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# On a conjecture of Artin and Tate

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## Introduction

Let $X$ be a projective smooth surface over the finite field $k$ of $q = p^a$ elements. It is known [6] that the zeta function of $X$ has the form

$$Z(X, t) = \frac{P_1(X, t)P_3(X, t)}{(1 - t)P_2(X, t)(1 - q^2t)}$$

where $P_r(X, t)$ is a polynomial with integer coefficients whose reciprocal roots have absolute value $q^{r/2}$. Let $\rho(X)$ be the rank of the Néron-Severi group $\mathrm{NS}(X)$ of $X$. Tate [19, p. 104] has conjectured the following:

(T) $\rho(X)$ is equal to the multiplicity of $q$ as a reciprocal root of $P_2(t)$.

The following refinement of (T) has been conjectured by Artin and Tate [20, (C)]:

(A-T) the Brauer group $\mathrm{Br}(X)$ of $X$ is finite and

$$P_2(X, q^{-s}) \sim \frac{(-1)^{\rho(X)-1}[\mathrm{Br}(X)] \det (D_i \cdot D_j)}{q^{\alpha(X)}(\mathrm{NS}(X): B)^2}(1 - q^{1-s})^{\rho(X)} \text{ as } s \to 1\ ,$$

where $[S]$ denotes the order of a set $S$, $\alpha(X) = \chi(X, O_X) - 1 + \dim(\mathrm{Pic\,Var}(X))$, $D_1, \cdots, D_\rho$ are independent elements of $\mathrm{NS}(X)$ and $B = \sum \mathbf{Z}D_i$ is the subgroup of $\mathrm{NS}(X)$ generated by the $D_i$.

In [20] it is shown that if (T) is assumed to hold for $X/k$, then it follows that $\mathrm{Br}(X)$ (non-$p$), the subgroup of $\mathrm{Br}(X)$ of elements of order prime to $p$, is finite and has the order predicted by (A-T). The proof of this makes heavy use of the known properties of $l$-adic étale cohomology. In attempting to extend this result to include the $p$-part, three essential difficulties arise. Firstly, the different components of the formula in (A-T) are described by two different cohomology theories; viz., $\mathrm{Br}(X)$ and $\mathrm{NS}(X)$ are described by the flat cohomology and $P_2(t)$ by the crystalline cohomology. Secondly, neither the flat nor the crystalline cohomology is as well understood as the étale cohomology. Thirdly, the $p$-part of (A-T) seems genuinely to have

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more content than the prime-to-$p$ part; for example, the term $q^{\alpha(X)}$ plays no role in the prime-to-$p$ case and the roots of $P_2(t)$ are $l$-adic units for all primes $l \neq p$.

In this paper we prove that (T) implies the whole of (A-T) when $p \neq 2$. In an earlier version of this paper we followed a program suggested by Grothendieck [9, p. 316] to relate flat and crystalline cohomology; viz., we defined flat homology groups and then showed how their Dieudonné modules were related to the crystalline cohomology groups. While the flat homology groups are undoubtedly essential for a full understanding of the relation between flat and crystalline cohomology, we have been able to simplify this paper by removing them except for one reference.

In Section 1 we review various results on the cohomologies of varieties in characteristic $p$, and in particular the results of Artin and Mazur, and Bloch comparing various of the $p$-adic cohomologies. In Section 2 we quote a Poincaré duality theorem for the flat cohomology of a surface and apply it to show that the Brauer group is self-dual. The next five sections demonstrate that, once one has the results quoted in the first two sections, Tate's paper [20] may be rewritten to include the $p$-part. We have tried to give a proof which simultaneously covers the $p$-case and the prime-to-$p$ case but there remain considerable, and interesting, differences between the two cases. In Section 8 we re-interpret our results in terms of the function field analogues of the conjectures of Birch and Swinnerton-Dyer.

In [20, p. 212] Tate suggested that the proving of the analogues for $l = p$ of his theorems "should furnish a good test for any $p$-adic cohomology theory, and might well serve as a guide for sorting out and unifying the various constructions which have been suggested and used". On this, we note that our proof uses flat cohomology, crystalline cohomology, and Witt vector cohomology, and that the close relation between flat cohomology and crystalline cohomology is becoming increasingly apparent.

In [12, 13], [14], and [1] it was shown that (T) implies (A-T) for products of curves, rational surfaces, and elliptic supersingular K3 surfaces respectively. (In each of these cases, one even knows that (T) is true.) In [12, 13] the explicit use of crystalline cohomology could be avoided, essentially because the crystalline cohomology groups of a curve are obvious, and then those for a product of curves can be deduced by assuming the Kunneth formula. With hindsight, one can see that this is what is done in [12, 13]. The use of crystalline cohomology could be avoided in [14] and [1] because of the special nature of the surfaces considered.

To convince the reader that this paper is non-vacuous, we list the surfaces for which Tate's conjecture (T) is known: (i) rational surfaces (trivial); (ii) certain hypersurfaces $X_0^n + X_1^n + X_2^n + X_3^n = 0$ (Tate [19]); (iii) abelian surfaces (Tate [21]); (iv) a product of two curves (Tate [21]); (v) any K3 surface with a pencil of elliptic curves (Artin and Swinnerton-Dyer [3]); (vi) any surface birationally equivalent to a surface for which (T) holds (trivial); (vii) any surface $X$ for which there exists a map $Y \to X$ of finite degree with $Y$ a surface for which (T) holds (trivial). Thus, for example, (T) also holds for unirational surfaces and Kummer surfaces. It is known for surfaces which lift to a surface in characteristic zero with $p_g = 0$.

## Notation

$X$ is always a variety over a perfect field $k$ of characteristic $p \neq 0$, $\bar{k}$ is the algebraic closure of $k$, $\Gamma$ the Galois group of $\bar{k}$ over $k$, and $\bar{X} = X \otimes_k \bar{k}$. All cohomology groups are relative to the flat (f. p. p. f.) topology unless otherwise stated. $\mu_n$ is the sheaf of $n^{\text{th}}$ roots of unity and $\mu(\infty) = \varinjlim \mu_n$, the limit being taken over all positive integers $n$. Then $H^r(X, \mu(\infty)) = \varinjlim H^r(X, \mu_n)$ and we define

$$H^r(X, T\mu) = \varprojlim H^r(X, \mu_n) .$$

$W_k$ is the ring of Witt vectors over $k$, $K_k$ the field of fractions of $W_k$, and $a \mapsto a^{(p)}$ the Frobenius automorphism of $W_k$. An $F$-isocrystal over $k$ is a finite-dimensional vector space $V$ over $K_k$ together with a bijective additive homomorphism $F$: $V \to V$ satisfying $F(av) = a^{(p)}F(v)$ for $a \in K_k$, $v \in V$. If $V$ is an $F$-isocrystal then $V(n)$ is the $F$-isocrystal which, as a vector space is $V$, but on which $F$ acts by $p^{-n}F$. Also we write $V^F = \{v \in V \mid F(v) = v\}$.

