Polarizations and Grothendieck's Standard Conjectures

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Abstract. We prove that Grothendieck's Hodge standard conjecture holds for abelian varieties in arbitrary characteristic if the Hodge conjecture holds for complex abelian varieties of CM-type. For abelian varieties with no exotic algebraic classes, we prove the Hodge standard conjecture unconditionally.

Contents

Introduction. In examining Weil's proofs (Weil 1948) of the Riemann hypothesis for curves and abelian varieties over finite fields, Grothendieck was led to state two "standard" conjectures (Grothendieck 1969), which imply the Riemann hypothesis for all smooth projective varieties over a finite field, essentially by Weil's original argument. Despite Deligne's proof of the Riemann hypothesis, the standard conjectures retain their interest for the theory of motives.

The first, the *Lefschetz standard conjecture* (Grothendieck 1969, §3), states that, for a smooth projective variety V over an algebraically closed field, the operators Λ rendering commutative the diagrams $(0 \leq r \leq 2n = 2\dim V)$

$$\begin{array}{ccc} H^r(V) & \xrightarrow[\approx]{L^{n-r}} & H^{2n-r}(V) \\ \downarrow \Lambda & & \downarrow L \\ H^{r-2}(V) & \xrightarrow[\approx]{L^{n-r+2}} & H^{2n-r+2}(V) \end{array}$$

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are algebraic. Here H is a Weil cohomology theory and L is cup product with the class of a smooth hyperplane section (L^{n-r} is assumed to be an isomorphism for $n \geq r$, and $L^{n-r} = (L^{r-n})^{-1}$ for $n < r$). This conjecture is known for curves (trivial), abelian varieties (Lieberman 1968, Kleiman 1968), surfaces and Weil cohomologies for which $\dim H^1(V) = 2\dim \mathrm{Pic}^0(V)$ (Grothendieck), generalized flag manifolds (trivial), complete intersections (trivial), and products of such varieties (see Kleiman 1994, 4.3). For abelian varieties, it is even known that the operator Λ is defined by a Lefschetz class, i.e., a class in the $\mathbb{Q}$-algebra generated by divisor classes (Milne 1999a, 5.9).

The second, the *Hodge standard conjecture* (Grothendieck 1969, §4), states that, for $r \leq n/2$, the bilinear form

$$(x, y) \mapsto (-1)^r \langle L^{n-2r} x \cdot y\rangle \colon P^r(V) \times P^r(V) \to \mathbb{Q}$$

is positive-definite. Here $P^r(V)$ is the $\mathbb{Q}$-space of primitive algebraic classes of codimension r modulo homological equivalence. In characteristic zero, $\mathrm{Hdg}(V)$ is a consequence of Hodge theory (Weil 1958). In nonzero characteristic, $\mathrm{Hdg}(V)$ is known for surfaces (Segre 1937; Grothendieck 1958). An important consequence of the Hodge standard conjecture for abelian varieties, namely, the positivity of the Rosati involution was proved in nonzero characteristic by Weil (1948, Théorème 38). Apart from these examples and the general coherence of Grothendieck's vision, there appears to have been little evidence for the conjecture in nonzero characteristic.

In fact, no progress seems to have been made on these conjectures since they were first formulated: the lists of known cases in Kleiman 1968 and in Kleiman 1994 are identical.

In this paper, we prove that the Hodge standard conjecture holds for abelian varieties in arbitrary characteristic if the Hodge conjecture holds for complex abelian varieties of CM-type.

Let $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$ be the category of motives based on abelian varieties over $\mathbb{F}$ using the numerical equivalence classes of algebraic cycles as correspondences. This is a Tannakian category (Jannsen 1992, Deligne 1990), and it is known that the Tate conjecture for abelian varieties over finite fields implies that it has all the major expected properties but one, namely, that the Weil forms coming from algebraic geometry are positive for the canonical polarization on $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$ (see Milne 1994, especially 2.47).

In Milne 1999b it is shown that the Hodge conjecture for complex abelian varieties of CM-type is stronger than (that is, implies) the Tate conjecture for abelian varieties over finite fields. Here, we show that the stronger conjecture also implies the positivity of the Weil forms coming from algebraic geometry (Theorem 3.1). As a consequence, we obtain the Hodge standard conjecture for abelian varieties over finite fields, and a specialization argument then proves it over any field of nonzero characteristic (Theorem 4.6).

Most of the arguments in the paper hold with "algebraic cycle" replaced by "Lefschetz cycle". We prove that the analogue of the Hodge standard conjecture holds (unconditionally) for Lefschetz classes on abelian varieties. In particular, the Hodge standard conjecture is true for abelian varieties without exotic (i.e., non-Lefschetz) algebraic classes (4.11, 4.12).

In preparation for proving these results, we study in §1 the polarizations on a category of Lefschetz motives, and in §2 the polarizations on a quotient Tannakian category.

Notations and Conventions. The symbol k always denotes an algebraically closed field, and all algebraic varieties over k are smooth and projective but not necessarily connected.

The algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted $\mathbb{Q}^{\mathrm{al}}$. We fix a p-adic prime on $\mathbb{Q}^{\mathrm{al}}$ and let $\mathbb{F}$ be the residue field.

By the Hodge conjecture for a variety V over $\mathbb{C}$, we mean the statement that, for all r, the $\mathbb{Q}$-space $H^{2r}(V,\mathbb{Q})\cap H^{r,r}$ is spanned by the classes of algebraic cycles.

For abelian varieties A and B, $\mathrm{Hom}(A,B)_{\mathbb{Q}}=\mathrm{Hom}(A,B)\otimes\mathbb{Q}$. An abelian variety A over $\mathbb{C}$ (or $\mathbb{Q}^{\mathrm{al}}$) is said to be of CM-type if, for each simple isogeny factor B of A, $\mathrm{End}(B)_{\mathbb{Q}}$ is a commutative field of degree $2\dim B$ over $\mathbb{Q}$. A polarization of A is an isogeny $A\to A^{\vee}$ from A to its dual of the form $a\mapsto[D_a-D]$ for some ample divisor D.

By the Tate conjecture for a variety V over a finite field $\mathbb{F}_q$ we mean the statement that, for all r, the order of the pole of the zeta function $Z(V,t)$ at $t=q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r on V (Tate 1994, 2.9). We say that a variety over $\mathbb{F}$ satisfies the Tate conjecture if all of its models over finite fields satisfy the Tate conjecture (equivalently, one model over a "sufficiently large" finite field).

We shall need the following categories of motives:

	Based on	Correspondences
$\mathbf{Mot}(\mathbb{F})$	(smooth projective) varieties over $\mathbb{F}$	algebraic cycles mod numerical equivalence
$\mathbf{Mot}(k;\mathcal{A})$	abelian varieties over k	algebraic cycles mod numerical equivalence
$\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$	abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$	(absolute) Hodge classes
$\mathbf{LM}(k)$	abelian varieties over k	Lefschetz classes
$\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})$	abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$	Lefschetz classes

Each is a Tannakian category with a natural structure of a Tate triple (Jannsen 1992; ibid.; Deligne and Milne 1982, §6; Milne 1999b, §1; ibid., §2).

