# THE BRAUER-MANIN OBSTRUCTION FOR CURVES

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## 1. Introduction

Let $X$ be a smooth projective variety defined over a number field $K$. A fundamental problem in arithmetic geometry is to determine whether or not $X$ has any $K$-rational points. In general this is a very difficult task, and so we look instead for necessary conditions for $X(K)$ to be non-empty. Embedding $K$ in each of its completions $K_v$ we have the *local condition* that $X(K_v)$ must be non-empty for every $v$. For some classes of varieties this is sufficient (in which case we say that the *Hasse Principle* holds), but often it is not. Manin [15] introduced subsets $X^{\mathbf{B}}$ of $\prod_v X(K_v)$ for every $B \subseteq \mathrm{Br}(X)$ such that

$$X(K) \subseteq X^{\mathbf{Br}} \subseteq X^{\mathbf{B}} \subseteq \prod_v X(K_v). \tag{1}$$

The sets $X^{\mathbf{B}}$ are defined in § 2. If $X^{\mathbf{Br}} = \emptyset$ then $X(K)$ must be empty. Manin's construction accounts for almost all known examples where the Hasse Principle fails. However it is difficult to calculate $X^{\mathbf{Br}}$ explicitly, especially if $\mathrm{Br}(X)$ is large, as it is for curves. The aim of this paper is to give a more tractable formulation for $X^{\mathbf{Br}}$ for curves (Theorem 1.2), and then use it to show that $X^{\mathbf{Br}} = \emptyset$ in several cases. In particular we show that if $\Gamma$ is the curve $3x^4 + 4y^4 = 19z^4$ first introduced by Bremner, Lewis and Morton in [2] then $\Gamma^{\mathbf{Br}} = \emptyset$ provided the Jacobian of $\Gamma$ has finite Tate-Shafarevich group. This partially answers a question of Skorobogatov [21]. Along the way we develop a test (Proposition 4.3) which in practice is extremely effective at showing that a curve has no rational points.

Our notation will be as follows. Let $K$ be a number field, and let $X$ be a smooth projective geometrically connected curve defined over $K$. For each prime $v$ let $K_v$ denote the completion of $K$ at $v$, let $k_v$ be the residue field for each finite $v$, and let $G_K = \mathrm{Gal}(\overline{K}/K)$ and $G_{K_v} = \mathrm{Gal}(\overline{K_v}/K_v)$. We shall assume that $\prod_v X(K_v) \neq \emptyset$. Let $J \cong \mathrm{Pic}^0(X)$ denote the Jacobian variety of $X$, and let $\mathbf{TS}(J)$ denote the Tate-Shafarevich group of $J$:

$$\mathbf{TS}(J) = \ker\left(H^1(K, J) \to \bigoplus_v H^1(K_v, J)\right).$$

We assume throughout that $\mathbf{TS}(J)$ is finite.

Finally let $\delta$ be the *index* of $X$, defined as the greatest common divisor of the degrees of finite field extensions $L/K$ for which $X$ has an $L$-rational point, or equivalently, as the smallest positive degree of a divisor on $X$. Clearly if $\delta > 1$ then $X(K) = \emptyset$. We prove in § 2 that in this case $X^{\mathbf{Br}} = \emptyset$ also.

**Theorem 1.1.** *Suppose $\mathbf{TS}(J)$ is finite. Then there is a subset $B \subseteq \mathrm{Br}(X)$ such that $X$ has index 1 iff $X^{\mathrm{B}} \neq \emptyset$. In particular, if $X$ has index greater than 1 then $X^{\mathbf{Br}} = \emptyset$.*

The remaining case occurs when $X$ has index 1. Let $\Delta$ be a fixed degree 1 divisor. We can use $\Delta$ to identify $\mathrm{Pic}^0$ and $\mathrm{Pic}^1$. We also have a map $\xi_\Delta \colon X(\overline{K}) \to J(\overline{K})$ given by $x \mapsto [x - \Delta]$. This is a translation of the standard embedding $f^P \colon X(\overline{K}) \to J(\overline{K})$ which sends a point $P$ to 0. However $\xi_\Delta$ is defined over $K$ even if $X$ has no $K$-rational point. We have the following diagram.

(2)
$$\begin{tikzcd} X(K) \arrow[r, "\xi_\Delta"] \arrow[d] & J(K) \arrow[d] \\ \prod_v X(K_v) \arrow[r, "\xi_\Delta"] & \prod_v J(K_v) \end{tikzcd}$$

Thus inside $\prod_v J(K_v)$ we have

$$X(K) \cong \xi_\Delta \left( \prod_v X(K_v) \right) \cap J(K).$$

Consider the topological closure of this set. Since $\xi_\Delta$ is a closed immersion [14, Proposition 2.3] we have

$$\overline{X(K)} \subseteq \xi_\Delta \left( \prod_v X(K_v) \right) \cap \overline{J(K)}$$

where $\overline{J(K)}$ denotes the closure of $J(K)$ inside the group $\prod_v \mathrm{Pic}(X_{K_v})$. Furthermore we know that $X^{\mathbf{Br}}$ is always a closed set [6, Proposition 2], so that $\overline{X(K)} \subseteq X^{\mathbf{Br}}$. Our main result is that these two closed sets containing $X(K)$ are actually equal.