For an abelian group $A$, $A_n$ and $A^{(n)}$ are respectively the kernel and cokernel of multiplication by $n$ on $A$, $A_{\text{tors}} = \varinjlim A_n$, and $TA = \varprojlim A_n = \text{Hom}(\mathbf{Q}/\mathbf{Z}, A)$. Also, for a prime $l$, we write $A(l) = \varinjlim A_{l^m}$, $T_l A = \varprojlim A_{l^m}$, and $V_l A = T_l A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$. Note that $TA$ and $T_l A$ are torsion-free.

In Sections 2 to 6, $X$ is a smooth projective surface over a finite field with $q$ elements. $\sigma$ then denotes the canonical generator of $\Gamma$ and if $A$ is a $\Gamma$-module then $A^\Gamma$ and $A_\Gamma$ are respectively the kernel and cokernel of $\sigma - 1$: $A \to A$. Note that, if $A$ is torsion, then $A^\Gamma$ and $A_\Gamma$ are the Galois cohomology groups $H^0(\Gamma, A)$ and $H^1(\Gamma, A)$.

## 1. Review of cohomology in characteristic $p$

Throughout this section, $X$ will be a smooth projective variety of dimension $d$ over a finite field $k$ of $q$ elements.

(1.1) *l-adic étale cohomology,* $l \neq p$ [10, 6]. The étale cohomology groups $H^r(\bar{X}, \mathbf{Q}_l)$ are finite-dimensional vector spaces over $\mathbf{Q}_l$ and are zero except for $0 \leqq r \leqq 2d$. Let $\Phi: \bar{X} \to \bar{X}$ be the Frobenius endomorphism, i.e., the endomorphism of $\bar{X}/\bar{k}$ which raises the coordinates of points to their $q^{\text{th}}$ power, and let $P_r(X, t) = \det(1 - \Phi t \mid H^r(\bar{X}, \mathbf{Q}_l)) = \prod_{i=1}^{b_r}(1 - a_{r,i}t)$. Then $P_r \in \mathbf{Z}[t]$ is independent of $l$, and $Z(X, t) = \prod_{r=0}^{2d} P_r(X, t)^{(-1)^{r+1}}$.

If $\sigma$ is the canonical generator of $\Gamma = \mathrm{Gal}(\bar{k}/k)$ then $\det(1 - \sigma t \mid H^r(\bar{X}, T_l\mu^{\otimes j}) \otimes \mathbf{Q}) = \prod_{i=1}^{b_r}(1 - q^j/a_{r,i})t$. In the case that $d = 2$, $j = 1$, $r = 2$, Poincaré duality allows this last equation to be written

$$\det(1 - \sigma t \mid H^2(\bar{X}, T_l\mu) \otimes \mathbf{Q}) = \prod_{i=1}^{b_2}\left(1 - \frac{a_{2,i}}{q}t\right).$$

(1.2) *Crystalline cohomology* [5, 11]. Let $H^r(X/W)$ and $H^r(\bar{X}/\bar{W})$ be the cohomology groups of the crystalline sites $X/W$ and $\bar{X}/\bar{W}$, where $W = W_k$, $\bar{W} = W_{\bar{k}}$. Then $H^r(X/W)_K = H^r(X/W) \otimes_W K$ is a finite-dimensional vector space over $K$ and is zero except for $0 \leqq r \leqq 2d$. Moreover $H^r(\bar{X}/\bar{W}) = H^r(X/W)_K \otimes_K \bar{K}$, where $\bar{K} = K_{\bar{k}}$. The absolute Frobenius on $X$ induces $(p)$-linear injective maps $F: H^r(X/W)_K \to H^r(X/W)_K$ and so $H^r(X/W)_K$ and $H^r(\bar{X}/\bar{W})_{\bar{K}}$ are isocrystals.

If $\Phi$ is the Frobenius endomorphism of $X/k$, then $\det(1 - \Phi t \mid H^r(X/W)_K) = P_r(X, t)$. It follows that the slopes of $H^r(\bar{X}/\bar{W})_{\bar{K}}$ are exactly the numbers $v(a_{r,1}), \cdots, v(a_{r,b_r})$ where $v$ is the $p$-adic valuation such that $v(q) = 1$ [7, p. 90].

(1.3) *Witt vector cohomology* [18, 4, 2]. Write $H^r(\bar{X}, W) = \varprojlim H^r(\bar{X}, W_n)$ and $H^r(\bar{X}, W)_{\bar{K}} = H^r(\bar{X}, W) \otimes \bar{K}$. Assume either that $p > d$ or that the assumptions of [2] apply. Then $H^r(\bar{X}, W)_{\bar{K}}$ is zero except for $0 \leqq r \leqq d$ and is isomorphic as an $F$-isocrystal to $(H^r(\bar{X}/\bar{W})_{\bar{K}})_{[0,1)}$, the part of the crystalline cohomology of slope $\lambda$ with $0 \leqq \lambda < 1$.

Write

$$H^r(\bar{X}, W)_t = \mathrm{Ker}\left(H^r(\bar{X}, W) \longrightarrow H^r(\bar{X}, W)_{\bar{K}}\right),$$

i.e., $H^r(\bar{X}, W)_t$ is the $p$-torsion in $H^r(\bar{X}, W)$. As $H^r(\bar{X}, W)$ is finitely generated over $\bar{W}[[V]]$ (see [4]), $H^r(\bar{X}, W)_t \otimes \bar{W}((V))$ has finite length over $\bar{W}((V))$. In the notation of (1.4) below, this length is equal to $\dim U^2(p^\infty)$ for $r = 2$. (See [2] under certain assumptions, and [15] in general.)

(1.4) *Flat cohomology* [4, 15]. Assume $p > d$. The flat cohomology group $H^r(\bar{X}, V_p\mu) = (\varprojlim H^r(\bar{X}, \mu_{p^r})) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is isomorphic to $H^r(\bar{X}/\bar{W})_{\bar{K}}(1)^F$. The corresponding statement over $k$ is also true. It follows that

$$\det(1-\sigma t \mid H^r(\bar{X}, V_p\mu)) = \prod_{v(a_{r,i})=1}\left(1-\frac{q}{a_{r,i}}t\right).$$

In the case that $r = 2 = d$ this may be written

$$\det(1-\sigma t \mid H^2(\bar{X}, V_p\mu)) = \prod_{v(a_{2,i})=1}\left(1-\frac{a_{2,i}}{q}t\right), \qquad (p \neq 2).$$

Let $X$ be a surface. For each $r$ and $n$ there is a unipotent connected quasi-algebraic group $U^r(p^n)$ and an étale group scheme $D^r(p^n)$ such that there are exact sequences $0 \to U^r(p^n)(S) \to \underline{H}^r(X_S, \mu_{p^n}) \to D^r(p^n)(S) \to 0$ for any affine perfect $k$-scheme $S$([1], [16]). The groups $U^r$ are zero except for $r = 2, 3$, and for large $n$, $U^r(p^n)$ is equal to a group scheme $U^r(p^\infty)$ which is independent of $n$. The dimensions of $U^2(p^\infty)$ and $U^3(p^\infty)$ are equal.

## 2. The auto-duality of Br($X$)

First we review what is known on the Poincaré duality of a surface.

