VALUES OF ZETA FUNCTIONS OF VARIETIES OVER FINITE FIELDS

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Introduction. The zeta function of a smooth projective variety X of dimension d over a finite field k of q elements has the form

$$\zeta(X, s) = \frac{P_1(X, q^{-s}) \cdots P_{2d-1}(X, q^{-s})}{P_0(X, q^{-s})P_2(X, q^{-s}) \cdots P_{2d}(X, q^{-s})}$$

where $P_i(X, t) = 1 + \cdots \in \mathbf{Z}[t]$. Therefore, for r an integer,

$$\zeta(X, s) \sim C_X(r)/(1 - q^{r-s})^{\rho_r} \quad \text{as} \quad s \to r$$

for some integer ρ_r and rational number $C_X(r)$. In this article we continue the investigation of $C_X(r)$ and ρ_r begun in [36], [38], [24], [3], [39], [32], and [22].

Before stating our results, we introduce some notations. The order of a group M is denoted by $[M]$, and $|\ |_\ell$ is the ℓ-adic valuation normalized so that $|\ell|_\ell^{-1} = \ell$. For $\alpha: M \to N$ a map of abelian groups, we let

$$z(\alpha) = [\mathrm{Ker}(\alpha)]/[\mathrm{Coker}(\alpha)]$$

when both orders are finite. If M and N are finitely generated over $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/m\mathbf{Z}$ and $\alpha \otimes 1_{\mathbf{Q}}: M \otimes_{\mathbf{Z}} \mathbf{Q} \to N \otimes_{\mathbf{Z}} \mathbf{Q}$ is an isomorphism, we let

$$\det(\alpha) = \Pi |\det(\alpha_\ell)|_\ell^{-1}$$

provided $\alpha_\ell \overset{df}{=} \alpha \otimes 1_{\mathbf{Z}_\ell}: M \otimes_{\hat{\mathbf{Z}}} \mathbf{Z}_\ell \to N \otimes_{\hat{\mathbf{Z}}} \mathbf{Z}_\ell$ is an isomorphism modulo torsion for almost all ℓ; in this case

$$z(\alpha) = [M_{\mathrm{tor}}]/[N_{\mathrm{tor}}]\det(\alpha)$$

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when all the terms are defined and finite. (Throughout, M_{tor} will denote the torsion subgroup of M.)

For $\ell \neq p$, the characteristic of k, $H^i(X, \mathbf{Z}_\ell(r))$ will denote the ℓ-adic étale cohomology group, and for $\ell = p$ it will denote the group defined in Section 1 below. Let $H^i(X, \hat{\mathbf{Z}}(r)) = \Pi_\ell H^i(X, \mathbf{Z}_\ell(r))$. There is an obvious element in $H^1(k, \hat{\mathbf{Z}})$ which defines, by cup-product, canonical maps

$$\epsilon^i : H^i(X, \hat{\mathbf{Z}}(r)) \to H^{i+1}(X, \hat{\mathbf{Z}}(r))$$

and we let

$$\chi(X, \hat{\mathbf{Z}}(r)) = \prod_{i \neq 2r, 2r+1} [H^i(X, \hat{\mathbf{Z}}(r))]^{(-1)^i} z(\epsilon^{2r})$$

when all the terms are defined and finite. A theorem of Gabber [12] implies that $H^i(X, \hat{\mathbf{Z}}(r))$ is finite for all $i \neq 2r, 2r+1$ and that $H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite. As the groups $H^i(X, \hat{\mathbf{Z}}(r))$ are finitely generated over $\hat{\mathbf{Z}}$, $\chi(X, \hat{\mathbf{Z}}(r))$ is defined if and only if $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite and $\det(\epsilon^{2r})$ is defined, in which case

$$\chi(X, \hat{\mathbf{Z}}(r)) = \prod_{i=0}^{2d+1} [H^i(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]^{(-1)^i} / \det(\epsilon^{2r}).$$

Let $\bar{k}$ be the algebraic closure of k, and let γ be the canonical generator of $\Gamma = \operatorname{Gal}(\bar{k}/k)$; let $\bar{X} = X \otimes_k \bar{k}$. It has been conjectured:

$SS(X, r, \ell)$: the minimal polynomial of γ acting on $H^{2r}(X, \mathbf{Q}_\ell(r))$ does not have 1 as a multiple root.

This conjecture is obviously true if $\dim H^{2r}(\bar{X}, \mathbf{Q}_\ell(r)) = 1$. It is also true for abelian varieties and the varieties whose cohomology can be expressed in terms of that of abelian varieties (cf. [10, 6.26]). For $\ell \neq p$, it is equivalent to the minimal polynomial of the Frobenius element acting on $H^{2r}(\bar{X}, \mathbf{Q}_\ell)$ not having q^r as a multiple root.

THEOREM 0.1. *The number $\chi(X, \hat{\mathbf{Z}}(r))$ is defined if and only if $SS(X, r, \ell)$ holds for all ℓ, in which case*

$$\zeta(X, s) \sim \pm \chi(X, \hat{\mathbf{Z}}(r)) q^{\chi(X, \mathcal{O}_X, r)} (1 - q^{r-s})^{-\rho_r} \quad as \quad s \to r$$

where

$$\chi(X, \mathcal{O}_X, r) = \Sigma\,(-1)^{i+j}(r-i)h^{ij} \qquad 0 \le i \le r, \qquad 0 \le j \le d,$$

$$h^{ij} = \dim H^j(X, \Omega^i).$$

The ℓ-part of (0.1), $\ell \neq p$, was proved in [3, 3.4] for $r > 2d$ and, in a slightly different form, in [32, Theorem 5] for all r.

THEOREM 0.2. *Let $d = 2r$, and assume $SS(X, r, \ell)$ holds for all ℓ. Then*

$$P_{2r}(X, q^{-s}) \sim \pm \frac{[H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}]}{q^{\alpha_r(X)}[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\mathrm{tor}}]^2} \det(\epsilon^{2r})(1-q^{r-s})^{\rho_r} \quad as \quad s \to r$$

where

$$\alpha_r(X) = s^{2r+1}(r) - s^{2r}(r) + \sum_{\mathrm{ord}_q(a_{2r,j}) < r} (r - \mathrm{ord}_q(a_{2r,j})),$$

$s^i(r) = \dim \underline{H}^i(X, \mathbf{Z}_p(r))$ (*as a perfect group scheme*), $\{a_{i1}, a_{i2}, \ldots\}$ *are the inverse roots of* $P_i(X, t)$, *and* $\mathrm{ord}_q(q^m u) = m$ *if* $|u|_p = 1$.

For small values of d, the expression for $\alpha_r(X)$ simplifies:

$$\alpha_1(X) = \chi(X, \mathcal{O}_X) - 1 + (1/2)\beta_1,$$

$$\beta_i = \dim_{\mathbf{Q}_\ell} H^i(X, \mathbf{Q}_\ell), \qquad \ell \neq p;$$

$$\alpha_2(X) = \chi(X, \mathcal{O}_X, 2) - 2 + (3/2)\beta_1 - \beta_2 + b_3,$$

$$b_3 = \sum_{j=1}^{\beta_3} |2 - \mathrm{ord}\, a_{3j}|.$$

There is a canonical non-degenerate skew-symmetric pairing on $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$, and therefore its order is a square or twice a square. In 1966, Swinnerton-Dyer obtained some squares when computing $\zeta(X, s)$, s near 2, for X the product of three elliptic curves over $\mathbf{Q}$ (see [35, p. 155]). These squares are still unexplained, but in the analogous situation where $\mathbf{Q}$ is replaced by a function field in one variable over a finite field, (0.2) provides an explanation.

In order to obtain a less formal expression for $C_X(r)$, we shall need to assume Tate's conjecture. Any subvariety Z of X of codimension r defines a class $c^r(Z) \in H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$, and the weak form of Tate's conjecture asserts the following:

$T'(X, r, \ell)$: the $\mathbf{Q}_\ell$-subspace of $H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$ generated by the classes of the form $c^r(Z)$ is exactly the space $H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma$ of classes left fixed by the action of $\Gamma = \mathrm{Gal}(\bar{k}/k)$.

PROPOSITION 0.3. *If $T'(X, 1, \ell)$ holds for one ℓ, then it holds for all ℓ; moreover, then $SS(X, 1, \ell)$ holds for all ℓ.*

For X of dimension 2, this is proved in [38] and [24], and the same arguments suffice in the general case (see Section 8). The conjecture $T'(X, r, \ell)$ is obviously true if $H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$ has dimension 1. It is known that $T'(X, 1, \ell)$ is true for products of abelian varieties, curves, and some other special varieties [37].

Let $K_r\mathcal{O}_X$ denote the sheaf for the étale topology on X associated with the presheaf $U \mapsto K_r(\Gamma(U, \mathcal{O}_X))$, where K_rA denotes the r^{th} Quillen K-group of a ring A. In particular $K_0\mathcal{O}_X = \mathbf{Z}$, and $K_1\mathcal{O}_X = \mathcal{O}_X^\times$. Write $K_0 = \mathbf{Z}$, $K_1 = \mathcal{O}_X^\times$, $K_2 = K_2\mathcal{O}_X/\{p\text{-torsion}\}$. It is expected that K_2A, for A a ring of characteristic p, has no p-torsion, and therefore that $K_2 = K_2\mathcal{O}_X$, but this has not been proved.[1] In Section 7 we define canonical maps $\delta^r: H^r(X, K_r)^\wedge \to TH^{r+2}(X, K_r)$ (étale cohomology groups) where, for any group M, $M^\wedge$ denotes the completion of M for the topology defined by the subgroups mM and $TM = \mathrm{Hom}(\mathbf{Q}/\mathbf{Z}, M)$. Let

$$\chi(X, K_r) = \det(\delta^r)^{(-1)^r} \prod_i [H^i(X, K_r)^\wedge_{\mathrm{tor}}]^{(-1)^i}$$

when all terms are defined and finite. The groups $H^i(X, \mathbf{Z})$ are torsion for $i \neq 0$ and finite for $i \neq 0, 2$; moreover, $H^0(X, \mathbf{Z}) \approx \mathbf{Z}$ and $H^2(X, \mathbf{Z}) \approx H^2(X, \mathbf{Z})_{\mathrm{cotor}} \oplus \mathbf{Q}/\mathbf{Z}$ where $H^2(X, \mathbf{Z})_{\mathrm{cotor}}$, the cotorsion subgroup of $H^2(X, \mathbf{Z})$, is finite. As $\delta^0: \hat{\mathbf{Z}} \to TH^2(X, \mathbf{Z}) = \hat{\mathbf{Z}}$ is the identity map, $\chi(X, \mathbf{Z})$ is always defined and

$$\chi(X, \mathbf{Z}) = \frac{[H^0(X, \mathbf{Z})_{\mathrm{tor}}][H^2(X, \mathbf{Z})_{\mathrm{cotor}}][H^4(X, \mathbf{Z})] \cdots}{[H^1(X, \mathbf{Z})][H^3(X, \mathbf{Z})] \cdots}.$$

[1] A. Suslin has recently shown (Torsion in K_2 of fields, preprint, *I.H.E.S.* 1983) that K_2 of a field of characteristic p has no p-torsion. It follows that the sheaf $K_2 = K_2\mathcal{O}_X$.

The groups $H^i(X, \mathcal{O}_X^\times)$ are torsion for $i \neq 1$, and are finite for $i \neq 1, 2, 3$; moreover, $H^1(X, \mathcal{O}_X^\times)$ is the finitely generated abelian group $\mathrm{Pic}(X)$ and $H^3(X, \mathcal{O}_X^\times) \approx H^3(X, \mathcal{O}_X^\times)_{\mathrm{cotor}} \oplus H^3(X, \mathcal{O}_X^\times)_{\mathrm{div}}$ where $H^3(X, \mathcal{O}_X^\times)_{\mathrm{cotor}}$ is finite and $H^3(X, \mathcal{O}_X^\times)_{\mathrm{div}}$ is divisible. Thus $\chi(X, \mathcal{O}_X^\times)$ is defined if and only if $H^2(X, \mathcal{O}_X^\times)$ is finite and $\det(\delta^1)$ is defined, in which case

$$\chi(X, \mathcal{O}_X^\times) = \frac{[H^0(X, \mathcal{O}_X^\times)][H^2(X, \mathcal{O}_X^\times)][H^4(X, \mathcal{O}_X^\times)] \cdots}{\det(\delta^1)[H^1(X, \mathcal{O}_X^\times)_{\mathrm{tor}}][H^3(X, \mathcal{O}_X^\times)_{\mathrm{cotor}}] \cdots}$$

THEOREM 0.4. (a) *As $s \to 0$, $\zeta(X, s) \sim \pm\chi(X, \mathbf{Z})(1 - q^{1-s})^{-1}$.*
(b) *If $T'(X, 1, \ell)$ holds for one ℓ, then $\chi(X, \mathcal{O}_X^\times)$ is defined and*

$$\zeta(X, s) \sim \pm\frac{q^{\chi(X,\mathcal{O}_X)}}{\chi(X, \mathcal{O}_X^\times)}(1 - q^{1-s})^{-\rho_1} \quad as \quad s \to 1,$$

where ρ_1 is the rank of $\mathrm{Pic}(X)$.

(c) *Assume that the order ρ_2 of the pole of $\zeta(X, s)$ at $s = 2$ is equal to the dimension of the subspace of $H^4(\bar{X}, \mathbf{Q}_p(2))$ generated by algebraic cycles. Then $\chi(X, K_2)$ is defined and*

$$\zeta(X, s) \sim C_0\chi(X, K_2)q^{\chi(X,\mathcal{O}_X,2)}(1 - q^{2-s})^{-\rho_2} \quad as \quad s \to 2$$

where C_0 is a rational number such that $|C_0|_p = 1$.

Except for the definition of the term $\det(\delta^1)$, part (b) of (0.4) was conjectured in [22], which also contains a proof of the non-p-part of (b). A weak form of the ℓ-part of (b), $\ell \neq p$, is also proved in [32, Corollary 7].

Unfortunately, $K_2\mathcal{O}_X$ is uniquely divisible by all primes $\ell \neq p$, and so it is not possible to express the number C_0 in part (c) in terms of it.

After (0.4) it is tempting to conjecture that, for $r > 2$,

$$\zeta(X, s) \sim C_0\chi(X, K_r)^{(-1)^r}q^{\chi(X,\mathcal{O}_X,r)}(1 - q^{r-s})^{-\rho_r} \quad \text{as} \quad s \to r$$

with C_0 a rational number such that $|C_0|_p = 1$. Conversations with Lichtenbaum have convinced me however that the true situation is probably more complicated. It is more natural to raise the following question (see the discussion at the end of Section 10 below).

Problem 0.5. Find a canonical complex of étale sheaves $\mathbf{Z}(r)$ such that $\chi(X, \mathbf{Z}(r))$ can be defined in the same way as for $\mathbf{Z}(0) \overset{df}{=} \mathbf{Z}$ and $\mathbf{Z}(1) = \mathcal{O}_X^\times[-1]$, and such that

$$\zeta(X, s) \sim \pm\chi(X, \mathbf{Z}(r)) q^{\chi(X, \mathcal{O}_X, r)}(1 - q^{r-s})^{-\rho_r} \quad \text{as} \quad s \to r.$$

There is some additional evidence that $\chi(X, \mathcal{O}_X, r)$ is the correct exponent of q to insert. The complex $\mathbf{Z}(r)$ should be defined in terms of the sheaves $K_i\mathcal{O}_X$, and it is expected that these are uniquely divisible by p if $i > \dim(X)$. Thus it should be true that

$$|\zeta(X, r)|_p^{-1} = q^{\chi(X, \mathcal{O}_X, r)}, \qquad r > \dim X.$$

In Section 10 we verify this formula.

When X has dimension 4, it is possible to make a conjecture for the behaviour of $P_4(X, q^{-s})$ near $s = 2$ that is closely analogous to the original conjecture of Artin and Tate [38] for surfaces, and it is possible to prove the conjecture under the analogous assumption.

THEOREM 0.6. *Let X have the dimension* 4; *then under the same hypothesis as in* (0.4c), $H^3(X, K_2)$ *is finite and*

$$P_4(X, q^{-s}) \sim \frac{C_0[H^3(X, K_2)]\det(\delta^2)}{q^{\alpha_2(X)}[H^2(\bar{X}, K_2)^\Gamma_{\mathrm{tor}}]}(1 - q^{2-s})^{\rho_2(X)} \quad as \quad s \to 2,$$

where C_0 is a rational number such that $|C_0|_p = 1$. Moreover, if the cycle map $c^2: CH^2(X) \otimes \mathbf{Z}_p \to H^4(X, \mathbf{Z}_p(2))$ is surjective, then

$$P_4(X, q^{-s}) \sim \frac{C_0[H^3(X, K_2)]\,|\det(Z_i \cdot Z_j)|}{q^{\alpha_2(X)}[A_p^2(X)_{\mathrm{tor}}]^2}(1 - q^{2-s})^{\rho_2(X)} \quad \text{as} \quad s \to 2$$

where $CH^2(X)$ is the Chow group of codimension 2 cycles on X, $\det(Z_i \cdot Z_j)$ *is the discriminant of the intersection pairing $CH^2(X) \times CH^2(X) \to \mathbf{Z}$, and $A_p^2(X)$ is the image of $CH^2(X)$ in $H^4(\bar{X}, \mathbf{Z}_p(2))$.*

When X is a surface, all of the above results are closely related to the conjecture of Artin and Tate [38, (C)], and the proofs of their ℓ-parts, $\ell \neq p$, and their p-parts follow the same general lines as those in [38] and [24] respectively. The latter require, however, an extensive development of the theory of the cohomology groups $H^i(X, \mathbf{Z}_p(r))$. This is carried out in the first three sections of the paper and the fifth. In the fourth section $\chi(X, \mathcal{O}_X, r)$ is expressed in terms of the slopes of the crystalline cohomology groups and the infinite parts of the cohomology groups of the truncated deRham-Witt complex. Theorems (0.1) and (0.2) are proved in the sixth section. The following two sections discuss the cohomology of K_r and Tate's conjecture respectively. In Section 9, Theorems (0.4) and (0.6) are proved, and it is shown that the determinants of ϵ^{2r} and δ^r are related to discriminants of intersection pairings on groups of algebraic cycles. Finally, the compatibility of (0.5) with the functional equation of $\zeta(X, s)$ is discussed in Section 10.

Notations. Except in Section 2, X will be a complete smooth variety of dimension d over a perfect field k of characteristic $p \neq 0$. In Sections 5, 6, 8, 9, and 10, k is finite, and in 6, 8, 9, and 10, X is projective. The ring of Witt vectors over k is denoted by W, and σ is its Frobenius automorphism, inducing $a \mapsto a^p$ on k. The absolute Frobenius map on X is denoted by F, and the relative Frobenius map with respect to a finite field k of $q = p^m$ elements is denoted by ϕ $(= F^m)$. Cohomology groups will be with respect to the étale topology, unless stated otherwise. The kernel and cokernel of multiplication by m on M are denoted respectively by M_m and $M^{(m)}$.

1. Cohomology of the sheaves $\nu_n(r)$: duality.

We begin by reviewing some of the definitions and results in [26] (see also [4], which gives a more polished exposition of the material in [26]). Throughout, X will be a smooth complete variety over a perfect field, although much of the material applies to schemes smooth and proper over a perfect scheme.

The Cartier operator C is a semilinear map from the sheaf of closed differentials $\Omega^r_{X/k,cl}$ on X to $\Omega^r_{X/k}$, and the sheaf $\nu(r)$ is defined to be the kernel of $1 - C: \Omega^r_{X/k,cl} \to \Omega^r_{X/k}$.

LEMMA 1.1. *The sequence of sheaves on* X

$$0 \to \nu(r) \to \Omega^r_{X/k,cl} \xrightarrow{1-C} \Omega^r_{X/k} \to 0$$

is exact (relative to the étale topology).

Proof. [26, 1.3].

Let Pf/k denote the category of perfect affine schemes over k, i.e., those of the form spec A with A a k-algebra such that $A = A^p$. A group in the category Pf/k will be called a *perfect group scheme* over k. Such a group scheme G is said to be *algebraic* if there exists an affine group scheme G_0 of finite type over k such that $G(A) = G_0(A)$ for all perfect k-algebras; one then calls G the perfection G_0^{pf} of G_0. Let $\mathcal{G}(p^\infty)$ denote the category of commutative algebraic perfect group schemes over k that are killed by a power of p. When Pf/k is endowed with the étale topology, $\mathcal{G}(p^\infty)$ becomes embedded as a full subcategory of the category of sheaves on Pf/k.

LEMMA 1.2. *The sheaf on $(Pf/k)_{et}$ associated with the presheaf $T \mapsto H^i(X_T, \nu(r))$ is represented by an object of $\mathcal{G}(p^\infty)$.*

Proof. This follows easily from (1.1) (see [26, 2.7]).

The perfect group scheme defined by the lemma will be denoted by $\underline{H}^i(X, \nu(r))$.

