Deligne's Notes on Symmetric Hermitian Spaces (Spring 1973)*

Abstract

"Ces notes, qui ne sont pas destineés à être publieés, sont avant tout un résumé d'une partie de la théorie classique des domaines hermitiens symétriques, et accessoirement une mise au goût du jour (ou à mon goût personnel). Elles sont suscitées par mon désir de comprendre les travaux récents de Mumford sur les compactifications des quotients des domaines hermitiens symétriques par des groupes arithmétiques."

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1 Dramatis Personæ

1.1 Review of symmetric riemannian domains

(Partial summary of Helgason 1962, Chap. IV, V.)

A preliminary result is the following (Helgason 1962, IV 1.2, p165).

THEOREM 1.1.1. *Let M and M' be differentiable manifolds with affine connections ∇ and ∇'. Assume that the torsion tensors T and T' and the curvature tensors ∇ and ∇' of M and M' are invariant under parallel :*

$$\nabla T = 0 = \nabla R, \quad \nabla T' = 0 = \nabla R'.$$

Let x and x' be points of M and M', and let $A\colon T_x \to T_{x'}$ be an isomorphism of the tangent spaces at x and x' sending T and R onto T' and R'. Then there exists a germ of an isomorphism $a\colon (M, \nabla, x) \to (M', \nabla', x')$ whose differential is A.

PROOF. Write $\exp_x\colon T_x \to M$ for the exponential map and $\log_x$ for its inverse. The germ must satisfy

$$a(y) = \exp_{x'}(A \log_x y),$$

and this formula defines a local isomorphism of (M, ∇) with (M', ∇'), as can be seen by writing the Cartan equations in "polar coordinates" (Helgason 1962, I §8, (6), (7), p47). □

REMARK 1.1.2. (i) Let (M, ∇) be such that $\nabla T = 0 = \nabla R$. The Cartan equations show also that the connection $\log_x(\nabla)$ on a neighbourhood of the origin in T_x is real analytic. There exists therefore on M a unique real analytic structure such that ∇ is analytic. The $\log_x$ form a system of local coordinates for it.

(ii) Under the hypotheses of (1.1.1), assume that M and M' are simply connected and geodesically complete. Then a can be analytically continued to a global isomorphism

$$\tilde{a} : (M, \nabla, x) \to (M', \nabla', x')$$

with differential A.

COROLLARY 1.1.3. *Let (M, ∇) be such that $\nabla T = 0 = \nabla R$ and M is simply connected and geodesically complete. Let $\gamma(t)$ be a geodesic:*

$$\nabla_t \dot{\gamma}(t) = 0.$$

There exists a unique one parameter group of automorphisms $\tau(\gamma, u)$ of M that induces translation $\gamma(t) \mapsto \gamma(t + u)$ on the geodesic γ and displaces the tangent bundle on γ by parallel transport.

PROOF. Let $\tau(\gamma, u)$ be the automorphism sending $\gamma(0)$ to $\gamma(u)$ and which has for differential

$$d\tau(\gamma, u)|_0\colon T_{\gamma(0)} \to T_{\gamma(u)}$$

the parallel transport along γ. Since

$$d\tau(\gamma, u)(\dot{\gamma}(0)) = \dot{\gamma}(u),$$

$\tau(\gamma, u)(\gamma(t)) = \gamma(t + u)$. If $X \in T_{\gamma(t)}$ is parallel to $X_0 \in T_{\gamma(0)}$, then $\tau(\gamma, u)(X)$ is parallel to $d\tau(\gamma, u)(X_0) // X$, from which the corollary follows. □

REMARK 1.1.4. Let (M, ∇) be such that[1] $\nabla T = 0 = \nabla R$ and M is simply connected. The existence of the $\tau(\gamma, u)$ in (1.1.3) is equivalent to M being geodesically complete. From that, and the fact that a Lie groupuscule generates a Lie group, one can deduce that M is an "espace étalé" over a similar (M_1, ∇_1) with M_1 simply connected and geodesically complete.

COROLLARY 1.1.5. *Let M have an affine connection ∇. The following conditions are equivalent:*

(i) $T = \nabla R = 0$;

(ii) for all $x \in M$, there exists a germ at x of an automorphism of (M, ∇) inducing $t \mapsto -t$ on the tangent space at x.

PROOF. (ii) $\Longrightarrow$ (i). The tensors T and ∇R are respected by s_x and are of odd degree, hence zero.
(i) $\Longrightarrow$ (ii). Apply (1.1.1) to $t \mapsto -t$. □

COROLLARY 1.1.6. *Let M be a riemannian manifold. The following conditions are equivalent:*

(i) $\nabla R = 0$;

(ii) the sectional curvature is invariant under parallel transport;

(iii) for all $x \in M$, $y \mapsto \exp_x(-\log_x(y))$ is the germ of an isometry.

PROOF. The equivalence of (i) and (iii) follows from (1.1.5), and (ii) $\Longrightarrow$ (i) because, in view of the symmetry properties of R, the sectional curvature determines R. □

DEFINITION 1.1.7. A riemannian manifold M satisfying the equivalent conditions (1.1.5) is said to be *locally symmetric*. It is *symmetric* if M is connected and for all $x \in M$ there exists an isometric involution s_x (*the symmetry with respect to* x) with x as an isolated fixed point (whence $ds_x = -1$). Such a manifold is complete; conversely, a complete simply connected manifold that is locally symmetric is symmetric. A simply connected locally symmetric manifold is étale over a symmetric manifold.

Let M be a symmetric space and let $\gamma(t)$ be a geodesic. Write s_t for the symmetry with respect to $\gamma(t)$. Then $s_t(\gamma(t+u)) = \gamma(t-u)$.

PROPOSITION 1.1.8. *(i) For any tangent vector X at $\gamma(t+u)$,*

$$s_t(X) /\!/ -X \tag{1.1.8.1}$$

(parallel transport along γ from $\gamma(t-u)$ to $\gamma(t+u)$).

(ii) We have

$$\tau(\gamma, u) = s_{u/2} \circ s_0, \tag{1.1.8.2}$$

and

$$s_t \circ \tau(\gamma, u) \circ s_t^{-1} = \tau(\gamma, -u). \tag{1.1.8.3}$$

[1] The original only has $\nabla T = \nabla R$.

PROOF. Let X_0 be the tangent vector at $\gamma(t)$ deduced from X by parallel transport. Applying s_t to $X//X_0$, we find that $s_t(X)// - X_0$, whence (1.1.8.1). From this, $s_{u/2} \circ s_0$ satisfies the condition characterizing $\tau(\gamma, u)$. Now (1.1.8.3) results from the fact that $s_t\gamma(t+u) = \gamma(t-u)$ by transport of structure. □

1.1.9. Let M be a symmetric riemannian space, and let $\mathfrak{g}$ be the Lie algebra of the identity component G of its Lie group of automorphisms. Let K be the stabilizer of a point $x \in M$, and let σ be the involution $g \mapsto s_x g s_x^{-1}$ of G. Since K injects into $O(T_x)$, the elements of K are fixed by σ. We have $M = G/K$. It follows from (1.1.8.3) that the set $\mathfrak{p} \subset \mathrm{Lie}(G)$ of infinitesimal generators of the one parameter subgroups $\tau(\gamma, u)$ (γ geodesic through u) is contained in the subspace of $\mathrm{Lie}(G)$ on which σ acts as -1. On the other hand, σ is the identity on $\mathfrak{k} = \mathrm{Lie}(K)$. Since $\mathfrak{p}$ maps onto T_x, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the subspaces of $\mathfrak{g}$ where σ acts as 1 or -1.

1.1.10 (CONVERSE). Let G be a Lie group and let K be a compact subgroup of G. Assume that

(a) $M = G/K$ is connected, and

(b) there exists an involution σ of G such that $\mathrm{Lie}(K) = \mathrm{Lie}(G)^\sigma$ and $K = K^\sigma$.

Then, M is symmetric riemannian for any invariant riemannian structure Q. If $\mathfrak{p}$ is the subspace of $\mathrm{Lie}(G)$ where $\sigma = -1$, the $\exp(\pi u)$ ($\pi \in \mathfrak{p}$) are the one parameter groups of isometries (1.1.3) relative to geodesics passing through the origin.

(i) Since M is homogeneous, it suffices to prove the existence of a symmetry with respect to the origin: take $\sigma\colon G/K \to G/K$.

(ii) We have

$$(\exp(\pi/2) \cdot \sigma \cdot \exp(\pi/2)^{-1})\sigma = \exp(\pi/2) \cdot \exp(-\sigma(\pi/2)) = \exp(\pi),$$

and so can apply (1.1.8.2).

REMARK 1.1.11. (i) The following conditions are equivalent:

(a) G acts faithfully on M;

(b) K acts faithfully on $\mathfrak{p} = \mathrm{Lie}(G)/\mathrm{Lie}(K) =$ tangent space to M at the origin.

When they hold, σ is the identity on K and is uniquely determined by K: the Killing form is nondegenerate on $\mathrm{Lie}(K)$, and $d\sigma$ is the reflection with respect to $\mathrm{Lie}(K)$.

(ii) To give Q is the same as to give a K-invariant Euclidian structure Q_0 on $\mathfrak{p}$ (the tangent space to M at the origin). After (1.1.3) and (1.1.10), the riemannian connection ($=$ parallel transport) is independent of the choice of Q.

REMARK 1.1.12. Pass to the Lie algebras. Let $\mathfrak{g}$ be a real Lie algebra, let σ be an involution of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition into ± 1 eigenspaces. We assume that $\mathfrak{k}$ is compact in $\mathfrak{g}$ (equivalently, the Killing form is negative definite on $\mathfrak{k}$), and that $\mathfrak{p}$ is a faithful representation of $\mathfrak{k}$ (cf. 1.1.11(i)). Let $\mathfrak{p}_0 \subset \mathfrak{p} = \mathfrak{k}^\perp$ be the kernel of the Killing form B, and let $\mathfrak{p} = \mathfrak{p}_0 \oplus \oplus_{i \neq 0} \mathfrak{p}_i$ be a decomposition of $\mathfrak{p}$ into sub-$\mathfrak{k}$-representations. Assume that $\mathfrak{p}_i$ is orthogonal to $\mathfrak{p}_j$ relative to B (this is in fact automatic).

(a) For $i = 0$ or $i \neq j$, $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$.

Indeed, since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and B is nondegenerate on $\mathfrak{k}$, it suffices to show that $B(\mathfrak{k}, [\mathfrak{p}_i, \mathfrak{p}_j]) = 0$. But

$$B(\mathfrak{k}, [\mathfrak{p}_i, \mathfrak{p}_j]) = B([\mathfrak{k}, \mathfrak{p}_i], \mathfrak{p}_j) \subset B(\mathfrak{p}_i, \mathfrak{p}_j) = 0.$$

(b) If $i \neq 0$ and $x \in \mathfrak{k}$ is orthogonal to $[\mathfrak{p}_i, \mathfrak{p}_i]$, then $[x, \mathfrak{p}_i] = 0$.

Indeed, $B|\mathfrak{p}_i$ is nondegenerate and

$$B([x, \mathfrak{p}_i], \mathfrak{p}_i) = B(x, [\mathfrak{p}_i, \mathfrak{p}_i]) = 0.$$

Let $\mathfrak{k}_i = [\mathfrak{p}_i, \mathfrak{p}_i]$, let $\mathfrak{k}_0$ be the orthogonal complement of $\oplus \mathfrak{k}_i$ in $\mathfrak{k}$, and let $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. We have:

(c) $\mathfrak{g} = \oplus \mathfrak{g}_i$ (direct sum of Lie algebras).

For $i \neq 0$, one checks immediately that $\mathfrak{k}_i$ is a Lie subalgebra. But $\mathfrak{k}_0$ is the centraliser of $\oplus_{i \neq 0} \mathfrak{p}_i$ in $\mathfrak{k}$, and therefore is also a Lie subalgebra. Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ is a subalgebra and, for $i \neq j$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$. As $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$, one deduces that the sum of the $\mathfrak{k}_i$, hence of the $\mathfrak{g}_i$, is direct. For $i \neq 0$, $[\mathfrak{p}_i, \mathfrak{p}_i] \neq 0$ (otherwise $B|\mathfrak{p}_i = 0$).

It is possible in the decomposition of $\mathfrak{p}$ to take the $\mathfrak{p}_i$ to be irreducible: let Q be a $\mathfrak{k}$-invariant quadratic form that is positive definite on $\mathfrak{p}$ and let $B(x, y) = Q(x, Hy)$. First decompose $\mathfrak{p}$ according to the eigenvalues of the symmetric operator H. On the components obtained (except $\mathfrak{p}_0$), B is definite, because it is a nonzero multiple of Q, and one continues the decomposition. In the final decomposition, no two of the $\mathfrak{p}_i$ $(i \neq 0)$ give the same representation, because the $\mathfrak{p}_i$ have different centralizers in $\mathfrak{k}$. This proves automatically that $\mathfrak{p}_i$ is orthogonal to $\mathfrak{p}_j$ under B. We have proved:

THEOREM 1.1.13. *The pair* $(\mathfrak{g}, \sigma)$ *admits a unique decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{i \neq 0} \mathfrak{g}_i, \quad \mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i,$$

such that:

(a) $\mathfrak{p}_0$ *is an abelian ideal in* $\mathfrak{g}_0$ *(euclidean case);* $\mathfrak{g}_0$ *is the semi-direct product of* $\mathfrak{k}_0$ *and* $\mathfrak{p}_0$*;*

(b) for $i \neq 0$, $\mathfrak{p}_i$ *is an irreducible representation of* $\mathfrak{k}_i$, $\mathfrak{k}_i \neq 0$, *and* $\mathfrak{k}_i = [\mathfrak{p}_i, \mathfrak{p}_i]$.

For $i \neq 0$, $B|\mathfrak{p}_i$ is either negative definite (in which case $\mathfrak{g}_i$ is compact) or positive definite (in which case $\mathfrak{g}_i$ is noncompact and $\mathfrak{k}_i$ is a maximal compact subalgebra).

For the second assertion, note that, up to a factor, there is only one $\mathfrak{k}_i$-invariant form on $\mathfrak{p}_i$.

1.1.14 (CLASSIFICATION). (a) The euclidean case corresponds to euclidean space. Here $\mathfrak{k}_0$ is any subalgebra of $\mathfrak{so}(\mathfrak{p}_0)$.

(b) If B is positive definite on $\mathfrak{p}$, $\mathfrak{g}$ is semisimple and σ is a Cartan involution. If $(\mathfrak{g}, \sigma)$ is indecomposable, then $\mathfrak{g}$ is noncompact and simple, and it is either absolutely simple or complex. Converse...

(c) The duality

$$\begin{aligned} \mathfrak{g} \mapsto & \mathfrak{g}^{(\sigma)} = \{x \in \mathfrak{g}_{\mathbb{C}} \mid \bar{x} = \sigma(x)\} \text{ (same } \sigma) \\ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \mapsto & \mathfrak{g}^{(\sigma)} = \mathfrak{k} \oplus i\mathfrak{p} \end{aligned}$$

interchanges the cases $B > 0$ and $B < 0$.

(b*) If $B < 0$, $\mathfrak{g}$ is compact and semisimple. If $(\mathfrak{g}, \sigma)$ is indecomposable, then $\mathfrak{g}$ is simple and compact, and σ is any involution, or $\mathfrak{g} \approx \mathfrak{g}_1 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_1$ simple and compact and σ is the involution $(x, y) \mapsto (y, x)$.

In the cases (b) and (b*), the $\mathrm{ad}\pi$ $(\pi \in \mathfrak{p})$ are semisimple, because π or $\iota\pi$ is in the Lie algebra of a compact group.

COROLLARY 1.1.15. *In addition to the hypotheses of (1.1.6), assume that G is connected, semisimple, and acts faithfully on M. Then,*

(i) G is the identity component of the group of isometries of M;

(ii) the identity component of K is the identity component of the group of isometries of $\mathfrak{p}$ (identified with the tangent space at the origin), endowed with Q_0 and the curvature tensor R.

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}'$ that of the group of isometries of M. Apply 1.1.9 to $\mathfrak{g}$ and $\mathfrak{g}'$; $\mathfrak{p}$ is the same for $\mathfrak{g}$ and $\mathfrak{g}'$ (it is the tangent space at the origin), and it contains no nontrivial abelian ideal. Therefore

$$\mathfrak{g}' = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] = \mathfrak{g}.$$

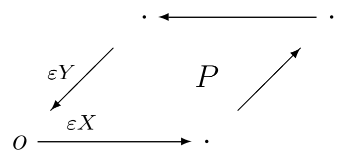
The assertion on K results from (1.1.1) or from an argument analogous to the preceding. □

1.1.16 (NORMALIZATION). As in Helgason 1962, we define the curvature tensor by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

(X, Y vector fields, $R(X, Y)$ a section of the fibre bundle "the Lie algebra of the linear group of the tangent bundle T_M" identified with $\mathrm{End}(T_M)$). If X and Y are two tangent vectors at o, and P is the infinitesimal parallelogram with sides ϵX and ϵY, then $1 -$

$R(\epsilon X, \epsilon Y)$ is the infinitesimal automorphism of the tangent bundle at o obtained by parallel transport along ∂P (mod ϵ^3):



PROPOSITION 1.1.17. *For X, Y in $\mathfrak{p}$,*

$$R(X,Y)(Z) = -[[X,Y],Z].$$

PROOF. Since $[X,Y] \in \mathfrak{k}$, the quadrilateral with geodesic sides $(0 \leq u \leq \varepsilon)$

$$\begin{aligned}
&\exp(uX)o,\\
&\exp(\varepsilon X)\exp(uY)\exp(-uY)o,\\
&(\exp(\varepsilon X)\exp(\varepsilon Y))\exp(-uX)\,(\exp(\varepsilon X)\exp(\varepsilon Y))^{-1},\\
&(\exp(\varepsilon X)\exp(\varepsilon Y)\exp(-\varepsilon X))\exp(-uY)(\exp(\varepsilon X)\exp(\varepsilon Y)\exp(-\varepsilon X))^{-1}
\end{aligned}$$

is closed (mod ε^3). After (1.1.6) and (1.1.1), we have

$$\begin{aligned}
(1 - R(\varepsilon X, \varepsilon Y))(Z) &= d(\exp(\varepsilon X)\exp(\varepsilon Y)\exp(-\varepsilon X)\exp(-\varepsilon Y))(Z)\\
&= d(\exp([\varepsilon X, \varepsilon Y]))(Z)\\
&= Z + [[\varepsilon X, \varepsilon Y], Z] \quad (\text{mod } \varepsilon^3),
\end{aligned}$$

whence the assertion. □

COROLLARY 1.1.18. *Under the hypotheses of (1.1.12), the identity component K^0 of K is the local holonomy group.*

COROLLARY 1.1.19. *Let $\mathfrak{s} \subset \mathfrak{p}$; then $\exp(\mathfrak{s})$ is totally geodesic if and only if $[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}] \subset \mathfrak{s}$.*

PROOF. The condition signifies that $[\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ is a Lie subalgebra (stable under σ). In a neighbourhood of 0, $\exp(\mathfrak{s})$ is then the corresponding symmetric riemannian subspace. □

THEOREM 1.1.20. *Let $H, X \in \mathfrak{p}$. Assume that X is closer (or further) from H that all its K^0-conjugates. Then $[H,X] = 0$.*

PROOF. By hypothesis, for $Z \in \mathfrak{k}$, $B(H, [Z,X]) = 0$. Therefore, $B([H,X], \mathfrak{k}) = 0$, and $[H,X] = 0$. □

COROLLARY 1.1.21. *The maximal commutative subalgebras of $\mathfrak{p}$ are conjugate among themselves under the identity component K^0 of K.*

PROOF. Use 1.1.20 and the fact that the centralizer of a general element in a commutative subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ coincides with the centralizer of $\mathfrak{a}$. □

REMARK 1.1.22. Assume that M has no compact factor ($B > 0$ on $\mathfrak{p}$). If[2] G is the adjoint real algebraic group with Lie algebra $\mathfrak{g}$, the maximal commutative subalgebras $\mathfrak{a}$ of $\mathfrak{p}$ are the Lie algebras of the maximal split tori S of G. The *Weyl group* W is the quotient of the normalizer of S in $\mathsf{G}(\mathbb{R})$ by its centralizer. We show in [?][3] that it is also the quotient of the normalizer of $\mathfrak{a}$ in K by its centralizer. The *rank* of M is the dimension of $\mathfrak{a}$.

