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V. CONJUGATES OF SHIMURA VARIETIES

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Introduction: In the first three sections we review the definition of a Shimura variety of abelian type, describe how certain Shimura varieties are moduli varieties for abelian varieties with Hodge cycles and level structure, and prove a result concerning reductive groups that will frequently enable us to replace one such group by a second whose derived group is simply connected.

To be able to discuss the results in the remaining sections both concisely and precisely, we shall assume throughout the rest of the introduction that a pair (G,X) defining a Shimura variety Sh(G,X) satisfies the following additional conditions (Deligne [2, 2.1.1.4, 2.1.1.5]):

(0.1) for any $h \in X$, the weight $w_h : \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$;

(0.2) ad $h(i)$ is a Cartan involution on $(G/w(\mathbb{G}_m))_{\mathbb{R}}$.

These conditions imply that for any special $h \in X$, the associated cocharacter $\mu = \mu_h$ factors through the Serre group: $\mu = \rho_\mu \circ \mu_{can}$, $\rho_\mu : S \to G$. Thus to any such h and any representation of G there is associated a representation of S, and hence an object in the category of motives generated by abelian varieties of CM-type over $\mathbb{C}$.

Consider the Taniyama group

$$1 \to S \to \underset{\sim}{T} \overset{\pi}{\to} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

$$\underset{\sim}{T}(\mathbb{A}^f) \underset{\pi}{\overset{sp}{\leftrightarrows}} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ , \ \pi \circ sp = 1 .$$

For any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, ${}^{\tau}S \overset{df}{=} \pi^{-1}(\tau)$ is an S-torsor with a distinguished $\mathbb{A}^f$-point $sp(\tau)$. If $h \in X$ is special, we can use ρ_μ, $\mu = \mu_h$, to transform the adjoint action of G on itself into an action of S on G. We can then use ${}^{\tau}S$ to twist G, and so define ${}^{\tau,\mu}G = {}^{\tau}S \times {}^{S}G$. Thus ${}^{\tau,\mu}G$ is a $\mathbb{Q}$-rational algebraic group such that ${}^{\tau,\mu}G(\overline{\mathbb{Q}}) = \{s.g \mid s \in {}^{\tau}S(\overline{\mathbb{Q}}) \ , \ g \in G(\overline{\mathbb{Q}})\}/\sim$ where $ss_1.g \sim s.\rho_\mu(s_1)g\rho_\mu(s_1)^{-1}$, all $s_1 \in S(\overline{\mathbb{Q}})$. Let $T \subset G$ be a $\mathbb{Q}$-rational torus through which h factors. Then ${}^{\tau,\mu}T \overset{df}{=} {}^{\tau}S \times {}^{S}T = T$, and so T is also a subgroup of ${}^{\tau,\mu}G$. Define ${}^{\tau}h$ to be the homomorphism $\mathbb{S} \to {}^{\tau,\mu}G$ with associated cocharacter $\tau\mu : \mathbb{G}_m \to T \subset {}^{\tau,\mu}G$, and let ${}^{\tau,\mu}X$ be the ${}^{\tau,\mu}G(\mathbb{R})$-

conjugacy class containing ${}^{\tau}h$. The point $sp(\tau)$ provides us with a canonical isomorphism $g \mapsto {}^{\tau,\mu}g \overset{df}{=} sp(\tau).g : G(\mathbb{A}^f) \to {}^{\tau,\mu}G(\mathbb{A}^f)$. The pair $({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ defines a Shimura variety, and the first part of the Langlands's conjecture states the following.

Conjecture C . (a) For any special $h \in X$, with $\mu_h = \mu$, there is an isomorphism $\phi_{\tau,\mu} : \tau\, Sh(G,X) \to Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ such that

$$\phi_{\tau,\mu}(\tau[h,1]) = [{}^{\tau}h,1]$$

$$\phi_{\tau,\mu} \circ \tau\, \mathcal{J}(g) = \mathcal{J}({}^{\tau,\mu}g) \circ \phi_{\tau,\mu}, \quad g \in G(\mathbb{A}^f)\ , \ \mathcal{J}(g) =$$
Hecke operator.

In order to compare the isomorphisms ϕ corresponding to two different special points, it is necessary to construct some isomorphisms. For this the following two lemmas are useful.

Lemma 0.3. Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is simply connected. Two elements of $H^1(\mathbb{Q},G)$ are equal if their images in $H^1(\mathbb{Q},G/G^{der})$ and $H^1(\mathbb{R},G)$ are equal.

Lemma 0.4. Let (G_1,X_1) and (G_2,X_2) define Shimura varieties, and suppose there are given:

$f_1 : G_1 \xrightarrow{\approx} G_2$ mapping X_1 into X_2 ;

$f_2 : G_1(\mathbb{A}^f) \xrightarrow{\approx} G_2(\mathbb{A}^f)$;

$\beta \in G_1(\mathbb{A}^f)$ such that $f_1 \circ \underset{\sim}{\mathrm{ad}}\ \beta^{-1} = f_2$.

Then $\phi \overset{\mathrm{df}}{=} Sh(f_1) \circ \mathcal{J}(\beta) : Sh(G_1, X_1) \xrightarrow{\approx} Sh(G_2, X_2)$ has the following properties:

$\phi[h, \beta^{-1}] = [f_1 \circ h, 1]$, all $h \in X$;

$\phi \circ \mathcal{J}(g) = \mathcal{J}(f_2(g)) \circ \phi$, all $g \in G_1(\mathbb{A}^f)$.

Moreover, if f_1 is replaced with $f_1 \circ \underset{\sim}{\mathrm{ad}}\ q$, $q \in G_1(\mathbb{Q})$, and β with βq, then ϕ is unchanged.

Let h and h' be special points of X with cocharacters μ and μ' . A direct calculation shows that $\rho_{\mu *}({}^{\tau}S)$ and $\rho_{\mu' *}({}^{\tau}S)$ have the same image in $H^1(\mathbb{R}, G)$, and they become equal in $H^1(\mathbb{Q}, G/G^{der})$ because ρ_{μ} and $\rho_{\mu'}$ define the same map to G/G^{der} . There is therefore a $\mathbb{Q}$-rational isomorphism $f : \rho_{\mu *}({}^{\tau}S) \to \rho_{\mu' *}({}^{\tau}S)$ which, because ${}^{\tau,\mu}G \overset{\mathrm{df}}{=} {}^{\tau}S \times {}^{S}G = {}^{\tau}S \times {}^{S}G \times {}^{G}G = \rho_{\mu *}({}^{\tau}S) \times {}^{G}G$, can be transferred into an isomorphism $f_1 : {}^{\tau,\mu}G \to {}^{\tau,\mu'}G$ which is uniquely determined up to composition with $\mathrm{ad}\,q$, $q \in {}^{\tau,\mu}G(\mathbb{Q})$; it maps ${}^{\tau,\mu}X$ into ${}^{\tau,\mu'}X$. Let $f_2 : {}^{\tau,\mu}G(\mathbb{A}^f) \to {}^{\tau,\mu'}G(\mathbb{A}^f)$ be $sp(\tau).g \mapsto sp(\tau).g$. Then there is a $\beta \in {}^{\tau,\mu}G(\mathbb{A}^f)$ satisfying $f_1 \circ \underset{\sim}{\mathrm{ad}}\ \beta^{-1} = f_2$ whose definition depends on the choice of f_1 : if f_1 is changed to $f_1 \circ \underset{\sim}{\mathrm{ad}}\ q$ then β is changed to βq . There is therefore a well-defined map $\phi(\tau;\mu',\mu) : Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) \to Sh({}^{\tau,\mu'}G, {}^{\tau,\mu'}X)$ such that $\phi(\tau;\mu',\mu) \circ \mathcal{J}({}^{\tau,\mu}g) = \mathcal{J}({}^{\tau,\mu'}g) \circ \phi(\tau;\mu',\mu)$.

<u>Conjecture</u> C. (b) For special $h, h' \in X$, the maps $\phi_{\tau,\mu}$ and $\phi_{\tau,\mu'}$ satisfy $\phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu} = \phi_{\tau,\mu'}$.

If τ fixes the reflex field $E(G,X)$ of $Sh(G,X)$, then Shimura's conjecture asserting the existence of a canonical model for $Sh(G,X)$ over $E(G,X)$ shows that $\tau\, Sh(G,X) \approx Sh(G,X)$ canonically. This suggests that, for τ fixing $E(G,X)$, there should exist a canonical isomorphism $\phi(\tau;\mu) : Sh(G,X) \to Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. Again (0.3) and the result in §3 enable one to show that, in this case, $\rho_{\mu *}({}^{\tau}S) \in H^1(\mathbb{Q},G)$ is trivial. This allows us to define an isomorphism $f_1 : (G,X) \xrightarrow{\approx} ({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ such that the conditions of (0.4) are satisfied for f_1, $f_2 = (g \mapsto sp(\tau).g)$, and a certain $\beta \in G(\mathbb{A}^f)$. Thus the canonical isomorphism $\phi(\tau;\mu)$ exists.

<u>Theorem</u> 0.5. Let $\tau \in Aut(\mathbb{C})$ fix $E(G,X)$.

(a) Let $h \in X$ be special and let $\mu = \mu_h$. Choose elements $a(\tau) \in {}^{\tau}S\,(\overline{\mathbb{Q}})$ and $c(\tau) \in \rho_{\mu *}({}^{\tau}S)(\mathbb{Q})$, and let $v \in G(\overline{\mathbb{Q}})$ and $\alpha \in G(\mathbb{A}^f)$ be such that $\rho_\mu(a(\tau)) = c(\tau)v$ and $\rho_\mu(sp(\tau)) = c(\tau)\alpha$. Then the element $[ad(v) \circ {}^{\tau}h, \alpha]$ of $Sh(G,X)$ is independent of the choice of $a(\tau)$ and $c(\tau)$.

(b) Assume that $Sh(G,X)$ has a canonical model; then conjecture C is true for τ and $Sh(G,X)$ if and only if

$$\tau[h,1] = [\mathrm{ad}v \circ {}^{\tau}h,\ \alpha]$$

for all special $h \in X$.

(c) If conjecture C is true for $Sh(G,X)$ and all τ fixing $E(G,X)$ then $Sh(G,X)$ has a canonical model $(M(G,X), M(G,X)_{\mathbb{C}} \xrightarrow[\approx]{f} Sh(G,X))$; moreover, $f\circ(\tau f)^{-1} = \phi(\tau,\mu)^{-1} \circ \phi_{\tau,\mu}$ for every μ corresponding to a special h .

Let A be an abelian variety over $\mathbb{C}$ with complex multiplication by a CM-field F (so that $V \overset{df}{=} H_1(A, \mathbb{Q})$ is of dimension 1 over F) . Write T for $\mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_m$, and let $h : \mathbb{S} \to T_{\mathbb{R}}$ be the homomorphism defined by the Hodge structure on V . The main theorem of complex multiplication describes the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$ on $Sh(T, \{h\})$ arising from its identification with a moduli variety. From conjecture C for $Sh(CSp(V), S^{\pm})$ one can deduce a description of the action of the whole of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\bigcup_{\tau \in \mathrm{Hom}(F,\overline{\mathbb{Q}})} Sh(T,\{{}^{\tau}h\}) \subset Sh(CSp(V),S^{\pm})$. This suggests a conjecture (conjecture CM) stated purely in terms of abelian varieties of CM-type.

Proposition 0.6. Conjecture CM is true if and only if conjecture C is true for all Shimura varieties of the form $Sh(CSp(V),S^{\pm})$.

It is possible to restate conjecture C for connected Shimura varieties. For this it is first necessary to show that, for a connected Shimura variety $Sh^{\circ}(G,G',X^{+})$, special $h, h' \in X^{+}$, and $\tau \in \mathrm{Aut}(\mathbb{C})$, there are maps

$$g \mapsto {}^{\tau,\mu}g : G(\mathbb{Q})^{+} \ (\mathrm{rel}\ G') \to {}^{\tau,\mu}G(\mathbb{Q})^{+} \ (\mathrm{rel}\ {}^{\tau,\mu}G')$$

$$\overset{\circ}{\phi}(\tau;\mu',\mu) : Sh^{\circ}({}^{\tau,\mu}G, {}^{\tau,\mu}G', X^{+}) \to Sh^{\circ}({}^{\tau,\mu'}G, {}^{\tau,\mu'}G', X^{+})$$

compatible with those defined for nonconnected Shimura varieties.

Conjecture C°

(a) For any special $h \in X^{+}$, with $\mu = \mu_h$, there is an isomorphism

$$\overset{\circ}{\phi}_{\tau,\mu} : \tau\, Sh^{\circ}(G,G',X^{+}) \to Sh^{\circ}({}^{\tau,\mu}G, {}^{\tau,\mu}G', {}^{\tau}X^{+})$$

such that $\overset{\circ}{\phi}_{\tau,\mu}(\tau[h]) = [{}^{\tau}h]$

$$\phi^{\circ}_{\tau,\mu} \circ \tau(\gamma\cdot) = {}^{\tau}\gamma\cdot \circ \phi^{\circ}_{\tau,\mu}, \ \gamma \in G(\mathbb{Q})^{+\wedge} \ (\mathrm{rel}\ G') .$$

(b) For h' a second special element and $\mu' = \mu_{h'}$,

$$\phi^{\circ}(\tau;\mu',\mu) \circ \phi_{\tau,\mu} = \phi_{\tau,\mu'} .$$

Proposition 0.7. Conjecture C is true for $Sh(G,X)$ if and only if conjecture C° is true for $Sh^{\circ}(G^{ad}, G^{der}, X^{+})$

Using 0.7) we prove the following.

Theorem 0.8. If conjecture C is true for all Shimura varieties of the form $Sh(CSp(V),S^{\pm})$ then it is true for all Shimura varieties of abelian type.

All of the above continues to make sense if the Taniyama group is replaced by the motivic Galois group (II.6) except that the maps $\phi(\tau;\mu',\mu)$ and $\phi(\tau;\mu)$ are (possibly) different and the conjectures have a (possibly) different meaning. We shall use a tilde to distinguish the objects associated with the motive Galois group from those associated with the Taniyama group. A new fact is that, almost by construction of the motivic Galois group, conjecture $\widetilde{CM}$ is true. Thus $(\widetilde{0.6})$ and $(\widetilde{0.8})$ show that conjecture $\tilde{C}$ is true for all Shimura varieties of CM-type. This has the following consequence.

Theorem 0.9. Let $Sh(G,X)$ be a Shimura variety of abelian type and let $M(G,X)$ be its canonical model. For any μ associated with a special h, there is an isomorphism $g \mapsto g' : G(\mathbb{A}^f) \to {}^{\tau,\mu}G(\mathbb{A}^f)$ such that, if $g' \in {}^{\tau,\mu}G(\mathbb{A}^f)$ is made to act on $\tau\, M(G,X)$ as $\tau(\mathcal{J}(g))$, then $\tau\, M(G,X)$ together with this action is a canonical model for $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$.

(0.9) is the original form Langlands's conjecture on Shimura varieties. (${}^{\tau,\mu}G$ is the same for the motivic Galois group and the Taniyama group.) Such a result was first proved for Shimura curves by Doi and Naganuma [1] and for Shimura varieties of primitive type A and C by Shih [2]. A theorem of Kazhdan [1] can be interpreted as saying that the conjugate $\tau\, Sh(G,X)$ of a compact Shimura variety is again a Shimura variety but unfortunately his method gives little information on the pair (G',X') to which the conjugate corresponds.

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Notations and conventions.

For Shimura varieties and algebraic groups we generally follow the notations of Deligne [2]. Thus a reductive algebraic group G is always connected, with derived group G^{der}, adjoint group G^{ad}, and centre $Z = Z(G)$. (We assume also that G^{ad} has no factors of type E_8). A central extension is an epimorphism $G \to G'$ whose kernel is contained in $Z(G)$, and a covering is a central extension such that G is connected and the kernel is finite. If G is reductive, then $\rho : \tilde{G} \to G^{der}$ is the universal covering of G^{der}.

A superscript $+$ refers to a topological connected component; for example $G(\mathbb{R})^+$ is the identity connected component of $G(\mathbb{R})$ relative to the real topology, and $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. For G reductive, $G(\mathbb{R})_+$ is the inverse image of $G^{ad}(\mathbb{R})^+$ in $G(\mathbb{R})$ and $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. In contrast to Deligne [2], we use the superscript ^ to denote both completions and closures since we wish to reserve the superscript - for certain negative components.

We write $Sh(G,X)$ for the Shimura variety defined by a pair (G,X) and $Sh^{\circ}(G, G', X^{+})$ for the connected Shimura variety defined by a triple (G, G', X^{+}). The canonical model of $Sh(G,X)$ is denoted by $M(G,X)$.

Vector spaces are finite-dimensional, number fields are of finite degree over $\mathbb{Q}$ (and usually contained in $\mathbb{C}$), and $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. If V is a vector space over $\mathbb{Q}$ and R is the $\mathbb{Q}$-algebra, we often write $V(R)$ for $V \otimes R$.

If $x \in X$ and $g \in G(\mathbb{A}^f)$ then $[x, g]$ denotes the element of $Sh(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^f)/Z(\mathbb{Q})^{\wedge}$ containing (x,g). The Hecke operator $[x,g] \mapsto [x,gg']$ is denoted by $\mathcal{J}(g')$. The symbol $A \overset{df}{=} B$ means A is defined to be B or that A equals B by definition.

For Galois cohomology and torsors (= principal homogeneous spaces) we follow the notations of Serre [1].

For the Taniyama group, we use the same notations as in III; we refer the reader particularly to III. 2.9.

If A is an abelian variety, then

$$V^{f}(A) \overset{df}{=} (\varprojlim \ker(n\colon A \to A)) \otimes \mathbb{Q}$$

depends functorially on the isogeny class of A. Throughout the article, an abelian variety will be regarded as an object in the category of abelian varieties up to isogeny.

1. Shimura varieties of abelian type.

A Shimura variety $Sh(G,X)$ is defined by a pair (G,X), comprising a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$, that satisfies the following axioms:

(1.1a) the Hodge structure defined on $\mathrm{Lie}(G_{\mathbb{R}})$ by any $h \in X$ is of type $\{(-1, 1), (0,0), (1, -1)\}$;

(1.1b) for any $h \in X$, $\mathrm{ad}\, h(i)$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$;

(1.1c) the group G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group. Then $Sh(G,X)$ has complex points $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^f)/Z(\mathbb{Q})^\wedge$, where Z is the centre of G and $Z(\mathbb{Q})^\wedge$ the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}^f)$.

A connected Shimura variety $Sh^o(G,G',X^+)$ is defined by a triple (G,G',X^+) comprising an adjoint group G over $\mathbb{Q}$, a covering G' of G, and a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ such that G and the $G(\mathbb{R})$-conjugacy class of X containing X^+ satisfy (1.1). The topology $\tau(G')$ on $G(\mathbb{Q})$ is that for which the images of the congruence subgroups of $G'(\mathbb{Q})$ form a fundamental system of neighbourhoods of the identity and $Sh^o(G,G',X^+)$ has complex points $\varprojlim \Gamma\backslash X^+$ where Γ runs over the arithmetic subgroups of $G(\mathbb{Q})^+$ that are open relative to the topology $\tau(G')$ (Deligne [2, 2.1.8]).

The relation between the two notions of Shimura variety is as follows: let (G,X) be as in the first paragraph and let X^+ be some connected component of X; then X^+ can be regarded as a $G^{ad}(\mathbb{R})^+$-conjugacy class of maps $\mathbb{S} \to G^{ad}_{\mathbb{R}}$ and $Sh^o(G^{ad}, G^{der}, X^+)$ can be identified with the connected component of $Sh(G,X)$ that contains the image of $X^+ \times \{1\}$.

