

Grothendieck’s standard conjecture of Lefschetz type over finite fields

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July 14, 2022; v1.1

Abstract

Grothendieck’s standard conjecture of Lefschetz type has two main forms: the weak form C and the strong form B . The weak form is known for varieties over finite fields as a consequence of the proof of the Weil conjectures. This suggests that the strong form of the conjecture in the same setting may be the most accessible of the standard conjectures. Here, as an advertisement for the conjecture, we explain some of its remarkable consequences.

Contents

1	Proof of Theorem 1	2
2	Almost-algebraic classes on abelian varieties	3
3	Proof of Theorem 2	6
4	Proof of Theorem 3	7
5	An unconditional variant	9
6	Statements implying the Lefschetz standard conjecture	10
	Bibliography	13

All algebraic varieties are projective, smooth, and connected (unless denoted by S). Let H be a Weil cohomology theory on the algebraic varieties over an algebraically closed field k . Let X be a variety over k of dimension d , and let $L : H^*(X) \rightarrow H^{*+2}(X)$ be the Lefschetz operator defined by a hyperplane section. The strong Lefschetz theorem states that

$$L^{d-i} : H^i(X) \rightarrow H^{2d-i}(X)$$

is an isomorphism for all $i \leq d$. The Lefschetz standard conjecture (in its strong form) states that L^{d-2i} induces an isomorphism on the \mathbb{Q} -subspaces of algebraic classes (see [Kleiman 1968](#)). We say that the Lefschetz standard conjecture holds for the algebraic varieties over a field k if the strong form holds for the classical Weil cohomology theories (ℓ -adic étale, de Rham in characteristic zero, crystalline in characteristic p).

An *almost-algebraic class* on an algebraic variety in characteristic zero is an absolute Hodge class that becomes algebraic modulo p for almost all p (see [1.3](#) below).

THEOREM 1. *The Lefschetz standard conjecture for algebraic varieties over finite fields implies the almost-Hodge conjecture for abelian varieties, i.e., all Hodge classes on complex abelian varieties are almost-algebraic.*

In the remaining statements, p is a fixed prime number and \mathbb{F} is an algebraic closure of the field \mathbb{F}_p of p elements.

THEOREM 2. *The Lefschetz standard conjecture for algebraic varieties over \mathbb{F} implies*

- (a) *the full Tate conjecture for abelian varieties over \mathbb{F} ;*
- (b) *the standard conjecture of Hodge type for abelian varieties in characteristic p .*

See §3 for the statement of the Tate conjecture and [Kleiman 1994](#), p. 16, for the statement of the Hodge standard conjecture.

THEOREM 3. *The full Tate conjecture for algebraic varieties over \mathbb{F} implies Grothendieck's standard conjectures over \mathbb{F} .*

We prove these theorems in the first four sections of the paper. In Section 5, we use a construction of Schäppi to give unconditional variants of our theorems, and in Section 6, we list some statements that imply the Lefschetz standard conjecture.

We refer to [Kleiman 1994](#) for the various forms, A, B, C, D , of the Lefschetz standard conjecture. We assume that the reader is familiar with the expository article [Milne 2020](#), cited as HAV. Throughout, \mathbb{Q}^{al} is the algebraic closure of \mathbb{Q} in \mathbb{C} .

Let A be an abelian variety over a number field K , and let w be a prime of K dividing p . We say that A has **very good reduction** at w if

- (a) A has good reduction at w , and
- (b) the Mumford-Tate group of $A_{\mathbb{C}}$ (a reductive group over \mathbb{Q}) is unramified at p .

Note that if K has finite degree over \mathbb{Q} , then A has very good reduction at almost all primes of K .

1 Proof of Theorem 1

In this section, X is an algebraic variety over a field k of characteristic zero.

1.1. Suppose first that k is algebraically closed. We let $B^r(X)$ denote the space of absolute Hodge classes of codimension r on X . Thus $B^r(X)$ is a finite-dimensional \mathbb{Q} -subspace of the adèlic cohomology group $H_{\mathbb{A}}^{2r}(X)(r)$ ([Deligne 1982](#), §3). Let $k \rightarrow k'$ be a homomorphism from k into a second algebraically closed field k' ; then the canonical map $H_{\mathbb{A}}^{2r}(X)(r) \rightarrow H_{\mathbb{A}}^{2r}(X_{k'})(r)$ induces an isomorphism $B^r(X) \rightarrow B^r(X_{k'})$ (ibid., 2.9). For an abelian variety over \mathbb{C} , every Hodge class is absolutely Hodge (ibid., Main Theorem 2.11).

1.2. Now allow k to be arbitrary of characteristic 0, and let k^{al} be an algebraic closure of k . Then $\text{Gal}(k^{\text{al}}/k)$ acts on $B^r(X_{k^{\text{al}}})$ through a finite quotient, and $B^r(X) \stackrel{\text{def}}{=} B^r(X_{k^{\text{al}}})^{\text{Gal}(k^{\text{al}}/k)}$.

1.3. We define an **almost-algebraic** class of codimension r on X to be an absolute Hodge class γ of codimension r such that there exists a cartesian square

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & X \\ \downarrow f & & \downarrow \\ S & \longleftarrow & \text{Spec}(k) \end{array}$$

and a global section $\tilde{\gamma}$ of $R^{2r}f_*\mathbb{A}(r)$ satisfying the following conditions,

- ◊ S is the spectrum of a regular integral domain of finite type over \mathbb{Z} ;
- ◊ f is smooth and projective;
- ◊ the fibre of \tilde{y} over $\text{Spec}(k)$ is γ , and the reduction of \tilde{y} at s is algebraic for all closed points s in a dense open subset of S .