[2]Underlined characters $\underline{G}$ in the orginal have been transcribed as san serif G.

[3]This denotes a gap in the original.

1.2 Review of symmetric hermitian spaces

(Partial summary of Helgason 1962, Chapt. VIII.)

A *hermitian manifold* is a differentiable manifold whose tangent bundle is endowed with a hermitian structure. We write J for multiplication by i on the tangent bundle, and ∇ for the riemannian connection.

PROPOSITION 1.2.1. *Let M be a hermitian manifold. The following conditions are equivalent:*

(i) $\nabla R = 0 = \nabla J$;

(ii) for all $x \in M$, $s_x \colon y \mapsto \exp_x(-\log_x y)$ is the germ at x of an automorphism of M.

PROOF. (i) $\Longrightarrow$ (ii). If $\nabla R = 0$, M is symmetric riemannian and s_x is an isometry. If $\nabla J = 0$, s_x respects J, because s_x respects J_x.
(ii) $\Longrightarrow$ (i). If (ii) is satisfied, every canonical tensor of odd degree is zero, because it is invariant by s_x. This applies to ∇R, ∇J, S (torsion of the complex structure), $d\Omega$ ($\Omega =$the 2-form which is the imaginary part of the hermitian structure form), and $\nabla\Omega$ (whose nullity follows also from those of S and $d\Omega$). □

One says that M is *locally symmetric* if the equivalent conditions of (1.2.1) are satisfied, and *symmetric* if it is connected and for all $x \in M$ there exists an involutive automorphism s_x of M (*the symmetry with respect to* x) with x as its isolated fixed point. Remark 1.1.4 can be transposed.

COROLLARY 1.2.2. *A locally symmetric hermitian manifold is Kählerian.*

COROLLARY 1.2.3. *Let M be a simply connected symmetric hermitian manifold, and let G be the identity component of its group of automorphisms; let x be a point of M, and let K be its stabilizer. There exists a homomorphism*[4] $u_x \colon U^1 \to K$ *such that $u_x(z)$ induces multiplication by z on T_x.*

PROOF. After [?] it suffices to show that, for $z \in U^1 \subset \mathbb{C}$, the curvature tensor R on T_x is invariant under the automorphism $v \mapsto zv$ of T_x. This assertion, which is true for any Kähler manifold, follows from:

(a) R is a 2-form with values in the Lie algebra of the unitary group;

(b) $R_{ijk\ell} = R_{k\ell ij}$.

□

REMARK 1.2.4. Let M be a symmetric hermitian manifold. The maps $\tau(\gamma, u)$ of (1.1.4) are the composites of symmetries, and are therefore automorphisms of M. Let G be the

[4]The original sometimes writes U^1 a sometimes U_1.

identity component of the Lie group of automorphisms of M, let $x \in M$, let K be the stabilizer of x, and let σ be the automorphism of G induced by s_x. As in (1.1.6),

$$M = G/K$$
$$\mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}.$$

The complex structure on M induces on $\mathfrak{p}$ a complex structure J, which is invariant by K. Conversely, if $M = G/K$ is symmetric and riemannian, every complex K-invariant structure on $\mathfrak{p}$ endows M with a hermitian symmetric structure.

REMARK 1.2.5. We return to the decomposition (1.1.10). If J_0 is a $\mathfrak{k}$-invariant complex structure on $\mathfrak{p}$, the $\mathfrak{p}_i$ are complex subspaces (because they are disjoint representations of $\mathfrak{k}$) and, for M hermitian symmetric and simply connected, we obtain a decomposition

$$M = M_0 \times \prod M_i \tag{1.2.5.1}$$

with M_0 euclidean and $(\mathfrak{g}_i, \sigma)$ indecomposable and semisimple. The duality (1.1.11) respects the symmetric hermitian character.

REMARK 1.2.6. Suppose that G is semisimple. In this case, G is again the identity component of the group of isometries of M, and G has the same Lie algebra $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ as the analogous group for the universal covering $\tilde{M}$ of M. We have therefore by (1.2.3) a map $u_x \colon U^1 \to K_x$. On $\mathfrak{p} \approx T_x$, $\mathrm{ad}u_x(z)$ induces multiplication by z. In particular, $\mathrm{ad}u_x(-1) = \sigma$.

PROPOSITION 1.2.7. *(i) K is the centralizer of $u(U^1)$ in G.*
(ii) G is adjoint, K is connected, and M is simply connected.
(iii) If $(\mathfrak{g}, \sigma)$ is indecomposable, $\mathfrak{g}$ is absolutely simple and $u(U^1)$ is the centre of K.

PROOF. (i) Let K' be the centralizer of $u(U^1)$. As $\sigma = \mathrm{ad}(u(-1))$, $\mathrm{Lie}(K') \subset \mathrm{Lie}(K)$ is compact, and K' is connected because it is the centralizer of a torus, whence $K' \subset K$. On the other hand, the representation of K on $\mathfrak{p}$ is $\mathbb{C}$-linear and faithful, so that $K \subset K'$.

(ii) After (i), K contains the centre Z of G. This last is therefore trivial. On applying this remark to the universal covering $\tilde{M}$ of M, one finds that $\tilde{M} = M$.

(iii) For an indecomposable $(\mathfrak{g}, \sigma)$, the representation of K on $\mathfrak{p}$ is simple; since $u(U^1) \subset K$, the commutant of K in $\mathrm{End}(\mathfrak{p})$ coincides with its commutant in $\mathrm{End}_{\mathbb{C}}(\mathfrak{p})$, and consists only of scalars. The centre of K is therefore reduced to $u(U^1)$. Since it is nonzero, one can read on [?] that $\mathfrak{g}$ it is absolutely simple. □

The space M is said to be *simple* if the condition (1.2.7(iii)) is satisfied.

PROPOSITION 1.2.8. *Let M be a symmetric riemann space, and let G, $x \in M$, K, and σ be as usual. Assume G is semisimple, K is connected, and $(\mathfrak{g}, \sigma)$ is indecomposable. The following conditions are equivalent:*

(i) the representation $\mathfrak{p}$ of $\mathfrak{k}$ is not absolutely simple;

(ii) the centre of $\mathfrak{k}$ is nonzero;

(iii) M *admits the structure of a symmetric hermitian space (compatible with its riemannian structure).*

The structure in (iii) is unique up to complex conjugation.

PROOF. As the representation $\mathfrak{p}$ of $\mathfrak{k}$ is simple, (i) is equivalent to the existence of a $\mathfrak{k}$-invariant complex structure. Since, up to a factor, there is only one $\mathfrak{k}$-invariant quadratic form on $\mathfrak{p}$, such a complex structure underlies a $\mathfrak{k}$-invariant hermitian structure. Therefore (ii) $\implies$ (i) $\iff$ (iii), and (iii) $\implies$ (ii) by (1.2.7(i)). The unicity results from (1.2.7(iii)). □

PROPOSITION 1.2.9. *The following conditions on a symmetric hermitian space are equivalent:*

(i) the space has curvature < 0*;*

(ii) G *is a product of simple noncompact groups.*

A symmetric hermitian space is called a *symmetric hermitian domain* if it satisfies the equivalent conditions of the proposition.

Symmetric hermitian domains correspond by duality to compact hermitian symmetric spaces.

1.2.10 (CLASSIFICATION). Let $M = G/K$ be a compact simple symmetric hermitian space. For $x \in M$, let $u_x \colon U^1 \to G$ be as in (1.2.3). According to (1.2.6), in its representation $\mathrm{ad}u_x$ on $\mathrm{Lie}(G)_{\mathbb{C}}$, U^1 acts only through the characters 1 (on $\mathfrak{k}_{\mathbb{C}}$) and z and $\bar{z}$ (on $\mathfrak{p}_{\mathbb{C}}$).

Conversely, if G is a compact adjoint simple group and $u \colon U^1 \to G$ has this property, the centralizer K of $u(U^1)$ is connected (it is the centralizer of a torus); it has for Lie algebra, the subspace fixed by the involution $\mathrm{ad}u(-1)$ of $\mathrm{Lie}\, G$, and G/K is symmetric hermitian.

The classification problem for M is therefore reduced to the following:

(*) let G be compact, adjoint, and simple; we have to classify the conjugacy classes of morphisms $u \colon U^1 \to G$ such that, in its representation $\mathrm{ad}u$ on $\mathrm{Lie}(G)_{\mathbb{C}}$, U^1 acts only through the three characters 1, z, and $\bar{z}$.

Let $T \subset G$ be a maximal torus, $\Phi \subset \mathrm{Hom}(T, U^1)$ the root system, and Σ a base for Φ (simple root system). Let α_0 be the longest root. The group $\mathrm{Hom}(U^1, T)$ can be identified with the dual of the free $\mathbb{Z}$-module $\mathrm{Hom}(T, U^1)$ by means of the pairing

$$\langle \alpha, v \rangle = \alpha \circ v \in \mathrm{Hom}(U^1, U^1) \cong \mathbb{Z}.$$

The morphism u has a unique conjugate $u' \colon U^1 \to T$ such that $\langle \alpha, u' \rangle \geq 0$ for $\alpha \in \Sigma$. The condition (*) is equivalent to[5]

$$\begin{aligned} &\langle \alpha, u' \rangle = 0,\ +1, \text{ or } \ -1 \text{ for } \alpha \in \Phi, \text{ or} \\ &\langle \alpha_0, u' \rangle = 1. \end{aligned}$$

[5]To see this, write the longest root as $\sum n(\alpha)\alpha$, $\alpha \in \Sigma$.

The morphism u' is characterized by the integers $m_\alpha = \langle \alpha, u' \rangle$ $(\alpha \in \Sigma)$, and the condition above signifies that $m_\alpha = 0$, except for a root α such that $n_\alpha = 1$, and for those α, $m_\alpha = 1$. In other words, u' is a minuscule weight of the root system dual to Φ. The conjugacy class of u is completely defined by the vertex of the Dynkin diagram D of G corresponding to α. If $D' = D \cup \{e\}$ is the extended Dynkin diagram, the possible vertices are the vertices of D in the orbit of e under $\mathrm{Aut}(D')$. There are $f - 1$ of them, where f denotes the connection index.

The list of possible vertices is given in the table in the appendix [?].

The classification in the noncompact case can be deduced from the above case by duality.

1.3 Filtrations and gradations

For more details on the results reviewed in this section, see Saavedra Rivano 1972.

1.3.1. Let V be a finite dimensional vector space over a field k. To give a gradation of type $\mathbb{Z}$ on V is equivalent to giving a representation $w\colon \mathbb{G}_m \to \mathrm{GL}(V)$: one passes from one to the other by means of the rule

$$w(\lambda) = \lambda^i v \text{ for } v \in V^i.$$

The associated increasing (resp. decreasing) filtration is defined by

$$W_i(V) = \oplus_{j \leq i} V^j$$
$$(\text{resp. } W^i(V) = \oplus_{j \geq i} V^j).$$

1.3.2. Let G be a reductive group over a field k of characteristic zero and let $w\colon \mathbb{G}_m \to G$ be a morphism. For any representation $\rho\colon G \to \mathrm{GL}(V)$ of G, $\rho \circ w\colon \mathbb{G}_m \to \mathrm{GL}(V)$ defines (according to 1.3.1) an increasing filtration W of V. This filtration is functorial in V and the functor $(V, \rho) \mapsto (V, W)$ is compatible with tensor products and duality:

$$W^i(V' \otimes V'') = \textstyle\sum_{j'+j''=i} W^{j'}(V') \otimes W^{j''}(V'')$$
$$W^i(V^*) = W^{-1-i}(V)^\perp.$$

Conversely, any functor $(V, \rho) \mapsto (V, W)$ compatible with tensor products and duality is defined by a morphism $w\colon \mathbb{G}_m \to G$. We shall make no use of this fact, and instead define a *filtration* W of G (or of the category of representations of G) to be a functor $(V, \rho) \mapsto (V, W)$ defined by a morphism $w\colon \mathbb{G}_m \to G$.

Let $\mathfrak{g} = Lie(G)$ be the space of the adjoint representation of G.

1.3.3 (Review). (i) $W_0(\mathfrak{g})$ is the Lie algebra of a parabolic subgroup $W_0(G)$ of F. It is the subgroup that respects the filtration W on every representation of G.

(ii) $W_{-1}(\mathfrak{g})$ is the Lie algebra of the unipotent radical $W_{-1}(G)$ of $W_0(G)$. It is the subgroup of $W_0(G)$ which, for every representation V of G, acts trivially on $Gr^W_\bullet(V)$.

(iii) The centralizer $Z(w)$ of w is a Levi subgroup of $W_0(G)$.

In (i) and (ii), the set of all representations of G can be replaced by a set (V_i, ρ_i) such that the intersection of the kernels $\mathrm{Ker}(\rho_i)$ is finite.

After (iii), we have

$$Z(w) \xrightarrow{\approx} W_0(G)/W_{-1}(G).$$

In particular, the composite

$$\bar{w}\colon \mathbb{G}_m \to W_0(G) \to W_0(G)/W_{-1}(G)$$

is central.

Let $w'\colon \mathbb{G}_m \to G$ define a filtration W'.

1.3.4 (REVIEW). The filtrations W and W' are equal if and only if $W_0(G) = W'_0(G)$ and $\bar{w} = \bar{w}'$. The morphisms w and w' are then conjugate under $W_{-1}(G)$.

Analogous statements are true for decreasing filtrations.

1.3.5 (APPLICATION). Let M and G be as before, and let $\mathfrak{g}$ be the Lie algebra of G. Fix an origin $o \in M$, and let $h = h_o$; let K =centralizer of h_o, $\sigma = \mathrm{ad} h_o(-i)$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding decomposition, and $J = \mathrm{ad}\ h_o(e^{2\pi i/8})|\mathfrak{p}$ the multiplication by i on $\mathfrak{p}$ identified with the tangent space to M at o.

The Hodge structure (relative to $o \in M$) of the adjoint representation is of type $\{(-1,1),(0,0),(1,-1)\}$. We have

$$\mathfrak{g}^{0,0} = \mathfrak{k}_{\mathbb{C}}, \quad \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1} = \mathfrak{p}_{\mathbb{C}}.$$

Put $\mathfrak{p}^+ = \mathfrak{g}^{1,-1}$ and $\mathfrak{p}^- = \mathfrak{g}^{-1,1}$. They are commutative Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$. If (H, ρ) is a representation of G with finite kernel and $\rho(\epsilon) = (-1)^n$, $F^0(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+$ is the Lie algebra of the parabolic subgroup $F^0(G_{\mathbb{C}})$ which respects the Hodge filtration on $H_{\mathbb{C}}$. Since the Hodge filtration determines the Hodge structure, the map

$$M = G/K \to G_{\mathbb{C}}/F^0(G_{\mathbb{C}}) \tag{1.3.5.1}$$

is injective. One checks immediately that it is a holomorphic open immersion (the *Borel embedding*). If G^* is the compact form of G whose Lie algebra is $\mathfrak{k} \oplus i\mathfrak{p}$, one shows that

$$G^*/K \xrightarrow{\approx} G_{\mathbb{C}}/F^0(G_{\mathbb{C}}),$$

so that (1.3.5.1) is the embedding of M into the dual domain.

1.4 Hodge structures

DEFINITION 1.4.1. A *Hodge structure of weight* n on a real finite dimensional vector space H is any one of the following:

(A) a bigradation of the complexification $H_{\mathbb{C}}$,

$$H_{\mathbb{C}} = \oplus_{p+q=n} H^{p,q},$$

such that $H^{q,p}$ is the complex conjugate of $H^{p,q}$;

(B) a finite decreasing filtration F on $H_{\mathbb{C}}$ such that

$$H_{\mathbb{C}} = F^p \otimes \bar{F}^q \text{ for } p+q=n+1;$$

(C) an action h of the real algebraic group[6] $\mathbb{S} = \mathbb{C}^\times$ on H such that $x \in \mathbb{R}^\times \subset \mathbb{C}^\times$ acts as multiplication by x^{-n}.

One passes from one definition to the others by means of the rules:

$$\begin{aligned} H^{p,q} &= F^p \cap \bar{F}^q \\ F^p &= \oplus_{i\geq p} H^{i,j} \\ h(z)v &= z^{-p}\bar{z}^{-q}v \text{ for } z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \text{ and } v \in H^{p,q}. \end{aligned}$$

Definition (B) makes obvious the complex structure, induced from that of the Grassmanian, on the variety of Hodge structures on H.

A *polarization* ψ of H is a bilinear form on H, invariant under $h(U^1)$, and such that the form

$$\psi(x, h(-i)y)$$

is symmetric and positive definite. Note that $h(-i)v = i^{p-q}v$ for $v \in H^{pq}$.

EXAMPLE 1.4.2. *Type* $(-1,0)+(0,-1)$. A complex structure on a real vector space H defines an action (by homotheties) of $\mathbb{C}^\times$ on H. This construction identifies the complex structures on H with the Hodge structures of type $(0,-1)+(-1,0)$. The mapping $H \mapsto H_{\mathbb{C}}/F^0(H_{\mathbb{C}})$ is a $\mathbb{C}$-linear isomorphism. A polarization of H is nothing but the imaginary part of a positive definite hermitian form. We make precise, that "hermitian" signifies for us that

$$\phi(\lambda x, \mu y) = \lambda\phi(x,y)\bar{\mu}.$$

1.4.3. Let M be a symmetric hermitian domain (1.2.9), and let G be the identity component of its group of automorphisms. For $x \in M$, we write $u_x\colon U^1 \to G$ for the morphism (cf. 1.2.6) sending z to the automorphism of M that fixes x and acts on the tangent space by multiplication by z.
One knows that the stabilizer of x is equal to the centralizer of u_x, so that the mapping $x \mapsto u_x$ identifies M to the set of conjugates of u_x.

1.4.4. Let G be the real adjoint algebraic group with Lie algebra $\mathfrak{g}$ (therefore $G = \mathsf{G}(\mathbb{R})^0$), and let $\pi\colon \mathsf{G}_1 \to \mathsf{G}$ be a connected algebraic covering of G. Assume that $h_x = u_x^2$ lifts to a morphism (again denoted h_x) of U^1 into G_1. The central element $\epsilon = h_x(-1)$ does not depend on the choice of x. It is trivial or of order 2. We define the real algebraic group G' to be the quotient of $\mathsf{G}_1 \times \mathbb{G}_m$ by $\{e, (\epsilon, -1)\}$. The composed mapping

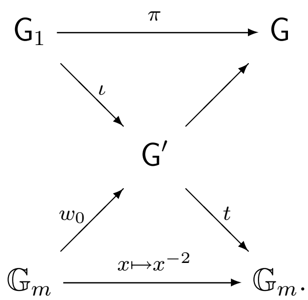
$$\iota : \mathsf{G}_1 \to \mathsf{G}_1 \times \mathbb{G}_m \to \mathsf{G}'$$

and the inverse w_0 of the composed map

$$w_0^{-1}\colon \mathbb{G}_m \to \mathsf{G}_1 \times \mathbb{G}_m \to \mathsf{G}'$$

[6] In the original, $\mathbb{S}$ is denoted $\underline{S}$.

fit into a commutative diagram



(1.4.4.1)

The composed diagonals are trivial and

$$t(\mathrm{Im}(g_1, \lambda)) = \lambda^2.$$

Just as G' is the quotient of $\mathsf{G}_1 \times \mathbb{G}_m$, the algebraic group $\mathbb{S}$ is the quotient of $U^1 \times \mathbb{G}_m$ by $\{e, (-1, -1)\}$. The morphism $(h_x, id)\colon U^1 \times \mathbb{G}_m \to \mathsf{G}_1 \times \mathbb{G}_m$ passes to the quotient and defines

$$h_x \colon \mathbb{S} \to \mathsf{G}'.$$

We have $th_x(z) = z\bar{z}$ and $h_x(\lambda^{-1}) = w(\lambda)$ for $\lambda \in \mathbb{R}^\times$.