We recall that the reflex field $E(G,X)$ of (G,X) is the subfield of $\mathbb{C}$ that is the field of definition of the $G(\mathbb{C})$ - conjugacy class of μ_h , any $h \in X$, (μ_h = restriction of $h_{\mathbb{C}}$ to $\mathbb{G}_m \times 1 \subset \mathbb{S}_{\mathbb{C}}$) and that $E(G,X^+)$ is defined to equal $E(G,X)$ if X^+ is a connected component of X (Deligne [2, 2.2.1]).

The following easy lemma will be needed in comparing the Shimura varieties defined by (G,X) and (G^{ad}, G^{der}, X^+) .

<u>Lemma 1.2</u>. Let $G_1 \to G$ be a central extension of reductive groups over $\mathbb{C}$; let M be a $G(\mathbb{C})$-conjugacy class of homomorphisms $\mathbb{G}_m \to G$ and let M_1 be a $G_1(\mathbb{C})$-conjugacy class lifting M . Then $M_1 \to M$ is bijective.

<u>Proof</u>. The map is clearly surjective and so it suffices to show that, for $\mu_1 \in M_1$ lifting $\mu \in M$, the centralizer of μ_1 is the inverse image of the centralizer of μ . Since the centralizer of μ_1 contains the center of G_1 , we only have to show the map on centralizers is surjective. We can construct a diagram

$$C \times G_2 \to G_1 \to G$$

in which the first map, and the composite $G_2 \to G$ are coverings. After replacing μ_1 and μ by multiples, we can assume μ_1 lifts to a homomorphism $(\mu',\mu''): \mathbb{G}_m \to C \times G_2$. Then the centralizer of (μ',μ'') maps into the centralizer of μ_1, and onto the centralizer of μ.

Let (G,X) be as in (1.1) with G adjoint and $\mathbb{Q}$-simple; if every $\mathbb{R}$-simple factor of $G_{\mathbb{R}}$ is of one of the types A, B, C, $D^{\mathbb{R}}$, $D^{\mathbb{H}}$, or E (in the sense of Deligne [2, 2.3.8]) then G will be said to be of that type. When G' is a covering of G, we say that (G,G') (or (G,G',X)) is of <u>primitive abelian type</u> if G is of type A, B, C, or $D^{\mathbb{R}}$ and G' is the universal covering of G, or if G is of type $D^{\mathbb{H}}$ and G' is the double covering described in Deligne [2, 2.3.8] (see Milne-Shih [1, Appendix]).

If (G,X) satisfies (1.1) and G is adjoint and $\mathbb{Q}$-simple, then there is a totally real number field F_o and an absolutely simple group G^s over F_o such that $G = \mathrm{Res}_{F_o/\mathbb{Q}} G^s$. For any embedding $v : F_o \hookrightarrow \mathbb{R}$, let $G_v = G^s \otimes_{F_o,v} \mathbb{R}$, and write I_c and I_{nc} for the sets of embeddings for which $G_v(\mathbb{R})$ is compact and noncompact. Let F be a quadratic totally imaginary extension of F_o and let $\Sigma = (\sigma_v)_{v \in I_c}$ be a set of embeddings $\sigma_v : F \hookrightarrow \mathbb{C}$ such that $\sigma_v | F_o = v$; we define h_Σ to be the Hodge structure on F (regarded as a vector space over $\mathbb{Q}$) such that $(F \otimes_{\mathbb{Q}} \mathbb{C})^{-1,0}$, $(F \otimes_{\mathbb{Q}} \mathbb{C})^{0,-1}$ and $(F \otimes_{\mathbb{Q}} \mathbb{C})^{0,0}$ are the direct summands of $F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(F,\mathbb{C})}$ corresponding to Σ, $\iota\Sigma$, and

$\{\sigma : F \hookrightarrow \mathbb{C} \mid \sigma|F_o \in I_{nc}\}$.

Proposition 1.3. Let G be a $\mathbb{Q}$-simple adjoint group and assume that (G,G',X) is of primitive abelian type. For any pair (F,Σ) as above there exists a diagram

$$(G,X) \longleftarrow (G_1,X_1) \hookrightarrow (CSp(V),S^{\pm})$$

such that $G_1^{ad} = G$, $G_1^{der} = G'$, and $E(G_1,X_1) = E(G,X)\ E(F^{\times},h_{\Sigma})$.

Proof. This is Deligne [2, 2.3.10].

Let (G,X) satisfy (1.1) with G adjoint, and let G' be a covering of G. We say that (G,G') or (G,G',X) is of abelian type if there exist pairs $(G_i,G_i')_i$ of primitive abelian type such that $G = \Pi G_i$ and G' is a quotient of the covering $\Pi G_i'$ of ΠG_i . If (G,X) satisfies (1.1), we say that G or (G,X) is of abelian type if (G^{ad},G^{der}) is of this type. Finally, we say that a Shimura variety $Sh^{o}(G,G',X^{+})$ or $Sh(G,X)$ is of abelian type if (G,G') or G is.

2. Shimura varieties as moduli varieties.

We shall want to make use of the notion of an absolute Hodge cycle on a variety (Deligne [3, 0.7]) and the important result (see I.2.11) that any Hodge cycle on an abelian variety is an absolute Hodge cycle. Let A be an abelian variety over an algebraically closed field $k \subset \mathbb{C}$; we shall always identify a Hodge cycle on A with its Betti realization. By this we mean the following. Let $V = H_1(A_{\mathbb{C}}, \mathbb{Q})$ (usual Betti homology) and note that V has a natural Hodge structure and that its dual $\check{V} = H^1(A, \mathbb{Q})$. If $H^1_{dR}(A)$ denotes the de Rham cohomology of A over k then there is a canonical isomorphism $H^1_{dR}(A) \otimes_k \mathbb{C} \xrightarrow{\approx} \check{V}(\mathbb{C})$. There is also a canonical isomorphism $V^f(A) \xrightarrow{\approx} V(\mathbb{A}^f)$. A Hodge cycle s on A is to be an element of some space $V^{\otimes m} \otimes \check{V}^{\otimes n}(p)$ such that:

(2.1a) s is of type $(0,0)$ for the Hodge structure defined by that on V;

(2.1b) there is an $s_{dR} \in (H^1_{dR}(A)^{\vee})^{\otimes m} \otimes H^1_{dR}(A)^{\otimes n}$ that corresponds to s under the isomorphism induced by $H^1_{dR}(A) \otimes_k \mathbb{C} \approx V(\mathbb{C})$ and $\mathbb{C} \approx 2\pi i\mathbb{C}$;

(2.1c) there is an $s_{et} \in V^f(A)^{\otimes m} \otimes (V^f(A)^{\vee})^{\otimes n} \otimes (\varprojlim \mu_n(k))^{\otimes p}$ that corresponds to s under the isomorphism induced by $V(\mathbb{A}^f) \approx V^f(A)$ and $2\pi i \hat{\mathbb{Z}} \xrightarrow{\exp} \varprojlim \mu_n(\mathbb{C})$.

Let τ be an automorphism of $\mathbb{C}$; then τA is an abelian variety over $\tau k \subset \mathbb{C}$ and the above-mentioned result of Deligne shows that τs is a well-defined Hodge cycle on τA: it has $(\tau s)_{dR} = s_{dR} \otimes 1 \in H_{dR}(\tau A) = H_{dR}(A) \otimes_{k,\tau} k$ and $(\tau s)_{et} = \tau s_{et}$.

Certain Shimura varieties can be described as parameter spaces for families of abelian varieties. Let (G,X) satisfy (1.1), and assume there is an embedding $(G,X) \hookrightarrow (CSp(V), S^{\pm})$ where V is a vector space over $\mathbb{Q}$, $CSp(V)$ is the group of symplectic similitudes corresponding to some non-degenerate skew-symmetric form ψ on V, and $S^{\pm}$ is the Siegel double space (in the sense of Deligne [2, 1.3.1]). There will be some family of tensors $(s_\alpha)_{\alpha \in J}$ in spaces of the form $V^{\otimes m} \otimes \check{V}^{\otimes n}(p)$ such that $G = Aut(V, (s_\alpha)) \subset GL(V) \times \mathbb{G}_m$ (see I, Prop. 3.1). We shall always take ψ to be one of the s_α; then the projection $G \to \mathbb{G}_m$ is defined by the action of G on ψ.

Consider triples $(A, (t_\alpha)_{\alpha \in J}, k)$ with A an abelian variety over $\mathbb{C}$, (t_α) a family of Hodge cycles on A, and k is an isomorphism $k\colon V^f(A) \xrightarrow{\approx} (V(\mathbb{A}^f))$ under which t_α corresponds to s_α for each $\alpha \in J$. We define $\mathcal{A}(G,X,V)$ to be the set of isomorphism classes of triples of this form that satisfy the following conditions:

(2.2a) there exists an isomorphism $H_1(A, \mathbb{Q}) \xrightarrow{\approx} V$ under which s_α corresponds to t_α for each $\alpha \in J$;

(2.2b) the map $\mathbb{S} \xrightarrow{h_A} GL(H_1(A, \mathbb{R}))$ defined by the Hodge structure on $H_1(A, \mathbb{R})$, when composed with the map $GL(H_1(A, \mathbb{R})) \to GL(V(\mathbb{R}))$ induced by an isomorphism as in (a), lies in X.

We let $g \in G(\mathbb{A}^f)$ act on a class $[A, (s_\alpha), k] \in \mathcal{A}(G,X,V)$ as follows: $[A, (t_\alpha), k]g = [A, (t_\alpha), g^{-1}k]$.

Proposition 2.3. There is a bijection $Sh(G,X) \xrightarrow{\approx} \mathcal{A}(G,X,V)$ commuting with the actions of $G(\mathbb{A}^f)$.

Proof: Corresponding to $[h,g] \in Sh(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)$, we choose A to be the abelian variety associated with the Hodge structure (V,h) . Thus $H_1(A, \mathbb{Q}) = V$ and the s_α can be regarded as Hodge cycles on A . As $V^f(A) = V(\mathbb{A}^f)$ we can define k to be $V^f(A) = V(\mathbb{A}^f) \xrightarrow{g^{-1}} V(\mathbb{A}^f)$. It is easily checked that the class $[A, (t_\alpha), k] \in \mathcal{A}(G,X,V)$ depends only on the class $[h,g] \in Sh(G,X)$. Conversely, let $(A, (t_\alpha), k)$ represent a class in $\mathcal{A}(G,X,V)$. We choose an isomorphism $f:H_1(A, \mathbb{Q}) \to V$ as in (2.2a) and define h to be $f\, h_A f^{-1}$ (cf. 2.2b) and g to be $V(\mathbb{A}^f) \xrightarrow{k^{-1}} V^f(A) \xrightarrow{f\otimes 1} V(\mathbb{A}^f)$. If f is replaced by qf, then (h,g) is replaced by $(\underset{\sim}{ad}(q) \circ h, qg)$, and $q \in G(\mathbb{Q})$.

Remark 2.4. The above proposition can be strengthened to show that $Sh(G,X)$ is the solution of a moduli problem over $\mathbb{C}$. Since the moduli problem is defined over $E(G,X)$, $Sh(G,X)$ therefore had model over $E(G,X)$ which, because of the main theorem of complex multiplication, is canonical. This is the proof of Deligne [2, 2.3.1] hinted at in the last paragraph of the introduction to that paper. Let K and K_1 be compact open subgroups of $G(\mathbb{A}^f)$ and $CSp(V)(\mathbb{A}^f)$ with K small and K_1 such that $Sh(G,X)_K \to Sh(CSp(V),S^{\pm})K_1$ is injective (see Deligne [1,1.15]). The pullback of the universal family of abelian varieties on $Sh(CSp(V),S^{\pm})_{K_1}$, constructed by Mumford, is universal for families of abelian varieties carrying Hodge cycles (t_α) and a level structure (mod K).

3. A result on reductive groups; applications.

The following proposition will usually be applied to replace a given reduction group by one whose derived group is simply connected.

Proposition 3.1. (cf. Langlands [3, p 228-29]). Let G be a reductive group over a field k of characteristic zero and let L be a finite Galois extension of k that is sufficiently large to split some maximal torus in G. Let $G' \to G^{der}$ be a covering of the derived group of G. Then there exists a central extension defined over k

$$1 \longrightarrow N \longrightarrow G_1 \longrightarrow G \longrightarrow 1$$

such that G_1 is a reductive group, N is a torus whose group of characters $X^*(N)$ is a free module over the group ring $\mathbb{Z}[Gal(L/k)]$, and $(G_1^{der} \longrightarrow G^{der}) = (G' \longrightarrow G^{der})$.

Proof: The construction of G_1 will use the following result about modules.

Lemma 3.2. Let $\mathcal{G}$ be a finite group and M a finitely generated $\mathcal{G}$-module. Then there exists an exact sequence of $\mathcal{G}$-modules $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ in which P_0 is free and finitely generated as a $\mathbb{Z}$-module and P_1 is a free $\mathbb{Z}[\mathcal{G}]$-module.

Proof: Write M_0 for M regarded as an abelian group, and choose an exact sequence

$$0 \to F_1 \to F_0 \to M_0 \to 0$$

of abelian groups with F_0 (and hence F_1) finitely generated and free. On tensoring this sequence with $\mathbb{Z}[\mathfrak{g}]$ we obtain an exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathbb{Z}[\mathfrak{g}] \otimes F_1 \longrightarrow \mathbb{Z}[\mathfrak{g}] \otimes F_0 \to \mathbb{Z}[\mathfrak{g}] \otimes M_0 \to 0$$

whose pull-back relative to the injection

$$(m \mapsto \Sigma\, g \otimes g^{-1}m) : \; M \hookrightarrow \mathbb{Z}[\mathfrak{g}] \otimes M_0$$

has the required properties.

We now prove (3.1). Let T be a maximal torus in G that splits over L and let T' be the inverse image of T under $G' \to G^{der} \subset G$; it is a maximal torus in G'. An application of (3.2) to the $\mathfrak{g} = \mathrm{Gal}(L/k)$-module $M = X_*(T)/X_*(T')$ provides us with the bottom row of the following diagram, and we define Q to be the fibred product of P_0 and $X_*(T)$ over M :

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_*(T') & = & X_*(T') & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & Q & \longrightarrow & X_*(T) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the terms of the middle row of the diagram are torsion-free, the $\mathbb{Z}$-linear dual of the sequence is also exact, and hence corresponds, via the functor X^*, to an exact sequence

$$1 \longrightarrow N \longrightarrow T_1 \longrightarrow T \longrightarrow 1$$

of tori. The map $X_*(T') \longrightarrow Q = X_*(T_1)$ corresponds to a map $T' \longrightarrow T_1$ lifting $T' \longrightarrow T$. Since the kernel of $T' \longrightarrow T_1$ is finite, the torsion-freeness of $P_0 = \operatorname{coker}(X_*(T') \longrightarrow X_*(T_1))$ thus implies that $T' \longrightarrow T_1$ is injective. On forming the pull-back of the above sequence of tori relative to $Z \hookrightarrow T$, where $Z = Z(G)$, we obtain an exact sequence

$$1 \longrightarrow N \longrightarrow Z_1 \longrightarrow Z \longrightarrow 1 .$$

As T' contains $Z' = Z(G')$, $T' \hookrightarrow T_1$ induces an inclusion $Z' \hookrightarrow Z_1$. The group G can be written as a fibred sum, $\tilde{G} = G *_{\tilde{Z}} Z$, where $\tilde{G}$ is the universal covering group of G^{der} and $\tilde{Z} = Z(\tilde{G})$ (Deligne [2,2.0.1]). We can identify G' with a quotient of $\tilde{G}$. Define $G_1 = \tilde{G} *_{\tilde{Z}} Z_1$. It is easy to check that $Z_1 \longrightarrow Z$ induces a surjection $G_1 \longrightarrow G$ with kernel $N \subset Z_1 = Z(G_1)$ and that $\tilde{G} \longrightarrow G_1$ induces an isomorphism $G' \xrightarrow{\approx} G_1^{der}$. Finally, we note that $X_*(N)$ is a free $\mathbb{Z}[\mathfrak{Y}]$-module and $X^*(N)$ is the $\mathbb{Z}$-linear dual of $X_*(N)$.

<u>Remark 3.3</u> (a) The torus N in (3.1) is a product of copies of $\operatorname{Res}_{L/k}\mathbb{G}_m$. Thus $H^1(k',N_{k'}) = 0$ for any field $k' \supset k$, and

the sequence $1 \to N(k') \to G_1(k') \to G(k') \to 1$ is exact.

(b) Let $\tilde{T}$ be the inverse image of T (or T') in $\tilde{G}$. Then the maps $\tilde{T} \to T' \hookrightarrow T_1$ and $Z_1 \hookrightarrow T_1$ induce an isomorphism $\tilde{T} *_{\tilde{Z}} Z_1 \xrightarrow{\approx} T_1$. Thus T_1 can be identified with a subgroup of G_1, and the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & T_1 & \longrightarrow & T & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

commutes. Obviously T_1 is a maximal torus in G_1.

Application 3.4. Let (G,X) satisfy (1.1), let $h \in X$ be special, and let T be a maximal torus such that h factors through $T_{\mathbb{R}}$. Let $G' \to G^{der}$ be some covering. Take k to be $\mathbb{Q}$ and L to split T, and construct $T_1 \subset G_1 \to G$ as above. Choose some $\mu_1 \in X_*(T_1)$ mapping to $\mu_h \in X_*(T)$. Then μ_1 obviously commutes with $\iota\mu_1$ and so defines a homomorphism $h_1\colon \mathbb{S} \to T_{\mathbb{R}} \subset G_{\mathbb{R}}$. We let X_1 be the $G(\mathbb{R})$- conjugacy class of maps containing h_1. The pair (G_1,X_1) satisfies (1.1) because, modulo centres, (G_1,X_1) and (G,X) are equal.

It is possible to choose μ_1 so that $E(G_1,X_1) = E(G,X)$. To prove this we first show that the image $\bar{\mu}_h$ of μ_h in M is fixed by $\mathrm{Aut}(\mathbb{C}/E(G,X))$, where $M = X_*(T)/X_*(T')$ is as in the proof of (3.1).

We have to show $\tau\mu_h - \mu_h$ lifts to an element of $X_*(T')$ for any $\tau \in \mathrm{Aut}(\mathbb{C}/E(G,X))$. Since $\tau\mu_h - \mu_h \in X_*(T^{der})$, where $T^{der} = T \cap G^{der}$, and $X_*(T^{der}) \to X_*(T^{ad})$ is injective, where T^{ad} is the image of T in G^{ad}, it suffices to show that the image of $\tau\mu_h - \mu_h$ in $X_*(T^{ad})$ lifts to $X_*(T')$ or, equivalently, to $X_*(G')$. Let $N = \{\mu_h^{ad} \mid h \in X\}$, where μ_h^{ad} is the composite $\mathbb{G}_m \xrightarrow{\mu_h} G \twoheadrightarrow G^{ad}$. Then N is a $G(\mathbb{C})$-conjugacy class of homomorphisms defined over $E = E(G,X)$. For any $\mu \in N$, the identity component of the pull-back of $G' \to G$ by μ is a covering $\pi: \mathbb{G}_m' \to \mathbb{G}_m$ that is independent of μ; it is therefore defined over E, and N lifts to a conjugacy class of N' of maps $\mathbb{G}_m' \to G'$ defined over E. Any two elements of N' restrict to the same element on $\mathrm{Ker}(\pi)$. Thus if $\mu' \in N'$ lifts $\mu \in N$, then $\tau\mu' - \mu'$ factors through $\mathbb{G}_m$ by a map that lifts $\tau\mu - \mu$.

