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AUTOMORPHISM GROUPS OF SHIMURA VARIETIES AND RECIPROCITY LAWS

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This paper has two purposes: the first is to extend Shimura's results, as described in his talk [2], to more general arithmetic quotients of bounded symmetric domains and the second is to interpret some of the results of Milne-Shih [3] as non-abelian reciprocity laws at the special points of the symmetric domain. The latter can be regarded as a non-abelian solution to Hilbert's twelfth problem.

Before describing our results in more detail, we review the classical case. A modular function (on the upper half-plane) of level N is said to be arithmetic if its Fourier expansion with respect to $e^{2\pi iz/N}$ has coefficients in $\mathbf{Q}(e^{2\pi i/N})$. The arithmetic modular functions form a field F' on which $GL_2(\mathbf{A})^+$ acts and there is an exact sequence of topological groups

$$1 \to \mathbf{Q}^\times \cdot GL_2(\mathbf{R})^+ \to GL_2(\mathbf{A})^+ \xrightarrow{\tau} \mathrm{Aut}(F') \to 1$$

(Shimura [3, 6.23]). The sequence generalizes the exact sequence

$$1 \to \mathbf{Q}^\times \cdot \mathbf{R}^{\times +} \to \mathbf{A_Q}^\times \to \mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q}) \to 1$$

of class field theory. We have $\mathbf{Q}^{\mathrm{ab}} \subset F'$ and for $u \in GL_2(\mathbf{A})^+$, the restriction of $\tau(u)$ to $\mathbf{Q}^{\mathrm{ab}}$ is given by $\det(u)^{-1} \in \mathbf{A_Q}^\times$ via the Artin reciprocity map. Moreover, let z be a special point of the upper half-plane, so that $\mathbf{Q}(z)$ is a quadratic imaginary extension of $\mathbf{Q}$. Let $\mathbf{Q}(z)^\times \hookrightarrow GL_2$ be the normalized embedding (ibid, p. 104). The reciprocity law at z asserts the following: if $f \in F'$ is defined at z then $f(z) \in \mathbf{Q}(z)^{\mathrm{ab}}$; for any $\nu \in \mathbf{A}^\times_{\mathbf{Q}(z)} \subset GL_2(\mathbf{A})^+$, ${}^{\tau(\nu)^{-1}}f(z) = [\nu]f(z)$, where $[\nu]$ is the image of ν in $\mathrm{Gal}(\mathbf{Q}(z)^{\mathrm{ab}}/\mathbf{Q}(z))$.

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More generally, one starts with a bounded symmetric domain X^+ and a reductive group G such that $G^{\mathrm{ad}}(\mathbf{R})^+$ is the identity component of the group of holomorphic automorphisms of X^+. The field F' consists of meromorphic functions on X^+ that are automorphic relative to a congruence subgroup of $G^{\mathrm{der}}(\mathbf{Q})$ and are arithmetic. In defining this last notion it is not always possible to use Fourier expansions. Instead one must first construct a family of varieties defined over number fields (a canonical model) and require an arithmetic automorphic function to be defined on one of the varieties in the family. Then, under certain assumptions, there is an exact sequence,

$$1 \to Z(\mathbf{Q})^\wedge G(\mathbf{R})_+ \to \mathcal{G}(G, X) \xrightarrow{\tau} \mathrm{Aut}(F'/E)$$

in which $\mathcal{G}(G, X)$ is a certain subgroup of $G(\mathbf{A})$, Z is the centre of G, $Z(\mathbf{Q})^\wedge$ is the closure of $Z(\mathbf{Q})$ in $Z(\mathbf{A})$, and E is a certain field associated with G and X (called the reflex field). The image of τ in $\mathrm{Aut}(F'/E)$ is an open subgroup of finite index. It is frequently possible to enlarge $\mathcal{G}(G, X)$ to make τ surjective. Associated with a special point z of X^+ there is a finite extension field $E(z)$ of E and a map $\eta: E(z)^\times \to G$ of $\mathbf{Q}$-rational algebraic groups such that $\eta(\mathbf{A}): \mathbf{A}_{E(z)}^\times \to G(\mathbf{A})$ factors through $\mathcal{G}(G, X)$. The reciprocity law at z asserts the following: if $f \in F'$ is defined at z then $f(z) \in E(z)^{\mathrm{ab}}$; if $\nu \in \mathbf{A}_{E(h)}^\times$ and $u = \eta(\nu)^{-1}$, then ${}^{\tau(u)}f(z) = [\nu] f(z)$, where $[\nu]$ is the image of ν in $\mathrm{Gal}(E(z)^{\mathrm{ab}}/E(z))$.

The above results were proved in Shimura [1], K. Miyake [1], and Shih [1] for a group whose derived group is simply connected and $\mathbf{Q}$-simple of type C, A, and B respectively. We prove them whenever G is classical and the canonical model is known to exist. (In this generality it is more natural to replace the exact sequence by an inclusion $\overline{\mathcal{E}}(G, X) \hookrightarrow \mathrm{Aut}(F'/E)$.) The essential step, which is carried out in Section 1, is to compute the automorphism group of a connected Shimura variety. An argument of T. Miyake allows us in Section 2 to show that the automorphism group of the variety is equal to the automorphism group of its function field. The above results, concerning Shimura's canonical models, can then be deduced without difficulty from Deligne's results [2] concerning his canonical models. This is done in Sections 3 and 4.

The non-abelian reciprocity laws proved in Section 5 take the following form. Recall that the Weil group of $E(z)$ over E is an extension

$$1 \to E(z)^\times \backslash \mathbf{A}_{E(z)}^\times \to W_{E(z)/E} \xrightarrow{\sigma} \mathrm{Hom}_E(E(z), \overline{\mathbf{Q}}) \to 1,$$

and that the map $E(z)^{\times}\backslash\mathbf{A}_{E(z)}^{\times} \to \mathrm{Gal}(E(z)^{ab}/E(z))$ of class field theory extends to a map $\nu \mapsto [\nu]: W_{E(z)/E} \to \mathrm{Hom}_E(E(z)^{\mathrm{ab}}, \overline{\mathbf{Q}})$. For any special $z \in X$ the abelian reciprocity law can be interpreted as stating the following: let $\nu \in E(z)^{\times}\backslash\mathbf{A}_{E(z)}^{\times}$ and let $\eta(\nu)$ be its image in $G(\mathbf{Q})\backslash G(\mathbf{A})$ under the map defined by η; then for any lifting $u = (u_\infty, u_f)$ of $\eta(\nu)$ to $G(\mathbf{A})$, ${}^{\tau(u_f^{-1})}f(z) = [\nu]f(\mathbf{ad}\, u_\infty \circ z)$. This makes sense because ${}^{\tau(a^{-1})}f(z) = f(\mathbf{ad}\, a \circ z)$ for $a \in G(\mathbf{Q})$. The non-abelian reciprocity law states the following: let C be the isotropy subgroup of $G_{\mathbf{R}}$ at z; the map $E(z)^{\times}\backslash\mathbf{A}_{E(z)}^{\times} \to G(\mathbf{Q})\backslash G(\mathbf{A})/C(\mathbf{R})$ defined by η extends to a map $\xi: W_{E(z)/E} \to G(\mathbf{Q})\backslash G(\mathbf{A})/C(\mathbf{R})$; for any $\nu \in W_{E(z)/E}$ and lifting $u = (u_\infty, u_f)$ of $\xi(\nu)$ to $G(\mathbf{A})$, ${}^{\tau(u_f^{-1})}f(z) = [\nu]f(\mathbf{ad}\, u_\infty \circ z)$.

We remark that there are now three notions of canonical model: Shimura's, and Deligne's connected and non-connected models. Only Shimura's model provides one with a family of geometrically irreducible varieties defined over number fields, but in proving theorems it seems to be easier to work first with Deligne's models.

Notations and conventions. For Shimura varieties and algebraic groups we generally follow the notations of Deligne [2]. Thus a reductive algebraic group G is always connected, with derived group G^{der}, adjoint group G^{ad}, and centre $Z = Z(G)$. A central extension is an epimorphism $G \to G'$ whose kernel is contained in $Z(G)$, and a covering is a central extension such that G is connected and the kernel is finite.

A superscript $+$ refers to a topological connected component; for example $G(\mathbf{R})^+$ is the identity connected component of $G(\mathbf{R})$ relative to the real topology, and $G(\mathbf{Q})^+ = G(\mathbf{Q}) \cap G(\mathbf{R})^+$. For G reductive, $G(\mathbf{R})_+$ is the inverse image of $G^{\mathrm{ad}}(\mathbf{R})^+$ in $G(\mathbf{R})$ and $G(\mathbf{Q})_+ = G(\mathbf{Q}) \cap G(\mathbf{R})_+$. In contrast to Deligne [2], we use the superscript $\wedge$ to denote both completions and closures.

We write $\mathrm{Sh}(G, X)$ for the Shimura variety defined by a pair (G, X) and $\mathrm{Sh}^{\circ}(G, G', X^+)$ for the connected Shimura variety defined by a triple (G, G', X^+). The canonical model of $\mathrm{Sh}(G, X)$ is denoted by $M(G, X)$.

Vector spaces are finite-dimensional, number fields are of finite degree over $\mathbf{Q}$ (and usually contained in $\mathbf{C}$), and $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. If V is a vector space over $\mathbf{Q}$ and R is a $\mathbf{Q}$-algebra, we often write $V(R)$ for $V \otimes R$.

We write $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/m\mathbf{Z}$, $\mathbf{A}^f = \mathbf{Q} \otimes \hat{\mathbf{Z}}$ for the ring of finite adèles of $\mathbf{Q}$, and $\mathbf{A} = \mathbf{R} \times \mathbf{A}^f$ for the ring of adèles of $\mathbf{Q}$. For E a number field, $\mathbf{A}_E{}^f$

and $\mathbf{A}_E$ denote $E \otimes_{\mathbf{Q}} \mathbf{A}^f$ and $E \otimes \mathbf{A}$. The group of idèles of E is $\mathbf{A}_E^{\times}$ and the idèle class group is $C_E = \mathbf{A}_E^{\times}/E^{\times}$.

We use $[*]$ to denote an equivalence class containing $*$; for example, if $x \in X$ and $g \in G(\mathbf{A}^f)$ then $[x, g]$ denotes the element of $\mathrm{Sh}(G, X) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}^f)/Z(\mathbf{Q})^{\wedge}$ containing (x, g).

All group actions are from the left throughout the paper.

We normalize the reciprocity isomorphism of class field theory so that a uniformizing parameter corresponds to the reciprocal of the (arithmetic) Frobenius element; we thus agree with Deligne [2] and Tate [2], but disagree with Langlands [1].