For a Tate triple $\mathbf{T}=(\mathbf{C},w,\mathbb{T})$, $\mathbf{C}_0$ is the quotient of $\mathbf{C}$ in which $\mathbb{T}$ has been identified with $\mathbb{1}$ (Deligne and Milne 1982, 5.8; also 2.5 below). The category of vector spaces over a field F is denoted $\mathbf{Vec}_F$. The Tate triple defined in ibid. 5.3, is denoted $\mathbf{V}$, and $\mathbf{V}_0$ denotes the corresponding quotient category (ibid., 5.9). Sometimes we use $\mathbf{C}$ to denote a Tate triple $(\mathbf{C},w,\mathbb{T})$.

For a finite separable extension of fields $K\supset F$, $(\mathbb{G}_m)_{K/F}$ is the torus over F obtained from $\mathbb{G}_{m/K}$ by restriction of scalars. The notation $X\approx Y$ means that X and Y are isomorphic, and $X\cong Y$ means that X and Y are canonically isomorphic (or that a particular isomorphism is given).

1. Polarizations on categories of Lefschetz motives

We refer to Deligne and Milne 1982 for the definitions of a Weil form (ibid., p. 165), a polarization on a Tannakian category over $\mathbb{R}$ (ibid., 4.10), and a graded polarization on a Tate triple over $\mathbb{R}$ (ibid., 5.12). We define a polarization on a Tannakian category

$\mathbf{C}$ (or Tate triple) over $\mathbb{Q}$ to be a polarization on $\mathbf{C}_{(\mathbb{R})}$. The canonical polarizations on $\mathbf{V}_0$ and $\mathbf{V}$ are denoted Π^{V_0} and Π^{V} (ibid., p. 185, p. 195).

A *morphism* $F\colon (\mathbf{C}_1, w_1, \mathbb{T}_1) \to (\mathbf{C}_2, w_2, \mathbb{T}_2)$ of Tate triples is an exact tensor functor $F\colon \mathbf{C}_1 \to \mathbf{C}_2$ preserving the gradations together with an isomorphism $F(\mathbb{T}_1) \cong \mathbb{T}_2$. Such a morphism is *compatible* with graded polarizations Π_1 and Π_2 on $\mathbf{T}_1$ and $\mathbf{T}_2$ (denoted $F\colon \Pi_1 \mapsto \Pi_2$) if

$$\psi \in \Pi_1(X) \Rightarrow F\psi \in \Pi_2(FX),$$

in which case, for any X homogeneous of weight n, $\Pi_1(X)$ consists of the sesquilinear forms $\psi\colon X \otimes \bar{X} \to \mathbb{1}(-n)$ such that $F\psi \in \Pi_2(FX)$. In particular, given F and Π_2, there exists at most one graded polarization Π_1 on $\mathbf{T}_1$ such that $F\colon \Pi_1 \mapsto \Pi_2$.

Let $\mathbf{Hod}_{\mathbb{R}}$ be the category of real Hodge structures: it is a Tannakian category over $\mathbb{R}$ with a natural Tate triple structure and a canonical graded polarization Π^{Hod} (ibid., 2.31, 5.2, 5.21). Betti cohomology defines a morphism of Tate triples

$$H\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}})_{(\mathbb{R})} \to \mathbf{Hod}_{\mathbb{R}}$$

and, because Lefschetz classes are Hodge, there is an evident morphism

$$J\colon \mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}).$$

PROPOSITION 1.1. *There are graded polarizations Π^{LCM} and Π^{CM} on $\mathbf{LCM}(\mathbb{Q}^{al})$ and $\mathbf{CM}(\mathbb{Q}^{al})$ (necessarily unique) such that*

$$\Pi^{LCM} \overset{J}{\mapsto} \Pi^{CM} \overset{H}{\mapsto} \Pi^{Hod}.$$

PROOF. We begin by reviewing the classification of the graded polarizations on a neutralized algebraic Tate triple $(\mathbf{T}, \omega)$. Such a pair $(\mathbf{T}, \omega)$ defines a triple (G, w, t) (ibid., 5.5). An element C of $G(\mathbb{R})$ is a Hodge element for $(\mathbf{T}, \omega)$ if $C^2 = w(-1)$, $t(C) = 1$, and the real form of $\mathrm{Ker}(t\colon G \to \mathbb{G}_m)$ defined by C is an anisotropic (= compact) group (ibid., p. 194). A Hodge element defines a graded polarization Π_C on $\mathbf{T}$, and every graded polarization on $\mathbf{T}$ arises from a Hodge element (ibid., 5.18).

Because of the uniqueness, in proving the proposition, we may replace $\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})$ and $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ with their subcategories based on a finite set of abelian varieties, and hence suppose them to be algebraic. We endow each with the Betti fibre functor ω_B, and let T' and S' be the corresponding algebraic groups. The functors J and H define homomorphisms

$$\mathbb{S} \to S'_{/\mathbb{R}} \to T'_{/\mathbb{R}}, \quad \mathbb{S} = (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}.$$

The image of $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ in $S'(\mathbb{R})$ is a Hodge element for the neutralized Tate triple $(\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}), \omega_B)$, and its image in $T'(\mathbb{R})$ is a Hodge element for $(\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}), \omega_B)$ (Milne 1999b, 2.5, 2.6; in both case $\mathrm{Ker}(t)$ is an anisotropic torus). The graded polarizations defined by these Hodge elements are evidently compatible with J and H. □

For an abelian variety A over $\mathbb{F}$, $\langle A\rangle^{\otimes}$ denotes the Tannakian subcategory of $\mathbf{LM}(\mathbb{F})$ generated by h_1A and $\mathbb{T}$.

LEMMA 1.2. *For any abelian variety A over $\mathbb{F}$, the Tate triple $(\langle A\rangle^{\otimes}, w, \mathbb{T})$ is polarizable.*

Proof. A Tate triple $\mathbf{T} = (\mathbf{C}, w, \mathbb{T})$ is polarizable if and only if $\mathbf{C}_0$ has a polarization of parity $\varepsilon =_{\text{df}} w(-1)$ (Deligne and Milne 1982, 5.13). If $A_1, \ldots, A_r$ are the distinct simple isogeny factors of A, then $\langle A\rangle_0^{\otimes}$ is equivalent to $\langle A_1\rangle_0^{\otimes} \otimes \cdots \otimes \langle A_r\rangle_0^{\otimes}$ (apply Milne 1999b, 1.8; Milne 1999a, 4.7). Therefore, it suffices to prove the lemma when A is simple (to see this apply Deligne and Milne 1982, 4.29, Deligne 1990, 5.13, and Proposition 2.1 below).

Recall that a Tannakian category $\mathbf{C}$ over a field F is determined up to a tensor equivalence inducing the identity map on its band B by its class in $H^2(F, B)$ (Saavedra 1972, III 2.3.3.1, III 3.2.6), and that a commutative band can be identified with an affine group scheme.

When A is a supersingular elliptic curve, the band of $\langle A\rangle^{\otimes}$ is $\mathbb{G}_m$, and its cohomology class in $H^2(\mathbb{R}, \mathbb{G}_m) = \mathrm{Br}(\mathbb{R})$ is the class of $\mathrm{End}(A)_{\mathbb{R}}$, which is nonzero. Therefore, $\langle A\rangle^{\otimes}$ is $\mathbb{G}_m$-equivalent to $\mathbf{V}$, which is polarizable.

