

Classification of the Mumford–Tate Groups of Rational Polarizable Hodge Structures

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May 10, 2023, v2.1

Abstract

Let G be the pro-algebraic group attached to the tannakian category of polarizable rational Hodge structures. We show that the quotient of G by its derived group is the Serre group, its derived group is the simply connected covering of its adjoint group, and its adjoint group is a product of specific simple algebraic groups. As the Mumford–Tate groups are exactly the algebraic quotients of G , this also describes them. ¹

Contents

1	Definitions	2
2	Cartan involutions and polarizations	3
3	Mumford–Tate groups	4
4	Tori as Mumford–Tate groups	6
5	Semisimple groups as Mumford–Tate groups	8
6	The classification	9
	References	11

Mumford and Tate originally defined their algebraic groups for complex abelian varieties. However, the group depends only on the Hodge structure attached to the abelian variety, and the notion was soon extended to all rational Hodge structures. The groups are of most interest when the Hodge structure is polarizable.

0.1. The Mumford–Tate group of a rational Hodge structure is an algebraic group G over \mathbb{Q} equipped with a cocharacter $\mu : \mathbb{G}_m \rightarrow G_{\mathbb{C}}$. The weight $w(\mu)$ of μ is the cocharacter $-\mu - \bar{\mu}$ of $G_{\mathbb{C}}$. In §3, we obtain the following criterion: a pair (G, μ) is the Mumford–Tate group of a polarizable rational Hodge structure if and only if it satisfies the following conditions:

- mt1: the weight $w(\mu)$ of μ is defined over \mathbb{Q} and is central;
- mt2: $\text{ad } \mu(-1)$ is a Cartan involution of $(G/w(\mathbb{G}_m))_{\mathbb{R}}$;
- mt3: μ generates G (i.e., if $H \subset G$ is such that $\mu(\mathbb{C}^{\times}) \subset H(\mathbb{C})$, then $H = G$).

¹v1, posted 27.12.20. v2, posted 09.05.23 improvements to exposition; v2.1, posted 10.05.23 minor fixes.

0.2. The polarizable rational Hodge structures form a tannakian category $\text{Hdg}_{\mathbb{Q}}$ over \mathbb{Q} with a canonical (forgetful) fibre functor. The corresponding Tannaka group G_{Hdg} is the pro-algebraic group over \mathbb{Q} having the Mumford–Tate groups as its algebraic quotients. Thus understanding the Mumford–Tate groups amounts to understanding G_{Hdg} . We obtain the following results:

- ◊ the quotient of G_{Hdg} by its derived group is the (well-known) Serre protorus S ;
- ◊ the derived group of G_{Hdg} is simply connected, and hence is the simply connected covering of the adjoint group of G_{Hdg} ;
- ◊ the simple factors of the adjoint group of G_{Hdg} are the groups of the form $(G)_{F/\mathbb{Q}}$, where F is a totally real number field and G is a geometrically simple algebraic group over F such that $(G)_{F/\mathbb{Q}}(\mathbb{R})$ has a compact maximal torus.

The article is largely expository because all of the intermediate results have long been available in the literature.

Notation and terminology

All vector spaces are finite dimensional. Complex conjugation on \mathbb{C} is denoted by $z \mapsto \bar{z}$ or $z \mapsto \iota z$. The terminology concerning algebraic groups is that of [Milne 2017](#). In particular, semisimple and reductive algebraic groups are connected, and an adjoint algebraic group is a semisimple group with trivial centre. The centre of G is denoted by $Z(G)$. When K/k is a finite extension of fields and G is an algebraic group over K , we let $(G)_{K/k}$ denote the algebraic group over k obtained from G by restriction of scalars.

1 Definitions

The *Deligne torus* S is defined to be $(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$. Thus

$$S(\mathbb{R}) = \mathbb{C}^\times, \quad S_{\mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_m.$$

The map $S(\mathbb{R}) \rightarrow S(\mathbb{C})$ induced by $\mathbb{R} \rightarrow \mathbb{C}$ is $z \mapsto (z, \bar{z})$. There are homomorphisms

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{w} & S & \xrightarrow{t} & \mathbb{G}_m, & t \circ w = -2, \\ \mathbb{R}^\times & \xrightarrow{a \mapsto a^{-1}} & \mathbb{C}^\times & \xrightarrow{z \mapsto z\bar{z}} & \mathbb{R}^\times. \end{array}$$

We denote the kernel of t by S^1 . Thus S^1 is a one-dimensional torus over \mathbb{R} with

$$S^1(\mathbb{R}) = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\} = \text{circle group } S^1.$$

There is a canonical isomorphism

$$S/w(\mathbb{G}_m) \rightarrow S^1, \quad z \pmod{\mathbb{R}^\times} \mapsto z/\bar{z},$$

with inverse $u \mapsto \sqrt{u} \pmod{\mathbb{R}^\times}$.