1. The automorphism group of a connected Shimura variety. Let G be a semi-simple $\mathbf{Q}$-rational adjoint group, G' a covering of G and X^+ a family of maps $\mathbf{C}^{\times} \to G(\mathbf{R})$. The topology on $G(\mathbf{Q})$ defined by G' is that for which the images of the congruence subgroups of $G'(\mathbf{Q})$ form a fundamental system of neighborhoods. Let $\Sigma = \Sigma(G')$ be the set of arithmetic subgroups Γ of $G(\mathbf{Q})^+$ that are open relative to this topology and are torsion-free. Under certain hypotheses (Deligne [2, 2.1.8]) (G, G', X^+) defines a connected Shimura variety $\mathrm{Sh}^{\circ}(G, G', X^+)$ equal to the projective limit of $\Gamma\backslash X^+$, $\Gamma \in \Sigma$. An automorphism of $\mathrm{Sh}^{\circ} = \mathrm{Sh}^{\circ}(G, G', X^+)$ is a morphism $\alpha: \mathrm{Sh}^{\circ} \to \mathrm{Sh}^{\circ}$ such that for any $\Gamma \in \Sigma$ there exist a $\Gamma' \in \Sigma$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}^{\circ} & \xrightarrow{\alpha} & \mathrm{Sh}^{\circ} \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \Gamma'\backslash X^+ & \xrightarrow{\alpha_\Gamma} & \Gamma\backslash X^+ \end{array}$$

with α_Γ an isomorphism. (The morphisms are required to be analytic; according to Borel [3, 3.10], they will then be algebraic.) Denote the group of automorphisms of Sh° by $\mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^{\circ})$. For a $\Gamma \in \Sigma$, the $\alpha \in \mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^{\circ})$ such that

$$(\mathrm{Sh}^{\circ} \xrightarrow{\alpha} \mathrm{Sh}^{\circ} \xrightarrow{\mathrm{can}} \Gamma\backslash X^+) = (\mathrm{Sh}^{\circ} \xrightarrow{\mathrm{can}} \Gamma\backslash X^+)$$

form a subgroup of $\mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^{\circ})$. The group $\mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^{\circ})$ is given the topology for which these subgroups form a fundamental system of neighborhoods of the identity element. In this section we give an explicit description of this topological group, assuming G is classical in the sense of Kneser [1, 2.3], i.e. it is not exceptional and neither ${}^3D_4$ nor ${}^6D_4$ occurs as a component of its universal covering group.

1.1. According to Weil [1] (see also Kneser [1, Chapter 2]), we can identify the classical group G with the identity component of the automorphism group of a semi-simple algebra with involution over $\mathbf{Q}$. The algebra with involution is a direct product of (L, σ) of the following types:

(A) L is a central simple algebra over a quadratic totally imaginary extension of a totally real field F_0, and σ is an involution of the second kind.

(B), ($\mathrm{D}^{\mathbf{R}}$) $L = \mathrm{End}_{F_0}(V)$, where V is a finite dimensional vector space over a totally real field F_0, and σ is defined by $q(\alpha x, y) = q(x, \alpha^\sigma y)$, where q is a non-degenerate quadratic form on V.

(C) $L = \mathrm{End}_B(\Lambda)$, where B is a quaternion algebra over a totally real field F_0, and Λ is a free left B-module of finite rank; σ is defined by $\varphi(\alpha x, y) = \varphi(x, \alpha^\sigma y)$, where φ is a B-valued form on Λ that is hermitian with respect to the main involution of B.

($\mathrm{D}^{\mathbf{H}}$) L is the same as in (C), but σ is defined by an anti-hermitian form φ.

We impose the following conditions: for type (A), $[L:F_0] \geq 2\cdot 3^2$; for types (B) and ($\mathrm{D}^{\mathbf{R}}$), $\dim_{F_0} V \geq 7$; and for type ($\mathrm{D}^{\mathbf{H}}$), $\mathrm{rank}_B \Lambda \geq 4$. Then for the given semi-simple adjoint group G, there is a unique (L, σ) such that the direct factors of (L, σ) satisfy the above conditions, and such that the identity component of the automorphism group of (L, σ) is G. We refer to (L, σ) as the algebra with involution corresponding to G.

1.2. Let (G, G', X^+) define a connected Shimura variety, with G classical. Let (L, σ) be the algebra with involution corresponding to G, and A the automorphism group of (L, σ). Since G is the identity component of A, inner automorphisms of A define an injection $\mathbf{ad}: A \to \mathrm{Aut}(G)$. In fact $\mathbf{ad}$ is an isomorphism. This follows from the fact that if G_1, G_2 are classical $\mathbf{Q}$-simple groups and (L_1, σ_1), (L_2, σ_2) are the corresponding algebras with involution, then any isomorphism of G_1 to G_2 is induced by a unique isomorphism of (L_1, σ_1) to (L_2, σ_2). Let $A(\mathbf{Q})^*$ be the group consisting of $\gamma \in A(\mathbf{Q})$ such that $\mathbf{ad}\gamma: G(\mathbf{Q}) \to G(\mathbf{Q})$ is a homeomorphism with respect to the topology defined by G'. If G is $\mathbf{Q}$-simple, then any $\mathbf{ad}\gamma$, $\gamma \in A(\mathbf{Q})$, lifts to an automorphism of $G'(\mathbf{Q})$. Therefore $A(\mathbf{Q})^* = A(\mathbf{Q})$ in this case.

Decompose $L \otimes_{\mathbf{Q}} \mathbf{R}$ into the direct product of simple algebras $L_1, \ldots, L_g$ over $\mathbf{R}$. Then σ induces an involution σ_ν on each factor L_ν. Let A_ν be the group of automorphisms of (L_ν, σ_ν). Denote by S the group of permutations τ of $\{1, \ldots, g\}$ such that (L_ν, σ_ν) is isomorphic to $(L_{\tau(\nu)}, \sigma_{\tau(\nu)})$ for all ν. Then we have an exact sequence

$$1 \to \Pi A_\nu(\mathbf{R}) \to A(\mathbf{R}) \to S \to 1.$$

Let G_ν be the identity component of the algebraic group A_ν over $\mathbf{R}$. Then $G_{\mathbf{R}} = \Pi G_\nu$.

For each ν there is a $G_\nu(\mathbf{R})^+$-conjugacy class $X_\nu{}^+$ of homomorphisms of $\mathbf{C}^\times$ into G_ν such that $X^+ = \Pi X_\nu{}^+$. The adjoint action of A (resp. A_ν) on G (resp. G_ν) induces an action of $A(\mathbf{R})$ (resp. $A_\nu(\mathbf{R})$) on the set of homomorphisms of $\mathbf{C}^\times$ into $G_{\mathbf{R}}$ (resp. G_ν), and we let $A(\mathbf{R})^1$ (resp. $A_\nu(\mathbf{R})^1$) be the subgroup that preserves X^+ (resp. $X_\nu{}^+$). For each ν, $G_\nu(\mathbf{R})^+$ is a subgroup of $A_\nu(\mathbf{R})^1$ of index at most 2, and $\mathbf{ad}: A_\nu(\mathbf{R})^1 \to \operatorname{Aut}(X_\nu)$ is an isomorphism. Furthermore, it is easy to see that we have the following exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi A_\nu(\mathbf{R}) & \longrightarrow & A(\mathbf{R}) & \longrightarrow & S & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \Pi A_\nu(\mathbf{R})^1 & \longrightarrow & A(\mathbf{R})^1 & \longrightarrow & S & \longrightarrow & 1.
\end{array}$$

It follows that $\mathbf{ad}: A(\mathbf{R})^1 \to \operatorname{Aut}(X^+)$ is an isomorphism. In fact, for $\gamma \in \operatorname{Aut}(X^+)$ there is a permutation τ of $\{1, \ldots, g\}$ such that γ is given componentwise by isomorphisms $X_\nu{}^+ \xrightarrow{\approx} X^+_{\tau(\nu)}$. Then $\tau \in S$. Let α be an element of $A(\mathbf{R})^1$ that maps to τ. Then $\mathbf{ad}\alpha^{-1} \circ \gamma \in \Pi\operatorname{Aut}(X_\nu)$. Hence $\mathbf{ad}\alpha^{-1} \circ \gamma = \Pi\mathbf{ad}\alpha_\nu = \mathbf{ad}(\Pi\alpha_\nu)$, $\alpha_\nu \in A_\nu(\mathbf{R})^1$. Therefore $\gamma \in \mathbf{ad}(A(\mathbf{R})^1)$.

Put $A(\mathbf{Q})^1 = A(\mathbf{Q})^* \cap A(\mathbf{R})^1$. Then $A(\mathbf{Q})^1$ acts on the set Σ, and therefore on the connected Shimura variety $\operatorname{Sh}^\circ(G, G', X^+)$. The inclusion $\mathbf{ad}: A(\mathbf{Q})^1 \hookrightarrow \operatorname{Aut}(\operatorname{Sh}^\circ(G, G', X^+))$ extends by continuity to the completion $A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\ G')$ of $A(\mathbf{Q})^1$ with respect to the topology defined by G'. Note that $G(\mathbf{Q})^+$ is a subgroup of $A(\mathbf{Q})^1$ of finite index.

1.3. THEOREM. *Under the above hypotheses,* $A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\ G')$ *is the full group of automorphisms of* $\operatorname{Sh}^\circ(G, G', X^+)$.

Let (L, σ) be the algebra with involution corresponding to G. Let U be the algebraic group over $\mathbf{Q}$ defined by $\{x \in L^\times \mid xx^\sigma = 1, Nx = 1\}$, where N denotes the reduced norm of L to its centre F. Then U is semisimple and $U^{\mathrm{ad}} = G$, see Weil [1, Theorem 2]. Let $\tilde{G}$ be the universal covering group of G and consider the covering $\tilde{G} \to U$.

1.4. Lemma. *Let Δ be an arithmetic subgroup of $U(\mathbf{Q})$ that is open with respect to the topology defined by $\tilde{G}$. Then for any natural number m, $\{\delta^m \mid \delta \in \Delta\}$ spans L over $\mathbf{Q}$.*

Proof. We follow the argument of Shimura [1, 6.6] and K. Miyake [1, 4.8]. Since the image of an arithmetic subgroup under an isogeny is arithmetic (Borel [2, 8.9]), we can assume that Δ is the image of a congruence subgroup $\tilde{\Delta}$ of $\tilde{G}(\mathbf{Q})$. Choose a rational prime $p \neq 2$ such that

(i) the congruence condition defining $\tilde{\Delta}$ does not involve p;
(ii) p splits completely in F, the centre of L;
(iii) $(L \otimes_{\mathbf{Q}} \mathbf{Q}_p, \sigma)$ splits, i.e. it is the direct product of algebras with involution over $\mathbf{Q}_p$ of the following types:

(A) $(M_n(\mathbf{Q}_p) \oplus M_n(\mathbf{Q}_p), \sigma)$, where $(X, Y)^\sigma = (Y^{\mathrm{tr}}, X^{\mathrm{tr}})$, $n \geq 3$.

(B), (D) $(M_n(\mathbf{Q}_p), \sigma)$, where $X^\sigma = X^{\mathrm{tr}}$, $n \geq 7$.

(C) $(M_{2n}(\mathbf{Q}_p), \sigma)$, where $X^\sigma = JX^{\mathrm{tr}}J^{-1}$,

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, n \geq 1.$$

Then $U(\mathbf{Q}_p)$ is isomorphic to the direct product of simple groups of the forms

(A) $SL_n(\mathbf{Q}_p)$, $n \geq 3$.
(B), (D) $SO_n(\mathbf{Q}_p)$, $n \geq 7$.
(C) $Sp_n(\mathbf{Q}_p)$, $n \geq 1$.