The category $\mathcal{G}(p^\infty)$ is equivalent to the category of unipotent commutative quasi-algebraic groups studied by Serre [34] in the case that k is algebraically closed. The identity component G^0 of an object G of $\mathcal{G}(p^\infty)$ has a composition series whose quotients are isomorphic to $\mathbf{G}_a^{pf}$, and the quotient $D = G/G^0$ is étale (i.e., D is the perfection of an étale group scheme). For D étale and in $\mathcal{G}(p^\infty)$, define

$$D^* = \operatorname{Hom}(D, \mathbf{Q}_p/\mathbf{Z}_p)$$

to be its Pontryagin dual, and for U connected in $\mathcal{G}(p^\infty)$ define

$$U^\vee = \operatorname{Ext}^1(U, \mathbf{Q}_p/\mathbf{Z}_p)$$

(Hom and Ext in the category of étale sheaves on Pf/k). Then D^* and $U^\vee$ are also objects of $\mathcal{G}(p^\infty)$ (e.g. $(\mathbf{G}_a^{pf})^\vee = \mathbf{G}_a^{pf}$), and the canonical maps $D \to D^{**}$ and $U \to U^{\vee\vee}$ are isomorphisms. These two autodualities can be combined as follows. Let $D^b(\mathcal{G}(p^\infty))$ be the full subcategory of the derived category of sheaves on Pf/k whose objects are the bounded complexes with cohomology in $\mathcal{G}(p^\infty)$. For $G^\cdot$ in $D^b(\mathcal{G}(p^\infty))$, let

$$G^{\cdot t} = R\operatorname{Hom}(G^\cdot, \mathbf{Q}_p/\mathbf{Z}_p).$$

LEMMA 1.3. *If $G^\cdot$ is in $D^b(\mathcal{G}(p^\infty))$ then so also is $G^{\cdot t}$, and the canonical map $G^\cdot \to G^{\cdot tt}$ is an isomorphism in $D^b(\mathcal{G}(p^\infty))$. Moreover there is an exact sequence*

$$0 \to U^{i+1}(G^{\cdot})^{v} \to H^{-i}(G^{\cdot t}) \to D^{i}(G^{\cdot})^{*} \to 0$$

where $U^i(G^{\cdot}) = H^i(G^{\cdot})^0$ *and* $D^i(G^{\cdot}) = H^i(G^{\cdot})/U^i(G^{\cdot})$.

Proof. This follows from the results recalled above and a vanishing theorem of Breen [8, 0.1], as is explained in most detail in [4, II].

The direct image (in the sense of derived categories) of $\nu(r)$ relative to $X \to \operatorname{spec} k$ is a bounded complex $\underline{H}^{\cdot}(X, \nu(r))$ such that $H^i(\underline{H}^{\cdot}(X, \nu(r))) = \underline{H}^i(X, \nu(r))$. Therefore, (1.2) shows that $\underline{H}^{\cdot}(X, \nu(r))$ lies in $D^b(\mathcal{G}(p^\infty))$.

The cohomology sequence of

$$0 \to \nu(d) \to \Omega^d_{X/k} \xrightarrow{1-C} \Omega^d_{X/k} \to 0$$

and the trace maps

$$H^d(X_T, \Omega^d) \xrightarrow{\approx} \Gamma(T, \mathcal{O}_T)$$

give rise to a trace map

$$\eta : \underline{H}^d(X, \nu(d)) \xrightarrow{\approx} \mathbf{Z}/p\mathbf{Z}.$$

This and the pairing

$$\omega, \omega' \mapsto \omega \wedge \omega' : \nu(r) \times \nu(d-r) \to \nu(d)$$

define a morphism

$$\underline{H}^{\cdot}(X, \nu(r)) \to R\operatorname{Hom}(H^{\cdot}(X, \nu(d-r)), \mathbf{Q}_p/\mathbf{Z}_p)[-d]$$

$$= \underline{H}^{\cdot}(X, \nu(d-r))^t[-d].$$

THEOREM 1.4. *The morphism*

$$\underline{H}^{\cdot}(X, \nu(r)) \to \underline{H}^{\cdot}(X, \nu(d-r))^t[-d]$$

defined above is an isomorphism. In particular, there are canonical isomorphisms

$$U^i(X, \nu(r)) \xrightarrow{\approx} U^{d+1-i}(X, \nu(d-r))^{v}$$

$$D^i(X, \nu(r)) \xrightarrow{\approx} D^{d-i}(X, \nu(d-r))^{*}$$

where $U^i(X, \nu(r)) = H^i(X, \nu(r))^0$ *and* $D^i(X, \nu(r)) = \underline{H}^i(X, \nu(r))/U^i(X, \nu(r))$.

Proof. [26, 2.4].

COROLLARY 1.5. *When k is finite there are canonical nondegenerate pairings of finite groups*

$$H^i(X, \nu(r)) \times H^{2d+1-i}(X, \nu(d-r)) \to H^{2d+1}(X, \nu(d)) \approx \mathbf{Z}/p\mathbf{Z}.$$

Proof. This can be proved the same way as (1.4) or else deduced from it; see [26, 1.9].

To extend these results to sheaves killed only by higher powers of p, it is necessary to replace the deRham complex by the deRham-Witt complex. At the time [26] was written the theory of this complex was still in a primitive form (only [6] was available) and so ad hoc definitions and arguments had to be used to obtain results strong enough for the intended applications. The next result suggests another approach to the sheaves $\nu(r)$.

LEMMA 1.6. *The sheaf* $\nu(r)$ *is the additive subsheaf of* $\Omega^r_{X/k}$ *locally generated by differentials of the form* $df_1/f_1 \wedge df_2/f_2 \wedge \cdots \wedge df_r/f_r$.

Proof. The proof mentioned in [26, 1.4] uses explicit calculation to obtain the analogous result for $\nu(r)_R = \mathrm{Ker}(1 - C: \Omega^r_{R/k,cl} \to \Omega^r_{R/k})$, where $R = \bar{k}[[t_1, \ldots, t_d]]$, and then applies Artin's approximation theorem to deduce the same result for a strictly Henselian subring of R. Another proof is given in [18, 0.2.4.2], and a stronger result is proved in [19, Section 1].

Let $(W_n\Omega^{\cdot}_X)_{n\geq 1}$ be the deRham-Witt pro-complex defined in [18]. In conflict with [18], I shall denote it by $W\Omega^{\cdot}_X$ (rather than $W.\Omega^{\cdot}_X$) and call it the deRham-Witt complex (rather than pro-complex). Recall that $W_n\Omega^{\cdot}_X$ is a complex

$$W_n\mathcal{O}_X \xrightarrow{d} W_n\Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} W_n\Omega^d_X$$

of coherent $W_n\mathcal{O}_X$-modules. The results of [18, I, 1.13, 1.14] allow us to regard the $W_n\Omega^i_X$ as sheaves for the étale topology on X (rather than the Zariski topology) and show that their étale and Zariski cohomology groups agree (cf. [27, III 3.7, II .17]). For $f \in \Gamma(U, \mathcal{O}_X^\times)$, let $\underline{f} = (f, 0, \ldots, 0)$ be its multiplicative representative in $\Gamma(U, W_n\mathcal{O}_X)$, and let $d\log f = d\underline{f}/\underline{f}$. Then $d\log$ is a homomorphism $\mathcal{O}_X^\times \to W_n\mathcal{O}_X$ and we define $\nu_n(r)$ to be the additive subsheaf of $W_n\Omega^r_X$ locally generated (for the étale topology) by sections of the form $d\log f_1 \wedge \cdots \wedge d\log f_r$. The maps $R: W_n\Omega_X^r \to W_{n-1}\Omega_X^r$ induce maps $\nu_n(r) \to \nu_{n-1}(r)$, and we write $\nu.(r)$ for the pro-system $(\nu_n(r))_n$. As $W_1\Omega_X^{\cdot} = \Omega_{X/k}^{\cdot}$ ([18, I1.3]), Lemma 1.6 shows that $\nu_1(r) = \nu(r)$. The map $d\log \wedge \cdots \wedge d\log : \mathcal{O}_X^\times \times \cdots \times \mathcal{O}_X^\times \to W_n\Omega_X^r$ factors through Sym $K_r\mathcal{O}_X$, the subsheaf of $K_r\mathcal{O}_X$ generated by symbols, and so we could also define $\nu_n(r)$ to be the image of Sym $K_r\mathcal{O}_X \to W_n\Omega_X^r$. Therefore, at least when $r \leq 2$, $\nu_n(r)$ is the sheaf defined in [26, 3.14], where the following result is suggested.

LEMMA 1.7. *There is a canonical exact sequence*

$$0 \to \nu_m(r) \to \nu_{m+n}(r) \to \nu_n(r) \to 0 \tag{1.7.1}$$

for all $m, n, r \geq 0$.

Proof. As multiplication by p is injective on $W\Omega_X^r$ ([18, I3.5]) it is also injective on $\nu.(r)$, and so there is an exact sequence

$$0 \to \nu.(r)/p^m\nu.(r) \to \nu.(r)p^{m+n}\nu.(r) \to \nu.(r)/p^n\nu.(r) \to 0.$$

The canonical map $\nu.(r)/p^n\nu.(r) \to \nu_n(r)$ is an isomorphism ([18, I5.7.5]), and the lemma follows.

LEMMA 1.8. *The sheaf on* $(Pf/k)_{et}$ *associated with the presheaf* $T \mapsto H^i(X_T, \nu_n(r))$ *is represented by an object of* $\mathcal{G}(p^\infty)$.

Proof. As $\mathcal{G}(p^\infty)$ is an abelian subcategory of the sheaves on Pf/k, closed under the formation of extensions, this follows from (1.2) and (1.7).

We write $\underline{H}^i(X, \nu_n(r))$ for the perfect group scheme defined by the lemma.

In order to define a trace map we need a strengthening of (1.1).

LEMMA 1.9. *There is an exact sequence*

$$0 \to \nu.(r) \to W\Omega_X^r \xrightarrow{1-F} W\Omega_X^r \to 0$$

of pro-sheaves on X.

Proof. [18, I5.7.2].

Define $H^i(X, \nu\cdot(r)) = \varprojlim H^i(X, \nu_n(r))$ and $H^i(X, W\Omega_X^r) = \varprojlim H^i(X, W_n\Omega_X^r)$.

PROPOSITION 1.10. *There is an exact sequence*

(1.10.1)

$$\cdots \to H^i(X, \nu.(r)) \to H^i(X, W\Omega^r_X) \xrightarrow{1-F} H^i(X, W\Omega^r_X) \to \cdots$$

Proof. An argument, as in the proof of [18, II 5.5], using (1.8) allows this to be deduced from (1.9).

As $\underline{H}^d(X, \nu(d)) \approx \mathbf{Z}/p\mathbf{Z}$ and $\underline{H}^i(X, \nu(d)) = 0$ for $i > d$, it follows from the cohomology sequence of (1.7.1) that $[\underline{H}^d(X, \nu_n(d))] \leq p^n$ and $\underline{H}^i(X, \nu_n(d)) = 0$ for $i > d$. On the other hand, the isomorphisms

$$H^d(X, W\Omega^d) \approx H^{2d}(X, W\Omega^{\cdot}) \approx H^{2d}_{crys}(X/W) \approx W \qquad [18, \text{II } 1.4]$$

lead (as for $\underline{H}^d(X, \nu(d))$) to a surjection $\underline{H}^d(X, \nu.(d)) \twoheadrightarrow \mathbf{Z}_p$. Thus the cohomology sequence of

$$0 \to \nu.(d) \xrightarrow{p^n} \nu.(d) \to \nu_n(d) \to 0$$

gives an isomorphism $\eta_n : \underline{H}^d(X, \nu_n(d)) \overset{\approx}{\to} \mathbf{Z}/p^n\mathbf{Z}$. In the next section we shall define the class of a point in $\underline{H}^d(X, \nu_n(d))$, and we normalize η_n so that it maps this class to 1. Now

$$\eta_n : \underline{H}^d(X, \nu_n(d)) \overset{\approx}{\to} \mathbf{Z}/p^n\mathbf{Z}$$

together with

$$\omega, \omega' \to \omega \wedge \omega' : \nu_n(r) \times \nu_n(d-r) \to \nu_n(d)$$

define a morphism

$$\underline{H}^{\cdot}(X, \nu_n(r)) \to \underline{H}^{\cdot}(X, \nu_n(d-r))^t[-d].$$

THEOREM 1.11. *The morphism*

$$\underline{H}^{\cdot}(X, \nu_n(r)) \to \underline{H}^{\cdot}(X, \nu_n(d-r))^t[-d]$$

defined above is an isomorphism. In particular, there are canonical isomorphisms

$$U^i(X, \nu_n(r)) \xrightarrow{\approx} U^{d+1-i}(X, \nu_n(d-r))^v$$

$$D^i(X, \nu_n(r)) \xrightarrow{\approx} D^{d-i}(X, \nu_n(d-r))^*$$

where $U^i(X, \nu_n(r)) = \underline{H}^i(X, \nu_n(r))^0$ *and* $D^i(X, \nu_n(r)) = \underline{H}^i(X, \nu_n(r))/U^i(X, \nu_n(r))$.

Proof. For $n = 1$, the theorem is just (1.4), and the general case follows by induction using the cohomology sequence of (1.7.1).

COROLLARY 1.12. *When k is finite, there are canonical nondegenerate pairings of finite groups,*

$$H^i(X, \nu_n(r)) \times H^{d+1-i}(X, \nu_n(d-r)) \to H^{d+1}(X, \nu_n(d)) \approx \mathbf{Z}/p^n\mathbf{Z}.$$

Proof. This can be deduced either from (1.5) or (1.11).

We now introduce the notations

$$H^i(X, (\mathbf{Z}/p^n\mathbf{Z})(r)) = H^{i-r}(X, \nu_n(r))$$

$$H^i(X, \mathbf{Z}_p(r)) = H^{i-r}(X, \nu.(r))$$

$$H^i(X, \mathbf{Q}_p(r)) = H^i(X, \mathbf{Z}_p(r)) \otimes \mathbf{Q}_p$$

$$H^i(X, (\mathbf{Q}_p/\mathbf{Z}_p)(r)) = \varinjlim H^{i-r}(X, \nu_n(r)).$$

(Of course, $H^i(X, F) = 0$ for $i < 0$). There are the interpretations

$$H^i(X, \mathbf{Z}_p(0)) = H^i(X, \mathbf{Z}_p) \text{ (étale cohomology)}$$

$$H^i(X, \mathbf{Z}_p(1)) = \varprojlim H^i(X, \mu_{p^n}) \text{ (flat cohomology)},$$

but there is no reason to believe this sequence continues. The notation is introduced only as a convenience—as we shall see, $H^i(X, \mathbf{Z}_p(r))$ plays a role similar to the group $H^i(X, \mathbf{Z}_\ell(r))$, $\ell \neq p$.

For future reference, we list some consequences of (1.11), (1.12), and the Poincaré duality theorems for étale cohomology.

THEOREM 1.13. *Assume k to be algebraically closed.*

(a) *There is a canonical surjection*

$$\underline{H}^i(X, (\mathbf{Z}/\ell^n\mathbf{Z})(r)) \to \underline{H}^{2d-i}(X, (\mathbf{Z}/\ell^n\mathbf{Z})(d-r))^*$$

whose kernel is zero if $\ell \neq p$ and is $U^{i-r}(X, (\mathbf{Z}/\ell^n\mathbf{Z}))$ if $\ell = p$.

(b) *There is a canonical surjection*

$$H^i(X, \mathbf{Z}_\ell(r)) \to \operatorname{Hom}(H^{2d-i}(X, \mathbf{Z}_\ell(d-r)), \mathbf{Z}_\ell)$$

whose kernel is the torsion subgroup of $H^i(X, \mathbf{Z}_\ell(r))$.

THEOREM 1.14. *Assume k to be finite, and let ℓ be any prime number.*

(a) *There is a canonical nondegenerate pairing of finite groups*

$$H^i(X, (\mathbf{Z}/\ell^n\mathbf{Z})(r)) \times H^{2d+1-i}(X, (\mathbf{Z}/\ell^n\mathbf{Z})(d-r)) \to \mathbf{Z}/\ell^n\mathbf{Z}.$$

(b) *There is a canonical surjection*

$$H^i(X, \mathbf{Z}_\ell(r)) \to \operatorname{Hom}(H^{2d+1-i}(X, \mathbf{Z}_\ell(d-r)), \mathbf{Z}_\ell)$$

whose kernel is the torsion subgroup of $H^i(X, \mathbf{Z}_\ell(r))$.

For the rest of this section we investigate the relation between $H^i(X, \mathbf{Z}_p(r))$ and the crystalline cohomology group $H^i_{\mathrm{crys}}(X/W)$. Throughout, k will be algebraically closed.

PROPOSITION 1.15. *For all i and r there is an exact sequence*

$$0 \to H^i(X, \mathbf{Q}_p(r)) \to H^i_{\mathrm{crys}}(X/W) \otimes \mathbf{Q}_p \xrightarrow{F-p^r} H^i_{\mathrm{crys}}(X/W) \otimes \mathbf{Q}_p \to 0$$

Proof. The spectral sequence

$$H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}(X, W\Omega^\bullet_X) = H^{i+j}_{\mathrm{crys}}(X/W)$$

degenerates when tensored with $\mathbf{Q}_p$ ([18, II.3]). Thus

$$H^i(X, W\Omega^r_X) \otimes \mathbf{Q}_p \xrightarrow{\approx} H^{i+r}_{\mathrm{crys}}(X/W) \otimes \mathbf{Q}_p.$$

Moreover, the action of F on the left hand term corresponds to that of $p^{-r}F$ on the right. Thus, on tensoring (1.10.1) with $\mathbf{Q}_p$, we obtain the above sequence except for the zeros. The quotient of $H^i(X, W\Omega^r_X)$ by its p-power torsion submodule $H^i(X, W\Omega^r_X)_t$ is a finitely generated W-module [18, II 2.13], and this implies that $1 - F$ is surjective on $H^i(X, W\Omega^r_X)/H^i(X, W\Omega^r_X)_t$. Therefore (1.10.1) breaks up into short exact sequences when tensored with $\mathbf{Q}_p$.

The relation between the integral groups $H^i(X, \mathbf{Z}_p(r))$ and $H^i_{\text{crys}}(X/W)$ is more subtle.

Let $W\Omega^{\geq r}_X$ denote the naive upper truncation

$$0 \to W\Omega^r_X \to W\Omega^{r+1}_X \to \cdots$$

of $W\Omega^{\cdot}_X$. Because of the formula $dF = pFd$ ([18, I 2.19]), the action of F on $W\Omega^r_X$ extends to an action on $W\Omega^{\geq r}_X$:

$$\begin{array}{ccccc} W\Omega^r_X & \xrightarrow{d} & W\Omega^{r+1}_X & \longrightarrow & \cdots \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF} & & \\ W\Omega^r_X & \xrightarrow{d} & W\Omega^{r+1}_X & \longrightarrow & \cdots \end{array}$$

LEMMA 1.16. *There is an exact sequence*

$$0 \longrightarrow \nu.(r) \longrightarrow W\Omega^{\geq r}_X \xrightarrow{1-F} W\Omega^{\geq r}_X \longrightarrow 0$$

of complexes of pro-sheaves on X.

Proof. As $1 - p^jF$ is an automorphism of $W\Omega^i_X$ for all $j \geq 1$ ([18, I 3.30)], this follows immediately from (1.9).

PROPOSITION 1.17. *There is an exact sequence*

(1.17.1)

$$\cdots \longrightarrow H^i(X, \nu.(r)) \longrightarrow H^i(X, W\Omega^{\geq r}_X) \xrightarrow{1-F} H^i(X, W\Omega^{\geq r}_X) \longrightarrow \cdots$$

Proof. Same as (1.10).

Remark 1.18. It would be interesting to know exactly when the maps $1 - F$ in (1.17.1) are surjective. If $H^i(X, W\Omega^{\geq r}_X)$ is complete for the V-adic topology and $H^i(X, W\Omega^{\geq r}_X)/VH^i(X, W\Omega^{\geq r}_X)$ is finitely generated as a W-module, then $1 - F$ will be surjective. Unfortunately, these conditions do not necessarily hold—see the example [18, p. 652] for which, nevertheless, $1 - F$ is surjective.[2]

Similar remarks apply to the maps $1 - F$ in (1.10.1).

There is a commutative diagram

$$\begin{array}{ccccccc} & & H^i(X, \mathbf{Z}_p(r)) & \to & H^{i-r}(X, W\Omega_X^{\geq r}) & \xrightarrow{1-F} & H^{i-r}(X, W\Omega_X^{\geq r}) \\ (1.18.1) & & & & \downarrow j & & \downarrow p^r j \\ 0 & \to & H^i_{\text{crys}}(X/W)^{F=p^r} & \to & H^i_{\text{crys}}(X/W) & \xrightarrow{p^r-F} & H^i_{\text{crys}}(X/W) \end{array}$$

in which j is defined by $W\Omega_X^{\geq r}[-r] \to W\Omega_X^{\cdot}$.