When A is not a supersingular elliptic curve, the centre of $\mathrm{End}(A)_{\mathbb{Q}}$ is a CM-field E (Tate 1968/69, p. 3), and the band of $\langle A\rangle_0^{\otimes}$ is the subtorus U of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ with $U(\mathbb{Q}) = \{a \in E^{\times} \mid a \cdot \bar{a} = 1\}$ (Milne 1999b, 1.8; Milne 1999a, 4.4). As $U_{/\mathbb{R}}$ is isomorphic to a product of copies of $U^1 =_{\text{df}} \{z \in \mathbb{C} \mid z\bar{z} = 1\}$ and $H^2(\mathbb{R}, U^1) = 0$, $\langle A\rangle_{0(\mathbb{R})}^{\otimes}$ is neutral. Because $U_{/\mathbb{R}}$ is anisotropic, $\langle A\rangle_{0(\mathbb{R})}^{\otimes}$ has a symmetric polarization, and so it has a polarization of parity ε if and only if ε is a square in $U(\mathbb{R})$ (Deligne and Milne 1982, 4.20(e)), but this is obviously true because $\varepsilon = (-1, \ldots, -1)$. □

A divisor D on an abelian variety A over k defines a pairing $\psi_D \colon h_1A \times h_1A \to \mathbb{T}$ which is a Weil form if D is very ample (Weil 1948, Théorème 38). Such a Weil form will be said to be *geometric*.

Remark 1.3. The geometric Weil forms are positive for Π^{LCM} and Π^{CM}.

Lemma 1.4. *If $(\mathbf{LM}(k), w, \mathbb{T})$ is polarizable, then it has a unique graded polarization for which the geometric Weil forms are positive.*

Proof. The uniqueness follows from the fact the $\mathbf{LM}(k)$ is generated by the objects h_1A.

Once a geometric Weil form ψ on h_1A has been fixed, the set of such forms is parametrized by the set

$$\{\alpha \in \mathrm{End}(A)_{\mathbb{Q}} \mid \alpha^{\psi} = \alpha,\ \alpha \text{ is totally positive}\}$$

(Mumford 1970, p. 208), and so the geometric Weil forms lie in a single compatibility class (Deligne and Milne 1982, 4.6). Therefore, if one geometric Weil form on h_1A is positive for a graded polarization Π, then all are.

To prove the existence, it suffices to prove the analogous statement for the quotient category $\mathbf{LM}(\mathbb{F})_0$, namely, that it has a polarization of parity $\varepsilon = w(-1)$ for which the geometric Weil forms are positive. Because of the uniqueness, it suffices to do this for each subcategory $\langle A\rangle_0^{\otimes}$ of $\mathbf{LM}(\mathbb{F})_0$.

Let Y be a simple object of Tannakian category $\mathbf{C}$ over $\mathbb{R}$. If ϕ is a Weil form on Y with parity (some) ε, then the Weil forms on Y with parity ε fall into exactly two compatibility classes, represented by ϕ and $-\phi$ (ibid. 4.8). An isotypic object in $\mathbf{C}$ of type Y can be written $W \otimes Y$ with W a finite-dimensional $\mathbb{R}$-vector space (regarded as an object of $\mathbf{C}^{\pi(\mathbf{C})}$), and the Weil forms on $W \otimes Y$ again fall into exactly two compatibility classes, represented by $\psi\otimes\phi$ and $-\psi\otimes\phi$ where ψ is any positive-definite bilinear form on W.

Let X denote h_1A regarded as an object of $\langle A\rangle^{\otimes}_{0(\mathbb{R})}$, and let $X\approx\bigoplus_i X_i$ be the decomposition of X into its isotypic components. The preceding remarks show that the Weil forms on X with parity $\varepsilon=w(-1)$ are parametrized by the $e\in\operatorname{Aut}(X)$ such that e acts as ±1 on each factor X_i.

Let $Z=\underline{\operatorname{Aut}}(\mathrm{id}_{\langle A\rangle^{\otimes}_{0(\mathbb{R})}})$, and let Π_0 be a polarization on $\langle A\rangle^{\otimes}_{0(\mathbb{R})}$ with parity ε. A $z\in Z(\mathbb{R})$ of order 2 defines a polarization $z\Pi_0$ of parity ε by the rule

$$\phi\in z\Pi_0(Y)\iff\phi_{z_Y}\in\Pi_0(Y),$$

and each polarization of parity ε is of this form for a unique z (ibid., 4.20d).

From the definition of the category of Lefschetz motives, it is clear that any $e\in\operatorname{Aut}(X)$ such that e acts on each X_i as ±1 is an element of order 2 in $Z(\mathbb{R})$, and, in fact, that these elements exhaust $Z(\mathbb{R})_2$ (Milne 1999a, 4.4). Therefore the map $\Pi\mapsto\Pi(X)$ from the set of polarizations on $\langle A\rangle^{\otimes}_{0(\mathbb{R})}$ of parity ε to the set of compatibility classes of Weil forms on X of parity ε is bijective.

Choose Π so that $\Pi(X)$ is the $\mathbb{R}$-span of the geometric Weil forms. Then the geometric Weil forms for each isogeny factor of A will also be positive for Π. □

Let $R^L\colon\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})\to\mathbf{LM}(\mathbb{F})$ be the reduction functor corresponding to the p-adic prime we have fixed on $\mathbb{Q}^{\mathrm{al}}$ (Milne 1999b, §5).

Proposition 1.5. *There exists a graded polarization Π^{LM} on $\mathbf{LM}(\mathbb{F})$ such that*

(a) *all geometric Weil forms are positive for Π^{LM};*
(b) *the reduction functor $R^L\colon\mathbf{LCM}(\mathbb{Q}^{al})\to\mathbf{LM}(\mathbb{F})$ is compatible with Π^{LCM} and Π^{LM}.*

Moreover, Π^{LM} is uniquely determined by each of the conditions (a) and (b).

Proof. Lemmas 1.2 and 1.4 show that there is a graded polarization Π^{LM} on $\mathbf{LM}(\mathbb{F})$ satisfying (a). It satisfies (b) because a polarization λ on A reduces to a polarization $\lambda_{\mathbb{F}}$ on the reduction $A_{\mathbb{F}}$ of A (Faltings and Chai 1990, I 1.10).

The next lemma shows that, if Π^{LM} satisfies (b), then it satisfies (a), which we know determines Π^{LM} uniquely. □

Lemma 1.6. *Let (A,λ) be a polarized abelian variety over $\mathbb{F}$. For some discrete valuation ring R containing the ring of Witt vectors $W(\mathbb{F})$ and finite over $W(\mathbb{F})$, there exists a polarized abelian scheme (B,μ) over R whose generic fibre has complex multiplication and whose special fibre is isogenous to (A,λ).*

Proof. Mumford 1970, Corollary 1, p. 234, allows us to assume that the polarization λ is principal, in which case we can apply Zink 1983, 2.7 (with $L=\mathbb{Q}$). □

Exercise 1.7. Show that the category of Lefschetz motives over an arbitrary k has a unique graded polarization Π for which all geometric Weil forms are positive (see 4.13 below).

2. Polarizations on quotients of Tannakian categories

We refer to Deligne 1989, §§5,6, and Deligne 1990, §8, for the theory of algebraic geometry in a Tannakian category $\mathbf{C}$. In particular, the *fundamental group* $\pi(\mathbf{C})$ of $\mathbf{C}$ is an affine group scheme "in" $\mathbf{C}$, such that, for any fibre functor ω,

$$\underline{\mathrm{Aut}}^{\otimes}(\omega) \cong \omega(\pi(\mathbf{C})).$$

The fundamental group acts on the objects of $\mathbf{C}$, and every fibre functor ω on $\mathbf{C}$ transforms the action of $\pi(\mathbf{C})$ on X into the natural action of $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on $\omega(X)$.