Using this isomorphism we can choose a $\mathbf{Z}_p$-order $\mathcal{O}_p$ in $L_p \overset{df}{=} L \otimes_{\mathbf{Q}} \mathbf{Q}_p$ invariant under σ such that $U(\mathbf{Z}_p) \overset{df}{=} \mathcal{O}_p^\times \cap U(\mathbf{Q}_p)$ is the direct product of groups of the forms

(A) $SL_n(\mathbf{Z}_p)$, $n \geq 3$.
(B), (D) $SO_n(\mathbf{Z}_p)$, $n \geq 7$.
(C) $Sp_n(\mathbf{Z}_p)$, $n \geq 1$.

For a σ-skew symmetric element w of $p\mathcal{O}_p$, the Cayley transformation of w lies in $U(\mathbf{Z}_p)$. Let $U' \subset U(\mathbf{Z}_p)$ be the set of Cayley transformations of such elements. Then it is easy to see that for any integer $m \geq 1$, $\{\alpha^m \mid \alpha \in U'\}$ spans L_p over $\mathbf{Q}_p$.

Let Δ_p (resp. $\tilde{\Delta}_p$) be the closure of Δ (resp. $\tilde{\Delta}$) in $U(\mathbf{Q}_p)$ (resp. $\tilde{G}(\mathbf{Q}_p)$). Then Δ_p is commensurable with $U(\mathbf{Z}_p)$. It follows that for any integer $m \geq 1$, there are $\alpha_1, \ldots, \alpha_k \in \Delta_p$ such that $\alpha_1{}^m, \ldots, \alpha_k{}^m$ form a basis for L_p. Lift α_i to elements β_i of $\tilde{\Delta}_p$ and apply the strong approximation theorem for $\tilde{G}$ to obtain elements b_i of $\tilde{G}(\mathbf{Q})$ that are in $\tilde{\Delta}$ and are close to β_i. Then $a_i{}^m$, where $a_i \in \Delta$ is the image of b_i, will be close to $\alpha_i{}^m$ in Δ_p, and $a_1{}^m, \ldots, a_k{}^m \in L$ will be linearly independent over $\mathbf{Q}_p$, and *a fortiori* over $\mathbf{Q}$. Thus the $\mathbf{Q}$-linear span of $a_1{}^m, \ldots, a_k{}^m$ is L.

1.5. *Remark.* Since any two arithmetic subgroups of $U(\mathbf{Q})$ are commensurable, Lemma 1.4 holds for any arithmetic subgroup Δ of $U(\mathbf{Q})$. As pointed out by S. Kudla, this result also follows from Borel's density theorem (Borel [1, Theorem 1]).

To complete the proof of Theorem 1.3, we have to show that for $\alpha \in \mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^\circ(G, G', X^+))$ and $\Gamma \in \Sigma$, there is $\beta \in A(\mathbf{Q})^1$ such that

$$(\mathrm{Sh}^\circ \xrightarrow{\alpha} \mathrm{Sh}^\circ \xrightarrow{\mathrm{can}} \Gamma\backslash X^+) = (\mathrm{Sh}^\circ \xrightarrow{\mathbf{ad}\beta} \mathrm{Sh}^\circ \xrightarrow{\mathrm{can}} \Gamma\backslash X^+).$$

By definition, there is a $\Gamma_1 \in \Sigma$ and an isomorphism $\alpha_1 : \Gamma_1\backslash X^+ \to \Gamma\backslash X^+$ such that the diagram

$$\begin{array}{ccc} \mathrm{Sh}^\circ & \xrightarrow{\alpha} & \mathrm{Sh}^\circ \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \Gamma_1\backslash X^+ & \xrightarrow{\alpha_1} & \Gamma\backslash X^+ \end{array}$$

commutes. As X^+ is simply connected, α_1 can be lifted to an element of $\mathrm{Aut}(X^+)$, say $\mathbf{ad}\beta(\beta \in A(\mathbf{R})^1)$. We first show that $\beta \in A(\mathbf{Q})$, i.e. it is an automorphism of (L, σ). Since $\beta \in A(\mathbf{R})$, it suffices to show that β maps L into L.

Let $\pi : U \to G$ be the covering map, and ℓ its degree. Choose arithmetic subgroups Δ, Δ_1 of $U(\mathbf{Q})$ that are open with respect to the topology defined by $\tilde{G}$ and such that $\pi(\Delta) \subset \Gamma$ and $\pi(\Delta_1) \subset \Gamma_1$. Let m be the index of $\pi(\Delta)$ in Γ. For $\delta_1 \in \Delta_1$, we have $\pi(\delta_1) \in \Gamma_1$ and $\mathbf{ad}\beta(\pi(\delta_1)) \in \Gamma$.

Note that $\beta(\delta_1{}^m) \in U(\mathbf{R})$ and $\pi(\beta(\delta_1{}^m)) = \mathbf{ad}\beta(\pi(\delta_1))^m \in \pi(\Delta)$. As the degree of π is ℓ, we see that $\beta(\delta_1{}^{m\ell}) \in \Delta$. Thus β maps $\{\delta_1{}^{m\ell} \mid \delta_1 \in \Delta_1\}$ into Δ. In view of Lemma 1.4, this proves that $\beta(L)$ is contained in L.

Next we show that $\mathbf{ad}\beta : G(\mathbf{Q}) \to G(\mathbf{Q})$ is a homeomorphism. Let $\Gamma' \in \Sigma$ be a normal subgroup of Γ. Then there are $\Gamma_1' \in \Sigma$ contained in Γ_1 and an isomorphism $\alpha_1' : \Gamma_1'\backslash X^+ \to \Gamma'\backslash X^+$ such that the diagram

$$\begin{array}{ccc}
\mathrm{Sh}^\circ & \xrightarrow{\alpha} & \mathrm{Sh}^\circ \\
\Big\downarrow{\scriptstyle \mathrm{can}} & & \Big\downarrow{\scriptstyle \mathrm{can}} \\
\Gamma_1'\backslash X^+ & \xrightarrow{\alpha_1'} & \Gamma'\backslash X^+ \\
\Big\downarrow{\scriptstyle q_1} & & \Big\downarrow{\scriptstyle q} \\
\Gamma_1\backslash X^+ & \xrightarrow{\alpha_1} & \Gamma\backslash X^+
\end{array}$$

commutes. Here q and q_1 denote the natural projections induced by the inclusions $\Gamma' \hookrightarrow \Gamma$ and $\Gamma_1' \hookrightarrow \Gamma_1$. Applying the above considerations to α_1', we see that there is $\beta' \in A(\mathbf{Q}) \cap A(\mathbf{R})^1$ such that α_1' lifts to $\mathbf{ad}\beta' \in \mathrm{Aut}(X^+)$. We have $\mathbf{ad}\beta'(\Gamma_1') = \Gamma'$. As both $\mathbf{ad}\beta$ and $\mathbf{ad}\beta'$ induce α_1, there is $\gamma \in \Gamma$ such that $\mathbf{ad}\beta = \mathbf{ad}\gamma\beta'$. Hence $\mathbf{ad}\beta(\Gamma_1') = (\mathbf{ad}\gamma \circ \mathbf{ad}\beta')(\Gamma_1') = \mathbf{ad}\gamma(\Gamma') = \Gamma'$, because Γ' is a normal subgroup of Γ. Since normal subgroups Γ' of Γ, $\Gamma' \in \Sigma$, form a basis for the neighborhoods of 1, this shows that $\mathbf{ad}\beta$ is continuous. The same argument applied to β^{-1} shows that β is a homeomorphism. Therefore $\beta \in A(\mathbf{Q})^* \cap A(\mathbf{R})^1 = A(\mathbf{Q})^1$ and $\mathbf{ad}\beta \circ \alpha^{-1} \in \mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^\circ)$ induces the identity map on $\Gamma\backslash X^+$.

1.6. The above techniques can be used to compute the automorphism group of a single variety $\Gamma\backslash X^+$. Let G be classical and $\Gamma \in \Sigma$. An automorphism α of $\Gamma\backslash X^+$ lifts to an automorphism $\mathbf{ad}\beta$ of X^+, $\beta \in A(\mathbf{R})^1$. The argument in the proof of 1.3 shows that $\beta \in A(\mathbf{Q}) \cap A(\mathbf{R})^1$. Conversely, any $\beta \in A(\mathbf{Q}) \cap A(\mathbf{R})^1$ such that $\mathbf{ad}\beta(\Gamma) = \Gamma$ defines an automorphism of $\Gamma\backslash X^+$. Thus $\mathrm{Aut}_{\mathbf{C}}(\Gamma\backslash X^+)$ is the quotient of $\{\beta \in A(\mathbf{Q}) \cap A(\mathbf{R})^1 \mid \mathbf{ad}\beta(\Gamma) = \Gamma\}$ by the subgroup of β such that $(X^+ \overset{\mathbf{ad}\beta}{\to} X^+ \overset{\mathrm{can}}{\to} \Gamma\backslash X^+) = (X^+ \overset{\mathrm{can}}{\to} \Gamma\backslash X^+)$.

In particular, if $A(\mathbf{Q}) = A(\mathbf{Q})^*$ (this is the case if G is $\mathbf{Q}$-simple or $G' = \tilde{G}$, the universal covering group of G), then every element of $\mathrm{Aut}_{\mathbf{C}}(\Gamma\backslash X^+)$ is induced by $\mathbf{ad}\beta$ with $\beta \in A(\mathbf{Q})^1$, hence can be extended to an element of $\mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^{\circ}(G, G', X^+))$. This is not true if $A(\mathbf{Q})^* \neq A(\mathbf{Q})$. Note also that we need to assume that Γ is torsion free. For example, if X^+ is the complex upper half plane, then the only automorphism of $SL_2(\mathbf{Z})\backslash X^+$ that can be extended to an automorphism of $\mathrm{Sh}^{\circ}(PGL_2, SL_2, X^+)$ is the identity.

1.7. The automorphism group of a non-connected Shimura variety $\mathrm{Sh}(G, X)$ (Deligne [2, 2.1]) is very large and complicated. However, the subgroup of automorphisms commuting with the Hecke operators is small: it consists only of the identity element if G is adjoint.

2. The automorphism group of a field of automorphic functions. Let (G, G', X^+) define a connected Shimura variety, and let Σ be as in Section 1. For $\Gamma \in \Sigma$, we let $\mathfrak{F}(\Gamma)$ be the function field of the algebraic variety $\Gamma\backslash X^+$ and we let $\mathfrak{F} = \varinjlim \mathfrak{F}(\Gamma)$. An element f of $\mathfrak{F}(\Gamma)$ can be identified with a meromorphic function $\tilde{f}$ on the bounded symmetric domain X^+ that is invariant under Γ. A meromorphic function on X^+ arising this way is said to be automorphic with respect to Γ. Thus $\mathfrak{F}$ can be regarded as the field of all meromorphic functions on X^+ that are automorphic with respect to some $\Gamma \in \Sigma$. When $\Gamma\backslash X^+$ is compact, the comparison Theorem (Shafarevich [1, VIII.3]) shows that $\mathfrak{F}(\Gamma)$ is equal to the field of meromorphic functions on X^+ invariant under Γ. The same is always true if G has no $\mathbf{Q}$-rational simple factors isomorphic to PGL_2 (Bailey-Borel [1, 10.12]). For the remaining case, an automorphic function is required to be meromorphic at cusps.