THEOREM 2.1. *Let $X$ be a projective smooth surface over a finite field $k$ of characteristic $p \neq 2$.*

(a) *For all $n$, the groups $H^r(X, \mu_n)$ are finite.*

(b) *There is a canonical non-degenerate pairing*

$$\langle\ ,\ \rangle: H^r(X, \mu_n) \times H^{5-r}(X, \mu_n) \longrightarrow \mathbf{Z}/n\mathbf{Z}.$$

(c) *The pairings in* (b) *are compatible for varying $n$, and with intersection products of divisors on $X$.*

(d) *If $\delta_n: H^r(X, \mu_n) \to H^{r+1}(X, \mu_n)$ is the boundary map arising from the exact sequence*

$$0 \longrightarrow \mu_n \longrightarrow \mu_{n^2} \longrightarrow \mu_n \longrightarrow 0$$

*then*

$$\langle y, \delta_n x\rangle + (-1)^r\langle x, \delta_n y\rangle = 0, \quad x \in H^r(X, \mu_n), \quad y \in H^{4-r}(X, \mu_n).$$

*Proof.* If $p \nmid n$ then $H^r(X, \mu_n)$ may be interpreted as an étale cohomology group, and the proofs of (a), (b), (c) are to be found in [10] and [22]. There is a canonical trace map $H^5(X, \mu_n^{\otimes 2}) \approx \mathbf{Z}/n\mathbf{Z}$ $(p \nmid n)$ and the pairing $\langle\ ,\ \rangle$ may be interpreted as the cup-product on Čech cohomology arising from the pairing $\mu_n \times \mu_n \to \mu_n^{\otimes 2}$. There is a standard formula for cup-products

$$\langle \delta_n x, y\rangle + (-1)^r\langle x, \delta_n y\rangle = \delta_n\langle x, y\rangle,$$

and $\delta_n\langle x, y\rangle = 0$ because the map $H^5(X, \mu_n^{\otimes 2}) \to H^5(X, \mu_{n^2}^{\otimes 2})$ is injective.

If $n = p^m$, then $H^r(X, \mu_n)$ may be identified with $H^{r-1}(X_{\mathrm{et}}, O_X^*/O_X^{*n})$. There is a sheaf $\nu_n(2)$ on $X_{\mathrm{et}}$ and a trace map $H^3(X_{\mathrm{et}}, \nu_n(2)) \cong \mathbf{Z}/n\mathbf{Z}$. Also there is a canonical pairing $O_X^*/O_X^{*n} \times O_X^*/O_X^{*n} \to \nu_n(2)$ and it is shown in [16] that the cup-product pairing

$$H^r(X, O_X^*/O_X^{*n}) \times H^{3-r}(X, O_X^*/O_X^{*n}) \longrightarrow \mathbf{Z}/n\mathbf{Z}$$

has all the properties required for the theorem.

*Remark* 2.2. The condition that $p \neq 2$ is only needed for (b), (c), (d) of the theorem, and only in the case that $p^2 \mid n$.

The following easy lemma will also be needed.

LEMMA 2.3. *Let $M$ be a discrete torsion abelian group, let $N$ be a profinite abelian group, and let $M \times N \to \mathbf{Q}/\mathbf{Z}$ be a continuous non-degenerate pairing. The exact annihilator of $N_{\text{tors}}$ is $M_{\text{div}}$, the group of divisible elements of $M$. Thus there is a non-degenerate pairing*

$$M/M_{\text{div}} \times N_{\text{tors}} \longrightarrow \mathbf{Q}/\mathbf{Z}\ .$$

THEOREM 2.4. *Let $X$ be as in* (2.1). *There is a skew-symmetric biadditive pairing*

$$\mathrm{Br}(X) \times \mathrm{Br}(X) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

*whose kernel consists exactly of the divisible elements of* $\mathrm{Br}(X)$.

*Proof.* From the Kummer sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \xrightarrow{\ n\ } \mathbf{G}_m \longrightarrow 0$$

we get a cohomology sequence,

$$\begin{aligned} 0 \longrightarrow \mathrm{Pic}\,(X)^{(n)} \longrightarrow H^2(X, \mu_n) \longrightarrow \mathrm{Br}(X) \xrightarrow{\ n\ } \mathrm{Br}(X) \\ \longrightarrow H^3(X, \mu_n) \longrightarrow H^3(X, G_m)_n \longrightarrow 0\ . \end{aligned}$$

On passing to the direct and inverse limits, this gives

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbf{Q}/\mathbf{Z} \longrightarrow H^2(X, \mu(\infty)) \longrightarrow \mathrm{Br}(X) \longrightarrow 0$$

and

$$0 \longrightarrow \varprojlim \mathrm{Br}(X)^{(n)} \longrightarrow H^3(X, T\mu) \longrightarrow TH^3(X, \mathbf{G}_m) \longrightarrow 0\ .$$

The pairings in (2.1) induce a non-degenerate pairing $H^2(X, \mu(\infty)) \times H^3(X, T\mu) \to \mathbf{Q}/\mathbf{Z}$, and the theorem follows from (2.3) by taking $M = H^2(X, \mu(\infty))$ and $N = H^3(X, T\mu)$. Indeed, $\mathrm{NS}(X) \otimes \mathbf{Q}/\mathbf{Z}$ is divisible and so $M/M_{\text{div}} \approx \mathrm{Br}(X)/\mathrm{Br}(X)_{\text{div}}$. Also $TH^3(X, \mathbf{G}_m)$ is torsion-free, and so $N_{\text{tors}} = (\varprojlim \mathrm{Br}(X)^{(n)})_{\text{tors}} = \mathrm{Br}(X)/\mathrm{Br}(X)_{\text{div}}$.

Let $x$ and $y$ be in $\mathrm{Br}(X)_n$. On unscrambling the above, one finds that under the pairing just defined, $(x, y)$ is mapped to $\langle x', \delta_n y' \rangle$ where $x'$ and $y'$ are elements of $H^2(X, \mu_n)$ mapping to $x$ and $y$ respectively. Thus the skew-symmetry follows from (2.1d).

*Remark* 2.5. It follows that the order of $\mathrm{Br}(X)$, if finite, is either a square or twice a square (without restriction on the characteristic).

### 3. Diagrams

We collect here some exact commutative diagrams which will be needed later.