When H is a closed subgroup of $\pi(\mathbf{C})$, we let $\mathbf{C}^H$ denote the full subcategory of $\mathbf{C}$ of objects on which the action of H is trivial. For example, for a Tate triple $(\mathbf{C}, w, \mathbb{T})$, $\mathbf{C}^{w(\mathbb{G}_m)}$ is the full subcategory of objects of weight 0. The functor $\mathrm{Hom}(\mathbb{1}, -)$ is a tensor equivalence $\mathbf{C}^{\pi(\mathbf{C})} \to \mathbf{Vec}_F$, $F = \mathrm{End}(\mathbb{1})$, which we denote γ^C. In particular, γ^C is an F-valued fibre functor on $\mathbf{C}^{\pi(\mathbf{C})}$; any other F-valued fibre functor on $\mathbf{C}^{\pi(\mathbf{C})}$ is isomorphic to γ^C by a unique isomorphism (trivial case of the main theorem of neutral Tannakian categories). When $\pi(\mathbf{C})$ is commutative, it lies in $\mathrm{Ind}(\mathbf{C}^{\pi(\mathbf{C})})$, and hence can be regarded as a group scheme in the usual sense.

Let $\mathbf{T} = (\mathbf{C}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Given a graded polarization Π on $\mathbf{T}$, there exists a morphism of Tate triples $\xi_\Pi \colon \mathbf{T} \to \mathbf{V}$ (well defined up to isomorphism) such that $\xi_\Pi \colon \Pi \mapsto \Pi^V$ (Deligne and Milne 1982, 5.20). Let ω_Π be the composite

$$\mathbf{C}^{w(\mathbb{G}_m)} \xrightarrow{\xi_\Pi} \mathbf{V}^{w(\mathbb{G}_m)} \xrightarrow{\gamma^V} \mathbf{Vec}_{\mathbb{R}};$$

it is a fibre functor on $\mathbf{C}^{w(\mathbb{G}_m)}$.

A criterion for the existence of a polarization.

PROPOSITION 2.1. *Let $\mathbf{T} = (\mathbf{C}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$, and let $\xi \colon \mathbf{T} \to \mathbf{V}$ be a morphism of Tate triples. There exists a graded polarization Π on $\mathbf{T}$ (necessarily unique) such that $\xi \colon \Pi \mapsto \Pi^V$ if and only if the real algebraic group $\underline{\mathrm{Aut}}^{\otimes}(\gamma^V \circ \xi | \mathbf{C}^{w(\mathbb{G}_m)})$ is anisotropic.*

PROOF. Let $G =_{\mathrm{df}} \underline{\mathrm{Aut}}^{\otimes}(\gamma^V \circ \xi | \mathbf{C}^{w(\mathbb{G}_m)})$.

Assume Π exists. The restriction of Π to $\mathbf{C}^{w(\mathbb{G}_m)}$ is a symmetric polarization, which the fibre functor $\gamma^V \circ \xi$ maps to the canonical polarization on $\mathbf{Vec}_{\mathbb{R}}$. This implies that G is anisotropic (Deligne 1972, 2.6).

For the converse, let X be an object of weight n in $\mathbf{C}_{(\mathbb{C})}$. A sesquilinear form $\psi \colon \xi(X) \otimes \overline{\xi(X)} \to \mathbb{1}(-n)$ arises from a sesquilinear form on X if and only if it is fixed by G. Because G is anisotropic, there exists a $\psi \in \Pi^V(\xi(X))$ fixed by G (ibid., 2.6), and we define $\Pi(X)$ to consist of all sesquilinear forms ϕ on X such that $\xi(\phi) \in \Pi^V(\xi(X))$. It is now straightforward to check that $X \mapsto \Pi(X)$ is a polarization on $\mathbf{T}$. □

COROLLARY 2.2. *Let $F \colon (\mathbf{C}_1, w_1, \mathbb{T}_1) \to (\mathbf{C}_2, w_2, \mathbb{T}_2)$ be a morphism of Tate triples, and let Π_2 be a graded polarization on $\mathbf{C}_2$. There exists a graded polarization Π_1 on $\mathbf{C}_1$ such that $F \colon \Pi_1 \mapsto \Pi_2$ if and only if the real algebraic group $\underline{\mathrm{Aut}}^{\otimes}(\gamma^V \circ \xi_{\Pi_2} \circ F | \mathbf{C}_1^{w(\mathbb{G}_m)})$ is anisotropic.*

Quotients of Tannakian categories. An exact tensor functor $q\colon \mathbf{C}\to\mathbf{Q}$ of Tannakian categories over F defines a morphism $\pi(q)\colon \pi(\mathbf{Q})\to q(\pi(\mathbf{C}))$ (Deligne 1990, 8.15.2), and $\pi(q)$ is a closed immersion if and only if every object in $\mathbf{Q}$ is a subquotient of an object in the image of q (this can be proved as Deligne and Milne 1982, 2.21(b), by working with bi-algebras "in" $\mathbf{Q}$).

DEFINITION 2.3. Let $q\colon \mathbf{C}\to\mathbf{Q}$ be an exact tensor functor, and let H be a closed subgroup of $\pi(\mathbf{C})$. We say that $(\mathbf{Q},q)$ is a *quotient of* $\mathbf{C}$ *by* H if $\pi(q)$ is an isomorphism of $\pi(\mathbf{Q})$ onto $q(H)$.

LEMMA 2.4. *Let* $\mathbf{C}$ *be Tannakian category over* F*, and let* $(\mathbf{Q},q)$ *be a quotient of* $\mathbf{C}$ *by a closed subgroup* H *of* $\pi(\mathbf{C})$*.*

(a) *The functor* $\omega_q =_{df} \gamma^Q\circ(q|\mathbf{C}^H)$ *is an* F*-valued fibre functor on* $\mathbf{C}^H$*; in particular,* $\mathbf{C}^H$ *is neutral.*
(b) *For* X,Y *in* $\mathbf{C}$*, there is a canonical functorial isomorphism*

$$\mathrm{Hom}_{\mathbf{Q}}(qX,qY)\cong\omega_q(\underline{\mathrm{Hom}}(X,Y)^H).$$

PROOF. (a) The functor ω_q is the composite of the exact tensor functor $q\colon\mathbf{C}^H\to\mathbf{Q}^{\pi(\mathbf{Q})}$ with the fibre functor γ^Q.

