebraic integer. The quotient map $H^n_p(X) \to \bar{H}^n_p(X)$ can be identified with the map

$$H^n(X, W\Omega^\bullet)_{\mathbb{Q}} \to H^n(X, W\mathcal{O}_X)_{\mathbb{Q}} \simeq H^n_p(X)_{[0,1[}$$

(Illusie 1979, II 3.5.3, p616).

PROPOSITION 6.7 *For all primes l (including $l = p$),*

$$R_l(\bar{h}^n(X)) \simeq \bar{H}^n_l(X).$$

PROOF. Clearly,

$$0 \to F^0_a h^n(X) \to h^n(X) \to \bar{h}^n(X) \to 0$$

is exact. On applying the exact functor R_l, this gives an exact sequence

$$0 \to F^0_b H^n_l(X) \to H^n_l(X) \to \bar{H}^n_l(X) \to 0$$

by (6.4). $\square$

THEOREM 6.8 *For all primes l (including $l = p$),*

$$A(X) \otimes \mathbb{Q}_l \simeq \mathrm{End}(\bar{H}^n_l(X))$$

(endomorphisms of $\bar{H}^n_l(X)$ as a Tate structure when $l \neq p$; endomorphisms of $\bar{H}^n_p(X)$ as an F-isocrystal when $l = p$).

PROOF. Follows from Proposition 6.7 and the fact that R_l defines isomorphisms

$$\mathrm{Hom}(M, N) \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq \mathrm{Hom}(R_l M, R_l N). \qquad \square$$

EXAMPLE 6.9 If $H^n(X, W\mathcal{O}_X)$ is torsion, then $A(X) = 0$. This is the case, for example, if X is a supersingular abelian surface, a supersingular $K3$ surface, or an Enriques surface (Illusie 1979, 7.1, 7.2, 7.3).

[11] For $l \neq p$, this is $Gr^0 H^n_l(X)$, the "composante pure de niveau n" of $H^n_l(X)$, of Grothendieck (1968, p162).

REMARK 6.10 It is possible to recover the rank of a motive M from its endomorphism algebra $\mathrm{End}(M)$. According to the Wedderburn theorems,

$$\mathrm{End}(M) = \prod\nolimits_j M_{r_j}(D_j)$$

with each D_j a division algebra over $\mathbb{Q}$. If Z_j is the centre of D_j, then

$$\mathrm{rank}(M) = \sum\nolimits_j r_j \cdot [Z_j\colon\mathbb{Q}] \cdot [D_j\colon Z_j]^{1/2}.$$

REMARK 6.11 Since $A(X)$ is a birational invariant, (6.4) and (6.10) show that the rank of $\bar{h}^n(X)$ ($n = \dim X$) is a birational invariant of connected smooth projective varieties. Hence the same is true of its p-adic realization, i.e.,

$$\mathrm{rank}\, H^n(X, W\mathcal{O}_X) = \mathrm{rank}\, H^0(X, W\Omega_X^n)$$

is a birational invariant of connected smooth projective varieties over a finite field. Of course, it is classical that

$$\dim H^n(X, \mathcal{O}_X) = \dim H^0(X, \Omega_X^n)$$

is a birational invariant (Hartshorne 1977, II Ex 8.8), but

$$\mathrm{rank}\, H^n(X, W\mathcal{O}_X) \neq \dim H^n(X, \mathcal{O}_X),$$

for example, when $n = 2$ and X is a supersingular abelian surface. Illusie (1979, II 2.18, p614) proves that $H^0(X, W\Omega_X^n)$ is of finite-type over W with F acting as an automorphism. The formal p-divisible group G with Cartier module $H^0(X, W\Omega_X^n)/\mathrm{torsion}$ has

$$\dim(G) = \mathrm{rank}(G) = \mathrm{rank}\, H^0(X, W\Omega_X^n)$$

(cf. ibid. II 4.4, p621) and so $\dim(G)$ and $\mathrm{rank}(G)$ are also birational invariants.

Explicit description of $A(X)$

6.12 Let X be a smooth projective variety over k, and let $(\alpha_i)_{1\le i\le\beta_n}$ be the family of eigenvalues of ϖ_X on $H_l^n(X)$. Then the family $S(X)$ of eigenvalues of $\varpi_{\bar{h}^n(X)}$ consists of the α_i for which α_i/q is not an algebraic integer. Therefore, by Milne 1994, 2.14–2.15, the semisimple $\mathbb{Q}$-algebra $A(X) = \mathrm{End}(\bar{h}(X))$ has the following description. Let $o_1, \ldots, o_s$ be the distinct orbits for the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $S(X)$ and let r_j be the multiplicity of o_j:

$$F(X) = \coprod\nolimits_j r_j o_j.$$

Then

$$\bar{h}^n(X) = \sum\nolimits_j r_j N_j$$

where N_j is a simple motive with Frobenius eigenvalues the elements of o_j, and

$$A(X) \simeq \prod\nolimits_j M_{r_j}(\mathrm{End}(N_j)).$$

Let $\alpha \in o_j$. Then $\mathrm{End}(N_j)$ is isomorphic to a central simple algebra D_j over $Z_j = \mathbb{Q}[\alpha]$ with invariants (at the primes v of $\mathbb{Q}[\alpha]$)

$$\mathrm{inv}_v(D_j) = \begin{cases} \frac{1}{2} & \text{if } v \text{ is real and } n \text{ is odd} \\ \frac{\mathrm{ord}_v(\alpha)}{\mathrm{ord}_v(q)} \cdot [\mathbb{Q}[\alpha]_v : \mathbb{Q}_p] & \text{if } v|p \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the degree $[\mathbb{Q}[\alpha]\colon\mathbb{Q}]$ is the order of o_i, and the degree $[D_j\colon\mathbb{Q}[\alpha]] = e^2$ where e is the least common denominator of the numbers $\mathrm{inv}_v(D_j)$.

Following Beilinson (2002, p37), the pessimists will be tempted to look for counterexamples to the above calculations in order to ruin the conjectures.

7 Base fields algebraic over a finite field

Let k be a subfield of $\mathbb{F}$, and assume that the Tate conjecture holds and numerical equivalence equals rational equivalence ($\mathbb{Q}$-coefficients) for finite subfields of k. When we define the various categories for k to be the 2-category direct limits of the categories for k' with k' running over the finite subfields of k, then these categories for k inherit the properties of the corresponding categories for k'.

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James S. Milne,
Alpine Retreat, New Zealand, and Ann Arbor, USA,
math@jmilne.org, `www.jmilne.org/math/`

Niranjan Ramachandran,
Mathematics Department, University of Maryland, College Park, MD 20742, USA,
atma@math.umd.edu, `www.math.umd.edu/~atma/`

A Appendix (not for publication)

We add some details.