An automorphism α of $\mathrm{Sh}^{\circ}(G, G', X^+)$ defines an automorphism α^* of $\mathfrak{F}$ over $\mathbf{C}$ by the rule: $\alpha^* f = f \circ \alpha^{-1}$. If $\mathrm{Aut}(\mathfrak{F}/\mathbf{C})$ is given its usual topology (Shimura [3, 6.3]) then $\alpha \mapsto \alpha^*$ is continuous.

2.1. PROPOSITION. *The map* $\alpha \mapsto \alpha^*: \mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^{\circ}(G, G', X^+)) \to \mathrm{Aut}(\mathfrak{F}/\mathbf{C})$ *is an isomorphism of topological groups.*

Proof. Since $\Gamma\backslash X^+$ is separated as an algebraic variety, any morphism $V \to \Gamma\backslash X^+$ is determined by its action on $\mathfrak{F}(\Gamma)$. Thus $\alpha \mapsto \alpha^*$ is injective. In proving that the map is surjective, we follow an argument of T. Miyake [1] which is based on the following result: any isomorphism $X^+ - Y_1 \to X^+ - Y$, where Y and Y_1 are proper analytic subsets of X^+, extends to an automorphism of X^+ (ibid, Section 4(I)).

Let α^* be an element of $\mathrm{Aut}(\mathfrak{F}/\mathbf{C})$. Then for any $\Gamma \in \Sigma$, there is a unique $\Gamma_1 \in \Sigma$ such that $\alpha^*(\mathfrak{F}(\Gamma)) = \mathfrak{F}(\Gamma_1)$, see Miyake [1, Section 4.(I)].

Thus the restriction of α^* to $\mathfrak{F}(\Gamma)$ defines a rational map $\alpha_\Gamma: \Gamma_1\backslash X^+ \to \Gamma\backslash X^+$. We show that α_Γ is an isomorphism. Let V and V_1 be Zariski-open subsets of $\Gamma\backslash X^+$ and $\Gamma_1\backslash X^+$ such that α_Γ is an isomorphism from V_1 to V. The inverse images Y and Y_1 of $(\Gamma\backslash X^+) - V$ and $(\Gamma_1\backslash X^+) - V_1$ are proper analytic subsets of X^+. Consider the coverings $X^+ - Y \to V$ and $X^+ - Y_1 \to V_1$. The argument of Miyake [1, Section 4.(III)] shows that $\alpha_\Gamma: V_1 \to V$ lifts to an isomorphism $\tilde{\alpha}_\Gamma: X^+ - Y_1 \to X^+ - Y$ which, according to the result recalled above, extends to an automorphism $\tilde{\alpha}_\Gamma$ of X^+. This $\tilde{\alpha}_\Gamma$ induces an isomorphism of $\Gamma_1\backslash X^+$ to $\Gamma\backslash X^+$ that agrees with α_Γ on V_1. Therefore (ibid, Lemma 1) α_Γ is defined everywhere and is an isomorphism from $\Gamma_1\backslash X^+$ to $\Gamma\backslash X^+$.

Thus for every $\Gamma \in \Sigma$ there are a unique $\Gamma_1 \in \Sigma$ and an isomorphism $\alpha_\Gamma: \Gamma_1\backslash X^+ \to \Gamma\backslash X^+$ such that $\alpha_\Gamma{}^*: \mathfrak{F}(\Gamma) \to \mathfrak{F}(\Gamma_1)$ is the restriction of α^* to $\mathfrak{F}(\Gamma)$. If $\Gamma' \in \Sigma$ is contained in Γ, then the corresponding $\Gamma_1{}'$ is contained in Γ_1, and the diagram

$$\begin{array}{ccc} \Gamma_1{}'\backslash X^+ & \xrightarrow{\alpha_\Gamma} & \Gamma'\backslash X^+ \\ \downarrow & & \downarrow \\ \Gamma_1\backslash X^+ & \xrightarrow{\alpha_\Gamma{}'} & \Gamma\backslash X^+ \end{array}$$

commutes. Therefore $\{\alpha_\Gamma \mid \Gamma \in \Sigma\}$ defines an automorphism α of $\mathrm{Sh}^\circ(G, G', X^+)$ whose image in $\mathrm{Aut}(\mathfrak{F}/\mathbf{C})$ is α^*.

3. The automorphism group of a field of arithmetic automorphic functions.

3.1. Throughout this section G will be assumed to be classical and $\mathrm{Sh}^\circ(G, G', X^+)$ will be assumed to have a canonical model $M^\circ(G, G', X^+)$ in the sense of Deligne [2, 2.7.10]. (For a list of those Shimura varieties known to have canonical models, see Deligne [2, 2.7.20].) Let $E = E(G, X^+)$ be the reflex field; then $M^\circ(G, G', X^+)$ is a scheme over $\overline{\mathbf{Q}}$ together with a left action of the middle term in the canonical extension

$$1 \to G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\ G') \to \mathcal{E}(G, G', X^+) \xrightarrow{\sigma} \mathrm{Gal}(\overline{\mathbf{Q}}/E) \to 1$$

(see Deligne [2, 2.5.9]). For any $\Gamma \in \Sigma$, $\Gamma\backslash M^\circ$ is a model for $\Gamma\backslash X^+$ over $\overline{\mathbf{Q}}$. Thus its function field $F(\Gamma)$ is a subfield of $\mathfrak{F}(\Gamma)$ linearly disjoint from $\mathbf{C}$

over $\overline{\mathbf{Q}}$ and such that $\mathbf{C}\cdot F(\Gamma) = \mathfrak{F}(\Gamma)$. We define F, *the field of arithmetic automorphic functions* on X^+ relative to (G, G') to be $\bigcup F(\Gamma) \subset \mathfrak{F}$.

We note that a Hilbert or Siegel modular function is arithmetic in this sense if and only if it is the quotient of two modular forms with algebraic Fourier coefficients (Shimura [4]).

3.2. For any $\Gamma, \Gamma' \in \Sigma$, there are only countably many isomorphisms $\alpha : (\Gamma'\backslash M^\circ)_{\mathbf{C}} \to (\Gamma\backslash M^\circ)_{\mathbf{C}}$ that extend to automorphisms of $\mathrm{Sh}^\circ(G, G', X^+)$. Thus each such α is defined over $\overline{\mathbf{Q}}$, and we can identify $\mathrm{Aut}_{\mathbf{C}}(\mathrm{Sh}^\circ(G, G', X^+))$ with its subgroup $\mathrm{Aut}_{\overline{\mathbf{Q}}}(M^\circ(G, G', X^+))$. Theorem 1.3 provides us with an isomorphism

$$A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\ G') \xrightarrow{\approx} \mathrm{Aut}_{\overline{\mathbf{Q}}}(M^\circ(G, G', X^+)).$$

The two actions of $G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\ G')$ on $M^\circ(G, G', X^+)$, arising from the actions of $A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\ G')$ and $\mathcal{E}(G, G', X^+)$, are the same. For any $\alpha \in \mathcal{E}(G, G', X^+)$,

$$\begin{array}{ccc} M^\circ(G, G', X^+) & \xrightarrow{\ \alpha\ } & M^\circ(G, G', X^+) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(\overline{\mathbf{Q}}) & \xrightarrow{\mathrm{Spec}\,(\sigma(\alpha)^{-1})} & \mathrm{Spec}\,(\overline{\mathbf{Q}}) \end{array}$$

commutes (Deligne [2, 2.7.10]). Thus we can define an action of $\mathcal{E}(G, G', X^+)$ on F by setting ${}^\alpha f = \sigma(\alpha) \circ f \circ \alpha^{-1}$ for any $f \in F$ and $\alpha \in \mathcal{E}(G, G', X^+)$. Note that if $\alpha \in G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\ G') \subset \mathcal{E}(G, G', X^+)$, then ${}^\alpha f = \alpha^* f$ (with the notation of Section 2).

3.3. **Theorem.** *The map* $\mathcal{E}(G, G', X^+) \to \mathrm{Aut}(F/E)$ *identifies* $\mathcal{E}(G, G', X^+)$ *with an open subgroup of* $\mathrm{Aut}(F/E)$ *of finite index.*

Proof. This follows from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\ G') & \longrightarrow & \mathcal{E}(G, G', X^+) & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Aut}(F/\overline{\mathbf{Q}}) & \longrightarrow & \mathrm{Aut}(F/E) & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E) & \to & 1 \end{array}$$

since we know from Sections 1 and 2 that the left hand vertical arrow identifies $G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\ G')$ with an open subgroup of $\mathrm{Aut}(F/\mathbf{Q})$ of finite index.

3.4. The reciprocity law at a special $h \in X^+$ (Deligne [2, 2.7.10c]) has the following interpretation. Let T be a **Q**-rational torus in G such that h factors through $T(\mathbf{R})$. In [2, 2.5.10] Deligne constructs an extension $\mathfrak{D}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/E(T, h))$ by $T(\mathbf{Q})$ and a commutative diagram

$$\begin{array}{ccccccccc} 1 \to & T(\mathbf{Q}) & \longrightarrow & \mathfrak{D} & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E(T,h)) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{E}(G, G', X^+) & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E) & \to 1 \end{array}$$

Let $\lambda \in \mathfrak{D}$ have images $\alpha(\lambda)$ in $\mathcal{E}(G, G', X^+)$ and $\sigma(\lambda)$ in $\mathrm{Gal}(\overline{\mathbf{Q}}/E(T, h))$. If $z = [h]$ is the point on $M(G, G'X^+)$ defined by h, then Deligne's reciprocity law says that $\alpha(\lambda)z = z$. Thus for any $f \in F$ that is defined at z, ${}^{\alpha(\lambda)}f(z) = \sigma(\lambda)(f(z))$.

3.5. Let (L, σ) be the algebra with involution corresponding as in 1.1 to G and let $A = \mathrm{Aut}(L, \sigma)$. According to Deligne [2, 2.7.16] there is a pair (G_1, X_1) satisfying the axioms for a Shimura variety and such that $(G_1{}^{\mathrm{ad}}, G_1{}^{\mathrm{der}}, X_1{}^+) = (G, G', X^+)$ and $E(G_1, X_1) = E(G, X^+)$. We assume that it is possible to choose (G_1, X_1) so that there is an action $\mathbf{ad}: A \to \mathrm{Aut}(G_1)$ of A on G_1 that is compatible with the action of A on G. (For this to be possible it is obviously necessary that the action of A on $Z(\tilde{G})$, the centre of $\tilde{G}$, induces an action on $Z(G')$; in 3.7 below we show that this condition is also sufficient.) Under this assumption we can apply the functor $-*_{G_1(\mathbf{Q})_+/Z_1(\mathbf{Q})}A(\mathbf{Q})^1$ to the first two terms of the sequence

$$1 \to G_1(\mathbf{Q})_+{}^{\wedge}/Z_1(\mathbf{Q})^{\wedge} \to G_1(\mathbf{A}^f)/Z_1(\mathbf{Q})^{\wedge} \to \overline{\pi}_0\pi(G_1) \to 1$$

and obtain an exact sequence

$$1 \to A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\,(G')) \to \frac{G_1(\mathbf{A}^f)}{Z_1(\mathbf{Q})^{\wedge}} *_{G_1(\mathbf{Q})_+/Z_1(\mathbf{Q})} A(\mathbf{Q})^1 \to \overline{\pi}_0\pi(G_1) \to 1$$

(cf. Deligne [2, 2.5.1]). On pulling-back relative to $\mathrm{Gal}(\overline{\mathbf{Q}}/E) \to \overline{\pi}_0\pi(G)$ we obtain the second row of the following exact commutative diagram

$$\begin{array}{ccccccccc} 1 \to & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{E}(G, G', X^+) & \xrightarrow{\sigma} & \mathrm{Gal}(\overline{\mathbf{Q}}/E) & \to 1 \\ & \downarrow & & \downarrow & & \| & \\ 1 \to & A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{E}^e(G, G', X^+) & \xrightarrow{\sigma} & \mathrm{Gal}(\overline{\mathbf{Q}}/E) & \to 1 \end{array}$$

This diagram is independent of the choice of (G_1, X_1). We give $\mathcal{E}^e(G, G', X^+)$ the topology for which $\mathcal{E}(G, G', X^+)$ is an open subgroup. The actions of $A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\, G')$ and $\mathcal{E}(G, G', X^+)$ on F combine to give an action of $\mathcal{E}^e(G, G', X^+)$ on F.