(b) From the various definitions and Deligne and Milne 1982, 1.6.4, 1.9,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Q}}(qX,qY) &\cong \mathrm{Hom}_{\mathbf{Q}}(\mathbb{1},\underline{\mathrm{Hom}}(qX,qY)^{\pi(\mathbf{Q})})\\ &\cong \mathrm{Hom}_{\mathbf{Q}}(\mathbb{1},(q\underline{\mathrm{Hom}}(X,Y))^{q(H)})\\ &\cong \mathrm{Hom}_{\mathbf{Q}}(\mathbb{1},q(\underline{\mathrm{Hom}}(X,Y)^H))\cong\omega_q(\underline{\mathrm{Hom}}(X,Y)^H).\end{aligned}$$

□

EXAMPLE 2.5. Let $(\mathbf{C},w,\mathbb{T})$ be a Tate triple. The functor $q\colon\mathbf{C}\to\mathbf{C}_0$ (Deligne and Milne 1982, 5.8) realizes $\mathbf{C}_0$ as a quotient of $\mathbf{C}$ by $\mathrm{Ker}(t)\subset\pi(\mathbf{C})$. In this case, the fibre functor ω_q on $\mathbf{C}^{\mathrm{Ker}(t)}$ is

$$X\mapsto\varinjlim_n\mathrm{Hom}(\bigoplus_{r=-n}^{r=n}\mathbb{1}(r),X).$$

REMARK 2.6. Let $(\mathbf{Q},Q)$ be a quotient of $\mathbf{C}$ by $H\subset\pi(\mathbf{C})$, and assume that $\mathbf{Q}$ is semisimple. Define $(\mathbf{C}/\omega_Q)'$ to be the category with one object $\bar{X}$ for each object X of $\mathbf{C}$ and with morphisms

$$\mathrm{Hom}_{(\mathbf{C}/\omega_Q)'}(\bar{X},\bar{Y})=\omega_Q(\underline{\mathrm{Hom}}(X,Y)^H).$$

There is a unique structure of an F-linear tensor category on $(\mathbf{C}/\omega_Q)'$ for which $q\colon X\mapsto\bar{X}$ is a tensor functor. With this structure, $(\mathbf{C}/\omega_0)'$ is rigid, and we define $\mathbf{C}/\omega_Q$ to be its pseudo-abelian hull. The functor Q factors through $q\colon\mathbf{C}\to\mathbf{C}/\omega_Q$, say, $Q=R\circ q$ with $R\colon\mathbf{C}/\omega_Q\to\mathbf{Q}$. Because $\mathbf{Q}$ is semisimple, every object in $\mathbf{Q}$ is a direct summand of an object in the image of Q. Therefore, R is essentially surjective, and (2.4(b)) shows that it is also full and faithful; hence it is a tensor equivalence.

REMARK 2.7. Given a closed subgroup H of $\pi(\mathbf{C})$ and an F-valued fibre functor ω on $\mathbf{C}$, there always exists a quotient $(\mathbf{Q},q)$ of $\mathbf{C}$ by H with $\omega_q\approx\omega$; moreover, $(\mathbf{Q},q)$ is unique up to equivalence. The proof is an exercise in gerbology.

Polarizations on quotients. The next proposition gives a criterion for a polarization on a Tate triple to pass to a quotient Tate triple.

PROPOSITION 2.8. *Let $\mathbf{T} = (\mathbf{C}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Let $(\mathbf{Q}, q)$ be a quotient of $\mathbf{C}$ by $H \subset \pi(\mathbf{C})$, and let ω_q be the corresponding fibre functor on $\mathbf{C}^H$. Assume $H \supset w(\mathbb{G}_m)$, so that $\mathbf{Q}$ inherits a Tate triple structure from that on $\mathbf{C}$, and that $\mathbf{Q}$ is semisimple. Given a graded polarization Π on $\mathbf{T}$, there exists a graded polarization Π' on $\mathbf{Q}$ such that $q\colon \Pi \mapsto \Pi'$ if and only if $\omega_q \approx \omega_\Pi | \mathbf{C}^H$.*

PROOF. $\Rightarrow$: Let Π' be such a polarization on $\mathbf{Q}$, and consider the exact tensor functors

$$\mathbf{C} \xrightarrow{q} \mathbf{Q} \xrightarrow{\xi_{\Pi'}} \mathbf{V}, \quad \xi_{\Pi'}\colon \Pi' \mapsto \Pi^V.$$

Both $\xi_{\Pi'} \circ q$ and ξ_Π are compatible with Π and Π^V and with the Tate triple structures on $\mathbf{C}$ and $\mathbf{V}$, and so $\xi_{\Pi'} \circ q \approx \xi_\Pi$ (Deligne and Milne 1982, 5.20). On restricting everything to $\mathbf{C}^{w(\mathbb{G}_m)}$ and composing with γ^V, we get an isomorphism $\omega_{\Pi'} \circ (q|\mathbf{C}^{w(\mathbb{G}_m)}) \approx \omega_\Pi$. Now restrict this to $\mathbf{C}^H$, and note that

$$\left(\omega_{\Pi'} \circ (q|\mathbf{C}^{w(\mathbb{G}_m)})\right)|\mathbf{C}^H = (\omega_{\Pi'}|\mathbf{Q}^{\pi(\mathbf{Q})}) \circ (q|\mathbf{C}^H) \cong \omega_q$$

because $\omega_{\Pi'}|\mathbf{Q}^{\pi(\mathbf{Q})} \cong \gamma^Q$.

$\Leftarrow$: The choice of an isomorphism $\omega_q \to \omega_\Pi|\mathbf{C}^H$ determines an exact tensor functor

$$\mathbf{C}/\omega_q \to \mathbf{C}/\omega_\Pi$$

(notations as in 2.6). As the quotients $\mathbf{C}/\omega_q$ and $\mathbf{C}/\omega_\Pi$ are tensor equivalent respectively to $\mathbf{Q}$ and $\mathbf{V}$, this shows that there is an exact tensor functor $\xi\colon \mathbf{Q} \to \mathbf{V}$ such that $\xi \circ q \approx \xi_\Pi$. Evidently $\underline{\mathrm{Aut}}^{\otimes}(\gamma^V \circ \xi|\mathbf{Q}^{w(\mathbb{G}_m)})$ is isomorphic to a subgroup of $\underline{\mathrm{Aut}}^{\otimes}(\gamma^V \circ \xi_\Pi|\mathbf{C}^{w(\mathbb{G}_m)})$. Since the latter is anisotropic, so also is the former (Deligne 1972, 2.5). Hence ξ defines a graded polarization Π' on $\mathbf{Q}$ (Proposition 2.1), and clearly $q\colon \Pi \mapsto \Pi'$. □

3. Polarizations on categories of motives over finite fields

If the Tate conjecture holds for all abelian varieties over $\mathbb{F}$, then the Tannakian category $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$ has as fundamental group the Weil number torus P (see, for example, Milne 1994, 2.26); moreover, there exist exactly two graded polarizations on $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$, and for exactly one of these (denoted Π^{Mot}) the geometric Weil forms on any supersingular elliptic curve are positive (ibid., 2.44).

If the Hodge conjecture holds for complex abelian varieties of CM-type, then the Tate conjecture holds for abelian varieties over $\mathbb{F}$ (Milne 1999b, 7.1), and, corresponding to the p-adic prime we have fixed on $\mathbb{Q}^{\mathrm{al}}$, there is a reduction functor $R\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}(\mathbb{F}; \mathcal{A})$, which realizes $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$ as the quotient of $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ by the closed subgroup P of the Serre group S. (A description of the inclusion $P \hookrightarrow S$ can be found, for example, in Milne 1994, 4.12.)

THEOREM 3.1. *If the Hodge conjecture holds for complex abelian varieties of CM-type, then $R\colon \Pi^{CM} \mapsto \Pi^{Mot}$ and all geometric Weil forms on all abelian varieties are positive for Π^{Mot}.*

PROOF. I claim that to prove the theorem it suffices to show:

(*) there exists a polarization Π on $\mathbf{Mot}(\mathbb{F};\mathcal{A})$ such that $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi$.