**Theorem 1.2.** *Suppose that $X$ has index 1, and that $\mathbf{TS}(J)$ is finite. Then*

$$X(K) \cong \xi_\Delta(\prod X(K_v)) \cap J(K) \subseteq \xi_\Delta(\prod X(K_v)) \cap \overline{J(K)} \cong X^{\mathbf{Br}}$$

*Here the closure of $J(K)$ is taken inside the group $\prod_v \mathrm{Pic}(X_{K_v})$.*

Note that the closure of $J(K)$ is the same as its profinite completion [12, I.6.14b]. If $\Delta'$ is any other degree 1 divisor then translation by $(\Delta - \Delta')$ gives a bijection between $\xi_\Delta(\prod X(K_v)) \cap J(K)$ and $\xi_{\Delta'}(\prod X(K_v)) \cap J(K)$, so that the description of $X^{\mathbf{Br}}$ is not canonical.

Serre showed [18, Theorem 3] that the topology induced on $J(K)$ by embedding it in $\prod J(K_v)$ is that defined by the subgroups of $J(K)$ of finite index. In particular, if $J(K)$ is a finite set then it has the discrete topology, and is unaltered on taking the completion. We thus have the following corollary to Theorem 1.2.

**Corollary 1.3.** *Suppose $J(K)$ and $\mathbf{TS}(J)$ are finite. Then $X(K) = \emptyset$ iff $X^{\mathbf{Br}} = \emptyset$.*

□

The intersections in Theorem 1.2(b) may still be difficult to calculate in general. However if we project from $K_v$ to the residue field $k_v$ in diagram 2 we are led to consider the conditions

(3)
$$\xi_\Delta \left( \prod X(k_v) \right) \cap J(K) \neq \emptyset$$

and

(4)
$$\xi_\Delta \left( \prod X(k_v) \right) \cap \overline{J(K))} \neq \emptyset$$

where the closure now takes place inside $\prod_v \mathrm{Pic}(X_{k_v})$. We must check that the map to $k_v$ is well defined (for example, is independent of any particular choice of projective embedding). This follows from the existence of a Néron model for $J$ [20, Chapter IV].

Conditions (3) and (4) are amenable to testing, since $J(K)$ is finitely generated, the $X(k_v)$ are finite, and the closure is easy to understand. If condition (3) fails then $X(K) = \emptyset$. In practice this provides a powerful method of showing that a curve has no $K$-rational points.

The closure of $J(K)$ in $\prod_v X(K_v)$ is mapped into its closure in $\prod_v X(k_v)$. Thus if condition (4) fails and $\mathbf{TS}(J)$ is finite then $X^{\mathbf{Br}} = \emptyset$. In §4 we show the following.

**Theorem 1.4.** *If $\mathbf{TS}(J)$ and $J(K)$ are finite then the following are equivalent:*

(a) $X(K) \neq \emptyset$.
(b) *The Brauer-Manin condition holds:* $X^{\mathbf{Br}} \neq \emptyset$.
(c) *Condition* (3) *holds:* $\xi_\Delta(\prod X(k_v)) \cap J(K) \neq \emptyset$.
(d) *Condition* (4) *holds:* $\xi_\Delta(\prod X(k_v)) \cap \overline{J(K))} \neq \emptyset$.

In § 5 we use these techniques to give some examples, including the curve $\Gamma$ mentioned above.

## 2. Reduction to Index One

In this section we define the set $X^{\mathbf{Br}}$, and then prove Theorem 1.1. For any field extension $L/K$ let $X(L)$ be the set of $L$-rational points of $X$, and let $\mathrm{Br}(X_L)$ denote the Brauer-Grothendieck group of $X$ over $L$, $H^2(X \underset{\mathrm{spec}\ K}{\times} \mathrm{spec}\ L, \mathbb{G}_{\mathrm{m}})$ (étale cohomology). For any prime $v$ of $K$ let

$$\begin{array}{rcl} \mathrm{ev}_v \colon \mathrm{Br}(X_{K_v}) \times X(K_v) & \to & \mathrm{Br}(K_v) \\ (b_v, x_v) & \mapsto & b_v(x_v) \end{array}$$

be the evaluation map sending $x_v$ to the fibre of $b_v$ at $x_v$. Let $\mathrm{inv}_v \colon \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ be the local invariant map. There is a pairing $\langle , \rangle$ [6, Lemma 1]

$$\begin{array}{rcl} \mathrm{Br}(X_K) \times \prod_v X(K_v) & \to & \mathbb{Q}/\mathbb{Z} \\ \langle b, (x_v) \rangle & = & \sum_v \mathrm{inv}_v\, \mathrm{ev}_v(b_v, x_v) \end{array}$$

where $(b_v)$ is the image of $b$ under $\mathrm{Br}(X_K) \hookrightarrow \prod_v \mathrm{Br}(X_{K_v})$. From the fundamental exact sequence [22, VII.9.6]

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

we see that if $x \in X(K)$ then $\langle b, x \rangle = 0$ for every $b \in \mathrm{Br}(X)$.