The mapping $x \mapsto h_x$ identifies M also with the set of conjugates of an h_x under G (or, which is the same thing, under $\mathsf{G}'(\mathbb{R})^0$).

1.4.5. Let n be an integer. A representation $\rho\colon G' \to \mathrm{GL}(H)$ of G' is of weight n if $(\rho \circ w_0)(\lambda)$ is multiplication by λ^n. Let ρ be of weight n. For all $x \in M$, $\rho \circ h_x \colon \mathbb{S} \to \mathrm{GL}(H)$ defines on H a Hodge structure of weight n — see (1.4.1(C)). We denote by F_x the corresponding Hodge filtration of $H_\mathbb{C}$. Let $\varphi\colon \mathbb{G}_m \to \mathbb{S}_\mathbb{C}$ be the homomorphism such that $z \circ \varphi$ is $x \mapsto x^{-1}$ and $\bar{z} \circ \varphi$ is trivial. The filtration F_x is the decreasing filtration associated with $\rho \circ h_x \circ \varphi$; it is amenable to the results of (1.3). In particular, $F_x^0(\mathrm{Lie}(\mathsf{G}'))$ is the Lie group of a parabolic subgroup $F_x^0(\mathsf{G}')$ of $\mathsf{G}'_\mathbb{C}$. We write $M^\vee$ for the corresponding flag manifold; it can be identified with the set of conjugates of $F_x^0(\mathsf{G}')$. The Borel map $\beta\colon M \to M^\vee$ is

$$x \mapsto F_x^0(\mathsf{G}').$$

THEOREM 1.4.6 (BOREL). *The mapping β is holomorphic. It identifies M with an open subspace of $M^\vee$.*

PROOF. *Injectivity of β.*

For (H, ρ, n) as in (1.4.5) and $y \in M^\vee$, we define a filtration F_y of $H_\mathbb{C}$ as follows: there is a $g \in G'(\mathbb{C})$ such that $y = g\beta(x)$, and we set $F_y = gF_x$. This definition does not depend on the choice of g, because if $g\beta(x) = \beta(x)$, g is in $F_x^0(G'(\mathbb{C}))$ and respects F_x. For $y \in M$, $F_{\beta(y)}$ is the Hodge filtration of F_y.

In particular, for $y \in M$, F_y depends only on $\beta(y)$. For all n and all representations ρ as in (1.4.5), the Hodge structure $\rho \circ h_y$ therefore depends only on $\beta(y)$. It follows that h_y, therefore y, depend only on $\beta(y)$.

β is a holomorphic open immersion.

Let $x \in M$, $\mathfrak{g}' = \mathrm{Lie}(G')$, $\mathfrak{k}'$ be the stabilizer of x, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}'$ in $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{g}'$. We have that $\mathrm{ad}h_x(-i) = -1$ on $\mathfrak{p}$. The tangent space to M at x can be identified with $\mathfrak{p}$, which has a complex structure. From the definition of h_x, we have

$\mathrm{ad}h_x(z) \cdot \pi = z^2\pi$, for $\pi \in \mathfrak{p}$ and $z \in U^1$, or

$\mathrm{ad}h_x(z) \cdot \pi = z\bar{z}^{-1}\pi$, for $\pi \in \mathfrak{p}$ and $z \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$.

At x, $d\beta$ can be identified with the natural map

$$\mathfrak{p} \to \mathfrak{g}'_{\mathbb{C}}/F^0(\mathfrak{g}'_{\mathbb{C}}). \quad (*)$$

We have $\mathfrak{k}'_{\mathbb{C}} = \mathfrak{g}'^{0,0}_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{g}'^{-1,1}_{\mathbb{C}} \oplus \mathfrak{g}'^{1,-1}_{\mathbb{C}}$, and $\mathfrak{g}'_{\mathbb{C}}/F^0(\mathfrak{g}'_{\mathbb{C}}) \approx \mathfrak{g}'^{-1,1}_{\mathbb{C}}$. The mapping $d\beta$ is therefore bijective. On $g'_{\mathbb{C}}/F^0(\mathfrak{g}'_{\mathbb{C}})$, $\mathrm{ad}h_x(z)$ acts as multiplication by $z\bar{z}^{-1}$. Since (*) commutes with the action of $h_x(z)$, $d\beta$ commutes with multiplication by $z\bar{z}^{-1}$ ($z \in \mathbb{C}^\times$), and is therefore $\mathbb{C}$-linear. This completes the proof. □

1.4.7. Let (H, ρ, n) be as (1.4.4). The formation of the Hodge structure $\rho \circ h_x$ on H is compatible with tensor products of representations and with passage to the dual representation. When ρ is trivial on G_1, it is purely of type $\left(\frac{n}{2}, \frac{n}{2}\right)$. The Hodge structure $\rho \circ h_x$ on H therefore satisfies the following condition:

$$\text{every } \mathsf{G}_1\text{-invariant tensor in } \otimes^k H \otimes^\ell H^\vee \text{ is of type } (m, m), m = n/2. \qquad (1.4.7.1)$$

PROPOSITION 1.4.8. *If the kernel of $\rho_1 = \rho|\mathsf{G}_1$ is finite, then $x \mapsto \rho \circ h_x$ is a holomorphic isomorphism of M with one of the connected components of the set of Hodge structures of weight n on H satisfying (1.4.7.1).*

PROOF. If (H, ρ) is a faithful representation of a reductive group G'', then G'' is the algebraic subgroup of $\mathrm{GL}(H)$ formed of the g fixing a suitable finite set of G''-invariant tensors. The Hodge structures considered on H are those defined by a morphism $h\colon U^1 \to \mathsf{G}_1/\operatorname{Ker}(\rho_1)$. The conjugates of h_x form one of the components of this set. □

REMARK 1.4.9. Because of 1.4.8, when $M^\vee$ is identified with a space of flags in $H_{\mathbb{C}}$, $\beta\colon M \to M^\vee$ becomes identified with

$$(\text{Hodge structure defined by } h) \mapsto (\text{Hodge filtration } F_h).$$

DEFINITION 1.4.10. Let (H, ρ, n) be as in (1.4.4). A G_1-invariant bilinear form ψ on H is a *polarization* of (H, ρ) if the following equivalent conditions are satisfied:

(i) for one $x \in M$, ψ is a polarization of $(H, \rho \circ h_x)$;

(ii) for all $x \in M$, ψ is a polarization of $(H, \rho \circ h_x)$.

The equivalence of (i) and (ii) results from the following formula: for $g \in \mathsf{G}_1(\mathbb{R})$,

$$\psi(a, h_{gx}(-i)b) = \psi(a, gh_x(-i)g^{-1}b) = \psi(g^{-1}a, h_x(-i)g^{-1}b).$$

A polarization is $(-1)^n$-symmetric.

Let $x \in M$; let σ be the Cartan involution $\mathrm{ad}h(-i)$ of G_1, and let $\mathsf{G}_1^{(\sigma)} \subset \mathsf{G}_1(\mathbb{C})$ be the compact subgroup

$$\mathsf{G}_1^{(\sigma)} = \{g \in \mathsf{G}_1(\mathbb{C}) \mid \bar{g} = \sigma(g)\}.$$

A form ψ on H is a polarization if and only if the hermitian form $\psi(a, h_x(-i)\bar{b})$ on $H_{\mathbb{C}}$ is symmetric, positive definite, and $\mathsf{G}_1^{(\sigma)}(\mathbb{R})$-invariant. Since $\mathsf{G}_1^{(\sigma)}$ is compact, any representation as above of G' is therefore polarizable.

REMARK 1.4.11. Let $\mathbb{R}(n)$ be the line $\mathbb{R}$ on which G' acts by multiplication by $t^n(g)$. For all $x \in M$, $\mathbb{R}(n)$ is of Hodge type $(-n, -n)$. If we regard a polarization ψ of a Hodge structure H of weight n as being a mapping $\psi\colon H \otimes H^\vee \to \mathbb{R}(-n)$, then it is G'-equivariant, and it is a morphism of Hodge structures.

REMARK 1.4.12. Fix $o \in M$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition of $\mathfrak{g}$. Let G^* be the compact form of G with Lie algebra $\mathfrak{k} \oplus i\mathfrak{p}$. We have already defined the dual $M_o^\vee$ of M as being the quotient G^*/K. One can show that

$$M_o^\vee = G^*/K \xrightarrow{\approx} \mathsf{G}'(\mathbb{C})/F_o^0(\mathsf{G}'(\mathbb{C})) = M^\vee,$$

which justifies the terminology.

1.4.13 (GENERALIZATION). In the preceding discussion, we have taken for G_1 a finite covering of G. More generally, let G_1 be a connected reductive real algebraic group endowed with a homomorphism $\pi\colon \mathsf{G}_1 \to \mathsf{G}$ whose kernel is compact. Moreover let $o \in M$ and let $\tilde{h}_o\colon U^1 \to \mathsf{G}_1$ be a lifting of h_o centralized by the $\mathrm{Ker}(\pi)$. For $x = g \cdot o$ in M and $\tilde{g}$ a lifting of g in $\mathsf{G}_1(\mathbb{R})$, put $\tilde{h}_x = \tilde{g}\tilde{h}_o\tilde{g}^{-1}$. This lifting of $\tilde{h}_o$is independent of the choice of $\tilde{g}$. The construction $x \mapsto \tilde{h}_x$ identifies M with the set of $\mathsf{G}_1(\mathbb{R})^0$-conjugates of h_o.

The element $\epsilon = h_x(-1)$ of G_1 is still central, and therefore is independent of x. Put $\mathsf{G}' = \mathsf{G}_1 \times \mathbb{G}_m/\{1, \epsilon\}$. This group fits again into a diagram (1.4.4.1) and, as in (1.4.4), one can extend $\tilde{h}_x$ to

$$h_x\colon \mathbb{S} \to \mathsf{G}'.$$

As in (1.4.5), we can define a representation of *weight* n of G'; for (H, ρ) a representation of weight n and $x \in M$, $\rho \circ h_x$ is a Hodge structure of weight n on H. We define a *polarization* of (H, ρ) as in (4.10); every representation is polarizable.

Conversely, let G_1 be a connected reductive real algebraic group, and let $h_o\colon U^1 \to \mathsf{G}_1$ be a morphism such that

(a) in the representation $z \mapsto \mathrm{ad}\ h_o(z)$ of U^1 on $\mathrm{Lie}(\mathsf{G}_1)_{\mathbb{C}}$, U^1 acts only through the characters $z\bar{z}^{-1}$, 1, $z^{-1}\bar{z}$;

(b) $\mathrm{ad}h_o(-1)$ is a Cartan involution.

Then the set M of conjugates of h_o under $\mathsf{G}_1(\mathbb{R})^0$ is a symmetric hermitian space, relative to which G_1 is of the type considered above.

REMARK 1.4.14. Later, we shall make use of the case where G_1 is $U^1 \times SL(2, \mathbb{R})$, and where h_o is the mapping whose components are the identity and h^{SL}, which sends $e^{i\theta}$ to the rotation of angle θ.

2 Cones

2.1 Homogeneous cones

(following Vinberg and Koecher)

2.1.1. Let V be a finite-dimensional real vector space. We say that a subset C of $V \smallsetminus \{0\}$ is a *cone* if it is open and if

$$\lambda C = C \text{ for all } \lambda \in \mathbb{R}^{+}.$$

Let $C' \subset V^{*}$ be the set of linear forms that are ≥ 0 on C. The *dual* C^{*} of $C \subset V$ is the interior of $C' \smallsetminus \{0\}$. We say that C is *projecting* [saillant] if $C^{*} \neq \emptyset$. We have $C \subset C^{**}$, with equality if and only if C is projecting and convex (or $C = V \smallsetminus \{0\}$). We say that C is *self-adjoint* if there exists a quadratic form $Q > 0$ on V for which the corresponding isomorphism $\approx$ transforms C into C^{*}.

2.1.2. Let $C \subset V$ be a projecting convex cone with dual C^{*}, and let G be the group of automorphisms of (V, C). Let dx and dx* be dual Haar measures on V and V^{*}. The measure on C,

$$\mu = \int_{C^{*}} e^{-\langle x, x^{*}\rangle} dx^{*} = \varphi(x) dx$$

is G-invariant, whence also the riemannian structure $g = d^{2} \log \circ \varphi$ on C (to show that $g > 0$, recall that

$$d^{2} \log \int \varphi_{u} du = \frac{\int \varphi_{u}\ d^{2}\ \log \varphi_{u} du}{\int \varphi_{u} du} + \frac{1}{2} \frac{\iint \varphi_{u}\varphi_{v}(\text{dlog } \ \varphi_{u} \text{ - dlog } \ \varphi_{v})^{2} du dv}{\iint\ \varphi_{u}\varphi_{v} du dv}.)$$

COROLLARY 2.1.3. *Let C be a connected projecting cone and let G be the Lie group of automorphisms of (V, C). The stabilizer of $e \in C$ in G is compact, and is a maximal compact subgroup if C is homogeneous.*

PROOF. For the second assertion, note that every (maximal) compact subgroup of G has a fixed point in V (argument de moyenne). □

COROLLARY 2.1.4. *Let C be an open subset of V such that $C^{*} \neq \emptyset$, and let G be a Lie group of automorphisms of (V, C). Suppose that there exists an $e \in C$ such that $G \cdot e$ is open. Then C is a connected cone, homogeneous under G.*

PROOF. We have $G \cdot e \subset C \subset C^{**}$, and C^{**} satisfies the hypotheses of (2.1.4). This allows us to replace C with C^{**}, i.e., to assume that C is a connected projecting cone. Let $r > 0$ be such that a ball of centre e and radius r (for the riemannian metric d defined by g) is contained in $G \cdot e$. Since g is invariant, for any sequence of points x_i with $e = x_0$ and $d(x_i, x_{i+1}) \leq r$, we have $x_n \in G \cdot e$. The result follows by noting that C is connected. □

COROLLARY 2.1.5. *Let G be a connected reductive group, (V, ρ) a representation of*[7] G, *$e \in V$, and G the identity component of $\mathsf{G}(\mathbb{R})$. Assume that the stabilizer of e in G is a compact maximal subgroup of G, and that $G \cdot e$ is open. Then $G \cdot e$ is a self-adjoint homogeneous cone. Conversely, the group of automorphisms of a self-adjoint cone is reductive.*

[7]The original has G.

PROOF. Let K be the stabilizer of e, σ the corresponding Cartan involution, and B a positive definite symmetric bilinear form on V such that $\rho(\sigma(g)) = {}^t\rho(g)^{-1}$. Consider the Cartan decompositions $\mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$ and $G = K \cdot \exp(\mathfrak{p})$. Let g_1, $g_2 \in G$, and put $\sigma(g_2)g_1 = p^2 k$ with $p \in \exp(\mathfrak{p})$. Then

$$\langle g_1 e, g_2 e\rangle = \langle \sigma(g_2)g_1 e, e\rangle = \langle p^2 e, e\rangle = \langle pe, pe\rangle > 0.$$

When we identify V with V^* using B, we find that $Ge \subset (Ge)^*$. The cone (Ge)* is stable under G : if $x \in (Ge)^*$ and $h \in G$, then for any $g \in G$,

$$\langle ge, hx\rangle = \langle \sigma(h)ge, x\rangle > 0.$$

On applying (2.1.4) to $(Ge)^*$, we find that $Ge = (Ge)^*$.

Conversely, if C is a self-adjoint cone, its automorphism group is stable under $g \mapsto {}^t g$, and is therefore reductive. □

2.2 Appendix: Jordan algebras

(following Vinberg, Koecher, Jacobson, and Tits).

2.2.1. Recall that a *Jordan algebra* A is an algebra with identity (not necessarily associative) whose multiplication $\circ$ satisfies the following identities:

(1) $a \circ b = b \circ a$, and (putting $a^2 = a \circ a$)

(2) $a^2 \circ (b \circ a) = (a^2 \circ b) \circ a$.

This definition is only correct when 2 is invertible (see Jacobson 1969), and it is necessary to add to (1) and (2) the identities required to make (1) and (2) remain true after an extension of scalars. These problems do not arise if A is an algebra of finite dimension over a field of characteristic zero: *we shall henceforth restrict ourselves to that case*.
For $a \in A$, we write R_a for multiplication by a:

$$R_a(x) = a \circ x.$$

Put

$$S(a, b, c, d) = (a \circ b) \circ (c \circ d) + (a \circ c) \circ (b \circ d) + (a \circ d) \circ (b \circ c)$$

(symmetric in a, b, c, d after (1))

$$T(a; b, c, d) = (a \circ (b \circ c)) \circ d + (a \circ (c \circ d)) \circ b + (a \circ (d \circ b)) \circ c$$

(symmetric in b, c, d after (1)).

By polarization and division by 2, (2) yields

(3) $S(a, b, c, d) = T(a; b, c, d)$,

which can also be written as

(4) $[R_a, R_{bc}] + [R_b, R_{ca}] + [R_c, R_{ab}] = 0.$

The identity

(5) $T(a; b, c, d) = T(b; a, c, d)$

deduced from (3) can also be written as

(6) $[R_a, R_b]$ is a derivation.

The powers $a^n (n \geq 0)$ of a are defined by recurrence:

$$a^0 = 1 \text{ and } a^{n+1} = a \circ a^n.$$

It is possible to prove by induction on $\ell = n + m$ that

(7) $a^n \circ a^m = a^{n+m}.$

If n or m equal 0 or 1, then (7) follows from the definition, from (1), and from the fact that 1 is an identity element. This proves (7) for $\ell \leq 3$; for $\ell = 4$, it is necessary to prove that

$$a^2 \circ a^2 = a^4,$$

but this is the special case $b = a^2$ of (2). Suppose that $\ell \geq 5$ and that (7) is true for $\ell' < \ell$. Apply (5) to (a^p, a^q, a^r, a^s) $(p, q, r, s > 0,\ p + q + r + s = \ell)$. In view of the induction hypothesis, we find that

$$a^p \circ a^{l-p} = a^q \circ a^{l-q}.$$

Take $q = r = s = 1$: one finds that (7) is true if $n + m = \ell$ and $n \leq \ell - 3$. Since it suffices to prove (7) for $n \leq m$, and when $\ell \geq 5$,

$$n \leq \ell - n \implies n \leq \ell - 3,$$

this completes the proof.

REMARK. It is easy to deduce from this proof that an algebra with identity satisfying (1) and (6) is a Jordan algebra if and only if $a^2 \circ a^2 = a^4$ (recall that we are in characteristic zero).

PROPOSITION 2.2.2 (VINBERG). *Let A be an algebra with identity (not necessarily associative) over a field of characteristic zero, provided with a "trace" form t such that*

(a) A satisfies (1) and (6);

(b) $t([R_a, R_b](c)) = 0$;

(c) the form $t(a \circ b)$ is nondegenerate.

Then A is a Jordan algebra.

REMARK. When $t(a) = \mathrm{Tr}(R_a)$, the condition (b) is automatic: for any derivation D of A, $\mathrm{Tr}(R_{Dc}) = \mathrm{Tr}([D, R_c]) = 0$.