Remark 1.4

Proof of "It follows...". Let V be a finite-dimensional vector space over a perfect field k equipped with a linear endomorphism α, and let k' be the splitting field of the characteristic polyonomial of α. Then $V_{k'}$ decomposes into a direct sum of generalized eigenspaces $V_{k'} = \bigoplus_{a\in I} V(a)$ for the set I of eigenvalues of α — here $V(a)$ is the subspace on which $\alpha - a$ is nilpotent. For any subset J of I stable under $\mathrm{Gal}(k'/k)$, the k'-space $V(J) \overset{\text{def}}{=} \bigoplus_{a\in J} V(a)$ is also stable under $\mathrm{Gal}(k'/k)$, and so it arises from k-subspace of V.

Proof of Lemma 1.6

It suffices to prove this with $\mathbb{Q}_\ell$ replaced by a finite ring Λ such that $\ell^m\Lambda = \Lambda$ for some m. Let $j\colon U \hookrightarrow X$ be an open immersion with X complete. Then (by definition), $H^i(U,\Lambda) = H^i(X, j_!\Lambda)$. There is a canonical map $j_!\Lambda \hookrightarrow \Lambda$, and hence a map $H^i_c(U,\Lambda) \to H^i(X,\Lambda)$ which we denote j_* (cf. Milne 1980, II 3.13).

A.1 *With the notations of (1.6), the following diagram commutes:*

$$\begin{array}{ccc} H^i(Y,\Lambda) & \xleftarrow{j'_*} & H^i_c(\pi^{-1}U,\Lambda) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi'^*} \\ H^i(X,\Lambda) & \xrightarrow{j_*} & H^i_c(U,\Lambda) \end{array}$$

PROOF. Choose compatible injective resolutions of $j_!\Lambda_X$ and Λ_X:

$$\begin{array}{ccc} j_!\Lambda_X & \longrightarrow & I_1^\bullet \\ \downarrow & & \downarrow \\ \Lambda_X & \longrightarrow & I_2^\bullet. \end{array}$$

Now pull-back by π^* to get the middle square of the following commutative diagram:

$$\begin{array}{ccccccc} j'_!\Lambda_Y & \xrightarrow{\simeq} & \pi^*(j_!\Lambda_X) & \xrightarrow{\sim} & \pi^*I_1^\bullet & \xrightarrow{\sim} & J_1^\bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda_Y & \xrightarrow{\simeq} & \pi^*\Lambda_X & \xrightarrow{\sim} & \pi^*I_2^\bullet & \xrightarrow{\sim} & J_2^\bullet. \end{array}$$

Here $\simeq$ denotes an isomorphism and $\sim$ a quasi-isomorphism; $J_1^\bullet$ and $J_2^\bullet$ are complexes of injectives. On applying $\Gamma(Y,-)$ we get the right hand square of the next commutative diagram:

$$\begin{array}{ccccc} \Gamma(X,I_1^\bullet) & \longrightarrow & \Gamma(Y,\pi^*I_1^\bullet) & \longrightarrow & \Gamma(Y,J_1^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(X,I_2^\bullet) & \longrightarrow & \Gamma(Y,\pi^*I_2^\bullet) & \longrightarrow & \Gamma(Y,J_2^\bullet). \end{array}$$

On omitting the middle column, and taking cohomology, we get

$$\begin{array}{ccc} H_c^i(U,\Lambda) & \longrightarrow & H_c^i(\pi^{-1}U,\Lambda) \\ \downarrow & & \downarrow \\ H^i(X,\Lambda) & \longrightarrow & H^i(Y,\Lambda) \end{array}$$

After a reflection, this becomes the required diagram. □

A.2 *The Poincaré dual of the diagram in (A.1) (with i replaced by $2\dim Y - i$) is the diagram in (1.6).*

PROOF. For π_* and π'_*, this is the definition. For j_* (and j'_*), it is the commutativity of the diagram

$$\begin{array}{ccccccc} H^i(U,\Lambda) & \times & H_c^{2\dim X - i}(U,\Lambda(\dim X)) & \to & H_c^{2\dim X}(U,\Lambda(\dim X)) & \simeq & \mathbb{Q}_l \\ \uparrow j^* & & \downarrow j_* & & & & \| \\ H^i(X,\Lambda) & \times & H^{2\dim X - i}(X,\Lambda(\dim X)) & \to & H^{2d}(X,\Lambda(\dim X) & \simeq & \mathbb{Q}_l \end{array}$$

□

Lefschetz hypersurface-section theorem (1.13)

Recall that the Lefschetz hypersurface-section theorem says that, for any hypersurface section Z of a projective variety X such that $U \stackrel{\text{def}}{=} X \smallsetminus Z$ is smooth, $H_l^i(X) \to H_l^i(Z)$ is injective if $i = \dim X - 1$ and an isomorphism if $i < \dim X - 1$. To prove this, use the exact sequence

$$\cdots \to H_c^i(U,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l) \to H^i(Z,\mathbb{Q}_l) \to \cdots$$

(Milne 1980, III 1.30) noting that $H_c^i(U,\mathbb{Q}_l)$ is dual to $H^{2d-i}(U,\mathbb{Q}_l(d))$, $d = \dim X$, (ibid. VI 11.2), which is zero for $2d - i > d$, i.e., $i < d$ (ibid. VI 7.1).

Proof of the “well-known” criterion in 4.6b

PROPOSITION A.3 *Let $\mathcal{D}$ and $\mathcal{E}$ be bounded nondegenerate t-categories. A t-exact functor $F\colon \mathcal{D} \to \mathcal{E}$ is an equivalence of categories if and only if*

(a) $F^{\heartsuit}\colon \mathcal{D}^{\heartsuit} \to \mathcal{E}^{\heartsuit}$ *is an equivalence of categories, and*
(b) F *induces isomorphisms* $\mathrm{Hom}_{\mathcal{D}}(A, B[i]) \to \mathrm{Hom}_{\mathcal{E}}(FA, FB[i])$ *for all* A, B *in* $\mathcal{D}^{\heartsuit}$ *and all* i.

PROOF. The necessity being obvious, we shall prove only the sufficiency. We define the length of an object K of a t-category to be the minimum of $(j - i)$ when (i, j) runs over the pairs such that $H^s(K) = 0$ unless $i \leq s \leq j$.

Proof that F is fully faithful. We have to prove that, for all K, L in $\mathcal{D}$, the map

$$\mathrm{Hom}_{\mathcal{D}}(K, L) \to \mathrm{Hom}_{\mathcal{E}}(FK, FL)$$

defined by F is bijective. This we shall do by induction on the sum of the lengths of K and L. When K and L have length zero, they are isomorphic to translates of objects in the heart, and so the statement is true by (a). Thus, we may assume that at least one of K or L has nonzero length — we assume L has nonzero length since the argument for K is similar. For some choice of s, both $\tau_{\le s}L$ and $\tau_{\ge s+1}L$ will have length less than that of L. There exists a distinguished triangle

$$\tau_{\le s}L \to L \to \tau_{\ge s+1}L \to (\tau_{\le s}L)[1]$$

(Gelfand and Manin 1996, IV 4.5, p279), which gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & \mathrm{Hom}_{\mathcal{D}}(K, \tau_{\le s}L[i]) & \to & \mathrm{Hom}_{\mathcal{D}}(K, L[i]) & \to & \mathrm{Hom}_{\mathcal{D}}(K, \tau_{\ge s+1}L[i]) & \to \cdots \\
\downarrow\simeq & \downarrow\simeq & & \downarrow & & \downarrow\simeq & \downarrow\simeq \\
\cdots \to & \mathrm{Hom}_{\mathcal{E}}(FK, F(\tau_{\le s}L[i])) & \to & \mathrm{Hom}_{\mathcal{E}}(FK, FL[i]) & \to & \mathrm{Hom}_{\mathcal{E}}(FK, F(\tau_{\ge s+1}L[i])) & \to \cdots
\end{array}$$

On applying the induction hypothesis and the five-lemma, we obtain the statement for K and L.