Cf. Serre 1974, 5.2, and Tate 1994, p.76. Usually, almost-algebraic classes are not required to be absolutely Hodge, but since we have a robust theory of absolute Hodge classes, it is natural to include it.

THEOREM 1.4. *Assume that the Lefschetz standard conjecture holds for algebraic varieties over finite fields. Then all absolute Hodge classes on abelian varieties over fields of characteristic zero are almost-algebraic.*

PROOF. It suffices to prove this with $k = \mathbb{C}$, where it becomes a question of showing that Hodge classes on abelian varieties are almost-algebraic. Let X be an algebraic variety of dimension d over \mathbb{C} , and let $L : H^*(X, \mathbb{Q}) \rightarrow H^{*+2}(X, \mathbb{Q}(1))$ be the Lefschetz operator on Betti cohomology defined by a hyperplane section. According to the strong Lefschetz theorem, the map $L^{d-i} : H^i(X, \mathbb{Q}) \rightarrow H^{2d-i}(X, \mathbb{Q})(d-i)$ is an isomorphism for $i \leq d$; let $\theta^i : H^{2d-i}(X, \mathbb{Q})(d-i) \rightarrow H^i(X, \mathbb{Q})$ denote the inverse isomorphism.

The isomorphism $\theta^i \otimes 1 : H_{\mathbb{A}}^{2d-i}(X)(d-i) \rightarrow H_{\mathbb{A}}^i(X)$ is absolutely Hodge (i.e., its graph is an absolute Hodge class). Consider a diagram as in 1.3. For a closed point s of S such that X and L have good reduction, $\theta^i \otimes 1$ specializes to the inverse of the isomorphism $L(s)^{d-i} : H_{\mathbb{A}}^i(X(s)) \rightarrow H_{\mathbb{A}}^{2d-i}(X(s))(d-i)$. As we are assuming the standard conjecture over \mathbb{F} , this inverse is algebraic (Kleiman 1994, 4-1, $\theta \Leftrightarrow B$). Hence θ^i is almost-algebraic.

Since this holds for all X and i , the Lefschetz standard conjecture holds for almost-algebraic classes on algebraic varieties over \mathbb{C} . As for algebraic classes, this implies that all Hodge classes on abelian varieties are almost-algebraic (HAV, Theorem 4). \square

Note that the theorem does not say that an absolute Hodge class becomes algebraic modulo p for any specific p . In the next section, we prove this in some cases.

2 Almost-algebraic classes on abelian varieties

Fix a prime number p , and let \mathbb{F} be an algebraic closure of \mathbb{F}_p . In the following, ℓ is a prime number $\neq p$.

Variation of algebraic classes over \mathbb{F}

PROPOSITION 2.1. *Let S be a complete smooth curve over \mathbb{F} and $f : X \rightarrow S$ an abelian scheme over S . Assume that the Lefschetz standard conjecture holds for X and ℓ -adic étale cohomology. Let t be a global section of the sheaf $R^{2r}f_*\mathbb{Q}_{\ell}(r)$; if t_s is algebraic for one $s \in S(\mathbb{F})$, then it is algebraic for all s .*

PROOF. For a positive integer n prime to p , let θ_n denote the endomorphism of X/S acting as multiplication by n on the fibres. By a standard argument (Kleiman 1968, p. 374), θ_n^* acts as n^j on $R^j f_*\mathbb{Q}_{\ell}$. As θ_n^* commutes with the differentials d_2 of the Leray

spectral sequence $H^i(S, R^j f_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}(X, \mathbb{Q}_\ell)$, we see that it degenerates at the E_2 -term and

$$H^r(X, \mathbb{Q}_\ell) \simeq \bigoplus_{i+j=r} H^i(S, R^j f_* \mathbb{Q}_\ell),$$

where $H^i(S, R^j f_* \mathbb{Q}_\ell)$ is the direct summand of $H^r(X, \mathbb{Q}_\ell)$ on which n acts as n^j . We let aH denote the \mathbb{Q} -subspace of a cohomology group H spanned by the algebraic classes.

Let $s \in S(\mathbb{F})$ and let $\pi = \pi_1(S, s)$. The inclusion $j_s : X_s \hookrightarrow X$ induces an isomorphism $j_s^* : H^0(S, R^{2r} f_* \mathbb{Q}_\ell) \rightarrow H^{2r}(X_s, \mathbb{Q}_\ell)^\pi$ preserving algebraic classes, and so

$$\dim aH^0(S, R^{2r} f_* \mathbb{Q}_\ell) \leq \dim aH^{2r}(X_s, \mathbb{Q}_\ell)^\pi. \quad (1)$$

Similarly, the Gysin map $j_{s*} : H^{2d-2r}(X_s, \mathbb{Q}_\ell) \rightarrow H^{2d-2r+2}(X, \mathbb{Q}_\ell)$, where $d = \dim(X/S)$, induces a map $H^{2d-2r}(X_s, \mathbb{Q}_\ell)^\pi \rightarrow H^2(S, R^{2d-2r} f_* \mathbb{Q}_\ell)$ preserving algebraic classes, and so

$$\dim aH^{2d-2r}(X_s, \mathbb{Q}_\ell)^\pi \leq \dim aH^2(S, R^{2d-2r} f_* \mathbb{Q}_\ell). \quad (2)$$

Because the Lefschetz standard conjecture holds for X_s (Kleiman 1968, 2A11),

$$\dim aH^{2r}(X_s, \mathbb{Q}_\ell)^\pi = \dim aH^{2d-2r}(X_s, \mathbb{Q}_\ell)^\pi. \quad (3)$$