Indeed, if $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi$, then every geometric Weil form is positive for Π (1.3, 1.6). In particular, the geometric Weil forms on a supersingular elliptic curve are positive, and so $\Pi = \Pi^{\mathrm{Mot}}$. This proves the claim.

We now prove (*). Let ω_R be the fibre functor on $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})^P$ defined by R (see 2.4(a)). According to (2.8), there exists a Π such that $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi$ if and only if $(\omega_R)_{(\mathbb{R})} \approx \omega_{\Pi^{\mathrm{CM}}}|\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})^P_{(\mathbb{R})}$, i.e., if and only if the $(S/P)_{\mathbb{R}}$-torsor

$$\wp = \underline{\mathrm{Hom}}^{\otimes}((\omega_R)_{(\mathbb{R})}, \omega_{\Pi^{\mathrm{CM}}}|\mathbf{CM}^P_{(\mathbb{R})})$$

is trivial.

Consider the diagrams

$$\begin{array}{ccccccc}
\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) & \xleftarrow{J} & \mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) & & S & \longrightarrow & T \\
\downarrow R & & \downarrow R^L & & \uparrow & & \uparrow \\
\mathbf{Mot}(\mathbb{F};\mathcal{A}) & \xleftarrow{I} & \mathbf{LM}(\mathbb{F}), & & P & \longrightarrow & L.
\end{array}$$

The first is a commutative diagram of Tate triples, and the second is the corresponding diagram of fundamental groups (all commutative; cf. Milne 1999b, §6).

According to (1.5), the analogue of (*) is true for R^L, and so (2.8) shows that the $(T/L)_{\mathbb{R}}$-torsor

$$\wp' = \underline{\mathrm{Hom}}^{\otimes}((\omega_{R^L})_{(\mathbb{R})}, \omega_{\Pi^{\mathrm{LCM}}}|\mathbf{LCM}^L_{(\mathbb{R})})$$

is trivial. But $\wp' \cong \wp\wedge^{S/P}T/L$, and so it remains to show that the map $H^1(\mathbb{R}, S/P) \to H^1(\mathbb{R}, T/L)$ is injective. From Milne 1999b, 6.1, we know that the map $S/P \to T/L$ is injective. Fix a CM-field $K \subset \mathbb{Q}^{\mathrm{al}}$ finite and Galois over $\mathbb{Q}$, and let S^K, P^K, T^K, and L^K be the corresponding quotients of S, P, T, and L (ibid.). When K is chosen to have degree at least 4 and contain a quadratic imaginary field Q in which p splits, then the map $S^K/P^K \to T^K/L^K$ admits a section[1] (Milne 1999c, 2.2), and so $H^1(\mathbb{R}, S^K/P^K) \to H^1(\mathbb{R}, T^K/L^K)$ is injective. This shows that $\wp^K(\mathbb{R})$ is nonempty, where $\wp^K = \wp \wedge^{S/P} S^K/P^K$. As $(S^K/P^K)(\mathbb{R})$ is compact, this implies that $\wp(\mathbb{R}) = \varprojlim \wp^K(\mathbb{R})$ is nonempty. □

4. The Hodge standard conjecture

Throughout this section, $\mathcal{S}$ will be a class of varieties over k satisfying the following condition:

[1]Let A and B be the abelian varieties defined in Milne 1999c, §1, and let $A_{\mathbb{F}}$ and $B_{\mathbb{F}}$ be their reductions. We have a diagram

$$\begin{array}{ccccc}
\mathbf{CM}^K(\mathbb{Q}^{\mathrm{al}}) & \xleftarrow{J} & \mathbf{LCM}^K(\mathbb{Q}^{\mathrm{al}}) & \longleftarrow & \langle A \times B\rangle^{\otimes} \\
\downarrow R & & \downarrow R^L & & \downarrow \\
\mathbf{Mot}^K(\mathbb{F};\mathcal{A}) & \xleftarrow{I} & \mathbf{LM}^K(\mathbb{F}) & \longleftarrow & \langle A_{\mathbb{F}} \times B_{\mathbb{F}}\rangle^{\otimes}.
\end{array}$$

of Tannakian categories, and correspondingly homomorphisms of groups

$$S^K/P^K \to T^K/L^K \to L(A\times B)/L(A_{\mathbb{F}} \times B_{\mathbb{F}}).$$

The exact commutative diagram of character groups ibid., 2.2, shows that the composite of these homomorphisms is an isomorphism.

(*): the projective spaces are in $\mathcal{S}$, and $\mathcal{S}$ is closed under passage to a connected component and under the formation of products and disjoint unions.

For example, $\mathcal{S}$ could be the class $\mathcal{T}$ of all varieties over k or the smallest class $\mathcal{A}$ satisfying (*) and containing the abelian varieties.

By a Weil cohomology theory on $\mathcal{S}$, we mean a contravariant functor $V \mapsto H^*(V)$ satisfying the conditions in Kleiman 1968, 1.2, (equal to the conditions (1)–(4) of Kleiman 1994, §3) except that we remember the Tate twists (Milne 1999a, Appendix). We say that such a cohomology theory is *good* if homological equivalence coincides with numerical equivalence on algebraic cycles with $\mathbb{Q}$-coefficients for all varieties in $\mathcal{S}$, and we say that it is *very good* if, in addition, the strong Lefschetz theorem holds:

for every connected variety V in $\mathcal{S}$ and map L defined by a smooth hyperplane section of V,

$$L^{n-r}\colon H^r(V) \to H^{2n-r}(V)(n-r),$$

is an isomorphism for $0 \leq r \leq n = \dim V$.

For a Weil cohomology theory H, π^r denotes the projection onto H^r, and when H satisfies the strong Lefschetz theorem, Λ, ${}^c\Lambda$, $*$, p^r denote the maps defined in Kleiman 1968, 1.4 (corrected in Kleiman 1994, §4).

PROPOSITION 4.1. *For all very good Weil cohomology theories H on $\mathcal{S}$, the operators Λ, ${}^c\Lambda$, $*$, p^r, and π^r are defined by algebraic cycles that (modulo numerical equivalence) depend only on L (not H).*

PROOF. Let H be a very good Weil cohomology theory on $\mathcal{S}$. Then the Lefschetz standard conjecture holds for all $V \in \mathcal{S}$ (Kleiman 1994, 5-1, 4-1(1)), and the proposition can be proved as in ibid., 5.4, (the Hodge standard conjecture is used there only to deduce that numerical equivalence coincides with homological equivalence on $V \times V$). □

Let $A^*_\sim(V)$ denote the $\mathbb{Q}$-algebra of algebraic classes on V modulo an admissible equivalence relation $\sim$, for example, numerical equivalence (num), or homological equivalence (hom) with respect to some Weil cohomology.

When there exists a very good Weil cohomology theory on $\mathcal{S}$, we define $\mathbf{Mot}(k;\mathcal{S})$ to be the category of motives based on $\mathcal{S}$ using the elements of $A^*_{\text{num}}(V \times V)$ as the correspondences and with the commutativity constraint modified using the π^r's given by (4.1). It is semisimple (Jannsen 1992), hence Tannakian (Deligne 1990), and it has a natural structure of a Tate triple.