A homomorphism $h : S \rightarrow G$ of real algebraic groups gives rise to cocharacters

$$\begin{array}{l} \mu_h : \mathbb{G}_m \rightarrow G_{\mathbb{C}}, \quad z \mapsto h_{\mathbb{C}}(z, 1), \quad z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times, \\ w_h : \mathbb{G}_m \rightarrow G, \quad w_h = h \circ w \quad (\text{weight homomorphism}). \end{array}$$

The following formulas are useful,

$$\begin{aligned} h_{\mathbb{C}}(z_1, z_2) &= \mu_h(z_1) \cdot \bar{\mu}_h(z_2); & h(z) &= \mu_h(z) \cdot \overline{\mu_h(z)} \\ h(i) &= \mu_h(-1) \cdot w_h(i); & w_h &= w(\mu_h). \end{aligned}$$

A *Hodge structure* on a real vector space V is a homomorphism $h : \mathbb{S} \rightarrow \mathrm{GL}_V$. Such a homomorphism determines a decomposition $V \otimes \mathbb{C} = \bigoplus V^{p,q}$, where $V^{p,q}$ is the subspace on which $h(z)$ acts as $z^{-p} \cdot iz^{-q}$. A *rational Hodge structure* (V, h) is a \mathbb{Q} -vector space V together with a Hodge structure h on $V \otimes \mathbb{R}$ such that w_h is defined over \mathbb{Q} . The *Tate Hodge structure* $\mathbb{Q}(m)$ is the \mathbb{Q} -subspace $(2\pi i)^m \mathbb{Q}$ of \mathbb{C} with $h(z)$ acting as multiplication by $(z\bar{z})^m$.

A *polarization* of a rational Hodge structure (V, h) of weight m is a morphism of Hodge structures

$$\psi : V \otimes V \rightarrow \mathbb{Q}(-m), \quad m \in \mathbb{Z},$$

such that

$$(x, y) \mapsto (2\pi i)^m \psi_{\mathbb{R}}(x, Cy) : V_{\mathbb{R}} \times V_{\mathbb{R}} \rightarrow \mathbb{R}$$

is symmetric and positive definite. Here $C \stackrel{\text{def}}{=} h(i)$ is the Weil operator.

NOTES. The conventions are those of [Deligne 1979](#).

2 Cartan involutions and polarizations

Let G be a connected algebraic group over \mathbb{R} , and let $\sigma_0 : g \mapsto \bar{g}$ denote complex conjugation on $G(\mathbb{C})$ with respect to G . A *Cartan involution* of G is an involution θ of G (as an algebraic group over \mathbb{R}) such that the group

$$G^{(\theta)}(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = \theta(\bar{g})\}$$

is compact. Then $G^{(\theta)}$ is a compact real form of $G_{\mathbb{C}}$, and θ acts on $G(\mathbb{C})$ as $\sigma_0 \sigma = \sigma \sigma_0$, where σ is complex conjugation on $G(\mathbb{C})$ with respect to $G^{(\theta)}$.

A connected algebraic group G over \mathbb{R} has a Cartan involution if and only if it has a compact real form, which is the case if and only if G is reductive. Any two Cartan involutions of G are conjugate by an element of $G(\mathbb{R})$.

Let C be an element of $G(\mathbb{R})$ whose square is central, so $\mathrm{ad}(C) \stackrel{\text{def}}{=} (g \mapsto CgC^{-1})$ is an involution. A *C-polarization* on a real representation V of G is a G -invariant bilinear form $\varphi : V \times V \rightarrow \mathbb{R}$ such that the form $\varphi_C : (x, y) \mapsto \varphi(x, Cy)$ is symmetric and positive definite.

THEOREM 2.1. *If $\mathrm{ad}(C)$ is a Cartan involution of G , then every finite dimensional real representation of G carries a C-polarization; conversely, if one faithful finite dimensional real representation of G carries a C-polarization, then $\mathrm{ad}(C)$ is a Cartan involution.*

PROOF. An \mathbb{R} -bilinear form φ on a real vector space V defines a sesquilinear form $\varphi' : (u, v) \mapsto \varphi_{\mathbb{C}}(u, \bar{v})$ on $V(\mathbb{C})$, and φ' is hermitian (and positive definite) if and only if φ is symmetric (and positive definite).