**Definition 2.1.** For any subset $B \subseteq \mathrm{Br}(X)$ let

$$X^{\mathbf{B}} = \left\{ (x_v) \in \prod_v X(K_v) \mid \langle b, (x_v) \rangle = 0 \quad \forall\, b \in B \right\}$$

and write $X^{\mathbf{Br}}$ for $X^{\mathrm{Br}(X)}$. Equation (1) is then immediate.

**Definition 2.2.** (a) If $\prod_v X(K_v) \neq \emptyset$ we say that the *local condition* holds for $X$.
(b) If $X^{\mathbf{Br}} \neq \emptyset$ we say that the *Brauer-Manin condition* holds for $X$.

The local and Brauer-Manin conditions are necessary conditions for the existence of a $K$-rational point on $X$. If the local condition is also sufficient we say that the *Hasse Principle* holds for $X$. If the Brauer-Manin condition is sufficient then in the terminology of Colliot-Thélène [6] we say that *the Brauer-Manin obstruction is the only obstruction* to the existence of a $K$-rational point on $X$.

We now prove Theorem 1.1, which follows easily from the results in [11]. The set $B$ is chosen so that the values of the pairing $\langle , \rangle$ are independent of the choice of the point

$(x_v)$. Thus on $B$ the pairing degenerates to a character $\varphi\colon B \to \mathbb{Q}/\mathbb{Z}$, which is explicitly described in [11].

*Proof of Theorem* 1.1. The Hochschild-Serre spectral sequence

$$H^r(G_K, H^s(X_{\overline{K}}, \mathbb{G}_\mathrm{m})) \implies H^{r+s}(X_K, \mathbb{G}_\mathrm{m})$$

yields

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X_K) \arrow[r] & \big(\mathrm{Pic}(X_{\overline{K}})\big)^{G_K} \arrow[r] & \mathrm{Br}(K) \arrow[d] \\ & H^3(G_K, \overline{K}^\times) & H^1(G_K, \mathrm{Pic}(X_{\overline{K}})) \arrow[l] & \mathrm{Br}(X_K) \arrow[l] \end{tikzcd} \tag{5}$$

since $H^1(X, \mathbb{G}_\mathrm{m}) = \mathrm{Pic}(X)$ and $H^2(X_{\overline{K}}, \mathbb{G}_\mathrm{m}) = 0$. (Sequence 5 can be obtained without using the spectral sequence; see [11, 2.3] or [10, § 2].) If $L$ is any local or global field then $H^3(G_L, \overline{L}^\times) = 0$ [4, VII 11.4]. The local points provide section maps $\mathrm{Br}(X_{K_v}) \to \mathrm{Br}(K_v)$ for every $v$, so that in the corresponding sequence for $K_v$, $\mathrm{Br}(K_v) \to \mathrm{Br}(X_{K_v})$ is injective. We thus have the following diagram [11, 2.7.1].

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Br}(K) \arrow[r] \arrow[d] & \mathrm{Br}(X_K) \arrow[r] \arrow[d] & H^1(G_K, \mathrm{Pic}(X_{\overline{K}})) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigoplus_v \mathrm{Br}(K_v) \arrow[r] \arrow[d, "\sum_v \mathrm{inv}_v"'] & \bigoplus_v \mathrm{Br}(X_{K_v}) \arrow[r] & \bigoplus_v H^1(G_{K_v}, \mathrm{Pic}(X_{\overline{K_v}})) \arrow[r] & 0 \\ & \mathbb{Q}/\mathbb{Z} & & & \end{tikzcd} \tag{6}$$

The injectivity of $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ follows from the injectivity of $\mathrm{Br}(K) \hookrightarrow \bigoplus_v \mathrm{Br}(K_v)$ and $\mathrm{Br}(K_v) \hookrightarrow \mathrm{Br}(X_{K_v})$. Note that we have by (5)

$$\big(\mathrm{Pic}(X_{\overline{K}})\big)^{G_K} = \mathrm{Pic}(X_K). \tag{7}$$

Let

$$\mathrm{Br}(X)' = \ker\left(\mathrm{Br}(X_K) \to \bigoplus\nolimits_v \mathrm{Br}(X_{K_v})\right).$$

This agrees with the notation in [11], as can be seen from Lemma 2.6 of that paper. Let

$$\mathbf{TS}(P) = \ker\left(H^1(G_K, \mathrm{Pic}(X_{\overline{K}})) \to \bigoplus\nolimits_v H^1(G_{K_v}, \mathrm{Pic}(X_{\overline{K_v}}))\right)$$

and define the set $B$ to be

$$B = \ker\left(\mathrm{Br}(X_K) \to \bigoplus\nolimits_v H^1(G_{K_v}, \mathrm{Pic}(X_{\overline{K_v}}))\right).$$

Equivalently, $B$ is the preimage of $\mathbf{TS}(P)$ in $\mathrm{Br}(X_K)$.