PROPOSITION 1.19. *The maps* $\phi^i: H^i(X, \mathbf{Z}_p(r)) \to H^i_{\text{crys}}(X/W)^{F=p^r}$ *defined by the above diagram are surjective for all* $i \leq r + 1$ *if the following conditions hold*:

(a) $H^i_{\text{crys}}(X/W)$ *is torsion-free for all* i;

(b) *the map* $H^i(X, \Omega^r_{X/k,cl}) \to H^i(X, \Omega^r_X)$ *induced by the inclusion* $\Omega^r_{X/k,cl} \hookrightarrow \Omega^r_X$ *is surjective for all* i;

(c) *the Hodge to deRham spectral sequence* $H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{dR}(X)$ *degenerates at* E_1.

Proof. Let $\Omega^{\leq r}_{X/k}$ denote the naive lower truncation

$$\mathcal{O}_{X/k} \to \cdots \to \Omega^r_{X/k} \to 0 \to \cdots$$

of $\Omega^{\cdot}_{X/k}$. The obvious map $\Omega^{\cdot}_{X/k} \to \Omega^{\leq r}_{X/k}$ induces maps $H^i_{dR}(X) \to H^i(X, \Omega^{\leq r}_{X/k})$.

LEMMA 1.20. *For a fixed* i *and* r *with* $i \leq r + 1$, *assume*

(a) $H^{i-r-1}(X, \Omega^r_{X/k,cl}) \to H^{i-r-1}(X, \Omega^r_{X/k})$ *is surjective*;

(b) $H^{i-1}_{dR}(X) \to H^{i-1}(X, \Omega^{\leq r-1}_{X/k})$ *is surjective.*

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[2] L. Illusie and M. Raynaud have shown (Les suites spectrales associeés au complexe de deRham-Witt, *Publ. Math. I.H.E.S.*, **57** (1983), 73-212) that, for X a complete nonsingular variety over an algebraically closed field, $1 - F: H^i(X, W\Omega^j_X) \to H^i(X, W\Omega^j_X)$ is surjective for all i and j. It follows that $1 - F: H^i(X, W\Omega_X^{\geq r}) \to H^i(X, W\Omega_X^{\geq r})$ is also surjective under the same hypotheses.

Then the map $H^i(X, (\mathbf{Z}/p\mathbf{Z})(r)) \to H^i_{dR}(X)$ *induced by* $\nu(r) \hookrightarrow \Omega^r_{X/k,cl} \to \Omega^{\cdot}_{X/k}$ *is injective.*

Proof. There is an exact sequence $0 \to \Omega^r_{cl}[-r] \to \Omega^{\cdot} \to C^{\cdot} \to 0$ in which

$$C^{\cdot} = (\mathcal{O}_X \to \Omega^1 \to \cdots \to \Omega^{r-1} \xrightarrow{0} \Omega^r/\Omega^r_{cl} \hookrightarrow \Omega^{r+1} \to \cdots).$$

Clearly, $H^j(X, C^{\cdot}) = H^j(X, \Omega^{\leq r-1})$ for $j \leq r$. From the sequence

$$H^{i-1}_{dR}(X) \to H^{i-1}(X, C^{\cdot}) \to H^{i-r}(X, \Omega^r_{cl}) \to H^i_{dR}(X)$$

and condition (b), we see that $H^{i-r}(X, \Omega^r_{cl}) \to H^i_{dR}(X)$ is injective. Now from the exact sequence (arising from (1.1))

$$H^{i-r-1}(X, \Omega^r_{cl}) \to H^{i-r-1}(X, \Omega^r) \to H^{i-r}(X, \nu(r)) \to H^{i-r}(X, \Omega^r_{cl})$$

and (a), we see that $H^{i-r}(X, \nu(r)) \to H^{i-r}(X, \Omega^r_{cl})$ is injective. Since $H^{i-r}(X, \nu(r)) = H^i(X, (\mathbf{Z}/p\mathbf{Z})(r))$, this proves the lemma.

LEMMA 1.21. *In addition to the hypotheses of* (1.20), *assume that* $H^i_{\text{crys}}(X/W)$ *is torsion-free. Then* $\phi^i : H^i(X, \mathbf{Z}_p(r)) \to H^i_{\text{crys}}(X/W)^{F=p^r}$ *is surjective.*

Proof. Let $\gamma \in H^i_{\text{crys}}(X, W)^{F=p^r}$; then (1.15) shows that $p^n\gamma \in \operatorname{Im}(\phi^i)$ for some n. Let n_0 be the least such n, and suppose $n_0 > 0$. Let $\gamma_0 \in H^i(X, \mathbf{Z}_p(r))$ be such that $\phi^i(\gamma_0) = p^n\gamma$. Clearly $\gamma_0 \notin pH^i(X, \mathbf{Z}_p(r))$ because if $\gamma_0 = p\gamma_1$, $\phi^i(\gamma_1) = p^{n_0-1}\gamma$. Now the diagram

$$\begin{array}{ccccc} H^i(X, \mathbf{Z}_p(r)) & \xrightarrow{p} & H^i(X, \mathbf{Z}_p(r)) & \longrightarrow & H^i(X, (\mathbf{Z}/p\mathbf{Z})(r)) \\ & & \downarrow & & \downarrow \cap \\ & & H^i_{\text{crys}}(X/W) & \longrightarrow & H^i_{dR}(X) \end{array}$$

provides a contradiction.

To complete the proof of the proposition, it remains to note that its conditions imply that the conditions of (1.21) hold for all i.

Example 1.22. Nygaard has shown ([18, II 3.11]) that $H^1(X, W\Omega^{\cdot}_X) \to H^1(X, W\mathcal{O}_X)$ is surjective. Therefore, for $i = 2$, $r = 1$, the map $j : H^1(X, W\Omega^{\geqq 1}_X) \to H^2(X, W\Omega^{\cdot}_X)$ of (1.18.1) is injective. As

$H^0(X, W\Omega_X^{\geq 1}) \to H^1(X, W\Omega_X^{\cdot})$ is always injective, $H^0(X, W\Omega_X^{\geq 1})$ is a finitely generated W-module and so (1.17) shows that $H^2(X, \mathbf{Z}_p(1)) \to H^1(X, W\Omega_X^{\geq 1})$ is injective. When $H^2_{\text{crys}}(X/W)$ is torsion-free, $H^1(X, W\mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is surjective [29, Section 2], and so the surjectivity of $H^1(X, W\Omega_X^{\cdot}) \to H^1(X, W\mathcal{O}_X)$ implies that of $H^1_{dR}(X) \to H^1(X, \mathcal{O}_X)$. Thus (1.21) in the case $i = 2$, $r = 1$ implies that $\phi^1: H^2(X, \mathbf{Z}_p(1)) \to H^2_{\text{crys}}(X/W)^{F=p}$ is an isomorphism whenever $H^2_{\text{crys}}(X/W)$ is torsion-free and $H^0(X, \Omega^1_{X/k,cl}) = H^0(X, \Omega^1_{X/k})$ (cf. [18, II 5.14]).

2. Cohomology of the sheaves $\nu_n(r)$: the cycle map. In this section X will be smooth and quasi-projective (but not necessarily complete) over a perfect field k.

The map $d\log: \mathcal{O}_X^{\times} \to \nu_n(1)$ induces a map $\text{Pic}(X) \to H^1(X, \nu_n(1))$ which, in the limit, becomes a cycle map $c^1: \text{Pic}(X) \to H^1(X, \nu.(1)) = H^2(X, \mathbf{Z}_p(1))$. Let $CH^r(X)$ denote the Chow group of algebraic cycles of codimension r on X modulo rational equivalence. We shall extend c^1 to a family of cycle maps $c^r: CH^r(X) \to H^r(X, \nu.(r)) = H^{2r}(X, \mathbf{Q}_p(r))$ that is compatible with intersection and cup products over the algebraic closure of k.

If $i: Z \hookrightarrow X$ is a closed immersion and F is a sheaf on X, $\underline{H}^r_Z(X, F)$ will denote $R^r i^! F$ where $i^! F$ is the subsheaf of F of sections with support on Z (see [27, p. 76, VI.5]). The crucial first step in the construction of the cycle maps is the following purity result.

PROPOSITION 2.1. *Let Z be a smooth closed subscheme of X of codimension r, and let $s \geq r$. Then $\underline{H}^i_Z(X, \nu_n(s)) = 0$ for $i \neq r, r+1$ and there is a canonical isomorphism*

$$\phi_{Z/X}: \nu_n(s-r)_Z \to \underline{H}^r_Z(X, \nu_n(s)).$$

Proof. Since Z is a local complete intersection in X, for any locally free sheaf F of $\mathcal{O}_X$-modules on X, $\underline{H}^i_Z(X, F) = 0$, $i \neq r$, ([16, III 8.7]). The sheaf $\Omega^r_{X/k,cl}$ is locally free when regarded as a sheaf on $X^{(p)} = (X, \mathcal{O}_X^p)$ and so it follows from (1.1) that $\underline{H}^i_Z(X, \nu(r)) = 0$ for $i \neq r, r+1$. An induction argument using (1.7) extends this result to all the sheaves $\nu_n(r)$.

Let $Y \supset Z$ be smooth of codimension t in X. For any locally free sheaf F on X, the spectral sequence $\underline{H}^i_Z(Y, \underline{H}^j(X, F)) \Rightarrow \underline{H}^{i+j}_Z(X, F)$ reduces to a single isomorphism $\underline{H}^{r-t}_Z(Y, \underline{H}^t_Y(X, F)) \xrightarrow{\approx} \underline{H}^r_Z(X, F)$.

In [16, III.8] there is defined a family of homomorphisms $\phi_{Z/X}:\Omega^{s-r}_{Z/k} \to \underline{H}^r_Z(X, \Omega^s_{X/k})$ with the following properties:

(a) $\phi_{Z/X}$ commutes with d.

(b) If Y is as above, then $\underline{H}^{r-t}_Z(\phi_{Y/X}) \circ \phi_{Z/Y} = \phi_{Z/X}$.

(c) Assume $r = 1$, and let $A = \mathcal{O}_{X,z}$ for some closed point $z \in Z$. Let $f = 0, f \in A$, be a local equation for Z near z. Then the stalk (in the Zariski sense) of $\underline{H}^1_Z(X, \Omega^s)$ at z is $\Omega^s_{A[f^{-1}]}/\Omega^s_A$, and $(\phi_{Z/X})_z$ is the map

$$\omega \mapsto \omega \wedge df/f:\Omega^{s-1}_{A/(f)} \to \Omega^s_{A[f^{-1}]}/\Omega^s_A.$$

We can regard the $\phi_{Z/X}$ as maps of sheaves for the étale topology on Z. From (a) it follows that $\phi_{Z/X}$ defines a map $\Omega^{s-r}_{Z/k,cl} \to \underline{H}^r_Z(X, \Omega^s_{X/k,cl})$. Moreover, the following diagram commutes:

$$\begin{array}{ccccccc} 0 \to \nu(s-r)_Z \to & \Omega^{s-r}_{Z/k,cl} & \xrightarrow{C-1} & \Omega^{s-r}_{Z/k} \to 0 \\ & \downarrow{\scriptstyle \phi_{Z/X}} & & \downarrow{\scriptstyle \phi_{Z/X}} \\ 0 \to \underline{H}^r_Z(X, \nu(s)) \to & \underline{H}^r_Z(X, \Omega^s_{X/k,cl}) & \xrightarrow{C-1} & \underline{H}^r_Z(X, \Omega^s_{X/k}). \end{array}$$

For $r = 1$, this follows from (c), and (b) shows that if it is true for $\phi_{Z/Y}$ and $\phi_{Y/X}$, then it is true for $\phi_{Z/X}$; but, locally, we can always find a sequence of smooth schemes

$$Z \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = X$$

and so this implies the general case. There is therefore a family of maps $\phi_{Z/X}:\nu(s-r)_Z \to \underline{H}^r_Z(X, \nu(s))$.

To show that these maps are isomorphisms is a local problem. Therefore an argument similar to the one just used allows us to assume that $r = 1$. Moreover, we can suppose that we are in the situation of (c) above, with A a strictly local ring. The diagram

$$\text{(2.1.1)} \quad \begin{array}{ccccccccc} 0 \to & \Omega^s_{A,cl} & \to & \Omega^s_{A[f^{-1}],cl} & \to & \underline{H}^1_Z(X, \Omega^s_{cl})_{\bar{z}} & \to 0 \\ & \downarrow{\scriptstyle C-1} & & \downarrow{\scriptstyle C-1} & & \downarrow{\scriptstyle C-1} & \\ 0 \to & \Omega^s_A & \to & \Omega^s_{A[f^{-1}]} & \to & \underline{H}^1_Z(X, \Omega^s)_{\bar{z}} & \to 0 \end{array}$$

and the snake lemma give an exact sequence

$$0 \to \nu(s)_A \to \nu(s)_{A[f^{-1}]} \to \underline{H}^1_Z(X, \nu(s))_{\bar{z}} \to 0.$$

We have therefore to show that

$$\omega \mapsto (\omega \wedge df/f): \nu(s-1)_B \to \nu(s)_{A[f^{-1}]}/\nu(s)_A,$$

where $B = A/(f)$, is an isomorphism. Clearly it is injective. Let $\hat{A}$ be the f-adic completion of A. Then $B = \hat{A}/(f)$, and if we can show that the map is surjective for $\hat{A}$, it will be surjective for A (because, if $\eta \in \nu(s)_{A[f^{-1}]}$ is equal to $\omega \wedge df/f + \eta'$ with $\eta' \in \Omega^s_{\hat{A}}$, then in fact η' will belong to $\Omega^s_{\hat{A}} \cap \Omega^s_{A[f^{-1}]} = \Omega^s_A$). Thus we can suppose that A is complete, and therefore ([16, p. 154]) that $A = B[[f]]$. An $\eta \in \Omega^s_{A[f^{-1}]}$ can be written uniquely as a finite sum

$$\eta = \frac{\alpha_1}{f} + \frac{\alpha_2}{f^2} + \cdots + \left(\frac{\beta_1}{f} + \frac{\beta_2}{f^2} + \cdots\right) \wedge df (\text{mod } \Omega^s_A)$$

with each $\alpha_i \in \Omega^s_B$ and each $\beta_i \in \Omega^{s-1}_B$. If $d\eta = 0$, then

$$\eta = d\theta + \sum_{i \geq 1} \frac{\alpha_{pi}}{f^{pi}} + \sum_{i \geq 0} \frac{\beta_{pi+1}}{f^{pi+1}} \wedge df (\text{mod } \Omega^s_A)$$

where

$$\theta = \pm \sum_{\substack{i \geq 1 \\ (p,i)=1}} \frac{\beta_{i+1}}{if^i} \quad \text{and} \quad d\alpha_{pi} = 0 = d\beta_{pi+1}.$$

Clearly

$$C(\eta) = \sum_{i \geq 1} \frac{C(\alpha_{pi})}{f^i} + \sum_{i \geq 0} \frac{C(\beta_{pi+1})}{f^{i+1}} \wedge df (\text{mod } \Omega^s_A),$$

and so $C(\eta) = \eta$ implies that $\eta = \beta_1 \wedge df/f$ where $d\beta_1 = 0$ and $C(\beta_1) = \beta_1$, i.e., that η is in the image of $\phi_{Z/X}$.

This completes the proof in the case that $n = 1$, and the same argument works without essential change in the general case.

COROLLARY 2.2. *With the notations of the proposition, $H^i_Z(X, \nu_n(s)) = 0$ for $i < r$, the map $H^0(Z, \nu_n(s-r)) \to H^r_Z(X, \nu_n(s))$ defined by $\phi_{Z/X}$ is an isomorphism, and there is an exact sequence*

$$0 \to H^1(Z, \nu_n(s-r)) \to H^1_Z(X, \nu_n(s)) \to H^0(Z, \underline{H}^{r+1}_Z(X, \nu_n(s))) \to \cdots$$

Proof. This follows from (2.1) and the spectral sequence

$$H^i(Z, \underline{H}^j_Z(X, \nu_n(s))) \Rightarrow H^{i+j}_Z(X, \nu_n(s)).$$

Remark 2.3. If in the statement of the proposition one takes $s < r$, then the conclusion becomes that $H^i_Z(X, \nu(s)) = 0$ for $i \leq r$. The crucial case in proving this is $r = 1$, $s = 0$, $i = 1$. The sequence of kernels in (2.1.1) is then $0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$ and so $H^1_Z(X, \nu(0))_{\bar{z}} = 0$.

Remark 2.4. Unfortunately, $\underline{H}^{r+1}_Z(X, \nu_n(s))$ is not usually zero. The diagram (2.1.1) shows that, in the case that $r = 1$, the stalk of $\underline{H}^1_Z(X, \nu(s))$ is

$$\underline{H}^1_Z(X, \nu(s))_{\bar{z}} = \operatorname{Coker}(C - 1 : \Omega^s_{A[f^{-1}],cl} \to \Omega^s_{A[f^{-1}]}).$$

The calculations in the proof of (2.1) show, for example, that α/f^p, $\alpha \in \Omega^s_B$, is not in the image of $C - 1$ if $d\alpha \neq 0$.

Let $\Lambda = \mathbf{Z}/p^n\mathbf{Z}$, and write $H^i_Z(X, \Lambda(r)) = H^{i-r}_Z(X, \nu_n(r))$. Then (2.2) and (2.3) show that, if Z is smooth of codimension r in X, $H^i_Z(X, \Lambda(s)) = 0$ for $i < 2r$, and there is a canonical isomorphism $\Lambda \to H^{2r}_Z(X, \Lambda(r))$.

We can now copy some of the material in [27, VI.9, VI.10].

LEMMA 2.5. *For any reduced closed subscheme Z of codimension r in X, $H^i_Z(X, \Lambda(s)) = 0$ for $i < 2r$ and all s.*

Proof. We know this already when Z is smooth, and the general case is proved by descending induction on r. If $r = \dim(X)$, then Z is smooth. In general, we can choose an open subset U of X such that $U \cap Z$ is smooth, U is dense in every irreducible component of Z of codimension r, and $X \supset U \supset X - Z$. The exact cohomology sequence of this last triple is

$$\cdots \to H^i_{X-U}(X, \Lambda(s)) \to H^i_Z(X, \Lambda(s)) \to H^i_{U\cap Z}(U, \Lambda(s)) \to \cdots$$

and the induction hypothesis shows that $H^i_{X-U}(X, \Lambda(s)) = 0$ for $i < 2(r+1)$. Thus the lemma is now obvious.

Let Z be an irreducible closed subvariety of X of codimension r, and choose an open subset as in the above proof. There are maps

$$\Lambda \xrightarrow{\approx} H^{2r}_{U \cap Z}(U, \Lambda(r)) \xleftarrow{\approx} H^{2r}_Z(X, \Lambda(r)) \to H^{2r}(X, \Lambda(r))$$

and we define the class, $c^r(Z)$, of Z to be the image of the element 1 of Λ in $H^{2r}(X, \Lambda(r))$. Clearly $c^r(Z)$ is independent of the choice of U.

Let $C^*(X)$ denote the graded group $\oplus\, C^r(X)$ where $C^r(X)$ is the group of algebraic r-cycles on X, and let $H^*(X, \Lambda)$, or simply $H^*(X)$, denote the cohomology ring $\oplus\, H^{2r}(X, \Lambda(r))$. The maps c^r extend by linearity to a homomorphism of graded groups (doubling degrees)

$$c^*: CH^*(X) \to H^*(X, \Lambda).$$

Note that in determining $c^r(Z)$ when Z is an r-cycle, we are allowed (because of (2.5)) to remove any closed subvariety of X of codimension $> r$.

LEMMA 2.6. *Let $\pi: Y \to X$ be a map of smooth quasi-projective varieties and let Z be an algebraic cycle on X. If, for every prime cycle Z' occurring in Z, $Y \times_X Z'$ is integral, then the cycle $\pi^* Z$ is defined and $c(\pi^* Z) = \pi^* c(Z)$.*

Proof. We can assume Z to be prime. Then the condition implies that after X and Y have been replaced by open subvarieties with complements of large codimension, both Z and $Y \times_X Z$ will be smooth, and $\pi^* Z = Y \times Z$. From the definition of $\phi_{Z/X}$, one sees that

$$\begin{array}{ccc} \pi^*(\phi_{Z/X}): \pi^* \nu_n(0)_Z & \longrightarrow & \pi^*(\underline{H}^r_Z(X, \nu_n(r))) \\ \downarrow & & \downarrow \\ \phi_{Y \times Z/Y}: \nu_n(0)_{Y \times Z} & \longrightarrow & \underline{H}^r_{Y \times Z}(Y, \nu_n(r))) \end{array}$$

commutes (the vertical maps are induced by the canonical maps $\pi^* W_n \mathcal{O}_Z \to W_n \mathcal{O}_{Y \times Z}$, $\pi^* W_n \Omega^r_X \to W_n \Omega^r_Y$). The lemma follows immediately from this.