3.6. THEOREM. *Under the above assumption there is an isomorphism of topological groups* $\mathcal{E}^e(G, G', X^+) \xrightarrow{\sim} \mathrm{Aut}(F/E)$.

Proof. The proof is the same as that of 3.3.

3.7. LEMMA. *With the notations of* 3.5, *assume that the action of A on $Z(\tilde{G})$ induces an action of A on the quotient $Z(G')$ of $Z(\tilde{G})$. Then there exists a (G_1, X_1) such that $({G_1}^{\mathrm{ad}}, {G_1}^{\mathrm{der}}, {X_1}^+) = (G, G', X^+)$, $E(G_1, X_1) = E(G, X^+)$, and the action of A on G lifts to an action on G_1.*

Proof. We have only to modify slightly the proof in Milne-Shih [3, Section 3]. Let T be a maximal torus in G and T' its inverse image in G'. Then $M \overset{df}{=} X_*(T)/X_*(T') \approx \mathrm{Hom}(X^*(Z(G')), \mathbf{Q}/\mathbf{Z})$ and so, by assumption, has a natural action by A/A°. The sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

of (ibid, Lemma 3.2) can then be chosen to have an action by A/A°. The centre of the group G_1 constructed in (ibid, 3.1) is a torus $Z(G_1)$ such that $X_*(Z(G_1)) = P_1$. It therefore carries an action by A/A° such that $Z(\tilde{G}) \to Z(G_1)$ is equivariant. Since $G_1 = \tilde{G} *_{Z(\tilde{G})} Z(G_1)$, it also carries an action by A. According to (ibid, 3.4) it is now possible to choose X_1 such that $E(G_1, X_1) = E(G, X^+)$.

4. The main theorems for canonical models in the sense of Shimura.

4.1. Let G be a reductive group and X a family of maps $\mathbf{C}^\times \to G(\mathbf{R})$ such that (G, X) satisfies the axioms for a Shimura variety (Deligne [2, 2.1.1]). Assume that $\mathrm{Sh}(G, X)$ has a canonical model $(M(G, X), \varphi : \mathrm{Sh}(G, X) \to M(G, X)_{\mathbf{C}})$ in the sense of Deligne [2, 2.2.5] defined over its reflex field $E = E(G, X)$. We begin this section by showing how to derive from $M(G, X)$ a canonical model in the sense of Shimura [1].

4.2. Let $\ell = \ell_{G,X}$ be the map $\pi_0 N_{E/\mathbf{Q}} q_M : \mathrm{Gal}(\overline{\mathbf{Q}}/E)^{\mathrm{ab}} = \pi_0\pi(\mathbf{G}_{mE}) \to \bar{\pi}_0\pi(G)$ explicitly defined in Deligne [2, 2.6.2.1]. According to Deligne [2, 2.6.3], $\sigma\epsilon = \ell(\sigma)\epsilon (\overset{df}{=} \epsilon\ell(\sigma)^{-1})$ for $\epsilon \in \pi_0(M(G, X)_{\overline{\mathbf{Q}}})$. Let $\mathfrak{k}$ be the subfield of $\overline{\mathbf{Q}}$ corresponding to the kernel of ℓ and define $\overline{\mathcal{E}}(G, X)$ to make the following diagram exact and commutative:

$$\begin{array}{ccccccc}
1 \to G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}}) & \longrightarrow & \dfrac{G(A^f)}{Z(\mathbf{Q})^\wedge} *_{G(\mathbf{Q})_+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+ & \longrightarrow & \bar{\pi}_0\pi(G) \to 1 \\
\Vert & & \uparrow & & \uparrow \ell \\
1 \to G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}}) & \longrightarrow & \overline{\mathcal{E}}(G, X) & \xrightarrow{\sigma} & \mathrm{Gal}(\mathfrak{k}/E) \to 1
\end{array}$$

Thus $\overline{\mathcal{E}}(G, X)$ is a quotient of $\mathcal{E}(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$. Let $\mathfrak{z}$ be the set of open compact subgroups of $\overline{\mathcal{E}}(G, X)$ and for any $S \in \mathfrak{z}$, let k_S be the subfield of $\mathfrak{k}$ defined by $\sigma(S)$. Let $\Gamma_S = S \cap G^{\mathrm{ad}}(\mathbf{Q})^+$, and choose a connected component X^+ of X. We shall construct a family (V_S, φ_S), $S \in \mathfrak{z}$, of models for the varieties $\Gamma_S \backslash X^+$, and a family of maps $(J_{TS}(x))$, $x \in \overline{\mathcal{E}}(G, X)$, analogous to those constructed in special cases in Shimura [1, 2.5], Miyake [1, 4.1], and Shih [1, 2.4], [2, 3].

4.3. The group Γ_S is open relative to the topology on $G^{\mathrm{ad}}(\mathbf{Q})$ defined by G^{der}, and is torsion free (and so belongs to Σ) if S is sufficiently small. Set $S_0 = S \cap G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G')$. Let $\mathrm{Sh}(G, X)^\circ$ be the connected component of $\mathrm{Sh}(G, X)$ containing the image of $X^+ \times \{1\}$. Thus $\Gamma_S \backslash X^+ \xrightarrow{\approx} S_0 \backslash \mathrm{Sh}(G, X)^\circ$. For some connected component $M(G, X)^\circ$ of $M(G, X)_{\overline{\mathbf{Q}}}$, φ induces an isomorphism $\mathrm{Sh}(G, X)^\circ \xrightarrow{\approx} M(G, X)_{\mathbf{C}}{}^\circ$, and we let e denote $M(G, X)^\circ$ regarded as a point on the pro-finite scheme $\pi_0(M(G, X))$. As e is fixed by $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathfrak{k})$, $M(G, X)^\circ$ is defined over $\mathfrak{k}$. The group $\overline{\mathcal{E}}(G, X)$ acts on $M(G, X)^\circ$ compatibly with the action of its quotient $\mathrm{Gal}(\mathfrak{k}/E)$ on $\mathfrak{k}$. Thus $V_S \overset{df}{=} S \backslash M(G, X)^\circ$ is a model of $S_0 \backslash M(G, X)^\circ$ rational over k_S. Define φ_S to be the composite,

$$\Gamma_S \backslash X^+ \xrightarrow{\approx} S_0 \backslash \mathrm{Sh}(G, X)^\circ \xrightarrow{\approx} S_0 \backslash M(G, X)_{\mathbf{C}}{}^\circ \xrightarrow{\approx} (V_S)_{\mathbf{C}}.$$

4.4. The compatibility of the actions of $\overline{\mathcal{E}}(G, X)$ on $M(G, X)^\circ$ and $\mathfrak{k}$ means that, for any $\alpha \in \overline{\mathcal{E}}(G, X)$,

$$\begin{array}{ccc} M(G,X)^{\circ} & \longrightarrow & M(G,X)^{\circ} \\ \downarrow & & \downarrow \\ \operatorname{spec} \mathfrak{k} & \xrightarrow{\operatorname{spec}(\sigma(\alpha)^{-1})} & \operatorname{spec} \mathfrak{k} \end{array} \qquad \text{(left action)}$$

commutes. We therefore have a map of $\mathfrak{k}$-schemes $J(\alpha): M(G, X)^{\circ} \to \sigma(\alpha)^{-1}M(G, X)^{\circ}$ such that $\sigma(\alpha) \circ J(\alpha) = \alpha$:

$$\begin{array}{ccccc} & & \xrightarrow{\qquad\qquad\alpha\qquad\qquad} & & \\ M(G,X)^{\circ} & \xrightarrow{J(\alpha)} & \sigma(\alpha)^{-1}M(G,X)^{\circ} & \xrightarrow{\sigma(\alpha)} & M(G,X)^{\circ} \\ & \searrow & \downarrow & & \downarrow \\ & & \operatorname{spec} \mathfrak{k} & \xrightarrow{\operatorname{spec}(\sigma(\alpha)^{-1})} & \operatorname{spec} \mathfrak{k} \end{array}$$

If S, $T \in \mathfrak{z}$ are such that $\alpha S \alpha^{-1} \subset T$, then $J(\alpha)$ defines a map on the quotients, $J_{TS}(\alpha): V_S \to T\backslash(\sigma(\alpha)^{-1}M(G, X)^{\circ}) = \sigma(\alpha)^{-1}V_T$. Thus, for $f \in k_T(V_T)$, $(\sigma(\alpha)^{-1}f) \circ J_{TS}(\alpha) = {}^{\alpha^{-1}}f \overset{df}{=} \sigma(\alpha)^{-1} \cdot f \circ \alpha$.

4.5. Let $h \in X^+$ be special, so that h factors through $H(\mathbf{R})$ for some $\mathbf{Q}$-rational torus $H \subset G$, and let μ be the cocharacter of $G_{\mathbf{C}}$ associated with h. By definition, the reflex field $E(h)$ is the smallest field over which μ is defined. Let η be the homomorphism of algebraic groups $E(h)^{\times} \xrightarrow{NR(\mu)} H \hookrightarrow G$ where, as in Deligne [2, 2.2.2], $NR(\mu)$ is the composite $E(h)^{\times} = \operatorname{Res}_{E(h)/\mathbf{Q}}(\mathbf{G}_{mE(h)}) \xrightarrow{\operatorname{Res}(\mu)} \operatorname{Res}_{E(h)/\mathbf{Q}}(H_{E(h)}) \xrightarrow{N_{E(h)/\mathbf{Q}}} H$. Let M be the $G(\mathbf{C})$-conjugacy class of maps $\mathbf{C}^{\times} \to G(\mathbf{C})$ associated with X, and let $q_m: \pi(\mathbf{G}_{mE}) \to \pi(G_E)$ and $N_{E/\mathbf{Q}}: \pi(G_E) \to \pi(G)$ be the maps defined in Deligne [2, 2.4]. It follows directly from the definitions that

$$\begin{array}{ccccccc} \mathbf{A}_{E(h)}^{\times} & & \xrightarrow{\qquad\eta\qquad} & & G(\mathbf{A}) & & \\ \downarrow{\scriptstyle N_{E(h)/E}} & & & & \downarrow & & \\ \mathbf{A}_E^{\times} & \longrightarrow & \pi(\mathbf{G}_{mE}) & \xrightarrow{N_{E/\mathbf{Q}} \circ q_m} & \pi(G) & \longrightarrow & \overline{\pi}_0\pi(G). \end{array}$$

commutes. Let $\nu \in \mathbf{A}_{E(h)}^{\times}$ and let $[\nu]$ and $[\eta(\nu)]$ denote respectively the images of ν and $\eta(\nu)$ in $\operatorname{Gal}(E(h)^{\mathrm{ab}}/E(h))$ and $\overline{\pi}_0\pi(G)$. The above diagram and Deligne [2, 2.6.3] show that $[\eta(\nu)]\,\epsilon = \ell_{G,X}(N_{E(h)/E}(\nu))\cdot\epsilon = [\nu]\,\epsilon$ for $\epsilon \in \pi_0(M(G, X)_{\overline{\mathbf{Q}}})$. The element $\eta(\nu) * 1 \in G(\mathbf{A}^f)/Z(\mathbf{Q})^\wedge * \cdots$ maps to $\ell_{G,X}(N_{E(h)/E}(\nu))$ and therefore lies in $\overline{\mathcal{E}(G, X)}$; we have therefore a map $\overline{\eta} = (\nu \mapsto \eta(\nu) * 1)\colon \mathbf{A}^\times_{E(h)} \to \overline{\mathcal{E}(G, X)}$. Moreover, $\sigma(\overline{\eta}(\nu)) = [\nu]^{-1}|_{\mathfrak{f}}$.