Proof that F is essentially surjective. We have to prove that every object L of $\mathcal{E}$ is isomorphic to an object of the form FK. This we shall do by induction on the length of L. When L has length 0, the assertion follows from (a), and so we may suppose that L has nonzero length. Again, for some s, $\tau_{\le s}L$ and $\tau_{\ge s+1}L$ have length less than that of L. There is a distinguished triangle

$$L' \to L \to L'' \xrightarrow{\alpha} L'[1]$$

with $L' = \tau_{\le s}L$ and $L'' = \tau_{\ge s+1}L$.

Axiom TR2 (ibid. IV 1.1, p239) applied twice shows that

$$T: \qquad L'' \xrightarrow{\alpha} L'[1] \to L[1] \to L''[1]$$

is also distinguished. By induction, there exist K' and K'' in $\mathcal{D}$ and isomorphisms $a'\colon FK' \to L'$ and $a''\colon FK'' \to L''$. Because F is fully faithful, there is a unique morphism $\beta\colon K'' \to K'[1]$ making

$$\begin{array}{ccc}
FK'' & \xrightarrow{F\beta} & FK'[1] \\
\downarrow a'' & & \downarrow a'[1] \\
L'' & \xrightarrow{\alpha} & L'[1]
\end{array}$$

commute. Axiom TR1 c) (ibid.) allows us to complete β to a distinguished triangle

$$T': \qquad K'' \xrightarrow{\beta} K'[1] \to K \to K''[1].$$

The triangle $F(T')$ is also distinguished, and axiom TR3 allows us to extend $(a'', a'[1])$ to a morphism of triangles $F(T') \to T$, which (ibid. IV 1.4a) is an isomorphism. In particular, L is isomorphic to $F(K[-1])$. □

The details for the proof of (c) of Theorem 5.3

We begin by listing some definitions. Initially, we allow k to be an arbitrary perfect field.

For the definition of a ***strict normal crossings divisor*** D in a variety V, we refer to de Jong 1996, 2.4. In particular, if $(D^i)_{i\in I}$ is the family of irreducible components of such a divisor D, then for each subset J of I, $D_J \overset{\text{def}}{=} \bigcap_{i\in J} D^i$ is a smooth subvariety of V of codimension $\#J$.

A ***strict compactification*** is an open immersion $j\colon U \hookrightarrow X$ from a smooth variety into a smooth projective variety such that the complement $X \smallsetminus U$ is a strict normal crossings divisor. A ***morphism*** of strict compactifications is a pair of morphisms making

$$\begin{array}{ccc} U_\bullet & \hookrightarrow & X_\bullet \\ \downarrow g & & \downarrow h \\ U'_\bullet & \hookrightarrow & X'_\bullet \end{array}$$

(cf. Huber 1995, 1.1.4). A simplicial object in the category of strict compactifications is called a ***simplicial strict compactification.***

A simplicial variety $X_\bullet$ is ***split*** if, for all n,

$$N(X_n) \overset{\text{def}}{=} X_n \smallsetminus \bigcup_{s\in S_n} s(X_{n-1})$$

is an open and closed subscheme of X_n. Here S_n is the set of degeneracy maps from X_{n-1} to X_n (Huber 1995, 1.1.1e; Deligne 1974b, 6.2.2).

A morphism of simplicial varieties $f\colon X_\bullet \to Y_\bullet$ is called a ***proper hypercovering*** if the adjunction maps

$$f_{n+1}\colon X_{n+1} \to (\operatorname{cosk}_n^{Y_\bullet} \operatorname{sk}_n X_\bullet)_{n+1}$$

are proper and surjective for all $n \geq -1$ (Huber 1995, 1.1.2). The significance of this notion is explained by the following proposition.

PROPOSITION A.4 *A proper hypercovering $f\colon X_\bullet \to Y_\bullet$ induces an isomorphism (of $\mathrm{Gal}(k^{al}/k)$-modules)*

$$H^n(Y_\bullet \otimes k^{al}, \mathcal{F}) \simeq H^n(X_\bullet \otimes k^{al}, f^*\mathcal{F})$$

for all torsion sheaves $\mathcal{F}$ on $(Y_\bullet)_{\text{et}}$.

PROOF. This follows from Saint-Donat 1973, 4.3.2 (see Huber 1995, 1.1.3). □

A morphism (g,h) of simplicial strict compactifications is ***proper*** if each of g and h is a proper hypercovering.

A ***simplicial resolution*** of V is a diagram

$$\begin{array}{ccc} U_\bullet & \overset{j}{\hookrightarrow} & X_\bullet \\ \downarrow f & & \\ V & & \end{array}$$

in which j is a simplicial strict compactification and f is a proper hypercovering of V by a split simplicial smooth variety. A ***morphism of simplicial resolutions*** is a proper morphism of simplicial strict compactifications compatible with the augmentations.

THEOREM A.5 *Let V be a variety over a perfect field k.*

(a) *There exists a simplicial resolution $(U_\bullet \hookrightarrow X_\bullet, f)$ of V.*

(b) *For any two simplicial resolutions a, b of V, there exists a third simplicial resolution c and morphisms $c \to a$, $c \to b$.*

(c) *Let $V \to W$ be a morphism from V to a second variety over k. Given simplicial resolutions $(U_\bullet \hookrightarrow X_\bullet, f)$ and $(U'_\bullet \hookrightarrow X'_\bullet, f')$ of V and W respectively, there exists a simplicial resolution $(U''_\bullet \hookrightarrow X''_\bullet, f'')$ of V and morphisms*

$$(U_\bullet \hookrightarrow X_\bullet, f) \leftarrow (U''_\bullet \hookrightarrow X''_\bullet, f'') \to (U'_\bullet \hookrightarrow X'_\bullet, f').$$

PROOF. Once the theorem of de Jong (stated below) is acquired, the method of proof sketched in Deligne 1974b, 6.2, applies in the present situation. (For (a), see also the statements in the introduction to de Jong 1996 and in Berthelot 1997, 6.3.) □

THEOREM A.6 (DE JONG 1996, 3.1) *Let V be a variety over a perfect field k, and let $Z \subset V$ be a proper closed subset. Then there exists a variety U over k with $\dim U = \dim V$, a dominant proper generically étale morphism*

$$f : U \to V,$$

and an open immersion $j : U \hookrightarrow X$ such that

(a) *X is a smooth projective variety, and*

(b) *the closed subset $j(f^{-1}(Z)) \cup (X \smallsetminus j(U))$ is a strict normal crossings divisor in X.*

Notations We now take k to be a finite field, and use the following notations:

$\mathcal{V}(k)$	category of varieties over k
$\tilde{\mathcal{V}}(k)$	category of strict compactifications
$\tilde{\mathcal{V}}^{\Delta}(k)$	category of simplicial objects in $\tilde{\mathcal{V}}(k)$

(For $\mathbb{F}$, one should define $\tilde{\mathcal{V}}^{\Delta}(\mathbb{F})$ to be $\varinjlim \tilde{\mathcal{V}}^{\Delta}(\mathbb{F}_q)$, not the category of simplicial objects in $\tilde{\mathcal{V}}(\mathbb{F})$.)