For the rest of this section, we assume k to be algebraically closed. Let $i: Z \hookrightarrow X$ be a smooth closed subvariety of X of codimension r. There is a homomorphism i_*, which will be called the Gysin homomorphism

$$z \mapsto z \cup c^r(Z): H^{2s}(Z, \Lambda(s)) \to H^{2(r+s)}(X, \Lambda(r + s)).$$

LEMMA 2.7. (a) *If* $Z \overset{i_1}{\hookrightarrow} Y \overset{i_2}{\hookrightarrow} X$ *are closed immersions and* Z *and* Y are smooth, *then* $(i_2 \circ i_1)_* = i_{2*} \circ i_{1*}$.

(b) *If* $i: Z \hookrightarrow X$ *is a closed immersion and* Z *is smooth, then* $i_*(i^*(x) \cup z) = x \cup i_* z$ *for any* $x \in H^*(X, \Lambda)$ *and* $z \in H^*(Z, \Lambda)$.

Proof. (a) This follows from property (b) of the maps $\phi_{Z/X}: \Omega^{s-r}_{Z/k} \to \underline{H}^r_Z(X, \Omega^s_{X/k})$ recalled in the proof of (2.1).

(b) The proof is the same as in the case of the étale groups; see [27, VI 6.5a].

PROPOSITION 2.8. *Let* E *be a vector bundle of rank* m *over* X, *and let* $P \to X$ *be the associated projective bundle. Let* $\xi \in H^2(P, \Lambda(1))$ *be the image of the canonical line bundle* $\mathcal{O}_P(1)$ *on* P *under the map* $\mathrm{Pic}(P) \to H^2(P, \Lambda(1))$ *defined by* $d\log$. *Then* $H^*(P)$ *is a free module over* $H^*(X)$ *with basis* $1, \xi, \ldots, \xi^{m-1}$

Proof. The standard argument using Mayer-Vietoris sequences reduces the question to the case that X is affine and E is trivial, and there it can be proved by a direct calculation using the explicit description of the cohomology of projective space over a ring [17, III.5].

The proposition shows that there is a unique sequence of elements $c_0(E), c_1(E), \ldots$, with $c_r(E) \in H^{2r}(X, \Lambda(r))$ such that

$$\Sigma\, c_r(E)\xi^{m-r} = 0, \qquad c_0(E) = 1, \qquad c_r(E) = 0 \quad \text{for} \quad r > m.$$

The element $c_r(E)$ is called the r^{th} Chern class of E.

THEOREM 2.9. (a) *If* $\pi: Y \to X$ *is a morphism of smooth quasi-projective varieties and* E *is a vector bundle on* E, *then* $c_r(\pi^{-1}(E)) = \pi^*(c_r(E))$ *for all* r.

(b) *If* E *is a line bundle on* X, *then* $c_1(E)$ *is the image of* E *under the map* $\mathrm{Pic}(X) \to H^2(X, \Lambda(1))$ *defined by* $d\log$.

(c) *If* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of vector bundles on* X, *then* $c_r(E) = \Sigma_{r_1+r_2=r}\, c_{r_1}(E')c_{r_2}(E'')$.

These three properties characterize the Chern classes uniquely.

Proof. The preceding results show that the hypotheses of [14, Theorem 1] are fulfilled.

The cycle maps $c^*: C^*(X) \to H^*(X, \Lambda)$, $\Lambda = \mathbf{Z}/p^n\mathbf{Z}$, define in the limit a cycle map $C^*(X) \to H^*(X, \mathbf{Z}_p) \overset{df}{=} \oplus H^{2r}(X, \mathbf{Z}_p(r))$, and hence a cycle map $C^*(X) \to H^*(X, \mathbf{Q}_p) = \oplus H^{2r}(X, \mathbf{Q}_p(r))$.

THEOREM 2.10. *The cycle map $c^*: C^*(X) \to H^*(X, \mathbf{Q}_p)$ factors through the Chow ring $CH^*(X)$, and is a homomorphism of rings; it is functorial for maps of smooth quasi-projective varieties.*

Proof. Formula 16 of [14] allows the cycle map to be expressed in terms of the Chern class maps, and so (2.10) follows from (2.9). (See also [27, VI.10], but note that the proof there of 10.6 is incorrect.)

Let $H^*_{\text{crys}}(X/W)$ denote the crystalline cohomology ring.

COROLLARY 2.11. *There is a canonical homomorphism $CH^*(X) \to H^*_{\text{crys}}(X) \otimes K$ that is functorial for maps of smooth quasi-projective varieties.*

Proof. As we saw in Section 1, there is a canonical map

$$H^*(X, \mathbf{Q}_p) \to H^*_{\text{crys}}(X/W) \otimes K.$$

Remark 2.12. I conjecture that there is an exact sequence

$$0 \to \nu_n(r) \to \bigoplus_{\dim(x)=d} i_* \nu_n(r-1)_x \to \bigoplus_{\dim(x)=d-1} i_* \nu_n(r-1)_x \to \cdots$$

of sheaves on X_{et}, generalizing that in [28, 4.3], and analogous to that of Bloch and Ogus [7]. Such a sequence would define a map $CH^r(X)/p^n CH^r(X) \to H^{2r}(X, \nu_n(r))$ whose composite with $C^r(X) \to CH^r(X)$ is c^r.

Remark 2.13. For $r \leq 2$, the $d\log$ map $K_r \mathcal{O}_X \to \nu_n(r)$ defines a cycle map $CH^r(X) = H^r(X_{zar}, K_r\mathcal{O}_X) \to H^r(X_{Zar}, \nu_n(r)) \to H^r(X_{et}, \nu_n(r))$. This agrees with that defined above and is automatically compatible with intersection and cup products (see [13]).

3. The dimension of $\underline{H}^i(X, \mathbf{Z}_p(r))$. Let $s^i(r)$ denote the dimension of the perfect pro-algebraic group scheme $\underline{H}^i(X, \mathbf{Z}_p(r)) = \underline{H}^{i-r}(X, \nu.(r))$. Thus $s^i(r) = \infty$ unless $\underline{H}^i(X, \mathbf{Z}_p(r))^0$ is algebraic, in which case $s^i(r)$ is the number of copies of $\mathbf{G}_a^{pf}$ occurring as quotients in a composition series for $\underline{H}^i(X, \mathbf{Z}_p(r))^0$. We shall show that $s^i(r)$ is finite for all i and r and shall calculate $\Sigma(-1)^i s^i(r)$ in terms of the deRham-Witt complex.

Let $W\Omega^{\leq r}$ denote the naive lower truncation

$$W\mathcal{O}_X \to W\Omega^1_X \to \cdots \to W\Omega^r_X \to 0$$

of $W\Omega_X^{\cdot}$. The formula $Vd = pdV$ ([18, I2.19]) shows that there is an endomorphism V of $W\Omega_X^{\leqq r}$:

$$\begin{array}{ccccccc} W\mathcal{O}_X & \longrightarrow & W\Omega_X^1 & \longrightarrow & \cdots & \longrightarrow & W\Omega_X^r \\ \downarrow{\scriptstyle p^r V} & & \downarrow{\scriptstyle p^{r-1} V} & & & & \downarrow{\scriptstyle V} \\ W\mathcal{O}_X & \longrightarrow & W\Omega_X^1 & \longrightarrow & \cdots & \longrightarrow & W\Omega_X^r. \end{array}$$

The noncommutative ring $W[[V]]$ (in which $aV = Va^\sigma$, $a \in W$) therefore acts on $H^i(X, W\Omega_X^{\leqq r})$, and it is shown in [18, II 2.11] that $H^i(X, W\Omega_X^{\leqq r})$ is a finitely generated $W[[V]]$-module for all i and r. More precisely, there is an exact sequence

$$0 \to H^i(X, W\Omega_X^{\leqq r})_t \to H^i(X, W\Omega_X^{\leqq r}) \to M \to 0$$

of $W[[V]]$-modules in which M is a free W-module of finite rank and $H^i(X, W\Omega_X^{\leqq r})_t$ is a finitely generated $(W/p^n W)[[V]]$-module for some n. Let $d^i(r)$ be the length of $H^i(X, W\Omega_X^{\leqq r})_t \otimes_{W[[V]]} W((V))$ as a $W((V))$-module. Thus $d^i(r)$ is finite and is equal to the number of copies of $k[[V]]$ occurring as quotients in a composition series for $H^i(X, W\Omega_X^{\leqq r})$.

PROPOSITION 3.1. *With the above notations*, $s^i(r) < \infty$ *for all* i *and* r, *and* $\Sigma(-1)^{i+1}s^i(r+1) = \Sigma(-1)^i d^i(r)$.

Proof. Let $W\Omega_X^{\geqq r+1}$ denote the naive upper truncation of $W\Omega_X^{\cdot}$. The exact sequence

$$0 \to W\Omega_X^{\geqq r+1}[-r-1] \to W\Omega_X^{\cdot} \to W\Omega_X^{\leqq r} \to 0$$

of pro-complexes gives rise to an exact sequence

$$\to H^{i+r}(X, W\Omega_X^{\cdot}) \to H^{i+r}(X, W\Omega_X^{\leqq r}) \xrightarrow{d} H^i(X, W\Omega_X^{\geqq r+1}) \to H^{i+r+1}(X, W\Omega_X^{\cdot}) \to$$

in which $H^j(X, W\Omega_X^{\cdot})$ is isomorphic to the crystalline cohomology group $H^j_{\text{crys}}(X/W)$ ([18, II 1.4]). In particular, the kernel and cokernel of d are finitely generated W-modules. It follows that $H^i(X, W\Omega_X^{\geqq r+1})$ is the extension of a free W-module of finite rank by a submodule $H^i(X, W\Omega_X^{\geqq r+1})_t$

that is killed by a power of p. The formula $dF = pFd$ ([18, I 2.19]) shows that there is an endomorphism F of $W\Omega_X^{\geqq r+1}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W\Omega_X^{r+1} & \longrightarrow & W\Omega_X^{r+2} & \longrightarrow & \cdots \\ & & \downarrow F & & \downarrow pF & & \\ 0 & \longrightarrow & W\Omega_X^{r+1} & \longrightarrow & W\Omega_X^{r+2} & \longrightarrow & \cdots \end{array}$$

The following diagram

$$\text{(3.1.1)}\qquad \begin{array}{ccccc} H^{i+r}(X, W\Omega_X^{\leqq r})_t & \xrightarrow{V} & H^{i+r}(X, W\Omega_X^{\leqq r})_t & \xrightarrow{d} & H^i(X, W\Omega_X^{\geqq r+1})_t \\ \downarrow V-1 & & & & \downarrow 1-F \\ H^{i+r}(X, W\Omega_X^{\leqq r})_t & & \xrightarrow{\qquad d \qquad} & & H^i(X, W\Omega_X^{\geqq r+1})_t \end{array}$$

commutes because of the formula $FdV = d$ ([18, I2.19]). It can be regarded, in a canonical way, as a diagram of perfect pro-algebraic group schemes and morphisms of such schemes. For α a morphism of perfect pro-algebraic group schemes, let

$$\chi(\alpha) = \dim(\mathrm{Ker}(\alpha)) - \dim(\mathrm{Coker}(\alpha))$$

when both these numbers are finite. If α is an endomorphism of M, we also write $\chi(\alpha \mid M)$ for $\chi(\alpha)$. The proof will proceed by comparing the χ's of the various maps in (3.1.1), but first we need an easy lemma.

LEMMA 3.2. (a) *Let $\gamma = \beta \circ \alpha$; if any two of $\chi(\alpha)$, $\chi(\beta)$, $\chi(\gamma)$ are defined, then so also is the third, and*

$$\chi(\gamma) = \chi(\alpha) + \chi(\beta).$$

(b) *Consider $\alpha: M_1 \to M_2$; if M_1 and M_2 have finite dimension then* $\chi(\alpha) = \dim(M_1) - \dim(M_2)$.

(c) *Let α be an endomorphism of M, and let $M' \subset M$ be such that $\alpha(M') \subset M'$; then*

$$\chi(\alpha \mid M) = \chi(\alpha \mid M') + \chi(\alpha \mid M/M').$$

Proof. (a) Use the exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\gamma) \to \mathrm{Ker}(\beta) \to \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\gamma) \to \mathrm{Coker}(\beta) \to 0.$$

(b) Obvious.

(c) Apply the snake lemma.

The map $V - 1: k[[V]] \to k[[V]]$ is an isomorphism of perfect pro-algebraic groups, and so $\chi(V - 1 | k[[V]]) = 0$. On the other hand, if M is a finitely generated torsion W-module, then (3.2b) shows that $\chi(V - 1 | M) = 0$. As $H^{i+r}(X, W\Omega_X^{\leq r})_t$ has a finite filtration whose quotients are isomorphic to $k[[V]]$ or are finitely generated over W, (3.2c) shows that $\chi(V - 1 | H^{i+r}(X, W\Omega_X^{\leq r})_t) = 0$.

Clearly $\chi(V | k[[V]]) = -1$, and so a similar argument to the above shows that $\chi(V | H^{i+r}(X, W\Omega_X^{\leq r})_t) = -d^{i+r}(r)$.

The remarks made at the start of the proof show that $\chi(d)$ is defined. Therefore, on applying (3.2a), we find that $\chi(1 - F | H^i(X, W\Omega_X^{\geq r+1})_t)$ is defined and equal to $d^{i+r}(r)$. It remains to relate $\chi(1 - F)$ to $s^i(r)$.

As we saw in (1.17), there is an exact sequence

$$\cdots \to \underline{H}^i(X, \nu.(r+1)) \to \underline{H}^i(X, W\Omega_X^{\geq r+1}) \xrightarrow{1-F} \underline{H}^i(X, W\Omega_X^{\geq r+1}) \to \cdots.$$

Recall that $\underline{H}^i(X, W\Omega_X^{\geq r+1})$ is an extension of a free W-module M of finite rank by $\underline{H}^i(X, W\Omega_X^{\geq r+1})_t$. In fact

$$\text{(3.2.1)} \qquad \chi(1 - F | \underline{H}^i(X, W\Omega_X^{\geq r+1})) = \chi(1 - F | \underline{H}^i(X, W\Omega_X^{\geq r+1})_t).$$

For this to be true, $\chi(1 - F | M)$ must be zero. Since we are computing the kernel and cokernel of $1 - F$ as a map of group schemes, we can assume that k is algebraically closed. Then there is a submodule M_0 of M admitting a W-basis of elements fixed by F and such that F is topologically nilpotent on M/M_0 (see for example, [11, IV]). On M/M_0, $1 - F$ has an inverse ΣF^n, and so $\chi(1 - F | M/M_0) = 0$. On M_0, $1 - F$ is surjective and has as kernel a free $\mathbf{Z}_p$-module of finite rank, and so $\chi(1 - F | M_0) = 0$. Thus (3.2.1) holds. From the preceding sequence we can extract exact sequences

$$0 \to \mathrm{Cok}(1 - F^{i-1}) \to \underline{H}^i(X, \nu.(r+1)) \to \mathrm{Ker}(1 - F^i) \to 0$$

where F^i denotes F acting on $\underline{H}^i(X, W\Omega_X^{\geq r+1})$. Thus

$$\begin{aligned}\Sigma(-1)^i s^{i+r+1}(r+1) &= \Sigma(-1)^i(\dim(\mathrm{Cok}(1 - F^{i-1})) + \dim(\mathrm{Ker}(1 - F^i))) \\ &= \Sigma(-1)^i \chi(1 - F^i)\end{aligned}$$

$$= \Sigma(-1)^i \chi(1 - F | H^i(X, W\Omega_X^{\geqq r+1})_t) \quad \text{by} \quad (3.2.1)$$

$$= \Sigma(-1)^i d^{i+r}(r).$$

Remark 3.3. If $1 - F: H^i(X, W\Omega_X^{\geqq r+1})_t \to H^i(X, W\Omega_X^{\geqq r+1})_t$ is surjective for $i = j - 1, j$, then the above argument shows that $s^{i+1}(r+1) = d^i(r)$. (Cf. (1.18)).

Example 3.4. Let X be a surface and consider the case $r = 0$. Then $s^i(1) = 0$ except for $i = 3$ ([1, p. 554]) and $d^i(0) = 0$ except for $i = 2$ ([33, Section 7, Proposition 4]). Thus in this case (3.1) becomes the equality $s^3(1) = d^2(0)$. There is another proof of this. Consider the morphism of flat sites $\pi: X_{fl} \to (\text{spec}\, k)_{fl}$. The sheaf $R^i\pi_*\mu_{p^n}$ is representable by a group scheme $\underline{H}^i(X, \mu_{p^n})$ of finite-type over k (Artin, unpublished; cf. [1, 3.1]) and it is easily seen that $\underline{H}^i(X, \nu_n(1)) = \underline{H}^{i+1}(X, \mu_{p^n})^{pf}$. On taking typical curves in the exact sequence

$$\cdots \to R^2\pi_*\mathbf{G}_m \xrightarrow{p^n} R^2\pi_*\mathbf{G}_m \to R^3\pi_*\mu_{p^n} \to \cdots$$

one obtains an exact sequence

$$\cdots \to H^2(X, W\mathcal{O}_X) \xrightarrow{p^n} H^2(X, W\mathcal{O}_X) \to TC(\underline{H}^3(X, \mu_{p^n})) \to 0$$

(see [2, II4.3]). In the limit, this becomes an isomorphism

$$H^2(X, W\mathcal{O}_X) \xrightarrow{\approx} TC(\varprojlim \underline{H}^3(X, \mu_{p^n})).$$

Since $TC(\mathbf{G}_a) = k[[V]]$ and the module of typical curves of a finite group scheme has finite length over W, it follows immediately from the isomorphism that $d^2(0) = s^3(1)$.

In this case, $\underline{H}^3(X, \mathbf{Z}_p(1))^0 = \underline{H}^3(X, (\mathbf{Z}/p^n\mathbf{Z})(1))^0$ for all n sufficiently large, and so the equality $d^2(0) = s^3(1)$ is the statement in the second paragraph of [24, 1.3]. The above proof is essentially the one I intended, but neglected, to include in [25].

Remark 3.5. It seems to be very difficult to compute the invariants $d^i(r)$ and $s^i(r)$ attached to X.

4. A formula for $\chi(X, \mathcal{O}_X, r)$. In this section, k will be algebraically closed. Let

$$\chi(X, \mathcal{O}_X, r) = \sum_{i=0}^{r} (-1)^i (r-i)\chi(X, \Omega_X^i) = \sum_{i,j} (-1)^{i+j}(r-i)h^{ij}$$

where, for any sheaf F of $\mathcal{O}_X$-modules on X, $\chi(X, F) = \Sigma(-1)^i \dim H^i(X, F)$, and $h^{ij} = \dim H^j(X, \Omega^i)$. We shall prove a formula for $\chi(X, \mathcal{O}_X, r)$ in terms of the crystalline cohomology groups of X generalizing that for $\chi(X, \mathcal{O}_X)$ $(= \chi(X, \mathcal{O}_X, 1))$ in [24, 7.1].

Let K be the field of fractions of W and let $K[F]$ be the noncommutative ring in which $Fa = a^\sigma F$, $a \in K$. For each positive rational number λ, define a $K[F]$-module

$$E^\lambda = K[F]/(F^r - p^s), \qquad \lambda = s/r \quad g.c.d.(r, s) = 1.$$

It has dimension r over K. Let M be a $K[F]$-module of finite dimension over K such that F acts as an injection. Then there is a decomposition

$$M \approx (E^{\lambda_1})^{m_1} \oplus \cdots \oplus (E^{\lambda_t})^{m_t}$$

with distinct λ_j's. The numbers $\lambda_1, \ldots, \lambda_t$ are called the slopes of M, and $m_j r_j$, where $\lambda_j = s_j/r_j$, is the multiplicity of λ_j. In the case that $M = H^i_{\text{crys}}(X/W) \otimes K$, the λ_j are called the slopes of $H^i_{\text{crys}}(X/W)$.

Proposition 4.1. *For all r,*

$$\chi(X, \mathcal{O}_X, r) = \sum_{\lambda_{ij} \leq r} (-1)^i m_{ij}(r - \lambda_{ij}) + \sum_i (-1)^i d^i(r-1)$$

where $\{\lambda_{i1}, \lambda_{i2}, \ldots\}$ is the set of slopes of $H^i_{\text{crys}}(X/W)$

m_{ij} is the multiplicity of λ_{ij}

$$d^i(r-1) = \text{length}_{W((V))} H^i(X, W\Omega^{\leq r-1}) \otimes_{W[[V]]} W((V)).$$

Proof. For M a $W[[V]]$-module, define

$$\chi_V(M) = \text{length}_W(\text{Ker}(V)) - \text{length}_W(\text{Coker}(V))$$

when both these numbers are finite.

LEMMA 4.2. *Let M be a finitely generated $W[[V]]$-module such that M/VM has finite length as a W-module, and assume that there is a σ-linear operator F on M such that $FV = p^r = VF$. Then*

$$\chi_V(M) = -\Sigma\, m_j(r - \lambda_j) - \mathrm{length}_{W((V))} M \otimes W((V))$$

where the λ_j are the slopes of $K \otimes M$ and m_j is the multiplicity of λ_j.