Hence,

$$\begin{aligned} \dim aH^0(S, R^{2r} f_* \mathbb{Q}_\ell) &\stackrel{(1)}{\leq} \dim aH^{2r}(X_s, \mathbb{Q}_\ell)^\pi \stackrel{(3)}{=} \dim aH^{2d-2r}(X_s, \mathbb{Q}_\ell)^\pi \\ &\stackrel{(2)}{\leq} \dim aH^2(S, R^{2d-2r} f_* \mathbb{Q}_\ell). \end{aligned}$$

The Lefschetz standard conjecture for X implies that

$$\dim aH^0(S, R^{2r} f_* \mathbb{Q}_\ell) = \dim aH^2(S, R^{2d-2r} f_* \mathbb{Q}_\ell),$$

and so the inequalities are equalities. Thus

$$aH^{2r}(X_s, \mathbb{Q}_\ell)^\pi = aH^0(S, R^{2r} f_* \mathbb{Q}_\ell),$$

which is independent of s . □

REMARK 2.2. The proof shows that t , when regarded as an element of $H^{2r}(X, \mathbb{Q}_\ell(r))$, is algebraic.

Weil classes

Fix a prime w of \mathbb{Q}^{al} dividing p . The residue field at w is an algebraic closure \mathbb{F} of \mathbb{F}_p . We refer to Deligne 1982 or HAV for facts on abelian varieties of Weil type.

PROPOSITION 2.3. *Assume that the Lefschetz standard conjecture holds for algebraic varieties over \mathbb{F} and ℓ -adic étale cohomology, some $\ell \neq p$. Let (A, ν) be an abelian variety over \mathbb{Q}^{al} of split Weil type relative to a CM field E , and let $t \in W_E(A) \subset H_{\mathbb{A}}^{2r}(A)$ be a Weil class on A . If A has good reduction at w to an abelian variety A_0 over \mathbb{F} , then the element $(t_\ell)_0$ of $H^{2r}(A_0, \mathbb{Q}_\ell)$ is algebraic.*

The proof will occupy the remainder of this subsection. In outline, it follows the proof of [Deligne 1982](#), Theorem 4.8, but requires a delicate reduction argument of André.

LEMMA 2.4. *Let (A, ν) be an abelian variety over \mathbb{Q}^{al} of split Weil type relative to E . Then there exists a connected smooth variety S over \mathbb{C} , an abelian scheme $f : X \rightarrow S$ over S , and an action ν of E on X/S such that*

- (a) *for some $s_1 \in S(\mathbb{C})$, $(X_{s_1}, \nu_{s_1}) \approx (A, \nu)_{\mathbb{C}}$;*
- (b) *for all $s \in S(\mathbb{C})$, (X_s, ν_s) is of split Weil type relative to E ;*
- (c) *for some $s_2 \in S(\mathbb{C})$, X_{s_2} is of the form $B \otimes_{\mathbb{Q}} E$ with $e \in E$ acting as $\text{id} \otimes e$.*

PROOF. See the proof of [Deligne 1982](#), 4.8. □

We shall need to use additional properties of the family $X \rightarrow S$ constructed by Deligne. For example, there is a local subsystem $W_E(X/S)$ of $R^{2r}f_*\mathbb{Q}$ such that $W_E(X/S)_s = W_E(X_s)$ for all $s \in S(\mathbb{C})$. Also, the variety B in (c) can be chosen to be a power of CM elliptic curve (so X_{s_2} is isogenous to a power of a CM elliptic curve).

The variety S has a unique model over \mathbb{Q}^{al} with the property that every CM-point $s \in S(\mathbb{C})$ lies in $S(\mathbb{Q}^{\text{al}})$. This follows from the general theory of Shimura varieties; or from the general theory of locally symmetric varieties (Faltings; [Peters 2017](#)); or (best) from descent theory ([Milne 1999a](#), 2.3) using that S is a moduli variety over \mathbb{C} and that the moduli problem is defined over \mathbb{Q}^{al} . The morphism f is also defined over \mathbb{Q}^{al} , and we will now simply write $f : X \rightarrow S$ for the family over \mathbb{Q}^{al} . There is a \mathbb{Q} -local subsystem $W_E(X/S)$ of $R^{2r}f_*\mathbb{Q}_{\ell}$ such that $W_E(X/S)_s = W_E(X_s)$ for all $s \in S(\mathbb{Q}^{\text{al}})$. The points s_1 and s_2 lie in $S(\mathbb{Q}^{\text{al}})$.

We now assume that E contains an imaginary quadratic field in which the prime p splits — this is the only case we shall need, and it implies the general case.

The family $X \rightarrow S$ (without the action of E) defines a morphism from S into a moduli variety M over \mathbb{Q}^{al} for polarized abelian varieties with certain level structures. Let \mathcal{M} be the corresponding moduli scheme over \mathcal{O}_w and \mathcal{M}^* its minimal compactification ([Chai and Faltings 1990](#)). Let S^* be the closure of S in \mathcal{M}^* .