We now use the fact that $X_*(N)$ is a free $\mathrm{Gal}(LE/E)$-module to deduce the existence of a $\mu_1 \in X_*(T_1)$ mapping to $\mu \in X_*(T)$ and whose image $\bar{\mu}_1$ in P_0 is fixed by $\mathrm{Aut}(\mathbb{C}/E)$. The map $G_1 \to G$ induces an isomorphism $W(G_1,T_1) \xrightarrow{\approx} W(G,T)$ of Weyl groups. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ and suppose $\tau\mu = \omega \circ \mu$ with $\omega \in W(G,T)$. If $\omega_1 \in W(G_1,T_1)$ maps to ω, then $\omega_1 \circ \mu_1$ maps to $\tau\mu$ in $X_*(T)$ and $\bar{\mu}_1 = \tau\bar{\mu}_1$ in P_0; thus $\omega_1 \circ \mu_1 = \tau\mu_1$. It follows that τ fixes $E(G_1,X_1)$, and so $E(G,X) \supset E(G_1,X_1)$. The reverse inclusion is automatic.

We can apply this to a triple (G,G',X^+) defining a connected Shimura variety. Thus there exists a pair (G_1,X_1) satisfying (1.1) and such that $(G_1^{ad},G_1^{der},X_1^+) \approx (G,G',X^+)$, $E(G_1,X_1) = E(G,X^+)$, and $X^*(Z(G_1))$ is a free $Gal(L/\mathbb{Q})$-module for some finite Galois extension L of $\mathbb{Q}$ (cf. Deligne [2, 2.7.16]). The last condition implies $G_1(k) \to G_1^{ad}(k) = G(k)$ is surjective for any field $k \supset \mathbb{Q}$.

Application 3.5. Let G be a reductive group over a field k of characteristic zero, and let $\rho: \tilde{G} \to G^{der} \subset G$ be the universal covering of G^{der}. When k is a local or global field and k' is a finite extension of k, there is a canonical norm map $N_{k'/k}: G(k')/\rho\tilde{G}(k') \to G(k)/\rho\tilde{G}(k)$ (Deligne [2,2.4.8]). We shall use (3.1) to give a more elementary construction of this map.

If G is commutative, $N_{k'/k}$ is just the usual norm map $G(k') \to G(k)$.

Next assume G^{der} is simply connected and let $T = G/G^{der}$. If in the diagram

$$\begin{array}{ccccccc} 1 \longrightarrow & G(k)/\tilde{G}(k') & \longrightarrow & T(k') & \longrightarrow & H^1(k',\tilde{G}) \\ & & & \Big\downarrow N_{k'/k} & & \\ 1 \longrightarrow & G(k)/\tilde{G}(k) & \longrightarrow & T(k) & \longrightarrow & H^1(k,\tilde{G}) \end{array}$$

the map $G(k')/\tilde{G}(k') \to H^1(k,\tilde{G})$ is a zero, we can define $N_{L'/k}$ for G to be the restriction of $N_{k'/k}$ for T.

When k is local and nonarchimedean then $H^1(k,\tilde{G}) = 0$, and so the map is zero. When k is local and archimedean we can suppose $k = \mathbb{R}$ and $k' = \mathbb{C}$; then $N_{\mathbb{C}/\mathbb{R}} : T(\mathbb{C}) \to T(\mathbb{R})$ maps into $T(\mathbb{R})^+$, and any element of $T(\mathbb{R})^+$ lifts to an element of $G(\mathbb{R})$ (even to an element of $Z(G)(\mathbb{R})$) . When k is global, we can apply the Hasse principle.

In the general case we choose an exact sequence

$$1 \longrightarrow N \longrightarrow G_1 \longrightarrow G \longrightarrow 1$$

as in (3.1) with G_1^{der} simply connected. From the diagram

$$\begin{array}{ccccccc} N(k') & \longrightarrow & G_1(k')/\tilde{G}(k') & \longrightarrow & G(k')/\rho\tilde{G}(k') & \longrightarrow & 1 \\ \downarrow N_{k'/k} & & \downarrow N_{k'/k} & & & & \\ N(k) & \longrightarrow & G_1(k)/\tilde{G}(k) & \longrightarrow & G(k)/\rho\tilde{G}(k) & \longrightarrow & 1 \end{array}$$

we can deduce a norm map for G .

Let k be a number field. If we take the restricted product of the norm maps for the completions of k , and form the quotient by the norm map for k , we obtain the map

$$N_{k'/k} \colon \ \pi(G_{k'}) \longrightarrow \pi(G_k)$$

of Deligne [2,2.4.0.1], where $\pi(G_k) = G(\mathbb{A}_k)/(G(k).\rho\tilde{G}(\mathbb{A}_k))$.

Application 3.6. Let G and G' be reductive groups over $\mathbb{Q}$ with adjoint groups having no factors over $\mathbb{Q}$ whose real points are compact. Assume G' is an inner twist of G , so that for some Galois extension L of $\mathbb{Q}$ there is an isomorphism $f : G_L \xrightarrow{\approx} G'_L$ such that, for all $\sigma \in Gal(L/\mathbb{Q})$,

$(\sigma f)^{-1} \circ f = \mathbf{ad}\,\alpha_\sigma$ with $\alpha_\sigma \in G^{ad}(L)$. We shall show that f induces a canonical isomorphism $\pi_0\pi(f)\colon\ \pi_0\pi(G) \to \pi_0\pi(G')$ with $\pi(-)$ defined as in Deligne [2,2.0.15] (not Deligne [1,2.3]).

If f is defined over $\mathbb{Q}$, for example if G is commutative, then $\pi_0\pi(f)$ exists because $\pi_0\pi$ is a functor.

Next assume that G^{der} is simply connected, and let $\bar{f}$ be the isomorphism from $T = G/G^{der}$ to $T' = G'/G'^{der}$ induced by f . A theorem of Deligne [1,2.4] shows that the vertical arrows in the following diagram are isomorphisms

$$\begin{array}{ccc} \pi_0\pi(G) & \xrightarrow{\ \pi_0\pi(f)\ } & \pi_0\pi(G') \\ \downarrow{\approx} & & \downarrow{\approx} \\ \pi_0\pi(T) & \xrightarrow{\ \pi_0\pi(\bar{f})\ } & \pi_0\pi(T') \end{array} .$$

We define $\pi_0\pi(f)$ to make the diagram commute.

In the general case we choose an exact sequence

$$1 \to N \to G_1 \to G \to 1$$

as in (3.1) with G_1^{der} simply connected. Note that $G_1^{ad} = G^{ad}$ so that we can use the same cocycle to define an inner twist $f_1\colon\ G_{1L} \to G'_{1L}$. The first case considered above allows us to assume f_1 lifts f . Remark (3.3a) shows that $\pi_0\pi(G_1) \to \pi_0\pi(G)$ is surjective, and we define $\pi_0\pi(f)$ to make the following diagram commute:

$$\begin{array}{ccccccc}
\pi_0\pi(N) & \longrightarrow & \pi_0\pi(G_1) & \longrightarrow & \pi_0\pi(G) & \longrightarrow & 1 \\
\downarrow \pi_0\pi(f|N) & & \downarrow \pi_0\pi(f) & & \vdots\ \pi_0\pi(f) & & \\
\pi_0\pi(N') & \longrightarrow & \pi_0\pi(G'_1) & \longrightarrow & \pi_0\pi(G') & \longrightarrow & L
\end{array}$$

Note that, if $f : G_L \to G'_L$ and $f' : G'_L \to G''_L$ define G' and G'' as inner twists of G and G', then $\pi_0\pi(f') \circ \pi_0\pi(f) = \pi_0\pi(f' \circ f)$. Also that if f is of the form $\underset{\sim}{\mathrm{ad}}q : G_L \to G_L$ with $q \in G^{ad}(L)$, then $\pi_0\pi(f) = \mathrm{id}$. In the case that G^{der} is simply connected this is obvious because $\underset{\sim}{\mathrm{ad}}q$ induces id on T , and the general case follows. On combining these two remarks we find that $\pi_0\pi(f)$ is independent of f , because f can only be replaced by $f \circ \underset{\sim}{\mathrm{ad}}q$ with $q \in G^{ad}(L)$, and $\pi_0\pi(f) \circ \pi_0\pi(\underset{\sim}{\mathrm{ad}}\, q) = \pi_0\pi(f)$.

§4. The conjectures of Langlands.

Let (G,X) satisfy (1.1). Before discussing the conjectures of Langlands concerning $Sh(G,X)$ we review some of the properties of (G,X) over $\mathbb{R}$.

Let $h \in X$ be special (in the sense of Deligne [2,2.2.4]), and let T be a $\mathbb{Q}$-rational maximal torus such that h factors through $T_{\mathbb{R}}$. Let $\mu = \mu_h$ be the cocharacter corresponding to h. According to (1.1b) $\underset{\sim}{ad}\, h(i)$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$, and hence on $G^{der}_{\mathbb{R}}$. Thus $\underline{g}^{der} = \underline{k} \oplus \underline{p}$ where $\underline{g}^{der} = \mathrm{Lie}(G^{der}_{\mathbb{R}}) = \mathrm{Lie}(G_{\mathbb{R}})^{der}$ and $\mathrm{Ad}\, h(i)$ acts as 1 on $\underline{k}$ and -1 on $\underline{p}$. According to (1.1a) there is a decomposition

$$\underline{g}_{\mathbb{C}} = \underline{c}_{\mathbb{C}} \oplus \underline{k}_{\mathbb{C}} \oplus \underline{p}^{+} \oplus \underline{p}^{-}$$

where $\underline{g} = \mathrm{Lie}(G_{\mathbb{R}})$, $\underline{c} = \mathrm{Lie}(Z(G)_{\mathbb{R}})$, $\underline{p}_{\mathbb{C}} = \underline{p}^{+} \oplus \underline{p}^{-}$, and $\mathrm{Ad}\,\mu(z)$ acts as z on $\underline{p}^{+}$ and $\bar{z}$ on $\underline{p}^{-}$. (Thus $\underline{g}^{0,0} = \underline{c}_{\mathbb{C}} + \underline{k}_{\mathbb{C}}$, $\underline{g}^{-1,1} = \underline{p}^{+}$, and $\underline{g}^{1,-1} = \underline{p}^{-}$.) As $T_{\mathbb{C}}$ is a maximal torus in $G_{\mathbb{C}}$, we also have a decomposition

$$\underline{g}_{\mathbb{C}} = \underline{t}_{\mathbb{C}} + \sum_{\alpha \in R} \underline{g}_{\alpha}$$

where $\underline{t} = \mathrm{Lie}(T_{\mathbb{R}})$ and $R \subset \underline{t}^{\vee}_{\mathbb{C}}$ is the set of roots of (G,T). A root α is said to be <u>compact</u> or <u>noncompact</u> according as $\underline{g}_{\alpha} \subset \underline{k}_{\mathbb{C}}$ or $\underline{g}_{\alpha} \subset \underline{p}_{\mathbb{C}}$.

<u>Remark 4.1</u>. If $Y \in \underline{g}_{\alpha}$ then $\mathrm{Ad}(\mu(-1))Y = \alpha(\mu(-1))Y = (-1)^{\langle\alpha,\mu\rangle}Y$. Since $\mathrm{Ad}\mu(-1)$ acts on $\underline{k}_{\mathbb{C}}$ as $+1$ and on $\underline{p}_{\mathbb{C}}$ as -1, this shows that α is compact or noncompact according as $\langle\alpha,\mu\rangle$ is

even or odd.

Note that $T^{der} \overset{df}{=} T \cap G^{der}$ is anisotropic because $\underline{t}^{der} \subset \underline{k}$. Let N be the normalizer of T in G and let $W = N(\mathbb{C})/T(\mathbb{C})$ be the Weyl group. As ι acts as -1 on $R \subset \underline{t}_{\mathbb{C}}^{der}$, it commutes with the action of any reflection s_α. Hence ι acts trivially on W and there is an exact cohomology sequence

$$1 \to T(\mathbb{R}) \to N(\mathbb{R}) \xrightarrow{ad} W \xrightarrow{\delta} H^1(\mathbb{R},T)$$

where, for $\omega \in W$ lifting to $w \in N(\mathbb{C})$, $\delta(\omega)$ is represented by $w^{-1}.\iota w \in \mathrm{Ker}(1 + \iota\colon T(\mathbb{C}) \to T(\mathbb{C}))$.

<u>Proposition 4.2</u>. The class $\delta(\omega)$ is represented by $(\omega^{-1}\mu)(-1)/\mu(-1) \in T(\mathbb{C})$.

<u>Proof</u>: Note that $\delta(\omega_1\omega_2) = \omega_2^{-1}\,\delta(\omega_1).\delta(\omega_2)$ while $(\omega_1\omega_2)^{-1}\mu(-1)/\mu(-1) = \omega_2^{-1}\,(\omega_1^{-1}\mu(-1)/\mu(-1)).(\omega_2^{-1}\mu(-1)/\mu(-1))$ and so it suffices to prove the proposition for a generator s_α of W.

We make the identifications $T(\mathbb{C}) = X_*(T) \otimes \mathbb{C}^\times$, $\underline{t}_{\mathbb{C}} = X_*(T) \otimes \mathbb{C}$, and $\underline{t}_{\mathbb{C}}^\vee = X^*(T) \otimes \mathbb{C}$. If $\check{\alpha}$ is a coroot and H_α is the element of $\underline{t}_{\mathbb{C}}$ corresponding to α, then $\exp \pi i H_\alpha = \check{\alpha}(-1)$. Let $X_\alpha \in \underline{g}_\alpha$ and $X_{-\alpha} \in \underline{g}_{-\alpha}$ be such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. As $\iota\alpha = -\alpha$, we have that $\iota H_\alpha = -H_\alpha$ and that $\iota X_\alpha = cX_{-\alpha}$ and $\iota X_{-\alpha} = dX_\alpha$ with $c,d \in \mathbb{C}$. The conditions $[X_\alpha, X_{-\alpha}] = H_\alpha$ and $\iota^2 = 1$ imply that $cd = 1$ and $\iota c.d = 1$, and so c is real and $d = c^{-1}$. If we replace X_α by aX_α then we must replace $X_{-\alpha}$ by $\frac{1}{a}X_{-\alpha}$ and c by a^2c. Thus, for a given α, there are two possibilities:

either X_α can be chosen so that $\iota X_\alpha = -X_{-\alpha}$ or X_α can be chosen so that $\iota X_\alpha = X_{-\alpha}$. In the first case α is compact and in the second it is noncompact.

Assume that α is compact; then the map $\underline{su}_2 \to \underline{g}$ such that $\begin{pmatrix}1&0\\0&-1\end{pmatrix} \mapsto H_\alpha$, $\begin{pmatrix}0&i\\0&0\end{pmatrix} \mapsto X_\alpha$, $\begin{pmatrix}0&0\\-i&0\end{pmatrix} \mapsto X_{-\alpha}$ lifts to a homomorphism $SU_2 \to G_{\mathbb{R}}$ (defined over $\mathbb{R}$). The image w of $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ in $G(\mathbb{R})$ represents s_α. Thus $\delta(s_\alpha) = 1$ in this case. On the other hand, $s_\alpha(\mu)-\mu = -\langle\alpha,\mu\rangle\alpha^\vee$, and so $s_\alpha\mu(-1)/\mu(-1) = \alpha^\vee(-1)^{-\langle\alpha,\mu\rangle} = 1$ (by 4.1).

If α is noncompact, then the map $s\ell_2 \to \underline{g}$ such that $\begin{pmatrix}0&-i\\i&0\end{pmatrix} \mapsto H_\alpha$, $\frac{1}{2}\begin{pmatrix}-i&1\\1&i\end{pmatrix} \mapsto X_\alpha$, $\frac{1}{2}\begin{pmatrix}i&1\\1&-i\end{pmatrix} \mapsto X_{-\alpha}$ lifts to a homomorphism $SL_2 \longrightarrow G_{\mathbb{R}}$. The image w of $\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ in $G(\mathbb{C})$ represents s_α. Then $w^{-1}.\iota w$ is the image of $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$, which is $\exp \pi i H_\alpha = \alpha^\vee(-1)$. On the other hand $s_\alpha\mu(-1)/\mu(-1) = \alpha^\vee(-1)^{-\langle\alpha,\mu\rangle} = \alpha^\vee(-1)$ (by 4.1).

<u>Corollary 4.3</u>. If the reflex field $E(G,X)$ of (G,X) is real then there exists an $n \in N(\mathbb{R})$ such that $\underset{\sim}{ad}(n)\circ\mu = \iota\mu$.

<u>Proof</u>: Since ι fixes $E(G,X)$ there is an element w in $G(\mathbb{C})$, which we can choose to lie in $N(\mathbb{C})$, such that $\iota\mu = \underset{\sim}{ad}(w)\circ\mu$. The proposition shows that the image of $\underset{\sim}{ad}\, w$ in $H^1(\mathbb{C}/\mathbb{R},T(\mathbb{C}))$ is represented by $(\iota-1)\mu(-1)$, and therefore is zero. Thus there is an $n \in N(\mathbb{R})$ representing $\underset{\sim}{ad}\, w$.

When the reflex field $E(G,X)$ is real and $Sh(G,X)$ has a canonical model over $E(G,X)$ then ι defines an antiholomorphic involution of $Sh(G,X)$. One of the conjectures of Langlands gives an explicit description of this involution.

Let h , as before, be special and let ${}^{\iota}h$ be the element of X corresponding to $\iota\mu$. If n is as in the corollary, then $\underset{\sim}{ad}(n)\circ h = {}^{\iota}h$. Since K_∞ is the centralizer of $h(i)$, and of ${}^{\iota}h(i)$, we see that n normalizes K_∞ . Thus $g \mapsto gn : G(\mathbb{R}) \to G(\mathbb{R})$ induces a map on the quotient $G(\mathbb{R})/K_\infty$, which we can transfer to X by means of the isomorphism $g \mapsto \underset{\sim}{ad}g\circ h : G(\mathbb{R})/K_\infty \xrightarrow{\approx} X$. Thus we obtain an antiholomorphic isomorphism $\eta = (\underset{\sim}{ad}g\circ h \mapsto \underset{\sim}{ad}(gn)\circ h : X\to X)$

<u>Conjecture B</u>. (Langlands [1, p. 418], [2, p. 2.7, Conjecture B], [3, p. 234]). The involution of $Sh(G,X)$ defined by ι is $[x,g] \mapsto [\eta(x),g]$.

<u>Remark 4.4</u>. The conjecture is true for all special h if it is true for one, and it follows from Deligne [1, 5.2] that to prove the conjecture it suffices to show $\iota[h, 1] = [\eta(h), 1]$ $(=[{}^{\iota}h, 1])$ for a single special h. Conjecture B is easy to prove if $Sh(G,X)$ is a moduli variety for abelian varieties over $E(G,X)$. (More generally, if it is a "moduli variety for motives", see (10.7). It is proved for all Shimura varieties of abelian type in Milne-Shih [1].

The conjecture of Langlands concerning conjugates of Shimura varieties is expressed in terms of the Taniyama group; thus let

$$1 \to S \to \underset{\sim}{T} \xrightarrow{\pi} Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1$$

be the extension, and $sp : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \underset{\sim}{T}(\mathbb{A}^f)$ the splitting, defined in (III. 3). For any $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, ${}^{\tau}S \overset{df}{=} \pi^{-1}(\tau)$ is a right S-torsor, and $sp(\tau) \in {}^{\tau}S(\mathbb{A}^f)$ defines a trivialization of ${}^{\tau}S$ over $\mathbb{A}^f$. (For any finite Galois extension L of $\mathbb{Q}$ and $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ we can also define an S^L-torsor ${}^{\tau}S^L$; it corresponds to the cohomology class $\gamma(\tau) \in H^1(L/\mathbb{Q}, S^L)$; (see III. 2.9).)