The exact sequence of sheaves

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \xrightarrow{\ n\ } \mathbf{G}_m \longrightarrow 0$$

yields exact cohomology sequences over $X$ and $\bar{X}$ respectively:

$$\begin{aligned} k^* \xrightarrow{\ n\ } k^* \longrightarrow H^1(X, \mu_n) \longrightarrow \mathrm{Pic}\,(X) \xrightarrow{\ n\ } \mathrm{Pic}\,(X) \\ \longrightarrow H^2(X, \mu_n) \longrightarrow \mathrm{Br}(X)_n \longrightarrow 0\,, \\ \bar{k}^* \xrightarrow{\ n\ } \bar{k}^* \xrightarrow{\ 0\ } H^1(\bar{X}, \mu_n) \longrightarrow \mathrm{Pic}\,(\bar{X}) \xrightarrow{\ n\ } \mathrm{Pic}\,(\bar{X}) \\ \longrightarrow H^2(\bar{X}, \mu_n) \longrightarrow \mathrm{Br}(\bar{X})_n \longrightarrow 0\,. \end{aligned}$$

The Hochschild-Serre spectral sequence for $\bar{X}/X$,

$$H^r(\Gamma, H^s(\bar{X}, \mu_n)) \Longrightarrow H^{r+s}(X, \mu_n)$$

reduces to short exact sequences

$$0 \longrightarrow H^{r-1}(\bar{X}, \mu_n)_\Gamma \longrightarrow H^r(X, \mu_n) \longrightarrow H^r(\bar{X}, \mu_n)^\Gamma \longrightarrow 0\,.$$

Out of these exact sequences, we get an exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & & \mathrm{Pic}\,(X)^{(n)} & \longrightarrow & (\mathrm{NS}(\bar{X})^{(n)})^\Gamma & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(\bar{X}, \mu_n)_\Gamma & \longrightarrow & H^2(X, \mu_n) & \longrightarrow & H^2(\bar{X}, \mu_n)^\Gamma & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & (\mathrm{Pic}\,(\bar{X})_n)_\Gamma & & \mathrm{Br}(X)_n & \longrightarrow & (\mathrm{Br}(\bar{X})_n)^\Gamma & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.1}$$

The columns come from the first two sequences and the rows from the third. Note that $\mathrm{Pic}\,(\bar{X})^{(n)} \approx \mathrm{NS}(\bar{X})^{(n)}$ because $\mathrm{Pic}^0\,(\bar{X})$ is divisible, and $H^1(\bar{X}, \mu_n) \approx \mathrm{Pic}\,(\bar{X})_n$ because $\bar{k}^*$ is divisible. All groups in the diagram are finite because $H^2(X, \mu_n)$ is finite.

On passing to the direct limit over positive integers $n$ we get an exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathrm{NS}(X)\otimes \mathbf{Q}/\mathbf{Z} & \longrightarrow & (\mathrm{NS}(\bar{X})\otimes \mathbf{Q}/\mathbf{Z})^{\Gamma} & & \\
 & & \downarrow & & \downarrow g & & \downarrow & & \\
(3.2)\quad 0 & \longrightarrow & H^1(\bar{X}, \mu(\infty))_{\Gamma} & \longrightarrow & H^2(X, \mu(\infty)) & \longrightarrow & H^2(\bar{X}, \mu(\infty))^{\Gamma} & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & & & \\
 & & (\mathrm{NS}(\bar{X})_{\mathrm{tors}})_{\Gamma} & & \mathrm{Br}(X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$\mathrm{NS}(X)$ replaces $\mathrm{Pic}\,(X)$ in upper-centre because $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}\,(X) \to \mathrm{NS}(X) \to 0$ is exact, and $\mathrm{Pic}^0(X)\otimes \mathbf{Q}/\mathbf{Z} = 0$ because $\mathrm{Pic}^0(X)$ is finite. $\mathrm{NS}(\bar{X})_{\mathrm{tors}}$ replaces $\mathrm{Pic}\,(\bar{X})_{\mathrm{tors}}$ in the lower-left because $\mathrm{Pic}^0(\bar{X})_{\Gamma} = H^1(\Gamma, \mathrm{Pic}^0(\bar{X})) = 0$ (Lang's theorem). We get all of $\mathrm{Br}(X)$ in lower-centre because $\mathrm{Br}(X)$ is a torsion group [8].

On passing to the inverse limit in (3.1), we get an exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathrm{Pic}\,(X)\otimes \hat{\mathbf{Z}} & \longrightarrow & \mathrm{NS}(X)\otimes \hat{\mathbf{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow h & & \downarrow \bar{h} & & \\
(3.3)\quad 0 & \longrightarrow & H^1(\bar{X}, T\mu)_{\Gamma} & \longrightarrow & H^2(X, T\mu) & \longrightarrow & H^2(\bar{X}, T\mu)^{\Gamma} & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Pic}\,(X)_{\mathrm{tors}} & & T(\mathrm{Br}(X)) & = & T(\mathrm{Br}(X)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The diagram stays exact because inverse limits preserve the exactness of sequences of finite groups. $\mathrm{NS}(X)\otimes\hat{\mathbf{Z}}$ occurs in the top right because $\mathrm{NS}(X)$ is defined to be the image of $\mathrm{Pic}\,(X)$ in $\mathrm{NS}(\bar{X})$, and the map $\mathrm{Pic}\,(X) \to \mathrm{Pic}\,(\bar{X})^{\Gamma} \to \mathrm{NS}(\bar{X})^{\Gamma}$ is surjective. To see that $\varprojlim (\mathrm{Pic}\,(\bar{X})_n)_{\Gamma} = \mathrm{Pic}\,(X)_{\mathrm{tors}}$, pass to the inverse limit in the exact sequence

$$(\mathrm{Pic}\,(\bar{X})_{\mathrm{tors}})^{\Gamma} \xrightarrow{\;n\;} (\mathrm{Pic}\,(\bar{X})_{\mathrm{tors}})^{\Gamma} \longrightarrow (\mathrm{Pic}\,(\bar{X})_n)_{\Gamma} \longrightarrow ((\mathrm{Pic}\,(\bar{X})_{\mathrm{tors}})_{\Gamma}\ ,$$

and note that $(\mathrm{Pic}(\bar{X})_{\mathrm{tors}})^{\Gamma} = \mathrm{Pic}(X)_{\mathrm{tors}}$ and that $(\mathrm{Pic}(\bar{X})_{\mathrm{tors}})_{\Gamma}$ is finite. Finally, the fact that $H^1(\bar{X}, T\mu)_{\Gamma}$ is finite implies that $h$ and $\bar{h}$ have the same cokernel.

## 4. Equivalent forms of Tate's conjecture

THEOREM 4.1. *Let $X$ be a smooth projective surface over a finite field $k$ of characteristic $\neq 2$. The following statements are equivalent:*

(a) (T) *holds for* $X/k$.

(b) *There is a prime $l$ ($l = p$ is allowed) such that* $\mathrm{Br}(X)(l)$ *is finite.*

(b′) *For all primes $l$,* $\mathrm{Br}(X)(l)$ *is finite.*

(c) *There is a prime $l$ such that* $\rho(X) = \mathrm{rank}_{\mathbf{Z}_l}(H^2(\bar{X}, T_l\mu)^{\Gamma})$.

(c′) *For all primes $l$,* $\rho(X) = \mathrm{rank}_{\mathbf{Z}_l}(H^2(\bar{X}, T_l\mu)^{\Gamma})$.

(d) *There is a prime $l$ such that* $\bar{h}_l\colon NS(X) \otimes \mathbf{Z}_l \to H^2(\bar{X}, T_l\mu)^{\Gamma}$ *is bijective* (*See* (3.3) *for the map* $\bar{h}$).

(d′) *The map* $\bar{h}\colon \mathrm{NS}(X) \otimes \mathbf{Z} \to H^2(\bar{X}, T\mu)^{\Gamma}$ *is bijective.*

(e) $\rho(X) = \dim_{\mathbf{Q}_p}(H^2(X/W_k)_K(1)^F)$.