LEMMA 2.5. *The complement of $S_{\mathbb{F}}^* \cap \mathcal{M}_{\mathbb{F}}$ in $S_{\mathbb{F}}^*$ has codimension at least two.*

PROOF. See André 2.4.2. □

Recall that s_1 and s_2 are points in $S(\mathbb{Q}^{\text{al}})$ such that $X_{s_1} = A$ and X_{s_2} is a power of a CM-elliptic curve. As A and the elliptic curve have good reduction, the points extend to points \mathfrak{s}_1 and \mathfrak{s}_2 of $S^* \cap \mathcal{M}$. Let $\bar{\mathcal{S}}$ denote the blow-up of S^* centred at the closed subscheme defined by the image of \mathfrak{s}_1 and \mathfrak{s}_2 , and let \mathcal{S} be the open subscheme obtained by removing the strict transform of the boundary $S^* \setminus (S^* \cap \mathcal{M})$. It follows from 2.5 that $S_{\mathbb{F}}$ is connected, and that any sufficiently general linear section of relative dimension $\dim(S) - 1$ in a projective embedding $\bar{\mathcal{S}} \hookrightarrow \mathbb{P}_{\mathcal{O}_w}^N$ is a projective flat \mathcal{O}_w -curve \mathcal{C} contained in \mathcal{S} with smooth geometrically connected generic fibre (André 2.5.1). Consider $(\mathcal{X}|\mathcal{C})_{\mathbb{F}} \rightarrow \mathcal{C}_{\mathbb{F}}$. After replacing $\mathcal{C}_{\mathbb{F}}$ by its normalization and pulling back $(\mathcal{X}|\mathcal{C})_{\mathbb{F}}$, we are in the situation of Proposition 2.1. The class t_{s_2} is algebraic because the Hodge conjecture holds for powers of elliptic curves (the \mathbb{Q} -algebra of Hodge classes is generated by divisor classes). Hence $(t_{s_2\ell})_0$ is algebraic, and 2.1 shows that $(t_{s_1\ell})_0$ is algebraic. This completes the proof of Proposition 2.3.

Absolute Hodge classes on abelian varieties

Again, w is a prime of \mathbb{Q}^{al} lying over p and ℓ is a prime number $\neq p$.

THEOREM 2.6. *Assume that the Lefschetz standard conjecture holds for algebraic varieties over \mathbb{F} . Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w to an abelian variety A_0 over \mathbb{F} , and let t be an absolute Hodge class on A . If A is CM or has very good reduction at w , then the class $(t_\ell)_0$ on A_0 is algebraic.*

PROOF. We first assume that A is CM, say, of type (E, Φ) . Let F be a CM-subfield of \mathbb{C} , finite and Galois over \mathbb{Q} , that splits E . We may suppose that F contains an imaginary quadratic field in which p splits.

For each subset Δ of $\text{Hom}(E, F)$ such that $|t\Delta \cap \Phi| = r = |t\Delta \cap \bar{\Phi}|$ for all $t \in \text{Gal}(F/\mathbb{Q})$, we let $A_\Delta = \prod_{s \in \Delta} A \otimes_{E, s} F$. There is an obvious homomorphism $f_\Delta : A \rightarrow A_\Delta$. The abelian variety A_Δ is of split Weil type, and every absolute Hodge class t on A can be written as a sum $t = \sum f_\Delta^*(t_\Delta)$ with t_Δ a Weil class on A_Δ (André 1992; HAV, Theorem 1). Thus the theorem in this case follows from Proposition 2.3.

We now consider the case that A has very good reduction. There exists an abelian scheme $f : X \rightarrow S$ over \mathbb{C} with S a connected Shimura variety, and a section γ of $R^{2r} f_* \mathbb{A}$ such that $(X, \gamma)_s = (A, t)$ (Deligne 1982, 6.1). As before, we may suppose that f is defined over \mathbb{Q}^{al} and that $s \in S(\mathbb{Q}^{\text{al}})$. There exists a point $s' \in S(\mathbb{Q}^{\text{al}})$ such that $(s')_0 = s_0$ in $S_0(\mathbb{F})$ and $X_{s'}$ is a CM abelian variety (Kisin, Vasiu, using that A has very good reduction). Now the theorem for $X_{s'}$ implies that $(t_{s\ell})_0$ is algebraic. \square

3 Proof of Theorem 2.

Fix an algebraic closure \mathbb{F} of \mathbb{F}_p , and let \mathbb{F}_q be the subfield of \mathbb{F} with q elements.

3.1. Let X be an algebraic variety over \mathbb{F}_q . For $\ell \neq p$, the Tate conjecture $T(X, \ell)$ states that the \mathbb{Q}_ℓ -vector space $H_\ell^{2*}(X)(*)^{\text{Gal}(\mathbb{F}/\mathbb{F}_q)}$ is spanned by algebraic classes, and the conjecture $S(X, \ell)$ states that the obvious map $H_\ell^{2*}(X)(*)^{\text{Gal}(\mathbb{F}/\mathbb{F}_q)} \rightarrow H_\ell^{2*}(X)(*)^{\text{Gal}(\mathbb{F}/\mathbb{F}_q)}$ is an isomorphism. The full Tate conjecture $T(X)$ states that, for all r , the pole of the zeta function $Z(X, t)$ at $t = q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles on X of codimension r . It is known (folklore) that, if $T(X, \ell)$ and $S(X, \ell)$ hold for a single ℓ , then the full Tate conjecture $T(X)$ holds, in which case $T(X, \ell)$ and $S(X, \ell)$ hold for all ℓ . See Tate 1994.

We say that one of these conjectures holds for an algebraic variety X over \mathbb{F} if it holds for all models of X over finite subfields of \mathbb{F} (it suffices to check that it holds for some model over a sufficiently large subfield).