Construction of the functor from $\tilde{\mathcal{V}}(k)$ to $C^b(\mathcal{M}_{\mathrm{num}}(k))$ The starting point for our construction is the functor from the category of smooth projective varieties over k to $C^b(\mathcal{M})$ that sends V to $h^*V \overset{\text{def}}{=} \bigoplus_i h^i(V)[-i]$ (this is denoted R in §5).

Consider a strict compactification $j : U \hookrightarrow X$ over k of dimension d, and let $D = X \smallsetminus j(U)$. By strictness, if $D = \bigcup_{i \in I} D^i$ is the decomposition of D into its irreducible components, then each of

$$D_1 = \bigsqcup_{i \in I} D^i, \quad D_2 = \bigsqcup_{(i.j) \in I \times I, i \neq j} D^i \cap D^j, \quad \ldots,$$

is smooth and projective and of dimension one less than its predecessor.

Let Z be a smooth subvariety of codimension e in a smooth projective variety V. The inclusion $j : Z \hookrightarrow V$ defines a map $h^*V \xrightarrow{j^*} h^*Z$ in $C^b(\mathcal{M}_{\mathrm{num}}(k))$. By Poincaré duality (applied to both V and Z), this gives a Gysin map

$$h^*Z(-e)[2e] \to h^*V$$

in $C^b(\mathcal{M}_{\mathrm{num}}(k))$.

We apply this remark to each step of the sequence

$$\cdots D_3 \overrightarrow{\overrightarrow{\longrightarrow}} D_2 \rightrightarrows D_1 \longrightarrow X$$

(each D_{i+1} is a smooth divisor in D_i, and each D_i is smooth and projective) to obtain

$$(h^* D_d)(-d)[2d] \; \begin{matrix}\longrightarrow \\ \vdots \\ \longrightarrow\end{matrix} \; \cdots \overrightarrow{\overrightarrow{\longrightarrow}} (h^* D_2)(-2)[4] \rightrightarrows (h^* D_1)(-1)[2] \longrightarrow h^* X.$$

We take the alternating sums of the individual maps and then form the total complex to obtain an object $h^*(U, X)$ of $C^b(\mathcal{M}_{\mathrm{num}}(k))$.

Then $h^*(U, X)$ is the complex

$$0 \to \underset{\text{degree } 0}{h^0(X)} \to \underset{\text{degree } 1}{h^1(X) \oplus h^0(D_1)(-1)} \to \underset{\text{degree } 2}{h^2(X) \oplus h^1(D_1)(-1) \oplus h^0(D_2)(-2)} \to$$

in which the maps are formed using the Gysin maps for the inclusions $D_{i+1} \hookrightarrow D_i$.

PROPOSITION A.7 *The map* $(U, X) \mapsto h^*(U, X)$ *is contravariant for morphisms in* $\tilde{\mathcal{V}}(\mathbb{F})$.

PROOF. Clear from the constructions. □

Construction of the functor from $\tilde{\mathcal{V}}^{\Delta}(k)$ to $C^+(\mathcal{M}_{\mathrm{num}}(k))$

For each object $(U_\bullet, X_\bullet)$ in $\tilde{\mathcal{V}}^{\Delta}(k)$ we shall construct an object $h^*(U_\bullet, X_\bullet)$ in $C^+(\mathcal{M}_{\mathrm{num}}(k))$. This complex need not be bounded, but will have bounded cohomology.

First, the simplicial object $(U_\bullet, X_\bullet)$ gives us a cosimplicial object of $C^b(\mathcal{M}_{\mathrm{num}}(k))$:

$$0 \to h^*(U_0, X_0) \rightrightarrows h^*(U_1, X_1) \overrightarrow{\overrightarrow{\longrightarrow}} h^*(U_2, X_2) \cdots$$

On taking the alternating sums of the individual maps, we obtain a bicomplex $e_{i,j}$ with $e_{i,*} = h^*(U_i, X_i)$. This bicomplex is concentrated in the region $i, j \geq 0$, and the associated single complex is an object $h^*(U_\bullet, X_\bullet)$ of $C^+(\mathcal{M}_{\mathrm{num}}(k))$.

PROPOSITION A.8 *The map* $(U_\bullet, X_\bullet) \mapsto h^*(U_\bullet, X_\bullet)$ *is contravariant for morphisms in* $\tilde{\mathcal{V}}^{\Delta}(k)$.

PROOF. Clear from the constructions. □

Construction of the functor from $\mathcal{V}(k)$ to $\mathcal{D}^b(\mathcal{M}_{\mathrm{num}}(k))$

Given a variety V over k, define $h^*(V) = \varinjlim h^*(U_\bullet, X_\bullet)$, where $(U_\bullet, X_\bullet)$ runs over the proper hypercovering $(U_\bullet, X_\bullet)$ of V (in fact, all the transition maps in the direct system are isomorphisms because they become isomorphisms after R_l has been applied). It follows from (c) of Theorem A.5 that the map $V \mapsto h^*(V)$ is contravariant for morphisms of varieties.

The details for the proof of (d) of Theorem 5.3d ($l \neq p$)

It seems to be considered "well-known" that Deligne's category $D^b_c(k, \mathbb{Q}_l) \simeq D^b(\mathcal{R}(k, \mathbb{Q}_l))$, but we do not know of a proof in the literature. Here we prove that it holds even on the level of the $\mathbb{Z}_l$-categories.

The triangulated category $D^b_{\text{ctf}}(X, \Lambda)$, Λ finite

(Deligne 1980, 1.1; Kiehl and Weissauer 2001, II 5.) Let X be a variety over a field k, and let Λ be a finite ring such that $p\Lambda = \Lambda$ if k has characteristic $p \neq 0$.

DEFINITION A.9 $D^b(X, \Lambda)$ is the derived category whose objects are the bounded complexes of sheaves of Λ-modules on X_{et};

$D^b_c(X, \Lambda)$ is the full subcategory of $D^b(X, \Lambda)$ of complexes $K^\bullet$ such that $H^r(K^\bullet)$ is constructible for all r; and

$D^b_{\text{ctf}}(X, \Lambda)$ is the full subcategory of $D^b_c(X, \Lambda)$ of complexes with finite Tor dimension. There are similar definitions with b omitted or replaced by $+$ or $-$.

By a bounded complex, we mean one with only finitely many nonzero terms. Thus $D^b(X, \Lambda)$ is the bounded derived category of the abelian category $\mathcal{S}(X, \Lambda)$ of sheaves of Λ-modules on X_{et} in the usual sense. In particular, it has a natural t-structure.