PROOF. It suffices to prove the polarized form (4) of (2). Put

$$\{a, b, c, d; e\} = t(([R_a, R_{bc}] + [R_b, R_{ca}] + [R_c, R_{ab}])(d) \cdot e).$$

On writing (4) in the form (3) and applying (6) under the form (5), we find that $\{a, b, c, d; e\}$ is symmetric in a, b, c, d. We have

$$\{a, b, c, d; e\} + \{a, b, c, e; d\} = t(([R_a, R_{bc}] + ...)(de)) = 0$$

by (b), so that $\{a, b, c, d; e\}$ is antisymmetric in d and e. It follows from this that the form $\{a, b, c, d; e\}$ is zero:

$$\{a, b, c, d; e\} = -\{a, b, c, e; d\} = \{a, b, d, e; c\} = -\{a, b, d, c; e\}.$$

After (c), e is arbitrary, and so we have (4). □

2.2.3. Let $\mathfrak{g}$ be a Lie algebra, and let σ be an involution of $\mathfrak{g}$ with corresponding decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let (V, ρ) be a faithful representation of $\mathfrak{g}$, and let $e \in V$, Suppose that

$\mathfrak{k}$ is the stabilizer of e and the mapping $\pi \mapsto \pi e$ of $\mathfrak{p}$ into V is bijective. (*)

Define on $\mathfrak{p}$ the composition law $\circ$ by the rule:

$$\pi \circ \pi' \cdot e = \pi'(\pi \cdot e). \tag{2.2.3.1}$$

Let ε be the element of $\mathfrak{p}$ such that $\varepsilon \cdot e = e$. The law $\circ$ satisfies the following identities:

$$\pi \circ \pi' = \pi' \circ \pi. \tag{2.2.3.2}$$

Indeed, $(\pi \circ \pi' - \pi' \circ \pi) \cdot e = [\pi, \pi']e = 0$, because $[\pi, \pi'] \in \mathfrak{k}$.

$$\varepsilon \text{ is an identity element.} \tag{2.2.3.3}$$

Indeed, $\pi \circ \varepsilon \cdot e = \pi e$. For $v \in V$, we have $\varepsilon v = v$.

$$\text{The identity (6).} \tag{2.2.3.4}$$

One checks, in fact, that

$$[R_\pi, R_{\pi'}] = \mathrm{ad}[\pi, \pi']. \tag{2.2.3.5}$$

2.2.4. Conversely, if an algebra with identity $(A, \circ)$ satisfies (1) and (6), and D is the algebra of derivations of A, it is possible to make $\mathcal{L}^0(A) = D \oplus A$ into a Lie algebra by putting

- for d, $d' \in D$, $[d, d']$ is in D, and equals the usual bracket;
- for $d \in D$, and $a \in A$, $[d, a]$ is in A; it is $d(a)$;

– for $a, b \in A$, $[a, b]$ is in D; it is $[R_a, R_b]$.

The involution $\sigma\colon (d, a) \mapsto (d, -a)$ is an automorphism, and $\mathcal{L}^0(A)$ acts on A by the rules:

$$\rho(d)a = d(a), \text{ and}$$
$$\rho(b)a = b \circ a.$$

This representation is faithful: the derivations d of A respect e: $d(e) = 0$; if $(d + a)x$ is identically zero, one finds on putting $x = e$ that $a = 0$; that $d = 0$ is then clear. The system $(\mathcal{L}^0(A), A, \rho, e, \sigma)$ is therefore of the type (2.2.3), and the construction (2.2.3) applies to the system gives $(A, \circ)$.

2.2.5. Let τ be the unique bilinear form on $\mathfrak{sl}(2)$ such that

(8) $[a, [b, c]] = \tau(a, b)c - \tau(c, a)b.$

We have

$$\tau(a, b) = 2 \times (\text{trace of the matrix } a \cdot b);$$

τ is an invariant symmetric bilinear form. For A a Jordan algebra and D the Lie algebra of its derivations, we put

$$\mathcal{L}(A) = D \oplus \mathfrak{sl}(2) \otimes A.$$

We can make $\mathcal{L}(A)$ into a Lie algebra by putting

– on D, [,] is the usual bracket;

– for $d \in D$, $u \in \mathfrak{sl}(2)$, and $a \in A$, we have

$$[d, u \otimes a] = u \otimes d(a);$$

– for $u, v \in \mathfrak{sl}(2)$, and a, $b \in A$, we have

$$[u \otimes a, v \otimes b] = \tau(u, v)[R_a, R_b] + [u, v] \otimes (a \circ b).$$

It is clear that for $d \in D$, $\mathrm{ad}d$ is a derivation. The Jacobi identity is therefore true for x, y, or z in D, and it remains to check it for a triple $(u \otimes a, v \otimes b, w \otimes c)$. We have

$$[u\otimes a, [v\otimes b, w\otimes c]] = -\tau(v, w)u\otimes[R_b, R_c](a) + [u, [v, w]]a\circ(b\circ c) + \tau(u, [v, w])[R_a, R_b].$$

The Jacobi identity then results from the following:

– $\tau(u, [v, w])$ is invariant under circular permutation of u, v, w (this expresses its invariance) and $\circ$ satisfies (4);

– $\circ$ satisfies (1) and τ satisfies (8).

For $\mathfrak{k}$ the algebra of diagonal matrices in $\mathfrak{sl}(2,\mathbb{R})$, we can identify $\mathcal{L}^0(A)$ with

$$D \oplus \mathfrak{k} \otimes A \subset \mathcal{L}(A)$$

by

$$(d,a) \mapsto d + (1/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes a.$$

The action of $\mathcal{L}^0(A)$ on A can be identified with the adjoint action of this subalgebra on A, identified with a subalgebra by $a \mapsto n \otimes a$, $n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2.2.6. Take for base field $\mathbb{R}$. Let V be a real vector space of finite dimension, $C \subset V$ a homogeneous self-adjoint open cone, and $e \in C$. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Aut}(V,C)$, $\mathfrak{k}$ the stablizer of $\mathfrak{g}$, $\mathfrak{p}$ its orthogonal complement, and σ the involution of $\mathfrak{g}$ having $\mathfrak{k}$ and $\mathfrak{p}$ as its eigenspaces with eigenvalues $+1$ and -1. The system $(\mathfrak{g}, V, e, \sigma)$ satisfies the hypotheses of (2.2.3), and σ is a Cartan involution; for a suitable positive definite quadratic form on V, σ is the restriction to $\mathfrak{g} \subset \mathrm{End}(V)$ of $u \mapsto -{}^t u$.

The construction (2.2.3) provides $\mathfrak{p}$ with a law $\circ$ satisfying (1) and (6), and the identity element ε acts by scalars on V.

2.2.7. Let t be the linear form
$$t(\pi) = \langle \pi e, e\rangle \text{ on } \mathfrak{p}.$$
Then

$$\begin{aligned} t([R_a, R_b](c)) &= \langle [[a,b],c] \cdot e, e\rangle \\ &= \langle c \cdot e, [a,b]e\rangle \\ &= \langle c \cdot e, 0\rangle \\ &= 0, \end{aligned}$$

by the $\mathfrak{k}$-invariance of $\langle , \rangle$. In addition, if $\pi \neq 0$,

$$t(\pi \circ \pi) = \langle \pi \circ \pi e, e\rangle = \langle \pi\pi e, e\rangle = \langle \pi e, \pi e\rangle > 0.$$

The form t satisfies the hypotheses of (2.2.2), and $(\mathfrak{p}, o)$ is a Jordan algebra.

2.2.8. Conversely, let A be a Jordan algebra equipped with a linear form t such that

$$t(x \circ (y \circ z) - (x \circ y) \circ z)) = 0, \text{ and}$$
$$t(x^2) > 0 \text{ for } x \neq 0.$$

Let D^+ be the algebra of derivations of A respecting t, and $\mathcal{L}^+(A)$ the subalgebra $D^+ \oplus A$ of $\mathcal{L}^0(A)$ provided with the induced involution σ. It acts on A through the representation ρ of (2.2.4). The involution σ is a Cartan involution, because relative to the quadratic form $t(x^2)$ on A, we have $\rho(\lambda^\sigma) = -{}^t\rho(\lambda)$. It is clear that $\lambda \in D$, and for $\lambda \in A$, we have

$$t(x \cdot (\lambda y)) = t((x\lambda) \cdot y)$$

by hypothesis. After [?] the orbit of e under the group of the Lie algebra $\mathcal{L}^0(A)$ is therefore a self-adjoint homogeneous cone.

The cone C is the set of $\exp(\rho(a)) \cdot e$, or the set of

$$\exp(R_a) \cdot e,\ a \in A.$$

After (7), the subalgebra generated by a is associative, and $\exp(R_a) \cdot e$ is $\exp(a)$, calculated in this subalgebra:

$$C = \{\exp(a) |\ a \in A\}.$$

In A, the subalgebra generated by an element is commutative, associative, and without nilpotent elements (because $t(x^2) \neq 0$ for $x \neq 0$) and such that $x^2 + y^2 \neq 0$ if $x, y \neq 0$ (because $t(x^2) > 0$). It is therefore a product of copies of $\mathbb{R}$, and in A, the exponentials are the squares of invertible elements. The cone C is therefore also the set of squares of invertible elements in A; its closure is the set of all the squares.

2.2.9. The classification of self-adjoint homogeneous cones therefore coincides with that of the Jordan algebras over $\mathbb{R}$ for which there exists a $t\colon A \to \mathbb{R}$ satisfying the conditions (2.2.8).

Complements on Jordan algebras

(A). By induction on n, define a^n ($a^0 = 1$, $a^1 = a$), and show that $a^p a^q$ depends only on $p + q$ ($p, q > 0$); it is a^{p+q}.

[Proof illegible.]

(B). $[R_{a^p}, R_{a^q}] = 0$.

PROOF. We have

$$[R_{a^p}, R_{a^{q+r}}] + [R_{a^q}, R_{a^{r+p}}] + [R_{a^r}, R_{a^{p+q}}] = 0.$$

For $\Delta_p = [R_{a^p}, R_{a^{n-p}}]$, this gives

(i) if $p + q + 2 = n$, $\Delta_p + \Delta_q + \Delta_r = 0$;

(ii) $\Delta_p = \Delta_{n-p}$.

$\Delta\colon \mathbb{Z}/n\mathbb{Z} \to A$, therefore $\Delta = 0$. □

2.3 Cones in $\mathbb{R}^2$

(following Hirzebruch and Mumford).

2.3.1. Let L be a free oriented $\mathbb{Z}$-module of rank 2, and let $e \in \Lambda^2 L$ be the class of the orientation. Let

$$\begin{cases} V_{\mathbb{Q}} = L \otimes \mathbb{Q} \\ V = L \otimes \mathbb{R}, \end{cases}$$

and let $C_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ be such that

(a) $x, y \in C_{\mathbb{Q}} \implies x + y \in C_{\mathbb{Q}}$;

(b) $x \in C_{\mathbb{Q}}$, $\lambda \in \mathbb{Q}$, $\lambda > 0 \implies \lambda x \in C_{\mathbb{Q}}$;

(c) there exists a linear form > 0 on $C_{\mathbb{Q}}$;

(d) $C_{\mathbb{Q}} \neq \emptyset$, and is not reduced to half-line.

Let C be the cone of linear combinations with real coefficients > 0 of the elements of $C_{\mathbb{Q}}$. We have $C_{\mathbb{Q}} = C \cap L_{\mathbb{Q}}$; C is the angular sector bounded by two half-lines ℓ_1 and ℓ_2; if $\ell_i \subset C$, ℓ_i is rational. We are interested in the locally finite decompositions of C into rational angular sectors. We shall identify such a triangulation to the following locally finite set $\mathcal{E}$ of elements of $L \cap C$: if $\mathcal{L}$ is the set of half-lines bounding the said angular sectors, and, for $\ell \in \mathcal{L}$, e_ℓ is the generator of $\ell \cap L$, $\mathcal{E}$ is the set of e_ℓ.

Index $\mathcal{E}$ by an interval (finite or infinite) I of $\mathbb{Z}$, the e_i turning in a positive direction when i increases: $e_i \wedge e_{i+1} = ae$ with $a > 0$. The interval I has a smallest (resp. largest) element i if and only if the first (resp. second) side ℓ of C is rational; $\ell \cap L$ is then generated by e_i.

We say that the triangulation $\mathcal{E}$ is *smooth* [lisse] if $e_i{\scriptstyle\wedge} e_{i+1} = e$. We shall consider only smooth triangulations, and we shall call them simply *triangulations*.

2.3.2 (CONSTRUCTION). (i) If $\mathcal{E}$ is a triangulation and $i, i+1 \in I$, we obtain a new triangulation by adjoining $e_i + e_{i+1}$ to $\mathcal{E}$ (*blowing up* the sector (e_i, e_{i+1})).

(ii) The conditions $e_i = e_{i-1} + e_{i+1}$ and $e_{i-1}{\scriptstyle\wedge} e_{i+1} = e$ are equivalent. If they are satisfied, one says that e_i is *exceptional*. One obtains then a new triangulation by omitting e_i from $\mathcal{E}$ (the *contraction* of e_i).

A triangulation is said to be *minimal* if it has no exceptional element.

PROPOSITION 2.3.3. *The half-lines e_i and e_{i+1} are not simultaneously exceptional.*

PROOF. Otherwise, $e_{i+1} - e_i$ and $e_i - e_{i+1}$ will be in $\mathcal{E}$, and C will not be projecting. $\square$

PROPOSITION 2.3.4. *If C is bounded by a basis (e_0, e_1) of L with $e_0{\scriptstyle\wedge} e_1 = e$, then every triangulation $\mathcal{E}$ of C can be obtained from $\{e_0, e_1\}$ by a finite sequence of blowing ups.*

PROOF. Because the sides of C are rational, $\mathcal{E}$ will be finite.

(a) If $e_0 + e_1 \notin \mathcal{E}$, then $\mathcal{E} = \{e_0, e_1\}$.

Let f and f' be the elements of $\mathcal{E}$ which immediately precede and follow $e_0 + e_1$:

$$f = \lambda e_0 + \mu e_1, \quad \lambda \geq \mu + 1;$$
$$f' = \lambda' e_0 + \mu' e_1, \quad \lambda' \leq \mu';$$
$$f{\scriptstyle\wedge} f' = e, \quad (\text{i.e., } \lambda\mu' - \lambda'\mu = 1).$$

Since $\lambda\mu' - \lambda'\mu \geq \lambda' + \mu + 1$, it follows that $\lambda' = \mu = 0$ and $e_0 = f$, $e_1 = f'$.

(b) Continue by induction on the number of elements of $\mathcal{E}$. If $\mathcal{E} = \{e_0, e_1\}$, we win. Otherwise, we write C' and C'' for the cones bounded by $(e_0, e_0 + e_1)$ and $(e_0 + e_1, e_1)$, and apply the induction hypothesis to $C' \cap \mathcal{E}$ and $C'' \cap \mathcal{E}$. $\square$

COROLLARY 2.3.5. *If $\mathcal{E}$ and $\mathcal{E}'$ are two triangulations with $\mathcal{E} \subset \mathcal{E}'$, $\mathcal{E}'$ can be obtained from $\mathcal{E}$ by a locally finite sequence of blowing ups.*

THEOREM 2.3.6. *The cone C has a unique minimal triangulation $\mathcal{E}(C)$. Every triangulation of C contains $\mathcal{E}(C)$.*

PROOF. (a) If C is rational, then C admits a triangulation.

We assume that C is bounded by e_0 and e_1, and we prove (a) by induction on the integer $a > 0$ such that $e_0 {\wedge} e_1 = ae$. If $a > 1$, there exists an $f \in L$ in the interior of the parallelogram bounded by e_0 and e_1, and we apply the induction hypothesis to the cones bounded by (e_0, f) and (f, e_1) respectively.

(b) C admits a triangulation.

It is clear that C admits a not necessarily smooth triangulation. The refinement provided by (a) proves (b).

(c) For all $i \in I$, there exists a finite interval $J = [i, k]$ (resp. $[k, i]$) such that for every triangulation $\mathcal{E}' \subset \mathcal{E}$, one of the e_j for $j \in J$ is in $\mathcal{E}'$.

It suffices to proved the nonresp. assertion. Endow V with a euclidean structure. For $j_1 \leq i \leq j_2$, the angle θ between e_{j_1} and e_{j_2} satisfies $\varepsilon \leq \theta \leq \pi - \varepsilon$ for a suitable $\varepsilon > 0.$ If $e_{j_1} \wedge e_{j_2} = e$, e_{j_2} is therefore of bounded length, and j_2 runs over a finite subset of I. It suffices to take k larger than the j_2.

(d) The intersection of two triangulations is a triangulation.

Let the triangulations be $\mathcal{E}'$ and $\mathcal{E}''$. The proof of (b) shows that there exists a triangulation $\mathcal{E}$ containing $\mathcal{E}'$ and $\mathcal{E}''$. After (2.3.5), $\mathcal{E}'$ (resp. $\mathcal{E}''$) is deduced from $\mathcal{E}$ by a sequence (perhaps infinite) of contractions "remove $e_{i(a)}$" (resp. "remove $e_{i''(a)}$) ($a \geq 0$). According to (2.3.3), when $e_{i'(a)} \neq e_{i''(0)}$, $e_{i'(a)}$ is not close to $e_{i''(0)}$. From this, $\mathcal{E} - e_{i''(0)}$, $\mathcal{E}' - e_{i''(0)}$, $\mathcal{E}''$ are again three triangulations, $\mathcal{E}' - e_{i''(0)}$ can be obtained from $\mathcal{E} - e_{i''(0)}$ by a sequence of contractions "remove $e_{i'(a)}$" (with $e_{i''(0)}$ excluded in case of failure) and $\mathcal{E}''$ can be obtained from $\mathcal{E} - e_{i''(0)}$ by a sequence of contractions "remove $e_{i''(a)}$" ($a > 0$). The sequence of contractions "remove by turns $e_{i'(a)}$ and $e_{i''(a)}$" results in $\mathcal{E}' \cap \mathcal{E}''$ which, after (c), is therefore a triangulation. □

THEOREM (2.3.6). [8]Let D be the convex envelope of $C \cap L$, and $\mathcal{E}(C) = \partial D \cap L$. Then $\mathcal{E}(C)$ is the unique minimal triangulation of C, and any triangulation of C contains $\mathcal{E}(C)$.

REMARK 2.3.7. Let $C^-(e)$ (resp. $C^+(e)$) be bounded by e and the first (resp. second) side of C. If $e \in \mathcal{E}(C)$, then $\mathcal{E}(C^{\pm}(e)) = \mathcal{E}(C) \cap C^{\pm}(e)$.

COROLLARY 2.3.8. *Let $\mathcal{E}$ be a triangulation of C. The following conditions are equivalent:*

(a) $\mathcal{E}$ is minimal;

(b) if $j > i + 1$, $e_i {\wedge} e_j = ae$, with $a \geq 2$;

(c) for i nonmaximal, $e_{i+1} - e_i \notin C$.

It is clear that (c) $\Longrightarrow$ (a), and, after (2.3.5), that (a) $\Longleftrightarrow$ (b). That (a) $\Longrightarrow$ (b) results from (2.3.7) and from the following applied to $C^+(e_i)$.

[8]The original numbers both theorems 2.3.6 — perhaps this theorem is not meant to be included.

COROLLARY 2.3.9. *Suppose that the first side of C is rational, bounded by e_0. The e_i of the minimal triangulation of C are defined by recurrence by*

$$e_i{}_\wedge e_{i+1} = 1,\ e_{i+1} \in C, \text{ and } \ e_i \notin C.$$

Let e_1 be defined by these formulas. Then e_1 is not exceptional in $\{e_0\} \cup \mathcal{E}(C(e_1))$ which is therefore a (the) minimal triangulation of C, and we conclude by recurrence.