THEOREM 3.2. *Assume that the Lefschetz standard conjecture holds for algebraic varieties over \mathbb{F} and ℓ -adic étale cohomology (some $\ell \neq p$). Then the full Tate conjecture holds for abelian varieties over finite fields of characteristic p .*

PROOF. In Milne 1999b, the Tate conjecture for abelian varieties over \mathbb{F} is shown to follow from the Hodge conjecture for CM abelian varieties over \mathbb{C} . However, the proof does not use that the Hodge classes are algebraic, but only that they become algebraic modulo p . Hence we can deduce from Proposition 2.6 that the Tate conjecture holds

for abelian varieties over \mathbb{F} and some ℓ . As the Frobenius map acts semisimply on the cohomology of abelian varieties (Weil 1948), this implies that the full Tate conjecture holds for abelian varieties over \mathbb{F} . \square

THEOREM 3.3. *Assume that the Lefschetz standard conjecture holds for algebraic varieties over \mathbb{F} and ℓ -adic étale cohomology (some $\ell \neq p$). Then Grothendieck's standard conjecture of Hodge type holds for abelian varieties over fields of characteristic p and the classical Weil cohomology theories.*

PROOF. In Milne 2002 the Hodge standard conjecture for abelian varieties in characteristic p is shown to follow from the Hodge conjecture for CM abelian varieties over \mathbb{C} . Again, the proof uses only that the Hodge classes become algebraic modulo p , and so the theorem follows from Proposition 2.6. \square

COROLLARY 3.4. *Assume that the Lefschetz standard conjecture holds for algebraic varieties over \mathbb{F} and ℓ -adic étale cohomology (some $\ell \neq p$). Then the conjecture of Langlands and Rapoport (1987, 5.e) is true for simple Shimura varieties of PEL-types A and C .*

PROOF. Langlands and Rapoport (ibid., §6) prove this under the assumption of the Hodge conjecture for CM abelian varieties and the Tate and Hodge standard conjectures for abelian varieties over \mathbb{F} . However, their argument does not use that Hodge classes on CM abelian varieties are algebraic, but only that they become algebraic modulo p . As this, together with the Tate and Hodge standard conjectures, are implied by the Lefschetz standard conjecture, so also is their conjecture. \square

4 Proof of Theorem 3.

Briefly, the Tate conjecture over \mathbb{F} implies the Lefschetz standard conjecture over \mathbb{F} , and hence the Hodge standard conjecture for abelian varieties (Theorem 2). Now form the category of abelian motives over \mathbb{F} : Grothendieck's standard conjectures hold for it. The full Tate conjecture implies that the category of abelian motives contains the motives of all algebraic varieties over \mathbb{F} , and so the Hodge standard conjecture holds for them also.

We now prove more precise statements.

PROPOSITION 4.1. *Let X be an algebraic variety over \mathbb{F} . If the Tate conjecture holds for X and some ℓ , then the Lefschetz standard conjecture holds for X and the same ℓ .*

PROOF. To prove the Lefschetz standard conjecture for X and a prime ℓ , it suffices to show that, for each $i \leq d \stackrel{\text{def}}{=} \dim(X)$, there exists an algebraic correspondence inducing an isomorphism $H_\ell^{2d-i}(X) \rightarrow H_\ell^i(X)$ (Kleiman 1994, 4-1, $\nu(X) \Leftrightarrow B(X)$). The inverse θ^i of the Lefschetz map $L^{d-i} : H_\ell^i(X) \rightarrow H_\ell^{2d-i}(X)(d-i)$ is an isomorphism $H_\ell^{2d-i}(X)(d-i) \rightarrow H_\ell^i(X)$ commuting with the action of the Galois group. Any algebraic class ν^i sufficiently close to the graph of θ^i will induce the required isomorphism. \square

PROPOSITION 4.2. *Let H be a Weil cohomology theory on algebraic varieties over an algebraically closed field k , and let X and Y be algebraic varieties over k . Assume that there exists an algebraic correspondence α on $X \times Y$ such that*

$$\alpha_* : H^*(X) \rightarrow H^*(Y)$$

is injective. If the Hodge standard conjecture holds for Y , then it holds for X .

PROOF. Apply [Kleiman 1968](#), 3.11, and [Saavedra Rivano 1972](#), VI, 4.4.2. \square

LEMMA 4.3. *Let X be an algebraic variety over \mathbb{F}_q . If $S(X \times X, \ell)$ holds for some ℓ , then the Frobenius endomorphism acts semisimply on the ℓ -adic étale cohomology of X .*

PROOF. The statement $S(X \times X, \ell)$ says that 1, if an eigenvalue of the Frobenius element acting on the ℓ -adic cohomology of $X \times X$, is semisimple. From the Künneth formula

$$H_\ell^r(X \times X) \simeq \bigoplus_{i+j=r} H_\ell^i(X) \otimes H_\ell^j(X)$$

and linear algebra, we see that this implies that all eigenvalues on $H_\ell^*(X)$ are semisimple. \square

It is conjectured that the Frobenius element always acts semisimply (Semisimplicity Conjecture).

Fix a power q of p and a prime $\ell \neq p$. Define a **Tate structure** to be a finite-dimensional \mathbb{Q}_ℓ -vector space with a linear (Frobenius) map ϖ whose characteristic polynomial lies in $\mathbb{Q}[T]$ and whose eigenvalues are Weil q -numbers, i.e., algebraic numbers α such that, for some integer m (called the weight of α), $|\rho(\alpha)| = q^{m/2}$ for every homomorphism $\rho : \mathbb{Q}[\alpha] \rightarrow \mathbb{C}$, and, for some integer n , $q^n \alpha$ is an algebraic integer. When the eigenvalues are all of weight m (resp. algebraic integers, resp. semisimple), we say that V is of **weight m** (resp. **effective**, resp. **semisimple**). For example, for any smooth complete variety X over k , $H_\ell^i(X)$ is an effective Tate structure of weight $i/2$ ([Deligne 1974](#)), which is semisimple if X is an abelian variety ([Weil 1948](#), no. 70).