Let $G \rightarrow \mathrm{GL}_V$ be a representation of G . If $\mathrm{ad}(C)$ is a Cartan involution of G , then $G^{(\mathrm{ad} C)}(\mathbb{R})$ is compact, and so there exists a $G^{(\mathrm{ad} C)}$ -invariant positive definite symmetric bilinear form φ on V . Then φ_C is $G(\mathbb{C})$ -invariant, and so

$$\varphi'(gu, (\sigma g)v) = \varphi'(u, v), \quad \text{for all } g \in G(\mathbb{C}), u, v \in V_{\mathbb{C}},$$

where σ is the complex conjugation on $G_{\mathbb{C}}$ with respect to $G^{(\text{ad}C)}$. Now $\sigma g = \text{ad}(C)(\bar{g}) = \text{ad}(C^{-1})(\bar{g})$, and so, on replacing v with $C^{-1}v$ in the equality, we find that

$$\varphi'(gu, (C^{-1}\bar{g}C)C^{-1}v) = \varphi'(u, C^{-1}v), \quad \text{for all } g \in G(\mathbb{C}), u, v \in V_{\mathbb{C}}.$$

In particular, $\varphi(gu, C^{-1}gv) = \varphi(u, C^{-1}v)$ when $g \in G(\mathbb{R})$ and $u, v \in V$. Therefore, $\varphi_{C^{-1}}$ is G -invariant. As $(\varphi_{C^{-1}})_C = \varphi$, we see that φ is a C -polarization.

For the converse, one shows that, if φ is a C -polarization on a faithful representation, then φ_C is invariant under $G^{(\text{ad}C)}(\mathbb{R})$, which is therefore compact. ■

2.2. Let G be an algebraic group over \mathbb{Q} , and let C be an element of $G(\mathbb{R})$ whose square is central. A C -polarization on a \mathbb{Q} -representation V of G is a G -invariant bilinear form $\varphi : V \times V \rightarrow \mathbb{Q}$ such that $\varphi_{\mathbb{R}}$ is a C -polarization on $V_{\mathbb{R}}$. In order to show that a \mathbb{Q} -representation V of G is polarizable, it suffices to check that $V_{\mathbb{R}}$ is polarizable. We prove this when C^2 acts as $+1$ or -1 on V , which are the only cases we shall need. Let $P(\mathbb{Q})$ (resp. $P(\mathbb{R})$) denote the space of G -invariant bilinear forms on V (resp. on $V_{\mathbb{R}}$) that are symmetric when C^2 acts as $+1$ or skew-symmetric when it acts as -1 . Then $P(\mathbb{R}) = P(\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R})$. The C -polarizations of $V_{\mathbb{R}}$ form an open subset of $P(\mathbb{R})$, whose intersection with $P(\mathbb{Q})$ consists of the C -polarizations of V .

NOTES. Theorem 2.1 is [Deligne 1972](#), 2.8. The exposition follows [Milne 2005](#), 1.20.

3 Mumford–Tate groups

Let (V, h) be a rational Hodge structure. Following [Deligne 1972](#), 7.1, we define the *Mumford–Tate group* of (V, h) to be the smallest algebraic subgroup G of GL_V such that $G_{\mathbb{R}} \supset h(\mathbb{S})$. We usually regard the Mumford–Tate group as a pair (G, h) . Note that G is connected, because otherwise we could replace it with its neutral component.

The rational Hodge structures form a tannakian category over \mathbb{Q} . Let (V, h) be a rational Hodge structure, and let $\langle V, h \rangle^{\otimes}$ be the tannakian subcategory generated by (V, h) . The Mumford–Tate group of (V, h) is the algebraic group attached to $\langle V, h \rangle^{\otimes}$ and the forgetful fibre functor.

The *special Mumford–Tate group* of (V, h) is defined to be the smallest algebraic subgroup G^1 of GL_V such that $G^1_{\mathbb{R}} \supset h(\mathbb{S}^1)$. It is a subgroup of the Mumford–Tate group G , and $G = G^1 \cdot w_h(\mathbb{G}_m)$.

Let G be a connected algebraic group over \mathbb{Q} and h a homomorphism $\mathbb{S} \rightarrow G_{\mathbb{R}}$. Consider the following conditions² on h :

MT1: the map $w_h : \mathbb{G}_{m\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is defined over \mathbb{Q} and $w_h(\mathbb{G}_m) \subset Z(G)$;

MT2: $\text{ad}(h(i))$ is a Cartan involution of $(G/w_h(\mathbb{G}_m))_{\mathbb{R}}$.

Note that (MT1) implies that $G/w_h(\mathbb{G}_m)$ is an algebraic group over \mathbb{Q} and that (MT2) implies that G is reductive.

THEOREM 3.1. *A pair (G, h) as above is the Mumford–Tate group of a polarizable rational Hodge structure if and only if it satisfies (MT1,2) and h generates G .*

²These are the conditions SV4 and SV2* of [Milne 2005](#), which, for a reductive group, coincide with the conditions (2.1.1.4) and (2.1.1.5) of [Deligne 1979](#).

This combines the next two propositions.