4.6. Theorem. (a) *For each* $S \in \mathfrak{z}$, (V_S, φ_S) *is a model of* $\Gamma_S\backslash X^+$ *over* k_S.

(b) *For any* $\alpha \in \overline{\mathcal{E}(G, X)}$ *and* $S, T \in \mathfrak{z}$ *such that* $\alpha S \alpha^{-1} \subset T$, $J_{TS}(\alpha)$ *is a map* $V_S \to \sigma(\alpha)^{-1} V_T$ *defined over* k_S. *The following hold:*

$J_{SS}(\alpha)$ *is the identity map if* $\alpha \in S$;
$(\sigma(\alpha)^{-1} J_{TS}(\beta)) \circ J_{SR}(\alpha) = J_{TR}(\beta\alpha)$;
$J_{TS}(\alpha) \circ \varphi_S = \varphi_T \circ \alpha$ *for all* $\alpha \in G(\mathbf{Q})_+$ *such that* $\alpha S \alpha^{-1} \subset T$.

(c) *Let* $z \in X^+$ *be special; for each* $S \in \mathfrak{z}$, $\varphi_S(z)$ *is rational over* $E(z)^{\mathrm{ab}}$, *and for every* $\nu \in \mathbf{A}^{f\times}_{E(z)}$,

$$[\nu]\,\varphi_S(z) = J_{ST}(\overline{\eta}(\nu))\,\varphi_T(z)$$

where $[\nu]$ *is the element of* $\mathrm{Gal}(E(z)^{\mathrm{ab}}/E(z))$ *corresponding to* ν *and* $T = \overline{\eta}(\nu)^{-1} S \overline{\eta}(\nu)$.

Proof. Both (a) and (b) follow directly from the definitions. Let $z \in X^+$ be special and let $\nu \in \mathbf{A}^{f\times}_{E(z)}$. One of the conditions for $M(G, X)$ to be a canonical model is that $[\nu][z, 1] = [z, \eta(\nu)^{-1}]$ (Deligne [2, 2.2.5]) where $[z, 1]$ is regarded as a point on $M(G, X)_{\overline{\mathbf{Q}}}$. Therefore $\overline{\eta}(\nu)[z, 1] \overset{df}{=} \sigma(\eta(\nu))[z, \eta(\nu)^{-1}] \overset{4.5}{=} [\nu]^{-1}[z, \eta(\nu)^{-1}] = [z, 1]$, and so $J(\overline{\eta}(\nu))[z, 1] = [\nu][z, 1]$, which implies (c).

4.7. It is sometimes possible to strengthen this theorem by enlarging $\overline{\mathcal{E}(G, X)}$ and $\mathfrak{z}$. Let A be the algebraic group associated with G^{ad} as in Section 1 and assume that the action of A on G^{ad} extends to an action on G(cf. 3.7). Since $G = \tilde{G} *_{Z(\tilde{G})} Z(G)$ it only necessary for this that the action of A/A° on $Z(\tilde{G})$ extends to $Z(G)$. Under this assumption we can construct, as in 3.5 and 4.2, extensions

$$\begin{array}{ccccccc}
1 \to A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\,G^{\mathrm{der}}) & \longrightarrow & \dfrac{G(\mathbf{A}^f)}{Z(\mathbf{Q})^\wedge} *_{G(\mathbf{Q})_+/Z(\mathbf{Q})} A(\mathbf{Q})^1 & \longrightarrow & \overline{\pi}_0\pi(G) & \to 1 \\
\Big\| & & \uparrow & & \uparrow & \\
1 \to A(\mathbf{Q})^{1\wedge}(\mathrm{rel}\,G^{\mathrm{der}}) & \longrightarrow & \overline{\mathcal{E}^e(G, X)} & \xrightarrow{\ \sigma\ } & \mathrm{Gal}(\mathfrak{f}/E) & \to 1
\end{array}$$

Let $\mathfrak{Z}$ be the set of compact open subgroups of $\overline{\mathcal{E}^e}(G, X)$. Then, exactly as before, we can construct a family $(V_S, \varphi_S, J_{ST}(\alpha))$ with the same properties as the family in 4.6 except that now S and T are allowed to lie in $\mathfrak{Z}$ and α in $\overline{\mathcal{E}(G, X)}$. For example if G^{ad} is $\mathbf{Q}$-simple of type C or A and we take (G, X) to be the pair denoted (G_0, X_0) in the appendix of Milne-Shih [1], then we obtain the families constructed in Shimura [1, 5.2] and K. Miyake [1, 4.1].

4.8. We return to the situation of 4.6. For any $S \in \mathfrak{z}$ the function field $F(S)$ of V_S is a subfield of the function field $\mathfrak{F}(\Gamma_S)$ of $\Gamma_S \backslash M(G, X)^{\circ}$ linearly disjoint from $\mathbf{C}$ over $\overline{\mathbf{Q}}$ and such that $\mathbf{C}.F(S) = \mathfrak{F}(\Gamma_S)$. We define F', the *field of arithmetic automorphic functions* on X^+ relative to G to be $\cup F(S) \subset \mathfrak{F}$. Thus $f \in \mathfrak{F}$ is in F' if and only if there is an $S \in \mathfrak{z}$ such that $f = g \circ \varphi_S$ for some $g \in k_S(V_S)$. We define an action of $\overline{\mathcal{E}(G, X)}$ on F' as follows: for $\alpha \in \overline{\mathcal{E}(G, X)}$, $f \in F'$, and $z \in M(G, X)^{\circ}$, we set ${}^{\alpha}f(z) = \sigma(\alpha) \circ f \circ \alpha^{-1}(z)$. Thus, if $f = g \circ \varphi_S$, then ${}^{\alpha}f = {}^{\alpha}g \circ \varphi_S = (\sigma(\alpha)g) \circ J_{ST}(\alpha^{-1}) \circ \varphi_T$ (cf. 4.4).

4.9. THEOREM. *If G^{ad} is classical then $\overline{\mathcal{E}(G, X)}$ is isomorphic to an open subgroup of* $\mathrm{Aut}(F'/E)$ *of finite index.*

Proof. The proof is the same as that of 3.4.

4.10. There is a reciprocity law at the special points: with the notation of 4.5, for any special $z \in X^+$ and $f \in F'$ defined at z, $f(z) \in E(z)^{\mathrm{ab}}$; moreover, for any $\nu \in \mathbf{A}_{E(z)}^{f\times}$, ${}^{\overline{\eta}(\nu)^{-1}}f$ is also defined at z, and $[\nu]f(z) = {}^{\overline{\eta}(\nu)^{-1}}f(z)$ (because ${}^{\overline{\eta}(\nu)^{-1}}f \overset{df}{=} [\nu] \circ f \circ \overline{\eta}(\nu)$, and $\overline{\eta}(\nu)z = z$; cf. the proof of 4.6c).

4.11. In the case that (as in 4.7) the action of A on G^{ad} extends to G we can replace $\overline{\mathcal{E}}(G, X)$ by $\overline{\mathcal{E}}^e(G, X)$ and (V_S, φ_S), $S \in \mathfrak{z}$, by the larger family (V_S, φ_S), $S \in \mathfrak{Z}$. The field $F' = \cup F(S)$ is unchanged. If further G^{ad} is classical then there is an isomorphism $\mathcal{E}^e(G, X) \xrightarrow{\approx} \mathrm{Aut}(F'/E)$ of topological groups.

4.12. In the special case that G^{der} is simply connected and the centre Z of G is a cohomologically trivial torus, the above results can be made more explicit. This will be so for example if G^{ad} is $\mathbf{Q}$-simple of type A, B, or C and (G, X) is taken to be the pair denoted (G_0, X_0) in the appendix of Milne-Shih [1]. (These are the cases studied in K. Miyake [1], Shih [1], and Shimura [1] respectively.)

Let T be the torus defined by the exact sequence

$$1 \to G^{\mathrm{der}} \to G \xrightarrow{\nu} T \to 1.$$

Since G^{der} is simply connected, the map

$$\pi_0\pi(G) = \pi_0(G(\mathbf{Q})\backslash G(\mathbf{A})) \to \pi_0(T(\mathbf{Q})\backslash T(\mathbf{A}))$$

induced by ν is an isomorphism (Deligne [1, 2.4]). As $G(\mathbf{R})_+ = \{g \in G(\mathbf{R}) \mid \nu(g) \in \nu(Z(\mathbf{R}))\}$ (see Milne-Shih [1, 3.1]) we see that $\bar{\pi}_0\pi(G) \overset{df}{=} \pi_0(G(\mathbf{R})_+)\backslash\pi_0\pi(G) = \nu(Z(\mathbf{R})\backslash\pi_0(T(\mathbf{Q})\backslash T(\mathbf{A})) = T(\mathbf{A})/T^c\nu(Z(\mathbf{R})$, where T^c is the closure of $T(\mathbf{R})^+T(\mathbf{Q})$ in $T(\mathbf{A})$. In many cases $\nu(Z(\mathbf{R})) \subset T(\mathbf{R})^+$ and so $\bar{\pi}_0\pi(G) = T(\mathbf{A})/T^c$.

The map $\ell\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) = \pi_0(\mathbf{A}_E{}^\times/E^\times) \to \bar{\pi}_0\pi(G)$ of 4.2 can be described as follows. Let $\mu_h\colon \mathbf{G}_m \to G_{\mathbf{C}}$ correspond to $h \in X$. The composite $\mu_X = \nu \circ \mu_h$ is independent of h, and is therefore defined over E. We have therefore a map $\lambda = (\mathrm{Res}_{E/\mathbf{Q}}\mathbf{G}_m \xrightarrow{\mathrm{Res}(\mu_X)} \mathrm{Res}_{E/\mathbf{Q}}T \xrightarrow{N_{E/\mathbf{Q}}} T)$. This gives a map $\pi_0(\mathbf{A}_E{}^\times/E^\times) \to \pi_0(T(\mathbf{A})/T(\mathbf{Q}))$ that, when composed with $\pi_0(T(\mathbf{A})/T(\mathbf{Q})) \to \pi_0(T(\mathbf{A})/T(\mathbf{Q}))/\nu(Z(\mathbf{R}))$, equals ℓ.