Suppose $b \in B$, and let $(b_v)$ be its image in $\bigoplus_v \mathrm{Br}(X_{K_v})$. By definition of $B$, $(b_v)$ is the image of a (uniquely determined) element $(a_v) \in \bigoplus_v \mathrm{Br}(K_v)$, and hence $\mathrm{ev}_v(b_v, x_v) = a_v$ for any $(x_v) \in \prod_v X(K_v)$. Thus $\langle v, (x_v)\rangle$ is independent of the choice of $x_v$, and therefore the pairing $\langle , \rangle$ determines a character $\varphi\colon B \to \mathbb{Q}/\mathbb{Z}$. In particular, for any $(x_v)$ we have that $\langle b, (x_v)\rangle = 0$ for all $b$ iff $\varphi = 0$. That is:

$$X^{\mathbf{B}} \neq \emptyset \iff \varphi = 0. \tag{8}$$

Observe that $\varphi$ can be viewed as a map $\mathbf{TS}(P) \to \mathbb{Q}/\mathbb{Z}$, and as such is exactly the connecting map supplied by the Snake Lemma. Indeed, we obtain a diagram

$$\begin{tikzcd} & & 0 \arrow[d] & \\ & & \mathbb{Z}/\delta \arrow[d] & \\ & & \mathbf{TS}(J) \arrow[d] \arrow[dr, "\chi"] & \\ 0 \arrow[r] & \mathrm{Br}(X)' \arrow[r] & \mathbf{TS}(P) \arrow[r, "\varphi"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \\ & & 0 & \end{tikzcd} \tag{9}$$

The column arises as follows: From the sequence

$$0 \longrightarrow \mathrm{Pic}^0(X_{\overline{K}}) \longrightarrow \mathrm{Pic}(X_{\overline{K}}) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

and the similar ones for $X_{\overline{K_v}}$ and using (7) and its local version we obtain

$$\begin{tikzcd} \mathrm{Pic}(X_K) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & H^1(G_K, \mathrm{Pic}^0(X_{\overline{K}})) \arrow[r] \arrow[d] & H^1(G_K, \mathrm{Pic}(X_{\overline{K}})) \arrow[r] \arrow[d] & 0 \\ \bigoplus_v \mathrm{Pic}(X_{K_v}) \arrow[r, "\deg"] & \bigoplus_v \mathbb{Z} \arrow[r, "0"] & \bigoplus_v H^1(G_{K_v}, \mathrm{Pic}^0(X_{\overline{K_v}})) \arrow[r] & \bigoplus_v H^1(G_{K_v}, \mathrm{Pic}(X_{\overline{K_v}})) \arrow[r] & 0 \end{tikzcd} \tag{10}$$

The existence of local points implies that $\mathrm{Pic}(X_{K_v}) \twoheadrightarrow \mathbb{Z}$, while the image of $\mathrm{Pic}(X) \to \mathbb{Z}$ (called the *period*) is exactly $\delta\mathbb{Z}$ [11, 2.5], so that the Snake Lemma gives the desired exact sequence.

The map $\chi$ can be given explicitly, in terms of the Cassels-Tate pairing:

$$\langle , \rangle_{CT} : \mathbf{TS}(J) \times \mathbf{TS}(J) \to \mathbb{Q}/\mathbb{Z}.$$

In [11, 2.11] it is shown that there exists an element $\beta \in \mathbf{TS}(J)$ of order $\delta$ such that for all $\alpha$

$$\chi(\alpha) = \langle \alpha, \beta \rangle_{CT}.$$

We are assuming that $\mathbf{TS}(J)$ is finite. Thus $\langle , \rangle_{CT}$ is non-degenerate, so that $\chi = 0$ iff $\beta = 0$, and hence

$$\chi = 0 \iff \delta = 1. \tag{11}$$

Thus by (8) and diagram (9) $X^B \neq \emptyset$ iff $\delta = 1$, which proves Theorem 1.1. □

The following result is stated in [16].