*Proof.* There is, for any prime $l$, an exact sequence (3.3)

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbf{Z}_l \xrightarrow{\bar{h}_l} H^2(\bar{X}, T_l\mu)^{\Gamma} \longrightarrow T_l(\mathrm{Br}(X)) \longrightarrow 0\ .$$

Also, $H^2(X, \mu_l)$ is finite (2.1) which implies that $\mathrm{Br}(X)_l$ is finite, which in turn implies that $\mathrm{Br}(X)(l)$ is finite if and only if $T_l\,\mathrm{Br}(X) = 0$. From this follows immediately the equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) and also (b′) $\Leftrightarrow$ (c′) $\Leftrightarrow$ (d′).

The equivalence of (e) to (c) with $l = p$ (i.e., the implications (c′) $\Rightarrow$ (e) $\Rightarrow$ (c)) follows from (1.4) and (3.3).

Now let $\rho'(X)$ be the multiplicity of $q$ as a reciprocal root of $P_2(X, t)$. From Section 1 it is clear that, for any prime $l$, $\rho'(X)$ is also equal to the multiplicity of 1 as an eigenvalue for $\sigma$ acting on $H^2(X, T_l\mu)$. Since the map $\bar{h}_l$ in (d) is always injective, we get the following inequalities holding for any $l$,

$$\rho(X) \leqq \mathrm{rank}_{\mathbf{Z}_l}(H^2(\bar{X}, T_l\mu)^{\Gamma}) \leqq \rho'(X)\ .$$

Thus if (a) holds, i.e., $\rho(X) = \rho'(X)$, then both inequalities are equalities, and (c′) holds.

To complete the proof of the theorem we have only to prove that, if any one of (b), (c) or (d) holds, then (a) holds. This is done in the next section.

*Remark* 4.2. It is conjectured that the Frobenius endomorphism $\Phi$ acts semi-simply on the groups $H^r(\bar{X}, \mathbf{Q}_l)$. If this is assumed, then automatically

$\rho'(X) = \text{rank}_{\mathbf{Z}_l}(H^2(\bar{X}, T_l\mu)^\Gamma)$ and (c) implies (a).

## 5. Another diagram

Let $\psi: A \to B$ be a homomorphism of $\hat{\mathbf{Z}}$-modules. Such a map breaks up into a product of homomorphisms $\psi_l: A_l \to B_l$ where $A_l$ is the $\mathbf{Z}_l$-module $A \otimes_{\mathbf{Z}} \mathbf{Z}_l$ etc. If the kernel and cokernel of $\psi_l$ are finite then we say that $\psi$ is an $l$-quasi-isomorphism and write $z_l(\psi) = ([\ker \psi_l])/([\text{coker}\, \psi_l])$. If $\psi$ is an $l$-quasi-isomorphism for all $l$, we call it a quasi-isomorphism and put $z(\psi)$ equal to the formal product $\prod_{l \text{ prime}} z_l(\psi)$ ("un nombre surnaturel"). For example, if the kernel and cokernel of $\psi$ are both finite then $\psi$ is a quasi-isomorphism and $z(\psi) = ([\ker \psi])/([\text{coker}\, \psi]) \in \mathbf{Q}$. These notations are similar to those adopted in [20, p. 207] and we assume that the reader is familiar with the elementary lemmas z. 1, z. 2, z. 3 and z. 4 stated there.

Consider the diagram:

$$\begin{array}{ccccc} \text{NS}(X) \otimes \hat{\mathbf{Z}} & \xrightarrow{e} & \text{Hom}(\text{NS}(X), \hat{\mathbf{Z}}) & \approx & \text{Hom}(\text{NS}(X) \otimes \mathbf{Q}/\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \\ \downarrow \bar{h} & & & & \uparrow g^* \\ H^2(\bar{X}, T\mu)^\Gamma & \xrightarrow{f} & H^2(\bar{X}, T\mu)_\Gamma & \xrightarrow{j} & \text{Hom}(H^2(X, \mu(\infty)), \mathbf{Q}/\mathbf{Z}) \end{array}$$

where the maps will be explained below.

The map $e$ is induced by intersection-products $\text{NS}(X) \times \text{NS}(X) \to \mathbf{Z}$. Recall that $\text{NS}(X)$ is defined to be the image of $\text{Pic}(X)$ in $\text{NS}(\bar{X})$, and that intersection-products define a non-degenerate pairing on $\text{NS}(\bar{X})/\text{NS}(\bar{X})_{\text{tors}}$. From this it follows that $e$ is a quasi-isomorphism and

$$z(e) = \frac{(-1)^{\rho-1}(\text{NS}(X): B)^2}{\det(D_i \cdot D_j)[\text{NS}(X)_{\text{tors}}]}$$

where $D_1, \cdots, D_\rho$ are independent elements of $\text{NS}(X)$ and $B$ is the subgroup generated by them. The number on the right is positive because, by the Hodge index theorem, if $D$ is defined by a hyperplane section of $X$, then $D \cdot D > 0$ and $D' \cdot D' \leq 0$ for any $D'$ orthogonal to $D$. Thus the bilinear form $(\text{NS}(X) \otimes_{\mathbf{Z}} \mathbf{R}) \times (\text{NS}(X) \otimes_{\mathbf{Z}} \mathbf{R}) \to \mathbf{R}$ is of type $(+, -, -, \cdots)$.

The isomorphism on the top row is obvious.

The map $j$ is the composite of the map $H^2(\bar{X}, T\mu)_\Gamma \xrightarrow{j_0} H^3(X, T\mu)$ coming from the Hochschild-Serre spectral sequence (cf. § 3) and the isomorphism $j_1: H^3(X, T\mu) \to \text{Hom}(H^2(X, \mu(\infty)), \mathbf{Q}/\mathbf{Z})$ coming from Poincaré duality. The map $j_0$ is injective, with cokernel $H^3(\bar{X}, T\mu)^\Gamma$, and so it follows from the next lemma that $j$ is a quasi-isomorphism with $z(j) = [\text{NS}(X)_{\text{tors}}]^{-1} q^{-s}$.

LEMMA 5.2. *The group* $H^3(\bar{X}, T\mu)^\Gamma$ *is finite, with order* $[H^3(\bar{X}, T\mu)^\Gamma] =$

$[\mathrm{NS}(X)_{\mathrm{tors}}]q^s$ *where* $s = \dim U^3(p^\infty)$.

*Proof.* $H^4(\bar{X}, T_l\mu) \approx \mathbf{Z}_l$ with $\sigma$ acting as $q^{-1}$ for $l \neq p$, and $H^4(\bar{X}, T_p\mu) = 0$. Thus $H^4(\bar{X}, T\mu)^\Gamma = 0$ and $H^3(\bar{X}, T\mu)_\Gamma \approx H^4(X, T\mu)$. Also $H^4(\bar{X}, T\mu)$ is dual to $H^1(X, \mu(\infty))$ (§ 2) and $H^1(X, \mu(\infty)) \approx \mathrm{Pic}\,(X)_{\mathrm{tors}}$. Thus $H^3(\bar{X}, T\mu)_\Gamma$ is finite and $[H^3(\bar{X}, T\mu)_\Gamma] = [\mathrm{Pic}\,(X)_{\mathrm{tors}}]$.