PROPOSITION 4.4. *Every effective semisimple Tate structure is isomorphic to a Tate substructure of $H_\ell^*(A)$ for some abelian variety A over \mathbb{F}_q .*

PROOF. We may assume that the Tate structure V is simple. Then V has weight m for some $m \geq 0$, and the characteristic polynomial $P(T)$ of ϖ is a monic irreducible polynomial with coefficients in \mathbb{Z} whose roots all have real absolute value $q^{m/2}$. According to Honda's theorem ([Honda 1968](#); [Tate 1968](#)), $P(T)$ is the characteristic polynomial of an abelian variety A over \mathbb{F}_{q^m} . Let B be the abelian variety over \mathbb{F}_q obtained from A by restriction of the base field. The eigenvalues of the Frobenius map on $H_\ell^1(B)$ are the m th-roots of the eigenvalues of the Frobenius map on $H_\ell^1(A)$, and it follows that V is a Tate substructure of $H_\ell^m(B)$. \square

THEOREM 4.5. *Let X be an algebraic variety over \mathbb{F} , and let ℓ be a prime $\neq p$. If the Frobenius map acts semisimply on $H_\ell^*(X)$ and the Tate conjecture holds for ℓ and all varieties of the form $X \times A$ with A an abelian variety, then the Hodge standard conjecture holds for X and ℓ .*

PROOF. According to [4.4](#), there exists an inclusion $H_\ell^*(X) \hookrightarrow H_\ell^*(A)$ of Tate structures with A an abelian variety. This map is defined by a cohomological correspondence on $X \times A$ fixed by the Galois group. Any algebraic correspondence sufficiently close to this correspondence defines an inclusion $H_\ell^*(X) \hookrightarrow H_\ell^*(A)$. Now we can apply [Proposition 4.2](#). \square

COROLLARY 4.6. *If the Tate and semisimplicity conjectures hold for all algebraic varieties over \mathbb{F} and some prime number ℓ , then both the full Tate and Grothendieck standard conjectures hold for all algebraic varieties over \mathbb{F} and all ℓ .*

PROOF. Immediate consequence of the theorem. \square

5 An unconditional variant

We use [Schäppi 2020](#) to replace some of the above statements by unconditional variants.

Characteristic zero

Let k be an algebraically closed field of characteristic zero, and fix an embedding $k \hookrightarrow \mathbb{C}$. Let H denote the Weil cohomology theory $X \rightsquigarrow H^*(X(\mathbb{C}), \mathbb{Q})$, and let $\text{Mot}_H(k)$ denote the category of motives defined using almost-algebraic classes as correspondences. It is a graded pseudo-abelian rigid tensor category¹ over \mathbb{Q} .

According to [Schäppi 2020](#), §3, the fibre functor² $\omega_H : \text{Mot}_H(k) \rightarrow \mathbb{Z}\text{-Vec}_{\mathbb{Q}}$ factors in a canonical way through a “universal” graded tannakian category $\mathcal{M}_H(k)$ over \mathbb{Q} ,

$$\text{Mot}_H(k) \xrightarrow{[-]} \mathcal{M}_H(k) \xrightarrow{\omega} \mathbb{Z}\text{-Vec}_{\mathbb{Q}},$$

where ω is a graded fibre functor.³

We define the **algebraic*** classes on an algebraic variety X over k to be the elements of $\text{Hom}(\mathbf{1}, [h(X)])$. The Lefschetz standard conjecture holds for algebraic* classes ([Schäppi 2020](#), §3; alternatively, apply Corollary 6.5 below).

Now ω_H is a functor from $\text{Mot}_H(k)$ into the category $\text{Hdg}_{\mathbb{Q}}$ of polarizable rational Hodge structures. This factors through $\mathcal{M}_H(k)$,

$$\text{Mot}_H(k) \xrightarrow{[-]} \mathcal{M}_H(k) \xrightarrow{\omega} \text{Hdg}_{\mathbb{Q}},$$

where ω is a functor of graded tannakian categories. Therefore algebraic* classes on X are Hodge classes relative to the given embedding of k into \mathbb{C} . It follows that Grothendieck’s standard conjecture of Hodge type holds for algebraic* classes. Moreover, all algebraic* classes on abelian varieties are absolutely Hodge ([Deligne 1982](#), 2.11).

The same proof as for almost-algebraic classes (see §1) shows that the Hodge conjecture holds for algebraic* classes on abelian varieties over \mathbb{C} , i.e., all Hodge classes on abelian varieties over \mathbb{C} are algebraic*. As a consequence, for abelian varieties satisfying the Mumford-Tate conjecture, the Tate conjecture holds for algebraic* classes.

Characteristic p

Fix a prime number p , and let \mathbb{F} denote an algebraic closure of \mathbb{F}_p . For $\ell \neq p$, we let $\text{Mot}_{\ell}(\mathbb{F})$ denote the category of motives over \mathbb{F} defined using algebraic classes modulo ℓ -adic homological equivalence as correspondences. It is a graded pseudo-abelian rigid tensor category over \mathbb{Q} .