Proof. The hypotheses imply that, for some n, M is the extension of a free W-module of finite rank by a $(W/p^n W)[[V]]$-module [6, III 2.4]. By considering a filtration of M, one reduces the problem to the following three cases: M has finite length over W; $M = k[[V]]$; M is free of finite rank over W and $K \otimes M \approx K[F]/(F^{mr} - p^{mr\lambda})$, $\lambda < r$. In the first case $\chi_V(M) = 0$, and in the second, $\chi_V(M) = -1$. In the third case $K \otimes M$ contains a lattice $M' = W[F, V]/(F^{m(r-\lambda)} - V^{m\lambda})$, and clearly $\chi_V(M) = \chi_V(M')$. But an elementary calculation using the basis $F^{m(r-\lambda)}, F^{m(r-\lambda)-1}, \ldots, 1, \ldots, V^{m\lambda - 1}$ shows that $\chi_V(M') = -mr(r - \lambda)$. As $K \otimes M'$ has slope λ and multiplicity mr, this completes the proof.

According to [18, II 2.11], the lemma applies to $H^i(X, W\Omega^{\leq r-1})$, and it is known ([18, II 3.5]) that the slopes of $H^i(X, W\Omega^{\leq r-1})$ are precisely the slopes λ of $H^i_{\mathrm{crys}}(X/W)$ with $\lambda < r$. Thus

$$\text{(4.2.1)} \qquad \chi_V(H^i(X, W\Omega^{\leq r-1})) = -\sum_{\lambda_{ij} \leq r} m_{ij}(r - \lambda_{ij}) - d^i(r-1).$$

Next, as multiplication by p and V are injective on $W\Omega^i_X$ ([18, I 3.5]), there is an exact sequence

$$0 \to W\Omega_X^{\leq r-1} \xrightarrow{V} W\Omega_X^{\leq r-1} \to W\Omega^{\leq r-1}/V(W\Omega_X^{\leq r-1}) \to 0$$

of pro-complexes, which gives rise to an exact sequence

$$\cdots \to H^i(X, W\Omega_X^{\leq r-1}) \xrightarrow{V} H^i(X, W\Omega_X^{\leq r-1})$$
$$\to H^i(X, W\Omega_X^{\leq r-1})/V(W\Omega_X^{\leq r-1})) \to \cdots.$$

From this we get that

$$\text{(4.2.2)} \qquad \Sigma(-1)^{i+1}\chi_V(H^i(X, W\Omega_X^{\leq r-1})) = \chi(X, W\Omega^{\leq r-1}/V).$$

Finally, there is an exact sequence of pro-complexes

$$\begin{array}{ccccccccc}
0 & & & & & & & & \\
\downarrow & & & & & & & & \\
W\Omega^{\leq i-1}/V & : & W\mathcal{O}_X/p^{i-1}V & \xrightarrow{d} & \cdots & \xrightarrow{d} & W\Omega^{i-1}/V & \longrightarrow & 0 \\
\downarrow & & \downarrow p & & & & \downarrow p & & \downarrow 0 \\
W\Omega^{\leq i}/V & : & W\mathcal{O}_X/p^{i}V & \xrightarrow{d} & \cdots & \xrightarrow{d} & W\Omega^{i-1}/pV & \longrightarrow & W\Omega^{i}/V \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^{\cdot} & : & W\mathcal{O}_X/p & \xrightarrow{d} & \cdots & \xrightarrow{d} & W\Omega^{i-1}/p & \longrightarrow & W\Omega^{i}/V \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

in which $C^{\cdot}$ is quasi-isomorphic to $\Omega_{X/k}^{\leq i}$ (see [18, I 3.20]) and so

$$\chi(X, \Omega_{X/k}^{\leq i}) = \chi(X, W\Omega^{\leq i}/V) - \chi(X, W\Omega^{\leq i-1}/V).$$

When summed over i, $0 \leq i \leq r-1$, this gives

$$\sum_{i=0}^{r-1} \chi(X, \Omega_{X/k}^{\leq i}) = \chi(X, W\Omega^{\leq r-1}/V).$$

But

$$\sum_{i=0}^{r-1} \chi(X, \Omega_{X/k}^{\leq i}) = \chi(X, \mathcal{O}_X, r)$$

and

$$\begin{aligned}
\chi(X, W\Omega^{\leq r-1}/V) &= \Sigma(-1)^{i+1}\chi_V(H^i(X, W\Omega_X^{\leq r-1})) \quad \text{by} \quad (4.2.2) \\
&= \sum_i (-1)^i \Big(\sum_{\lambda_{ij} \leq r} m_{ij}(r - \lambda_{ij}) \\
&\qquad + d^i(r-1)\Big) \quad \text{by} \quad (4.2.1)
\end{aligned}$$

and so the proof is complete.

Remark 4.3. When X is defined over a finite field, then the slopes of $H^i_{\text{crys}}(X/W)$ are precisely the numbers $\text{ord}_q(a_{ij})$, where $\{a_{i1}, a_{i2}, \ldots\}$ is the set of eigenvalues of ϕ on $H^i_{\text{crys}}(X/W) \otimes \mathbf{Q}_p$ (cf. (5.2) below). On applying this remark and (3.1), we find that the formula in (4.1) can be rewritten,

$$\chi(X, \mathcal{O}_X, r) = \sum_{\text{ord}_q(a_{ij}) \le r} (-1)^i (r - \text{ord}_q(a_{ij})) - \Sigma(-1)^i s^i(r)).$$

Remark 4.4. From (4.1) we find that

$$\begin{aligned}
(-1)^r \chi(X, \Omega^r_X) &= \chi(X, \mathcal{O}_X, r+1) - 2\chi(X, \mathcal{O}_X, r) + \chi(X, \mathcal{O}_X, r-1) \\
&= \sum_{r-1 < \lambda_{ij} \le r+1} (-1)^i m_{ij}(r - \lambda_{ij}) \\
&\quad - 2 \sum_{r-1 < \lambda_{ij} \le r} (-1)^i m_{ij}(r - \lambda_{ij}) \\
&\quad + \sum_i (-1)^i (d^i(r) - d^i(r-1)) \\
&\quad - \sum_i (-1)^i (d^i(r-1) - d^i(r-2)) \\
&= \sum_{r-1 < \lambda_{ij} \le r+1} (-1)^i m_{ij}(r - \lambda_{ij}) \\
&\quad - 2 \sum_{r-1 < \lambda_{ij} \le r} (-1)^i m_{ij}(r - \lambda_{ij}) \\
&\quad + \sum_i (-1)^i \delta^i(r) - \sum_i (-1)^i \delta^i(r-1),
\end{aligned}$$

where $\delta^i(r) = \text{length}_{W((V))} H^i(X, W\Omega^r) \otimes_{W[[V]]} W((V))$.

This formula has also been found by R. Crew (Etale p-covers in characteristic p, Compositio Math. to appear).

5. The action of γ on $H^i(\bar{X}, \mathbf{Q}_p(r))$. In this section k is a finite field with $q = p^m$ elements. The canonical generator $a \mapsto a^q$ of $\text{Gal}(\bar{k}/k)$ is denoted by γ. The rings of Witt vectors over k and $\bar{k}$ are denoted by W and $\bar{W}$ respectively and their fields of fractions by K and $\bar{K}$.

Let L be a field containing $\mathbf{Q}_p$. Then $L \otimes_{\mathbf{Q}_p} K$ is a Galois (ring) extension of L with Galois group generated by $1 \otimes \sigma$, which we will again denote σ. By an F-isocrystal over $L \otimes K$ we mean a free $L \otimes K$-module of finite rank together with an injective σ-linear map $F: M \to M$. The endomorphism F^m of M is $L \otimes K$-linear. Let $\bar{M} = \bar{K} \otimes_K M$ and let γ and F act on $\bar{M}$ as $\gamma \otimes 1$ and $1 \otimes F$.

LEMMA 5.1. *Let $\lambda \in \mathbf{Q}$ and assume that L contains an element p^λ. Then*

$$M_\lambda \overset{df}{=} \operatorname{Ker}(F - p^\lambda : \bar{M} \to \bar{M})$$

is finite-dimensional over L, and

$$\det(1 - \gamma t \mid M_\lambda) = \Pi(1 - (q^\lambda/a)t)$$

where the product is over those eigenvalues of F^m on M such that $\operatorname{ord}_q(a) = \lambda$.

Proof. It is easy to show, using the classification of F-isocrystals over $\bar{K}$, that $F - p^\lambda$ is surjective on $\bar{M}$. Therefore an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of F-isocrystals over $L \otimes K$ gives rise to an exact sequence

$$0 \to M'_\lambda \to M_\lambda \to M''_\lambda \to 0.$$

Thus the lemma will be true for M if it is true for M' and M''. Moreover, in proving the lemma for M we can replace L by a larger field and therefore we can assume that F^m possesses an eigenvector e in M generating a free submodule of M. These two remarks, and induction on the rank of M, allow us to assume that M has rank 1.

Let $e \in M$ be a basis for M over $L \otimes K$. Then $Fe = \alpha e$, some $\alpha \in L \otimes K$, and the sole eigenvalue of F^m acting on M is $a = \sigma^{m-1}(\alpha) \cdots \sigma(\alpha)\alpha = N_{L \otimes K/L}(\alpha)$. An arbitrary element of M can be written $\beta \otimes e$ with $\beta \in L \otimes \bar{K}$, and

$$F(\beta \otimes e) = p^\lambda(\beta \otimes e) \Leftrightarrow \sigma(\beta)/\beta = p^\lambda/\alpha.$$

Therefore $M_\lambda = 0$ unless $|\alpha/p^\lambda|_p = 1$, i.e., $\mathrm{ord}_q(a) = \lambda$, in which case

$$M_\lambda = \{b\beta_0 \otimes e \mid b \in L\}$$

where β_0 is an element of $L \otimes \bar{K}$ such that $\sigma(\beta_0)/\beta_0 = p^\lambda/\alpha$. (Note that there does exist such an element β_0 because $W^\times/(\sigma - 1)\bar{W}^\times = 0$, as can be seen by examining $\sigma - 1$ on the natural filtration of $\bar{W}^\times$). In particular, M_λ has dimension zero or one over L according as $\mathrm{ord}_q(a) \neq \lambda$ or $\mathrm{ord}_q(a) = \lambda$. In the second case,

$$\begin{aligned}\gamma(\beta_0 \otimes e) &= \sigma^m(\beta_0 \otimes e) = \sigma^m(\beta_0) \otimes e \\ &= \frac{\sigma^m(\beta_0)}{\sigma^{m-1}(\beta_0)} \frac{\sigma^{m-1}(\beta_0)}{\sigma^{m-2}(\beta_0)} \cdots \frac{\sigma(\beta_0)(\beta_0 \otimes e)}{\beta_0} \\ &= (q^\lambda/a)(\beta_0 \otimes e).\end{aligned}$$

Remark 5.2. Let M be an F-isocrystal over K, and let L be an extension of $\mathbf{Q}_p$ containing all s^{th} roots of p for a suitably large integer s. From the classification of F-isocrystals over $\bar{K}$, it is clear that $(L \otimes M)_\lambda$ has dimension equal to the multiplicity of λ as a slope of M. Thus the lemma allows us to recover the result of Manin [23, Theorem 2.2] that the multiplicity of λ as a slope of M is equal to the number of eigenvalues a of F^m on M such that $\mathrm{ord}_q(a) = \lambda$.

Remark 5.3. We shall need (5.1) only in the case that M is an F-isocrystal over K and λ is an integer. Then (5.1) says that

$$\det(1 - \gamma t \mid \bar{M}^{F=p^\lambda}) = \prod_{\mathrm{ord}_q(a)=\lambda} (1 - (q^\lambda/a)t).$$

PROPOSITION 5.4. *Let X be a complete smooth variety over k; then*

$$\det(1 - \gamma t \mid H^i(\bar{X}, \nu.(r)) \otimes \mathbf{Q}_p) = \prod_{\mathrm{ord}_q(a_{i+r,j})=r} (1 - (q^r/a_{i+r,j})t)$$

where $\prod(1 - a_{i+r,j}t) = \det(1 - F^m t \mid H^{i+r}_{\mathrm{crys}}(X/W) \otimes \mathbf{Q}_p)$.

Proof. As we have already observed (1.15), the exact sequence

$$0 \to \nu.(r) \to W\Omega^r_{\bar{X}} \xrightarrow{1-F} W\Omega^r_{\bar{X}} \to 0$$

leads to an exact sequence

$$0 \to H^i(\bar{X}, \nu.(r)) \otimes \mathbf{Q}_p \to H^{i+r}_{\text{crys}}(\bar{X}/\bar{W}) \otimes \bar{K} \xrightarrow{F-p^r} H^{i+r}_{\text{crys}}(\bar{X}/\bar{W}) \otimes \bar{K} \to 0.$$

On applying (5.3) with $M = H^{i+r}_{\text{crys}}(X/W) \otimes K$ and $\lambda = r$, we obtain the proposition.

From now on we write ϕ for F^m.

Remark 5.5. If X is projective, the eigenvalues of ϕ acting on $H^i_{\text{crys}}(\bar{X}/\bar{W}) \otimes \mathbf{Q}_p$ are the same as those of ϕ acting on $H^i(\bar{X}, \mathbf{Q}_\ell)$, any $\ell \neq p$, (see [21]). In particular, they have absolute value $q^{i/2}$.

Remark 5.6. We can restate (5.4) as

$$\det(1 - \gamma t \,|\, H^i(\bar{X}, \mathbf{Q}_p(r))) = \prod_{\text{ord}_q(a_{ij}) = r} (1 - (q^r/a_{ij})t)$$

where the a_{ij} are the eigenvalues of ϕ on $H^i(X/W) \otimes \mathbf{Q}_p$. This is close to the analogous result for $\ell \neq p$:

$$\det(1 - \gamma t \,|\, H^i(\bar{X}, \mathbf{Q}_\ell(r)) = \prod (1 - (q^r/a_{ij})t)$$

where the a_{ij} are the eigenvalues of ϕ on $H^i(\bar{X}, \mathbf{Q}_\ell)$.

6. Proofs of (0.1) and (0.2). We now assume that X is projective so that there exist algebraic integers $a_{i1}, a_{i2}, \ldots$ that can be unambiguously defined as the eigenvalues of $\phi = F^m$ on $H^i(\bar{X}, \mathbf{Q}_\ell)$ any $\ell \neq p$ or as the eigenvalues of ϕ on $H^i_{\text{crys}}(X/W) \otimes \mathbf{Q}_p$. As in the introduction, $P_i(X, t) = \prod_j (1 - a_{ij}t)$.

If M is a Γ-module, we let M^Γ and M_Γ denote the kernel and cokernel respectively of $\gamma - 1 : M \to M$, and we always use f to denote the map $M^\Gamma \to M_\Gamma$ induced by the identity map on M. Recall the following elementary lemma from [38, z.4].

LEMMA 6.1. *Let M be a Γ-module that is a finitely generated $\mathbf{Z}_\ell$-module. Then $z(f)$ is defined if and only if the minimal polynomial of γ on $\mathbf{Q}_\ell \otimes M$ does not have 1 as a multiple root, in which case*

$$z(f) = \Big| \prod_{a_i \neq 1} (a_i - 1) \Big|_\ell$$

where $\{a_1, a_2, \ldots\}$ is the set of eigenvalues of γ on $\mathbf{Q}_\ell \otimes M$.

LEMMA 6.2. *Let f be the map $H^i(\bar{X}, \mathbf{Z}_\ell(r))^\Gamma \to H^i(\bar{X}, \mathbf{Z}_\ell(r))_\Gamma$ induced by the identity map. Then $z(f)$ is defined if and only if the minimal polynomial of γ on $H^i(\bar{X}, \mathbf{Q}_\ell(r))$ does not have 1 as a multiple root, in which case*

$$z(f) = \Big| \prod_{a_{ij} \neq q^r} (1 - a_{ij}/q^r) \Big|_p \Big| \prod_{\mathrm{ord}_q(a_{ij}) < r} q^r/a_{ij} \Big|_p q^{s^i(r)}, \qquad \ell = p,$$

$$z(f) = \Big| \prod_{a_{ij} \neq q^r} (1 - a_{ij}/q^r) \Big|_\ell, \qquad \ell \neq p.$$

Proof. The case $\ell \neq p$ is an immediate consequence of (5.6) and (6.1), since $|q^r/a_{ij}|_\ell = 1$. For $\ell = p$, we use the sequence from Section 1,

$$0 \to U^i \to \underline{H}^i(X, \mathbf{Z}_p(r)) \to D^i \to 0$$

in which U^i is a connected perfect algebraic group scheme and D^i is pro-étale. The map $\gamma - 1 : U^i(\bar{k}) \to U^i(\bar{k})$ is surjective because it is étale and U^i is connected. Therefore, on applying a snake lemma argument, we can get an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^i(\bar{k})^\Gamma & \longrightarrow & H^i(\bar{X}, \mathbf{Z}_p(r))^\Gamma & \longrightarrow & D^i(\bar{k})^\Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H^i(\bar{X}, \mathbf{Z}_p(r))_\Gamma & \longrightarrow & D^i(\bar{k})_\Gamma & \longrightarrow & 0 \end{array}$$

Clearly $z(f')$ is defined and equals $[U^i(k)] = q^{s^i(r)}$. As $D^i(\bar{k})$ is a finitely generated $\mathbf{Z}_p$-module and $D^i(\bar{k}) \otimes \mathbf{Q}_p = H^i(\bar{X}, \mathbf{Q}_p(r))$, on applying (6.1) to it we find that $z(f'')$ is defined if and only if the minimal polynomial of γ on $H^i(\bar{X}, \mathbf{Q}_p(r))$ does not have 1 as a multiple root, and then

$$z(f'') = \Big|\prod(1 - q^r/a_{ij})\Big|_p,$$

where the product is over the a_{ij} such that $\mathrm{ord}_q(a_{ij}) = r$ but $a_{ij} \neq q^r$. This can be rewritten

$$z(f'') = \Big|\prod(a_{ij}/q^r - 1) q^r/a_{ij}\Big|_p = \Big|\prod(1 - a_{ij}/q^r)\Big|_p.$$

Note that

$$|1 - a_{ij}/q^r|_p = \begin{cases} |a_{ij}/q^r|_p & \text{if} \quad \mathrm{ord}_q(a_{ij}) < r \\ 1 & \text{if} \quad \mathrm{ord}_q(a_{ij}) > r. \end{cases}$$

Thus

$$z(f'') = |\prod_{a_{ij} \neq q^r} (1 - a_{ij}/q^r)|_p | \prod_{\mathrm{ord}_q(a_{ij}) < r} q^r/a_{ij}|_p$$

On combining these results for f' and f'', we obtain the required result for f.

PROPOSITION 6.3. *Let $f: H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma \to H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ be the map induced by the identity map. When $i \neq 2r$, $z(f)$ is always defined, and when $i = 2r$, $z(f)$ is defined if and only if $SS(X, r, \ell)$ holds for all ℓ. If $z(f)$ is defined then*

$$z(f) = |\prod_{a_{ij} \neq q^r} (1 - a_{ij}/q^r)^{-1}| q^e$$

where $e = s^i(r) - \Sigma_{\mathrm{ord}_q(a_{ij}) < r} (r - \mathrm{ord}_q(a_{ij}))$.

Proof. The Riemann hypothesis and (5.6) show that, when $i \neq 2r$, 1 is not an eigenvalue of γ on $H^i(\bar{X}, \mathbf{Q}_\ell(r))$ and therefore the formulas in (6.2) hold for all ℓ; moreover $H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma$ is torsion. The theorem in [12] implies that $H^i(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ is finite, and now (6.2) shows that $H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ is finite. Clearly $z(f)$ is defined.

The case $i = 2r$ can be proved by a similar argument.

The formula in the proposition can be proved by multiplying together the formulas for the different ℓ in (6.2).

COROLLARY 6.4. *Assume $i \neq 2r$; then $H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma$ and $H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ are both finite, and*

$$\frac{[H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]}{[H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]} = |P_i(X, q^{-r})^{-1}| q^e$$

where $e = s^i(r) - \Sigma_{\mathrm{ord}_q(a_{ij}) < r} (r - \mathrm{ord}_q(a_{ij}))$.

Proof. In this case $z(f) = [H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]/[H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]$.

The isomorphism $\gamma^\alpha \mapsto \alpha : \mathrm{Gal}(\bar{k}/k) \to \hat{\mathbf{Z}}$ defines a canonical element $\theta_0 \in H^1(k, \hat{\mathbf{Z}})$. We can pull θ_0 back to an element θ of $H^1(X, \hat{\mathbf{Z}})$, and then cup-product with θ defines a map $H^i(X, \hat{\mathbf{Z}}(r)) \to H^{i+1}(X, \hat{\mathbf{Z}}(r))$ which we denote ϵ^i.