For $l \neq p$, $H^3(\bar{X}, T_l\mu)$ is a finitely-generated $\mathbf{Z}_l$-module and $\det(1 - \sigma t \mid H^3(\bar{X}, T_l\mu) \otimes \mathbf{Q}_l) = \prod (1 - (q/a_{3,i})t)$. An easy computation using [20, Lemma z. 4] shows that

$$[H^3(\bar{X}, T_l\mu)_\Gamma] = [H^3(\bar{X}, T_l\mu)^\Gamma] \mid \prod (1 - q/a_{3,i}) \mid_l^{-1}$$

where $|\ |_l$ is the valuation with $|\ |_l = 1/l$.

For $p$, there is an exact sequence,

$$0 \longrightarrow U^3(p^\infty)(\bar{k}) \longrightarrow H^3(\bar{X}, T_p\mu) \longrightarrow \varprojlim D(p^n)(\bar{k}) \longrightarrow 0\ .$$

The group $\varprojlim D(p^n)(\bar{k})$ is a finitely-generated $\mathbf{Z}_p$-module and its tensor product with $\mathbf{Q}_p$ is isomorphic to $H^3(\bar{X}, V_p\mu)$. A computation similar to the above, using that

$$\det(1 - \sigma t \mid H^3(\bar{X}, V_p\mu)) = \prod\nolimits_{v(a_{3,i})=1} (1 - q/a_{3,i})\ , \qquad U^3(p^\infty)(\bar{k})_\Gamma = 0\ ,$$
$$U^3(p^\infty)(\bar{k})^\Gamma = U^3(p^\infty)(k)\ ,$$

shows that

$$[H^3(\bar{X}, T_p\mu)_\Gamma] = [H^3(\bar{X}, T_p\mu)^\Gamma] \mid \prod\nolimits_{v(a_{3,i})=1} (1 - q/a_{3,i}) \mid_p^{-1} q^{-s}\ .$$

On combining these equations for all primes, we get

$$[H^3(\bar{X}, T\mu)_\Gamma] = \pm[H^3(\bar{X}, T\mu)^\Gamma] \prod (1 - q/a_{3,i}) \mid \prod\nolimits_{v(a_{3,i})\neq 1} (1 - q/a_{3,i}) \mid_p q^{-s}$$

Since $P_3(X, t) = P_1(X, qt)$ we have that

$$\prod (1 - q/a_{3,i}) \mid \prod\nolimits_{v(a_{3,i})\neq 1} (1 - q/a_{3,i}) \mid_p = \frac{\prod (1 - a_{1,i})}{\prod a_{1,i}} \mid \prod\nolimits_{v(a_{1,i})\neq 0} (1 - a_{1,i}^{-1}) \mid_p$$
$$= [\mathrm{Pic\,Var}\,(X)(k)]$$

as $\prod a_{1,i} = q^{\dim(\mathrm{Pic\,Var}(X))} = \mid \prod_{v(a_{1,i})\neq 0} (1 - a_{1,i}^{-1}) \mid_p$. Thus

$$[H^3(\bar{X}, T\mu)^\Gamma] = \frac{[\mathrm{Pic}\,(X)_{\mathrm{tors}}]}{[\mathrm{Pic\,Var}\,(X)_{\mathrm{tors}}]} q^s = [\mathrm{NS}(X)_{\mathrm{tors}}]q^s\ .$$

The map $g^*$ is the dual of the map $g$ in (3.2). Thus $g$ is an $l$-quasi-isomorphism for a given prime $l$ if and only if $\mathrm{Br}(X)(l)$ is finite.

The map $\bar{h}$ is as in (3.3). It is an $l$-quasi-isomorphism for a given prime $l$ if and only if $T_l\,\mathrm{Br}(X) = 0$, i.e., if and only if $\mathrm{Br}(X)(l)$ is finite.

The map $f$ is induced by the identity map on $H^2(\bar{X}, T\mu)$. By [20, z. 4] it is an $l$-quasi-isomorphism for a given prime $l$ if and only if the multiplicity of 1 as an eigenvalue for $\sigma$ acting on $H^2(\bar{X}, T_l\mu)$ is equal to the $\mathbf{Z}_l$-rank of $H^2(X, T_l\mu)^\Gamma$, i.e., in the notation of Section 4, if and only if $\rho'(X) = \operatorname{rank}_{\mathbf{Z}_l} H^2(\bar{X}, T_l\mu)^\Gamma$.

Finally we need:

LEMMA 5.3. *The diagram at the start of this section commutes.*

*Proof.* This amounts to saying that the following diagram commutes:

$$\begin{array}{ccccc}
\operatorname{Pic}(X) & \times & \operatorname{NS}(X) & \longrightarrow & \mathbf{Z} \\
\Big| & & \downarrow & & \Big| \\
\Big| & & H^2(\bar{X}, \mu_n)_\Gamma & & \Big| \\
\downarrow & & \downarrow & & \downarrow \\
H^2(X, \mu_n) & \times & H^3(X, \mu_n) & \longrightarrow & \mathbf{Z}/n\mathbf{Z}\,.
\end{array}$$

The top pairing is intersection-product, the bottom pairing is cup-product, and the maps should, by now, be obvious. The commutativity of this diagram is exactly what we meant in (2.1) by the compatibility of cup and intersection products.

Now assume that $\operatorname{Br}(X)(l)$ is finite for some given prime $l$; i.e., assume (4.1b′). Then all maps in the diagram, except possibly $f$, are $l$-quasi-isomorphisms. It follows that $f$ must be an $l$-quasi-isomorphism which, after the remark preceding (5.3), implies that $\rho'(X) = \operatorname{rank}_{\mathbf{Z}_l} H^2(X, T_l\mu)^\Gamma = \rho(X)$, i.e., that (T) holds.

This completes the proof of Theorem 4.1.

*Remark* 5.4. The conditions of the theorem are also equivalent to the map $h\colon \operatorname{Pic}(X) \otimes \hat{\mathbf{Z}} \to H^2(X, T\mu)$ being an isomorphism, and to the cycle map in crystalline cohomology defining an isomorphism $\operatorname{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}_p \to H^2(X/W)_K(1)^F$.

## 6. The Artin-Tate conjecture

THEOREM 6.1. *Let $X$ be a smooth projective surface over a finite field $k$ of characteristic $\neq 2$. If* (T) *holds for $X/k$ then so also does* (A-T).

*Proof.* We must show that $\operatorname{Br}(X)$ is finite and that

$$\prod_{a_{2,i} \neq q} \left(1 - \frac{a_{2,i}}{q}\right) = \frac{(-1)^{\rho(X)-1}[\operatorname{Br}(X)] \det(D_i \cdot D_j)}{q^{\alpha(X)}(\operatorname{NS}(X) \colon B)^2}\,.$$

Since we are assuming (T), all of the maps in the diagram (5.1) are quasi-isomorphisms and, since the diagram commutes,

$$z(e) = z(\bar{h})z(f)z(j)z(g^*) .$$

We have that:

$$\begin{aligned} z(e) &= \frac{(-1)^{\rho-1}(\mathrm{NS}(X)\colon B)^2}{\det(D_i \cdot D_j)[\mathrm{NS}(X)_{\mathrm{tors}}]} , \\ z(\bar{h}) &= 1 , \\ z(f)^{-1} &= \prod\nolimits_{a_2\, i \neq q}\left(1 - \frac{a_{2,i}}{q}\right) \Big| \prod\nolimits_{v(a_{2,i})\neq 1}\left(\frac{a_{2,i}}{q}\right)\Big|_p , \\ z(j) &= \frac{q^{-s}}{[\mathrm{NS}(X)_{\mathrm{tors}}]} , \\ z(g^*) &= [\mathrm{Br}(X)] . \end{aligned}$$

Since all of these are rational numbers except possibly the last, we see that the last must also be a rational number, i.e., that $[\mathrm{Br}(X)]$ is finite.