As Z is cohomologically trivial, $G(\mathbf{Q}) \to G^{\mathrm{ad}}(\mathbf{Q})$ is surjective.

Thus
$$\frac{G(\mathbf{A}^f)}{Z(\mathbf{Q})^\wedge} *_{G(\mathbf{Q})_+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+ = \frac{G(\mathbf{A}^f)}{Z(\mathbf{Q})^\wedge} = \frac{G(\mathbf{A})_+}{Z(\mathbf{Q})^\wedge G(\mathbf{R})_+}.$$

There is a diagram

$$\begin{array}{ccccccccc}
1 & \to & G(\mathbf{Q})_+{}^\wedge G(\mathbf{R})_+ & \longrightarrow & G(\mathbf{A})_+ & \longrightarrow & \bar{\pi}_0\pi(G) & \to & 1 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \ell} & & \\
1 & \to & G(\mathbf{Q})_+{}^\wedge G(\mathbf{R})_+ & \longrightarrow & \mathcal{G}(G, X) & \xrightarrow{\sigma} & \mathrm{Gal}(\mathfrak{k}/E) & \to & 1.
\end{array}$$

which, when the four left-most terms are divided by $Z(\mathbf{Q})^\wedge G(\mathbf{R})_+$, becomes the diagram in 4.2. By definition, $\mathcal{G}(G, X) = \{g \in G(\mathbf{A})_+ \mid \nu(g) \in T^c \cdot \lambda(\mathbf{A}_E{}^\times) \cdot \nu(Z(\mathbf{R}))\}$. Note that for any special $h \in X$, the map η of 4.5 defines a homomorphism $\mathbf{A}_{E(h)}^\times \to \mathcal{G}(G, X)$.

We can now identify $\mathfrak{z}$ with the set of subgroups S of $\mathcal{G}(G, X)$ such that $Z(\mathbf{Q})^\wedge G(\mathbf{R})_+ \subset S$ and $S/Z(\mathbf{Q})^\wedge G(\mathbf{R})^+$ is open and compact in $\mathcal{G}(G, X)/Z(\mathbf{Q})^\wedge G(\mathbf{R})_+ (= \mathcal{E}'(G, X))$, and we can substitute $\mathcal{G}(G, X)$ for $\overline{\mathcal{E}}(G, X)$ in the statement of Theorem 4.6. From 4.6, 4.9, and 4.10 we can conclude the following.

4.13. **Theorem.** *There is a continuous homomorphism* $\tau\colon \mathcal{G}(G, X) \to \mathrm{Aut}(F'/E)$ *with the following properties.*

(a) *The kernel of* τ *is* $Z(\mathbf{Q})^\wedge G(\mathbf{R})_+$.
(b) *For* $\alpha \in \mathcal{G}(G, X)$, $\tau(\alpha)|_{\mathfrak{k}} = \sigma(\alpha)$.
(c) *For any* $\alpha \in G(\mathbf{Q})_+^\wedge \subset \mathcal{G}(G, X), f \in F'$, *and* $z \in M(G, X)^\circ$, $^{\tau(\alpha)}f(z) = f(\alpha^{-1}z)$.
(d) *For any special* $z \in X^+$ *and* $f \in F'$, *defined at* z, $f(z) \in E(z)^{ab}$; *if* $\nu \in \mathbf{A}_{E(h)}^\times$ *and* $u = \eta(\nu)^{-1}$, *then* $^{\tau(u)}f$ *is also defined at* z, *and* $^{\tau(u)}f(z) = [\nu]f(z)$.
(e) *The map* τ *defines a topological isomorphism of* $\mathcal{G}(G, X)/Z(\mathbf{Q})^\wedge G(\mathbf{R})_+$ *onto an open subgroup of* $\mathrm{Aut}(F'/E)$. *For every* $S \in \mathfrak{z}$, F' *is an infinite Galois extension of* k_S *and* $\tau(S) = \mathrm{Gal}(F'/k_S)$.

4.14. If, in addition to the assumptions of 4.12, we suppose that the action of A on G^{ad} extends to an action on G then we can extend $\mathcal{G}(G, X)$ to a group $\mathcal{G}^e(G, X)$ having $\overline{\mathcal{E}}^e(G, X)$ as a quotient group. In this case there is an exact sequence

$$1 \to Z(\mathbf{Q})^\wedge G(\mathbf{R})_+ \to \mathcal{G}^e(G, X) \to \mathrm{Aut}(F'/E) \to 1$$

of topological groups.

4.15. Unlike Deligne's connected canonical model, Shimura's canonical model depends on the pair (G, X) with G a reductive group and not merely the associated triple (G^{ad}, G^{der}, X^+) (because the field $\mathfrak{k}$ depends on G; consider for example the case of a torus). It would perhaps be most natural to define the canonical model associated with a triple (G, G', X^+) to be a family $(V_S, \varphi_S, J_{ST}(\alpha))$ in which S is an open compact subgroup of $\mathcal{E}(G, G', X^+)$, $\alpha \in \mathcal{E}(G, G', X^+)$, and $V_S = S\backslash M^\circ(G, G', X^+)$ is a model for $\Gamma_S\backslash X^+$ where $\Gamma_S = S \cap G(\mathbf{Q})^+$.

5. Non-abelian reciprocity laws and solutions to Hilbert's twelfth problem.

5.1. Throughout this section (G, X) will satisfy the axioms for a Shimura variety and will be of abelian type in the sense of Milne-Shih [1, Section 1]. We also assume that the weight w of any $h \in X$ is defined over $\mathbf{Q}$ and that $\mathbf{ad}h(i)$ is a Cartan involution on $(G/w(\mathbf{G}_m))_{\mathbf{R}}$ (see Deligne [2, 2.1.1.4, 2.1.1.5]). These conditions ensure that, for any special $h \in X$, the cocharacter μ associated with h factors through the Serre group:

$$\mathbf{G}_m \xrightarrow{\mu_{can}} S_{\mathbf{C}} \xrightarrow{\rho_\mu} G_{\mathbf{C}} \qquad (\rho_\mu\ \mathbf{Q}\text{-rational}).$$

See Milne-Shih [2, Section 1].

5.2. Let h be a special element of X and let $T \subset G$ be a $\mathbf{Q}$-rational torus such that h factors through $T_{\mathbf{R}}$. Fix a field $E \subset \overline{\mathbf{Q}}$ of finite degree over $E(G, X)$ and let $E'(h) = E(h).E$. We define η' to be the composite

$$E'(h)^{\times} = \mathrm{Res}_{E'(h)/\mathbf{Q}}(\mathbf{G}_m) \xrightarrow{\mathrm{Res}(\mu)} \mathrm{Res}_{E'(h)/\mathbf{Q}}(T_{E(h)}) \xrightarrow{N_{E(h)/\mathbf{Q}}} T \hookrightarrow G$$

(so $\eta' = \eta$ if $E'(h) = E$; see 4.5). Let $W_{E'(h)/E}$ be the Weil group of $E'(h)$ over E (see Tate [2]). There is an exact commutative diagram

$$\begin{array}{ccccccc} 1 \to & \mathbf{A}_{E'(h)}^{\times}/E'(h)^{\times} & \longrightarrow & W_{E'(h)/E} & \longrightarrow & \mathrm{Hom}_E(E'(h), \overline{\mathbf{Q}}) \to 1 \\ & \downarrow & & \downarrow & & \| \\ 1 \to & \mathrm{Gal}(E'(h)^{\mathrm{ab}}/E'(h)) & \longrightarrow & \mathrm{Hom}_E(E'(h)^{\mathrm{ab}}, \overline{\mathbf{Q}}) & \longrightarrow & \mathrm{Hom}_E(E'(h), \overline{\mathbf{Q}}) \to 1. \end{array}$$

For $\alpha = (\alpha_\infty, \alpha_f) \in G(\mathbf{Q})$, the element $[\mathbf{ad}\alpha_\infty \circ h, \alpha_f] \in \mathrm{Sh}(G, X)$ depends only on the class of α in $G(\mathbf{Q})\backslash G(\mathbf{A})/C(\mathbf{R})$, where C is the centralizer of h in $G_{\mathbf{R}}$. Let $M(G, X)$ be a weakly canonical model for $\mathrm{Sh}(G, X)$, and identify $\mathrm{Sh}(G, X)$ with $M(G, X)_{\mathbf{C}}$. Then, as $[h, 1]$ is rational over $E(h)^{\mathrm{ab}}$, $\sigma[h, 1]$ is well-defined for $\sigma \in \mathrm{Hom}_E(E'(h)^{\mathrm{ab}}, \overline{\mathbf{Q}})$.

5.3. THEOREM. *There is a map* $\xi: W_{E'(h)/E} \to G(\mathbf{Q})\backslash G(\mathbf{A})/C(\mathbf{R})$ *with the following properties:*

(a) *for any* $\nu \in W_{E'(h)/E}$, $[\mathbf{ad}(\xi(\nu)_\infty) \circ h, \xi(\nu)_f] = [\nu^{-1}][h, 1]$, *where* $[\nu]$ *is the image of* ν *in* $\mathrm{Hom}_E(E'(h)^{\mathrm{ab}}, \overline{\mathbf{Q}})$;

(b) *the diagram*

$$\begin{array}{ccc} \mathbf{A}_{E'(h)}^{\times} & \hookrightarrow & W_{E'(h)/E} \\ \downarrow \eta' & & \downarrow \xi \\ G(\mathbf{A}^f) & \longrightarrow & G(\mathbf{Q})\backslash G(\mathbf{A})/C(\mathbf{R}) \end{array}$$

commutes;

(c) *for any* $\nu \in W_{E'(h)/E}$, *and* $\epsilon \in \pi_0(M(G, X)_{\overline{\mathbf{Q}}})$, $\xi(\nu)_f\epsilon \equiv [\nu]\epsilon$ *in* $G(\mathbf{Q})\backslash\pi_0(M(G, X)_{\overline{\mathbf{Q}}})$.

