II. Algebraic Functions

( Tata, spring 1965)

Ref: Chevalley: Introduction to the Theory of Algebraic Functions of One Variable 1951
(see also Weil: Review of the above, Bull A.M.S 51951, Vol 57 384-98)
Weil: Sur les Courbes Algébriques et les Variétés qui s’en déduisent 1948
Walker: Algebraic Curves 1950
Weyl: Die Idee der Riemannschen Fläche 1923
Serre: Groupes Algébriques et Corps de Classes 1959.

1. Function Fields

$k$ will be a field with algebraic closure $\overline{k}$.

defn: A function field in $r$ variables over $k$ is a $k$-extension field $K$ of transcendence degree $n$ over $k$.

Space $K = k(x_1, \ldots, x_r) + A = k[x_1, \ldots, x_r]$. The kernel of the obvious surjection

$k[x_1, \ldots, x_r] \twoheadrightarrow k[x_1, \ldots, x_r] = A$

is a prime ideal $\mathfrak{p}$, say $\mathfrak{p} = (f_1, \ldots, f_s)$, of $k[x_1, \ldots, x_r]$

The variety of $\mathfrak{p}$ in $A_r(\overline{k})$ is

$V = \{ a = (a_1, \ldots, a_n) \in \overline{k}^n \mid f_i(a) = 0, i = 1, \ldots, s \}$

defn: We can identify $A$ with the set of polynomials functions (weffs in $k$) from $V$ to $k$.

defn: Any elt of $A$ can be written $f$ where $f \in k[x_1, \ldots, x_r]$ associate to $f$ the function

$a \mapsto f(a) : V \to k$

By the defn of $f$, this map is well defined and is clearly an epimorphism.

Now since $f \in k[x_1, \ldots, x_r]$ vanishes on all of $V$.

By the Nullstellensatz, $f \equiv 0$, so $\mathfrak{p} = (0)$, and we have shown the map to be injective.

Kas called the field of rational functions on $V$.
Any $t = f(y) \in K$ may be regarded as the function $x \mapsto f(x)/g(x)$ from the nonempty subset $V$ where $g \neq 0$ to $\overline{K}$. (i.e., $t$ defines a “rational map” $V \to A(\overline{K})$)

The local ring of a point $a \in V$ is

$$\mathcal{O}_a = \{ h \in K | \exists f, g \in A \text{ with } h = f/g, g(a) \neq 0 \}$$

$\mathcal{O}_a$ is a maximal local ring with maximal ideal $\mathfrak{m}_a = \{ h \in K | f(a) = 0 \text{ for any } f \in \mathcal{O}_a \}$. If we define $\mathcal{O}_a = \mathfrak{m}_a \cap A = \{ f \in A | f(a) = 0 \}$ then $\mathcal{O}_a$ is a prime ideal of $A$. $\mathcal{O}_a = A_{\mathfrak{m}_a}$.

Let $a = (a_1, ..., a_r) \in V$; we can consider the field $k(a_1, ..., a_r)$. If $\sigma \in \text{Gal}(\overline{K}/k)$ then $\sigma \cdot a = (\sigma(a_1), ..., \sigma(a_r)) \in V$, because $V$ is defined by polynomials with coefficients in $k$. We say that $a \in \overline{A}$ and $\mathcal{O}_a$ are conjugate points.

Note that $\mathcal{O}_{\overline{a}} = \sigma \mathcal{O}_a$, for $\overline{a} = \sigma \cdot a$ i.e., $f(a) = 0$ $\iff$ $f(\sigma(a)) = 0$. Thus $\mathcal{O}_a$ depends only on the conjugacy class of $a$.

**Proof:** $a \mapsto \mathcal{O}_a$ is a surjection from the points of $V$ to the maximal ideals of $A$. Two points correspond to the same ideal iff they are conjugate.

**Proof:** $\mathcal{O}_a$ maximal: if $f(a) \neq 0$, then for any $g \in A$

$$g = (g - \frac{g(a)}{f(a)}) \cdot f(a) + f(a)$$

$i.e.$, $\mathcal{O}_a$, together with any elt $f \notin \mathcal{O}_a$, generate $A$. Surjection: let $m$ be maximal in $A$; by the Nullstellensatz, $m$ has a zero $a \in V$; then $m \subset \mathcal{O}_a \in A$.