COROLLARY 2.3.10. *If C and D have the same second side ℓ_2, $\mathcal{E}(C)$ and $\mathcal{E}(D)$ coincide in a neighbourhood of ℓ_2.*

It suffices to consider the case in which $C \subset D$ and the first side of C is rational, bounded by e_0: $C = D^+(e_0)$. Let $\mathcal{E} = \mathcal{E}(D^+(e_0)) \cup \mathcal{E}(D^-(e_0))$. We shall show that $\mathcal{E}(D)$ is obtained from $\mathcal{E}$ by a *finite* number of contractions. It is possible to contract e_i only after having contracted the e_j between e_i and e_0 (e_0 included) and, after (2.3.6) (cf. 2.3.6(c)), the process must stop.

2.3.11. If r is a real number, we write $\{r\}$ for the smallest integer such that $\{r\} \geq r$. Define r_i and a_i inductively,

$$\begin{aligned} r_1 &= r; \\ a_i &= \{r_i\},\ i \geq 1; \\ r_{i+1} &= 1/a_i - r_i,\ i \geq 1. \end{aligned}$$

Stop if $a_i = r_i$. With Hirzebruch, we will call $(a_0, a_1, ...)$ the continued fraction expansion of r. Take care that it is not the classical notion. We have

$$r = a_0 - 1/a_1 - 1/a_2 \ - \cdots$$

$a_i \geq 2$ for $i > 0$, and the a_i can not be all equal to 2 for $i \geq i_0$, because $2 - 1/2 - 1/\cdots = 1$.

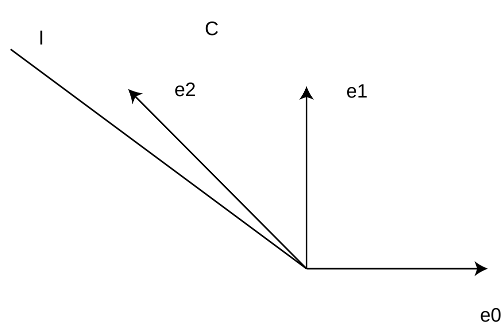
PROPOSITION (ALSO 2.3.11.). Let (e_0, e_1) be a basis of L with $e_0{}_\wedge e_1 = e$, let r be a real number, and let ℓ be a half-line which can be written in the basis

$$\ell : x + ry = 0,\ x < 0.$$

Let C be the cone bounded by e_1 and ℓ, $\mathcal{E}(C) = (e_1, e_2, ...)$ the minimal triangulation of C, and define a_i by

$$e_{i-1}{}_\wedge e_{i+1} = a_i e \quad (i \geq 1).$$

Then $(a_1, a_2, ...)$ is the expansion (2.3.11) of r as a continued fraction.



We apply (2.3.9). Since $e_1 \wedge e_2 = 1$, we can write $e_2 = -e_0 + ae_1$, $a_1 e = e \wedge e_2 = ae$; a is the smallest integer such $-e_0 + ae_1 \in C$; in other terms, $a_1 = \{r\}$. In the basis (e_1, e_2), the equation of ℓ is $x + r_2 y = 0$, and we continue by induction.

COROLLARY 2.3.12. *Two numbers r and r', with continued fraction expansions $(a_1, a_2, ...)$ and $(a'_1, a'_2, ...)$, can be transformed into each other by means of a substitution $r \mapsto \frac{ar+b}{cr+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, if and only if $a_{i_1+k} = a'_{i_2+k}$ for suitable i_1 and i_2.*

PROOF. This is a translation of (2.3.10). □

PROPOSITION 2.3.13. *The following conditions are equivalent:*

(a) the group of automorphisms of $(L \text{ orientated}, C)$ is nontrivial (it is then cyclic of infinite order);

(b) there exists a rational indefinite quadratic form Q not representing 0 such that C is one of the two connected components of $Q > 0$.

PROOF. (b) $\Longrightarrow$ (a): Follows from the Dirichlet unit theorem for the real quadratic field attached to Q.

(a) $\Longrightarrow$ (b): The sides of C are eigenvectors for every automorphism σ. Therefore, their direction coefficients are the roots of a quadratic equation. Moreover, they can not be rational. □

2.3.14. Suppose that the equivalent conditions of (2.3.13) are satisfied. The minimal triangulation $\mathcal{E}(C)$ is stable for all automorphisms of (L, C); the numbers a_i such that $e_{i-1} \wedge e_{i+1} = a_i e$ therefore form a periodic sequence. After (2.3.10) and (2.3.11), the period can be calculated as follows:

(a) In any oriented basis of L, write the equation of the line supporting the second side of C in the form $x + \iota y = 0$.

(b) Form the continued fraction expansion $(b_0, b_1, \ldots)$ of r. For i sufficiently large, the sequence b_{i+h} is periodic; it has the same period as the sequence of a_i.

COROLLARY 2.3.15. *The expansion of a irrational quadratic number as a continued fraction is periodic, except for the initial terms.*

REMARK 2.3.16. If $(a_1, \ldots, a_r)$ is the period of the sequence (a_i), the generator σ of the group of automorphisms of (L oriented, C) such that $\sigma e_i = e_{i+r}$ has for matrix relative to the basis (e_0, e_1), the product

$$\begin{pmatrix} & -1 \\ 1 & a_1 \end{pmatrix} \circ \begin{pmatrix} & -1 \\ 1 & a_2 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} & -1 \\ 1 & a_r \end{pmatrix}.$$

If σ_i is defined by $\sigma_i(e_{i-1}) = e_i$ and $\sigma_i(e_i) = e_{i+1}$, we have, in fact,

$$\sigma = \sigma_r \circ \sigma_{r-1} \circ \cdots \circ \sigma_1$$

and σ_i and $\begin{pmatrix} & -1 \\ 1 & a_1 \end{pmatrix}$ relative to the basis (e_{i-1}, e_i).

3 Study at infinity

3.1 Roots

(following Harish-Chandra; cf. Helgason 1962, Chapter VIII)

3.1.1. Let M be a symmetric hermitian domain (1.2.9) of rank r (1.1.22), and let G, G, G_1, and G' be as in (1.4.13). Fix $o \in M$. The point o defines $h_o : U^1 \to \mathsf{G}_1$. We write σ for the Cartan involution $\mathrm{ad} h_o(-i)$. Let $\mathfrak{g}$ and $\mathfrak{g}_1$ be the Lie algebras of G and G_1, and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$$

be the decompositions of $\mathfrak{g}$ and $\mathfrak{g}_1$ defined by σ. The algebra $\mathfrak{k}_1$ is the Lie algebra of the centralizer K_1 of h_o, and $\mathfrak{p}$ can be identified with the tangent space to M at o. Its complex structure J is $\mathrm{ad} h_o(e^{2\pi i/8})|\mathfrak{p}$. The morphism h_o defines a Hodge structure of weight 0 on the space $\mathfrak{g}_1$ of the adjoint representation. When we put $\mathfrak{p}^+ = F^1(\mathfrak{g}_{\mathbb{C}})$ and $\mathfrak{p}^- = \overline{\mathfrak{p}^+}$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

3.1.2. We write h^{SL} for the morphism

$$h^{\mathrm{SL}} \colon U^1 \to \mathrm{SL}(2, \mathbb{R}), \quad e^{i\theta} \mapsto \text{rotation of angle } \theta.$$

Let $s \geq 0$, and write h^{USL} (or, if this notation is ambiguous, $h^{\mathrm{USL}s}$) for the map

$$h^{\mathrm{USL}} \colon U^1 \to U^1 \times SL(2, \mathbb{R})^s, \quad x \mapsto (x, h^{\mathrm{SL}}(x), \ldots).$$

3.1.3. Let $\mathfrak{a}$ be a maximal commutative subalgebra of $\mathfrak{p}$. The following theorem, which is fundamental for the study of Cayley transformations, will be proved in (3.1.19).

THEOREM 3.1.4. *There exists a morphism*

$$\varphi : U^1 \times \mathrm{SL}(2, \mathbb{R})^r \to \mathsf{G}_1$$

such that

(a) $d\varphi$ *induces an isomorphism of the standard Cartan subalgebra of* $\mathfrak{sl}(2, \mathbb{R})^r$ *(diagonal matrices) with* $\mathfrak{a}$*;*

(b) $h_o = \varphi \circ h^{USLr}$.

REMARK 3.1.5. We have

$$d\varphi(\mathfrak{sl}(2, \mathbb{R})^r) = \mathfrak{a} + J\mathfrak{a} + [\mathfrak{a}, J\mathfrak{a}];$$

this can be checked by reduction to the case of the group $U^1 \times \mathrm{SL}(2, \mathbb{R})^r$.

3.1.6. The irreducible real representations of U^1 are the following:

ρ_0: the trivial representation of U^1;

ρ_k $(k > 0)$: the representation of dimension two of U^1, whose complexification is the sum of the representations of dimension one with characters z^k and $\bar{z}^k$.[9]

Let σ_1 be the obvious (two-dimensional) representation of $\mathrm{SL}(2,\mathbb{R})$ and let σ_k be its k^{th} symmetric power. The irreducible real representations of $\mathrm{SL}(2,\mathbb{R})$ are the σ_k $(k \geq 0)$. These representations are absolutely irreducible. Every real irreducible representation of $U^1 \times SL(2,\mathbb{R})^r$ can therefore be written as an external tensor product

$$\rho_i \otimes \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_r}.$$

We decompose the adjoint representation $\mathfrak{g}$ into irreducible representations of $U^1 \times \mathrm{SL}(2,\mathbb{R})^r$. On the space of the representation σ_r, the subgroup U^1 of $\mathrm{SL}(2,\mathbb{R})$ acts by the characters $z^p\bar{z}^q$ ($p+q=r$, $p \geq 0$, $q \geq 0$), that is, by the characters z^i ($-r \leq i \leq r$, $r \equiv i$ mod 2). Since U^1 acts on $\mathfrak{g}_{\mathbb{C}}$ only through the characters z^2, 1, z^{-2}, the representations $\rho_i \otimes \sigma_{j_1} \otimes ... \otimes \sigma_{j_r}$ which occur in $\mathfrak{g}$ satisfy

$$i + \sum j_i = 0 \text{ or } 2.$$

They are among the following:

(a$_i$) $\rho_0 \otimes (\sigma_0 \otimes ... \otimes \sigma_2 \otimes ... \otimes \sigma_0)$ (σ_2 in the i^{th} place, $1 \leq i \leq r$).

Let $\mathfrak{g}_i$ be the image by $d\varphi$ of the Lie algebra of the i^{th} factor of $\mathrm{SL}(2,\mathbb{R})$. Then $\mathfrak{g}_i \subset \mathfrak{g}$ is the unique subrepresentation of $\mathfrak{g}$ of this type: if there were another v, then the vectors of v fixed by $\mathfrak{a}$ would be in $\mathfrak{p}$, and $\mathfrak{a}$ would not be a maximal commutative subalgebra.

(b$_{i,j}$) $\rho_0 \otimes (\sigma_1$ in the i^{th} and j^{th} places with σ_0's elsewhere).

(c$_i$) $\rho_1 \otimes (\sigma_1$ in the i^{th} place and σ_0's elsewhere).

(d) $\rho_2 \otimes$ (trivial).

(e) $\rho_0 \otimes$ (trivial).

Let R be a system of roots. A root α in R is said to *long* if, for all roots β in the same irreducible component of R as α, we have $\|\beta\| \leq \|\alpha\|$.

COROLLARY 3.1.7. *If M is simple, then the relative root system of $\mathfrak{g}$ is of type C or BC. The long roots correspond to the $\mathfrak{g}_i \subset \mathfrak{g}$.*

PROOF. Let $\pm\alpha_i$ be the roots of $\mathfrak{a}$ in $\mathfrak{g}_i$. There exist in the image by φ of the i^{th} factor $\mathrm{SL}(2,\mathbb{R})$ an element w that normalizes $\mathfrak{a}$, transforms α_i into $-\alpha_i$, and respects the α_j for $j \neq i$. Let s_i be the reflection $\mathrm{ad}w$ of $\mathfrak{a}$. Then s_i belongs to the Weyl group W. The α_i are therefore orthogonal to each other. After (3.1.6), the other roots are among the following:

(b)$_{i,j}$ gives the roots $\frac{1}{2}(\pm\alpha_i \pm \alpha_j)$;

(c)$_i$ gives the roots $\pm\frac{1}{2}\alpha_i$.

The Weyl group W is generated by the reflexions with respect to the roots, and therefore by the s_i and by the symmetries which interchange two roots and fix the others. Since M is simple, W acts irreducibly on $\mathfrak{a}$. After Lemma 3.1.8 below, W contains the symmetric group acting on $\mathfrak{a}$ by permuting the the α_i. From this, we see that W is the group of

[9] Thus $\rho\colon e^{i\theta} \mapsto \begin{pmatrix} \cos\ k\theta & \sin\ k\theta \\ -\sin\ k\theta & \cos\ k\theta \end{pmatrix}$.

automorphisms of $\mathfrak{a}$ respecting the set of the $\pm\alpha_i$, the $\frac{1}{2}(\pm\alpha_i \pm \alpha_j)$ are all roots, and either none of the $\pm\frac{1}{2}\alpha_i$ is a root, or all the $\pm\frac{1}{2}\alpha_i$ are roots. In the first case, the relative root system is reduced, and admits for simple roots

$$\frac{1}{2}(\alpha_1 - \alpha_2), \ldots, \frac{1}{2}(\alpha_{r-1} - \alpha_r), \ldots, \alpha_r.$$

It is of type C. In the second case, half the long roots (the α_i) are again roots, and it is of type BC. □

LEMMA 3.1.8. *Let L be a finite set, and let T be a set of transpositions of L. If all the elements of L are equivalent for the equivalence relation generated by "the transposition (a, b) is in T", then T generates the symmetric group of L.*

PROOF. The proof is left to the reader.[10] □

3.1.9. We denote by w^{SL} the morphism

$$w^{\mathrm{SL}}\colon \mathbb{G}_m \to \mathrm{SL}(2, \mathbb{R}), \quad \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

For $s \leq r$, we denote by $w_s\colon \mathbb{G}_m \to \mathsf{G}_1$ the composite map

$$w_s = \varphi \circ (1, w^{\mathrm{SL}}, ..., w^{\mathrm{SL}}, e, ..., e)$$

(w^{SL} occurs s times), and we write ${}_sW$ for the corresponding filtration of G_1 (1.3). These notations are only useful for M simple, *to which case we now restrict ourselves*. We have

$$\langle \alpha_i, w_s \rangle = \begin{cases} 2 \text{ if } i \leq s \\ 0 \text{ if } i > s \end{cases}$$

so that w_s kills all the simple roots (3.1.7), except $\frac{1}{2}(\alpha_s - \alpha_{s+1})$ (for $0 < s < r$) or α_r (for $s = 2$). For $0 < s \leq r$, ${}_sW(\mathsf{G}_1)$ is therefore a maximal parabolic subgroup of G_1, and each conjugacy class of maximal parabolics in G_1 is obtained in this way exactly once.

The group G' is a quotient of $\mathbb{G}_m \times \mathsf{G}_1$. We write w'_s for the morphism $w'_s\colon \mathbb{G}_m \to \mathsf{G}'$ defined by $x \mapsto (x^{-1}, w_s(x))\colon \mathbb{G}_m \to \mathbb{G}_m \times \mathsf{G}_1$. We again write ${}_sW$ for the corresponding filtration of G'.

DEFINITION 3.1.10. A filtration W of G' (1.3) is said to be *Cayley* if for all representations (V, ρ) of weight n of G' and all $x \in M$, (V, F_x, W) is a mixed Hodge structure.

Let W be a filtration of G', defined by $w'\colon \mathbb{G}_m \to \mathsf{G}'$. If W is Cayley, then tw' is $\lambda \mapsto \lambda^{-2}$, i.e., $w = w_0^{-1}w'$ takes its values in G_1. One sees this by testing (3.1.10) on $\mathbb{R}(1)$ (1.4.11). If $(\mathbb{R}(1), F, W)$ is a mixed Hodge structure, then $\mathbb{R}(1)$ is in fact purely of W-filtration -2.

Let (V_i, ρ_i) be a family of representations of G', and let n_i be the weight of (V_i, ρ_i).

PROPOSITION 3.1.11. *Suppose that the intersection of the kernels $\mathrm{Ker}(\rho_i)$ is finite. If for one $x \in M$, the (V_i, F_x, W) are all mixed Hodge structures, then W is Cayley.*

[10] $\langle T \rangle$ contains all the transpositions.

PROOF. (a) *Independence of* $x \in M$. Suppose (V, F_x, W) is a mixed Hodge structure; we shall prove that for all $y \in M$, (V, F_y, W) is again a mixed Hodge structure. The parabolic subgroup $W_0(G)$ acts transitively on M (because[11] $G = W_0(G) \cdot K$ with K compact maximal). There therefore exists $g \in W_0(G)$ such that $y = gx$. Then $F_y = gF_x$, and g respects W; g induces an isomorphism of $(Gr_i^W(V), Gr_i^W(F_x))$ with $(Gr_i^W(V), Gr_i^W(F_y))$. That (V, F, W) is a mixed Hodge structure signifies that, for all i, $(Gr_i^W(V), Gr_i^W(F))$ is a Hodge structure of weight i, and assertion follows.

(b) Let $\mathcal{E}$ be the set of isomorphism classes of representations (H, τ) such that (H, F_x, W) is a mixed Hodge structure. It follows from the lemmas below that

1) $\tau_1 \oplus \tau_2 \in \mathcal{E} \iff \tau_1 \in \mathcal{E}, \tau_2 \in \mathcal{E}$;

2) $\tau_1 \in \mathcal{E}, \tau_2 \in \mathcal{E} \Longrightarrow \tau_1 \otimes \tau_2 \in \mathcal{E}$; $\tau \in \mathcal{E} \Longrightarrow \tau^\vee \in \mathcal{E}$;

3) $\tau^{\otimes n} \in \mathcal{E} \quad (n > 0) \Longrightarrow \tau \in \mathcal{E}$.

It follows formally from these stabilities that, if the (V_i, ρ_i) are in $\mathcal{E}$, then every representation is in $\mathcal{E}$. □

LEMMA (3.1.12.1). Let F_i be a filtration of the complexification of a vector space V_i, $i = 1, 2$. Then $(V_1 \oplus V_2, F_1 \oplus F_2)$ is a Hodge structure of weight n if and only if the (V_i, F_i) are.

PROOF. Obvious. □

LEMMA (3.1.12.2.). Let (V_i, F_i) be as above, with $V_i \neq \{0\}$. Then $(V_1 \otimes V_2, F_1 \otimes F_2)$ is a Hodge structure of weight n if and only if the (V_i, F_i) are Hodge structures of weight n_i for some integers with $n_1 + n_2 = n$.

PROOF. If a vector space X is endowed with two filtrations G_1 and G_2, then there always exists a bigradation $X^{a,b}$ of X such that $G_1^\alpha(X) = \oplus_{a \leq \alpha} X^{a,b}$ and $G_1^\beta(X) = \oplus_{b \leq \beta} X^{a,b}$. Apply this fact to $(V_{i\mathbb{C}}, F_i, \bar{F}_i)$; the hypothesis assures us that when $V_1^{a,b} \neq 0$ and $V_2^{c,d} \neq 0$, we have $a + b + c + d = n$. From this, when $V_1^{a,b} \neq 0$, $a + b$ has a fixed value n_1, and a similar statement holds for V_2. The lemma follows. nzl □

LEMMA (3.1.12.3.). Let (V_1, F_1) and (V_2, F_2) be as above, and let W_i be an increasing filtration of V_i. Then $(V_1 \oplus V_2, F_1 \oplus F_2, W_1 \oplus W_2)$ is a mixed Hodge structure of weight n if and only if the (V_i, F_i, W_i) are.