Let G, X, h, μ, T be as at the start of this section. As $T^{ad}_{\mathbb{R}}$ is anisotropic, $\mu^{ad} \overset{df}{=} (\mathbb{G}_m \xrightarrow{\mu} T \longrightarrow T^{ad})$ satisfies (III. 1.1) and so factors into $\mathbb{G}_m \xrightarrow{\mu_{can}} S \xrightarrow{\rho_\mu} T^{ad} \subset G^{ad}$. Thus S acts on G, and we can use ${}^{\tau}S$ to twist G: we define ${}^{\tau}G$ (or ${}^{\tau,\mu}G$) to be ${}^{\tau}S \times {}^{S}G$. (If $L \supset \mathbb{Q}$ splits T then there is an isomorphism $f : G_L \xrightarrow{\approx} {}^{\tau}G_L$ such that $\sigma f = f \circ \underset{\sim}{ad}\, \gamma_\sigma(\tau, \mu^{ad})$.) Note that the action of S on T is trivial, and so $T = {}^{\tau}S \times {}^{S}T \subset {}^{\tau}G$. Define ${}^{\tau}h$ to be the homomorphism $\mathbb{S} \to {}^{\tau}G_{\mathbb{R}}$ associated with $\mathbb{G}_m \xrightarrow{\tau\mu} T_{\mathbb{C}} \subset {}^{\tau}G_{\mathbb{C}}$, and ${}^{\tau}X$ (or ${}^{\tau,\mu}X$) to be the $G(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to {}^{\tau}G_{\mathbb{R}}$ containing ${}^{\tau}h$. The element $sp(\tau) \in {}^{\tau}S(\mathbb{A}^f)$ provides a canonical isomorphism $g \mapsto sp(\tau).g : G(\mathbb{A}^f) \longrightarrow {}^{\tau}G(\mathbb{A}^f)$, which we write as $g \mapsto {}^{\tau}g$ (or $g \mapsto {}^{\tau,\mu}g$). Langlands has shown [3, p. 231] that $({}^{\tau}G, {}^{\tau}X)$ satisfies (1.1); he asserts [3, p. 233] that if h' is a second special point of X and $\mu' = \mu_{h'}$, then there is an isomorphism

$$\phi(\tau;\mu',\mu) : Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) \longrightarrow Sh({}^{\tau,\mu'}G, {}^{\tau,\mu'}X)$$

such that

$$\phi(\tau;\mu',\mu) \circ \mathcal{J}({}^{\tau,\mu}g) = \mathcal{J}({}^{\tau,\mu'}g) \circ \phi(\tau;\mu',\mu) .$$

Conjecture C. (Langlands [3, p. 232-33]) (a) For any special $h \in X$ there is an isomorphism

$$\phi_\tau = \phi_{\tau,\mu_h} : \tau\mathrm{Sh}(G,X) \xrightarrow{\approx} \mathrm{Sh}({}^\tau G, {}^\tau X)$$

such that

$$\phi_\tau(\tau[h,1]) = [{}^\tau h, 1]$$
$$\phi_\tau \circ \tau \mathcal{J}(g) = \mathcal{J}({}^\tau g) \circ \phi_\tau \text{ , all } g \in G(\mathbb{A}^f).$$

(b) If h is a second special element of X and $\mu = \mu_h$, $\mu' = \mu_{h'}$, then

$$\begin{array}{ccc} \tau\mathrm{Sh}(G,X) & \xrightarrow{\phi_{\tau,\mu}} & \mathrm{Sh}({}^{\tau,\mu}G, {}^{\tau,\mu}X) \\ & \searrow^{\phi_{\tau,\mu'}} & \downarrow \phi(\tau;\mu',\mu) \\ & & \mathrm{Sh}({}^{\tau,\mu'}G, {}^{\tau,\mu}X) \end{array}$$

commutes.

Remark 4.5. For a given h there is at most one map $\phi_{\tau,\mu}$ satisfying the conditions in part (a) of the conjecture (this follows from Deligne [1, 5.2]).

We note one consequence of conjecture C. Assume that $\mathrm{Sh}(G,X)$ has a canonical model $(M(G,X),\ f : M(G,X)_{\mathbb{C}} \xrightarrow{\approx} \mathrm{Sh}(G,X))$, and let $h \in X$ be special with associated cocharacter μ. Then for any automorphism τ of $\mathbb{C}$, $\tau M(G,X)$ is defined over $\tau E(G,X)$, and obviously $\tau E(G,X) = E({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. Moreover, if we make

${}^{\tau}g \in {}^{\tau}G(\mathbb{A}^f)$ act on $\tau M(G,X)$ as $\tau\mathcal{J}(g)$, then $(\tau M(G,X),\ \tau M(G,X)_{\mathbb{C}} \xrightarrow{\phi_{\tau,\mu}\circ\tau f} Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ satisfies the condition, relative to h, to be a canonical model. Part (b) of conjecture C shows that everything is essentially independent of h, and so $\tau M(G,X)$ is a canonical model for $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. For the sake of reference, and because it is the original form of conjecture C, we state another conjecture which is a weak form of this consequence.

<u>Conjecture A</u>. (Langlands [1, p. 417], [2, p. 2.5]) Assume that $Sh(G,X)$ has a canonical model $(M(G,X),f)$, and let h be some special point of X with associated cocharacter μ. Then there exists an isomorphism $g \mapsto g' : G(\mathbb{A}^f) \to {}^{\tau,\mu}G(\mathbb{A}^f)$ such that, if $g' \in {}^{\tau}G(\mathbb{A}^f)$ is made to act on $\tau M(G,X)$ as $\tau(\mathcal{J}(g))$, then $\tau M(G,X)$ is a canonical model for $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$.

<u>Remark 4.6</u>. Conjecture A appears to depend on the choice of h. One can, however, use the maps $\phi(\tau;\mu',\mu)$ to show that if the conjecture is true with one special point h then it is true with any special h.

We shall need to use several properties of the maps $\phi(\tau;\mu',\mu)$. Thus we prove them for the Shimura varieties of interest to us, namely those of abelian type. We begin by defining the maps in an easy case.

Let (G,X) satisfy (1.1). Assume:

(4.7a) for all special $h \in X$ and all $\tau \in \mathrm{Aut}(\mathbb{C})$,

$$(\tau-1)(\iota+1)\mu_h = 0 = (\iota+1)(\tau-1)\mu_h ;$$

(4.7b) if h is special and $\rho_h : S \to G$ is the map defined by μ_h (see III.1) then the element $\gamma(\tau,\mu) \overset{df}{=} \rho_h(\gamma(\tau))$ of $H^1(\mathbb{Q},G)$ is independent of h.

Now fix two special points h and h' of X and let $\mu = \mu_h$ and $\mu' = \mu_{h'}$. We write $'G$ for ${}^{\tau,\mu}G$, $"G$ for ${}^{\tau,\mu'}G$, etc.. Let L be some large finite Galois extension of $\mathbb{Q}$ and let $a(\tau)$ be a section to $\underset{\sim}{T}^L \to \mathrm{Gal}(L^{ab}/\mathbb{Q})$. Then there are defined $\beta(\tau) \in S(\mathbb{A}_L^f)$, $\beta(\tau,\mu) \overset{df}{=} \rho_h(\beta(\tau)) \in G(\mathbb{A}_L^f)$, and $\beta(\tau,\mu') \overset{df}{=} \rho_{h'}(\beta(\tau)) \in G(\mathbb{A}_L^f)$, and cocycles $\gamma_\sigma(\tau)$, $\gamma_\sigma(\tau,\mu) \overset{df}{=} \rho_h(\gamma_\sigma(\tau))$, and $\gamma_\sigma(\tau,\mu')$. Moreover there are maps $f' = (g \mapsto a(\tau).g) : G_L \xrightarrow{\approx} {}'G_L$, $f" = (g \mapsto a(\tau).g) : G_L \xrightarrow{\approx} {}"G_L$, and $f = f" \circ f'^{-1} : {}'G_L \to {}"G_L$. According to (4.7b) there is a $v \in G(L)$ such that $\gamma_\sigma(\tau,\mu') = v^{-1}.\ \gamma_\sigma(\tau,\mu).\sigma v$. The map $f_1 = f \circ \underset{\sim}{ad}\, f'(v^{-1}) : {}'G_L \xrightarrow{\approx} {}"G_L$ is defined over $\mathbb{Q}$ and sends $'X$ into $"X$. It therefore defines an isomorphism $Sh(f_1) : Sh('G,'X) \xrightarrow{\approx} Sh("G,"X)$.

As $B \overset{df}{=} \beta(\tau,\mu')\ v^{-1}\ \beta(\tau,\mu)^{-1}$ is fixed by $\mathrm{Gal}(L/\mathbb{Q})$ it lies in $G(\mathbb{A}^f)$, and hence $'B \overset{df}{=} {}^{\tau,\mu}B = f'(\beta(\tau,\mu)^{-1}\ \beta(\tau,\mu')\ v^{-1})$ lies in $'G(A^f)$. We define $\phi(\tau;\mu',\mu)$ to be the composite $Sh(f_1) \circ \mathcal{J}('B)$. Thus

$$\phi(\tau;\mu',\mu)\ [x,'g] = [f_1 \circ x,\ "(Bg)]$$

Evidently,

$$\phi(\tau;\mu',\mu) \circ \mathcal{J}('g) = \mathcal{J}("g) \circ \phi(\tau;\mu',\mu)\ .$$

Replace $a(\tau)$ by $a(\tau)u$ with $u \in S^L(L)$, and let $u_1 = \rho_h(u)$ and $u_2 = \rho_{h'}(u)$. This forces the following changes:

f'	f	$\gamma_\sigma(\tau,\mu)$	v^{-1}	$\beta(\tau)$
$f' \circ \underset{\sim}{ad}\, u_1$	$f \circ \underset{\sim}{ad}(f'(u_2u_1^{-1}))$	$u_1^{-1}\, \gamma_\sigma(\tau,\mu)\sigma u_1$	$u_2^{-1}v^{-1}u_1$	$\beta(\tau)u$.

Thus f_1 and B are unchanged, and so also is $\phi(\tau;\mu',\mu)$. If v^{-1} is replaced by $v^{-1}\, u^{-1}$ where $u \in G(L)$ satisfies $u = \gamma_\sigma(\tau,\mu).\sigma u$, then $[\underset{\sim}{ad}\, f'(u^{-1}) \circ x\ ,\ f'(u^{-1})g] = [\ x,g]$ for any $[x,g] \in Sh('G,'X)$ because $f'(u) \in {}'G(\mathbb{Q})$. Again $\phi(\tau;\mu',\mu)$ is unchanged, and is therefore well-defined.

Example 4.8. Let $(G,X) = (CSp(V),\ S^{\pm})$. For $h \in X$ special, we can use $\rho_h : S \to CSp(V)$ to define an action of S on V . Let ${}^{\tau,\mu}V = {}^{\tau}S \times^{S} V$; clearly ${}^{\tau,\mu}G = CSp({}^{\tau,\mu}V)$. The element $sp(\tau) \in S(\mathbb{A}^f)$ defines an isomorphism $sp(\tau,\mu) : V(\mathbb{A}^f) \to {}^{\tau,\mu}V(\mathbb{A}^f)$. Under the bijections $Sh(G,X) \xrightarrow{\approx} \mathcal{A}(G,X,V)$ defined in (2.3), $\phi(\tau;\mu',\mu)$ corresponds to the map $[A,t,\ sp(\tau,\mu) \circ k] \to [A,t,sp(\tau,\mu')\circ k]$.

Example 4.9. Suppose $h' = \underset{\sim}{ad}\, q \circ h$ with $q \in G(\mathbb{Q})$. Then $B = q$ and $v^{-1} = q$. Thus $\phi(\tau;\mu',\mu)$ is the map

$$[x,'g] \mapsto [f \circ \underset{\sim}{ad}\ f'\ (q) \circ x\ ,\ ''(qg)]\ .$$

Note that, even without the assumption (4.7), this expression gives a well-defined map.

To be able to apply the above discussion, we need to know when (4.7) holds. Clearly (4.7a) is valid if G is an adjoint group or if there is a map $(G,X) \to (CSp(V),\ S^{\pm})$ such that the kernel of $G \to CSp(V)$ is finite.

<u>Lemma 4.10</u>. The pair (G,X) satisfies (4.7) if it is of abelian type and G is adjoint.

<u>Proof</u>. We can assume G to be $\mathbb{Q}$-simple. There is a diagram

$$(G,X) \longleftarrow (G_1,X_1) \longrightarrow (CSp(V),\ S^{\pm})$$

such that $G_1^{ad} = G$, $G_1^{der} = \tilde{G}$, and $G_1 \rightarrow CSp(V)$ has finite kernel (cf. 1.4). We shall prove (4.7b) holds for (G_1,X_1) . To show that the two classes $\gamma(\tau,\mu)$ and $\gamma(\tau,\mu')$ are equal in $H^1(\mathbb{Q},G_1)$ it suffices to show they have the same images in $H^1(\mathbb{Q},G_1/G_1^{der})$ and in $H^1(\mathbb{R},G_1)$ (see 7.3) . The first is obvious since μ and μ' map to the same element of $X_*(G_1/G_1^{der})$. For the second we use (III.3.14). Thus $\gamma = ((\tau-1)\mu)(-1)$ and $\gamma' = ((\tau-1)\mu')(-1)$ represent the images of $\gamma(\tau,\mu)$ and $\gamma(\tau,\mu')$ in $H^1(\mathbb{R},G_1)$. For any $z \in G(\mathbb{C})$ we write $z(\mu)$ for $\underset{\sim}{ad}\ z \circ \mu$. A direct calculation shows that if $\mu' = x(\mu)$, $x \in G(\mathbb{R})$, then

$$x^{-1}\gamma'\ x\ \gamma^{-1} = (x^{-1} \, . \, \tau x - \tau)\mu\ (-1) \ .$$

Let T be a maximal $\mathbb{Q}$-rational torus in G such that μ factors through $T(\mathbb{C})$, and let N be the normalizer of T . If $w \in N(\mathbb{C})$ then

$$\iota w.\gamma.\ w^{-1}.\gamma^{-1} = (\iota w.\ w^{-1})\ [(w-1)(\tau-1)\mu(-1)]$$

According to (4.2), $\iota w.\ w^{-1} = (\iota c.\ c^{-1})\ [(w-1)\mu(-1)]$ for some $c \in T(\mathbb{C})$.

Thus

$$\iota w.\gamma.w^{-1}.\ \gamma^{-1} = (\iota c.\ c^{-1})\ [(w-1)\tau\mu(-1)]\ .$$

If we choose w to act on the roots of (G,T) as $x^{-1}.\ \tau x.\ \tau^{-1}$, then $(w-1)\tau\mu(-1) = (x^{-1}\tau x-\tau)\ \mu(-1)$, and it follows that $x\ \gamma'\ x^{-1} = \iota(c^{-1}w).\gamma.(c^{-1}w)^{-1}$, which completes the proof.

Lemma 4.11. Let (G,G',X^+) define a connected Shimura variety and assume (G,X) is of abelian type. Then there exists a map $(G_0,X_0) \longrightarrow (G,X)$ such that $G_0^{ad} = G$, $G_0^{der} = G'$, $G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ is surjective, and (G_0,X_0) satisfies (4.7).

Proof. Clearly the lemma is true for a product if it is true for each factor, and is true for (G,G',X^+) if it is true for $(G,\tilde{G},X^+)$. Thus we can assume G is $\mathbb{Q}$-simple and $G' = \tilde{G}$. Choose (G_1,X_1) as in the proof of (7.10). Let L be a finite Galois extension of $\mathbb{Q}$ that splits $Z(G_1)$. There exists a subjective map $M \longrightarrow X^*(Z(G_1))$ with M a finitely-generated free $\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})]$-module. Let $Z(G_1) \hookrightarrow Z$ be the corresponding map of tori, and define $G_0 = \tilde{G} *_{Z(\tilde{G})} Z$ (see Deligne [2,2.0.1]). The map $Z(G_1) \hookrightarrow Z$ induces an inclusion $G_1 \hookrightarrow G_0$, and we define X_0 to be the composite of X_1 with this inclusion. Then (G_0,X_0) satisfies (4.7) because (G_1,X_1) does, and $G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ is surjective because $Z(G_0) = Z$ and $H^1(\mathbb{Q},Z) = 0$.

Let (G,G',X^+) and (G_0,X_0) be as in (4.11), and let h and h' be special elements of X^+. Write $\mu = \mu_h$ and $\mu' = \mu_{h'}$. The map

$$\phi(\tau;\mu',\mu) \ : \ Sh('G_0,'X_0) \xrightarrow{\approx} Sh(''G_0,''X_0)$$

induces an isomorphism

$$\phi^0(\tau;\mu',\mu) \ : \ Sh^0('G,'G','X) \longrightarrow Sh^0(''G,''G',''X) \ .$$

(As before, we have substituted ' and " for the superscripts τ,μ and τ,μ'.) The usual argument shows that ϕ^0 is independent of (G_0,X_0). Moreover, the surjectivity of $G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ shows that

$$\phi^0(\tau;\mu',\mu) \circ {}'\gamma. = {}''\gamma. \circ \phi^0(\tau;\mu',\mu)$$

for all $\gamma \in G(\mathbb{Q})^{+\wedge}$ (rel G') where $\gamma.$ denotes the canonical left action on Sh^0. (For the fact that $\gamma \mapsto \gamma' = {}^{\tau,\mu}\gamma$ maps $G(\mathbb{Q})^{+\wedge}$ into ${}'G(\mathbb{Q})^{+\wedge}$, see 8.1.)

<u>Proposition 4.12</u>. Let (G,X) be such that (G^{ad},X) is of abelian type. Then there is a unique family of isomorphisms

$$\phi(\tau;\mu',\mu) \ : \ Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) \longrightarrow Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X) \ ,$$

$\tau \in Aut(\mathbb{C})$, $\mu = \mu_h$, $\mu' = \mu_{h'}$ with h and h' special, such that:

(a) $\phi(\tau;\mu',\mu) \circ \mathcal{J}('g) = \mathcal{J}(''g) \circ \phi(\tau;\mu',\mu)$, $g \in G(\mathbb{A}^f)$;

(b) $\phi(\tau;\mu'',\mu') \circ \phi(\tau;\mu',\mu) = \phi(\tau;\mu'',\mu)$;

(c) if h and h' belong to the same connected component X^+ of X , then $\phi(\tau;\mu',\mu)$ restricted to the connected component of $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ is the map $\phi^0(\tau;\mu',\mu)$ defined above;

(d) if $h' = \underset{\sim}{ad}(q) \circ h$ with $q \in G(\mathbb{Q})$, then $\phi(\tau;\mu',\mu)$ is the map defined in (4.9).

Proof. There is clearly at most one family of maps with these properties. To show the existence one uses the standard technique for extending a map from the connected component of a variety to the whole variety (see Deligne [2, 2.7], or § 9).

Remark 4.13. In the case that τ fixes $E(G,X)$, we define in (7.8) below a map $\phi(\tau;\mu) : Sh(G,X) \longrightarrow Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. On comparing the two definitions one finds that

$$\phi(\tau;\mu',\mu) = \phi(\tau;\mu') \circ \phi(\tau;\mu)^{-1} .$$

Remark 4.14. Let $h' = \underset{\sim}{ad}(q) \circ h$ with $q \in G(\mathbb{Q})$, and assume part (a) of conjecture C holds. One checks directly that $\phi = \phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu}$ has the following properties:

$$\phi(\tau[h',1]) = \phi(\tau[h,q^{-1}]) = [{}^{\tau,\mu'}h',1]$$
$$\phi \circ \tau\mathcal{J}(g) = \mathcal{J}({}^{\tau,\mu'}g) \circ \phi .$$

Thus $\phi = \phi_{\tau,\mu'}$, and part (b) of the conjecture holds (for μ and μ').

§5. A cocycle calculation. (cf. IV C).

Let A be an abelian variety over $\mathbb{C}$ of CM-type, so that there is a product F of CM-fields acting on A in such a way that $H_1(A,\mathbb{Q})$ is a free F-module of rank 1. Assume that there is a homogeneous polarization $[\psi]$ on A whose Rosati involution stabilizes $F \subset \mathrm{End}(A)$ and induces ι on it. Let $F_0 = \{f \in F \mid \iota f = f\}$; thus F_0 is a product of totally real fields. Note that the Hodge structure h on $V = H_1(A,\mathbb{Q})$ is compatible with the action of F. Let $\psi \in [\psi]$ be a polarization of A (or (V,h)) ; for any choice of an element $f \in F^{\times}$ with $\iota f = -f$ there exists a unique F-Hermitian form ϕ on V such that $\psi(x,y) = \mathrm{Tr}_{F/\mathbb{Q}}(f\phi(x,y))$ (see I.4.6).