**Corollary 2.3.** *If $X$ has genus 1 then $X(K) = \emptyset$ iff $X^{\mathbf{Br}} = \emptyset$.*

*Proof.* Let $X$ be a genus 1 curve. We prove the contrapositive result. Thus, we assume that $X^{\mathbf{Br}} \neq \emptyset$, and must show that $X(K) \neq \emptyset$. Applying Theorem 1.1 we see that $X$ has index 1. Let $\Delta$ be a divisor of degree 1. The Riemann-Roch Theorem implies that $\ell(\Delta) = 1$ and the result follows by Lemma 4.1(a) below. □

## 3. Index One

In the remainder of this paper we shall assume that $X$ has index 1, with fixed degree 1 divisor $\Delta$. Our goal is to prove Theorem 1.2. We must show that there is a bijection of sets

$$X^{\mathbf{Br}} \cong \xi_\Delta(\prod X(K_v)) \cap \overline{J(K)}.$$

The strategy is to reformulate the basic pairing $\langle,\rangle$ in terms of the dual of diagram (6), and then use duality results of Tate and Lichtenbaum.

We first refine diagram (6. From diagram (10) we see that

$$H^1(K, J) \cong H^1(G_K, \mathrm{Pic}(X_{\overline{K}}))$$

and similarly for $H^1(K_v, J)$, while from (9) we have $\mathbf{TS}(J) \cong \mathbf{TS}(P)$. Thus we have

(12)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & \mathrm{Br}(X)' \arrow[d] & \mathbf{TS}(J) \arrow[d] & \\
0 \arrow[r] & \mathrm{Br}(K) \arrow[r] \arrow[d] & \mathrm{Br}(X_K) \arrow[r] \arrow[d] & H^1(K, J) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_v \mathrm{Br}(K_v) \arrow[r] \arrow[d, "\sum_v \mathrm{inv}_v"'] & \bigoplus_v \mathrm{Br}(X_{K_v}) \arrow[r] \arrow[d] & \bigoplus_v H^1(Kv, J) \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Q}/\mathbb{Z} \arrow[d] & T \arrow[d] & W \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $T$ and $W$ are the indicated cokernels.

Let $(x_v) \in \prod_v X(K_v)$. The evaluation maps $\mathrm{ev}_v(\_, x_v)\colon \bigoplus \mathrm{Br}(X_{K_v}) \to \bigoplus_v \mathrm{Br}(K_v)$ induce $\beta_{(x_v)}\colon \mathrm{Br}(X_K)/\mathrm{Br}(X)' \to \mathbb{Q}/\mathbb{Z}$, and by definition of the pairing $\langle,\rangle$ we have $(x_v) \in X^{\mathbf{Br}}$ iff $\beta_{(x_v)}(\overline{b}) = 0$ for every $b \in \mathrm{Br}(X_K)$, that is, iff $\beta_{(x_v)}$ is the zero map. Let $\beta((x_v)) = \beta_{(x_v)}$. Then

$$X^{\mathbf{Br}} = \ker\left(\prod_v X(K_v) \xrightarrow{\beta} \left(\frac{\mathrm{Br}(X_K)}{\mathrm{Br}(X)'}\right)^*\right)$$

We now take duals. Write $(\_)^*$ for the exact contravariant functor $\mathrm{Hom}(\_, \mathbb{Q}/\mathbb{Z})$ from profinite to torsion abelian groups.

We first use Tate duality on the third column of diagram (12). The dual sequence begins [12, p. 102]

$$0 \to \overline{J(K)} \to \textstyle\prod_v' J(K_v)$$

where $\overline{J(K)}$ is the closure of $J(K)$ in $\prod_v' J(K_v)$, which is equal to its profinite completion. The prime on the product indicates that when $v$ is a real prime we must take $J(K_v)$ modulo the connected component of 0.

Lichtenbaum in [10, Theorems 1,4] defines a pairing

$$\begin{array}{rcl} \psi\colon \mathrm{Br}(X_{K_v}) \times \mathrm{Pic}(X_{K_v}) & \to & \mathbb{Q}/\mathbb{Z} \\ (b_v, [x_v]) & \mapsto & \mathrm{inv}_v \mathrm{ev}_v(b_v, x_v) \end{array}$$

which induces an isomorphism $\mathrm{Pic}(X_{K_v})^* \to \mathrm{Br}(X_{K_v})$ for every finite $v$. Again, if $v$ is Archimedean then we must take $\mathrm{Pic}(X_{K_v})$ modulo the connected component of 0. On taking the dual of the second and third columns of diagram 12 we thus get the following.