Proof. To construct ξ we shall need to use the motivic Galois group associated with the category of abelian varieties over $\mathbf{Q}$ that are of potential CM-type (see, Deligne-Milne [1]). This is an extension

$$1 \to S \to \mathbf{M} \xrightarrow{\pi} \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to 1$$

of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by the Serre group S together with a splitting $\mathbf{M}(\mathbf{A}^f) \xleftarrow{sp} \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over $\mathbf{A}^f$. For $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)$, let ${}^{\tau}S$ be the S-torsor $\pi^{-1}(\tau)$ and let ${}^{\tau}G$ be the G-torsor ${}^{\tau}S \times {}^{S}G$. According to Milne-Shih [3, 7.2, Section 10], ${}^{\tau}G$ is trivial. Choose an $a(\tau) \in {}^{\tau}S(\overline{\mathbf{Q}})$ and a $c(\tau) \in {}^{\tau}G(\mathbf{Q})$ and set

$$\rho(a(\tau)) = c(\tau)\nu, \qquad \nu \in G(\overline{\mathbf{Q}}),$$

$$\rho(sp(\tau)) = c(\tau)\alpha, \qquad \alpha \in G(\mathbf{A}^f)$$

with $\rho = \rho_\mu$ as in 5.1. Let $\omega_\tau \in G(\mathbf{C})$ normalize $T(\mathbf{C})$ and be such that $\mathbf{ad}\omega_\tau \circ \mu = \tau\mu$. Then $\nu\omega_\tau \in G(\mathbf{R})$ (ibid, 7.7, Section 10) and $\tau[h, 1] = [\mathbf{ad}(\nu\omega_\tau) \circ h, \alpha]$ (ibid, 7.11, Section 10). For $\nu \in W_{E'(h)/E}$ we choose a $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)$ whose restriction to $E'(h)^{\mathrm{ab}}$ is $[\nu^{-1}]$, and we set $\xi(\nu)$ equal to the class of $(\nu\omega_\tau, \alpha)$ in $G(\mathbf{Q})\backslash G(\mathbf{A})/C(\mathbf{R})$. Obviously $\xi(\nu)$ is well-defined, and $[\nu^{-1}][h, 1] = [\mathbf{ad}(\xi(\nu)_\infty) \circ h, \xi(\nu)_f]$.

Let $\nu \in \mathbf{A}^{\times}_{E(h)}$ and identify $\tau = [\nu^{-1}]$ with an element of $\mathrm{Gal}(E'(h)^{\mathrm{ab}}/E'(h))$. Then $\tau\mu = \mu$ because $E(h) \subset E'(h)$ is the field of definition of μ, and we can take $\omega_\tau = 1$. According to Milne-Shih [2, 3.10], ${}^{\tau}S$ is trivial, so that we can take $a(\tau) \in {}^{\tau}S(\mathbf{Q})$ and $c(\tau) \in \rho(a(\tau))$. Moreover $\alpha = \eta'(\nu_f)$, and so $\xi(\nu) \overset{df}{=} (1, \eta'(\nu_f)) \equiv (\eta'(\nu_\infty), \eta'(\nu_f))$ because $\eta'(\nu_\infty) \in C(\mathbf{R})$.

As $G(\mathbf{A}^f)$ acts transitively on $\pi_0(M(G, X)_{\overline{\mathbf{Q}}})$ and its action factors through an abelian quotient and commutes with the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$, it suffices to check (c) for a single $\epsilon \in \pi_0(M(G, X)_{\overline{\mathbf{Q}}})$. But if we take ϵ to be the class of $[h, 1]$ then the formula follows from (a).

5.4. Let $\tilde{\xi}(\nu) \in G(\mathbf{A})$ represent $\xi(\nu)$. The real approximation theorem allows us to assume $\tilde{\xi}(\nu)_\infty \in G(\mathbf{R})_+$. Part (c) of the theorem can be strengthened to read: $\tilde{\xi}(\nu)_f\epsilon = [\nu]\epsilon$. From this it follows that $\overline{\xi}(\nu)_f \overset{df}{=} \tilde{\xi}(\nu)_f * 1 \in G(\mathbf{A}^f)/Z(\mathbf{Q})^\wedge * \ldots$ lies in $\overline{\mathcal{E}}(G, X)$, and that $\sigma(\overline{\xi}(\nu)_f) = [\nu]|_{\mathfrak{f}}$.

5.5. Corollary. *Part* (c) *of Theorem* 4.6 *can be replaced by the following: let* $h \in X^+$ *be special; for each* $S \in \mathfrak{z}$, $\varphi_S(h)$ *is rational over* $E(h)^{\mathrm{ab}}$ *and for every* $\nu \in W_{k_S E(h)/k_S}$

$$[\nu]\varphi_S(\mathbf{ad}\tilde{\xi}(\nu)_\infty \circ h) = J_{ST}(\bar{\xi}(\nu)_f)(\varphi_T(h))$$

where $T = \bar{\xi}(\nu)_f{}^{-1}S\bar{\xi}(\nu)_f$ *and* $\tilde{\xi}(\nu)_\infty$ *and* $\bar{\xi}(\nu)_f$ *correspond to some lifting* $\tilde{\xi}(\nu)$ *of* $\xi(\nu)$ *with* $\tilde{\xi}(\nu)_\infty \in G(\mathbf{R})_+$.

Proof. We have $\bar{\xi}(\nu)_f[h, 1] \overset{df}{=} \sigma(\bar{\xi}(\nu)_f)[h, \bar{\xi}(\nu)_f{}^{-1}]$, and $\sigma(\bar{\xi}(\nu)_f) = [\nu]$ and $[\nu][h, 1] = [\mathbf{ad}\xi(\nu)_\infty \circ h, \xi(\nu)_f]$. Thus $\bar{\xi}(\nu)_f{}^{-1}[\mathbf{ad}\xi(\nu)_\infty \circ h, 1] = [h, 1]$ and so $J(\bar{\xi}(\nu)_f)[h, 1] = [\nu][\mathbf{ad}\xi(\nu)_\infty \circ h, 1]$, which implies the formula.

5.6. COROLLARY. *Let* $z \in X^+$ *be special and let* $f \in F'$ *be defined at* z*; then for any* $\nu \in W_{E(h)/E}$, ${}^u f$, *where* $u = \bar{\xi}(\nu)_f{}^{-1}$, *is defined at* z *and* ${}^u f(z) = [\nu]f(\mathbf{ad}\tilde{\xi}(\nu)_\infty \circ z)$.

Proof. ${}^u f(z) = [\nu] \circ f \circ \bar{\xi}(\nu)_f(z) = [\nu]f(\mathbf{ad}\tilde{\xi}(\nu)_\infty \circ z)$ because $\bar{\xi}(\nu)_f[z, 1] = [\mathbf{ad}\tilde{\xi}(\nu)_\infty \circ z, 1]$ (see the above proof).

5.7. The last corollary can be regarded as a non-abelian solution to Hilbert's twelfth problem: it describes the action of the Weil group on the special values of certain functions and therefore determines the fields they generate; this field is an abelian extension of a finite extension of the base field.

5.8. We shall prove a non-abelian version of the reciprocity law stated in Deligne [2, 2.7.10c]. Let $h \in X$ be special and let $\rho = \rho_\mu$ be the map $S \to G$ associated to h as in 5.1. Let ${}_E\mathbf{M}$ be the pullback of the motivic Galois group relative to $\mathrm{Gal}(\overline{\mathbf{Q}}/E) \hookrightarrow \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Then we have an extension

$$1 \to S \to {}_E\mathbf{M} \overset{\pi}{\to} \mathrm{Gal}(\overline{\mathbf{Q}}/E) \to 1$$

together with a splitting ${}_E\mathbf{M}(\mathbf{A}^f) \overset{sp}{\leftarrow} \mathrm{Gal}(\overline{\mathbf{Q}}/E)$. Write $\mathbf{M}$ for ${}_E\mathbf{M}$ and let S act on $\mathbf{M} \times G$ by $s(m, g) = (ms^{-1}, \rho(s)g)$ and define $\mathbf{M}G$ to be the quotient scheme $S\backslash\mathbf{M} \times G$. The splitting sp allows us to define a map $\mathbf{M}G(\mathbf{A}^f) \to G(\mathbf{A}^f) \times \mathrm{Gal}(\mathbf{Q}/E)$: to $[m, g]$ we associate $(\rho(sp(\tau)^{-1}m)g, \tau)$, where τ is the image of m in $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$. If $q \in \mathbf{M}G(\mathbf{Q})$ maps to $(\alpha, \tau) \in G(\mathbf{A}^f) \times \mathrm{Gal}(\overline{\mathbf{Q}}/E)$ we define $q[h, 1] = \tau[h, \alpha]$.

5.9. PROPOSITION. *Let* $h \in X$ *be special, and let* $q = [m, g] \in \mathbf{M}G(\mathbf{Q})$. *If* τ *is the image of* m *in* $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$, *then* ${}^q h \overset{df}{=} \mathbf{ad}g^{-1} \circ {}^\tau h$ *is in* X, *and* $q[h, 1] = [{}^q h, 1]$.

Proof. Let L be some large finite Galois extension of $\mathbf{Q}$ containing $E(h)$. Using that the motivic Galois group, and therefore ${}_E\mathbf{M}$, is a pro-system, we have an extension

$$1 \to S^L \to {}_E\mathbf{M}^L \to \mathrm{Gal}(L^{\mathrm{ab}}/E) \to 1$$

with a splitting ${}_E\mathbf{M}^L(A^f) \overset{sp}{\leftarrow} \mathrm{Gal}(L^{\mathrm{ab}}/E)$, see Milne-Shih [2, Section 2]. Replace ${}_E\mathbf{M}$ in the above discussion by its quotient ${}_E\mathbf{M}^L$. Choose a section $\tau \mapsto a(\tau)$ to ${}_E\mathbf{M}^L(L) \to \mathrm{Gal}(L^{\mathrm{ab}}/E)$ that is a morphism of pro-algebraic schemes. After possibly multiplying g by an element of $S^L(L)$, we can assume $q = [a(\tau), g]$. Let $\sigma(a(\tau)) = a(\tau)\gamma_\sigma(\tau)$ with $\gamma_\sigma(\tau) \in S^L(L)$; since $\sigma g = [\sigma a(\tau), \sigma g] = q$, we have $\sigma g = \rho(\gamma_\sigma(\tau))g$. The formula of Milne-Shih [3, 7.13, Section 10] states that $\tau[h, 1] = [\mathbf{ad}\nu \circ {}^\tau h, \nu\rho(sp(\tau)^{-1}a(\tau))^{-1}]$, where ν is any element of $G(L)$ such that $\sigma\nu = \nu\rho(\gamma_\sigma(\tau))$. Clearly g^{-1} is such an element, and so $\tau[h, 1] = [{}^q h, g^{-1}\rho(sp(\tau)^{-1}a(\tau))^{-1}]$. On multiplying on the right with $\rho(sp(\tau)^{-1}a(\tau))g$ we obtain the formula $q[h, 1] = [{}^q h, 1]$.

5.10. If one knew the conjecture of Langlands [1, p. 232-33] (equivalently, conjecture *CM* or *D* of Milne-Shih [3]) it would be possible to replace the motivic Galois group by the Taniyama group. The above results would then be more explicit.

5.11. Let N be the normalizer of T in G. It would be interesting to know under what conditions every element $\xi(\nu)$ is represented by an element of $N(\mathbf{A})$ and further that there exists a commutative diagram

$$\begin{array}{ccccccc}
1 \to & E(h)^\times \backslash \mathbf{A}^\times_{E(h)} & \longrightarrow & W_{E(h)/E} & \longrightarrow & \mathrm{Hom}_E(E(h), \overline{\mathbf{Q}}) & \to 1 \\
& \downarrow \eta & & \downarrow \xi & & \| & \\
1 \to & T(\mathbf{Q})\backslash T(\mathbf{A}) & \longrightarrow & N(\mathbf{Q})\backslash N'/N' \cap C(\mathbf{R}) & \longrightarrow & \mathrm{Hom}_E(E(h), \overline{\mathbf{Q}}) & \to 1
\end{array}$$

with $N' \subset N(A)$. The existence of such a diagram would provide a partial answer to the problem mentioned in Tate [1, p. 200].

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