Let Σ be the set of embeddings $F_0 \hookrightarrow \mathbb{C}$; then

$$H_1(A,\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\sigma\in\Sigma} V_\sigma \ , \text{ where } V_\sigma = V \otimes_{F_0,\sigma} \mathbb{C} \ .$$

Moreover:

V_σ is a free $F \otimes_{F_{0,\sigma}} \mathbb{C}$-module of rank 1;

$V_\sigma = V_\sigma^+ \oplus V_\sigma^-$, where $h(z)$ acts as z on V_σ^+ and ιz on V_σ^- ;

ϕ defines a Hermitian form ϕ_σ on V_σ such that $\phi_\sigma > 0$ on V_σ^+ and $\phi_\sigma < 0$ on V_σ^- .

Let τ be an automorphism of $\mathbb{C}$ and let $V' = H_1(\tau A,\mathbb{Q})$. The action of F on A induces an action of F on τA , and $[\psi]$ gives rise to a homogeneous polarization $[\tau\psi]$ on τA . Thus there is a decomposition $H_1(\tau A,\mathbb{C}) = \oplus_{\sigma\in\Sigma} V'_\sigma$ where the V'_σ have similar structures to the V_σ .

Our purpose is to construct an isomorphism $\theta : H_1(A,\mathbb{C}) \longrightarrow H_1(\tau A,\mathbb{C})$ that is $F \otimes \mathbb{C}$-linear and takes $[\psi]$ to $[\tau\psi]$. It will suffice to define θ on each component V_σ of $H_1(A,\mathbb{C})$.

As $H_1(A,\mathbb{C})$ is canonically dual to the de Rham cohomology group $H^1_{dR}(A)$, and $H^1_{dR}(\tau A) = H^1_{dR}(A) \otimes_{\mathbb{C},\tau} \mathbb{C}$, we see that $H_1(\tau A,\mathbb{C}) = H_1(A,\mathbb{C}) \otimes_{\mathbb{C},\tau} \mathbb{C}$. Under this identification, the two actions of F correspond, and $\tau\psi$ corresponds to ψ.

Fix a $\sigma \in \Sigma$, and consider V_σ and V'_σ. Since F_0 acts on $V_\sigma \otimes_{\mathbb{C},\tau} \mathbb{C}$ through $\tau\sigma$, we see that we must have $V'_\sigma = V_{\tau^{-1}\sigma} \otimes_{\mathbb{C},\tau} \mathbb{C}$. There is an $F_0 \otimes \mathbb{C}$-linear isomorphism $\theta_1 : V_\sigma \xrightarrow{\approx} V'_\sigma$ and, since F acts on V^+_σ and V^-_σ through distinct enbeddings $F \hookrightarrow \mathbb{C}$, exactly one of the following must hold:

$$(+) \quad \theta_1 : V^+_\sigma \xrightarrow{\approx} V'^+_\sigma \ , \ \theta_1 : V^-_\sigma \xrightarrow{\approx} V'^-_\sigma \ ;$$

$$(-) \quad \theta_1 : V^+_\sigma \xrightarrow{\approx} V'^-_\sigma \ , \ \theta_1 : V^-_\sigma \xrightarrow{\approx} V'^+_\sigma \ .$$

Choose a basis for V_σ compatible with the decomposition $V_\sigma = V^+_\sigma \oplus V^-_\sigma$ and define θ_σ to be θ_1 in case $(+)$ and to be the composite of θ_1 with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ in case $(-)$. Then θ_σ is an $F \otimes_{F_0,\sigma} \mathbb{C}$-linear isomorphism $V_\sigma \to V'_\sigma$ taking ϕ_σ to a multiple of ϕ'_σ.

<u>Lemma 5.1</u>. With the above notations, there exists an isomorphism $\theta : H_1(A,\mathbb{C}) \longrightarrow H_1(\tau A,\mathbb{C})$ such that:

(a) $\theta \circ f = f \circ \theta$ for all $f \in F$;

(b) $\theta(\tau[\psi]) = [\psi]$;

(c) $\iota\theta = \theta.(\tau\mu(-1)/\mu(-1))$

Proof. Define $\theta = \oplus_\sigma \theta_\sigma$ and note that $\iota\theta_\sigma = \theta_\sigma$ in case (+) while $\iota\theta_\sigma = -\theta_\sigma$ in case (-) . On the other hand, $\mu(-1)$ acts as $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ on $V_\sigma = V_\sigma^+ \oplus V_\sigma^-$ and $\tau\mu(-1) = \mu(-1)$ in case (+) while $\tau\mu(-1) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ in case (-) .

We shall need a slightly more precise result.

Proposition 5.2. Let A be an abelian variety over $\mathbb{C}$ that is of CM-type, and let τ be an automorphism of $\mathbb{C}$. There exists an isomorphism $\theta : H_1(A,\mathbb{C}) \longrightarrow H_1(\tau A,\mathbb{C})$ such that:

(a) $\theta(s) = \tau s$ for all Hodge cycles s on A ;

(b) $\iota\theta = \theta.\gamma$ where γ is the class in $H^1(\mathbb{R},MT(A))$ represented by $\tau\mu(-1)/\mu(-1)$. ($MT(A)$ = Mumford-Tate group of A.)

Proof. Note that, if we let

$$P(R) = \{\theta : H_1(A,R) \xrightarrow{\approx} H_1(\tau A,R) \mid \theta \text{ satisfies } (5.2a)\}$$

for any $\mathbb{Q}$-algebra R , then P is a right $MT(A)$-torsor. Proposition (5.2) describes the class of $P_{\mathbb{R}}$ in $H^1(\mathbb{R},MT(A))$. The lemma shows that image of the class in $H^1(\mathbb{R},T)$ is correct, where T is the subtorus of $F^\times$ of elements whose norm to F_0 lies in $\mathbb{Q}^\times$.

We shall complete the proof of the proposition by showing that $H^1(\mathbb{R}, MT(A)) \to H^1(\mathbb{R}, T)$ is injective.

The norm map N_{F/F_0} defines a surjection $T \to \mathbb{G}_m$, and we define ST and SMT to make the rows in the following diagram exact:

$$\begin{array}{ccccccccc} 1 & \to & SMT & \to & MT & \to & \mathbb{G}_m & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & ST & \to & T & \to & \mathbb{G}_m & \to & 1 \end{array} .$$

This diagram gives rise to an exact commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}^\times & \to & H^1(\mathbb{R}, SMT) & \to & H^1(\mathbb{R}, MT) & \to & 0 \\ \| & & \downarrow & & \downarrow & & \\ \mathbb{R}^\times & \to & H^1(\mathbb{R}, ST) & \to & H^1(\mathbb{R}, T) & \to & 0 \end{array} .$$

Note that ST (and hence SMT) is anisotropic over $\mathbb{R}$, and that for an anisotropic torus S', $H^1(\mathbb{R}, S') = \mathrm{Ker}(S'(\mathbb{C}) \xrightarrow{2} S'(\mathbb{C}))$. Thus $H^1(\mathbb{R}, SMT) \to H^1(\mathbb{R}, ST)$ is injective, and the five-lemma shows that $H^1(\mathbb{R}, MT) \to H^1(\mathbb{R}, T)$ is injective.

Remark 5.3. Let A, F, and $V = H_1(A, \mathbb{Q})$ be as in the first paragraph. Then h can be regarded as a map $h : \mathbb{S} \to F^\times(\mathbb{R})$ (thinking of $F^\times$ as a $\mathbb{Q}$-rational torus). It is clear from the discussion preceeding (5.1) that τA is the abelian variety corresponding to $(V, {}^\tau h)$, where ${}^\tau h$ is the map $\mathbb{S} \to F^\times(\mathbb{R})$ with associated cocharacter $\tau\mu_h \in X_*(F^\times)$.

§6. Conjugates of abelian varieties of CM-type.

Let A be an abelian variety of CM-type over $\mathbb{C}$, let $V = H_1(A,\mathbb{Q})$, and let h be the (natural) Hodge structure on V. Fix some family $(s_\alpha)_{\alpha\in J}$ of tensors such that the Mumford-Tate group $MT(A)$ of A is $\mathrm{Aut}(V,(s_\alpha))$ (see I.3). The canonical map $S \xrightarrow{\rho} MT(A)$ induces an action of S on $(V,(s_\alpha))$ and, for any automorphism τ of $\mathbb{C}$, we define $({}^\tau V,({}^\tau s_\alpha)) = {}^\tau S \times^S (V,(s_\alpha))$. The element $sp(\tau) \in {}^\tau S(\mathbb{A}^f)$ defines an isomorphism

$$v \mapsto sp(\tau).v: \ (V(\mathbb{A}^f),(s_\alpha)) \xrightarrow{\approx} ({}^\tau V(\mathbb{A}^f),({}^\tau s_\alpha)),$$

which we shall again denote by $sp(\tau)$.

Lemma 6.1. There is an isomorphism $f : (H_1(\tau A,\mathbb{Q}),(\tau s_\alpha)) \xrightarrow{\approx} ({}^\tau V,({}^\tau s_\alpha))$

Proof. Let P_A be the functor such that, for any $\mathbb{Q}$-algebra R, $P_A(R)$ is the set of isomorphisms $(H_1(A,R),(s_\alpha)) \xrightarrow{\approx} (H_1(\tau A,R),(\tau s_\alpha))$. Clearly P_A is representable, and is a right $MT(A)$-torsor. Since $P_A \times^{MT(A)} (H_1(A,\mathbb{Q}),(s_\alpha)) = (H_1(\tau A,\mathbb{Q}),(\tau s_\alpha))$, to prove the lemma it suffices to show that P_A is isomorphic to the $MT(A)$-torsor $\rho_*({}^\tau S)$. We shall show this simultaneously for all abelian varieties (over $\mathbb{C}$, of CM-type) whose Mumford-Tate groups are split by a fixed finite Galois extension L of $\mathbb{Q}$.

According to (III.1.7), $S^L = \varprojlim MT(A)$, and it will suffice to show that the two S^L-torsors $P = \varprojlim P_A$ and ${}^\tau S^L$ are isomorphic. As $H^1(\mathbb{Q},S^L)$ satisfies the Hasse principle (III. 1.5) this only has to be shown locally. The isomorphisms

$H_1(A,\mathbb{Q}_\ell) = V^f(A) \xrightarrow[\approx]{\tau} V^f(\tau A) = H_1(\tau A,\mathbb{Q}_\ell)$ show P to be trivial over $\mathbb{Q}_\ell$, while $sp(\tau)e^{\tau}S^L(\mathbb{Q}_\ell)$ shows ${}^{\tau}S^L_{\mathbb{Q}_\ell}$ to be trivial. Finally (III. 3.14) and (5.2) show ${}^{\tau}S^L_{\mathbb{R}}$ and $P_{\mathbb{R}}$ define the same cohomology class in $H^1(\mathbb{R},S^L)$ and are therefore isomorphic.

Note that f is uniquely determined up to right multiplication by an element of $MT(A)(\mathbb{Q})$.

<u>Conjecture CM(first form)</u>. The isomorphism f of (6.1) can be chosen to make the following diagram commute:

$$\begin{array}{ccc} V^f(A) & \xrightarrow{\tau} & V^f(\tau A) \\ \| & & \downarrow f\otimes 1 \\ V(\mathbb{A}^f) & \xrightarrow{sp(\tau)} & {}^{\tau}V(\mathbb{A}^f) \end{array}$$

We next restate the conjecture in a form that is closer to the usual statements of the main theorem of complex multiplication. Let $T = MT(A)$, and choose a polarization ψ for (V,h) which we shall assume to be one of the s_α. From the inclusion $(T,\{h\}) \hookrightarrow (CSp(V),S^{\pm})$ we obtain, as in §2, a bijection

$$Sh(T,\{h\}) \xrightarrow{\approx} \mathcal{A}(T,\{h\},V)$$

where $\mathcal{A}(T,\{h\},V)$ consists of certain isomorphism classes of triples $(A',(t_\alpha),k)$.

The torus T continues to act on ${}^{\tau}V$, and in fact $T = Aut({}^{\tau}V,({}^{\tau}s_\alpha))$. One of the ${}^{\tau}s_\alpha$ is ${}^{\tau}\psi$, which is a polarization for $({}^{\tau}V,{}^{\tau}h)$, where ${}^{\tau}h$ is the homomorphism $\mathbb{S} \to T$ corresponding to $\tau\mu_h$. Thus we have an inclusion

$(T,\{{}^{\tau}h\}) \hookrightarrow (CSp({}^{\tau}V), S^{\pm})$ and, as before, a bijection

$$Sh(T,\{{}^{\tau}h\}) \xrightarrow{\approx} \mathcal{A}(T,\{{}^{\tau}h\},{}^{\tau}V) .$$

We define $\chi_{\tau} : \mathcal{A}(T,\{h\},V) \longrightarrow \mathcal{A}(T,\{{}^{\tau}h\},{}^{\tau}V)$ to be the mapping that sends $[A',(t_{\alpha}),k]$ to the class $[\tau A',(\tau t_{\alpha}),{}^{\tau}k]$ where ${}^{\tau}k$ is the composite $V^{f}(\tau A) \xrightarrow{\tau^{-1}} V^{f}(A) \xrightarrow{k} V(\mathbb{A}^{f}) \xrightarrow{sp(\tau)} {}^{\tau}V(\mathbb{A}^{f})$. Lemma (6.1) shows that $[\tau A',(\tau t_{\alpha}),{}^{\tau}k]$ satisfies condition (2.1a) to lie in $\mathcal{A}(T,\{{}^{\tau}h\},{}^{\tau}V)$ and (5.3) shows that it satisfies (2.1b).

<u>Conjecture CM (second form)</u>. The following diagram commutes:

$$\begin{array}{ccc}
[h,g] & Sh(T,\{h\}) \xrightarrow{\approx} & \mathcal{A}(T,\{h\},V) \\
\downarrow & \downarrow \approx & \chi_{\tau} \downarrow \approx \\
[{}^{\tau}h,g] & Sh(T,\{{}^{\tau}h\}) \xrightarrow{\approx} & \mathcal{A}(T,\{{}^{\tau}h\},{}^{\tau}V)
\end{array}$$

It is easy to check that the two forms of the conjecture are equivalent.

<u>Remark 6.2</u>. When τ fixes the reflex field $E(T,\{h\})$, then conjecture CM becomes the main theorem of complex multiplication (see Milne-Shih [1, 2.6]).

<u>Example 6.3</u>. Let F be a CM-field and Σ a CM-type for F . Let A be an abelian variety (an actual abelian variety - not an isogeny class of abelian varieties!) of type (F,Σ) . Then $H_1(A,\mathbb{Z})$ is a locally free module of rank one over the ring of integers O_F in F , and hence defines an element $I(A)$ of $Pic(O_F)$. Consider

$$\begin{array}{c} (S^L(\mathbb{A}^f_L)/S^L(L))^{Gal(L/\mathbb{Q})} \\ \downarrow \\ 1 \longrightarrow T(\mathbb{A}^f)/T(\mathbb{Q}) \longrightarrow (T(\mathbb{A}^f_L)/T(L))^{Gal(L/\mathbb{Q})} \longrightarrow H^1(L/\mathbb{Q},T) \end{array}$$

where $T = Res_{F/\mathbb{Q}}\, \mathbb{G}_m$, L is a (sufficiently large) finite Galois extension of $\mathbb{Q}$, and the vertical map is induced by the canonical map $\rho : S^L \to T$. As $H^1(L/\mathbb{Q},T) = 0$, the image of $\bar{\beta}(\tau)$ in $(T(\mathbb{A}^f_L)/T(L))^{Gal(L/\mathbb{Q})}$ arises from an element $\beta'(\tau) \in T(\mathbb{A}^f)/T(\mathbb{Q})$. This defines an ideal class $I(\tau) \in Pic(O_F)$, and the conjecture predicts that $I(\tau A) = I(\tau)\, I(A)$.

Remark 6.4. Let A be an abelian variety of potential CM-type defined over a number field k . Conjecture CM would imply that the zeta function of A is an alternating product of L-series associated to complex representations of the Weil group of k . Deligne has proved this result without, however, proving the conjecture (cf. IV.).

§ 7. Conjecture C, conjecture CM, and canonical models.

Let $(M(G,X),\ f\colon M(G,X)_{\mathbb{C}} \xrightarrow{\approx} Sh(G,X))$ be a canonical model for $Sh(G,X)$ (Deligne [2, 2.2.5]) and, for each automorphism τ of $\mathbb{C}$ fixing $E(G,X)$, set $\psi_\tau = f \circ (\tau f)^{-1}$. These isomorphisms $\psi_\tau\colon \tau Sh(G,X) \to Sh(G,X)$ satisfy the following conditions:

(7.1a) $\psi_{\tau_1\tau_2} = \psi_{\tau_1} \circ (\tau_1\psi_{\tau_2}),\ \tau_1,\tau_2 \in \mathrm{Aut}(\mathbb{C}/E(G,X))$;

(7.1b) $\psi_\tau \circ \tau(\mathcal{J}(g)) = \mathcal{J}(g) \circ \psi_\tau,\ \tau \in \mathrm{Aut}(\mathbb{C}/E(G,X)),\ g \in G(\mathbb{A}^f)$;

(7.1c) let $h \in X$ be special and assume that τ fixes the reflex field $E(h)$ of h; then $\psi_\tau(\tau[h,1]) = [h, \tilde{r}(\tau)]$. (Here $\tilde{r}(\tau) \in G(\mathbb{A}^f)$ represents $r_E(T,h)(\tau) \in T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge$ where T is some $\mathbb{Q}$-rational torus such that h factors through $T_{\mathbb{R}}$ and $r_E(T,h)$ is the reciprocity morphism (Deligne [2, 2.2.3]).) Note that the family (ψ_τ) is uniquely determined by (G,X): if $(M(G,X)', f')$ is a second canonical model, there is an isomorphism $q\colon M(G,X)' \longrightarrow M(G,X)$ such that $f' = f \circ q_{\mathbb{C}}$, and so $f' \circ (\tau f')^{-1} = f \circ (\tau f)^{-1} = \psi_\tau$. Moreover, descent theory shows that every family (ψ_τ) satisfying (7.1) arises from a canonical model for $Sh(G,X)$.

If τ fixes $E(G,X)$ and $M(G,X)$ is a canonical model for $Sh(G,X)$, then $\tau M(G,X) = M(G,X)$ is again canonical model for $Sh(G,X)$, and so conjecture A suggests that we should have $Sh(G,X) \xrightarrow{\approx} Sh({}^\tau G, {}^\tau X)$. We shall prove this. Thus let (G,X) be any pair satisfying (1.1) and let $h \in X$ be special. Choose a

$\mathbb{Q}$-rational maximal torus T in G such that h factors through $T_{\mathbb{R}}$, and let $\mu = \mu_h$. If τ is an automorphism of $\mathbb{C}$ that fixes $E(G,X)$ then $\tau\mu$ and μ have the same weight; thus $(1 + \iota)\tau\mu = (1 + \iota)\mu$, and (see III. 3.18) there is a well-defined cohomology class $\gamma(\tau,\mu) \in H^1(\mathbb{Q}, T)$.

Lemma 7.2. The image of $\gamma(\tau,\mu)$ in $H^1(\mathbb{Q},G)$ is trivial.

Proof: After replacing (G,X) with the pair (G_1,X_1) constructed in (3.4), we can assume G^{der} is simply connected. Let $H = G/G^{der}$ and let μ' be the composite of μ with $G \to H$. As $\tau\mu$ is conjugate to μ, $\tau\mu' = \mu'$ and (III. 3.10) shows that $\gamma(\tau,\mu')$ is trivial.