Note that both sides of the equation are positive and so it remains to prove that

$$q^{\alpha(X)} = \Big|\prod\nolimits_{v(a_{2,i})\neq 1}\left(1 - \frac{a_{2,i}}{q}\right)\Big|_p q^s ,$$

i.e., that

$$\alpha(X) = \sum\nolimits_{v(a_{2,i})<1}\left(1 - v(a_{2,i})\right) + s .$$

This we do in the next section.

## 7. Calculation of $\alpha(X)$

Since it requires no extra effort, we prove a slightly more general result than necessary.

PROPOSITION 7.1. *Let $X$ be a smooth projective variety over an algebraically closed field of characteristic $p$. Then*

$$\chi(X, O_X) = \sum\nolimits_{\lambda_{r,i}\leq 1}(-1)^r m_{r,i}(1 - \lambda_{r,i}) + \sum\nolimits_r (-1)^r d_r$$

*where the $\lambda_{r,i}$ are the slopes of $H^r(X/W)_K$, $m_{r,i}$ is the multiplicity of $\lambda_{r,i}$, and $d_r = \mathrm{length}_{W((V))}(H^r(X, W)_t \otimes_{W[[V]]} W((V)))$.*

Before proving this we need to make some computations concerning modules over the ring $W[V]$. If $M$ is such a module, then the kernel and cokernel of $V\colon M \to M$ are $W$-modules ($V$ is $(p^{-1})$-linear) and we define

$$\chi(M) = \mathrm{length}_W(\ker(V)) - \mathrm{length}_W(\mathrm{coker}(V))$$

provided both numbers on the right are finite.

LEMMA 7.2.

(a) *If $M$ has finite length as a $W$-module then $\chi(M) = 0$.*

(b) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $W[V]$-modules, then $\chi(M)$ is defined if $\chi(M')$ and $\chi(M'')$ are defined and $\chi(M) = \chi(M') + \chi(M'')$.*

(c) *If $M$ is a module over the Dieudonné ring $W[F, V]$ which is free and finitely-generated when regarded as a $W$-module, then $\chi(M) = -\sum m_i(1 - \lambda_i)$ where the $\lambda_i$ are the slopes of $M$ and $m_i$ is the multiplicity of $\lambda_i$.*

(d) *Let $M$ be a finitely-generated $p$-torsion $W[[V]]$-module. Then $\chi(M) = -\operatorname{length}_{W((V))} M \otimes_{W[[V]]} W((V))$.*

*Proof.* (a) This follows from counting lengths in the exact seqence,

$$0 \longrightarrow \ker(V) \longrightarrow M \xrightarrow{V} M \longrightarrow \operatorname{coker}(V) \longrightarrow 0 .$$

(b) This is a trivial application of the serpent lemma.

(c) If $M = W[F, V]/(F^m - p^{m\lambda})$ then a trivial calculation shows that $\chi(M) = -m(1 - \lambda)$. But any $W[F, V]$-module $M$ which is free as a $W$-module contains a submodule $M'$ which is a direct sum of such modules and is such that $M/M'$ is of finite length over $W$. Thus (c) now follows from (a) and (b).

(d) If $M = W[[V]]$ then clearly $\chi(M) = -1 = -(\text{length } W((V)))$. But every finitely-generated torsion $W[[V]]$-module has a finite composition series whose quotients are $W[[V]]$ and modules of finite length over $W$. Moreover the number of factors $W[[V]]$ is equal to $\operatorname{length}_{W((V))} M$. Thus (d) now follows from (a) and (b).

*Proof* (of 7.1). There is a long exact sequence [18]

$$\cdots \longrightarrow H^r(X, W) \xrightarrow{V} H^r(X, W) \longrightarrow H^r(X, O_X) \longrightarrow H^{r+1}(X, W) \longrightarrow \cdots .$$

Thus $\chi(X, O_X) = \sum (-1)^{r+1}\chi(H^r(X, W))$. But, for each $r$, there is an exact sequence

$$0 \longrightarrow H^r(X, W)_t \longrightarrow H^r(X, W) \longrightarrow H^r(X, W)' \longrightarrow 0$$

where $H^r(X, W)_t$ is a module as in (7.2 d) and $H^r(X, W)'$ is a module as in (7.2 c) with slopes $\{\lambda_{r,i} \mid \lambda_{r,i} < 1\}$ (see § 1). Thus

$$\begin{aligned}\chi(H^r(X, W)) &= \chi(H^r(X, W)_t) + \chi(H^r(X, W)') \\ &= -d_r - \textstyle\sum_{\lambda_{r,i} \leq 1} (1 - \lambda_{r,i})m_{r,i} .\end{aligned}$$

On combining this with the previous equation, we get the required formula.

COROLLARY 7.3. *Let $X$ be a projective smooth variety over a finite field $k$. Then*

$$\chi(X, O_X) = \sum_{v(a_{r,i})\leq 1} (-1)^r(1 - v(a_{r,i})) + \sum_{r=1}^{d} (-1)^r d_r \, .$$

*Proof.* Since $H^r(\bar{X}, O_{\bar{X}}) = H^r(X, O_X) \otimes_k \bar{k}$ and $H^r(\bar{X}, \bar{W}) = H^r(X, W) \otimes_W \bar{W}$ we see that two of the terms are unchanged when $X$ is replaced by $\bar{X}$. The fact that

$$\sum_{v(a_{r,i})\leq 1} (-1)^r(1 - v(a_{r,i})) = \sum_{\lambda_{r,i}\leq 1} (-1)^r m_{r,i}(1 - \lambda_{r,i})$$

follows from (1.2).

*Remark* 7.4. Let $X$ be as in (7.3). Then

$$\sum_{v(a_{0,i})\leq 1} (1 - v(a_{0,i})) = 1 \, ,$$
$$\sum_{v(a_{1,i})\leq 1} (1 - v(a_{1,i})) = \dim(\text{Pic Var}(X)) \, .$$

For let $d' = \dim(\text{Pic Var}(X))$; then

$$\sum_{v(a_{1,i})\leq 1} (1 - v(a_{1,i})) = \sum_{i=1}^{d'} (1 - v(a_{1,i})) + (1 - v(\bar{a}_{1,i})) \quad \text{where } a_{1,i}\bar{a}_{1,i} = q$$
$$= \sum_{i=1}^{d'} (2 - v(a_{1,i}a_{1,i})) = d' \, .$$

Moreover, if $X$ is a surface, then $\sum_{v(a_{r,i})<1} (1 - v(a_{r,i})) = 0$ for $r > 2$, as $\{a_{3,i}\} = \{qa_{1,i}\}$ and $\{a_{4,i}\} = \{q^2\}$. Thus, in this case, the identity in (7.3) reduces to

$$\chi(X, O_X) = 1 - \dim(\text{Pic Var}(X)) + \sum_{v(a_{2,i})<1} (1 - v(a_{2,i})) + d_2 \, .$$

But (cf. § 1), $d_2 = s$, and so this may be re-written

$$\chi(X, O_X) - 1 + \dim(\text{Pic Var}(X)) = \sum_{v(a_{2,i})<1} (1 - v(a_{2,i})) + s \, .$$

This completes the proof of Theorem 6.1.