According to [Schäppi 2020](#), §3, the graded tensor functor $\omega_{\ell} : \text{Mot}_{\ell}(\mathbb{F}) \rightarrow \mathbb{Z}\text{-Vec}_{\mathbb{Q}_{\ell}}$ factors in a canonical way through a graded tannakian category $\mathcal{M}_{\ell}(\mathbb{F})$,

$$\text{Mot}_{\ell}(k) \xrightarrow{[-]} \mathcal{M}_{\ell}(\mathbb{F}) \xrightarrow{\omega} \mathbb{Z}\text{-Vec}_{\mathbb{Q}},$$

¹tensor category (functor) = symmetric monoidal category (functor)

² $\mathbb{Z}\text{-Vec}_{\mathbb{Q}}$ is the category of finite-dimensional \mathbb{Z} -graded vector space over \mathbb{Q} .

³fibre functor = exact faithful tensor functor

where ω is a graded fibre functor. Unfortunately, we do not know that $\text{End}(\mathbf{1}) = \mathbb{Q}$ in $\mathcal{M}_\ell(\mathbb{F})$, only that it is a subfield of \mathbb{Q}_ℓ .⁴

Let X be an algebraic variety over \mathbb{F} . We define the **algebraic*** classes on X to be the elements of $\text{Hom}(\mathbf{1}, [h(X)])$. As before, the Lefschetz standard conjecture holds for algebraic* classes. Therefore Proposition 2.1 holds unconditionally for algebraic* classes: let $f : X \rightarrow S$ be as in the proposition, and let t be a global section of the sheaf $R^{2r}f_*\mathbb{Q}_\ell(r)$; if t_s is algebraic* for one $s \in S(\mathbb{F})$, then it is algebraic* for all s .

REMARK 5.1. Until it is shown that $\text{End}(\mathbf{1}) = \mathbb{Q}$ in $\mathcal{M}_\ell(\mathbb{F})$, this category is of only modest interest. For abelian motives, what is needed is a proof of the rationality conjecture (Milne 2009, 4.1).⁵

Mixed characteristic

Fix a prime w of \mathbb{Q}^{al} dividing p and a prime number $\ell \neq p$. Theorem 2.6 holds unconditionally for algebraic* classes: let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w to an abelian variety A_0 over \mathbb{F} , and let t be an absolute Hodge class (e.g., an algebraic* class) on A ; if A is CM or has very good reduction, then $(t_\ell)_0$ is an algebraic* class on A_0 . The proof is the same as before, using the * version of Proposition 2.1.

We deduce, as in the proof of Theorem 3.2, that the Tate conjecture holds for algebraic* classes on abelian varieties over \mathbb{F} , i.e., that ℓ -adic Tate classes on abelian varieties over \mathbb{F} are algebraic*.

Let $\mathcal{M}'_H(\mathbb{Q}^{\text{al}})$ denote the tannakian subcategory of $\mathcal{M}_H(\mathbb{Q}^{\text{al}})$ generated by abelian varieties with good reduction at w . There is a canonical tensor functor $\mathcal{M}'_H(\mathbb{Q}^{\text{al}}) \rightarrow \mathcal{M}_\ell(\mathbb{F})$.

6 Statements implying the Lefschetz standard conjecture

Conjecture D and the Lefschetz standard conjecture

Let H be a Weil cohomology theory. The next statement goes back to Grothendieck.

PROPOSITION 6.1. *Assume that H satisfies the strong Lefschetz theorem. Conjecture D(X) implies $A(X, L)$ (all L); in the presence of the Hodge standard conjecture, $A(X, L)$ (one L) implies D(X).*

PROOF. Conjecture D(X) says that the pairing

$$x, y \mapsto \langle x \cdot y \rangle : A_H^i(X) \times A_H^{d-i}(X) \rightarrow A_H^d(X) \simeq \mathbb{Q} \quad (4)$$

is nondegenerate for all $i \leq d \stackrel{\text{def}}{=} \dim(X)$. Therefore, $\dim A_H^i(X) = \dim A_H^{d-i}(X)$. As the map $L^{d-2i} : A_H^i(X) \rightarrow A_H^{d-i}(X)$ is injective, it is surjective, i.e., $A(X, L)$ holds. The converse is equally obvious. \square

⁴André's category of motivated classes in characteristic p has the same problem.

⁵Let A be an abelian variety over \mathbb{Q}^{al} with good reduction to an abelian variety A_0 over \mathbb{F} ; the cup product of the specialization to A_0 of any absolute Hodge class on A with a product of divisors of complementary codimension lies in \mathbb{Q} .

COROLLARY 6.2. *Conjecture $D(X \times X)$ implies $B(X)$.*

PROOF. Indeed, $A(X \times X, L \otimes 1 + 1 \otimes L)$ implies $B(X)$ (Kleiman 1968, Theorem 4-1). \square

REMARK 6.3. If Conjecture $D(X \times X)$ holds whenever X is an abelian scheme over a complete smooth curve over \mathbb{C} , then the Hodge conjecture holds for abelian varieties.

Does Conjecture C imply Conjecture B?

Kleiman (1994) states eight versions of Grothendieck’s standard conjecture of Lefschetz type. He proves that six of the eight are equivalent and that a seventh is “practically equivalent” to the others, but he states that the eighth version, Conjecture C, “is, doubtless, truly weaker”. In this subsection we examine whether Conjecture C is, in fact, equivalent to the remaining conjectures.

Let H be a Weil cohomology theory on the algebraic varieties over an algebraically closed field k . Assume that H satisfies conjecture C, and let $\text{Mot}_H(k)$ denote the category of motives defined using algebraic classes modulo homological equivalence as the correspondences. It is a graded pseudo-abelian rigid tensor category over \mathbb{Q} equipped with a graded tensor functor $\omega_H : \text{Mot}_H \rightarrow \mathbb{Z}\text{-Vec}_Q$, where Q is the coefficient field of H .