Recall that a sheaf F of Λ-modules on X_{et} is ***constructible*** if X has a finite partition into locally closed subvarieties X_i such that the restrictions $F|X_i$ of F are locally constant with finite stalks. The constructible sheaves form a thick subcategory of $\mathcal{S}(X, \Lambda)$, and so $D^b_c(X, \Lambda)$ is a t-subcategory of $D^b(X, \Lambda)$.

For any complex $K^\bullet$ in $D^b(X, \Lambda)$, there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded-above complex of flat sheaves of Λ-modules (e.g., Milne 1980, VI 8.3). If $P^\bullet$ can be chosen to be bounded, then $K^\bullet$ is said to have finite Tor dimension.[12] The objects of $D^b_{\text{ctf}}(X, \Lambda)$ are the complexes in $D^b(X, \Lambda)$ that are quasi-isomorphic to bounded complexes of flat sheaves of Λ-modules with constructible cohomology. The category $D^b_{\text{ctf}}(X, \Lambda)$ is a triangulated subcategory of $D^b_c(X, \Lambda)$, but it is not in general a t-subcategory because the standard truncation operator $\tau_{\leq n}$ does not respect finiteness of the Tor-dimension.

Recall that a ***perfect complex*** of sheaves Λ-modules is a bounded complex of constructible flat sheaves of Λ-modules.

PROPOSITION A.10 *If Λ is principal, i.e., every ideal is principal, then every complex in $D^b_{\text{ctf}}(X, \Lambda)$ is quasi-isomorphic to a perfect complex of Λ-modules.*

PROOF. We are given that the complex is quasi-isomorphic to a bounded complex $P^\bullet$ of flat sheaves with constructible cohomology, and have to show that the sheaves in $P^\bullet$ can themselves can be chosen to be constructible. When Λ is local, this is proved in Kiehl and Weissauer 2001, II 5.2.[13] As Λ is a product of local rings, the general case follows easily. □

PROPOSITION A.11 *For varieties X, the categories $D^b_{\text{ctf}}(X, \Lambda_r)$ are stable under the four operations Rf_*, f^*, $Rf_!$, $Rf^!$ and the internal operations $\otimes^L$ and $\mathcal{H}om$.*

PROOF. See Deligne 1977, pp233-261; Deligne 1980, p148. □

Sheaves of $\mathbb{Z}_\ell$-modules

(Jouanolou 1977.)[14] Let X be a variety over a field k, and let ℓ be a prime such that $\ell k = k$.

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[12]Equivalently, there exists a d such that $H^i(K^\bullet \otimes^L L) = 0$ for $i < d$ and all sheaves L (cf. Milne 1980, p263).

[13]Obviously, the category in the statement should be $D^b_{\text{ctf}}(X, \mathfrak{o}_r)$, not $D_{ctf}(X, \mathfrak{o}_r)$.

[14]Also Freitag and Kiehl 1988, I 12.

DEFINITION A.12 (ibid. 3.1.1) An ***ℓ-adic sheaf*** on X is an inverse system $(F_r)_{r\in\mathbb{N}}$ sheaves of abelian groups on X_{et} such that, for each $r \geq 0$,

(a) the sheaf F_r is killed by ℓ^{r+1}, and

(b) the transition map $F_r \to F_{r-1}$ induces an isomorphism

$$F_r/\ell^r F_r \simeq F_{r-1}.$$

We shall need to consider a somewhat larger class of sheaves. Let $\mathcal{S}^{\mathbb{N}(\ell)} = \mathcal{S}(X_{\mathrm{et}})^{\mathbb{N}(\ell)}$ denote the category of inverse systems of ℓ-torsion[15] sheaves of abelian groups on X_{et} indexed by $\mathbb{N}$. This is an abelian category with kernels and cokernels computed on each component. For F in $\mathcal{S}^{\mathbb{N}(\ell)}$ and $t \geq 0$, let $F[t]$ be the inverse system with $F[t]_r = F_{r+t}$. An object F is ***null*** if the map $F[t] \to F$ is zero for some t. The full subcategory of null objects is a thick abelian subcategory of $\mathcal{S}^{\mathbb{N}(\ell)}$, and we let $\overline{\mathcal{S}}^{\mathbb{N}(\ell)}$ denote the quotient of $\mathcal{S}^{\mathbb{N}(\ell)}$ by this subcategory. Then $\overline{\mathcal{S}}^{\mathbb{N}(\ell)}$ is an abelian category with the same objects as $\mathcal{S}^{\mathbb{N}(\ell)}$, but

$$\mathrm{Hom}_{\overline{\mathcal{S}}^{\mathbb{N}(\ell)}}(F, F') = \varinjlim_{t} \mathrm{Hom}_{\mathcal{S}^{\mathbb{N}(\ell)}}(F[t], F')$$

(ibid. 2.4).

DEFINITION A.13 (ibid. 3.2.2) An object F of $\mathcal{S}^{\mathbb{N}(\ell)}$ is an ***AR***[16]***-ℓ-adic sheaf*** on X if

(a) for each $r \geq 0$, the sheaf F_r is killed by ℓ^{r+1}, and

(b) there exists an ℓ-adic sheaf on X that becomes isomorphic to F in $\overline{\mathcal{S}}^{\mathbb{N}(\ell)}$.

Condition (b) is equivalent to:

> there exists an ℓ-adic sheaf F' on X and a homomorphism $F' \to F$ in $\mathcal{S}^{\mathbb{N}(\ell)}$ whose kernel and cokernel are null.

REMARK A.14

$$\mathbb{Z}_\ell\text{-sheaf} \quad = \quad \text{AR-}\ell\text{-adic sheaf}$$

We let $\mathcal{S}(X, \mathbb{Z}_\ell)$ (resp. $\bar{\mathcal{S}}(X, \mathbb{Z}_\ell)$) denote the full subcatgory of $\mathcal{S}^{\mathbb{N}(\ell)}$ of ℓ-adic sheaves (resp. of $\bar{\mathcal{S}}^{\mathbb{N}(\ell)}$ of $\mathbb{Z}_\ell$-sheaves). A $\mathbb{Z}_\ell$-sheaf $F = (F_r)_{r\geq 0}$ is ***constructible*** if each F_r is constructible, and $\mathcal{S}_c(X, \mathbb{Z}_\ell)$ and $\bar{\mathcal{S}}_c(X, \mathbb{Z}_\ell)$ denote the corresponding categories of constructible sheaves.

Note that if F is an ℓ-adic sheaf, then $F[t] \simeq F$, and so, for any G in $\mathcal{S}^{\mathbb{N}(\ell)}$,

$$\mathrm{Hom}_{\mathcal{S}^{\mathbb{N}(\ell)}}(F, G) \simeq \mathrm{Hom}_{\overline{\mathcal{S}}^{\mathbb{N}(\ell)}}(F, G).$$

Therefore, the quotient functor $\mathcal{S}^{\mathbb{N}(\ell)} \to \bar{\mathcal{S}}^{\mathbb{N}(\ell)}$ defines an equivalence $\mathcal{S}(X, \mathbb{Z}_\ell) \to \bar{\mathcal{S}}(X, \mathbb{Z}_\ell)$.