Let $w \in G(\mathbb{C})$ normalize $T(\mathbb{C})$ and be such that $\tau\mu = \mathrm{ad}\,w \circ \mu$. According to (III. 3.14), the image of $\gamma(\tau,\mu)$ in $H^1(\mathbb{R},G)$ is represented by $\tau\mu(-1)/\mu(-1) = (\mathrm{ad}\,w \circ \mu)(-1)/\mu(-1)$ which (see 4.2) is also represented by $w \cdot \iota w^{-1}$; it is therefore trivial. The lemma is now a consequence of the following easy result.

Sublemma 7.3. Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is simply connected. An element γ of $H^1(\mathbb{Q},G)$ is trivial if its images in $H^1(\mathbb{Q}, G/G^{der})$ and $H^1(\mathbb{R}, G)$ are trivial.

We continue with the notations of the second paragraph of this section; thus $h \in X$ is special, $\mu = \mu_h$, and τ fixes $E(G,X)$. Choose an element $a(\tau) \in {}^{\tau}S(\bar{\mathbb{Q}})$ and let $f\colon G_{\bar{\mathbb{Q}}} \to {}^{\tau}G_{\bar{\mathbb{Q}}}$

be the isomorphism $g \longmapsto a(\tau).g$. It will often be convenient to regard f as being defined over L where L is some sufficiently large finite Galois extension of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$. Let $\beta(\tau) = sp(\tau)^{-1}\, a(\tau) \in S(\mathbb{A}_L^f)$ and let $\beta(\tau, \overline{\mu})$ be the image of $\beta(\tau)$ in $T^{ad}(\mathbb{A}_L^f)$ under the map $\rho_{\overline{\mu}} : S \to T^{ad}$ defined by $\mu^{ad} \stackrel{df}{=} \overline{\mu} \stackrel{df}{=} (\mathbb{G}_m \stackrel{\mu}{\to} T \to T^{ad})$. Recall (III.3.18) that we have also defined an element $\overline{\beta}(\tau,\mu) \in T(\mathbb{A}_L^f)/T(L)T(\mathbb{Q})^{\wedge}$. Since $\beta(\tau,\overline{\mu})$ and $\overline{\beta}(\tau,\mu)$ have the same image in $T(\mathbb{A}_L^f)/Z(\mathbb{A}_L^f)\, T(L)\ T(\mathbb{Q})^{\wedge}$ we can choose an element $\tilde{\beta}(\tau,\mu) \in T(\mathbb{A}_L^f)$ that lifts both $\beta(\tau,\overline{\mu})$ and $\overline{\beta}(\tau,\mu)$; it is determined up to multiplication by an element of $Z(\mathbb{A}_L^f) \cap T(L)\ T(\mathbb{Q})^{\wedge} = Z(L)\ Z(\mathbb{Q})^{\wedge}$. (Note that $T(\mathbb{Q})Z(\mathbb{Q})^{\wedge} = T(\mathbb{Q})^{\wedge}$ because $T^{ad}(\mathbb{Q})$ is a discrete subgroup of $T^{ad}(\mathbb{A}^f)$.) Let $\sigma\tilde{\beta}(\tau,\mu) = \tilde{\beta}(\tau,\mu)\gamma_\sigma$; then (γ_σ) is a 1-cocycle representing $\gamma(\tau,\mu) \in H^1(\mathbb{Q},T)$. We have $\sigma f = f \circ \underset{\sim}{ad}\,\gamma_\sigma$. The lemma shows that there is an element $v \in G(\overline{\mathbb{Q}})$ such that $\gamma_\sigma = v^{-1}.\sigma v$ for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. We define an isomorphism $f_1 : G \to {}^{\tau}G$ and an element $\beta_1(\tau,\mu) \in G(\mathbb{A}^f)$ by the formulas:

$$f_1 = f \circ \underset{\sim}{ad}\, v^{-1} \tag{7.4a}$$

$$\beta_1(\tau,\mu) = \tilde{\beta}(\tau,\mu)v^{-1} \tag{7.4b}$$

<u>Remark</u> 7.5. In the above we have had to choose an $a(\tau)$, $\tilde{\beta}(\tau,\mu)$, and v. For example, if $a(\tau)$ is replaced by $a(\tau)u$, $u \in S(L)$, then $\beta(\tau,\overline{\mu})$ is replaced by $\beta(\tau,\overline{\mu})\ \rho_{\overline{\mu}}(u)$. We show that the cosets defined by $\beta_1(\tau,\mu)$ and $\beta_1(\tau,\mu)^{-1}$ in $G(\mathbb{Q}) \backslash G(\mathbb{A}^f)/Z(\mathbb{Q})^{\wedge}$ are independent of all choices.

Consider the exact commutative diagram

$$\begin{array}{ccccc} 1 \to T(\mathbb{Q})\backslash T(\mathbb{A}^f) & \longrightarrow & (T(L)\backslash T(\mathbb{A}_L^f))^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},T(L)) \\ \downarrow & & \downarrow & & \downarrow \\ 1 \to G(\mathbb{Q})\backslash G(\mathbb{A}^f) & \longrightarrow & (G(L)\backslash G(\mathbb{A}_L^f))^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},G(L)) \end{array}$$

in which the vertical arrows are induced by the inclusion $T \hookrightarrow G$. On dividing by $Z(\mathbb{Q})^\wedge$ we obtain

$$\begin{array}{ccccc} 1 \longrightarrow T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge & \longrightarrow & (T(L)\backslash T(\mathbb{A}_L^f)/T(\mathbb{Q})^\wedge)^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},T(L)) \\ \downarrow & & \downarrow & & \downarrow \\ 1 \longrightarrow G(\mathbb{Q})\backslash G(\mathbb{A}^f)/Z(\mathbb{Q})^\wedge & \longrightarrow & (G(L)\backslash G(\mathbb{A}_L^f)/Z(\mathbb{Q})^\wedge)^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},G(L)). \end{array}$$

Lemma 7.2. shows that the image of $\bar{\beta}(\tau,\mu)$ (or $\bar{\beta}(\tau,\mu)^{-1}$) under the middle vertical arrow lies in $G(\mathbb{Q})\backslash G(\mathbb{A}^f)/Z(\mathbb{Q})^\wedge$; it is represented by $\beta_1(\tau,\mu)$.

Remark 7.6. Everything is much simpler when μ satisfies (III. 1.1). Then there is a map $\rho_\mu : S \to T$ and we can choose $\tilde{\beta}(\tau,\mu) = \beta(\tau,\mu) \overset{\mathrm{df}}{=} \rho_\mu(\beta(\tau))$. A change in the choices of $a(\tau)$ and v forces the following changes:

$a(\tau)$	$\beta(\tau,\mu)$	γ_σ	v	f	$\beta_1(\tau,\mu)$
$a(\tau)u_0$	$\beta(\tau,\mu)u_2$	$u_2^{-1}\gamma_\sigma \sigma u_2$	$u_3 v u_2$	$f \circ \underset{\sim}{\mathrm{ad}}\, u_2$	$\beta_1(\tau,\mu)u_3^{-1}$

$u_0 \in S(L)$, $u_2 = \rho_\mu(u_0) \in T(L)$, $u_3 \in G(\mathbb{Q})$.

We shall abuse notation by writing ${}^\tau h$ also for the map $\mathbb{S} \to G_{\mathbb{R}}$ associated with $\tau\mu$: $\mathbb{G}_m \to G_{\mathbb{C}}$; thus ${}^\tau h$ (in the sense of § 4) = $f \circ {}^\tau h$(this sense).

Lemma 7.7. Regard v as an element of $G(\mathbb{C})$; then $\underset{\sim}{ad}v \circ {}^{\tau}h \in X$.

Proof. Let $w \in G(\mathbb{C})$ normalize $T(\mathbb{C})$ and be such that $\tau\mu = \underset{\sim}{ad}w \circ \mu$. Then (see the proof of 7.2) v^{-1} and w represent the same cocycle, and so $vw \in G(\mathbb{R})$. Hence $adv \circ {}^{\tau}h = \underset{\sim}{ad}v \circ \underset{\sim}{ad}w \circ h \in X$.

Since $\underset{\sim}{ad}v \circ {}^{\tau}h \in X$, and $f_1 \circ \underset{\sim}{ad}v \circ {}^{\tau}h = {}^{\tau}h \in {}^{\tau}X$, we see that $f_1 : G \xrightarrow{\approx} {}^{\tau}G$ defines an isomorphism $Sh(f_1):Sh(G,X) \xrightarrow{\approx} Sh({}^{\tau}G, {}^{\tau}X)$.

Proposition 7.8. Let $\phi(\tau;\mu)$ be the map

$$Sh(f_1) \circ \mathcal{J}(\beta_1(\tau,\mu)) : Sh(G,X) \xrightarrow{\approx} Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X).$$

Then $\phi(\tau;\mu)$ is independent of the choices of $a(\tau)$, $\tilde{\beta}(\tau,\mu)$, and v; moreover

$$\phi(\tau;\mu)[\underset{\sim}{ad}v \circ {}^{\tau}h, \beta_1(\tau,\mu)^{-1}] = [{}^{\tau}h,1]$$

$$\phi(\tau;\mu) \circ \mathcal{J}(g) = \mathcal{J}({}^{\tau,\mu}g) \circ \phi(\tau;\mu).$$

Proof. The formula $\phi(\tau;\mu)[x,g] = [f_1 \circ x, f_1(g\beta_1(\tau,\mu))]$ shows immediately that $\phi(\tau;\mu)$ maps $[\underset{\sim}{ad}v \circ {}^{\tau}h, \beta_1(\tau,\mu)^{-1}]$ to $[{}^{\tau}h,1]$ and that $\phi(\tau;\mu)\,\mathcal{J}(g) = \mathcal{J}(g')\,\phi(\tau;\mu)$ with $g' = f_1(\beta_1^{-1}g\beta_1) = f \circ \underset{\sim}{ad}\,\beta(\tau,\bar{\mu})(g) = {}^{\tau}g$. The independence assertion is a consequence of this, the following lemma, and Deligne [1,5.2].

<u>Lemma 7.9.</u> The element $[\underset{\sim}{ad}v \circ {}^{\tau}h,\ \beta_1(\tau,\mu)^{-1}] \in Sh(G,X)$ is independent of the choices of $a(\tau)$, $\tilde{\beta}(\tau,\mu)$, and v.

<u>Proof</u>. Suppose that, after a change in the choices of $a(\tau)$, $\tilde{\beta}(\tau,\mu)$ and v, the elements β_1 and v are replaced by β_1' and v'. Remark (7.5) shows that $(\beta_1')^{-1} = u\beta_1^{-1}z$ with $u \in G(\mathbb{Q})$ and $z \in Z(\mathbb{Q})^\wedge$; moreover $\underset{\sim}{ad}(\beta_1' v') = \underset{\sim}{ad}(\beta(\tau,\bar{\mu})u_1) = \underset{\sim}{ad}(\beta_1 v u_1)$ with $u_1 \in T^{ad}(L)$. Thus $\underset{\sim}{ad}(z^{-1}\beta_1 u^{-1}v') = \underset{\sim}{ad}(\beta_1 v u_1)$ and, on cancelling the β_1, we find $\underset{\sim}{ad}(u^{-1}v') = \underset{\sim}{ad}(vu_1)$. Hence $[\underset{\sim}{ad}v' \circ {}^{\tau}h,\ (\beta_1')^{-1}] = [\underset{\sim}{ad}v' \circ {}^{\tau}h,\ u\beta_1^{-1}z] = [\underset{\sim}{ad}u^{-1}v' \circ {}^{\tau}h,\ \beta_1^{-1}] = [\underset{\sim}{ad}v \circ \underset{\sim}{ad}u_1 \circ {}^{\tau}h,\ \beta_1^{-1}] = [\underset{\sim}{ad}v \circ {}^{\tau}h,\ \beta_1^{-1}]$ because ${}^{\tau}h$ maps into $T(\mathbb{R})$ and $u_1 \in T^{ad}(L)$.

<u>Remark 7.10.</u> Under the hypothesis of (7.6), the map $\phi(\tau;\mu)$ becomes $[x,g] \longmapsto [f \circ \underset{\sim}{ad}v^{-1} \circ x,\ f(v^{-1} g\beta(\tau,\mu))]$ and the element in (7.9) becomes $[\underset{\sim}{ad}v \circ {}^{\tau}h,\ v\beta(\tau,\mu)^{-1}]$. Both can be directly shown to be independent of all choices.

<u>Proposition 7.11.</u> Assume that $Sh(G,X)$ has a canonical model and let (ψ_τ), $\tau \in Aut(\mathbb{C}/E(G,X))$, be the corresponding family of maps as in (7.1) above. Conjecture C is true for $Sh(G,X)$ and a particular $\tau \in Aut(\mathbb{C}/E(G,X))$ if

$$\psi_\tau(\tau[h,1]) = [\underset{\sim}{ad}v \circ {}^{\tau}h,\ \beta_1(\tau,\mu)^{-1}] \qquad (7.12)$$

holds for all special $h \in X$.

Proof: Note that Lemma 7.9 shows (7.12) makes sense. Define $\phi_{\tau,\mu} = \phi(\tau;\mu) \circ \psi_\tau$. Then

$$\begin{aligned}\phi_{\tau,\mu}(\tau[h,1]) &= \phi(\tau;\mu)[\underset{\sim}{\mathrm{ad}}\,v \circ {}^\tau h,\ \beta_1(\tau,\mu)^{-1}] && \text{by (7.12)}\\ &= [{}^\tau h,1] && \text{by (7.8)}\end{aligned}$$

Moreover,

$$\begin{aligned}\phi_{\tau,\mu} \circ (\tau\mathfrak{J}(g)) &= \phi(\tau;\mu) \circ \mathfrak{J}(g) \circ \psi_\tau\\ &= \mathfrak{J}({}^{\tau,\mu}g) \circ \phi_{\tau,\mu} && \text{by (7.8)}\end{aligned}$$

Thus $\phi_{\tau,\mu}$ satisfies condition (a) of conjecture C. Let h' be a second special point and let $\mu' = \mu_{h'}$. Then

$$\begin{aligned}\phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu} &= \phi(\tau;\mu',\mu) \circ \phi(\tau;\mu) \circ \psi_\tau\\ &= \phi_{\tau,\mu'}\end{aligned}$$

because $\phi(\tau;\mu',\mu) = \phi(\tau;\mu') \circ \phi(\tau;\mu)^{-1}$ (4.13).

Remark 7.13 In certain situations, (7.12) simplifies. For example, under the hypothesis of (7.6) it becomes

$$\psi_\tau(\tau[h,1]) = [\underset{\sim}{\mathrm{ad}}\,v \circ {}^\tau h,\ v\,\beta(\tau,\mu)^{-1}] \tag{7.13a}$$

(see 7.10). On the other hand, if we identify $Sh(G,X)$ with $M(G,X)_{\mathbb{C}}$, then (7.12) becomes

$$\tau[h,1] = [\underset{\sim}{\mathrm{ad}}\,v \circ {}^\tau h,\ \beta_1(\tau,\mu)^{-1}] \tag{7.13b}$$

If $\gamma(\tau,\bar\mu) \overset{df}{=} \rho_{\bar\mu}(\gamma(\tau))$ is trivial in $H^1(\mathbb{Q}, \rho_{\bar\mu}(S))$ then there exists a $u \in S(L)$ such that $\rho_{\bar\mu}(u)^{-1}(\sigma\rho_{\bar\mu}(u)) = \gamma_\sigma \pmod{Z(G)}$.

After replacing $a(\tau)$ with $a(\tau)u$ one finds that f is defined over $\mathbb{Q}$, that $\tilde{\beta}(\tau,\mu)$ can be chosen to lie in $T(\mathbb{A}^f)$, and consequently that $v = 1$. Thus (7.12) becomes

$$\psi_\tau(\tau[h,1]) = [{}^\tau h, \tilde{\beta}(\tau,\mu)^{-1}] \qquad (7.13c)$$

Finally, if τ fixes $E(h)$ then the hypothesis of (7.6) is satisfied, $\gamma(\tau)$ is trivial in $H^1(\mathbb{Q}, S^{E(h)})$, and (7.12) can be written

$$\psi_\tau(\tau[h,1]) = [h, \beta(\tau,\mu)^{-1}] = [h, \tilde{r}(\tau)] \qquad (7.13d)$$

(see III.3.10), which is one of the defining conditions for $M(G,X)$ to be canonical model (see 7.1c).

<u>Proposition 7.14.</u> Assume that conjecture C is true for $Sh(G,X)$ and all $\tau \in Aut(\mathbb{C}/E(G,X))$; then $Sh(G,X)$ has a canonical model and the maps ψ_τ (as in 7.1) satisfy $\psi_\tau = \phi(\tau;\mu_h)^{-1} \circ \phi_{\tau,\mu_h}$ for any special $h \in X$; equation (7.12) is true for all τ fixing $E(G,X)$ and all special $h \in X$.

<u>Proof:</u> Choose a special h and set $\psi_\tau = \phi(\tau;\mu)^{-1} \circ \phi_{\tau,\mu}$ with $\mu = \mu_h$. Arguments reverse to those in the proof of (7.11) show that ψ_τ is independent of h, that $\psi_\tau \circ \tau\mathcal{J}(g) = \mathcal{J}(g) \circ \psi_\tau$, and that $\psi_\tau(\tau[h,1]) = [\underset{\sim}{ad}v \circ {}^\tau h, \beta_1(\tau,\mu)^{-1}]$. To complete the proof it must be shown that $\psi_{\tau_1\tau_2} = \psi_{\tau_1} \circ (\tau_1\psi_{\tau_2})$, but it can be checked directly that the two maps agree at the point $\tau_1\tau_2[h,1]$, and this implies they agree everywhere.

Corollary 7.15. In addition to the assumption of (7.14) suppose that $E(G,X) \subset \mathbb{R}$. Then conjecture B is true for $Sh(G,X)$.

Proof. If we identify $Sh(G,X)$ with $M(G,X)_{\mathbb{C}}$ then (7.14) and (7.13) show that $\iota[h,1] = [{}^{\iota}h, \tilde{\beta}(\iota,\mu)^{-1}]$ for any special h, where $\tilde{\beta}(\iota,\mu)$ has been chosen to be in $T(\mathbb{A}^f)$. But, according to (III. 3.9), $\overline{\beta}(\iota,\mu) = 1$ and so $\beta(\iota,\mu) \in T(\mathbb{Q})$. Thus $\iota[h,1] = [{}^{\iota}h, \tilde{\beta}(\iota,\mu)^{-1}] = [{}^{\iota}h, 1]$, which implies conjecture B (4.4).

We come now to the relation between conjectures C and CM. Let A be an abelian variety of CM-type, let $V = H_1(A,\mathbb{Q})$, let h be the natural Hodge structure on V, and let ψ be a Riemann form for A. If T is the Mumford-Tate group of A then we have an embedding $(T,h) \hookrightarrow (CSp(V), S^{\pm})$.

Proposition 7.16. Conjecture CM is true for A and a given $\tau \in Aut(\mathbb{C})$ if and only if (7.12) holds for $Sh(CSp(V),S^{\pm})$, h, and τ.