PROPOSITION 6.5. *For any sheaf F of $\hat{\mathbf{Z}}$-modules on X, the map*

$$x \mapsto \theta \cup x : H^i(X, F) \to H^{i+1}(X, F)$$

is equal to the composite of the obvious maps

$$H^i(X, F) \to H^i(\bar{X}, \bar{F}), \qquad H^i(\bar{X}, \bar{F}) \to H^i(\bar{X}, \bar{F})_\Gamma$$

with the map $H^i(\bar{X}, \bar{F})_\Gamma \to H^{i+1}(X, F)$ arising from the Hochschild-Serre spectral sequence.

Proof. This can be proved easily using the description of the first map on the level of complexes in [31, 1.2].

PROPOSITION 6.6. *The determinant of $\epsilon^{2r} : H^{2r}(X, \hat{\mathbf{Z}}(r)) \to H^{2r+1}(X, \hat{\mathbf{Z}}(r))$ is defined if and only if $SS(X, r, \ell)$ holds for all ℓ, in which case $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ is finite, $z(\epsilon^{2r})$ is defined, and*

$$z(f) = z(\epsilon^{2r}) \frac{[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]}{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}$$

$$= \frac{1}{\det(\epsilon^{2r})} \frac{[H^{2r}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}]}{[H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}]} \frac{[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]}{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}$$

with f as in (6.3).

Proof. Consider the diagram

$$\begin{array}{ccc} H^{2r}(X, \hat{\mathbf{Z}}(r)) & \xrightarrow{\epsilon^{2r}} & H^{2r+1}(X, \hat{\mathbf{Z}}(r)) \\ \downarrow i & & \uparrow j \\ H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma & \xrightarrow{f} & H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma, \end{array}$$

which commutes because of (6.5). As $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ and $H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ are finite [12], $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ is finite. Therefore, the observations in the introduction show that $z(\epsilon^{2r})$ is defined if and only if $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite and $\det(\epsilon^{2r})$ is defined. Clearly $z(i)$ is defined (see (6.4)), and

$$z(i) = [H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma].$$

Also $z(j)$ is defined, and

$$z(j) = [H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]^{-1}.$$

Thus $z(\epsilon^{2r})$ is defined if and only if $z(f)$ is defined, and we know $z(f)$ is defined if and only if $SS(X, r, \ell)$ holds for all ℓ. Supposing these three equivalent statements to be true, we find that

$$\begin{aligned} z(f) &= z(i)^{-1}z(\epsilon^{2r})z(j)^{-1} \\ &= \frac{1}{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}\, z(\epsilon^{2r})[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma] \end{aligned}$$

as required.

We are now in a position to prove (0.1). It follows from (6.4) that $H^i(X, \hat{\mathbf{Z}}(r))$ is finite except possibly for $i = 2r, 2r+1$, and (using that $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite [12]) that $H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite. From (6.6) it follows that, if $\det(\epsilon^{2r})$ is defined, then $SS(X, r, \ell)$ holds for all ℓ and, conversely, that if $SS(X, r, \ell)$ holds for all ℓ, then $\chi(X, \hat{\mathbf{Z}}(r))$ is defined. Moreover,

$$0 \to H^{i-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma \to H^i(X, \hat{\mathbf{Z}}(r)) \to H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma \to 0$$

shows that

$$\chi(X, \hat{\mathbf{Z}}(r)) = \prod_{i \neq 2r} \left(\frac{[H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]}{[H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}\right)^{(-1)^i} \frac{[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]}{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]} \cdot z(\epsilon^{2r}).$$

which (6.4) and (6.6) show equals

$$= \prod_{i \neq 2r} (P_i(q^{-r}))^{(-1)^{i+1}} z(f) q^d$$

where

$$d = \sum_{i \neq 2r} (-1)^i s^i(r) + \sum_{i \neq 2r} (-1)^{i+1} \sum_{\operatorname{ord}_q(a_{ij}) < r} (r - \operatorname{ord}_q(a_{ij})).$$

Now (6.3) and (4.3) allow us to rewrite this as

$$\chi(X, \hat{\mathbf{Z}}(r)) = (\lim_{s \to r} \zeta(X, s)(1 - q^{r-s})^{\rho_r}) q^{-\chi(X, \mathcal{O}_X, r)}.$$

This completes the proof of (0.1).

Before proving (0.2), we need a lemma.

LEMMA 6.7. *Let $d = 2r$; then*

$$\frac{1}{[H^{2r}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}]} \, \frac{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}{[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]} = \frac{q^{-s^{2r+1}(r)}}{[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}^\Gamma]^2}$$

Proof. From (1.13a), we have an exact sequence

$$0 \to U^{2r+1}(\bar{k}) \to H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r)) \to H^{2r-1}(\bar{X}, (\mathbf{Q}/\mathbf{Z})(r))^* \to 0$$

where $U^{2r+1} = \underline{H}^{2r+1}(X, \mathbf{Z}_p(r))^0$. As U^{2r+1} is connected, $U^{2r+1}(\bar{k})_\Gamma = 0$ and so there is an exact sequence

$$0 \to U^{2r+1}(k) \to H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma \to (H^{2r-1}(\bar{X}, (\mathbf{Q}/\mathbf{Z})(r))_\Gamma)^* \to 0.$$

On passing to the direct limit in

$$0 \to H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))^{(n)} \to H^{2r-1}(\bar{X}, (\mathbf{Z}/n\mathbf{Z})(r)) \to H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_n \to 0$$

we obtain an exact sequence

$$0 \to H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r)) \otimes \mathbf{Q}/\mathbf{Z} \to H^{2r-1}(\bar{X}, (\mathbf{Q}/\mathbf{Z})(r)) \to H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}} \to 0.$$

As $H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ is finite, clearly $(H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r)) \otimes \mathbf{Q}/\mathbf{Z})_\Gamma = H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma \otimes \mathbf{Q}/\mathbf{Z}$ is zero. Thus

$$H^{2r-1}(\bar{X}, (\mathbf{Q}/\mathbf{Z})(r))_\Gamma \xrightarrow{\approx} (H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}})_\Gamma.$$

But $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite, and so

$$[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\text{tor}}] = [(H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\text{tor}})_{\Gamma}] = [H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}]q^{-s^{2r+1}(r)}.$$

The lemma now follows from the exact sequence

$$0 \to H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_{\Gamma} \to H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}} \to H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\text{tor}} \to 0.$$

We now prove (0.2). From (6.3), (6.6), and (6.7), we find that

$$\prod_{a_{2r,j}\neq q^r}(1 - a_{2r,j}/q^r) \overset{(6.3)}{=} z(f)^{-1}q^e, e = s^{2r}(r) - \sum_{\text{ord}_q(a_{2r,j})<r}(r - \text{ord}_q(a_{2r,j}))$$

$$\overset{(6.6)}{=} \frac{[H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]}{[H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]} \frac{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_{\Gamma}]}{[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}]} \det(\epsilon^{2r})q^e$$

$$\overset{(6.7)}{=} \frac{[H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]}{[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\text{tor}}]^2} \det(\epsilon^{2r})q^{e-s^{2r+1}(r)}$$

which is the formula in (0.2).

We now obtain another expression for $\alpha_r(X)$.

LEMMA 6.8. *Let $d = 2r$; then*

$$\chi(X, \mathcal{O}_X, r) = \sum_{i=0}^{d-1}(-1)^i b_i + \sum_{\text{ord}_q(a_{2r,j})\leq r}(r - \text{ord}_q(a_{2r,j}))$$

$$+ \Sigma(-1)^{i+1}s^i(r)$$

where $b_i = \sum_j |r - \text{ord}_q(a_{ij})|$ *for all* i

$$b_i = (r - i/2)\beta_i, \qquad i \leq r.$$

Proof. We have to reinterpret the terms in (4.3). Because of Poincaré duality, the inverse roots of $P_{2d-i}(X, t)$ are $\{a_{2d-i,j} = q^d/a_{ij}\}$. As $d = 2r$, $\text{ord}_q(a_{2d-i,j}) \leq r$ if and only if $\text{ord}_q(a_{ij}) \geq r$. Hence, for a fixed i,

$$\sum_{\mathrm{ord}_q(a_{ij}) \geq r} (r - \mathrm{ord}_q(a_{ij})) + \sum_{\mathrm{ord}_q(a_{2d-i,j}) \leq r} (r - \mathrm{ord}_q(a_{2d-i,j})) = 0.$$

Thus, for a fixed i,

$$\sum_{\mathrm{ord}_q(a_{ij}) \leq r} (r - \mathrm{ord}_q(a_{ij})) + \sum_{\mathrm{ord}_q(a_{2d-i,j}) \leq r} (r - \mathrm{ord}_q(a_{2d-i,j})) = b_i,$$

and, the sum over all i and j,

$$\sum_{\mathrm{ord}_q(a_{ij}) \leq r} (-1)^i (r - \mathrm{ord}_q(a_{ij})) = \sum_{i=0}^{d-1} (-1)^i b_i + (-1)^d \sum_{\mathrm{ord}_q(a_{2r,j}) \leq r} (r - \mathrm{ord}_q(a_{2r,j})).$$

It remains to show that $b_i = (r - i/2)\beta_i$ for $i \leq r$. Note that

$$\begin{aligned} q^{\Sigma_j [r - \mathrm{ord}_q(a_{ij}))} &= \pm \prod_j q^r / a_{ij} \\ &= \pm q^{\beta_i r} / (\text{coeff. of } t^{\beta_i} \text{ in } P_i(X, t)) \\ &= q^{\beta_i (r - i/2)}. \end{aligned}$$

In particular, when $i \leq r$, $\mathrm{ord}_q(a_{ij}) \leq r$ (because $\bar{a}_{ij}$ is also an algebraic integer, and $a_{ij}\bar{a}_{ij} = q^i$) and so

$$\sum_j |r - \mathrm{ord}_q(a_{ij})| = \Sigma(r - \mathrm{ord}_q(a_{ij})) = \beta_i(r - i/2).$$

PROPOSITION 6.9. *For $d = 2$, $\alpha_1(X) = \chi(X, \mathcal{O}_X) - 1 + 1/2\beta_1$. For $d = 4$, $\alpha_2(X) = \chi(X, \mathcal{O}_X, 2) - 2 + 3/2\beta_1 - \beta_2 + b_3$.*

Proof. Let $d = 2$; then

$$\chi(X, \mathcal{O}_X, 1) = \beta_0 - 1/2\beta_1 + \sum_{\mathrm{ord}_q(a_{2j}) \leq 1} (1 - \mathrm{ord}_q(a_{2j})) + s^3(1)$$

while

$$\alpha_1(X) = \sum_{\mathrm{ord}_q(a_{2j}) \leq 1} (1 - \mathrm{ord}_q(a_{2j})) + s^3(1),$$

which immediately gives the required formula as $\chi(X, \mathcal{O}_X, 1) = \chi(X, \mathcal{O}_X)$.

Let $d = 4$; then

$$\chi(X, \mathcal{O}_X, 2) = 2\beta_0 - 3/2\beta_1 + \beta_2 - b_3 + \sum_{\operatorname{ord}_q(a_{4j}) \le 2} (2 - \operatorname{ord}_q a_{4j})$$

$$+ s^3(2) - s^4(2) + s^5(2) - s^6(2),$$

while

$$\alpha_2(X) = \sum_{\operatorname{ord}_q(a_{4j}) \le 2} (2 - \operatorname{ord}_q(a_{4j})) + s^5(2) - s^4(2).$$

Thus,

$$\alpha_2(X) = \chi(X, \mathcal{O}_X, 2) - 2 + 3/2\beta_1 - \beta_2 + b_3 - s^3(2) + s^6(2).$$

For large N, there is a sequence

$$0 \to \underline{H}^i(\bar{X}, \mathbf{Z}_p(2))^0 \to \underline{H}^i(\bar{X}, (\mathbf{Z}/p^N\mathbf{Z})(2))^0 \to H^{i+1}(\bar{X}, \mathbf{Z}_p(2))^0 \to 0$$

that is exact modulo finite groups. As $\underline{H}^2(\bar{X}, \mathbf{Z}_p(2))^0 = 0$ and $\underline{H}^7(\bar{X}, \mathbf{Z}_p(2))^0 = 0$, we have

$$s^3(2) = \dim \underline{H}^2(\bar{X}, (\mathbf{Z}/p^N\mathbf{Z})(2))$$

$$s^6(2) = \dim \underline{H}^6(\bar{X}, (\mathbf{Z}/p^N\mathbf{Z})(2)),$$

but (1.11) shows that these two numbers are equal.

PROPOSITION 6.10. *If $H^i(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ and $H^{2d+2-i}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ are both finite, then they are canonically dual; moreover, the pairing on $H^{d+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is skew-symmetric.*

Proof. The exact sequence

$$0 \to H^{i-1}(X, \hat{\mathbf{Z}}(r)) \otimes \mathbf{Q}/\mathbf{Z} \to H^{i-1}(X, (\mathbf{Q}/\mathbf{Z})(r)) \to H^i(X, \hat{\mathbf{Z}}(r))_{\text{tor}} \to 0$$

shows that

$$H^i(X, \hat{\mathbf{Z}}(r))_{\text{tor}} = H^{i-1}(X, (\mathbf{Q}/\mathbf{Z})(r))/H^{i-1}(X, (\mathbf{Q}/\mathbf{Z})(r))_{\text{div}}.$$

Now the duality between $H^{i-1}(X, (\mathbf{Q}/\mathbf{Z})(r))$ and $H^{2d+2-i}(X, \hat{\mathbf{Z}}(r))$ (see (1.14)) induces a duality between $H^{i}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ and $H^{2d+2-i}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ (see [24, 2.3]). The skew-symmetry of the pairing on $H^{d+1}(X, \hat{\mathbf{Z}}(r))$ can be proved as in [24, 2.4].

Remark 6.11. If in (0.1) and (0.2) the hypotheses are weakened so that $SS(X, r, \ell)$ holds for any one or some primes ℓ, then one obtains correspondingly weakened conclusions.

7. The cohomology of K_r, $r \leq 2$. For any regular scheme Y, we write $\underline{K}_r Y$ for the presheaf $U \mapsto K_r(\Gamma(U, \mathcal{O}_U))$ on Y_{et} and $K_r\mathcal{O}_Y$ for the sheaf associated with $\underline{K}_r Y$. If $Y = \operatorname{spec} A$, we also write $\underline{K}_r A$ for $\underline{K}_r Y$ and $K_r A$ for $K_r\mathcal{O}_Y$.

PROPOSITION 7.1. *Let Y be a regular scheme of finite type over a field k. Then $H^j(Y, K_r\mathcal{O}_Y)$ is torsion for $j > r$.*

Proof. For $r = 1$, this was proved by Grothendieck [15, 1.4] using the exact sequence

$$0 \to \mathcal{O}_Y^\times \to i_* k(Y)^\times \to \underline{\mathrm{Div}} \to 0,$$

and our proof will be along the same lines.

Let y be a point of Y and i the inclusion map $i: y \hookrightarrow Y$; let P be a presheaf on y ($=\operatorname{spec} k(y)$), and let i_*P be the direct image of P. There is a canonical map $a(i_*P) \to i_*(aP)$ from the sheaf associated with i_*P to the direct image of the sheaf associated with P. If P commutes with direct limits of rings, then for any $z \in \overline{\{y\}}$, the map on stalks $a(i_*P)_{\bar{z}} \to (i_*P)_{\bar{z}}$ is $P(k(y)^{un}) \to P(\overline{k(y)})^G$ where $\overline{k(y)}$ is the separable closure of $k(y)$ and $k(y)^{un} = \overline{k(y)}^G$ is the maximal subextension of $\overline{k(y)}$ unramified over z. When $P = \underline{K}_r k(y)$, then the kernel and cokernel of this map are torsion, as follows from the statement: for any finite Galois extension $j: F \hookrightarrow E$ of fields, there exist maps $j_*: K_rF \to K_rE$, $j^*: K_rE \to K_rF$ such that $j^* \circ j_* = [E:F]$ and $j_* \circ j^*(\alpha) = \Sigma\sigma\alpha$, $\sigma \in \mathrm{Gal}(E/F)$. Thus, $a(i_*\underline{K}_r k(y)) \to i_*K_rk(y)$ has a kernel and cokernel that are torsion.

The theorem of Quillen [30, 5.11] shows that there is an exact sequence of presheaves,

$$0 \to \underline{K}_r Y \to \bigoplus_{\dim(y)=d} i_*\underline{K}_r k(y) \to \bigoplus_{\dim(y)=d-1} i_*\underline{K}_{r-1} k(y) \to \cdots$$

On applying the functor a to this sequence, we get an exact sequence

$$0 \to K_r \mathcal{O}_Y \to \bigoplus a(i_* \underline{K}_r k(y)) \to \bigoplus a(i_* \underline{K}_{r-1} k(y)) \to \cdots.$$

A standard argument [15, 1.1], using that Galois cohomology groups are torsion except in dimension 0, shows that $H^j(Y, i_* K_m k(y))$ is torsion for $j > 0$. Hence $H^j(Y, a(i_* \underline{K}_r k(y))$ is torsion for $j > 0$, and the proposition follows from this.

Now define $K_0 = \mathbf{Z}$, $K_1 = \mathcal{O}_Y^\times$, $K_2 = K_2 \mathcal{O}_Y/\text{torsion}$.

COROLLARY 7.2. *Suppose $r \leq 2$; then $H^j(Y, K_r)$ is torsion for $j > r$.*

Proof. Obvious.

It is known [28, Section 3] that the d log map

$$\{f, g\} \mapsto d \log \underline{f} \wedge d \log \underline{g} \colon K_2 \mathcal{O}_X \to \nu_n(2)$$

factors through K_2.

PROPOSITION 7.3. *For $r \leq 2$, there is an exact sequence*

$$0 \longrightarrow K_r \xrightarrow{p^n} K_r \xrightarrow{d \log} \nu_n(r) \longrightarrow 0$$

of sheaves on X_{et}.

Proof. As K_r is torsion-free, it suffices (because of (1.7.1)) to prove this for $n = 1$. The sequences for $r = 0, 1$ are

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X^\times \xrightarrow{p} \mathcal{O}_X^\times \xrightarrow{d \log} \nu(1) \longrightarrow 0$$

which are obviously exact. The definitions of K_2 and $\nu_1(2)$ have been chosen so as to make the sequence exact at the end-points. Consider

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_2 \mathcal{O}_X & \longrightarrow & a i_* \underline{K}_2 k(\eta) & \longrightarrow & \bigoplus i_* k(x)^\times \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
0 & \longrightarrow & K_2 \mathcal{O}_X & \longrightarrow & a i_* \underline{K}_2 k(\eta) & \longrightarrow & \bigoplus i_* k(x)^\times \\
 & & \downarrow & & \downarrow & & \\
 & & \nu(2) & \longrightarrow & a i_* \nu(2)_{k(\eta)} & &
\end{array}$$

where η is the generic point of X. Bloch has shown (unpublished)[3] that the sequence of groups

$$K_2k(\eta) \xrightarrow{p} K_2k(\eta) \xrightarrow{d\log} \nu(2)_{k(\eta)}$$

is exact, and this implies that the middle column of the diagram becomes exact once one divides the top two terms by their p-torsion subgroups. A diagram chase now shows that the first column becomes exact after the same modification, and this completes the proof.

COROLLARY 7.4. *For $r \leq 2$, there are exact sequences*

$$0 \to H^i(X, K_r)^{(p^n)} \to H^i(X, \nu_n(r)) \to H^{i+1}(X, K_r)_{p^n} \to 0.$$

For the rest of this section, we assume X to be projective and k to be finite. If M is an abelian group, $M^\wedge$ denotes the completion $\varprojlim M^{(n)}$ of M relative to the topology defined by the subgroups $\{nM\}$.

PROPOSITION 7.5. *For $r \leq 1$, there are exact sequences*

$$0 \to H^i(X, K_r)^\wedge \to H^{i+r}(X, \hat{\mathbf{Z}}(r)) \to TH^{i+1}(X, K_r) \to 0.$$

For $r = 2$, there are exact sequences

$$0 \to H^i(X, K_r)^\wedge \to H^{i+r}(X, \mathbf{Z}_p(r)) \to TH^{i+1}(X, K_r) \to 0.$$

Proof. The p-part is obtained by passing to the limit in the sequences in (7.4) (note that $H^i(X, \nu_n(r))$ is finite). The ℓ-part, $\ell \neq p$, is obtained similarly.

COROLLARY 7.6. *Let r be ≤ 2.*

(a) *For $i > r + 2$, $H^i(X, K_r)$ is finite, and*

$$H^i(X, K_r) \xrightarrow{\approx} H^{i+r}(X, \hat{\mathbf{Z}}(r)), \qquad r \leq 1$$

$$H^i(X, K_2) \xrightarrow{\approx} H^{i+2}(X, \mathbf{Z}_p(r)), \qquad r = 2$$

[3] The proof will appear in: S. Bloch and K. Kato, p-adic étale cohomology.