PROPOSITION 3.2. *A pair (G, h) as above is the Mumford–Tate group of a rational Hodge structure if and only if h satisfies (MT1) and h generates G .*

PROOF. If (G, h) is the Mumford–Tate group of a Hodge structure (V, h) , then certainly h generates G . The weight homomorphism w_h is defined over \mathbb{Q} because (V, h) is a rational Hodge structure. Let $Z(w_h)$ denote the centralizer of w_h in G . For any $a \in \mathbb{R}^\times$, $w_h(a) : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ is a morphism of real Hodge structures, and so it commutes with the action of $h(\mathbb{S})$. Hence $h(\mathbb{S}) \subset Z(w_h)_{\mathbb{R}}$. As h generates G , this implies that $Z(w_h) = G$.

Conversely, suppose that (G, h) satisfies the conditions. For any faithful representation $\rho : G \rightarrow \mathrm{GL}_V$ of G , the pair $(V, h \circ \rho)$ is a rational Hodge structure, and (G, h) is its Mumford–Tate group. ■

PROPOSITION 3.3. *Let (G, h) be the Mumford–Tate group of a rational Hodge structure (V, h) . Then (V, h) is polarizable if and only if h satisfies (MT2).*

PROOF. Let $C = h(i)$. For notational convenience, assume that (V, h) has a single weight m . Let G^1 be the special Mumford–Tate group of (V, h) . Then $C \in G^1(\mathbb{R})$, and a pairing $\psi : V \times V \rightarrow \mathbb{Q}(-m)$ is a polarization of the Hodge structure (V, h) if and only if $(2\pi i)^m \psi$ is a C -polarization of V for G^1 in the sense of §2. It follows from (2.1) and (2.2) that a polarization ψ for (V, h) exists if and only if $\mathrm{ad}(C)$ is a Cartan involution of $G^1_{\mathbb{R}}$. Now $G^1 \subset G$ and the quotient map $G^1 \rightarrow G/w_h(\mathbb{G}_m)$ is an isogeny, and so $\mathrm{ad}(C)$ is a Cartan involution of G^1 if and only if it is a Cartan involution of $G/w_h(\mathbb{G}_m)$. ■

COROLLARY 3.4. *The Mumford–Tate group of a polarizable rational Hodge structure is reductive.*

PROOF. Immediate consequence of Proposition 3.3. ■

There is a canonical homomorphism $h_{\mathrm{Hg}} : \mathbb{S} \rightarrow (G_{\mathrm{Hg}})_{\mathbb{R}}$ corresponding to the functor $-\otimes \mathbb{R}$ from polarizable rational Hodge structures to polarizable real Hodge structures.

COROLLARY 3.5. *For any reductive group G over \mathbb{Q} and homomorphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying (MT2,4), there is a unique homomorphism $\rho : G_{\mathrm{Hg}} \rightarrow G$ such that $\rho_{\mathbb{R}} \circ h_{\mathrm{Hg}} = h$.*

PROOF. Immediate consequence of Theorem 3.1. ■

REMARK 3.6. Let (V, h) be a rational Hodge structure, and let $\mu = \mu_h$. Then $h(z) = \mu_h(z) \cdot \bar{\mu}_h(z)$ and so μ_h determines h . A cocharacter μ of $G_{\mathbb{C}}$ is of the form μ_h if and only if μ commutes with $\bar{\mu}$. The Mumford–Tate group of (V, h) is the smallest algebraic subgroup G of GL_V such that $G_{\mathbb{C}} \supset \mu_h(\mathbb{G}_m)$. As $h(i) = \mu(-1) \cdot w_h(i)$, we see that (0.1) is simply a restatement of Theorem 3.1.

From now on, we say “ (G, h) or (G, μ) is a Mumford–Tate group” to mean that the pair is the Mumford–Tate group of a *polarizable* rational Hodge structure. We say that G is a Mumford–Tate group if there exists an h such that (G, h) is a Mumford–Tate group.

NOTES. Theorem 3.1 is Proposition 1.6 of Milne 1994. The exposition follows Milne 2013, §6.

4 Tori as Mumford–Tate groups

A number field E is a *CM field* if it is a totally imaginary quadratic extension of a totally real field. Let \mathbb{Q}^{cm} be the union of the CM-subfields of \mathbb{Q}^{al} . Then \mathbb{Q}^{cm} is the largest Galois extension of \mathbb{Q} in \mathbb{Q}^{al} such that complex conjugation is in the centre of $\text{Gal}(\mathbb{Q}^{\text{cm}}/\mathbb{Q})$.

LEMMA 4.1. *Let T be a torus over \mathbb{Q} and μ a cocharacter of T over \mathbb{Q}^{al} . The following conditions on μ are equivalent:*

- (a) *the weight of μ is defined over \mathbb{Q} and μ is defined over a CM field;*
- (b) *for all $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$,*

$$(\sigma - 1)(\iota + 1)\mu = 0 = (\iota + 1)(\sigma - 1)\mu.$$

PROOF. The first equality in (b) says that the weight of μ is defined over \mathbb{Q} , and then the second says that $\sigma\mu = \iota\sigma\mu$ for all $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$, i.e., that μ is defined over \mathbb{Q}^{cm} . ■

The equivalent conditions of the lemma are called the *Serre condition*. When T is split by a CM field, the Serre condition simply says that the weight of μ is defined over \mathbb{Q} .