PROPOSITION 6.4. *Assume that H satisfies the strong Lefschetz theorem in addition to Conjecture C. If ω_H is conservative, then H satisfies the Lefschetz standard conjecture.*

PROOF. Let $L : H^r(X) \rightarrow H^{r+2}(X)(1)$ be the Lefschetz operator defined by a hyperplane section of X . By assumption

$$L^{d-2i} : H^{2i}(X)(i) \rightarrow H^{2d-2i}(X)(d-i) \quad (5)$$

is an isomorphism for all $2i \leq d \stackrel{\text{def}}{=} \dim(X)$. As ω_H is conservative,

$$l^{d-2i} : h^{2i}(X)(i) \rightarrow h^{2d-2i}(X)(d-i) \quad (6)$$

is an isomorphism for all $2i \leq d$. On applying the functor $\text{Hom}(\mathbf{1}, -)$ to this isomorphism, we get an isomorphism

$$L^{d-2i} : A_H^i(X) \rightarrow A_H^{d-i}(X).$$

Thus, Conjecture $A(X, L)$ is true. \square

COROLLARY 6.5. *Assume that H satisfies the strong Lefschetz theorem and Conjecture C. If $\text{Mot}_H(k)$ is tannakian, then H satisfies Conjecture B.*

PROOF. Fibre functors on tannakian categories are conservative. \square

Proposition 6.1 shows that a Weil cohomology theory satisfying both the strong Lefschetz theorem and Conjecture D also satisfies Conjecture B. Here we prove a stronger result.

PROPOSITION 6.6. *Suppose that there exists a Weil cohomology theory \mathcal{H} satisfying both the strong Lefschetz theorem and Conjecture D . Then every Weil cohomology theory H satisfying the strong Lefschetz theorem and Conjecture C also satisfies Conjecture B.*

PROOF. Let \mathcal{H} and H be Weil cohomology theories satisfying the strong Lefschetz theorem and assume that \mathcal{H} (resp. H) satisfies Conjecture D (resp. Conjecture C). Then \mathcal{H} satisfies the Lefschetz conjecture (6.1), in particular, Conjecture C . Let $\text{Mot}_{\text{num}}(k) = \text{Mot}_{\mathcal{H}}(k)$ be the category of motives defined using algebraic cycles modulo numerical equivalence as correspondences. Then Mot_{num} is a semisimple tannakian category over \mathbb{Q} (Jannsen, Deligne), and there is a quotient functor $q : \text{Mot}_H \rightarrow \text{Mot}_{\text{num}}$. For each M in Mot_H , the map $\text{End}(M) \rightarrow \text{End}(qM)$ is surjective with kernel the radical of the ring $\text{End}(M)$, and this radical is nilpotent (Jannsen 1992).

The conditions on \mathcal{H} imply that it satisfies Conjecture B (Proposition 6.1). This means that for each $i \leq d \stackrel{\text{def}}{=} \dim(X)$, there exists a morphism $h_{\text{num}}^{2d-i}(X)(d-i) \rightarrow h_{\text{num}}^i(X)$ inducing the inverse of the map

$$L^{d-i} : \mathcal{H}_{\text{num}}^i(X) \rightarrow \mathcal{H}_{\text{num}}^{2d-i}(X)(d-i).$$

Write α for the morphism $h^i(X) \rightarrow h^{2d-i}(X)(d-i)$ in $\text{Mot}_H(k)$ inducing the isomorphism

$$L^{d-i} : H^i(X) \rightarrow H^{2d-i}(X)(d-i). \quad (7)$$

According to the last paragraph, there exists a morphism $\beta : h^{2d-i}(X)(d-i) \rightarrow h^i(X)$ such that $q(\beta \circ \alpha) = \text{id}_{h_{\text{num}}^i(X)}$. Now $\beta \circ \alpha = 1 + n$ in $\text{End}(h^i(X))$, where n is nilpotent. On replacing β with $(1 - n + n^2 - \dots) \circ \beta$, we find that $\beta \circ \alpha = 1$ in $\text{End}(h^i(X))$. Hence the inverse of the map (7) is algebraic, as required. \square

PROPOSITION 6.7. *If there exists one Weil cohomology theory satisfying the strong Lefschetz theorem and Conjecture D , then every Weil cohomology theory satisfying Conjecture D also satisfies the strong Lefschetz theorem*

PROOF. If there exists a Weil cohomology theory satisfying the strong Lefschetz theorem and Conjecture D , then in $\text{Mot}_{\text{num}}(k)$,

$$l^{d-i} : h^i(X) \rightarrow h^{2d-i}(X)(d-i)$$

is an isomorphism for $i \leq d$. Let H be a Weil cohomology theory satisfying Conjecture D . On applying H to this isomorphism, we get an isomorphism

$$L^{d-i} : H^i(X) \rightarrow H^{2d-i}(X)(d-i). \quad \square$$

REMARK 6.8. Because Mot_{num} is Tannakian, there exists a field Q of characteristic zero and a Q -valued fibre functor ω . Then $\mathcal{H} : X \rightsquigarrow \bigoplus_i \omega(X, \pi_i, 0)$ is a Weil cohomology theory satisfying Conjecture D . It remains to show that ω can be chosen so that \mathcal{H} satisfies the strong Lefschetz theorem. This comes down to showing that $l^{d-i} : h^i(X) \rightarrow h^{2d-i}(X)(d-i)$ is an isomorphism in $\text{Mot}_{\text{num}}(k)$.

REMARK 6.9. Every Weil cohomology theory satisfying the weak Lefschetz theorem also satisfies the strong Lefschetz theorem (Katz and Messing 1974, Corollaries to Theorem 1).

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