PROOF. Obvious because of (3.1.12.1)[12]. □

LEMMA (3.1.12.4.). Let (V_i, F_i, W_i) be as above, with $V_i \neq \{0\}$. Then $(V_1 \otimes V_2, F_1 \otimes F_2, W_1 \otimes W_2)$ is a mixed Hodge structure if and only if, for a suitable n, $(V_1, F_1, W_1[n])$ and $(V_2, F_2, W_2[-n])$ are mixed Hodge structures.

PROOF. This follows from (3.1.12.2)[13], and completes the proof of Proposition 3.1.11. □

[11] The original has $W^0(G)$.
[12] The original has 3.1.11.1.
[13] The original has 3.1.11.2.

PROPOSITION 3.1.13. *Suppose that M is simple. The Cayley filtrations of G' are the conjugates of the filtrations ${}_sW$ of (3.1.9).*

PROOF. We first prove that ${}_sW$ is a Cayley filtration. Define ${}_s\varphi\colon U^1 \times \mathrm{SL}(2,\mathbb{R}) \to \mathsf{G}_1$ to be the composite of

$$U_1 \times \mathrm{SL}(2,\mathbb{R}) \frac{\text{on } U^1\colon (\mathrm{id}, e, \dots, e, h^{\mathrm{SL}}, \dots, h^{\mathrm{SL}})}{\text{on } \mathrm{SL}\colon (e, \mathrm{id}, \dots, \mathrm{id}, e, \dots, e)} \to U_1 \times \mathrm{SL}(2,\mathbb{R})^r \xrightarrow{\varphi} G$$

(on SL, s copies of id, $r-s$ copies of e). Write w^{USL} for the morphism $(1, w^{\mathrm{SL}})\colon \mathbb{G}_m \to U^1 \times \mathrm{SL}(2,\mathbb{R})$; then we have

$$\begin{aligned} {}_s\varphi \circ h^{\mathrm{USL}} &= h_o \quad \text{and} \\ {}_s\varphi \circ w^{\mathrm{USL}} &= w_s. \end{aligned}$$

This reduces the problem to the case of the group $U^1 \times SL(2,\mathbb{R})$. We shall apply the criterion (3.1.11) to this group and to the representations of weight -1 of the corresponding group G' whose restrictions to $U^1 \times \mathrm{SL}(2,\mathbb{R})$ are respectively,

(a) $\rho_1 \otimes \sigma_0$: here the space V of the representation is purely of W-filtration -1, and (V, F_x) is a Hodge structure of weight -1.

(b) $\rho_0 \otimes \sigma_1$: here (V, F, W) is a mixed Hodge structure, of type $\{(-1,-1),(0,0)\}$.

This expresses that $W^{-2}(V) \neq F^0(V)$ (these are one-dimensional subspaces, one real and the other not), and so $W^{-2}(V) \cap F^0(V) = 0$ and $W^{-2}(V) + F^0(V) = V$.

Conversely, let W be Cayley filtration, defined by $w'\colon \mathbb{G}_m \to \mathsf{G}'$. Let $w = w'w_o^{-1}\colon \mathbb{G}_m \to \mathsf{G}_1$. After replacing W by a conjugate filtration, we can assume that the image of w is in the split torus of A of the Lie algebra $\mathfrak{a}$. Let α be a long root of A, and let $\mathfrak{g}^\alpha \subset \mathfrak{g}$, $\mathfrak{g}^\alpha \approx \mathfrak{sl}(2)$, be the corresponding subalgebra. Consider $\mathfrak{g}$ as a representation of $U^1 \times \mathrm{SL}(2,\mathbb{R})^r$ (by $\mathrm{ad}\varphi$); then $\mathfrak{g}^\alpha$ is a direct factor of $\mathfrak{g}$. After (3.1.12.3), $(\mathfrak{g}^\alpha, F_\alpha, W)$ is therefore a mixed Hodge structure. The Hodge structure F of $\mathfrak{g}^\alpha$ is of type $(-1,1)$, $(0,0)$, $(1,-1)$; if $k = \langle \alpha, w \rangle$, then either $k = 0$ or the nonzero $Gr_k^W(\mathfrak{g}^\alpha)$ correspond to $n = k$, 0, or $-k$, and are of dimension one. The Hodge numbers h^{pq} of $(\mathfrak{g}^\alpha, F_\alpha, W)$ therefore satisfy the equations:

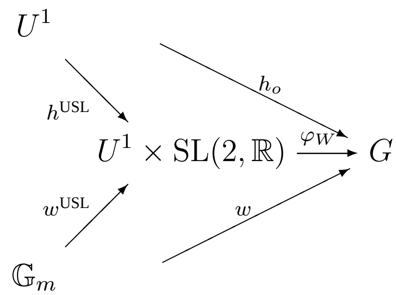
$$\begin{aligned} \sum_q h^{pq} &= 1 \quad (p = -1, 0, 1) \\ \sum_{p+q=\ell} h^{pq} &= 1 \quad (\ell = -k, 0, k). \end{aligned}$$

This forces $k = \pm 2$. We therefore have $\langle \alpha, w \rangle = 0$ or ± 2, and w is conjugate to one of the w_s. □

3.1.14 (CANONIFICATION). Let W be a Cayley filtration of G'. Since $G = K \cdot {}_sW(G)$, W is then conjugate to an ${}_sW$ by an element of K. There exists therefore $w\colon \mathbb{G}_m \to \mathsf{G}_1$ which defines W and is such that $\mathrm{Im}(dw) \subset \mathfrak{p}$ (the conjugates of w_s by $k \in K$ have this property). In other words,

$$\sigma w = w^{-1}. \tag{3.1.14.1}$$

This w is unique; moreover, it follows that $W_{-1}(\mathfrak{g}) \cap \mathfrak{p} = \{0\}$ (the elements of $\mathfrak{p}$ are semisimple). This w, and h, factor canonically through $U^1 \times \mathrm{SL}(2,\mathbb{R})$: there exists a unique φ_W rendering the following diagram commutative:



3.1.15. Since $(\mathfrak{g}, F_0)$ is a Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$, the nonzero Hodge numbers h^{pq} of the mixed Hodge structure $(\mathfrak{g}, F_0, W)$ satisfy $|p|, |q| \leq 1$. In particular, if $\mathfrak{g}^i$ is the gradation of $\mathfrak{g}$ defined by w, then $\mathfrak{g} = \oplus_{|i|\leq 2}\mathfrak{g}^i$. We put:

$$\mathfrak{g}^{even} = \oplus\mathfrak{g}^{2i} = \text{centralizer of } w(-1);$$
$$\mathfrak{g}_t^0 = [\mathfrak{g}^2, \mathfrak{g}^{-2}] \text{ and } \mathfrak{g}_t = \mathfrak{g}^{-2} \oplus \mathfrak{g}_t^0 \oplus \mathfrak{g}^2;$$
$$\mathfrak{g}_h = \text{the orthogonal complement of } \mathfrak{g}_t^0 \text{ in } \mathfrak{g}^0.$$

One checks that $\mathfrak{g}_t$ is a Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}_h \subset \mathfrak{g}^0$, we have $[\mathfrak{g}_h, \mathfrak{g}^2] \subset \mathfrak{g}^2$. The invariance of the Killing form B shows that

$$B([\mathfrak{g}_h, \mathfrak{g}^2], \mathfrak{g}^{-2}) = B(\mathfrak{g}_h, [\mathfrak{g}^2, \mathfrak{g}^{-2}]) = B(\mathfrak{g}_h, \mathfrak{g}_t^0) = 0.$$

Since B puts $\mathfrak{g}^i$ in duality with $\mathfrak{g}^{-i}$, we have $[\mathfrak{g}_h, \mathfrak{g}^2] = 0$. Likewise, $[\mathfrak{g}_h, \mathfrak{g}^{-2}] = 0$, and therefore $[\mathfrak{g}_h, \mathfrak{g}_t] = 0$.

The algebra $\mathfrak{g}_t$ is stable under σ, therefore reductive, and it is generated by its nilpotents, therefore semisimple. Its centralizer is therefore contained in its orthogonal complement, and $\mathfrak{g}_h$ is the centralizer of $\mathfrak{g}_t$ in $\mathfrak{g}^0$. In particular, it is a Lie subalgebra of $\mathfrak{g}$. This proves the assertion (i) of the following proposition. The verification of (ii) is left to the reader.

PROPOSITION 3.1.16. *(i) The algebra* $\mathfrak{g}^{even} = \mathfrak{g}_t \times \mathfrak{g}_h$ *(isomorphism of Lie algebras).*
(ii) $d\varphi_W$ *sends the Lie algebra of* $\mathrm{SL}(2,\mathbb{R})$ *(resp.* U^1*) into* $\mathfrak{g}_t$ *(resp.* $\mathfrak{g}_h$*).*

We will see later that $\mathfrak{g}_t$ corresponds to a symmetric hermitian tube domain.

3.1.17. We now examine the relation between the absolute and relative systems of roots of G. As in (3.1.6)[14], we write $\mathfrak{g}_i$ for the image by $d\varphi$ of the Lie algebra of the i^{th} factor $\mathrm{SL}(2,\mathbb{R})$ of $U^1 \times SL(2,\mathbb{R})^r$.

Let $\mathfrak{t}_1 = [\mathfrak{a}, J\mathfrak{a}] = \mathfrak{k} \cap \oplus\mathfrak{g}_i$, and let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal commutative subalgebra containing $\mathfrak{t}_1$. The complexifications of $\mathfrak{a}$ and $\mathfrak{t}_1$ are both Cartan subalgebras of $\oplus\mathfrak{g}_i$. They are therefore conjugate by an element c of $\varphi(\mathrm{SL}(2,\mathbb{C})) \subset G_{\mathbb{C}}$: $c\mathfrak{a}_{\mathbb{C}}c^{-1} = \mathfrak{t}_{1\mathbb{C}}$. Conjugation by c transforms the set of roots of $\mathfrak{a}$ into the set of nonzero restrictions to $\mathfrak{t}_1$ of the roots of $\mathfrak{t}$.

[14]The original has 3.1.3.

We read off from (3.1.3) that $\oplus\mathfrak{g}_i$ is the sum of $\mathfrak{a}$ and the radicial subspaces of $\mathfrak{g}$ relative to the long roots of $\mathfrak{a}$. The analogous statement for $\mathfrak{t}_1 = c\mathfrak{a}c^{-1}$ shows that $\oplus\mathfrak{g}_i$ is determined by $\mathfrak{t}_1$, and therefore is normalized by the normalizer of $\mathfrak{t}_1$. In particular, the $\mathfrak{g}_i$ are stable under $\mathfrak{t}$. Let $\pm\tilde{\alpha}_i$ be the corresponding roots. Then $H_{\tilde{\alpha}_i}$ is in $\mathfrak{g}_i$, therefore in $\mathfrak{t}_1$ and $\tilde{\alpha}_i$ is zero on the orthogonal complement of $\mathfrak{t}_1$ in $\mathfrak{t}$. We have $c(\alpha_i) = \pm\tilde{\alpha}_i|\mathfrak{t}_1$. In the adjoint representation of $\mathfrak{g}_i$ on $\mathfrak{g}$, the representation σ_2 of $\mathfrak{g}_i$ occurs only once ($\mathfrak{g}_i \subset \mathfrak{g}$), and the σ_k ($k > 2$) not at all. After (3.1.18)[15] below, $\tilde{\alpha}_i$ is a long root. The $\tilde{\alpha}_i$ are long and orthogonal in pairs, therefore strongly orthogonal. They are noncompact (not roots of $\mathfrak{t}$ in $\mathfrak{k}$).

LEMMA 3.1.18. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra, $\mathfrak{t}_{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and Φ the root system of $\mathfrak{t}_{\mathbb{C}}$. For $\alpha \in \Phi$, let $\mathfrak{g}_{\alpha\mathbb{C}}$ be the subalgebra $\approx \mathfrak{sl}(2,\mathbb{C})$ of $\mathfrak{g}_{\mathbb{C}}$ generated by the radicial subspaces e_α and $e_{-\alpha}$. Then α is long if and only if, in the adjoint representation of $\mathfrak{g}_{\alpha\mathbb{C}}$ on $\mathfrak{g}$, the representation σ_2 appears only once (for $\mathfrak{g}_{\alpha\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$) and σ_k ($k > 2$) not at all.*

It suffices to check this in the rank 2 case, where it can be done case by case.

3.1.19. PROOF OF 3.1.4.[16] Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. It contains the image of dh, and is a Cartan subalgebra of $\mathfrak{g}$ (because the centralizer of the image of dh is $\mathfrak{k}$). Let Φ be the root system for $\mathfrak{t}_{\mathbb{C}}$ acting on $\mathfrak{g}$, and define e_α, $\mathfrak{g}_{\alpha\mathbb{C}}$ as in (3.1.6)[?].

Let Φ^+, Φ^-, $\Phi^0 \subset \Phi$ be the roots occurring in $\mathfrak{p}^+$, $\mathfrak{p}^-$, $\mathfrak{k}_{\mathbb{C}}$. Since $\mathfrak{p}^+$ is commutative, in Φ^+, orthogonality is equivalent to strong orthogonality. For $\alpha \in \Phi^+$, $\mathfrak{g}_\alpha = \mathfrak{g} \cap \mathfrak{g}_{\alpha,\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

For each root $\alpha \in \Phi^+$, choose $E_\alpha \in e_\alpha$, and put $X_\alpha = E_\alpha + \overline{E}_\alpha$. It is a generator of the complex subspace of dimension one, $\mathfrak{g}_\alpha \cap \mathfrak{p} \subset \mathfrak{p}$.

LEMMA (3.1.19.1). Consider $\Delta' \subset \Phi^+$ and let $s = \#\Delta'$. The following conditions are equivalent:

(i) The X_α ($\alpha \in \Delta'$) commute in pairs, and the centralizer of $\mathfrak{a}' = \oplus\mathbb{R}\, X_\alpha$ in $\mathfrak{p}$ is the sum of $\mathfrak{a}'$ and of the centralizer of $\oplus\mathfrak{g}_\alpha$ in $\mathfrak{p}$.

(ii) The centralizer $\mathfrak{z}$ of $\mathfrak{a}'$ in $\mathfrak{g}$ is the sum of $\mathfrak{a}'$ and of the centralizer of $\oplus\mathfrak{g}_\alpha$ in $\mathfrak{g}$.

(iii) The $\alpha \in \Delta'$ are long roots, orthogonal in pairs.

PROOF. It is clear that (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i). Admit (i). From

$$[X_\alpha, X_\beta] = [E_\alpha, E_{-\beta}] + [E_{-\alpha}, E_\beta] = 0,$$

we see that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. There exists therefore,

$$\varphi' \colon \mathrm{SL}(2,\mathbb{R})^s \to G$$

such that $d\varphi'$ identifies $\mathfrak{sl}(2, \mathbb{R})^s$ to $\oplus\mathfrak{g}_\alpha$. The Hodge structure of $\mathfrak{sl}(2, \mathbb{R})$ induced by that on $\mathfrak{g}$ comes from

$$h' \colon U^1 \to \mathrm{SL}(2,\mathbb{R})^s \quad \text{(diagonal)},$$

[15]The original has 3.1.5.
[16]The original has 3.1.1

and there exists φ extending φ':

$$h' \colon U^1 \times \mathrm{SL}(2,\mathbb{R})^s \to G$$

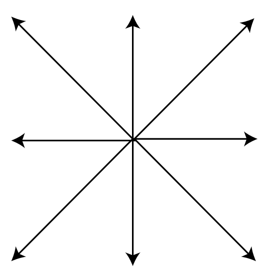
satisfying the condition (b) of (3.1.1). Decompose $\mathfrak{g}$ as in (3.1.6)[17]. The hypothesis (i) ensures that the representation (a)$_i$ of (3.1.6) appears only once, therefore that the representation σ_2 of $\mathfrak{g}_\alpha$ appears only once, and σ_k ($k > 2$) not at all. By (3.1.18), α is long, and (i) $\Longrightarrow$(iii).

Consider the condition

$$\Delta' \text{ satisfies the equivalent conditions of } (3.1.7[9?].1). \tag{*}$$

If $\Delta' \subset \Phi^+$ satisfies (*), we have $s = \dim(\mathfrak{a}') \leq r$. We shall prove in (3.1.19.3) below that, if Δ' is maximal, then $s = 2$. The preceding proof then shows (3.1.4). □

LEMMA (3.1.19.2.). Let $\Delta' \subset \Phi^+$ satisfy (*), and let Q be the set of roots $\beta \in \Phi^+$ (strongly) orthogonal to Δ' and $\beta \in Q$. If $\Delta \cup \{\beta\}$ does not satisfy (*), there exists in Φ a subsystem of roots of type B_2 as follows:



In particular, β is shorter than any other root of Q, and is not minimal in Q for any ordering of the roots.

PROOF. Let $\mathfrak{z}$ be the centraliser of $\oplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha$ in $\mathfrak{g}_\mathbb{C}$. By (*), under the form 3.1.19.1(i), the hypothesis on β is equivalent to:

- in the representation $\mathfrak{z}$ of $\mathfrak{g}_\beta$, there is a nontrivial irreducible representation $\mathfrak{x} \subset \mathfrak{z}$ of odd dimension other than $\mathfrak{g}_\beta \subset \mathfrak{z}$. Moreover, since $\mathfrak{t}$ normalize $\mathfrak{z}$, we may suppose that $\mathfrak{x}$ is stable under $\mathfrak{t}$. If E_γ is the dominant vector of $\mathfrak{x}$ ($[E_\beta, E_\gamma] = 0$), the situation is necessarily that described in the lemma. □

LEMMA (3.1.19.3.). The maximal subsets $\Delta \subset \Phi^+$ satisfying (*) have r elements.

PROOF. After (3.1.19.2), it suffices to show that Δ has $s < r$ elements, $Q \neq \emptyset$. The centralizer of $\mathfrak{a}'$ in $\mathfrak{p}$ in fact strictly contains $\mathfrak{a}$, since $\dim \mathfrak{a}' = s < r$. The centralizer of $\oplus \mathfrak{g}_\alpha$ in $\mathfrak{p}$ is therefore nontrivial. The corresponding roots are in Q. □

COROLLARY 3.1.20. *The maximal subsets $\Delta \subset \Phi^+$ satisfying (*) are conjugate under the Weyl group W of $\mathfrak{t}$ in $\mathfrak{k}$. If M is simple, every permutation of Δ is induced by an element of W.*

[17]The original has 3.1.3.

PROOF. Let $\mathfrak{t} \subset \mathfrak{k}$ be maximal, and let $\Delta \subset \Phi^+$ be a maximal subset satisfying (*). Giving Δ is equivalent to giving $\mathfrak{t}_1 = \mathfrak{t} \cap \oplus \mathfrak{g}_\alpha$: $\pm\Delta$ is the set of long roots which are zero on the orthogonal complement of $\mathfrak{t}_1$. If $\mathfrak{a}$ is a maximal commutative subalgebra in $\oplus \mathfrak{g}_\alpha \cap \mathfrak{p}$, then $\mathfrak{t}_1 = [\mathfrak{a}, J\mathfrak{a}]$. Since the maximal commutative subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is unique up to K-conjugation, $\mathfrak{a}$ determines $\mathfrak{t}_1$ in K, and $\mathfrak{t}_1$ determines $\mathfrak{t}$ up to conjugation by the centralizer of $\mathfrak{t}_1$, $(\mathfrak{t}, \mathfrak{t}_1)$, and therefore $(\mathfrak{t}, \Delta)$, is unique up to K-conjugation. This proves the first assertion.