Proof. Write (G,X) for $(CSp(V), S^{\pm})$. Recall (2.3) that there is a bijection $Sh(G,X) \xrightarrow{\approx} \mathcal{A}(G,X,V)$ where $\mathcal{A}(G,X,V)$ consists of certain isomorphism classes of triples (A',t,k). Let $\mu = \mu_h$; we define $\chi_{\tau,\mu} : \mathcal{A}(G,X,V) \longrightarrow \mathcal{A}({}^{\tau,\mu}G, {}^{\tau,\mu}X, {}^{\tau,\mu}V)$ to be the map $[A',t,k] \longmapsto [\tau A',\tau t,{}^{\tau}k]$ where ${}^{\tau}k$ is the composite $V^f(\tau A') \xrightarrow{\tau^{-1}} V^f(A') \xrightarrow{k} V(\mathbb{A}^f) \xrightarrow{sp(\tau)} {}^{\tau,\mu}V(\mathbb{A}^f)$. Clearly there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(T,\ h,\ V) & \hookrightarrow & \mathcal{A}(G,\ X,\ V). \\ \downarrow \chi_\tau & & \downarrow \chi_{\tau,\mu} \\ \mathcal{A}(T, {}^\tau h, {}^{\tau,\mu}V) & \hookrightarrow & \mathcal{A}({}^{\tau,\mu}G, {}^{\tau,\mu}X, {}^{\tau,\mu}V) \end{array}$$

where χ_τ is as defined in §6. On the other hand, as the canonical model for $Sh(G,X)$ is the moduli variety, $\tau: Sh(G,X) \to Sh(G,X)$ corresponds to the map $\tau : \mathcal{A}(G,X,V) \to \mathcal{A}(G,X,V)$ such that $[A',t,k] \mapsto [\tau A', \tau t, \tau k]$ (where $\tau k = k \circ \tau^{-1}$). It is easily verified that $\phi(\tau;\mu)$ corresponds to the map $[A',t,k] \longrightarrow [A',t,sp(\tau) \circ k]$; thus $\phi(\tau;\mu) \circ \tau$ corresponds to $\chi_{\tau,\mu}$. Since $\phi(\tau;\mu)$ is an isomorphism, (7.12) is equivalent to the equation $\phi(\tau;\mu)(\tau[h,1]) = [{}^\tau h,1]$, or, to the assertion that $\chi_{\tau,\mu}$ maps the triple corresponding to $[h,1]$ to the triple corresponding to $[{}^\tau h,1]$. But this is precisely the second form of conjecture CM.

<u>Corollary</u> 7.17. Conjecture CM is true if and only if conjecture C is true for all Shimura varieties of the form $Sh(CSp(V), S^{\pm})$.

<u>Proof</u>. Combine (7.16) with (7.11) and (7.14).

<u>Remark</u> 7.18. The same arguments as above show that conjecture CM implies conjecture C for Shimura varieties of the form $Sh(G,X)$ when (G,X) embeds into $(CSp(V), S^{\pm})$. We shall show, however, that conjecture C for Shimura varieties of the form $Sh(CSp(V), S^{\pm})$ implies conjecture C for all Shimura varieties

of abelian type, see Theorem 9.8. Thus, at least for these varieties, conjecture C is equivalent to a statement involving nothing more than abelian varieties of CM-type.

Remark 7.19. It is easy to verify conjecture CM in the case that $\tau = \iota$ (cf. 4.4). On combining this remark with (6.2), we find that conjecture CM is true whenever τ fixes the maximal real subfield of $E(G,X)$. In particular, conjecture CM is true for elliptic curves. Now (7.16) shows that conjecture C is true for $Sh(GL_2, S^{\pm})$. (Cf. Shimura [1, 6.9]).

Even if τ does not fix the maximal totally real subfield of $E(G,X)$, conjecture CM still holds in some cases. Such examples are given in Milne-Shih [1, 2.7]. They arise naturally when one analyzes conjecture B, for details see (ibid, §6).

§ 8. Statement of conjecture C°.

Let (G,X) satisfy (1.1), let $h \in X$ be special, and let $\mu = \mu_h$. Recall that there is a unique homomorphism $\rho_{\bar{\mu}} : S \to G^{ad}$ such that $\rho_{\bar{\mu}} \circ \mu_{can} = \bar{\mu} \overset{df}{=} \mu^{ad}$; then $\rho_{\bar{\mu}}$ defines an action of S on G, and we write ${}^{\tau}G$ for ${}^{\tau}S \times^{S} G$ and $g \mapsto {}^{\tau}g : G(\mathbb{A}^f) \to {}^{\tau}G(\mathbb{A}^f)$ for $g \mapsto sp(\tau).g$.

<u>Lemma 8.1</u>. The isomorphism $g \mapsto {}^{\tau}g : G(\mathbb{A}^f) \to {}^{\tau}G(\mathbb{A}^f)$ maps the subgroup $G(\mathbb{Q})^{+\wedge}$ of $G(\mathbb{A}^f)$ into ${}^{\tau}G(\mathbb{Q})^{+\wedge}$ and $G(\mathbb{Q})^{\wedge}_{+}$ into ${}^{\tau}G(\mathbb{Q})^{\wedge}_{+}$.

<u>Proof</u>. Choose an element $a(\tau) \in {}^{\tau}S(L)$ for some finite Galois extension L of $\mathbb{Q}$, and let $f : G_L \to {}^{\tau}G_L$ be the isomorphism $g \to a(\tau).g$. In (3.6) we have defined an isomorphism $\pi_0\pi(f) : \pi_0\pi(G) \to \pi_0\pi({}^{\tau}G)$ and it is easily checked that the following diagram commutes:

$$\begin{array}{ccc} g \mapsto {}^{\tau}g : G(\mathbb{A}^f) & \to & {}^{\tau}G(\mathbb{A}^f) \\ \downarrow & & \downarrow \\ \pi_0\pi(f) : \pi_0\pi(G) & \to & \pi_0\pi({}^{\tau}G) \end{array}$$

Since the kernels of the two vertical arrows in this diagram are $G(\mathbb{Q})^{+\wedge}$ and ${}^{\tau}G(\mathbb{Q})^{+\wedge}$ (Deligne [2, 2.5.1]), $g \mapsto {}^{\tau}g$ maps the first group into the second. Clearly $g \mapsto {}^{\tau}g$ maps $Z(G)(\mathbb{Q})$ into $Z({}^{\tau}G)(\mathbb{Q})$ and so it maps $G(\mathbb{Q})^{\wedge}_{+} = G(\mathbb{Q})^{+\wedge} Z(\mathbb{Q})$ into ${}^{\tau}G(\mathbb{Q})^{\wedge}_{+} = {}^{\tau}G(\mathbb{Q})^{+\wedge}({}^{\tau}Z(\mathbb{Q}))$.

<u>Lemma 8.2.</u> Let (G,G',X^+) define a connected Shimura variety, let $h \in X$ be special, and let $\mu = \mu_h$. Then there exists a unique isomorphism $g \mapsto {}^\tau g : G(\mathbb{Q})^{+\wedge}(\text{rel } G') \longrightarrow {}^\tau G(\mathbb{Q})^{+\wedge}(\text{rel } {}^\tau G')$ with the following property: for any map $(G_1,X_1) \longrightarrow (G,X)$ such that $G_1^{ad} = G$ and G_1^{der} is a covering of G', the diagram

$$\begin{array}{ccc} g \mapsto {}^\tau g : G_1(\mathbb{Q})^\wedge_+ & \longrightarrow & {}^\tau G_1(\mathbb{Q})^\wedge_+ \\ \downarrow & & \downarrow \\ g \mapsto {}^\tau g : G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & {}^\tau G(\mathbb{Q})^{+\wedge}(\text{rel } G') \end{array}$$

commutes.

<u>Proof</u>. According to (3.4) we can choose a (G_1,X_1), as in the statement of the lemma, such that $Z(G_1)$ is a torus having trivial cohomology. Then $G_1(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ is surjective, and the equality

$$G(\mathbb{Q})^{+\wedge}(\text{rel } G') = G_1(\mathbb{Q})^\wedge_+ *_{G_1(\mathbb{Q})_+} G(\mathbb{Q})^+$$

(Deligne [2, 2.1.6.2]) shows that $G_1(\mathbb{Q})^\wedge_+ \longrightarrow G(\mathbb{Q})^{+\wedge}(\text{rel } G')$ is surjective. Thus we can define $g \mapsto {}^\tau g$ to be the map induced by its namesake on $G_1(\mathbb{Q})^\wedge_+$.

Let $(G_2,X_2) \longrightarrow (G,X)$ be a second map as in statement of the lemma and define G_3 to be the identity component of $G_2 \times_G G_1$. There is an X_3 for which there are maps $(G_3,X_3) \longrightarrow (G_1,X_1)$ and $(G_3,X_3) \longrightarrow (G_2,X_2)$. Since $\text{Ker}(G_3 \longrightarrow G_2) = \text{Ker}(G_1 \longrightarrow G)$,

$G_3(\mathbb{Q}) \longrightarrow G_2(\mathbb{Q})$ is surjective and the image of $G_3(\mathbb{Q})^{\wedge}_{+}$ is dense in $G_2(\mathbb{Q})^{\wedge}_{+}$. Clearly the maps $g \mapsto {}^{\tau}g$ for G_3, G_1, and G are compatible, as are the same maps for G_3 and G_2. This forces the maps $g \mapsto {}^{\tau}g$ for G_2 and G to be compatible.

When necessary, we shall denote the map defined in the lemma by $\gamma \mapsto {}^{\tau,\mu}\gamma$.

Recall that any $\gamma \in G(\mathbb{Q})^{+\wedge}$ (rel G') defines an automorphism $\gamma.$ of $Sh^{\circ}(G,G',X^{+})$ which, when $\gamma \in G(\mathbb{Q})^{+}$, is equal to the family of maps $\underset{\sim}{ad}\,\gamma : \Gamma\backslash X^{+} \to \gamma\Gamma\gamma^{-1}\backslash X^{+}$.

<u>Conjecture C°.</u> Let (G,G',X^{+}) define a connected Shimura variety and let τ be an automorphism of $\mathbb{C}$.

a) For any special $h \in X^{+}$, with $\mu = \mu_h$, there is an isomorphism

$$\phi^{\circ}_{\tau} = \phi^{\circ}_{\tau,\mu} : \tau Sh^{\circ}(G,G',X^{+}) \to Sh^{\circ}({}^{\tau}G,{}^{\tau}G',{}^{\tau}X^{+})$$

such that

$$\phi^{\circ}_{\tau}(\tau[h]) = [{}^{\tau}h]$$
$$\phi^{\circ}_{\tau} \circ \tau(\gamma.) = {}^{\tau}\gamma. \circ \phi^{\circ}_{\tau}, \quad \gamma \in G(\mathbb{Q})^{+\wedge} \text{ (rel } G')\,.$$

b) If $h' \in X^{+}$ is a second special element and $\mu' = \mu_{h'}$, then

$$\begin{array}{ccc} \tau Sh^\circ(G,G',X^+) & \xrightarrow{\phi^\circ_{\tau,\mu}} & Sh^\circ({}^{\tau,\mu}G, {}^{\tau,\mu}G', {}^{\tau,\mu}X^+) \\ & \searrow^{\phi^\circ_{\tau,\mu'}} & \downarrow \phi^\circ(\tau;\mu',\mu) \\ & & Sh^\circ({}^{\tau,\mu'}G, {}^{\tau,\mu'}G', {}^{\tau,\mu'}X^+) \end{array}$$

commutes.

(For $\phi^\circ(\tau;\mu',\mu)$, see §4.)

§ 9. Reduction of the proof of conjecture C to the case of the symplectic group.

Let (G,X) satisfy (1.1), let $\gamma \in G^{ad}(\mathbb{Q})$, and let $h \in X$ be special. If the image of γ in $G^{ad}(\mathbb{R})$ lifts to an element of $G(\mathbb{R})$, then $h' = \underset{\sim}{ad}\,\gamma \circ h$ is also a special point of X. Write $\mu = \mu_h$, $\mu' = \mu_{h'}$, and choose an $a(\tau) \in {}^{\tau}S^{L}(L)$ for some finite Galois extension of $\mathbb{Q}$. Then
$f_1 = (a(\tau).g \mapsto a(\tau).\, gqq^{-1})$ is $\mathbb{Q}$-rational isomorphism ${}^{\tau,\mu}G \to {}^{\tau,\mu'}G$ which is independent of the choice of $a(\tau)$ and maps ${}^{\tau,\mu}X$ into ${}^{\tau,\mu'}X$.

<u>Lemma 9.1.</u> With the above notations, the composite

$$Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) \xrightarrow{\tau\gamma.} Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) \xrightarrow{\phi(\tau;\mu',\mu)} Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X)$$

is equal to $Sh(f_1)$.

<u>Proof</u>. If γ lifts to an element of $G(\mathbb{Q})$, this is immediate from the definition of $\phi(\tau;\mu',\mu)$ (see 4.12d). Since we can always find a group with the same adjoint and derived groups as G, but with cohomologically trivial centre, this shows that the two maps agree on a connected component of $Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X)$. To complete the proof we only have to note that both maps transfer the action of $\mathcal{J}(g)$ on $Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X)$ into the action of $\mathcal{J}(f_1(g))$ on $Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X)$.

<u>Lemma 9.2</u>. Suppose conjecture C is true for (G,X) and let $h \in X$ be special with $\mu = \mu_h$. Then for any $\gamma \in G^{ad}(\mathbb{Q})^+$,

$$\phi_{\tau,\mu} \circ \tau(\gamma.) = {}^{\tau}\gamma. \circ \phi_{\tau,\mu} \quad .$$

<u>Proof</u>. Note that the image of γ in $G^{ad}(\mathbb{R})$, being in $G(\mathbb{R})^+$, lifts to $G(\mathbb{R})$. Let $h' = \underset{\sim}{ad}\, \gamma \circ h$ and $\mu' = \mu_{h'}$, and consider the diagram

$$\begin{array}{ccccc}
\tau\,Sh(G,X) & \xrightarrow{\phi_{\tau,\mu}} & Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) & & \\
\downarrow{\scriptstyle \tau(\gamma.)} & & \downarrow{\scriptstyle {}^{\tau}\gamma.} & & \\
\tau Sh(G,X) & \xrightarrow{\phi_{\tau,\mu}} & Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) & & \Big\downarrow{\scriptstyle Sh(f_1)} \\
 & \searrow{\scriptstyle \phi_{\tau,\mu'}} & \downarrow{\scriptstyle \phi(\tau;\mu',\mu)} & & \\
 & & Sh({}^{\tau,\mu'}G, {}^{\tau,\mu'}X) & &
\end{array}$$

Since we are assuming that the bottom triangle commutes, it suffices to show that the diagram commutes with the lower $\phi_{\tau,\mu}$ removed. But clearly

$$Sh(f_1) \circ \phi_{\tau,\mu}(\tau[h,1]) = [{}^{\tau}h',1] = \phi_{\tau,\mu'} \circ \tau(\gamma.) \quad (\tau[h,1]),$$

$$Sh(f_1) \circ \phi_{\tau,\mu} \circ \tau\mathcal{J}(g) = \mathcal{J}({}^{\tau,\mu'}ad\ \gamma(g)) \circ Sh(f_1) \circ \phi_{\tau,\mu},$$

$$\phi_{\tau,\mu'} \circ \tau(\gamma.) \circ \tau\mathcal{J}(g) = \mathcal{J}({}^{\tau,\mu'}ad\ \gamma(g)) \circ \phi_{\tau,\mu'} \circ \tau(\gamma.) \quad ,$$

which completes the proof.

Remark 9.3. If, in (9.2), γ lifts to $\delta \in G(\mathbb{Q})$, then the statement of the lemma becomes $\phi_{\tau,\mu} \circ \mathcal{J}(\delta^{-1}) = \mathcal{J}(^{\tau}\delta^{-1}) \circ \phi_{\tau,\mu}$, which is part of (a) of conjecture C.

Proposition 9.4. Let (G,X) satisfy (1.1) and let X^{+} be one connected component of X. Then conjecture C is true for $Sh(G,X)$ if and only if conjecture C° is true for $Sh^{\circ}(G^{ad},G^{der},X^{+})$.

Proof. Assume conjecture C and let $h \in X^{+}$ be special. Then $\phi_{\tau,\mu}$, with $\mu = \mu_h$, maps $\tau[h,1]$ to $[^{\tau}h,1]$ and therefore it maps $Sh^{\circ}(G,G',X^{+})$ into $Sh^{\circ}(^{\tau}G,{}^{\tau}G',{}^{\tau}X^{+})$. We can therefore define $\phi^{\circ}_{\tau,\mu}$ to be the restriction of $\phi_{\tau,\mu}$ to $Sh^{\circ}(G,G',X^{+})$. Part (a) of conjecture C° follows from part (a) of conjecture C and (9.2), while part (b) of conjecture C° follows from part (b) of conjecture C.

Next assume conjecture C° holds for $Sh^{\circ}(G^{ad},G^{der},X^{+})$. Suppose that, for special $h \in X^{+}$, we have extended $\phi^{\circ}_{\tau,\mu}$, $\mu = \mu_h$, to a map $\phi_{\tau,\mu} : \tau Sh(G,X) \to Sh(^{\tau}G,{}^{\tau}X)$ satisfying $\phi_{\tau,\mu} \circ \tau\mathcal{J}(g) = \mathcal{J}(^{\tau}g) \circ \phi_{\tau,\mu}$. Then $\phi_{\tau,\mu}(\tau[h,1]) = [^{\tau}h,1]$ and, for $\mu' = \mu_{h'}$ with $h' \in X^{+}$, $\phi_{\tau,\mu'} = \phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu}$, because the maps ϕ° have the corresponding properties. If h' is a special element of X, but $h' \notin X^{+}$, we write $h' = \underset{\sim}{ad}\, q \circ h$ with $h \in X^{+}$ and $q \in G(\mathbb{Q})$, and define $\phi_{\tau,\mu'}$ to be $\phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu}$. We have already noted in (4.14) that this map automatically satisfies part (a) of conjecture C. That the entire family, (ϕ_{τ,μ_h}), $h \in X$ special, satisfies part b of conjecture C follows easily from the definitions and from (4.12b).

It remains to see how to extend $\phi^{\circ}_{\tau,\mu}$. For this we use Deligne [2, 2.7.3]. Write τSh for $\tau Sh(G,X)$ and ${}^{\tau}Sh$ for $Sh({}^{\tau}G,{}^{\tau}X)$. Recall (Deligne [2, 2.1.16]) that $G(\mathbb{A}^f)$ acts transitively on $\pi_0(\tau Sh)$ $(= \tau\, \pi_0(Sh))$ and that the stabilizer of $\tau e \overset{df}{=} \tau Sh^{\circ}(G^{ad},G^{der},X^+)$ is $G(\mathbb{Q})^{\wedge}_{+}$. Similarly ${}^{\tau}G(\mathbb{A}^f)$ acts transitively on $\pi_0({}^{\tau}Sh)$ and the stabilizer of ${}^{\tau}e$ is ${}^{\tau}G(\mathbb{Q})^{\wedge}_{+}$. We have compatible isomorphisms $G(\mathbb{A}^f) \longrightarrow {}^{\tau}G(\mathbb{A}^f)$ and $\pi_0(\tau Sh) \longrightarrow \pi_0({}^{\tau}Sh)$ (see the proof of 8.1). Thus giving a morphism $\tau Sh \rightarrow {}^{\tau}Sh$ that is compatible with these two morphisms is equivalent to giving a morphism $\tau e \longrightarrow {}^{\tau}e$ that is equivariant for the actions of the stabilizers of τe and ${}^{\tau}e$. But $\phi^{\circ}_{\tau,\mu}$ is such a morphism.

<u>Lemma 9.5</u>. Suppose that (G,X) and (G',X') satisfy (1.1) and that there is a map $(G,X) \longrightarrow (G',X')$ with $G \rightarrow G'$ injective. If conjecture C is true for $Sh(G',X')$ then it is also true for $Sh(G,X)$.