(b) *For* $i = r+1, r+2$, $H^i(X, K_r)$ *is torsion, and* $H^i(X, K_r)_m$ *is finite for all integers m.*

(c) $H^0(X, K_1)$ *is finite*; $H^i(X, K_2)_{\text{tor}}$ *is finite for* $i = 0, 1$, *and* $H^i(X, K_2)_{\text{tor}} \cap H^i(X, K_2)_{\text{div}} = 0$.

Proof. We have seen in Section 6 that $H^i(X, \hat{\mathbf{Z}}(r))$ is finite for $i \neq 2r$, $2r+1$. As $TH^{i+1}(X, K_r)$ is torsion-free, this shows that $TH^{i+1}(X, K_r) = 0$ and $H^i(X, K_r)^\wedge \approx H^{i+r}(X, \mathbf{Z}_p(r))$ or $H^{i+r}(X, \hat{\mathbf{Z}}(r))$ for $i \neq r, r+1$. Since $H^i(X, K_r)$ is torsion for $i > r$, $TH^i(X, K_r) = 0$, $i \neq r+1, r+2$, implies that $H^i(X, K_r)$ is finite for $i > r+2$. This proves (a), and (b) is obvious.

For (c), $H^0(X, K_1) = k^*$, which is finite. As $TH^i(X, K_2) = 0$ for $i = 0, 1$, $H^i(X, K_2)_{\text{tor}}$ can have no p-divisible subgroup, and so the remaining assertions of (c) are obvious.

Remark 7.7. (a) If $SS(X, r, \ell)$ holds for all ℓ, then

$$H^{r+2}(X, K_r)_{\text{div}} = (\mathbf{Q}/\mathbf{Z})^{\rho_r}, \qquad r = 0, 1,$$

$$H^4(X, K_2)_{\text{div}} = (\mathbf{Q}_p/\mathbf{Z}_p)^{\rho_r}, \qquad r = 2,$$

where ρ_r is the order of the pole of $\zeta(X, s)$ at $s = r$ (because then $T_\ell H^{r+2}(X, K_r) = H^{2r+1}(X, \mathbf{Z}_\ell(r))/\text{torsion} = H^{2r}(\bar{X}, \mathbf{Z}_\ell(r))_\Gamma/\text{torsion} \approx \mathbf{Z}_\ell^{\rho_r}$).

(b) Consider

$$\begin{array}{ccc} CH^r(X) \otimes \mathbf{Z}_\ell & \xrightarrow{c^r} & H^{2r}(\bar{X}, \mathbf{Z}_\ell(r))^\Gamma \\ \downarrow a & & \uparrow d \\ H^r(X, K_r)^\wedge \otimes \mathbf{Z}_\ell & \xhookrightarrow{b} & H^{2r}(X, \mathbf{Z}_\ell(r)) \end{array}$$

where c^r is the cycle map, and a is defined by

$$CH^r(X) = H^r(X_{\text{Zar}}, K_r) \xrightarrow{a_0} H^r(X_{et}, K_r).$$

Note that b is injective with cokernel $T_\ell H^{r+1}(X, K_r)$ and d is surjective with finite kernel. Thus if $CH^r(X) \otimes \mathbf{Q}_\ell \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma$ is surjective (i.e. Tate's conjecture $T'(X, r, \ell)$ holds), then $T_\ell H^{r+1}(X, K_r) = 0$ and the ℓ-primary component $H^{r+1}(X, K_r)(\ell)$ of $H^{r+1}(X, K_r)$ is finite. For $r \leq 1$, a_0 is an isomorphism, and so the converse assertion also holds.

It would be interesting to know if

$$a_0 : H^r(X_{\mathrm{Zar}}, K_r) \otimes \mathbf{Z}_p \to H^r(X_{et}, K_r) \otimes \mathbf{Z}_p$$

is an isomorphism also for $r = 2$. For X a surface it is: there is an isomorphism $H^2(X_{\mathrm{Zar}}, K_2) \otimes \mathbf{Z}_p \to \pi_1(X)^{ab} \otimes \mathbf{Z}_p$ (see [28, Section 8]), and $\pi_1(X)^{ab} \otimes \mathbf{Z}_p$ is dual to $H^1(X, \mathbf{Q}_p/\mathbf{Z}_p)$, which in turn is dual to $H^2(X, \nu.(2)) = H^2(X_{et}, K_2)$.

PROPOSITION 7.8. *For $r \leq 1$, there are exact sequences*

$$0 \to H^i(X, K_r) \otimes \mathbf{Q}/\mathbf{Z} \to H^{i+r}(X, (\mathbf{Q}/\mathbf{Z})(r)) \to H^{i+1}(X, K_r)_{\mathrm{tor}} \to 0.$$

For $r = 2$, there are exact sequences

$$0 \to H^i(X, K_r) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^{i+r}(X, (\mathbf{Q}_p/\mathbf{Z}_p)(r)) \to H^{i+1}(X, K_r)_{\mathrm{tor}} \to 0.$$

Proof. The p-part is obtained by passing to the direct limit in the sequences in (7.4). The ℓ-part, $\ell \neq p$, can be proved similarly.

PROPOSITION 7.9. *Assume that $0 \leq r, d - r \leq 2$. For all i, there is a canonical pairing*

$$H^i(X, K_r)_{\mathrm{tors}} \times H^{d+2-i}(X, K_{d-r})_{\mathrm{tors}} \to \mathbf{Q}/\mathbf{Z}$$

whose left and right kernels consist exactly of the divisible elements in each group.

Proof. The nondegenerate pairing (1.14)

$$H^{i+r}(X, \hat{\mathbf{Z}}(r)) \times H^{2d+1-i-r}(X, (\mathbf{Q}/\mathbf{Z})(d-r)) \to \mathbf{Q}/\mathbf{Z}$$

induces a nondegenerate pairing

$$H^{i+r}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}} \times H^{2d+1-i-r}(X, (\mathbf{Q}/\mathbf{Z})(d-r))/D \to \mathbf{Q}/\mathbf{Z}$$

where D is the divisible subgroup of $H^{2d+1-i-r}(X, (\mathbf{Q}/\mathbf{Z})(d-r))$ (see [24, 2.3]). From (7.5) it is clear that $H^{i+r}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}} = H^i(X, K_r)_{\mathrm{tor}}/(H^i(X, K_r)_{\mathrm{tor}})_{\mathrm{div}}$, and from (7.8) it is clear that

$$H^{2d+1-i-r}(X, (\mathbf{Q}/\mathbf{Z})(d-r))/D = H^{d+2-i}(X, K_{d-r})_{\mathrm{tor}}/(H^{d+2-i}(X, K_{d-r})_{\mathrm{tor}})_{\mathrm{div}}.$$

Remark 7.10. The most interesting case of the above result is $d = 4$, $r = 2$, $i = 3$. It then says that there is a canonical pairing

$$H^3(X, K_2) \times H^3(X, K_2) \to \mathbf{Q}/\mathbf{Z} \tag{7.10.1}$$

whose kernels are equal to the divisible subgroup of $H^3(X, K_2)$. As in the case $d = 2$, $r = 1$, $i = 2$ (see [24, 2.4]) one can show that the pairing is skew-symmetric. If Tate's conjecture $T(X, 2, p)$ holds, then (7.10.1) is a nondegenerate skew-symmetric pairing of finite groups.

8. Complements on Tate's conjecture. In this section, X will be projective and k finite. Recall that there are cycle maps

$$c_\ell^r : CH^r(X) \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$$

compatible with intersection and cup products and such that c_ℓ^1 is the obvious map on $\mathrm{Pic}(X)$. Define

$$A_\ell^r(X) = CH^r(X)/CH_\ell^r(X), \qquad CH_\ell^r(X) = \mathrm{Ker}(c_\ell^r)$$

$$N^r(X) = CH^r(X)/CH_{\mathrm{num}}^r(X),$$

$$CH_{\mathrm{num}}^r(X) = \{Z \mid Z \cdot Z' = 0 \text{ all } Z' \in CH^{d-r}(X)\}$$

$$B_\ell^r(X) = c_\ell^r(CH^r(X))\mathbf{Q}_\ell = \text{subspace of } H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$$

generated by algebraic cycles.

We shall need to consider two forms of Tate's conjecture [36].

$T'(X, r, \ell) : B_\ell^r(X) = H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma$.

$T(X, r, \ell)$: the dimension of $B_\ell^r(X)$ is the multiplicity of q^r as an inverse root of $P_{2r}(X, t)$.

Remark 8.1. It has been conjectured that $CH_\ell^r(X)$ is independent of ℓ and that $A_\ell^r(X) \otimes \mathbf{Q}_\ell \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$ is injective. When these statements hold, as they do for $r = 0, 1, d$, the conjecture $T(X, r, \ell)$ is independent of ℓ.

Recall that there is also the following conjecture:

$SS(X, r, \ell)$: 1 is not a multiple root of the minimal polynomial of γ acting on $H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$.

This is equivalent to the map $f: H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))_\Gamma$ induced by the identity map being an isomorphism. For $\ell \neq p$, it is also equivalent to q^r not being a multiple root of the minimal polynomial of ϕ acting on $H^{2r}(\bar{X}, \mathbf{Q}_\ell)$.

PROPOSITION 8.2. *For all r and ℓ,*

$$T(X, r, \ell) \Leftrightarrow (T'(X, r, \ell) \text{ and } SS(X, r, \ell)).$$

Proof. There are inclusions

$$B_\ell^r(X) \subset H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^1 \subset H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))_1$$

where

$$H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^1 = \operatorname{Ker}(\gamma - 1)$$

$$H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))_1 = \cup \operatorname{Ker}(\gamma - 1)^N.$$

Moreover,

$$B_\ell^r(X) = H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^1 \Leftrightarrow T'(X, r, \ell)$$

$$H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^1 = H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))_1 \Leftrightarrow SS(X, r, \ell)$$

$$B_\ell^r(X) = H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))_1 \Leftrightarrow T(X, r, \ell).$$

The proposition is now obvious.

COROLLARY 8.3. *For all r and ℓ,*

$$(T'(X, r, \ell) \text{ and } T(X, d - r, \ell)) \Rightarrow T(X, r, \ell).$$

Proof. Since $H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$ is dual to $H^{2d-2r}(\bar{X}, \mathbf{Q}_\ell(r))$, $SS(X, d - r, \ell)$ implies $SS(X, r, \ell)$.

PROPOSITION 8.4. *There are the following implications*:

$$(T'(X, r, \ell) \text{ and } CH_\ell^r(X) = CH_{\text{num}}^r(X)) \Rightarrow T(X, d - r, \ell) \Rightarrow CH_\ell^r(X) = CH_{\text{num}}^r(X).$$

Proof. The hypothesis $CH^r_\ell(X) = CH^r_{\text{num}}(X)$ implies that $c^r: CH^r(X) \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$ factors through $N^r(X)$. Because the intersection pairing $N^r(X) \otimes \mathbf{Q} \times N^{d-r}(X) \otimes \mathbf{Q} \to \mathbf{Q}$ is nondegenerate, elements of $N^r(X) \otimes \mathbf{Q}$ that are linearly independent over $\mathbf{Q}$ will have images in $H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))$ that are linearly independent over $\mathbf{Q}_\ell$. There is therefore an injection $N^r(X) \otimes \mathbf{Q}_\ell \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma$ which $T'(X, r, \ell)$ implies is an isomorphism.

Assume the first statement, and consider the maps

$$
\begin{array}{ccccc}
CH^{d-r}(X) \otimes \mathbf{Q}_\ell & \xrightarrow{f \circ c} & H^{2d-2r}(\bar{X}, \mathbf{Q}_\ell(d-r))_\Gamma & \xrightarrow{\approx} & (H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma)^* \\
& {\scriptstyle c}\searrow & \uparrow {\scriptstyle f} & & \xrightarrow{\approx} (N^r(X) \otimes \mathbf{Q}_\ell)^*. \\
& & H^{2d-2r}(\bar{X}, \mathbf{Q}_\ell(d-r))^\Gamma & &
\end{array}
$$

The right-most isomorphism is the linear dual of that just noted; the other isomorphism is defined by Poincaré duality; f is induced by the identity map on $H^{2d-2r}(\bar{X}, \mathbf{Q}_\ell(d-r))$ and c is the cycle map. The nondegeneracy of the intersection product shows that the composite $CH^{d-r}(X) \otimes \mathbf{Q}_\ell \to (N^r(X) \otimes \mathbf{Q}_\ell)^*$ is surjective. Hence $f \circ c$, and f are also surjective. But the domain and range of f have the same dimension, and so f is an isomorphism: condition $SS(X, d-r, \ell)$ holds. It now also follows that c is surjective: condition $T'(X, d-r, \ell)$ holds. Now (8.2) shows that $T(X, d-r, \ell)$ holds.

Assume $T(d-r, \ell)$ holds. Then $SS(d-r, \ell)$ holds, and so Poincaré duality induces a nondegenerate pairing

$$H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma \times H^{2d-2r}(\bar{X}, \mathbf{Q}_\ell(d-r))^\Gamma \to \mathbf{Z}_\ell.$$

But $T(d-r, \ell)$ implies that $H^{2d-2r}(\bar{X}, \mathbf{Q}_\ell(d-r))^\Gamma = B^{d-r}_\ell(X)$. As $c(A^{d-r}_\ell(X))$ is dense in $B^{d-r}_\ell(X)$, this implies that $CH^r_\ell(X) \supset CH^r_{\text{num}}(X)$. But we always have $CH^r_{\text{num}}(X) \supset CH^r_\ell(X)$, and so this implies that we have equality: $CH^r_\ell(X) = CH^r_{\text{num}}(X)$.

Remark 8.5. For $r = 0, 1, d$, $CH^r_\ell(X) = CH^r_{\text{num}}(X)$ and so, for these values of r, $T'(X, r, \ell)$ implies $T(X, r, \ell)$. Moreover (see (8.1)), $T(X, r, \ell)$ is independent of ℓ. Thus, for $r = 0, 1$, or d, if $T'(X, r, \ell)$ holds for one ℓ, then $T(X, r, \ell)$ holds for all ℓ. This completes the proof of (0.3).

Remark 8.6. For any eigenvalue a of γ on $H^i(\bar{X}, \mathbf{Q}_\ell(r))$, let $H^i(\bar{X}, \mathbf{Q}_\ell(r))_a = \cup \operatorname{Ker}((\gamma - a)^n)$. By Poincaré duality, a^{-1} occurs as an eigenvalue of γ on $H^{2d-i}(\bar{X}, \mathbf{Q}_\ell(d - r))$. Clearly

$$H^i(\bar{X}, \mathbf{Q}_\ell(r))_a \otimes H^{2d-i}(\bar{X}, \mathbf{Q}_\ell(d - r))_{a^{-1}} \subset H^{2d}(\bar{X} \times \bar{X}, \mathbf{Q}_\ell(d))_1,$$

and so if $T(X \times X, \ell, d)$ holds, then a is not a multiple root of the minimal polynomial of γ on $H^i(\bar{X}, \mathbf{Q}_\ell(r))$: $T(X \times X, \ell, d)$ implies γ acts semisimply on the cohomology groups $H^i(\bar{X}, \mathbf{Q}_\ell(r))$ (for $\ell \neq p$, this is equivalent to ϕ acting semisimply on $H^i(\bar{X}, \mathbf{Q}_\ell)$).

In particular we see that if $T(X, \ell, r)$ holds for all X and r, then ϕ acts semisimply on $H^i(\bar{X}, \mathbf{Q}_\ell)$ for all i, and $CH^r_\ell(X) = CH^r_{\text{num}}(X)$ for all r and X.

Remark 8.7. As was pointed out to the author by S. Bloch, the strong Lefschetz theorem shows that if $T(X, 1, \ell)$ holds for all surfaces X, then $T(X, d - 1, \ell)$ holds for all varieties of dimension d (ℓ fixed, $\ell \neq p$). The proof is by induction on d. Let $i: Y \hookrightarrow X$ be a smooth hyperplane section of X (if necessary, we can enlarge the finite field k), and consider

$$H^{2d-4}(\bar{X}, \mathbf{Q}_\ell(d - 2)) \xrightarrow{i^*} H^{2d-4}(\bar{Y}, \mathbf{Q}_\ell(d - 2)) \xrightarrow{i_*} H^{2d-2}(\bar{X}, \mathbf{Q}_\ell(d - 1)).$$

The composite is $x \mapsto i_* i^*(x) = x \cup c^1(Y)$ (see [27, VI 11.6d])—call this map L. Then the strong Lefschetz theorem states that the composite L^{d-1}

$$H^2(\bar{X}, \mathbf{Q}_\ell) \xrightarrow{L^{d-2}} H^{2d-4}(\bar{X}, \mathbf{Q}_\ell(d - 2)) \xrightarrow{L} H^{2d-2}(\bar{X}, \mathbf{Q}_\ell(d - 1))$$

is an isomorphism. In particular L, and therefore i_*, are surjective. With the notation of the proof of (8.2), $i_*: H^{2d-4}(\bar{Y}, \mathbf{Q}_\ell(d - 2))_1 \to H^{2d-2}(\bar{X}, \mathbf{Q}_\ell(d - 1))_1$ is also surjective. By induction $CH^{d-2}(Y) \to H^{2d-4}(\bar{Y}, \mathbf{Q}_\ell(d - 2))_1$ is surjective, and so this proves $T(X, d - 1, \ell)$.

9. Proofs of (0.4) and (0.6), and additional remarks. Let $r \leq 1$. Then, as we saw in (7.5), there are exact sequences

$$0 \to H^i(X, K_r)^\wedge \to H^{i+r}(X, \hat{\mathbf{Z}}(r)) \to TH^{i+1}(X, K_r) \to 0.$$

As $TH^i(X, K_r)$ is torsion-free, there are isomorphisms

$$H^i(X, K_r)^{\wedge}_{\text{tor}} \to H^{i+r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}. \tag{9.0.1}$$

LEMMA 9.1. *Let $r \leq 1$ and, in the case that $r = 1$, assume that $T'(X, 1, \ell)$ holds for at least one ℓ. Then $H^{r+1}(X, K_r)$ is finite, and $z(\delta^r)$ and $z(\epsilon^{2r})$ are defined; moreover,*

$$z(\delta^r) = [H^{r+1}(X, K_r)]z(\epsilon^{2r}).$$

Proof. It is obvious that $T(X, 0, \ell)$ holds for all ℓ, and we have seen in (8.5) that the hypothesis in the lemma implies that $T(X, 1, \ell)$ holds for all ℓ. Thus, by (6.6), $z(\epsilon^{2r})$ is defined and $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}$ is finite. By (7.7b), $H^{r+1}(X, K_r)(\ell)$ is finite, and so

$$H^{r+1}(X, K_r)^{\wedge}_{\text{tor}} = H^{r+1}(X, K_r)_{\text{tor}} = H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}.$$

As $H^{r+1}(X, K_r)$ is torsion, it is finite.

Consider

$$\begin{array}{ccc} H^r(X, K_r)^{\wedge} & \xrightarrow{\delta^r} & TH^{r+2}(X, K_r) \\ \downarrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ H^{2r}(X, \hat{\mathbf{Z}}(r)) & \xrightarrow{\epsilon^{2r}} & H^{2r+1}(X, \hat{\mathbf{Z}}(r)). \end{array}$$

The cokernel of i is $TH^{r+1}(X, K_r)$ which, by the preceding remark, is zero. The kernel of j is $H^{r+1}(X, K_r)$. Thus $z(i)$, $z(\epsilon^{2r})$, and $z(j)$ are all defined, and therefore $z(\delta^r)$ is defined and

$$z(\delta^r) = z(i)z(\epsilon^{2r})z(j) = z(\epsilon^{2r})[H^{r+1}(X, K_r)].$$

We now prove (0.4)(a) and (b). Lemma 9.1, (9.0.1), and (0.1) show that $\chi(X, K_r)$ is defined under the hypotheses of (0.1)(a), (b). Moreover,

$$\chi(X, \hat{\mathbf{Z}}(r)) = \prod_{i \neq 2r, 2r+1} [H^i(X, \hat{\mathbf{Z}}(r))]^{(-1)^i} z(\epsilon^{2r})$$

$$= \prod_{i \neq r, r+1} [H^i(X, K_r)^{\wedge}_{\text{tor}}]^{(-1)^{i+r}} z(\delta^r)[H^{r+1}(X, K_r)]^{-1}$$

$$= \prod_i [H^i(X, K_r)^{\wedge}_{\text{tor}}]^{(-1)^{i+r}} \det(\delta^r)$$

$$= \chi(X, K_r)^{(-1)^r}$$

Thus (0.4)(a) and (b) follow from (0.1).

Part (c) of (0.4) is proved similarly (note that the hypothesis in (0.4c) is $T(X, 2, p)$).

Conjecture $T(X, \ell, r)$ says that

$$CH^r(X) \otimes \mathbf{Q}_\ell \to H^{2r}(\bar{X}, \mathbf{Q}_\ell(r))^\Gamma$$

is surjective. Assume the integral cycle map factors through the Chow group, and consider

$$CH^r(X) \otimes \mathbf{Z}_\ell \to H^{2r}(X, \mathbf{Z}_\ell(r)).$$

Clearly, $T(X, \ell, r)$ implies that the cokernel of this map is finite; we then denote the determinant of the map by $t(X, \ell, r)$, i.e., $t(X, \ell, r)$ is the order of the cokernel of $CH^r(X) \otimes \mathbf{Z}_\ell \to H^{2r}(X, \mathbf{Z}_\ell(r))/\text{torsion}$.