## 8. The conjectures of Birch and Swinnerton-Dyer

We state things only for the simplest case. The notations are the same as those of [20].

THEOREM 8.1. *Let $E$ be an elliptic curve over a function field $K$ in one variable with finite constant field of characteristic $p \neq 2$. The following statements are equivalent*:

(a) *The L-series $L(E, s)$ of $E$ has a zero of order $r$ at $s = 1$, where $r$ is the rank of $E(K)$ (see [20, A]).*

(b) *For some prime $l$, the $l$-primary component of the Tate-Šafarevič group Ш$(E/K)$ of $E$ over $K$ is finite.*

(c) Ш$(E/K)$ *is finite, and*

$$L^*(E, s) \sim \frac{[Ш] \det \langle a'_i, a_j \rangle}{[E(K)_{\mathrm{tors}}]^2}(s-1)^r \text{ as } s \longrightarrow 1$$

(see [20, B]).

*Proof.* This follows from (4.1) and (6.1) and the known equivalence of conjectures (B) and (C) of [20] in the case of elliptic surfaces.

*Remark* 8.2 (a). There exist many curves for which it is trivial to check that the statements of (8.1) hold. For example let $E$ be the curve defined by the equation $Y^2 = X(X-1)(X-T)$ over the field $K = k(T)$, $k$ finite of characteristic $p \neq 2, 3$. Consider $\bar{E}$ over $\bar{K} = \bar{k}(T)$. It has bad reduction only at $T = 0, 1, \infty$ and, in the notation of [17, Thm. 3], $\beta = 1, 1, 2$ at these points. Thus (loc. cit.)

$$\operatorname{rank}(E(\bar{K})) + \operatorname{rank}(T_l Ш(\bar{E}/\bar{K})) = -4 + 1 + 1 + 2 = 0\,,$$

and $Ш(\bar{E}/\bar{K})(l)$ is finite. This implies that $Ш(E/K)$ is finite. (In fact the minimal model of $E$ over $\mathbf{P}^1$ must be a rational surface.)

(b) Artin and Swinnerton-Dyer [1] have shown that the statement (b) of (8.1) holds for curves $E$ over $k(T)$ whose equations have the form

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$$

where $a_i$ is a polynomial of degree $\leqq 2i$.

(c) Let $\bar{E}$ be an elliptic curve over an algebraically closed field $C$ of characteristic $p$ and of transcendence degree at least one over $\mathbf{F}_p$. Then there exists a function field $K \subset C$, of transcendence degree 1 over its finite constant field, and an elliptic curve $E$ over $K$ such that:

(i) $E \otimes_K C \approx \bar{E}$,

(ii) Statement (8.1 b) holds for $E/K$.

Indeed, if the $j$-invariant $j(\bar{E})$ of $\bar{E}$ is algebraic over $\mathbf{F}_p$ then any $K$ containing $\mathbf{F}_p(j)$, and any constant $E$ over $K$ such that $E \otimes C = \bar{E}$, will do. Otherwise

$$E\colon Y^2 + (j-1728)XY = X^3 - 36(j-1728)^3X - (j-1728)^5$$

over $K = \mathbf{F}_p(j)$ will do (by the last remark).

Thus, in order to prove that the statements in (8.1) hold for all elliptic curves over function fields, it suffices to show that at least one statement is preserved under finite extensions of the function field.

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BIBLIOGRAPHY

[1] M. ARTIN, Supersingular K3 surfaces, Ann. Sc. ENS, t. 7, (1974), 543-567.
[2] M. ARTIN and B. MAZUR, Formal groups arising from algebraic varieties, preprint.

[3] M. ARTIN and H. SWINNERTON-DYER, The Šafarevič-Tate conjecture for pencils of elliptic curves on $K3$ surfaces, Inv. Math. **20** (1973), 249-266.

[4] S. BLOCH, Algebraic $K$-theory and crystalline cohomology, preprint.

[5] P. BERTHELOT, Cohomologie crystalline, *Lecture Notes in Mathematics*, 407, Springer, Berlin, 1974.

[6] P. DELIGNE, La conjecture de Weil I, Publ. Math. IHES **43**, 1974, 273-307.

[7] M. DEMAZURE, Lectures on $p$-divisible groups, *Lecture Notes in Mathematics* 302, Springer, Berlin, 1972.

[8] A. GROTHENDIECK, Le groupe de Brauer, *Dix Exposés sur la Cohomologie des Schémas*, North-Holland, Amsterdam, 1968.

[9] ———, Crystals and the de Rham cohomology of schemes, *Dix Exposés sur la Cohomologie des Schémas*, North-Holland, Amsterdam, 1968.

[10] A. GROTHENDIECK, M. ARTIN, and J. VERDIER, Séminaire de Géométrie Algébrique 4, 5, *Lecture Notes in Mathematics*, Nos. 269, 270, 305... Springer, Berlin 1972, 1973...

[11] N. KATZ and W. MESSING, Some consequences of the Riemann hypothesis for varieties over finite fields, Inv. Math. **23**, (1974) 73-77.

[12] J. MILNE, Extensions of abelian varieties defined over a finite field, Inv. Math. **5**, (1968), 63-84.

[13] ———, The Tate-Šafarevič group of a constant abelian variety, Inv. Math. **6** (1968), 91-105.

[14] ———, The Brauer group of a rational surface. Inv. Math. **11** (1970), 304-307.

[15] ———, Flat Homology, in preparation.

[16] ———, Duality in the flat cohomology of a surface, in preparation.

[17] M. RAYNAUD, Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes, *Dix Exposés sur la Cohomologie des Schémas*, North-Holland, Amsterdam, 1968.

[18] J-P. SERRE, Sur la topologie des variétés algébriques en caractéristique $p$, *Symposium Internacional de Topologia Algebrica*, Mexico, 1956.

[19] J. TATE, Algebraic cycles and poles of zeta functions, *Arithmetical Algebraic Geometry*, Harper and Row, New York, 1965.

[20] ———, On a conjecture of Birch and Swinnerton-Dyer and a geometric analogue, *Dix Exposés sur la Cohomologie des Schémas*, North-Holland, Amsterdam, 1968.

[21] ———, Endomorphisms of abelian varieties over finite fields, Inv. Math. **2** (1966), 134-144.

[22] J. VERDIER, A duality theorem in the étale cohomology of schemes, *Proceedings of a conference on local fields*, Springer, Berlin, 1967.

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