(13)
$$\begin{tikzcd}
 & \left(\frac{\mathrm{Br}(X)}{\mathrm{Br}(X)'}\right)^* & & \left(\frac{H^1(K,J)}{\mathbf{TS}(J)}\right)^* \arrow[ll] & 0 \arrow[l] \\
\prod_v \mathrm{Br}(X_{K_v})^* \arrow[ur] & & \prod_v X(K_v) \arrow[ul, dotted, "\beta"'] \arrow[dr, "\xi_\Delta"] & & \\
 & \mathrm{Pic}(X_{K_v})^{**} \arrow[ul, "\psi", "\cong"'] \arrow[uu] & & \prod_v' J(K_v) \arrow[ll, "i+\Delta"'] \arrow[uu] & \\
 & T^* \arrow[u] & & \overline{J(K)} \arrow[ll] \arrow[u] &
\end{tikzcd}$$

The map $\prod_v' J(K_v) \to \mathrm{Pic}(X_{K_v})^{**}$ is just induced by inclusion [10, §5], and we translate it by $\Delta$ to identify $\mathrm{Pic}^0$ and $\mathrm{Pic}^1$. Thus $X^{\mathbf{Br}}$ is the subset of $\prod_v X(K_v)$ which maps into the image of $T^*$ in $\prod_v \mathrm{Br}(X_{K_v})^*$. After a small diagram chase we see that inside $\prod_v \mathrm{Br}(X_{K_v})^*$ the image $\prod_v' J(X_{K_v}) \cap T^*$ comes from $\overline{J(K)}$. That is,

$$X^{\mathbf{Br}} \cong \xi(\prod_v X(K_v)) \cap \overline{J(K)}.$$

## 4. Calculating $X^{\mathbf{Br}}$

The intersections in Theorem 1.2 are still difficult to work with. In this section we consider the more easily tested conditions (3) and (4). In particular, the topological closure is also much more easily dealt with over the finite fields. The simple observation that an *effective* divisor of degree 1 must actually be a $K$-rational point generalizes as follows.

**Lemma 4.1.** *Let $D$ be any divisor of degree* 1. *Then $X(K) \neq \emptyset$ if any of the following hold.*

(a) $\ell_K(D) > 0$
(b) *There exists a finite prime $v$ with $\ell_{K_v}(D) > 0$.*
(c) *There exist infinitely many finite primes $v$ with $\ell_{k_v}(D) > 0$.*

*Proof.* (a) Let $F \in K(X)^\times$ with $\mathrm{div}(F) + D \geq 0$, and let $D' = D + \mathrm{div} F$. Then $\deg(D') = 1$ and $D'$ is effective, which is only possible if $D = 1 \cdot P$ for some $P \in X(K)$.

(b) This follows from (a), since $\ell_{K_v}(D) = \ell_K(D)$. The dimension of the vector bundle remains unchanged on tensoring with $\mathbb{Q}_p$. See [9, III.9.9.3].

(c) Assume $X$ is defined over $\mathbb{Q}$. We must show that $\ell_{\mathbb{F}_p}(D) = \ell_{\mathbb{Q}}(D)$ almost always. We consider $X$ as a scheme over spec $(\mathbb{Z})$. A Weil divisor $\sum n_i \cdot P_i$ will correspond to a Cartier divisor for all $p$ with $X_p$ non-singular, hence for almost all $p$. Thus we may discard the finitely many primes for which $X$ has singular fibres, and let $D$ be a Cartier divisor, and $\mathcal{L}$ the associated invertible sheaf, so that $\ell_{\mathbb{F}_p}(D) = \dim H^0(X, \mathcal{L}_p)$. The dimension of $H^0(X, \mathcal{L}_p)$ is essentially an upper semicontinuous function of $p$ [9, III, 12.8], and is hence is almost everywhere equal to the generic dimension. □

**Lemma 4.2.** *Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of discrete topological spaces, let $A$ be a set, and suppose $f\colon A \to \prod_{i=1}^{\infty} A_i$ is a map of sets. Let $f_i\colon A \to A_i$ be the composite of $f$ with the $i$-th projection, and endow $\prod_{i=1}^{\infty} A_i$ with the product topology. Let $B_i \subseteq A_i$. Then the following are equivalent.*

(a) $\overline{f(A)} \cap \prod_{i=1}^{\infty} B_i \neq \emptyset$.
(b) *There exists $b = (b_i) \in \prod_{i=1}^{\infty} B_i$ such that for every $n \in \mathbb{N}$ there exists $a_n \in A$ with $f_i(a_n) = b_i$ for $i = 1, 2, \dots, n$.*

*Proof.* (*a*) $\implies$ (*b*) Fix $b = (b_i) \in \overline{f(A)} \cap \prod_{i=1}^{\infty} B_i$. For every $n$ let $U_n = \{b_1\} \times \cdots \times \{b_n\} \times \prod_{i>n} B_i$. This is an open neighbourhood of $b$, and hence for every $n$ there exists $a_n \in A$ with $f(a_n) \in U_n \cap f(A)$. That is, $f_i(a_n) = b_i$ for $i = 1, 2, \dots, n$.