The normalizer of $\mathfrak{a}$ in K normalizes $\mathfrak{t}_1$; since the $\mathfrak{t}$ containing $\mathfrak{t}_1$ are conjugate under the centralizer of $\mathfrak{a}$ in K ($\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_1^\perp$, with $\mathfrak{t}_1^\perp$ a maximal torus in its centralizer), the normalizer of $\mathfrak{a}$ and $\mathfrak{t}$ in K maps onto the Weyl[18] group of $\mathfrak{a}$. It permutes the $\mathfrak{g}_\alpha$, therefore also the $\alpha \in \Delta$, as it permutes the pairs of opposite long roots of $\mathfrak{a}$, and the second assertion follows from (3.1.4)[?]. □

3.2 Study of $\mathrm{SL}(2, \mathbb{R})$

3.2.1. In this number we make explicit the fundamental case where M is the unit disk.

Let V be a real vector space of dimension 2. The set of complex structures on V has two connected components; two complex structures are in the same component if and only if they define the same orientation of V, and they are then conjugate under the group $\mathrm{SL}(V)$. We choose one of the components, and denote it by X. The complex structures in X will be said to be positive, and others negative.

Let G be the connected Lie group $\mathrm{PSL}(V)$, let G be the algebraic group $\mathrm{PGL}(V)$, and let G_1 be its double covering $\mathrm{SL}(V)$. Let J_o be a positive complex structure on V and let $u_0 \colon U^1 \to G$ be the morphism sending z to the image in $G(\mathbb{R})^+$ of multiplication by $\pm\sqrt{z}$ (relative to J_o). Via u_0, U^1 acts on $\mathfrak{g}_\mathbb{C} = \mathrm{Lie}(G)_\mathbb{C}$ only through the characters $1, z, \bar{z}$; the centralizer of u_0 is reduced to $u_0(U^1)$ and $G/u_0(U^1)$ is therefore a symmetric hermitian domain M.

The square u_o^2 of u_o lifts to a homomorphism $h_o \colon U^1 \to \mathrm{SL}(V)$ sending z to multiplication by z (relative to J_o). The group G' of (I.4.4) is here $\mathrm{GL}(V)$.

The map $x \mapsto h_x$ identifies M with a set of Hodge structures on V of type $\{(-1,0),(0,-1)\}$ or, what comes down to the same thing (1.4.2), to a set of complex structures on V. We denote by o the point of M that defines h_o and J_o. We have

(a) M is identified with the set of positive complex structures on V;

(b) the Hodge filtration F defined by a complex structure J is characterized by

$$F^0(V) = \{a + ib \mid a, b \in V, a + Jb = 0\}.$$

We have $V^{0,-1} = F^0V$ and $V^{-1,0} = \bar{F}^0V$.[19] The dual domain $M^\vee$ of M is the complex projective line whose points are the lines in $V_\mathbb{C}$, and the Borel embedding (1.4.6, 1.4.9) is $x \mapsto F_x^0(V)$.

[18] The original has Weil.

[19] $V \xrightarrow{\approx} V_\mathbb{C}/F^0$, $a \mapsto a + i0$; $ia \equiv ia + (Ja - ia) \equiv Ja$.

Write $M^\vee_{\mathbb{R}}$ for the real projective line whose points are the lines in $V_{\mathbb{C}}$ that are equal to their conjugates (that is, the complexification of a line of V). An element x of $M^\vee$ defines a Hodge structure of type $\{(-1,0),\ (0,-1)\}$ on V if and only if it does not lie on $M^\vee_{\mathbb{R}}$: for x nonreal, $V_{\mathbb{C}} = F^0_x \oplus \bar{F}^0_x$. The real line $M^\vee_{\mathbb{R}}$ cuts the riemann sphere $M^\vee$ into two half-planes: one, M, corresponds to the positive complex structures, and the other, M^-, to the negative complex structures.

3.2.2. Let x in $M^\vee$ correspond to a line X in $V_{\mathbb{C}}$, and let N be the unipotent radical of the parabolic subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ respecting X:

$$n \in N \iff (nX = X \text{ and } x \text{ acts trivially on } V_{\mathbb{C}}/X.$$

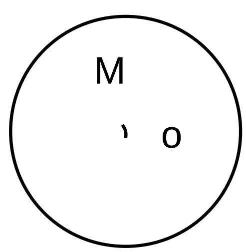
The points $y \neq x$ of $M^\vee$ form a principal homogeneous space for N.

3.2.3. Let x and y be points of M, and let $\bar{y}$ be defined by the line $\bar{F}^0_y(V)$. Then x and $\bar{y}$ are respectively in M and M^-. In particular, $x \neq \bar{y}$. After (3.2.2) applied to $\bar{o}$, for x in M (or more generally for $x \neq \bar{o}$), there is therefore a unique element $\rho(x)$ in $\bar{F}^1_o(\mathfrak{g})$ such that $x = \exp(\rho(x)) \cdot o$.

Write this in coordinates. Let e^+ be a generator of $F_o(V)$, and let e^- be its complex conjugate. Relative to the basis (e^+, e^-) of $V_{\mathbb{C}}$, $F^1_o(\mathfrak{g})$ is the set of matrices $n_\lambda = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$, $\lambda \in \mathbb{C}$, and the image by $\exp(n_\lambda)$ of the line F^0V is generated by $e^+ + \lambda e^-$. This line is real if and only if there exists a μ such that

$$e^+ + \lambda e^- = \mu(e^+ + \lambda e^-)^- = \mu(e^- + \bar{\lambda} e^+) = \bar{\lambda}\mu e^+ + \mu e^-,$$

that is, if $|\lambda| = 1$. If $M^\vee$ is identified, using ρ, to the one-point compacification of $\bar{F}^1_o(\mathfrak{g})$, and $\bar{F}^1_o(\mathfrak{g})$ is given the coordinate λ, then the subset $M^\vee_{\mathbb{R}}$ of $M^\vee$ is identified with the circumference of the unit circle $|\lambda| = 1$. The interior $|\lambda| < 1$ of the circle corresponds to M (because o in M corresponds to $\lambda = 0$) and the exterior to M^-. In this model of $M^\vee$, $\bar{F}^1_o(G)$ acts by translations, and $\bar{F}^0_o(G)$ acts by affine transformations; the stabilizer K of o in G acts by rotations:



(3.2.3.1)

The circle is $M^\vee_{\mathbb{R}}$, M^- is the exterior, and $\bar{o}$ is at infinity.

Application: the Harish-Chandra embedding

Let M be a symmetric hermitian domain, and let G be the identity component of its group of automorphisms. Fix a point o in M, and let F be the corresponding Hodge filtration on $\mathfrak{g}_{\mathbb{C}}$. The parabolic subgroups F^0G and $\bar{F}^0G$ are opposite, and the map

$$n \mapsto \exp(n) \cdot h_o$$

is a holomorphic isomorphism of $\bar{F}^1(\mathfrak{g})$ with the largest cell of $M^\vee = G(\mathbb{C})/F^0(G(\mathbb{C}))$.

THEOREM 3.2.5 (HARISH-CHANDRA). *The set M is contained in the largest cell of $M^\vee$. The map*

$$\zeta\colon M \to \bar{F}^1(\mathfrak{g}),\quad h \mapsto n \textit{ such that } h = \exp(n)\cdot h_o$$

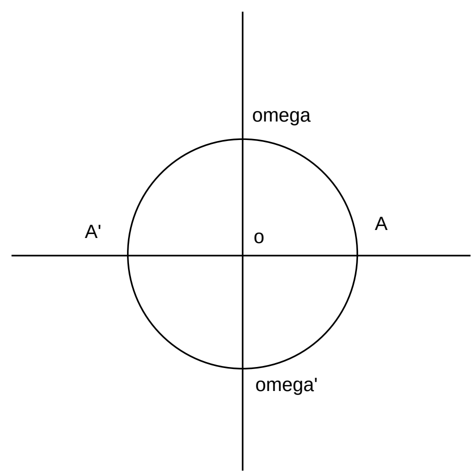
identifies M with a bounded open subset of $\bar{F}^1(\mathfrak{g})$.

PROOF. We return to the notations of (3.1.1). Since $M = \exp(\mathfrak{p})\cdot h_o$ and $\mathfrak{p}$ is the union of the $k\mathfrak{a}k^{-1}$ ($k \in K$), it suffices to prove that for α in $\mathfrak{a}$, $\exp(\alpha)\cdot h_o$ is in the largest cell and that the n for which $\exp(\alpha)\cdot h_o = \exp(n)\cdot h_o$ are bounded. Theorem 3.1.4 reduces this theorem to the case of $\mathrm{SL}_2(\mathbb{R})$: write $\alpha = d\varphi((0,\alpha_1,\alpha_2,...,\alpha_r))$. From the study made above of $\mathrm{SL}_2(\mathbb{R})$, we know there exist n_i in $\bar{F}^1(\mathfrak{sl}_2(\mathbb{R}))$ such that $\exp(\alpha^{-1})\cdot\exp(n_i) \in F^0(\mathrm{SL}_2(\mathbb{C}))$ and $|n_i| < 1$ for a suitable norm. Take $n = d\varphi((0,n_1,...,n_r))$. □

COROLLARY 3.2.6. *For $x, y \in M$, the parabolic subgroups $F^0_x(G)$ and $\bar{F}^0_x(G)$ are opposite.*

3.2.7. Return to the notations of (3.2.1)[20]. The nontrivial Cayley filtrations of G correspond to the filtrations W of V such that $Gr^W_i(V) \neq 0$ for $i = 0, -2$. They are all conjugate. For $x \in M$, the mixed Hodge structure (V, F_x, W) is of type $\{(0,0),(-1,-1)\}$.

The line $W_{-2}(V)$ defines a point $w \in M^\vee_{\mathbb{R}}$. We write ω' for the symmetric point with respect to o. We have $\omega' = h_o(\pm i)(\omega)$. Return to the figure (3.2.3.1).



(3.2.7.1)

On the sphere $M^\vee$, the three circles (or lines) $(\omega', o, \omega, \bar{o})$, $(o, A, \bar{o}, A')$, (ω', A, ω, A') are orthogonal.

On the riemann sphere S, the two configurations $C = (\gamma_1,\gamma_2,\gamma_3)$ formed of three orthogonal circles are isomorphic. By reduction to the standard case, we deduce that if $o \in \gamma_1\cap\gamma_2$, there exists a unique one parameter group $\varphi(z)$ ($z \in U^1$) of automorphisms of S leaving $\gamma_1\cap\gamma_2$ and γ_3 stable, and acting on the tangent space at o as multiplication by z. We call it the *group of rotations around o.*

Let $c'\colon U^1 \to G$ be the group of rotations around A' (relative to the three circumferences (3.2.7.1)). Lift c'^2 to $c\colon U^1 \to G_1$, and put $c = c(e^{2\pi i/8})$. It is the *Cayley transformation* relative to o and W. It cyclicly permutes $\omega', o, \omega, \bar{o}$, fixes A and A', and transforms M into the half-plane bounded by $(A', o, A, \bar{o})$ and containing ω.

[20] That is, to the unit disk

3.2.8. After (3.2.2), $M^\vee - \{\omega\}$ is a principal homogeneous space under $W_{-2}(G_{\mathbb{C}})$. Likewise, $M^\vee_{\mathbb{R}} \setminus \{\omega\}$ is a principal homogeneous space under $W_{-2}(G)$.

Let N be the element of $W_{-2}(\mathfrak{g})$ such that $\exp(N)\cdot\omega' = A$.

PROPOSITION 3.2.9. *(i)* $\exp(iN)\cdot\omega = h_o$.

(ii) $M = \{\exp(\lambda N)\cdot\omega' \mid \operatorname{Im}(\lambda) > 0\} = \{\exp((\lambda - i)N)h_o \mid \operatorname{Im}(\lambda) > 0\}$.

(iii) $M = \{\exp(\lambda c(N)\cdot h_o| \, |\lambda| < 1\}$.

PROOF. The Cayley transformation maps ω to $\infty = h_o^{-1}$, and $\exp(N)$ into the translation which sends h_o to A. It therefore transforms $\exp(iN)$ into the translation which sends h_o to ω. Since c sends ω', h_o onto h_o, ω, this proves (i).

The other assertions are verified similarly. □

PROPOSITION 3.2.10. *Let (H,ρ) be a representation of G' of weight n, ψ a polarization of H , and W the weight filtration of H. Let*

$$P^{n+i} \subset Gr^W_{n+i}(H) \text{ be the kernel of } N^i\colon Gr^W_{n+i}(H) \to Gr^W_{n-i}(H),$$

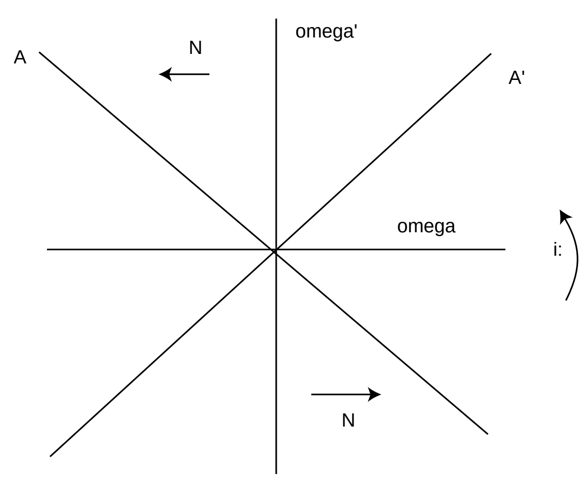
and again write ψ for the pairing between $Gr^W_{n+i}(H)$ and $Gr^W_{n-i}(H)$ defined by ψ. Then, for $x \neq 0$ in P^{n+i},

$$\psi(N^i x, x) > 0.$$

PROOF. Take $H = V$. That ψ is a polarization signifies that ψ is alternating and that

$$\psi(x, \, -J_0 x) > 0$$

for J_0 the complex structure defined by $o \in M$. Since $h_0(e^{2\pi i/8})$ permutes (ω, A', ω', A) cyclicly, the lines which define ω, A', ω', A and the transvection N, which fix ω and send ω' onto A, are as in the following figure:



The assertion follows. The general case can be deduced by taking tensor products. □

3.2.11 (VARIANT). Starting from the group $\mathsf{G}_1 = U^1 \times SL(2,\mathbb{R})$, let $h^{\mathrm{USL}} : U^1 \to G_1$ be as in [?], let $w^{\mathrm{USL}}\colon \mathbb{G}_m \to \mathsf{G}_1$ as in [?], and let G' be as in [?]. Let $h_o\colon \mathbb{S} \to G'$ be deduced from h^{USL} as in [?], and let w' be deduced from w^{USL} as in [?].

For any representation (H, ρ) of G', $\rho \circ h_o$ is a Hodge structure on H. Let F_o be the Hodge filtration, and W the increasing filtration defined by $\rho \circ w'$. Then (H, F_o, W) is a mixed Hodge structure; this follows easily from the criterion [?].

The morphism $\rho \circ w'$ defines not only W, but also a gradation of H, and hence an isomorphism of H with $Gr^W(H)$. Via this isomorphism, the Hodge structure of $Gr_i^W(H)$ is defined by the action of the factor U^1 of G_1. Write C for the Weil operator $h(-i)$ on $Gr^W(H)$; it is $\rho((-i, e))$.

Let $N \in W_{-2}(\mathfrak{sl}(2, \mathbb{R})$ as in [?], relative to h^{SL} and w. Keep the notations of [?]. If ψ is a polarization of H, we again have

$$\psi(N^i x, Cx) > 0 \text{ for } x \in P^{n+i}, x \neq 0.$$

3.3 Cayley transformations

3.3.1. Let M be a symmetric hermitian domain, G the identity component of its group of automorphisms, and W a Cayley filtration.

Choose $o \in M$ and let $w \colon \mathbb{G}_m \to \mathsf{G}$ be the morphism defining W and such that $\sigma w = w$ ($\sigma = \mathrm{ad}\, h_o(\pm i)$). Let $\varphi_W \colon U^1 \times SL(2, \mathbb{R}) \to \mathsf{G}$ be as in [?].

We apply the results of (3.2) to the group $\mathrm{SL}(2, \mathbb{R})$ endowed with h_o^{SL} and w^{SL}. Let $c \in \mathrm{SL}(2, \mathbb{R})$ be the Cayley transformation, and let $N \in W_{-2}(\mathfrak{sl}(2, \mathbb{R}))$ be as in [?].

The parabolic subgroup $W_0(\mathsf{G})$ has for Levi subgroup the centralizer Z of w, and the decomposition $\mathfrak{g}^0 = \mathfrak{g}_t^0 \oplus \mathfrak{g}_h$ can be integrated into an isogeny

$$\mathsf{Z}_t^0 \times \mathsf{Z}_h \to \mathsf{Z}.$$

The decomposition $\mathfrak{g}^{\text{even}} = \mathfrak{g}_t \oplus \mathfrak{g}_h$ can be integrated into an isogeny

$$\mathsf{G}_t \times \mathsf{Z}_h \to \mathsf{G}^{\text{even}}.$$

The morphism φ_W lifts to $\mathsf{G}_t \times Z_h$; it sends U^1 into Z_h and $\mathrm{SL}(2, \mathbb{R})$ into G_t. Note that Z (resp. [?]) is the intersection of $\mathsf{Z}(\mathbb{R})$ (resp. [?]) with G.

3.3.2. Since $W^0(G)$ is a parabolic subgroup of K (the centralizer of $o \in M$), which is compact and maximal, we have $W^0(G) \cdot K = G$. We also have

$$W^0(G) \cap K = W^0(G) \cap \sigma(W^0(G)) \cap K = Z \cap K,$$

which implies that $g \mapsto g \cdot o$ identifies M with $W^0(G)/(Z \cap K)$.

LEMMA 3.3.3. *The point $c^{-1}o = \exp(-iN)o$ of $M^\vee$ is fixed by Z_t.*

PROOF. On $\mathfrak{g}_t$, the filtrations F and W are deduced from h^{SL} and w^{SL} via the adjoint representation of $\mathrm{SL}(2, \mathbb{R})$. Indeed, $\mathfrak{g}_t$ commutes with $\varphi_W(U^1)$. After (3.2), we have therefore on $\mathfrak{g}_t$

$$e^{-1}(F) = \exp(-iN)(F) = \sigma(W).$$

Since o is fixed under $F^0(\mathsf{G}_\mathbb{C})$, $c^{-1}o$ is fixed under $\sigma(W^0(\mathfrak{g}_t))$, and in particular, is fixed under $\mathfrak{g}_t^0$.

For $g \in Z_t$, we have therefore

$$\begin{aligned} g \cdot o &= g \cdot \exp(iN) \cdot \exp(-iN) \cdot o \\ &= \exp(i\ (\mathrm{ad} g)(N))g\ \exp(-iN) \cdot o \\ &= \exp(i\ (\mathrm{ad} g)(N) - iN) \cdot o. \end{aligned}$$

In particular, the maximal compact subgroup $Z_t \cap K$ of Z_t is the centralizer of N in the adjoint representation of Z_t on $W_{-2}(\mathfrak{g})$. □

LEMMA 3.3.4. *The Z_t-orbit of N is open.*

PROOF. In any representation of $\mathrm{SL}(2,\mathbb{R})$, $\mathfrak{g}_t$ is the sum of k adjoint representations and of ℓ trivial representation, for some k and ℓ. As $Z_t \cap K$ is the intersection of the centralizers of w and N, it is the centralizer of $\mathrm{SL}(2,\mathbb{R})$. Therefore $\dim(Z_t \cap K) = \ell$. Moreover, $W_{-2}(\mathfrak{g}) = \mathfrak{g}_t^{-2}$ is of dimension k, and $\dim \mathfrak{g}_t^o = k + \ell$. Therefore,

$$\dim(W_{-2}(\mathfrak{g})) = \dim(Z_t) - \dim(Z_t \cap K)$$

and (3.3.4) follows. □

3.3.5. Apply [?]. The Z_t-orbit of N is a self-adjoint cone $C \subset W_{-2}(\mathfrak{g})$. The cone C is again the orbit of N for the adjoint action of $W^0(G)$ on $W_{-2}(\mathfrak{g})$, because $W_{-2}(\mathfrak{g})$ is central in $W_{-1}(\mathfrak{g})$ and commutes with Z_h. The cone C therefore depends only on W, and not on the origin $o \in M$ chosen. We have

$$\begin{aligned} M &= W^0(G) \cdot o \\ &= W_{-1}(G) \cdot Z_h \cdot Z_t \cdot o \\ &= W_{-1}(G) \cdot Z_h \cdot \exp(iC - iN) \cdot o \\ &= \exp(W_{-1}(\mathfrak{g}) + iC - iN) \cdot Z_h \cdot o. \end{aligned}$$

Taking account of the stabilizer of o, we therefore have the following result.