PROPOSITION A.15 *For a field k and its Galois group Γ, the functor*

$$F = (F_r) \mapsto \varprojlim F_r(k^{\mathrm{sep}}) \colon \bar{\mathcal{S}}_c(k, \mathbb{Z}_\ell) \to \mathcal{R}(\Gamma, \mathbb{Z}_\ell)$$

is an equivalence of categories.

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[15] That is, killed by some power of ℓ.

[16] AR=Artin-Rees.

PROOF. We can replace $\bar{\mathcal{S}}_c(k,\mathbb{Z}_\ell)$ with $\mathcal{S}_c(k,\mathbb{Z}_\ell)$. Thus, we have to show that $\mathcal{R}(\Gamma,\mathbb{Z}_\ell)$ is naturally equivalent to the category of inverse systems $(M_r)_{r\in\mathbb{N}}$ of finite abelian groups equipped with a continuous action of Γ satisfying (a) $\ell^{r+1}M_r = 0$, and (b) $M_r/\ell^r M_r \simeq M_{r-1}$. This follows easily equivalence of the following conditions on a $\mathbb{Z}_\ell$-module M:

(a) M is finitely generated;
(b) $M/\ell M$ is finite and $\bigcap \ell^n M = 0$;
(c) $M/\ell M$ is finite and $M \simeq \varprojlim M/\ell^n M$. □

The triangulated category $D^b_{\mathrm{c}}(X,\mathbb{Z}_\ell)$

(Deligne 1980.)[17] Let X be a variety over a field k, and let ℓ be a prime such that $\ell\mathcal{O}_X = \mathcal{O}_X$. Let $\Lambda_r = \mathbb{Z}/\ell^{r+1}\mathbb{Z}$. We have triangulated categories $D^-(X,\Lambda_r)$ and triangulated functors[18]

$$C^\bullet \mapsto \Lambda_r \otimes^L_{\Lambda_{r+1}} C^\bullet \colon D^-(X,\Lambda_{r+1}) \to D^-(X,\Lambda_r). \tag{11}$$

DEFINITION A.16 (ibid. 1.1.2)
$D^-(X,\mathbb{Z}_\ell)$ is the 2-category inverse limit of the categories $D^-(X,\Lambda_r)$;
$D^b_c(X,\mathbb{Z}_\ell)$ is the 2-category inverse limit of the categories $D^b_{\mathrm{ctf}}(X,\Lambda_r)$.

More explicitly, an object of $D^-(X,\mathbb{Z}_\ell)$ is a family $(K^\bullet_r,\phi_r)_{r\geq 0}$ of complexes $K^\bullet_r$ in $D^-(X,\Lambda_r)$ and isomorphisms

$$\phi_r\colon \Lambda_r \otimes^L_{\Lambda_{r+1}} K^\bullet_{r+1} \to K^\bullet_r, \quad r \geq 0,$$

in $D^-(X,\Lambda_r)$; a morphism $(K^\bullet_r,\phi_r) \to (L^\bullet_r,\phi'_r)$ is a family of morphisms

$$\psi_r \in \mathrm{Hom}_{D^-(X,\Lambda_r)}(K^\bullet_r, L^\bullet_r), \quad r \geq 0,$$

making the squares

$$\begin{array}{ccc} \Lambda_r \otimes^L_{\Lambda_{r+1}} K^\bullet_{r+1} & \xrightarrow{\mathrm{id}\otimes\psi_{r+1}} & \Lambda_r \otimes^L_{\Lambda_{r+1}} L^\bullet_{r+1} \\ \downarrow \phi_r & & \downarrow \phi'_r \\ K^\bullet_r & \xrightarrow{\psi_r} & L^\bullet_r \end{array}$$

commute in $D^-(X,\Lambda_r)$ (cf. Kiehl and Weissauer 2001, p94). For $K^\bullet = (K^\bullet_r)_{r\geq 0}$ in $D^-(X,\mathbb{Z}_\ell)$, define

$$\Lambda_r \otimes^L K^\bullet = K^\bullet_r.$$

The transition functors (11) don't preserve bounded complexes unless they have finite Tor dimension, and so the 2-category inverse limit of the categories $D^b(X,\Lambda_r)$ and $D^b_c(X,\Lambda_r)$ are not bounded.

The natural inclusion identifies $D^b_c(X,\mathbb{Z}_\ell)$ with the full subcategory of $D^-(X,\mathbb{Z}_\ell)$ of inverse systems (K_r,ϕ_r) with K_r in $D^b_{\mathrm{ctf}}(X,\Lambda_r)$.

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[17] Also Kiehl and Weissauer 2001, II 5.

[18] By this, we mean a pair consisting of an additive functor $F\colon\mathcal{C} \to \mathcal{C}'$ and an isomorphism of functors $F \circ T \simeq T' \circ F$ that transform distinguished triangles into distinguished triangles. Often, this is called an exact functor of triangulated categories (e.g., Verdier 1977, p4) or simply a functor of triangulated categories (Kashiwara and Schapira 1990, p38, except that they get the definition wrong).

PROPOSITION A.17 *(a) For $K^\bullet = (K^\bullet_r, \phi_r)$ in $D^b_c(X, \mathbb{Z}_\ell)$, the inverse systems*

$$\mathcal{H}^i(K^\bullet) \stackrel{\text{def}}{=} (\mathcal{H}^i(K^\bullet_r))_{r\geq 0}$$

are constructible $\mathbb{Z}_\ell$-sheaves; conversely, every torsion-free constructible $\mathbb{Z}_\ell$-sheaf $F = (F_r)_{r\geq 0}$ defines an object $(\cdots \to 0 \to F_r \to 0 \to \cdots)_{r\geq 0}$ of $D^b_c(X, \mathbb{Z}_\ell)$.

(b) An object $K^\bullet$ of $D^-(X, \mathbb{Z}_\ell)$ lies in $D^b_c(X, \mathbb{Z}_\ell)$ if $\Lambda_0 \otimes^L K^\bullet$ is bounded with constructible cohomology.

PROOF. Deligne 1980, 1.1.2. □

REMARK A.18 A constructible $\mathbb{Z}_\ell$-sheaf F with torsion also defines an object of $D^b_c(X, \mathbb{Z}_\ell)$. The sheaf

$$\tilde{F}_r \stackrel{\text{def}}{=} \tau_{\geq -1}((\Lambda_k \otimes F) \otimes^L_{\Lambda_k} \Lambda_r$$

is independent of k provided ℓ^{k-r} kills the torsion in F, and the $\tilde{F}_r$'s form an inverse system defining an object $\tilde{F}$ of $D^b_c(X, \mathbb{Z}_\ell)$ (Deligne 1980, p148). Moreover,

$$\mathcal{H}^0(\tilde{F}) \simeq F$$
$$\mathcal{H}^i(\tilde{F}) \simeq 0$$

in $\bar{\mathcal{S}}^{\mathbb{N}(\ell)}$ (Kiehl and Weissauer 2001, II 6.2).