PROPOSITION 4.2. *If there exists a very good Weil cohomology theory on $\mathcal{S}$, then all good Weil cohomology theories on $\mathcal{S}$ are very good.*

PROOF. Let $V \in \mathcal{S}$ be connected of dimension n, and let Z be a smooth hyperplane section of V. Then $l =_{\text{df}} \Delta_V(Z) \in A^{n+1}(V \times V)$ is a morphism $l\colon h(V) \to h(V)(1)$. Consider the morphisms

$$l^{n-r}\colon h^r(V) \to h^{2n-r}(V)(n-r), \quad 0 \leq r \leq n.$$

A good Weil cohomology theory H on $\mathcal{S}$ defines a fibre functor ω_H on $\mathbf{Mot}(k;\mathcal{S})$, and $\omega_H(l^{n-r})$ is $L^{n-r}\colon H^r(V) \to H^{2n-r}(V)(n-r)$. If H is very good, then l^{n-r} is an isomorphism, which in turn implies that L^{n-r} is an isomorphism for every good Weil cohomology theory. □

Proposition 4.3. *When $k = \mathbb{F}$, there exists a very good Weil cohomology theory on $\mathcal{A}$ (and therefore the conclusions of Propositions 4.1 and 4.2 hold for $\mathcal{A}$).*

Proof. For all $\ell \neq p$, the ℓ-adic étale cohomology theory satisfies the strong Lefschetz theorem (Deligne 1980). Let S be a finite set of abelian varieties over k. When $\mathcal{A}$ is replaced by the smallest class $\mathcal{A}(S)$ containing S and satisfying (*), then there exist ℓ for which ℓ-adic étale cohomology theory is good (Clozel 1999; see also Milne 1999c, B.2). Therefore (see the proof of Proposition 4.2), for the varieties in $\mathcal{A}(S)$,

$$l^{n-r}\colon h^r(V) \to h^{2n-r}(V)(n-r)$$

is an isomorphism for $0 \leq r \leq n = \dim V$. Since S was arbitrary, this shows that l^{n-r} is an isomorphism for all connected varieties in $\mathcal{A}$, which implies that every good Weil cohomology theory on $\mathcal{A}$ is very good. But, any fibre functor ω on $\mathbf{Mot}(\mathbb{F};\mathcal{A})$ defines a good cohomology theory with $H^r(V) = \omega(h^r V)$. □

Define $P^r_\sim(V)$ to be the $\mathbb{Q}$-subspace of $A^r_\sim(V)$ on which L^{n-2r+1} is zero, and let θ^r be the bilinear form

$$(x,y) \mapsto (-1)^r \langle L^{n-2r} x \cdot y \rangle \colon P^r_\sim(V) \times P^r_\sim(V) \to \mathbb{Q}, \quad r \leq n/2.$$

As originally stated (Grothendieck 1969), the Hodge standard conjecture asserts that these pairings are positive-definite when $\sim$ is ℓ-adic homological equivalence. Kleiman (1994, §5) states the conjecture for any Weil cohomology theory. When the pairings θ^r are positive-definite with $\sim$ equal to numerical equivalence, we shall say that the *numerical Hodge standard conjecture* holds. Note the Hodge standard conjecture for a good Weil cohomology theory coincides with the numerical Hodge standard conjecture.

Remark 4.4. In the presence of the Lefschetz standard conjecture, the Hodge standard conjecture for a Weil cohomology H is false unless homological equivalence coincides with numerical equivalence, in which case it coincides with the numerical Hodge standard conjecture (Kleiman 1994, 5-1).

Assume that there exists a very good Weil cohomology theory on $\mathcal{S}$, so that $\mathbf{Mot}(k;\mathcal{S})$ is defined. Let $V \in \mathcal{S}$ be connected of dimension n, and let $p^r(V)$ be the largest subobject of

$$\mathrm{Ker}(l^{n-2r+1}\colon h^{2r}(V)(r) \to h^{2n-2r+2}(V)(n-r+1))$$

on which $\pi =_{\mathrm{df}} \pi(\mathbf{Mot}(k;\mathcal{S}))$ acts trivially. Then[2]

$$\gamma^{\mathrm{Mot}}(p^r(V)) = P^r_{\mathrm{num}}(V)$$

and there is a pairing

$$\vartheta^r\colon p^r(V) \otimes p^r(V) \to \mathbb{1},$$

also fixed by π, such that $\gamma^{\mathrm{Mot}}(\vartheta^r) = \theta^r$.

[2]Recall (§2) that γ^{Mot} is the "unique" fibre functor on $\mathbf{Mot}(k;\mathcal{S})^\pi$.

Proposition 4.5. *Assume there exists a very good Weil cohomology theory on $\mathcal{S}$. Then the numerical Hodge standard conjecture holds for all $V \in \mathcal{S}$ if and only if there exists a polarization Π on $\mathbf{Mot}(k;\mathcal{S})$ for which the forms ϑ^r are positive.*

Proof. $\Rightarrow$: If the numerical Hodge standard conjecture holds for all $V \in \mathcal{S}$, then there is a canonical polarization Π^{Mot} on $\mathbf{Mot}(k;\mathcal{S})$ for which the bilinear forms

$$\varphi^r\colon h^r(V)\otimes h^r(V) \overset{\mathrm{id}\otimes *}{\to} h^r(V)\otimes h^{2n-r}(V)(n-r) \to h^{2n}(V)(n-r) \cong \mathbb{1}(-r)$$

are positive (cf. Saavedra 1972, VI 4.4) — here $V \in \mathcal{S}$ is connected of dimension n and $*$ is defined by any smooth hyperplane section of V. The restriction of $\varphi^{2r}\otimes \mathrm{id}_{\mathbb{1}(2r)}$ to the subobject $p^r(V)$ of $h^{2r}(V)(r)$ is the form ϑ^r, which is therefore positive for Π^{Mot} (Deligne and Milne 1982, 4.11b).

$\Leftarrow$: Let Π be a polarization on $\mathbf{Mot}(k;\mathcal{S})$ for which the forms ϑ^r are positive. There exists a morphism of Tate triples $\xi\colon \mathbf{Mot}(k;\mathcal{S})_{(\mathbb{R})} \to \mathbf{V}$ such that $\xi\colon \Pi \mapsto \Pi^V$; in particular, for X of weight 0 and $\phi \in \Pi(X)$, $(\gamma^V \circ \xi)(\phi)$ is a positive-definite symmetric form on $(\gamma^V\circ\xi)(X)$ (Deligne and Milne 1982, p. 195). The restriction of $\gamma^V\circ\xi$ to $\mathbf{Mot}(k;\mathcal{S})^{\pi}_{(\mathbb{R})}$ is (uniquely) isomorphic to γ^{Mot}, and so $\theta^r = \gamma^{\mathrm{Mot}}(\vartheta^r)$ is positive-definite. □

Theorem 4.6. *Let k be an algebraically closed field. If the Hodge conjecture holds for complex abelian varieties of CM-type, then*

(a) *numerical equivalence coincides with ℓ-adic étale homological equivalence on abelian varieties over k (all $\ell \neq char(k)$), and*
(b) *the Hodge standard conjecture holds for abelian varieties over k and all good Weil cohomologies on $\mathcal{A}$ (for example, for the ℓ-adic étale cohomology, $\ell \neq char(k)$).*

Proof. *(a) for $k = \mathbb{F}$.* For an abelian variety A over a finite field, the Frobenius endomorphism acts semisimply on the ℓ-adic étale cohomology (Weil 1948). Hence, the Tate conjecture implies that numerical equivalence coincides with ℓ-adic étale homological equivalence (see, for example, Tate 1994, 2.7), and our assumption implies that the Tate conjecture holds (Milne 1999b, 7.1).