A rational Hodge structure (V, h) is said to be of *CM-type* if it is polarizable and its Mumford–Tate group is commutative (hence a torus). When (V, h) is simple, this means that $\text{End}(V, h)$ is either a CM-field or \mathbb{Q} .

We have the following criterion.

PROPOSITION 4.2. *Let T be a torus over \mathbb{Q} and μ a cocharacter of T over \mathbb{Q}^{al} . Then (T, μ) is a Mumford–Tate group if and only if*

- (a) *the weight of μ is defined over \mathbb{Q} ,*
- (b) *T is split by a CM field,*
- (c) *μ generates T .*

PROOF. When μ generates T , the condition (a)+(b) is equivalent to (a) of Lemma 4.1; on the other hand, the condition (mt1)+(mt2) is equivalent to (b) of Lemma 4.1. Thus, the proposition is a restatement of Theorem 3.1 for the case of tori. ■

Let E be a CM subfield of \mathbb{Q}^{al} . Then $(\mathbb{G}_m)_{E/\mathbb{Q}}$ is a torus with character group $\mathbb{Z}^{\text{Hom}(E, \mathbb{Q}^{\text{al}})}$, and we define S^E to be the quotient of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ such that

$$X^*(S^E) = \{\lambda \in \mathbb{Z}^{\text{Hom}(E, \mathbb{Q}^{\text{al}})} \mid \lambda(\sigma) + \lambda(\iota\sigma) = \text{constant}, \sigma \in \text{Hom}(E, \mathbb{Q}^{\text{al}})\}.$$

Define μ^E to be the cocharacter of S^E such that

$$\langle \lambda, \mu^E \rangle = \lambda(\sigma_0), \quad \text{all } \lambda \in X^*(S^E),$$

where σ_0 is the given embedding of E into \mathbb{Q}^{al} . If $E \subset E' \subset \mathbb{Q}^{\text{al}}$, then the norm map defines a homomorphism $S^{E'} \rightarrow S^E$ carrying $\mu^{E'}$ to μ^E . We set

$$(S, \mu_{\text{can}}) = \varprojlim (S^E, \mu^E).$$

The pair (S, μ_{can}) is called the *Serre group*. Note that $X^*(S)$ is the set of all locally constant functions $\lambda : \text{Gal}(\mathbb{Q}^{\text{cm}}/\mathbb{Q}) \rightarrow \mathbb{Z}$ such that

$$\lambda(\sigma) + \lambda(\iota\sigma) = -m$$

for some integer m (called the *weight* of λ).

THEOREM 4.3. (a) *The cocharacter μ^E of S^E satisfies the Serre condition.*

(b) *Let T be a torus over \mathbb{Q} and μ a cocharacter satisfying the Serre condition. Then there is a unique homomorphism $\rho_\mu : S \rightarrow T$ such that $(\rho_\mu)_\mathbb{Q} \circ \mu_{\text{can}} = \mu$.*

(c) *We have*

$$(S, \mu_{\text{can}}) = \varprojlim (T, \mu), \quad (1)$$

where the limit runs over the pairs (T, μ) such that μ satisfies the Serre condition and generates T .

PROOF. (a) The weight of μ is defined over \mathbb{Q} and T is split by a CM field.

(b) For $\chi \in X^*(T)$ and $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$, define

$$f_\chi(\sigma) = \langle \sigma^{-1}\chi, \mu \rangle.$$

Then

$$\chi \mapsto f_\chi : X^*(T) \rightarrow X^*(S) \quad (2)$$

is a $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ -equivariant homomorphism, and so corresponds to a homomorphism

$$\rho : S \rightarrow T.$$

For σ_0 the given inclusion of E into \mathbb{Q}^{al} , $f_\chi(\sigma_0) = \langle \mu, \chi \rangle$, i.e., $\langle \mu_{\text{can}}, f_\chi \rangle = \langle \mu, \chi \rangle$, which shows that $(\rho_\mu)_\mathbb{Q} \circ \mu_{\text{can}} = \mu$.

(c) For every (T, μ) , we have defined an injective homomorphism (2), and $X^*(S)$ is union of their images. ■

COROLLARY 4.4. *The polarizable rational Hodge structures of CM-type form a tannakian category, and S is the pro-algebraic group attached to the forgetful fibre functor.*

PROOF. For any rational Hodge structures X and Y , $\text{MT}(X \oplus Y) \subset \text{MT}(X) \times \text{MT}(Y)$, and so $X \oplus Y$ is of CM type if X and Y are. The category of polarizable rational Hodge structures of CM-type is the directed union of the categories $\langle X \rangle^\otimes$ with X of CM-type, and is therefore tannakian. Correspondingly, the pro-algebraic group attached to the forgetful fibre functor is $\varprojlim (\text{MT}(X), \mu_X)$, which, according to 4.2 and (1), is equal to (S, μ_{can}) . ■

The functor sending a rational Hodge structure (V, h) to the real Hodge structure $(V \otimes \mathbb{R}, h)$ defines a homomorphism $h_{\text{can}} : \mathbb{S} \rightarrow S$; its associated cocharacter is μ_{can} .