(*b*) $\implies$ (*a*) Suppose $b = (b_i) \in \prod_{i=1}^{\infty} B_i$ and $a_n \in A$ satisfy the conditions above. Let $U = U_1 \times \cdots \times U_n \times \prod_{i>n} A_i$ be any basic open neighbourhood of $b$. Then $U$ contains $V = \{b_1\} \times \cdots \times \{b_n\} \times \prod_{i>n} A_i$. But $f(a_n) \in V$ so every open neighbourhood of $b$ meets $f(A)$, and thus $b \in \overline{f(A)} \cap \prod_{i=1}^{\infty} B_i$. □

The topology on $X(K_v)$ induces the discrete topology on $X(k_v)$, and thus we may apply Lemma 4.2 to condition (4). Write $A = J(K)$, $B_v = \xi_\Delta X(k_v)$, and let $f_v$ be the natural projection $f_v\colon K \to k_v$. Then (4) holds iff there exists $(x_{v_i}) \in \prod_{v_i} X(k_{v_i})$ and a sequence of points $a_n \in J(K)$ with $a_n \equiv \xi(x_{v_i})$ in $k_{v_i}$ for all $i < n$.

*Proof of Theorem* 1.4 We always have (a) $\implies$ (b) $\implies$ (d) and (a) $\implies$ (c) $\implies$ (d). To see that (d) $\implies$ (a) note that any sequence in $J(K)$ must contain a constant subsequence. Thus there exists $a \in J(K)$ with $a \equiv \xi(x_v)$ in $k_v$ for every $v$. Thus $a - [\Delta_v] = [x_v] > 0$ in $\operatorname{Pic}(X_{k_v})$, so $\ell_{k_v}(\Delta_v) > 0$. The result now follows from Lemma 4.1(c). □

The final difficulty in checking conditions (3) and (4) is that the Jacobian may not be easy to calculate. We introduce the following variant. Let $A$ be any abelian variety for which there is a nonconstant map $\theta\colon X \to A$ defined over $K$. After composing $\theta$ with a translation, we may assume that $\theta$ maps some geometric point $P$ to 0. Now $\xi_\Delta$ is just a translation of the standard embedding $f^P\colon X(\overline{K}) \to J(\overline{K})$ which maps $P$ to 0, and thus the universal property of the Jacobian [14, 6.1] implies that there exists a map $\lambda\colon J \to A$ with $\theta = \lambda \circ \xi_\Delta$. Hence $\lambda$ maps $\xi_\Delta(\prod X(k_v)) \cap \overline{J(K)}$ into $\theta(\prod X(k_v)) \cap \overline{A(K)}$. Thus if this intersection is empty, $X^{\mathbf{Br}}$ must be too.

In practice we would like a finite test. Suppose for convienence that $X$ is defined over $\mathbb{Q}$. If take only a finite product of finite fields then the toplogy is discrete, and taking the closure has no effect. Thus we have the following.

**Proposition 4.3.** *Let $X$ be defined over $\mathbb{Q}$ with index 1, and assume that $\mathbf{TS}(J)$ is finite. Let $\theta\colon X \to A$ be as above, and let $S$ be a finite set of prime numbers. If*

$$\theta(\prod_{p\in S} X(\mathbb{F}_p)) \cap A(\mathbb{Q}) = \emptyset$$

*then $X^{\mathbf{Br}} = \emptyset$ and hence $X(\mathbb{Q}) = \emptyset$.*

## 5. Examples

In this section we consider two examples. The more interesting example is the Bremner-Lewis-Morton curve $\Gamma : 3x^4 + 4y^4 = 19z^4$ over $\mathbb{Q}$. Skorobogatov [21] asked if $\Gamma^{\mathbf{Br}} = \emptyset$ for this curve. We show it is, provided $\mathbf{TS}(J)$ is finite. Unfortunately $\Gamma$ has an infinite

Jacobian, so Corollary 1.3 does not apply. However Proposition 4.3 can be applied after some work. This example also shows that in Corollary 1.3 finiteness of the Jacobian is sufficient but not necessary to produce a trivial Brauer-Manin obstruction.

**Theorem 5.1.** *The projective curve*

$$\Gamma : 3x^4 + 4y^4 = 19z^4$$

*has the following properties:*

(a) *There are points on* $\Gamma$ *over every* $p$*-adic field* $\mathbb{Q}_p$*, and over* $\mathbb{R}$.
(b) *The index of* $\Gamma$ *is 1.*
(c) *The Jacobian of* $\Gamma$ *over* $\mathbb{Q}$ *is infinite.*
(d) *Nonetheless, if* $\mathbf{TS}(J)$ *is finite, then* $\Gamma^{\mathbf{Br}}$ *is empty (and hence* $\Gamma$ *has no rational points).*

*Proof.* Parts (a) and (b) are proved by Bremner, Lewis and Morton [2, II(a)], who also show that $\Gamma$ has no rational points. Cassels provides an alternative proof of this last statement is given in [5, Appendix], and also notes that (c) holds.

To prove (d) we show that the condition (4) fails for $\Gamma$. The Jacobian is isogenous to a product of 3 elliptic curves [5]. We work with just one of these.