(b) for $k = \mathbb{F}$. Since the Hodge standard conjecture holds in characteristic zero, there is a polarization Π on $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ for which the forms

$$\varphi^r\colon h^r(A)\otimes h^r(A) \to \mathbb{1}(-r)$$

are positive for all abelian varieties A of CM-type over $\mathbb{Q}^{\mathrm{al}}$. Clearly, Π is the polarization Π^{CM} defined in Proposition 1.1. Let Z be the hyperplane section of A used in the definition of φ^r. Because $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi^{\mathrm{Mot}}$ (Theorem 3.1), the form

$$\varphi^r\colon h^r(A_{\mathbb{F}})\otimes h^r(A_{\mathbb{F}}) \to \mathbb{1}(-r)$$

defined by the reduction $Z_{\mathbb{F}}$ of Z on $A_{\mathbb{F}}$ is positive for Π^{Mot}. As in the proof of (4.5), this implies that $\vartheta^r\colon p^r(A_{\mathbb{F}})\otimes p^r(A_{\mathbb{F}}) \to \mathbb{1}$ is positive for Π^{Mot} and that $A_{\mathbb{F}}$ satisfies the numerical Hodge standard conjecture. Because of (1.6), the pair

$$(A_{\mathbb{F}}, Z_{\mathbb{F}} \text{ modulo numerical equivalence})$$

is arbitrary, and so the numerical Hodge standard conjecture holds for all abelian varieties over $\mathbb{F}$.

(a) for arbitrary k. For an abelian variety A of dimension n over k, consider the commutative diagram:

$$\begin{array}{ccccc} H^{2r}(A,\mathbb{Q}_\ell(r)) \times & H^{2n-2r}(A,\mathbb{Q}_\ell(n-r)) & \xrightarrow{\cup} & H^{2n}(A,\mathbb{Q}_\ell(n)) \cong & \mathbb{Q}_\ell \\ \uparrow cl & \uparrow L^{n-2r}\circ cl & & & \bigcup \\ P^r(A) \quad \times & P^r(A) & & \xrightarrow{\theta} & \mathbb{Q} \end{array}$$

Here $P^r(A)$ denotes the group of primitive algebraic classes modulo ℓ-adic homological equivalence. There is a similar diagram for a smooth specialization $A_{\mathbb{F}}$ of A to an abelian variety over $\mathbb{F}$. The specialization maps on the cohomology groups are bijective and hence they are injective on the P's. Since the pairings are compatible, this implies the Hodge standard conjecture for A and ℓ-adic étale cohomology. Since the Lefschetz standard conjecture is known for abelian varieties, this in turn implies that numerical equivalence coincides with ℓ-adic homological equivalence for A (Kleiman 1994, 5-4).

(b) for arbitrary k. In the last step we proved that the ℓ-adic étale Weil cohomology is a good Weil cohomology theory on $\mathcal{A}$ and that the ℓ-adic étale Hodge standard conjecture holds for abelian varieties over k. It follows that the numerical Hodge standard conjecture holds for abelian varieties over k. □

Corollary 4.7. *If the Hodge conjecture holds for complex abelian varieties of CM-type, then, for every k such that there exists a very good Weil cohomology theory on $\mathcal{A}$,* $\mathbf{Mot}(k;\mathcal{A})$ *has a polarization (necessarily unique) for which the forms ϑ^r are positive.*

Proof. Apply 4.5 and 4.6. □

Remark 4.8. It is possible to prove directly that, if the geometric Weil forms are positive for a polarization Π on $\mathbf{Mot}(k;\mathcal{A})$, then the forms φ^r are also positive for Π. Indeed, by assumption $\varphi^1 \in \Pi(A)$. The restriction of the form $\otimes^r\varphi^1$ on $\otimes^r h^1(A)$ to $H^r(A) \cong \wedge^r H^1(A)$ is a positive rational multiple of φ^r (see the proof of Kleiman 1968, 3.11), which is therefore positive for Π.

Remark 4.9. Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$ that is Galois over $\mathbb{Q}$ and properly contains a quadratic imaginary number field in which p splits. The preceding arguments can be modified to show that, if the Hodge conjecture holds for all complex abelian varieties with reflex field contained in K, then the conclusions of Theorem 4.6 hold for all abelian varieties over $\mathbb{F}$ whose endomorphism algebra is split by K.

Remark 4.10. If the Tate conjecture holds for all varieties over $\mathbb{F}$, then $\mathbf{Mot}(\mathbb{F};\mathcal{T}) = \mathbf{Mot}(\mathbb{F};\mathcal{A})$ (see, for example, Milne 1994, 2.7). Unfortunately, it does not[3] appear that this equality can be used to deduce the Hodge standard conjecture for all varieties over $\mathbb{F}$ from knowing it for abelian varieties over $\mathbb{F}$.

Remark 4.11. Most of the preceding arguments hold with "algebraic cycle" replaced by "Lefschetz cycle" (cf. Milne 1999a, §5). Let A be an abelian variety over k. Recall that, for any Weil cohomology theory, if a Lefschetz class a on A is not

[3]Contrary to what was asserted in the first version of this manuscript.

homologically equivalent to zero, then there exists a Lefschetz class b on A of complementary dimension such that $\langle a \cdot b\rangle \neq 0$; in particular, homological equivalence on Lefschetz classses is independent of the Weil cohomology theory, and coincides with numerical equivalence (ibid. 5.2).

Let $D^r(A)$ be the $\mathbb{Q}$-space of Lefschetz classes on A of codimension r modulo numerical equivalence, and let $DP^r(A)$ be the $\mathbb{Q}$-subspace on which L^{n-2r+1} is zero. The argument in (4.8) shows that the forms φ^r are positive for the canonical polarization Π on $LM(\mathbb{F})$. Hence (cf. the proof of 4.5), the bilinear forms

$$(x, y) \mapsto (-1)^r \langle L^{n-2r} x \cdot y\rangle \colon DP^r(A) \times DP^r(A) \to \mathbb{Q}$$

are positive-definite for $r \leq n/2$. In other words, the Lefschetz analogue of the Hodge standard conjecture holds unconditionally for abelian varieties over $\mathbb{F}$. A specialization argument (as in the proof of 4.6) extends the statement to arbitrary k.

Remark 4.12. Recall that a Hodge, Tate, or algebraic class on a variety is said to be *exotic* if it is not Lefschetz. Remark 4.11 shows that the Hodge standard conjecture holds unconditionally for abelian varieties with no exotic algebraic classes. For examples (discovered by Lenstra, Spiess, and Zarhin) of abelian varieties over $\mathbb{F}$ with no exotic Tate classes, and hence no exotic algebraic classes, see Milne 1999c, A.7.

Solution 4.13 (Solution to Exercise 1.7). Lemma 1.2 can be proved over an arbitrary k by a similar case-by-case argument; then Exercise 1.7 follows from Lemma 1.4. More elegantly, it follows from (4.11) and the Lefschetz analogue of (4.5).

Remark 4.14. Grothendieck (1969) stated: "Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry." Should the Hodge conjecture remain inaccessible, even for abelian varieties of CM-type, Theorem 4.6 suggests a possible approach to proving the Hodge standard conjecture for abelian varieties, namely, improve the theory of absolute Hodge classes (Deligne 1982) sufficiently to remove the hypothesis from the theorem.

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