NOTES. Everything in this section has been known to the experts since the 1960s — see, for example, Serre 1968. For a detailed account, see my notes *Complex Multiplication*.

5 Semisimple groups as Mumford-Tate groups

Let G be an algebraic group over \mathbb{R} and $h : \mathbb{S} \rightarrow G$ a homomorphism of weight 0. Then $\iota\mu_h = -\mu_h$, and so $\mu_h : (\mathbb{G}_m)_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ arises from a homomorphism $u : \mathbb{G}_m \rightarrow G$ over \mathbb{R} . As h has weight 0, it factors through $\mathbb{S}/w(\mathbb{G}_m)$, and u is the composite

$$\mathbb{S}^1 \simeq \mathbb{S}/w(\mathbb{G}_m) \xrightarrow{h} G.$$

In this way we get a one-to-one correspondence between homomorphisms $h : \mathbb{S} \rightarrow G$ of weight 0 and homomorphisms $u : \mathbb{S}^1 \rightarrow G$. If $h \leftrightarrow u$, then

$$\begin{aligned} h(z) &= u(z/\bar{z}), & z \in \mathbb{C}^{\times}, \\ u(z) &= h(\sqrt{z}), & z \in U^1. \end{aligned}$$

Note that $h(i) = u(-1)$, so (MT2) becomes the condition that $\text{ad } u(-1)$ is a Cartan involution.

LEMMA 5.1. *Let G be a simple algebraic group over \mathbb{R} (so, in particular, adjoint). If G is an inner form of its compact form, then it is geometrically simple.*

PROOF. If $G_{\mathbb{C}}$ is not simple, say, $G_{\mathbb{C}} = G_1 \times G_2$, then $G = (G_1)_{\mathbb{C}/\mathbb{R}}$ and any inner form of G is also the restriction of scalars of a complex group, but such a group cannot be compact (look at a subtorus). ■

PROPOSITION 5.2. *Let G be a simple algebraic group over \mathbb{R} . Then G admits a homomorphism $u : \mathbb{S}^1 \rightarrow G$ such that $\text{ad } u(-1)$ is a Cartan involution if and only if G contains a compact maximal torus (in which case, G is geometrically simple).*

PROOF. \Rightarrow : Any maximal torus containing $u(\mathbb{S}^1)$ is compact.

\Leftarrow : Let T be a compact maximal torus of $G_{\mathbb{R}}$. Choose a maximal compact subgroup of $G_{\mathbb{R}}$ containing T , and let θ be the corresponding Cartan involution. A root of (G, T) is compact or noncompact according as θ acts as -1 or $+1$ on it. A homomorphism u such that $\langle \alpha, u \rangle$ is even or odd according as α is compact or noncompact has the property that $\text{ad}(u(-1))$ is a Cartan involution of $G_{\mathbb{R}}$. ■

REMARK 5.3. It is possible to read off from the classification of geometrically simple algebraic groups over \mathbb{R} , a list of the groups satisfying the equivalent conditions of Proposition 5.2.

THEOREM 5.4. *An adjoint algebraic group G over \mathbb{Q} is a Mumford–Tate group if and only if $G_{\mathbb{R}}$ contains a compact maximal torus.*

PROOF. As G is adjoint, to give a homomorphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying (MT1,2) is the same as giving a homomorphism $u : \mathbb{S}^1 \rightarrow G_{\mathbb{R}}$ such that $\text{ad } u(-1)$ is a Cartan involution. If $G_{\mathbb{R}}$ contains a maximal torus, then the proof of 5.2 shows how to construct such a u . A general u will generate G , and so G is a Mumford–Tate group by Theorem 3.1. Conversely, if G is a Mumford–Tate group, then Proposition 5.2 shows that $G_{\mathbb{R}}$ contains a compact maximal torus. ■

REMARK 5.5. Let G be an adjoint algebraic group over \mathbb{Q} . Then G has a canonical decomposition

$$G = (G_1)_{F_1/\mathbb{Q}} \times \cdots \times (G_n)_{F_n/\mathbb{Q}},$$

where each F_i is a subfield of \mathbb{Q}^{al} and G_i is geometrically simple (Milne 2017, 24.4). If the simple factors of $(G_i)_{F_i/\mathbb{Q}}$ over \mathbb{R} are geometrically simple, then F_i is totally real. Thus, G is a Mumford-Tate group if and only if its simple factors (over \mathbb{Q}) are the groups of the form $(H)_{F/\mathbb{Q}}$, where F is a totally real number field and H is a geometrically simple group over F such that, for every $\rho : F \rightarrow \mathbb{R}$, ρH is on the list hinted at in 5.3.