The map $\theta : (x : y : z) \mapsto (-12y^2z : 36x^2y : z^3)$ sends a point on curve $\Gamma$ to a point on the elliptic curve

$$E : y^2z = x^3 - 684xz^2.$$

The map is well defined over $\mathbb{Q}$ or $\mathbb{F}_p$ with $p \neq 2$, 3, 19, since at least one image co-ordinate is always non-zero.

We need a description of the Mordell-Weil group of $E$.

**Lemma 5.2.** *The elliptic curve* $E$ *has good reduction for* $p \neq 2, 3, 19$ *and has Mordell-Weil group*

$$E(\mathbb{Q}) = \mathbb{Z} \times \mathbb{Z}/2.$$

*generated by the points* $P = (-3 : 45 : 1)$ *of infinite order, and* $T = (0 : 0 : 1)$ *of order* 2.

*Proof.* The conductor of $E$ is $2^5 \cdot 3^2 \cdot 19^2$. By Silverman [19, X.6.1] or explicit calculation, $E(\mathbb{Q}) = \mathbb{Z}^{r_i} \times \mathbb{Z}/2$, for some $r \geq 1$. The rank can be calculated by 2-descent using the algorithm described in [8, III.6, page 84] (there are no difficulties), or read from the table in [1].

Finally we must show that $P$ is not a proper multiple of a point of smaller height. Cremona [8, III.5.1] gives the following explicit bound due to Silverman for the curve $y^2 = x^3 + dx$:

$$h(Q) \leq \hat{h}(Q) + \frac{1}{2}\log(|d|) + 4.479 \tag{14}$$

for any point $Q = (a/c^2, b/c^3)$ on the curve, where $h(Q)$ is the naive height $\log\max\{|a|, c^2\}$ and $\hat{h}$ is the canonical height.

Suppose $P = n \cdot Q + T$ where $T$ is a torsion element and $|n| \geq 2$, so that $\hat{h}(Q) \leq \frac{1}{4}\hat{h}(P)$. This gives $|a| < 3050$ and $|c| < 56$, but a computer search for points up to this height bound yields only points of the form $n \cdot P$ and $n \cdot P + T$. □

After some experimentation, we find that can apply Proposition 4.3 with the finite set of primes $\{17, 31\}$. To complete the proof of Theorem 5.1 it is thus enough to show that no $R = n \cdot P + \epsilon \cdot T \in E(\mathbb{Q})$ with $n \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$ can be in

$$\theta\Gamma(\mathbb{F}_{17}) \cap \theta\Gamma(\mathbb{F}_{31}) \tag{15}$$

For convenience we use affine co-ordinates, and denote the point at infinity by 0.

**p = 17** Over $\mathbb{F}_{17}$ $P \equiv (-3,-6)$ of order 4. We have the following table of values of $n \cdot P + \epsilon \cdot T$.

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $\epsilon=0$ | 0 | (-3,-6) | (2,0) | (-3,6) |
| $\epsilon=1$ | (0,0) | (7,3) | (-2,0) | (7,-3) |

A quick search shows that $\theta\Gamma(\mathbb{F}_{17}) = \{(7,\pm 3),(10,\pm 5)\}$, so that $n$ must be odd and $\epsilon = 1$.

**p = 31** Over $\mathbb{F}_{31} P \equiv (-3,14)$ has order 8. We thus have only to consider the points $P+T = (11,10)$, $3 \cdot P + T = (-1,1)$, $5 \cdot P + T = (-1,-1)$ and $7 \cdot P + T = (11,-10)$. We find that $\theta\Gamma(\mathbb{F}_{31}) = \{0,(2,\pm 2),(4,\pm 5),(7,\pm 9),(8,0),(10,\pm 9),(14,\pm 9),(19,\pm 1),(28,\pm 4)\}$. Hence the intersection (15) is empty. □

We give a genus 2 example. The following curve was considered in [7].

**Example 5.3.** Let C be the smooth projective model of the affine curve

$$C : s^2 = 2(t^3+7)(t^3-7).$$

Then $C$ has points locally everywhere, has index 1, but has no global points. Moreover $C^{\mathbf{Br}} = \emptyset$, so that the Brauer-Manin obstruction is the only obstruction for $C$.

*Proof.* Coray and Manoil [7, Proposition 4.6, p 183] showed that $C$ has index 1 but fails the Hasse Principle. They also observed that $C$ maps into the elliptic curve

$$E : y^2 = x^3 - 392$$

via $(s,t) \mapsto (2t^2, 2s)$, and that $E(\mathbb{Q})$ consists only of the point at infinity. After replacing the Jacobian with $E$ we can apply Theorem 1.4. □

Proposition 4.3 is very effective at showing that a curve has no rational points; I have been unable to produce an example of a curve $X$ without a global point for which condition (4) still holds. If none exist then we would be close to a proof that the Brauer-Manin obstruction is the only one for curves.

Further interesting curves can be found in [3].

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