THEOREM 3.3.6. *The mapping*

$$x, g \mapsto \exp(x) \cdot g \cdot o$$

identifies $(W_{-1}(\mathfrak{g}) + iC - iN) \times (Z_h / Z_h \cap K)$ *with* M.

This theorem is an avatar of the theory of Cayley transformations.

3.3.7. The subgroup $Z_h \cap K$ of Z_h is the centralizer of $\varphi_W | U^1 \colon U^1 \to Z_h$. Via φ_W, U^1 acts on $\mathrm{Lie}(Z_h)_\mathbb{C}$ only by the characters $z^{-1}\bar{z}$, 1, $z\bar{z}^{-1}$, and $\mathrm{ad}\varphi_W(\pm i)$ is a Cartan involution. The quotient $M_\infty = Z_h / Z_h \cap K$ is therefore a symmetric hermitian domain.

The projection $\pi : M \to M_\infty$ made plain by (3.3.6) can be described as follows.

First description: Identify Z_h with a subgroup of $W_0(G)/W_{-1}(G)$. For $x \in M$, h_x and W define

$$\varphi_W^x \colon U^1 \times \mathrm{SL}(2,\mathbb{R}) \to G,$$

and the image of U^1 is in $W_0(G)$; its image in $W_0(G)/W_{-1}(G)$ is even in Z_h, whence we get a morphism

$$h_{\pi(x)}\colon U^1 \to Z_h.$$

The domain M_∞ can be identified with the set of conjugates of this morphism, and $x \mapsto h_{\pi(x)}$ is the projection of M onto M_∞.

Second description: Let G_1 and G' be as in [?]. For a representation V of G' with finite kernel, and for any representation of G_1, M can be identified to a space of Hodge structures on V. The graded space $Gr_W(V)$ is a representation of $W_0(G')/W_{-1}(G')$, and the Hodge structure $Gr_W(F_x)$ on $Gr_W(V)(x \in M)$ is defined by $h_{\pi(x)}$: the mapping $M \to M_\infty$ can be identified with the mapping that associates with a Hodge filtration F_x of V $(x \in M)$ the Hodge filtration $Gr_W(F_x)$ of $Gr_W(V)$. This description makes clear that π is holomorphic.

3.3.8. Put

$$M(W) = W_{-1}(G_\mathbb{C}) \cdot M \subset M^\vee.$$

In the description above, it is the set of filtrations F on V conjugate under $\mathsf{G}'(\mathbb{C})$ to a filtration $F_x \quad (x \in M)$, and inducing on $Gr_W(V)$ the same filtration as some F_x. The map π extends to

$$\pi\colon M(W) \to M_\infty,$$

and π makes $M(W)$ fibre in homogeneous spaces under $W_{-1}(G_\mathbb{C})$. The stabilizer in $W_{-1}(G_\mathbb{C})$ of $x \in M(W)$ is the subgroup

$$W_{-1}(G_\mathbb{C}) \cap F^0(G_\mathbb{C})$$

with Lie algebra $F^0(W_{-1}(\mathfrak{g}_\mathbb{C}))$.

3.3.9. Put $M(W)' = M(W)/W_{-2}(G_\mathbb{C})$, whence we have a factorization of π :

$$M(W) \overset{\pi_2}{\to} M(W)' \overset{\pi_1}{\to} M(\infty).$$

For $x \in M(W)$, the image $F^0(Gr_W^{-1}(\mathfrak{g}_\mathbb{C}))$ of $F^0(W_{-1}(\mathfrak{g}_\mathbb{C}))$ in $W_{-1}(\mathfrak{g}_\mathbb{C})/W_{-2}(\mathfrak{g}_\mathbb{C})$ depends only on $\pi(x)$. The projection π_1 therefore makes $M(W)'$ into principal homogeneous space over $M(\infty)$, with group $Gr_{-1}^W(\mathfrak{g}_\mathbb{C})/F^0_{\pi_1(x)}(Gr_{-1}^W(\mathfrak{g}_\mathbb{C}))$.

The projection π_2 makes $M(W)$ a principal homogeneous space over $M(W)'$ for the group $W_{-2}(\mathfrak{g}_\mathbb{C})$. For each $y \in M(W)'$, the intersection $M \cap \pi_2^{-1}(y)$ is a translate of $W_{-2}(\mathfrak{g}) + iC$.

For each $x \in M_\infty$, $Gr_{-1}^W(\mathfrak{g})$ is equipped with a Hodge structure, and, in particular, an operator $h_x(-i)$.

THEOREM 3.3.10. *For $u \in Gr_{-1}^W(\mathfrak{g})$, we have*

$$[u, h_x(-i)u] \in -\bar{C}.$$

The bracket is nonzero unless $u = 0$.

PROOF. We use [?], and the fact that $-B$ is a polarization of $\mathfrak{g}$. For $x = \pi(o)$ we have

$$-B([N, \sigma u], h_{\pi(0)}(-i)\sigma u) > 0, \text{ for } u \neq 0, \text{ or}$$

$$-B(N, \sigma[u, h_{\pi(o)}(-i)u]) > 0.$$

□

COROLLARY 3.3.11. *$W_{-2}(\mathfrak{g})$ is the centre of $W_{-1}(\mathfrak{g})$.*

3.3 Tube domains

.[21]

3.3.1. Let M be a simple symmetric hermitian domain of rank r, and let G be the identity component of its group of automorphisms. Keep the notations of (1.3.2), (1.3.5) (with $G' = G$), (3.1.2) and (3.1.13). Then G is the real adjoint algebraic group with Lie algebra $\mathfrak{g}$. We write D for the Dynkin diagram of $\mathsf{G}_{\mathbb{C}}$. Since G is an inner form of its compact form, complex conjugation acts on D by the opposite involution. The morphisms $h_x \colon U^1 \to G$ extend to morphisms $\mathbb{G}_m \to G_{\mathbb{C}}$. As was explained in [?], the conjugacy class of this defines a vertex e of D.

PROPOSITION 3.3.2. *The following conditions are equivalent:*

(i) e *is fixed by the opposite involution of* D*, i.e.,* h *and* h^{-1} *are conjugate under* $G(\mathbb{C})$*;*

(ii) $\mathsf{G}(\mathbb{R})$ *is disconnected;*

(iii) $\varphi|U^1$ *is trivial;*

(iv) $Gr^W_{\pm 1}(\mathrm{Lie}(G)) = 0$*;*

(v) the relative system of roots of G *is of type* C*.*

PROOF. The equivalence of (i) and (ii) results from the more precise statement that follows. □

PROPOSITION 3.3.3. *(i) If* e *is not fixed by the opposite involution, then* $\mathsf{G}(\mathbb{R})$ *is connected.*

(ii) If e *is fixed by the opposite involution, then* $\mathsf{G}(\mathbb{R})$ *has two components. The elements of the connected component not containing the identity transform* h *into a* G*-conjugate of* h^{-1}*.*

Recall that a real linear algebraic group K is said to be *compact* if $K(\mathbb{R})$ is compact and each connected component of $K(\mathbb{C})$ has a real point.
We shall use the following facts:

(a) $K \mapsto K(\mathbb{R})$ is an equivalence of categories from the category of real linear algebraic groups to the category of compact Lie groups.

(b) If the involution σ of a real connected reductive algebraic group G induces on $\mathrm{Lie}(\mathsf{G})$ a Cartan involution, then the algebraic subgroup K of G of fixed points of σ is compact, and $\mathsf{K}(\mathbb{R})$ is a maximal compact subgroup of $\mathsf{G}(\mathbb{R})$. In particular, $\pi_0(\mathsf{K}) \approx \pi_0(\mathsf{G}(\mathbb{R}))$.

Let K' be the centralizer of $h(-i)$ in G. It is compact, and $\mathsf{K}'(\mathbb{R})$ is a maximal compact subgroup of $\mathsf{G}(\mathbb{R})$. Since K' normalizes the centre of $K'(\mathbb{R})^0$, $k' \in K'$ fixes h or conjugates it to h^{-1}.

The centralizer of h in G is connected, because it is the centralizer of a torus. The centralizer of h in $\mathsf{G}(\mathbb{R})$ is a compact form, and so is also connected; it is K.

h and h^{-1} are conjugate under $G(\mathbb{C})$ is and only if e is fixed under the opposite involution; from that:

[21] The original numbers both subsections 3.3.

REMARK 3.3.4. One verifies on 3.1.3 that, for $s \neq 0, r$, one never has $Gr^{W_s}_{\pm 1}(Lie\ G) = 0$.

REMARK 3.3.5. Let $Z \in \mathfrak{g}$ be an infinitesimal generator of $h(U^1)$. The conditions of 3.[?].2 are also equivalent to:

(vi) Z is contained in a simple subalgebra $\mathfrak{s}$ of rank 3 of $\mathfrak{g}$.

It is clear that (iii) $\Longrightarrow$(vi). Admit (vi). The subalgebra $\mathfrak{s}$ is stable under σ. It can not be in $\mathfrak{k}$ because Z would not then be central. There therefore exists $x \neq 0$ in $\mathfrak{s} \cap \mathfrak{p}$. We may suppose (after conjugation) that $x \in \mathfrak{a}$. We then have $Z = \lambda[x, Jx]$, whence $Z \in [a, Ja] \subset d\varphi(\mathfrak{sl}(s, \mathbb{R})^r)$, whence (iii).[22]

3.3.6. When the equivalent conditions of [?] are satisfied, Theorem [?] furnishes an isomorphism between M and the tube domain $W_{-2}(\mathfrak{g}) + iC \subset W_{-2}(\mathfrak{g}_{\mathbb{C}})$.

[22]The following was deleted from the original. Suppose M satisfies the equivalent conditions of 3.?.2. There then exists

$$\varphi\colon SL(2, \mathbb{R}) \to G$$

such that

(a) Let $h_0\colon U^1 \to SL(2, \mathbb{R})$: $e^{i\theta} \mapsto$rotation by θ. Then $h = \varphi \circ h_0$.

(b) Let $w_0\colon \mathbb{G}_m \to SL(2, \mathbb{R})$: $\lambda \mapsto \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$. Then the filtration W_r of G is defined by φw_0.

(c) $\varphi(-1) = e$.

Until the end of this number, we will keep the preceding notations (and abandon those of 1.3.2), and we will simply denote W_r by W. Write $\mathfrak{g}^i$ for the gradation of g defined by φw_0.

4 Quotients

4.1 General Remarks

4.1.1. Let M be a symmetric hermitian domain, let G be its group of automorphisms, and let $\Gamma \subset G$ be an arithmetic subgroup. We are interested in the case that G/Γ is noncompact. There then exists a semisimple group $\mathsf{G}_{\mathbb{Q}}$ over $\mathbb{Q}$, and an isomorphism of G with the group deduced from $\mathsf{G}_{\mathbb{Q}}$ by extension of scalars to $\mathbb{R}$, such that Γ is commensurable with $\mathsf{G}(\mathbb{Z})$ (for any integral structure on $\mathsf{G}_{\mathbb{Q}}$). We shall assume in the following that $\mathsf{G}_{\mathbb{Q}}$ is $\mathbb{Q}$-simple.

4.1.2. I do not know of an *a priori* proof of the following facts, which can be checked by inspecting the Dynkin diagrams.

(a) When $P_{\mathbb{Q}}$ is a maximal parabolic of $\mathsf{G}_{\mathbb{Q}}$, then $P_{\mathbb{R}}$ is a product of maximal parabolic subgroups of the simple factors of G. In particular, $P_{\mathbb{R}}$ corresponds to a Cayley filtration W.

(b) The decomposition $P_{\mathbb{R}}/R_u(P_{\mathbb{R}}) \sim Z_h \times Z_t$ (isogeny) is defined over $\mathbb{Q}$.

4.1.3. Fix $P_{\mathbb{Q}}$ as above, and let

$$\Gamma_P = \Gamma \cap P(\mathbb{R})$$
$$\Gamma_P^1 = \mathrm{Ker}(\Gamma_P \to Z_t).$$

The quotient $M(W)/\Gamma_P^1$ is without doubt always in a natural way an algebraic variety (cf. the following number). We have in any case:

(a) The quotient M_∞/Γ_P^1 of the symmetric hermitian domain M_∞ by the arithmetic group that is the image of Γ_P^1 in Z_h, is an algebraic variety. Assume that Γ is sufficiently small so that Γ_P^1 acts freely on M_∞. Then we have:

(b) $M(W)'/\Gamma_P^1$ is a fibre bundle over M_∞/Γ_P^1. It is a principal homogeneous space for the fibre space of complex tori,

$$Gr_{-1}^W(\Gamma)\backslash Gr_{-1}^W(\mathfrak{g})/F_y^0(Gr_{-1}^W(\mathfrak{g}))$$

(F_y^0 is a holomorphic function of y on M_∞). This last is polarizable: after [?], any rational linear form on $W_{-2}(\mathfrak{g})$, positive on $\bar{C}$ defines a polarization of it. It is therefore an abelian scheme.

To give $M(W)'/\Gamma'_P$ an algebraic structure, it suffices to construct a multisection; such a section will define a trivialization of a power of the principal homogeneous space considered. We construct it as follows:

(α) Take $w : \mathbb{G}_m \to G_{\mathbb{Q}}$ which defines W;

(β) Take $o \in M$ such that ad $h_o(-i)(w) = w^{-1}$. The theorem [?] then shows that there is a mapping

$$M_\infty/\Gamma_P^1 \cap \{\text{centralizer of } \ w\} \to M(W)/\Gamma'_P$$

that defines the multisection sought.

(c) $M(W)/\Gamma'_P$ is a fibre bundle over $M(W)'/\Gamma'_P$; in fact, a principal homogeneous space for the torus (= product of copies of $\mathbb{G}_m$). Here, defining the algebraic structure poses a problem that I have not resolved.

4.2 Example: Siegel

4.2.1. Let $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank $2g$ and let ψ be an alternating from of discriminant one on $H_{\mathbb{Z}}$. Let M be the space of Hodge structures of type $\{(-1,0),(0,-1)\}$ on $H_{\mathbb{Z}}$ for which ψ is a polarization. It can also be described as the set of complex structures J on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$ such that $\psi_{\mathbb{R}}$ is the imaginary part of a positive definite hermitian form ([?]). The quotient $M/\operatorname{Sp}(H_{\mathbb{Z}})$ is the space of moduli of polarized abelian varieties in the principal series.

4.2.2. The weight filtrations are here defined by the isotropic filtrations of $H_{\mathbb{Z}}$ of the form

$$\{0\} \subset W_{-2}(H_{\mathbb{Z}}) \subset W_{-1}(H_{\mathbb{Z}}) \subset H_{\mathbb{Z}}$$

(with W_{-2} and W_{-1} orthogonal complements of each other). Fix W and put $\dim Gr^W_{-1}(H_{\mathbb{Z}}) = 2g_0$.

4.2.3. The form ψ induces on $Gr^W_{-1}(H_{\mathbb{Z}})$ an alternating form $\bar{\psi}$ of discriminant one. We have:

(a) M_∞ is the space of Hodge structures of type $\{(-1,0),\ (0,-1)\}$ on $Gr^W_{-1}(H_{\mathbb{Z}})$ for which $\bar{\psi}$ is a polarization.

(b) $M(W)$ is the space of maximal isotropic subspaces F^0 of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ such that $(H_{\mathbb{Z}}, F, W)$ is a mixed Hodge structure and $\bar{\psi}$ is a polarization of $Gr^W_{-1}(F)$.

4.2.4. Let $\Gamma \subset \operatorname{Sp}(H_{\mathbb{Z}})$ be the subgroup respecting the filtration W and acting trivially on $Gr^W_0(H_{\mathbb{Z}})$ (and $Gr^W_{-2}(H_{\mathbb{Z}})$). We propose to give a modular interpretation of the quotient $M(W)/\Gamma$. Put $X = Gr^W_0(H_{\mathbb{Z}})$. The quotient $M(W)/\Gamma$ classifies the objects consisting of:

(a) a mixed Hodge structure $(V_{\mathbb{Z}}, W, F)$ of type $\{(-1,-1),(-1,0),(0,-1),\ (0,0)\}$ with $V_{\mathbb{Z}}$ of rank $2g$.

(b) an isomorphism $X \xrightarrow{\approx} Gr^W_0(V_{\mathbb{Z}})$.

(c) an alternating form ψ on $V_{\mathbb{Z}}$, such that

(c_1) W_{-1} and W_{-2} are orthogonal to one another;

(c_2) F^0 is maximal isotropic;

(c_3) ψ induces a polarization of $Gr^W_{-1}(V_{\mathbb{Z}}, F)$.

4.2.5. After [?], an object (a), endowed with (b), can be identified with a two-term complex

$$\kappa\colon X \to G$$

where G is an extension of an abelian variety A of dimension g_0 by a torus T. Via this identification, (c) becomes an isomorphism φ between the one-motive κ and its Cartier dual κ^*, such that ${}^t\varphi = -\varphi$, and which induces a polarization $\bar{\varphi}$ on A.

Note that the isomorphism φ identifies T with the torus whose character group is X. The system (κ, φ) is determined by the following data:

(α) the principally polarized abelian scheme $(A, \bar{\varphi})$;

(β) the map $\delta\colon X \to A$ (deduced from $X \to G$). This map defines the extension G of $A \cong_{\bar{\varphi}} A^*$ by $T = \mathcal{H}om(X, \mathbb{G}_m)$;

(γ) the lifting $\tilde{\delta}\colon X \to G$ of δ; this lifting is required to satisfy the symmetry condition. For $\tilde{\delta}_o$ a permitted lifting, the other liftings can be written $\tilde{\delta}_o \cdot s$ for

$$s\colon X \to \mathrm{Hom}(X, \mathbb{G}_m)$$

defined by a symmetric bilinear form on X with values in $\mathbb{G}_m$.

4.2.6. The objects $(A, \bar{\varphi}, \delta, \bar{\delta})$ of (4.2.5) are purely algebraic, and $M(W)/\Gamma$ is, in the analytic category, their coarse moduli space. Their coarse moduli space over $\mathbb{Q}$ is an algebraic variety over $\mathbb{Q}$ admitting $M(W)/\Gamma$ as its set of complex points.

4.2.7. The "dévissage" [?] is also algebraic:

(a) The mapping π from $M(W)/\Gamma$ to the coarse moduli scheme of principally polarized abelian schemes of dimension g_0 sends $(A, \bar{\varphi}, \delta, \tilde{\delta})$ to $(A, \bar{\varphi})$.

(b) The quotient $(M(W)'/\Gamma)$ classifies the $(A, \bar{\varphi}, \delta)$ and π_1 is $(A, \bar{\varphi}, \delta, \bar{\delta}) \mapsto (A, \bar{\varphi}, \delta)$.

4.2.8. The group $\mathrm{GL}(X)$ acts on $M(W)/\Gamma$ by transport of structure via its action on X: $\sigma \in GL(X)$ acts by

$$(A, \bar{\varphi}, \delta, \tilde{\delta}) \mapsto (A, \bar{\varphi}, \delta\sigma^{-1}, \tilde{\delta}\sigma^{-1}).$$

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