NOTES. In their 2012 monograph, Green, Griffiths, and Kerr (IV.A.3) claim to show that an adjoint group over \mathbb{Q} is a Mumford-Tate group if and only if it has an anisotropic maximal torus. Patrikis pointed out that this statement is false and gave the correct statement.

6 The classification

We describe the structure of the pro-algebraic group G_{Hg} attached to the tannakian category of polarizable rational Hodge structures and the forgetful fibre functor.

LEMMA 6.1 (MUMFORD 1969, p. 348). *Let G be a connected algebraic group over \mathbb{Q} and T a maximal torus in $G_{\mathbb{R}}$. Then there exists a maximal torus T_0 in G and an $a \in G(\mathbb{R})$ such that $T_{0\mathbb{R}} = aT a^{-1}$.*

PROOF. According to the real approximation theorem (Milne 2017, 25.70), $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. If $a \in T(\mathbb{R})$ is a regular element, then T is the centralizer of a , and a has an open neighbourhood U in $G(\mathbb{R})$ such that the centralizer of every $a' \in U$ is a conjugate of T . If $a' \in U \cap G(\mathbb{Q})$, then the centralizer of a' is a conjugate of T defined over \mathbb{Q} , as required. ■

PROPOSITION 6.2 (MUMFORD 1969, p. 348). *Let G be an adjoint group over \mathbb{Q} and $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ a homomorphism satisfying (MT1,2). There exists a $g \in G(\mathbb{R})$ and a torus $T_0 \subset G$ such that $\text{ad}(g) \circ h$ factors through $T_{0\mathbb{R}}$.*

PROOF. Let K be the centralizer of h in $G_{\mathbb{R}}$ (so K is an algebraic subgroup of $G_{\mathbb{R}}$). Let T be a maximal torus of K . As $h(\mathbb{S})$ is contained in the centre of K , $h(\mathbb{S}) \subset T$. If T' is a torus in $G_{\mathbb{R}}$ containing T , then T' centralizes h and so $T' \subset K$; therefore T is maximal in $G_{\mathbb{R}}$. According to the lemma, there exists a maximal torus T_0 of G such that $T_{0\mathbb{R}} = gTg^{-1}$ for some $g \in G(\mathbb{R})$. Now $\text{ad}(g) \circ h$ factors through $T_{0\mathbb{R}}$. ■

PROPOSITION 6.3. *Let G be a reductive group over a field k of characteristic zero, and let L be a finite Galois extension of k splitting G . Let $G' \rightarrow G^{\text{der}}$ be a finite covering of the derived group of G . Then there exists a central extension*

$$1 \rightarrow N \rightarrow G_1 \rightarrow G \rightarrow 1$$

such that G_1 is reductive, N is a product of copies of $(\mathbb{G}_m)_{L/k}$, and

$$(G_1^{\text{der}} \rightarrow G^{\text{der}}) = (G' \rightarrow G^{\text{der}}).$$

PROOF. See Milne and Shih 1982, 3.1. ■

THEOREM 6.4 (MILNE 1994, 1.28). *Let H be a semisimple algebraic group over \mathbb{Q} and $\bar{h} : \mathbb{S}/\mathbb{G}_m \rightarrow H_{\mathbb{R}}^{\text{ad}}$ a homomorphism such that $\text{ad}(\bar{h}(i))$ is a Cartan involution. Then there exists a reductive group G with $G^{\text{der}} = H$ and a homomorphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ lifting \bar{h} and satisfying (MT1,2).*

PROOF. Assume first that \bar{h} is “special”, i.e., that it factors through $T_{\mathbb{R}}$ for some maximal torus T in H^{ad} . The hypothesis on h implies that $T_{\mathbb{R}}$ is anisotropic, and so T splits over a CM-field L , which we may choose to be Galois over \mathbb{Q} . According to Proposition 6.3, there exists a central extension defined over \mathbb{Q}

$$1 \longrightarrow N \longrightarrow G \longrightarrow H^{\text{ad}} \longrightarrow 1$$

such that $G^{\text{der}} = H$ and N is a product of copies of $(\mathbb{G}_m)_{L/\mathbb{Q}}$. There is a maximal torus $T' \subset G$ mapping onto T (Milne 2017, 17.20). Since T' is its own centralizer, it contains N , which is therefore the kernel of $T' \rightarrow T$. Hence $X_*(T') \rightarrow X_*(T)$ is surjective, and we can choose $\mu \in X_*(T')$ mapping to $\mu_{\bar{h}} \in X_*(T)$. The weight $w(h) \stackrel{\text{def}}{=} -\mu - i\mu$ of μ lies in $X_*(N)$. Because $X_*(N)$ is an induced Galois module, its cohomology groups are zero; in particular, the zeroth Tate group

$$H_{\text{Tate}}^0(\text{Gal}(\mathbb{C}/\mathbb{R}), X_*(N)) \stackrel{\text{def}}{=} \frac{X_*(N)^{\text{Gal}(\mathbb{C}/\mathbb{R})}}{(\iota + 1)X_*(N)} = 0.$$

Clearly $\iota w = w$, and so there exists a $\mu_0 \in X_*(N)$ such that $(\iota + 1)\mu_0 = w$. When we replace μ with $\mu + \mu_0$, then we find that the weight becomes 0; in particular, it is defined over \mathbb{Q} . Choose h so that $h(z) = \mu(z) \cdot \overline{\mu(z)}$.