Let Δ^r denote the discriminant of the intersection pairing

$$(Z, Z') \mapsto (Z \cdot Z') : N^r(X) \times N^{d-r}(X) \to \mathbf{Z}.$$

PROPOSITION 9.2. *Assume both $T(X, \ell, r)$ and $T(X, \ell, d - r)$ hold; then*

$$\det(\epsilon_\ell^{2r}) = |\Delta^r|_\ell^{-1} t(X, \ell, r)^{-1} t(X, \ell, d - r)^{-1}$$

where ϵ_ℓ^{2r} is the ℓ-component of ϵ^{2r}.

Proof. Let C_ℓ^r and C_ℓ^{d-r} denote the images of $CH^r(X) \otimes \mathbf{Z}_\ell$ and $CH^{d-r}(X) \otimes \mathbf{Z}_\ell$ in $H^{2r}(X, \mathbf{Z}_\ell(r))$ and $H^{2d-2r}(X, \mathbf{Z}_\ell(d - r))$. We saw in (8.4) that, under the above hypotheses, $C_\ell^r/\{\text{torsion}\} = N_\ell^r$ and $C_\ell^{d-r}/\{\text{torsion}\} = N_\ell^{d-r}$. Therefore, the discriminant of the pairing

$$C_\ell^r \times C_\ell^{d-r} \to \mathbf{Z}_\ell$$

defined by the intersection product is $|\Delta^r|_\ell^{-1}$.

Consider

$$\begin{array}{ccc} C_\ell^r & \longrightarrow & (C_\ell^{d-r})' \\ \downarrow & & \uparrow \\ H^{2r}(X, \mathbf{Z}_\ell(r)) \xrightarrow{\epsilon^{2r}} H^{2r+1}(X, \mathbf{Z}_\ell(r)) & \xrightarrow{\approx} & H^{2d-2r}(X, \mathbf{Z}_\ell(d-r))' \end{array}$$

in which $M' = \mathrm{Hom}(M, \mathbf{Z}_\ell)$. A direct calculation of the determinants of the four nonisomorphisms in this diagram proves the proposition.

COROLLARY 9.3. *If* $CH^r(X) \otimes \mathbf{Z}_\ell \to H^{2r}(X, \mathbf{Z}_\ell(r))$ *and* $CH^{d-r}(X) \otimes \mathbf{Z}_\ell \to H^{2d-2r}(X, \mathbf{Z}_\ell(d-r))$ *are both surjective, then*

$$\det(\epsilon_\ell^{2r}) = |\Delta^r|_\ell^{-1}.$$

Proof. Immediate consequence of the proposition.

We now prove (0.6). From (0.2) we know that

$$|P_4(X, q^{-s})|_p^{-1} \sim \frac{[H^5(X, \mathbf{Z}_p(2))_{\mathrm{tor}}]\det(\epsilon_p^4)}{q^{\alpha_2(X)}[H^4(\bar{X}, \mathbf{Z}_p(2))_{\mathrm{tor}}^\Gamma]^2}(1 - q^{2-s})^{\rho_2} \quad \text{as} \quad s \to 2.$$

But, under the hypotheses of (0.6), $H^3(X, K_2)$ is finite of order $[H^5(X, \mathbf{Z}_p(2))]$, and the first vertical map in

$$\begin{array}{ccc} H^2(X, K_2)^\wedge & \xrightarrow{\delta^2} & TH^4(X, K_2) \\ \downarrow \approx & & \uparrow \\ H^4(X, \mathbf{Z}_p(2)) & \xrightarrow{\epsilon^4} & H^5(X, \mathbf{Z}_p(2)) \end{array}$$

is an isomorphism and the second has a finite kernel. It follows that $\det(\delta^2) = \det(\epsilon^4)$. Finally, $H^4(\bar{X}, \mathbf{Z}_p(2))_{\mathrm{tor}} = H^2(\bar{X}, K_2)^\wedge_{\mathrm{tor}}$, which completes the proof of the first assertion in (0.6).

The second assertion follows from the first and (9.3), since if $CH^2(X) \otimes \mathbf{Z}_p \to H^4(X, \mathbf{Z}_p(2))$ is surjective, so also is $CH^2(X) \otimes \mathbf{Z}_p \to H^4(\bar{X}, \mathbf{Z}_p(2))^\Gamma$.

Remark 9.4. It would be interesting to know exactly when $CH^r(X) \otimes \hat{\mathbf{Z}} \to H^{2r}(X, \hat{\mathbf{Z}}(r))$ is surjective. For $r = d$ one in fact knows more: $CH^d(X)$ is finitely generated and $CH^d(X) \to H^{2d}(X, \hat{\mathbf{Z}}(d))$ is injective with dense image. As $H^1(X, \mathbf{Q}/\mathbf{Z})$ is dual to both $H^{2d}(X, \hat{\mathbf{Z}}(d))$ and the abelian fundamental group $\pi_1(X)^{ab}$, this statement is equivalent to the canonical map $CH^d(X) \to \pi_1(X)^{ab}$ being injective with dense image (see [20]).

10. Compatibility of (0.5) with the functional equation; afterthoughts. The functional equation for $\zeta(X, s)$ is

$$\zeta(X, d-s) = q^{(d/2-s)\chi}\zeta(X, s)$$

where χ is the self-intersection number of the diagonal on $X \times X$ (see [27, VI.12.6]). The number χ can also be expressed as

$$\begin{aligned}\chi = \chi_\ell &\overset{df}{=} \Sigma(-1)^i \dim_{\mathbf{Q}_\ell} H^i(\bar{X}, \mathbf{Q}_\ell), \qquad \ell \neq p,\\ = \chi_{\text{crys}} &\overset{df}{=} \Sigma(-1)^i \dim_K H^i_{\text{crys}}(X/W) \otimes K\\ = \chi_{dR} &\overset{df}{=} \Sigma(-1)^i \dim_k H^i_{dR}(X).\end{aligned}$$

(The last representation, $\chi = \chi_{dR}$, can be deduced from the preceding one; see [5, p. 7-34]). Thus

$$\lim_{s \to r} \frac{\zeta(X, d-s)}{\zeta(X, s)} = q^{(d/2-r)\chi}.$$

We shall compare this result with (0.5).

LEMMA 10.1. *If r and r' are integers such that $r + r' = d$, then*

$$\chi(X, \mathcal{O}_X, r) - \chi(X, \mathcal{O}_X, r') = \frac{1}{2}(r - r')\chi_{dR}.$$

(If $t < 0$, we set $\chi(X, \mathcal{O}_X, t) = 0$.)

Proof. Duality shows that

$$\dim H^j(X, \Omega^i) = \dim H^{d-j}(X, \Omega^{d-i})$$

and therefore that

$$\chi(X, \Omega^i) = (-1)^d \chi(X, \Omega^{d-i}).$$

From this one calculates that

$$\chi_{dR} = \begin{cases} 2 \sum_{i=0}^{d/2-1} (-1)^i \chi(X, \Omega^i) + (-1)^{d/2} \chi(X, \Omega^{d/2}), & d \text{ even} \\ 2 \sum_{i=0}^{[d/2]} (-1)^i \chi(X, \Omega^i) & d \text{ odd.} \end{cases}$$

In proving the lemma we can suppose that $r \geq r'$, and therefore that $r \geq d/2$. Then

$$\chi(X, \mathcal{O}_X, r) - \chi(X, \mathcal{O}_X, r') = \sum_{i=0}^{r'} (r - r')(-1)^i \chi(X, \Omega^i)$$

$$+ \sum_{r'+1}^{r} (r - i)(-1)^i \chi(X, \Omega^i).$$

The second term of this sum can be expressed in terms of the $\chi(X, \Omega^i)$ for $i \leq d/2$. When this is done, and the result compared with expressions for χ_{dR}, one obtains the result.

Proposition 10.2. *For all r,*

$$\lim_{s \to r} \frac{\zeta(X, d - s) q^{-\chi(X, \mathcal{O}_X, d-r)}}{\zeta(X, s) q^{-\chi(X, \mathcal{O}_X, r)}} = 1.$$

Proof. This follows immediately from the preceding calculations.

Remark 10.3. The hypothetical complexes $\mathbf{Z}(r)$ and $\mathbf{Z}(d - r)$ should therefore satisfy

$$\chi(X, \mathbf{Z}(r)) = \chi(X, \mathbf{Z}(d - r)).$$

This should be explained by a duality theorem.

Proposition 10.4. *If* $r > \dim X$, $|\zeta(X, r)|_p^{-1} = q^{\chi(X, \mathcal{O}_X, r)}$.

Proof. The first calculation above shows that

$$\zeta(X, r) = \zeta(X, d - r)q^{(r-d/2)\chi}.$$

As $d - r < 0$, $|1 - a_{ij}q^{d-r}|_p = 1$ for all i and j, and so $|\zeta(X, d - r)|_p = 1$; thus $|\zeta(X, r)|_p^{-1} = q^{(r-d/2)\chi}$. For $r > d$, (10.1) shows that $\chi(X, \mathcal{O}_X, r) = \chi(r - d/2)$, which completes the proof.

Remark 10.5. As mentioned in the introduction, (10.4) is consistent with (0.5) and the expectation that K_r is uniquely divisible by p for $r > d$.

Afterthought 10.6. O. Gabber has pointed out to the author that it is possible to compute the sign of $\lim_{s\to r} \zeta(X, s)/(1 - q^{r-s})^{\rho_r}$ for X smooth and projective. Since, by definition, all other terms in the formulas in (0.1), (0.2), (0.4), and (0.6) are positive, this means that in those formulas the term $\pm$ can be replaced with a definite sign.

Note: (a) for s large and real, $\zeta(X, s)$ is positive;

(b) as s passes an integer point on the real axis, the sign of $\zeta(X, s)$ changes exactly when $\zeta(X, s)$ has an odd zero or pole at the point;

(c) the sign does not change as s passes an odd integer.

To prove (c) we use the nondegenerate pairing

$$\psi : H^i(\bar{X}, \mathbf{Q}_\ell) \times H^i(\bar{X}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$$

given by the strong Lefschetz theorem. (We are identifying $\mathbf{Q}_\ell$ with $\mathbf{Q}_\ell(1)$). This pairing is symmetric if i is even and skew-symmetric if i is odd. Let $H^i(\bar{X}, \mathbf{Q}_\ell)_1 = \cup \operatorname{Ker}(\phi/q^{i/2} - 1)^N$. Then ψ is nondegenerate on $H^i(\bar{X}, \mathbf{Q}_\ell)_1$ and so, if i is odd, this space must have even dimension: $q^{i/2}$ occurs as an eigenvalue of ϕ on $H^i(\bar{X}, \mathbf{Q}_\ell)$ an even number of times. Similarly $-q^{i/2}$ occurs an even number of times.

It follows that the sign in question is $(-1)^{\Sigma b_m}$ where b_m is the order of the zero or pole of $\zeta(X, s)$ at $2m$, and the sum is over all m such that $2m > r$ and $2m \geq 2d - r$.

Afterthought 10.7. S. Lichtenbaum (talk at Journées Arithmétiques, Noordwijkerhout, July, 1983) predicts the existence of a complex $\mathbf{Z}(r)$ of sheaves on X_{et} having, among others, the following properties:

(a_n) for all n prime to p, there is an isomorphism in the derived category,

$$\mu_n^{\otimes r}[-1] \xrightarrow{\sim} (Z(r) \xrightarrow{n} \mathbf{Z}(r));$$

(b) $H^i(X, \mathbf{Z}(r))$ is finite for $i \neq 2r, 2r+1, 2r+2$; $H^{2r}(X, \mathbf{Z}(r))$ is finitely generated; $H^{2r+2}(X, \mathbf{Z}(r))$ is torsion;

(c) $H^{2r+1}(X, \mathbf{Z}(r))$ is finite.

In the present context it is natural also to predict

(a_p) there is an isomorphism in the derived category

$$\nu_m(r)[-r-1] \xrightarrow{\sim} (\mathbf{Z}(r) \xrightarrow{p^m} \mathbf{Z}(r)).$$

The complexes $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{Z}(1) = \mathcal{O}_X^\times[-1]$ satisfy conditions (a) and (b). Condition (c) for $\mathbf{Z}(0)$ is obvious, but for $\mathbf{Z}(1)$ it is the statement that $H^2(X, \mathcal{O}_X^\times)$ is finite which, as we have seen, is equivalent to Tate's conjecture. Results announced by Gabber and Suslin show that, for n prime to p, there is an exact sequence

$$0 \to \mu_n^{\otimes r} \to K_{2r-1}\mathcal{O}_X \xrightarrow{n} K_{2r-1}\mathcal{O}_X \to 0$$

of sheaves on X_{et}. Since we know there is an exact sequence

$$0 \to K_2\mathcal{O}_X \xrightarrow{p^m} K_2\mathcal{O}_X \to \nu_m(2) \to 0,$$

it at least seems plausible that there should exist such a complex $\mathbf{Z}(2) = (Z^1 \to Z^2)$ with

$$H^2(Z^1 \to Z^2) = K_2\mathcal{O}_X$$

$$H^1(Z^1 \to Z^2) = K_3\mathcal{O}_X/(\text{symbolic part})$$

as predicted by Lichtenbaum.

Now assume that there exists a complex $\mathbf{Z}(r)$ satisfying (a), (b), and (c). From (a) we get exact sequences

$$\cdots \to H^i(X, \mathbf{Z}(r)) \xrightarrow{n} H^i(X, \mathbf{Z}(r)) \to H^i(X, (\mathbf{Z}/n\mathbf{Z})(r)) \to \cdots$$

for all n which, in the limit, become

$$0 \to H^i(X, \mathbf{Z}(r))^\wedge \to H^i(X, \hat{\mathbf{Z}}(r)) \to TH^{i+1}(X, \mathbf{Z}(r)) \to 0.$$

Note that (b) and (c) imply

$$H^i(X, \mathbf{Z}(r))^\wedge = H^i(X, \mathbf{Z}(r)), \qquad i \neq 2r, 2r+2,$$

$$H^{2r}(X, \mathbf{Z}(r))^\wedge = H^{2r}(X, \mathbf{Z}(r)) \otimes \hat{\mathbf{Z}}$$

$$H^{2r+2}(X, \mathbf{Z}(r))^\wedge = H^{2r+2}(X, \mathbf{Z}(r))_{\text{cotor}} \text{ (a finite group)}$$

$$TH^{i+1}(X, \mathbf{Z}(r)) = 0, \qquad i \neq 2r+1.$$

Let $\epsilon^{2r}: H^{2r}(X, \hat{\mathbf{Z}}(r)) \to H^{2r+1}(X, \hat{\mathbf{Z}}(r))$ be as in Section 6, and define δ^r so as to make the following diagram commute:

$$\begin{array}{ccc} H^{2r}(X, \mathbf{Z}(r)) \otimes \hat{\mathbf{Z}} & \xrightarrow{\delta^r} & TH^{2r+2}(X, \mathbf{Z}(r)) \\ \downarrow i & & \uparrow j \\ H^{2r}(X, \hat{\mathbf{Z}}(r)) & \xrightarrow{\epsilon^{2r}} & H^{2r+1}(X, \hat{\mathbf{Z}}(r)). \end{array}$$

Condition (c) implies that i is an isomorphism and that j is surjective with finite kernel $H^{2r+1}(X, \mathbf{Z}(r))$. Thus $z(\delta^r)$ is defined provided $z(\epsilon^{2r})$ is defined, and then

$$z(\delta^r) = z(\epsilon^{2r}) z(j) = z(\epsilon^{2r})[H^{2r+1}(X, \mathbf{Z}(r))].$$

Define

$$\chi(X, \mathbf{Z}(r))$$

$$= \prod_{i \neq 2r, 2r+2} [H^i(X, \mathbf{Z}(r))]^{(-1)^i} \frac{[H^{2r}(X, \mathbf{Z}(r))_{\text{tor}}][H^{2r+2}(X, \mathbf{Z}(r))_{\text{cotor}}]}{\det(\delta^r)}.$$

The observations just made show that

$$H^i(X, \mathbf{Z}(r))_{\text{tor}} = H^i(X, \hat{\mathbf{Z}}(r))_{\text{tor}}, \qquad i \neq 2r+2,$$

$$H^{2r+2}(X, \mathbf{Z}(r))_{\text{cotor}} = H^{2r+2}(X, \hat{\mathbf{Z}}(r))_{\text{tor}},$$

and

$$\frac{[H^{2r}(X, \mathbf{Z}(r))_{\text{tor}}]}{\det(\delta^r)} = z(\delta^r) = z(\epsilon^{2r})[H^{2r+1}(X, \mathbf{Z}(r))]$$

$$= \frac{[H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]}{\det(\epsilon^{2r})}.$$

Thus

$$\chi(X, \mathbf{Z}(r))$$

$$= \prod_{i \neq 2r, 2r+2} [H^i(X, \hat{\mathbf{Z}}(r))]^{(-1)^i} \frac{[H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]}{\det(\epsilon^{2r})} [H^{2r+2}(X, \hat{\mathbf{Z}}(r))_{\text{tor}}]$$

$$= \chi(X, \hat{\mathbf{Z}}(r))$$

and it follows from (0.1) that

$$\zeta(X, s) \sim \pm\chi(X, \mathbf{Z}(r)) q^{\chi(X, \mathcal{O}_X, r)} (1 - q^{r-s})^{-\rho_r} \quad \text{as} \quad s \to r.$$

We remark that if one assumes only (a) and (c) and that $z(\epsilon^{2r})$ is defined, then the same argument shows that

$$\zeta(X, s) \sim \pm\chi'(X, \mathbf{Z}(r)) q^{\chi(X, \mathcal{O}_X, r)} (1 - q^{r-s})^{-\rho_r} \quad \text{as} \quad s \to r$$

where

$$\chi'(X, \mathbf{Z}(r)) = \prod_i [H^i(X, \mathbf{Z}(r))^{\wedge}_{\text{tor}}]^{(-1)^i} \det(\delta^r)^{-1}.$$

Since $K_{2r-1}\mathcal{O}_X[-r]$ satisfies the non-p-part of (a), this shows that, under the assumptions that the non-p-part of $z(\epsilon^{2r})$ is defined and the non-p-part of $H^{r+1}(X, K_{2r-1}\mathcal{O}_X)$ is finite,

$$\zeta(X, s) \sim C(\chi(X, K_{2r-1})(\text{non-}p))^{(-1)^r} q^{\chi(X, \mathcal{O}_X, r)} (1 - q^{r-s})^{-\rho_r}$$

as $s \to r$ where $|C|$ is a power of p and $\chi(X, K_{2r-1})$ is defined analogously to $\chi'(X, \mathbf{Z}(r))$.

It should be noted that although $\det(\delta^r)$ is defined as the determinant of a mapping of $\hat{\mathbf{Z}}$-modules, it is possible to realize it as the determinant of a mapping of finitely generated $\mathbf{Z}$-modules: let

$$T' = \{x \in TH^{2r+2}(X, \mathbf{Z}(r)) \mid mx \in \delta^r(H^{2r}(X, \mathbf{Z}(r))) \text{ some } m \neq 0\};$$

then T' is finitely generated and $\det(\delta^r)$ is the determinant of the map $H^{2r}(X, \mathbf{Z}(r)) \to T'$ induced by δ^r (cf. [22, 2.5, 2.6]). The approach to the determinant (or regulator) term in $\chi(X, \mathbf{Z}(r))$ adopted in this paper avoids the double dualizing in [22].

Of course, it should be possible to interpret $\det(\delta^r)$ in terms of intersections of algebraic cycles. For this it is natural to make the following assumption: (d) there exists a cycle map $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$ such that the composite

$$CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))^\wedge \to H^{2r}(X, \hat{\mathbf{Z}}(r)) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is the product of the ℓ-adic étale cycle maps with the p-adic cycle map defined in Section 2.

Now assume (a), (b), and (d) (but not (c)). If $T(X, r, \ell)$ holds for all ℓ, then it is possible to prove by the same methods as in (9.1) that (c) holds. Moreover then the cokernel of $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$ will be torsion, with finite ℓ-primary component for all ℓ. As we are assuming that $H^{2r}(X, \mathbf{Z}(r))$ is finitely generated, this means that the cokernel will be finite, and we denote its order by $t(X, r)$. Now assume also that $T(X, d - r, \ell)$ holds for all ℓ. Then it is possible to prove, as for (9.2), that

$$\det(\delta^r) = \Delta^r t(X, r)^{-1} t(X, d - r)^{-1}$$

where Δ^r is the discriminant of the intersection pairing

$$(Z, Z') \mapsto (Z \cdot Z'): N^r(X) \times N^{d-r}(X) \to \mathbf{Z}.$$

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