There is a natural translation functor on $D^-(X, \mathbb{Z}_\ell)$, namely,

$$(K_r, \phi_r)[1] = (K_r[1], \phi_r[1]),$$

and a natural notion of distinguished triangle, namely, (A, B, C, a, b, c) is distinguished if and only if $(A_r, B_r, C_r, a_r, b_r, c_r)$ is distinguished in $D^-(X, \Lambda_r)$ for all r. However, in verifying the axioms for a triangulated category, one runs into the problem that one may not be able to complete the commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \quad \text{(distinguished triangle)} \\ \downarrow & & \downarrow & & & & \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \quad \text{(distinguished triangle)} \end{array}$$

to a morphism of distinguished triangles unless one knows that the sets $\mathrm{Hom}_{\mathcal{D}^-(X,\Lambda_r)}(C_r, C'_r)$ are finite.

THEOREM A.19 *If the cohomology groups $H^i(k, \mathbb{Z}/\ell\mathbb{Z})$ are finite for all i, then, for all perfect complexes of sheaves of Λ_r-modules $K^\bullet$ and $L^\bullet$ on X, the group $\mathrm{Hom}_{D^-(X,\Lambda_r)}(K^\bullet, L^\bullet)$ is finite.*

PROOF. This follows from the finiteness theorems in Deligne 1977, pp233–251; cf. Kiehl and Weissauer 2001, Appendix D. □

PROPOSITION A.20 *If the field k is separably closed or algebraic over a finite field, then $D^b_c(X, \mathbb{Z}_\ell)$ is a triangulated category.*

PROOF. The groups $H^i(k, \mathbb{Z}/\ell\mathbb{Z})$ are obviously[19] finite for $i \leq 1$, and they are zero for $i > 1$ (Serre 1964, II §3). According to (A.10, A.19), $D^b_{\mathrm{c}}(X, \mathbb{Z}_\ell)$ is a 2-category inverse limit of triangulated categories in which the Hom groups are finite, and it is straightforward to verify that the natural structures on such a limit category make it into a triangulated category (Beilinson et al. 1982, 2.2.15). □

The t-category $D^b_{\mathrm{c}}(k, \mathbb{Z}_\ell)$

Let Γ be a profinite group. For a finite ring Λ, we let $\mathcal{R}'(\Gamma, \Lambda)$ denote the category of continuous representations of Γ on Λ-modules (not necessarily finitely generated). Then $\mathcal{R}'(\Gamma, \Lambda)$ is an abelian category, and we let $D^-(\Gamma, \Lambda) = D^-(\mathcal{R}'(\Gamma, \Lambda))$.

DEFINITION A.21 $D^-(\Gamma, \mathbb{Z}_\ell)$ is the 2-category inverse limit of the categories $D^-(\Gamma, \Lambda_r)$, $\Lambda_r = \mathbb{Z}/\ell^r\mathbb{Z}$;

$D^b_c(\Gamma, \mathbb{Z}_\ell)$ is the full subcategory of $D^-(\Gamma, \mathbb{Z}_\ell)$ of inverse systems $(K^\bullet_r)$ such that $K^\bullet_0$ is bounded with finite cohomology groups.

DEFINITION A.22 Let R be a ring.

(a) A complex of R-modules $K^\bullet$ is ***perfect*** if it is bounded and each K^i is finitely presented and flat.[20]
(b) A complex of objects of $\mathcal{R}'(\Gamma, \Lambda)$ (resp. $\mathcal{R}(\Gamma, \mathbb{Z}_\ell)$) is ***perfect*** if it is perfect as a complex of Λ-modules (resp. $\mathbb{Z}_\ell$-modules).

LEMMA A.23 *Every object of $D^b_c(\Gamma, \mathbb{Z}_\ell)$ is isomorphic to an inverse system $(K^\bullet_r)_{r\geq 0}$ in which $K^\bullet_r$ is a perfect complex of objects in $\mathcal{R}'(\Gamma, \Lambda)$ and the complexes $K^\bullet_r$ are uniformly bounded.*

PROOF. Exercise (cf. Kiehl and Weissauer 2001, p99). □

LEMMA A.24 *Every object of $D^b(\mathcal{R}(\Gamma, \mathbb{Z}_\ell))$ is isomorphic to a perfect complex of objects of $\mathcal{R}(\Gamma, \mathbb{Z}_\ell)$.*

PROOF. Apply Milne and Ramachandran 2004, 1.5. □

PROPOSITION A.25 *The map $(K^\bullet_r) \mapsto \varprojlim K^\bullet_r$ defines an equivalence of categories $D^b_c(\Gamma, \mathbb{Z}_\ell) \to D^b(\mathcal{R}(\Gamma, \mathbb{Z}_\ell))$.*

PROOF. Let $P(\Gamma, \mathbb{Z}/\ell^*)$ be the full subcategory of $D^b_c(\Gamma, \mathbb{Z}_\ell)$ of uniformly bounded inverse systems of perfect complexes, and let $P(\Gamma, \mathbb{Z}_\ell)$ be the full subcategory of $D^b(\mathcal{R}(\Gamma, \mathbb{Z}_\ell))$ of perfect complexes. Consider

$$\begin{array}{ccc} P(\Gamma, \mathbb{Z}/\ell^*) & \longrightarrow & P(\Gamma, \mathbb{Z}_\ell) \\ \downarrow a & & \downarrow b \\ D^b_c(\Gamma, \mathbb{Z}_\ell) & \longrightarrow & D^b(\mathcal{R}(\Gamma, \mathbb{Z}_\ell)) \end{array}$$

Lemma A.23 shows that a is an equivalence, and Lemma A.24 shows that b is an equivalence. It remains to show that $(K^\bullet_r) \to \varprojlim K^\bullet_r \colon P(\Gamma, \mathbb{Z}/\ell^*) \to P(\Gamma, \mathbb{Z}_\ell)$ is an equivalence, but $K^\bullet \mapsto (K^\bullet/\ell^{r+1}K^\bullet)_r$ provides a quasi-inverse.[21] □

[19] The only open subgroups of $\widehat{Z}$ are those of the form $m\widehat{Z}$ with m a nonzero natural number, from which it follows that the only closed subgroups are those of the form $m\widehat{Z}$ with m a nonzero supernatural number.

[20] Equivalently, finitely presented and projective, Bourbaki 1989, II 5.2.

[21] To prove this, show that the morphisms in $P(\Gamma, \mathbb{Z}_\ell)$ are homotopy classes of morphisms of complexes.