For a general \bar{h} , there will exist a $\bar{g} \in H^{\text{ad}}(\mathbb{R})$ such that $\text{ad} \bar{g} \circ \bar{h}$ is special (6.2). Construct G and h as in the last paragraph corresponding to $\text{ad} \bar{g} \circ \bar{h}$. Because $H^1(\mathbb{R}, N) = H^1(L \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{G}_m) = 0$, the element \bar{g} will lift to an element $g \in G(\mathbb{R})$, and we take the pair $(G, \text{ad}(g^{-1}) \circ h)$.

For the pair (G, h) we have constructed, the centre of G is split by a CM-field, h satisfies (MT1), and $\text{ad}(h(i))$ is a Cartan involution on G^{ad} . Let T be the subtorus of G/G^{der} generated by h . Then $T_{\mathbb{R}}$ is anisotropic, and when we replace G with the inverse image of T , we obtain a pair (G, h) satisfying (MT1,2). ■

THEOREM 6.5. (a) *The quotient of G_{Hg} by its derived group is the Serre group.*

(b) *A semisimple algebraic group G over \mathbb{Q} is a quotient of $G_{\text{Hg}}^{\text{der}}$ if and only if G^{ad} is a Mumford–Tate group.*

(c) *The adjoint group of G_{Hg} is a product of groups of the form $(G)_{F/\mathbb{Q}}$ with F a totally real number field and G an algebraic group over F such that, for all embeddings ρ of F into \mathbb{R} , ρG is a simple algebraic group over \mathbb{R} with a compact maximal torus.*

PROOF. (a) A polarizable rational Hodge structure is of CM-type if and only if $G_{\text{Hg}}^{\text{der}}$ acts trivially on it. Now 4.4 implies that the inclusion of the category of CM Hodge structures into the full category of polarizable rational Hodge structures induces an isomorphism $G/G^{\text{der}} \rightarrow S$.

(b) Let G be a quotient of $G_{\text{Hg}}^{\text{der}}$. The image of $Z(G_{\text{Hg}}^{\text{der}})$ in G is of multiplicative type, and therefore is central (Milne 2017, 12.38). Thus G^{ad} is a quotient of $(G_{\text{Hg}}^{\text{der}})^{\text{ad}} = G_{\text{Hg}}^{\text{ad}}$, and so is a Mumford–Tate group. Conversely, suppose that G^{ad} is a Mumford–Tate group,

and let $\bar{h} : \mathbb{S} \rightarrow G_{\mathbb{R}}^{\text{ad}}$ be a homomorphism satisfying (MT1,2). According to Proposition 6.4, there exists a reductive group G' with $G'^{\text{der}} = G$ and a homomorphism $h' : \mathbb{S} \rightarrow G'_{\mathbb{R}}$ lifting \bar{h} and satisfying (MT1,2). Let $\rho : G_{\text{Hg}} \rightarrow G'$ be the homomorphism given by Corollary 3.5. Then ρ maps $(G_{\text{Hg}})^{\text{der}}$ onto $(G')^{\text{der}} = G$.

(c) Apply 5.4 and 5.5. ■

COROLLARY 6.6. *The pro-algebraic group G^{der} is simply connected.*

PROOF. Immediate consequence of (b) of the theorem. ■

REMARK 6.7. The group G_{Hg} is not a product of $G_{\text{Hg}}^{\text{der}}$ and S because this would imply that, for a semisimple algebraic group H over \mathbb{Q} , every homomorphism $h : \mathbb{S} \rightarrow H_{\mathbb{R}}^{\text{ad}}$ satisfying (MT1,2) lifts to a homomorphism $\mathbb{S} \rightarrow H_{\mathbb{R}}$ satisfying the same conditions, but this is not true.

REMARK 6.8. No description, even conjectural, is known for the essential image of the functor from the category of motives over \mathbb{C} to the category of polarizable rational Hodge structures. However, when one replaces the category of all motives with the subcategory generated by the motives of abelian varieties such a description is known (Theorem 1.27 of Milne 1994). Note that these Hodge structures need be neither of CM-type nor of weight 1. The proof of Theorem 6.5 is a (simpler) variant of the proof of that theorem.

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