<u>Proof</u>. According to Deligne [1, 1.15.1] the map $Sh(G,X) \longrightarrow Sh(G',X')$ is injective. A special point h of X maps to a special point h' of X', and the map $\phi_{\tau,\mu_{h'}}$ sends $\tau[h,1]$ to $[{}^{\tau}h,1] \in Sh({}^{\tau}G,{}^{\tau}X) \subset Sh({}^{\tau}G',{}^{\tau}X')$. It therefore sends $\tau[h,g]$ to $[{}^{\tau}h,{}^{\tau}g] \in Sh({}^{\tau}G,{}^{\tau}X)$ for any $g \in G(\mathbb{A}^f)$, which implies that it maps $\tau Sh(G,X)$ into $Sh({}^{\tau}G,{}^{\tau}X)$. We define ϕ_{τ,μ_h} to be the restriction of $\phi_{\tau,\mu_{h'}}$ to $\tau Sh(G,X)$.

<u>Lemma 9.6</u>. If conjecture C° is true for $Sh^\circ(G,G',X^+)$, and G'' is a quotient of G', then conjecture C° is true for $Sh^\circ(G,G'',X^+)$.

<u>Proof</u>. This follows immediately from the general fact that $Sh^\circ(G,G'',X^+)$ is the quotient of $Sh^\circ(G,G',X^+)$ by the kernel of the surjective map

$$G(\mathbb{Q})^{+\wedge} \text{ (rel } G') \longrightarrow G(\mathbb{Q})^{+\wedge} \text{ (rel } G'') .$$

<u>Lemma 9.7</u>. If conjecture C° is true for $Sh^\circ(G_i,G_i',X_i^+)$, $i = 1,\ldots,n$, then the conjecture is true for $Sh^\circ(\Pi G_i,\Pi G_i',\Pi X_i^+)$.

<u>Proof</u>. Easy.

<u>Theorem 9.8</u>. If conjecture C is true for all varieties of the form $Sh(CSp(V), S^{\pm})$ then it is true for all Shimura varieties of abelian type.

<u>Proof</u>. If conjecture C is true for varieties of the form $Sh(CSp(V), S^{\pm})$ then (1.4), (9.5), and (9.4) show that conjecture C° is true for all connected Shimura varieties of primitive abelian type. Then (9.6) and (9.7) show that conjecture C° is true for all connected Shimura varieties of abelian type. Finally (9.4) then implies that conjecture C is true for all Shimura varieties of abelian type.

<u>Corollary 9.9</u>. Conjecture CM implies that conjecture C is true for all Shimura varieties of abelian type.

<u>Proof</u>. Combine (7.17) with (9.8).

§ 10. Application of the motivic Galois group.

Let M be an extension of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by the Serre group S,

$$1 \longrightarrow S \longrightarrow M \xrightarrow{\tilde{\pi}} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \tag{10.1}$$

together with a splitting $\widetilde{sp}$ over $\mathbb{A}^f$, in the sense defined in (III.2). This means, in particular, that the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on S by inner automorphisms in M is the algebraic action described in (III 1.8), and that (10.1) is the projective limit of a system of extensions

$$1 \longrightarrow S^L \longrightarrow M^L \longrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow 1 \tag{10.2}$$

over fields L finite and Galois over $\mathbb{Q}$ where $\mathbb{Q} \subset L \subset L^{ab} \subset \bar{\mathbb{Q}} \subset \mathbb{C}$. We assume in addition that the right S-torsor ${}^{\tau}\tilde{S} \overset{df}{=} \tilde{\pi}^{-1}(\tau)$ is isomorphic to the S-torsor ${}^{\tau}S$ arising from the Taniyama group (see §4). Since the existence of $\widetilde{sp}(\tau)$ implies that ${}^{\tau}\tilde{S}_{\mathbb{Q}_\ell}$ is trivial for all ℓ, (III 1.5) shows that the assumption holds if ${}^{\tau}S_{\mathbb{R}} \simeq {}^{\tau}\tilde{S}_{\mathbb{R}}$ as $S_{\mathbb{R}}$-torsors.

Let A be an abelian variety over $\mathbb{Q}$ of potential CM-type and let $V = H_1(A(\mathbb{C}),\mathbb{Q})$. We identify the Mumford-Tate group $MT(A)$ of A with the algebraic group $\mathrm{Aut}(V,(s_\alpha))$, where (s_α) is the family of all Hodge cycles on A (and its powers etc). There is a canonical map $\rho : S \to MT(A)$ (and $S = \varprojlim MT(A)$). As in (6.1) we let P_A be the $MT(A)$-torsor such that, for any $\mathbb{Q}$-algebra R,

$$P_A(R) = \{a : H_1(A,R) \xrightarrow{\approx} H_1(\tau A,R) \mid a(s_\alpha) = \tau s_\alpha , \text{ all } \alpha\} .$$

The proof of (6.1) shows that there exists an S-equivariant morphism ${}^\tau S \longrightarrow P_A$. Note that, as $H_1(A,\mathbb{A}^f) = V^f(A)$ and $H_1(\tau A,\mathbb{A}^f) = V^f(\tau A)$, $P_A(\mathbb{A}^f)$ contains a canonical element, namely τ , the map induced by letting τ act on the points of finite order of A . As in § 6, we let $({}^\tau V,({}^\tau s_\alpha)) = {}^\tau S \times {}^S(V,(s_\alpha))$.

<u>Lemma 10.3</u>. The following are equivalent:

(a) there exists an S-equivariant morphism $p : {}^\tau\tilde{S} \to P_A$ such that $p(\widetilde{sp}(\tau)) = \tau$;

(b) there exists an isomorphism $f : (H_1(\tau A,\mathbb{Q}), (\tau s_\alpha)) \to ({}^\tau V, ({}^\tau s_\alpha))$ such that

$$\begin{array}{ccc} V^f(A) & \xrightarrow{\ \tau\ } & V^f(\tau A) \\ \| & & \downarrow f\otimes 1 \\ V(\mathbb{A}^f) & \xrightarrow{\ \widetilde{sp}(\tau)\ } & {}^\tau V(\mathbb{A}^f) \end{array}$$

is commutative.

<u>Proof</u>. First note that if p_o is one S-equivariant morphism ${}^\tau S \to P_A$ then the other such morphisms are of the form $m \circ p_o$, $m \in MT(A)(\mathbb{Q})$, and that if f_o is one isomorphism $(H_1(\tau A,\mathbb{Q}), \tau s_\alpha) \to ({}^\tau V, ({}^\tau s_\alpha))$ then the others are of the form $f_o \circ m$, $m \in MT(A)(\mathbb{Q})$. Choose a p_o and define f_o to be the inverse of

$$s.v \longmapsto p_o(s)(V) \ : \ {}^\tau V \longrightarrow H_1(\tau A,\mathbb{Q}) \ .$$

Then $(f_o\otimes 1)^{-1}\circ\widetilde{sp}(\tau)$ is the map $v \longmapsto p_o(\widetilde{sp}(\tau))(v)$. The equivalence of (a) and (b) now obvious.

Note that (b) of the lemma says that, if in the statement of conjecture CM, T is replaced by M, then the conjecture becomes true for A.

By the motivic Galois group we shall mean the group associated with the Tannakian category of (absolute Hodge) motives generated by the abelian varieties over $\mathbb{Q}$ of potential CM-type; ie. the group called the Serre group in (II.6). It is an extension of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ by S in the sense of the first paragraph of this section (see II.6, and IV, especially B).

Proposition 10.4. (a) If, in the statement of conjecture CM, T is replaced by the motivic Galois group, then the conjecture becomes true for all abelian varieties over $\mathbb{Q}$ of potential CM-type.

(b) Let M be an extension of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ by S as in the first paragraph of this section. If conjecture CM becomes true for all abelian varieties over $\mathbb{Q}$ of potential CM-type when T is replaced by M, then M is isomorphic to the motivic Galois group (as extensions of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ by S with splittings over $\mathbb{A}^f$).

Proof. (a) It is shown in (II.6) that the motivic Galois group satisfies (a) of (10.3).

(b) Fix a finite Galois extension L of $\mathbb{Q}$ such that $L\subset\bar{\mathbb{Q}}$. For each abelian variety A over $\mathbb{Q}$ of potential CM-type whose Mumford-Tate group is split by L, (10.3) gives us an

S-equivariant morphism $p_A : {}^{\tau}\tilde{S}^L \longrightarrow P_A$ such that $p_A(\widetilde{sp}(\tau)) = \tau$ (here ${}^{\tau}\tilde{S}^L$ is the inverse image of τ in (10.2)). On passing to the inverse limit over A, we obtain an isomorphism $p : {}^{\tau}\tilde{S}^L \longrightarrow {}^{\tau}\tilde{\tilde{S}}^L$ such that $p(\widetilde{sp}(\tau)) = \widetilde{\widetilde{sp}}(\tau)$, where ${}^{\tau}\tilde{\tilde{S}}$ and $\widetilde{\widetilde{sp}}(\tau)$ refer to all motivic Galois group. Choose sections $\tilde{a}$ and $\tilde{\tilde{a}}$ to π for M and the motivic Galois group, and define $\tilde{\beta}(\tau)$ and $\tilde{\tilde{\beta}}(\tau)$ in $S^L(\mathbb{A}_L^f)$ by the formulas (see III.2.9)

$$\widetilde{sp}(\tau)\,\tilde{\beta}(\tau) = \tilde{a}(\tau)$$

$$\widetilde{\widetilde{sp}}(\tau)\,\tilde{\tilde{\beta}}(\tau) = \tilde{\tilde{a}}(\tau)$$

Since $p(\tilde{a}(\tau)) = \tilde{\tilde{a}}(\tau)a$, $a \in S^L(L)$, it follows that

$$\tilde{\beta}(\tau) \equiv \tilde{\tilde{\beta}}(\tau) \mod S^L(L) .$$

This implies (b) (by (III.2.7, 2.9)).

For the remainder of this article, M will denote the motivic Galois group. We retain the notations of the first paragraph of this section.

Much of sections §4 - §9 of this paper remains valid when the Taniyama group T is replaced by the motivic Galois group M. In particular, there are isomorphisms

$$\tilde{\phi}(\tau;\mu',\mu'') : Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) \longrightarrow Sh({}^{\tau,\mu'}G, {}^{\tau,\mu'}X)$$

as in (4.12) except now defined relative to M. The isomorphisms

$$\tilde{\phi}(\tau;\mu) \;:\; Sh(G,X) \longrightarrow Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$$

of (7.8) are not defined in the same generality because, in their definition, we have used that $b(\tau,\mu)$ is defined whenever μ satisfies (III.3.3) (rather than III.1.1). The alternative definition (see 7.6, 7.10) is, however, valid and provides a map $\tilde{\phi}(\tau;\mu)$ when (G,X) satisfies (0.1) and (0.2).

Theorem 10.5. If, in the statement of conjecture C, the Taniyama group is replaced with the motivic Galois group, then the conjecture becomes true for all Shimura varieties of abelian type.

Proof. As in (7.17) one proves that conjecture CM implies that conjecture C is true for Shimura varieties of the form $Sh(CSp(V),S^{\pm})$, and as in (9.8) that this implies that conjecture C is true for all Shimura varieties of abelian type.

Corollary 10.6. Conjecture A is true for all Shimura varieties of abelian type.

Proof. We remark that, because the S-torsors $\pi^{-1}(\tau)$ and $\tilde{\pi}^{-1}(\tau)$ defined by T and M are isomorphic, so also are the pairs $({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ defined by the two groups. (The maps $g \longmapsto {}^{\tau,\mu}g \;:\; G(\mathbb{A}^f) \longrightarrow {}^{\tau,\mu}G(\mathbb{A}^f)$ could however, differ.) The discussion preceding the statement of conjecture A in § 4 therefore shows that the corollary follows from (10.5).

Corollary 10.7. If (G,X) is of abelian type and satisfies (0.1) and (0.2), then conjecture B is true for $Sh(G,X)$.

Proof. The torsor ${}^{\iota}\tilde{S}$ is trivial because $\iota : A(\mathbb{C}) \longrightarrow \iota A(\mathbb{C})$ is a homeomorphism and therefore induces a map $H_1(A,\mathbb{Q}) \rightarrow H_1(\iota A,\mathbb{Q})$ which can be shown to map the Hodge cycle s_α to ιs_α. Thus in (0.5) (whose relevant part is implied by (7.14)) we can take $a(\iota) \in {}^{\iota}S(\mathbb{Q})$, $c(\iota) = \rho_\mu(a(\iota))$, $v = 1$, and $\alpha = \rho_\mu(\tilde{\beta}(\iota)^{-1})$ where $\tilde{\beta}(\iota)$ is defined by $\tilde{sp}(\tau)\,\tilde{\beta}(\iota) = a(\iota)$. Hence (10.5) implies $\iota[h,1] = [{}^{\iota}h, \rho_\mu(\tilde{\beta}(\iota)^{-1})]$. But conjecture CM in its original form is true when $\tau = \iota$ (see 7.19) and this implies $\tilde{\beta}(\iota) \equiv \beta(\iota) \bmod S^L(L)$. Since $\beta(\iota) \equiv 1 \bmod S^L(L)$ by (III.3.9), we have $\iota[h,1] = [{}^{\iota}h,1]$.

Remark 10.8. In (Milne-Shih [1]) conjecture B is proved for all Shimura varieties of abelian type. There is a good reason why it is easy to prove conjecture B under the assumption of (0.1) and (0.2): these conditions should imply $Sh(G,X)$ is a moduli variety for motives.

Remark 10.9. Theorem 10.5 together with the proof of (7.14) show that $Sh(G,X)$ has a canonical model whenever (G,X) is of abelian type and satisfies (0.1) and (0.2). Presumably if the maps $\tilde{\phi}(\tau,\mu)$ were defined (using M) for all Shimura varieties of abelian type, then one would recover the main theorem of Deligne [2], but there seems little point in this (except that it would give a proof not involving $E_E(G,G', X^+)$).

Deligne has conjectured the following:

Conjecture D. The Taniyama group and the motivic Galois group are isomorphic (as extensions of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by S together with a splitting over $\mathbb{A}^f$) .

See IV, where the two groups are shown to be isomorphic as extensions of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by S . (It therefore remains to show that the isomorphism can be chosen to carry sp into $\widetilde{sp}$.)

Deligne also suggested that his conjecture D should be equivalent to Langlands's conjecture C. We prove:

Proposition 10.10. Conjecture D is true if and only if conjecture C is true for all Shimura varieties of abelian type (equivalently, for all Shimura varieties of the form $\mathrm{Sh}(\mathrm{CSp}(V),S^{\pm}))$.

Proof. If conjecture D is true, then (10.5) shows that conjecture C is true for Shimura varieties of abelian type. Conversely, if conjecture C is true for varieties of the form $\mathrm{Sh}(\mathrm{CSp}(V),S^{\pm})$ then (7.17) shows that conjecture CM is true, and (10.4b) that conjecture D is true.

Remark 10.11. Let $L \subset \overline{\mathbb{Q}}$ be a finite Galois extension of $\mathbb{Q}$ and let K be subfield of L . We write ${}_K T^L$ and ${}_K M^L$ for the pull-backs of T^L and M^L relative $\mathrm{Gal}(L^{ab}/K) \hookrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q})$. Assume that L is a CM-field. If A is an abelian variety of CM-type whose Mumford-Tate group is split by L , then the reflex field of A is contained in L , and the main theorem of complex multiplication shows that conjecture CM is true for all such A and all τ fixing L (cf. 6.2). Thus an obvious variant of (10.4b) shows that ${}_LT^L \simeq {}_LM^L$ as extensions of $\mathrm{Gal}(L^{ab}/L)$ by S^L with splittings over $\mathbb{A}^f$. Since conjecture CM is known to be true for $\tau = \iota$, it is therefore also true for any τ fixing the maximal totally real subfield K of L; thus ${}_KT^L \simeq {}_KM^L$ (as extensions ...). The results of Shih[1] (see also Milne-Shih [1]) often allow one to replace K in this isomorphism by a subfield of L over which L has degree 4.

For example, let F_o be a totally real field of finite degree over $\mathbb{Q}$ and let F_1 and F_2 be distinct quadratic totally imaginary extensions of F_o. Let $F_3 = F_1 \otimes_{F_o} F_2$ and choose a subset Σ_o of $I_o \overset{df}{=} \mathrm{Hom}(F_o, \mathbb{R})$. For each $\sigma \in I_o$ choose extensions σ_1 and σ_2 of σ_o to F_1 and F_2. Let

$$I_1 = \mathrm{Hom}(F_1, \mathbb{C}) \;,\; \Sigma_1 = \{\sigma_1 | \sigma \in \Sigma_o\}$$

$$I_2 = \mathrm{Hom}(F_2, \mathbb{C}) \;,\; \Sigma_2 = \{\sigma_2 | \sigma \notin \Sigma_o\}$$

$$I_3 = \mathrm{Hom}(F_3, \mathbb{C}) \;,\; I_1 \times_{I_o} I_2$$

$$\Sigma_3 = \{(\sigma_1, \sigma_2) | \sigma \in I_o\} \cup \{(\iota\sigma_1, \sigma_2) | \sigma \in \Sigma_o\} \cup \{(\sigma_1, \iota\sigma_2) | \sigma \notin \Sigma_o\} .$$

Define E_i to be the subfield of $\mathbb{C}$ of elements fixed by $\{\tau \in \mathrm{Aut}(\mathbb{C}) | \tau\, \Sigma_i \subset \Sigma_i\}$, $i = 0, 1, 2, 3$. Then E_o is totally real and $E_3 = E_1E_2$ is a CM-field. In general, $[E_3 : E_o] = 4$. When E_3 is Galois over $\mathbb{Q}$, ${}_{E_o}T^{E_3} \simeq {}_{E_o}M^{E_3}$ (as extensions ...).

Remark 10.12 Our original approach to the results of this section was a little more elementary. We showed directly that there exists a compatible family of maps

$$e^{-L} : \mathrm{Gal}(L/\mathbb{Q}) \longrightarrow S^L(\mathbb{A}^f)/S^L(\mathbb{Q})$$

such that if M' is the extension and splitting defined by

$$\tau \longmapsto \bar{b}(\tau)\bar{e}(\tau) : \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow S^L(\mathbb{A}_L^f)/S^L(L)$$

with $\bar{b}$ as in (III.3.11) (cf. III.2.7) then conjecture CM holds for M'. Thus, in all of the above, the motivic Galois group can be replaced by M'. Of course (10.4b) shows that M' is isomorphic to the motivic Galois group (as extensions ...).

REFERENCES

Deligne, P.
1. Travaux de Shimura, Sém. Bourbaki Février 71, Exposé 389, Lecture Notes in Math., 244, Springer, Berlin, 1971.

2. Variétés de Shimura: interpretation modulaire, et techniques de construction de modéles canoniques. Proc. Symp. Pure Math., A.M.S. 33 (1979) part 2, 247-290.

3. Valeurs de fonctions L et périodes d'intégrales. Proc. Symp. Pure Math., A.M.S., 33 (1979) part 2, 313-346.

Doi, K. and Naganuma, H.
1. On the algebraic curves uniformized by arithmetical automorphic functions. Ann. Math. 86 (1967) 449-460.

Kazhdan, D.
1. On arithmetic varieties. Lie Groups and their representations, Halsted, New York, 1975. 158-217.

Langlands, R.
1. Some contemporary problems with origins in the Jugendtraum. Proc. Symp. Pure Math., A.M.S. 28(1976) 401-418.

2. Conjugation of Shimura varieties (preliminary version of [3]).

3. Automorphic representations, Shimura varieties, and motives. Ein Märchen. Proc. Symp. Pure Math., A.M.S., 33 (1979) part 2, 205-246.

Milne, J. and Shih, K.-y.
1. The action of complex conjugation on a Shimura variety, Annals of Math, 113 (1981) 569-599.

Serre, J.-P.
1. Cohomologie Galoisienne, Lecture Notes in Math. 5, Springer, Berlin, 1964.

Shih, K.-y.
1. Anti-holomorphic automorphisms of arithmetic automorphic function fields, Ann. of Math. 103 (1976) 81-102.

2. Conjugations of arithmetic automorphic function fields, Invent. Math. 44 (1978) 87-102.

Shimura, G.
1. Introduction to the arithmetic theory of automorphic functions. Princeton Univ. Press 1971.