THEOREM A.26 *Let k be a separably closed field or a field algebraic over a finite field. There is a canonical equivalence of triangulated categories*

$$D_c^b(k, \mathbb{Z}_\ell) \to D^b(\mathcal{R}(\Gamma, \mathbb{Z}_\ell)).$$

PROOF. Let k^{sep} be the separable closure of k used in the definition of $\mathcal{R}(k, \mathbb{Z}_\ell)$, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. The functor $F \mapsto F(k^{\mathrm{sep}}) \overset{\mathrm{def}}{=} \varinjlim F(k')$ (limit over subfields k' of k^{sep} finite over k) is an equivalence from the category of sheaves on $(\mathrm{Spec}\, k)_{\mathrm{et}}$ to the category of sets endowed with a continuous action of Γ carrying $D_c^b(k, \mathbb{Z}_\ell)$ to $D_c^b(\Gamma, \mathbb{Z}_\ell)$. Therefore the theorem follows from Proposition A.25. □

COROLLARY A.27 *There is a natural t-structure on $D_c^b(k, \mathbb{Z}_\ell)$ whose heart is canonically equivalent to the category of $\mathbb{Z}_\ell$-sheaves on* $\mathrm{Spec}\, k$.

PROOF. The category $D^b(\mathcal{R}(\Gamma, \mathbb{Z}_\ell))$ has a natural t-structure whose heart is $\mathcal{R}(k, \mathbb{Z}_\ell)$. □

The details for the proof of (d) of Theorem 5.3d ($l = p$)

The Raynaud ring is the graded W-algebra $R = R^0 \oplus R^1$ generated by F and V in degree 0 and d in degree 1, subject to the relations $FV = p = VF$, $Fa = \sigma a \cdot F$, $aV = V \cdot \sigma a$, $ad = da$ ($a \in W$), $d^2 = 0$, and $FdV = d$; in particular, R^0 is the Dieudonné ring $W_\sigma[F, V]$ (Illusie 1983, 2.1). A graded R-module is nothing more than a complex

$$M^\bullet = (\cdots \to M^i \xrightarrow{d} M^{i+1} \to \cdots)$$

of W-modules whose components M^i are modules over R^0 and whose differentials d satisfy $FdV = d$. We define T to be the functor of graded R-modules such that $(TM)^i = M^{i+1}$ and $T(d) = -d$. It is exact and defines a self-equivalence $T\colon D_c^b(R) \to D_c^b(R)$ where $D_c^b(R)$ is defined below.

The abelian category of graded R-modules with maps of degree zero) is not noetherian (the R-module U_0 defined below is not noetherian; Ekedahl 1986, p.18) and it does not have a natural tensor product (the ring R is not commutative). Ekedahl has isolated a nice subcategory of coherent R-modules which is better behaved.

A graded R-module M is ***elementary*** (ibid., p.10) if it admits a finite filtration with quotients of the following types:

Type I a W-module of finite length concentrated in degree zero;

Type II a finitely generated free W-module (concentrated in degree zero) on which $d = 0$. Via the action of F, V, it is an F-crystal with slopes concentrated in the interval $[0, 1)$;

Type III the module $U_i := U_i^0 + U_i^1$ ($i \in \mathbb{Z}$) concentrated in degrees $0, 1$ with $U_i = \prod_{n \geq 0} kV^n, U_i^1 = \prod_{n \geq i} kdV^n$ with the convention that $dV^n = F^{-n}d$ for $n \leq 0$ and F, d, V operate by left multiplication.

An R-module M is said to be ***coherent*** if it has a finite filtration with quotients (degree) shifts of elementary modules.

Consider the subcategory $D_c^b(R)$ of $D(R)$ consisting of bounded complexes of R-modules with cohomology modules are coherent (Illusie 1983, 2.4.6). This category is a triangulated category with several nonstandard t-structures (Illusie 1983, Ekedahl 1986). Ekedahl has defined an internal tensor product operation $*_R$ and internal Homs on R-modules (Illusie 1983, 2.6.1). The derived functor $*_R^L$on $D^b(R)$ is compatible with coherence and thus $D_c^b(R)$ becomes a rigid tensor triangulated category. The identity object

$\mathbb{1}$ for the tensor product is $\mathbb{1} = (W, F = \sigma, V = p\sigma^{-1}, d = 0)$ in degree zero. The functor $M \mapsto DM \stackrel{\text{def}}{=} R\,\mathrm{Hom}(M, \mathbb{1})$ on $D^b_c(R)$ gives a duality: $D(D(M))$ is naturally isomorphic to M.

The category $\mathcal{D}(k, \mathbb{Q}_p)$

Let R be the Raynaud ring. For a smooth proper variety V over k, the de Rham-Witt complex $W\Omega^\bullet$ (Illusie 1983) is a graded (étale) sheaf of R-modules on V. The complex of R-modules with components

$$H^i(V, W\mathcal{O}_V) \to H^i(V, W\Omega^1) \to \cdots H^i(V, W\Omega^j) \to \cdots$$

gives an object $R\Gamma(V, W\Omega^\bullet)$ of the derived category of R-modules. A key theorem of Illusie and Raynaud (ibid. 3.1.1) is that $R\Gamma(V, W\Omega^\bullet)$ belongs to $D^b_c(R)$. Ekedahl (1985)Ekedahl 1985 has proved Poincaré duality and Künneth theorems for $R\Gamma(V, W\Omega^\bullet)$.

The $\mathbb{Z}_p(r)$-cohomology groups of V (Milne 1986b, p309)

$$H^i(V, \mathbb{Z}_p(r)) \stackrel{\text{def}}{=} H^{i-r}(V, \nu_\bullet(r)),$$

admit a very natural description in $D^b_c(R)$ (Milne and Ramachandran 2005):

$$H^i(V, \mathbb{Z}_p(r)) \simeq \mathrm{Hom}(\mathbb{1}, R\Gamma V(r)[i]).$$

A graded R-module M can be viewed as a complex

$$\cdots \to M^i \to M^{i+1} \to \cdots$$

of R^0-modules (or $W_\sigma[F]$-modules). A bounded complex of graded R-modules gives a double complex of R^0-modules. The formation of the associated simple complex gives rise to a triangulated functor $s\colon D^b(R) \to D^b(W_\sigma[F])$ of triangulated categories. Now we may replace $W_\sigma[F]$ with $B_\sigma[F]$ where $B = W \otimes \mathbb{Q}$. If $M \in D^b_c(R)$, then $s(M) \in D^b_{\mathcal{R}}(B_\sigma[F])$ where $\mathcal{R} = \mathcal{R}(k, \mathbb{Q}_p)$ is the category of F-isocrystals over k.

PROPOSITION A.28 *The functor $s\colon D^b_c(R) \otimes \mathbb{Q} \to D^b_{\mathcal{R}}(B_\sigma[F])$ is fully faithful.*

PROOF. The assertion is obvious if the $H^i(M)$ are, up to a degree shift, of type II. But a coherent R-module differs from shifts of modules of type II by modules that are W-torsion. □

PROPOSITION A.29 *For any smooth projective variety X over a finite field k, there is a natural isomorphism*

$$H^n(s(R\Gamma X)) \simeq H^n(X/W) \otimes \mathbb{Q}.$$

PROOF. The left hand side is the direct sum $\bigoplus_{i+j=n}(H^i(X, W\Omega^j) \otimes \mathbb{Q}), p^j F)$ of the de Rham-Witt cohomology groups. The result follows from Illusie and Raynaud 1983, (1.5) on p. 75 (also quoted in Illusie 1983, Theorem 3.2.4). □