Finally, since $\mathcal{O}_a = \mathcal{O}_{\overline{a}}$, then have

$$k(a_1, ..., a_r) \cong k[x_1, ..., x_r]/\mathfrak{m}_a \cong k[a_1', ..., a_r']$$

Thus, $\exists k$-isom $\sigma: k(a_1, ..., a_r) \to k(a_1', ..., a_r')$ which can be extended to $\sigma: K \to \overline{K}$.

**Note:** More generally, we have a surjection from the set of all subvarieties of $V$ onto the prime ideals of $A$. 
\textbf{Pf.} \quad A = A_{\text{max}}

\textbf{Proposition 1.} \quad \text{For any } m \geq 1, \exists \gamma_m \in \mathbb{N} \text{ s.t. } \gamma_m h \in A. \text{ The ideal generated by the } \gamma_m \text{'s is all of } A, \text{ hence } \exists \mathbf{f} = (f_1, \ldots, f_r) \in A \text{ s.t. } 1 = \sum \gamma_m f_m. \text{ But then } \ h = \sum \gamma_m f_m. \ h = \sum f_i, f_i \in A.

\textbf{Note.} \quad \text{The integral closure of } A \text{ is the intersection of all valuation rings containing } A; \text{ thus it follows that if } A \text{ is not integrally closed then not all } A_m \text{ are valuation rings.}

\textbf{2. Results on Commutative Algebra}

Let \( k = k[x_1, \ldots, x_r] \) be a fin field in n variables \( k \),
\[ A = k[x_1, \ldots, x_r] \]

\textbf{Normalization Lemma.} \quad \exists y_1, \ldots, y_n \in A \text{ s.t. } A \text{ is integral over } B = k[y_1, \ldots, y_n] \cong k[y_1, \ldots, y_n].

\( B \rightarrow A \) corresponds to a rational map of varieties, \( A^n(k) \rightarrow V \subset A^r(k) \).

We have:
\[ \sum_{i=1}^{r} y_i(x_1^{e_1} + f_{1i}y_1x_1^{e_1-1} + \cdots + f_{ri}y_rx_1^{e_1-1} + \cdots) = 0, \quad i = 1, \ldots, r \]

Let \( a_1, \ldots, a_m \in A^n(k) \), then \( b \mapsto (a_1, \ldots, a_m) \) if \( b \) satisfies \( x_1^{e_1} + f_{1i}(a) x_1^{e_1-1} + \cdots = 0, \quad i = 1, \ldots, r \).

The map is a (finite one) map, as can be seen from \( A/B \) integral. Can think of \( V \) as lying over \( A^n(k) \) in a finite number of sheets.

\textbf{Dimension Theorem.} \quad If \( y_1, \ldots, y_m \in A \) is a chain of prime ideals of \( A \), then \( m \leq n \), \( \text{the chain is maximal if } m = n \).

Corresponding to such a maximal chain we get \( \mathbf{f} = \mathbf{f}_0, \ldots, \mathbf{f}_r, \mathbf{f}_{r+1}, \ldots, \mathbf{f}_n \subset k[x_1, \ldots, x_r] \).

\( V = V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_n \rightarrow \emptyset \)
Moreover, \( \text{tr} \deg_k (A/x_i) = n-i \)

In particular, \( \text{tr} \deg_k (A/x_n) = 0 \) if \( k[x_1, \ldots, x_n]/F_n = A/F_n \) is an algebraic extension of \( k \).

Let \( a_1, \ldots, a_r \in k \) be \( \mathfrak{m} \). \( k[x_1, \ldots, x_r]/F_n \equiv k[a_1, \ldots, a_r] \)

then \( a_1, \ldots, a_r \) is an algebraic zero of \( F_n \) — thus the dimension theorem implies the Nullstellensatz.

**Normalization Theorem:** The integral closure of \( A \) in a finite \( K \)-extension of \( K \) is a finite \( A \)-module.

We say that \( V \) is normal if \( A \) is integrally closed. Note that normalization removes singularities in codimension one and all the local rings of \( \text{dim } 1 \) of \( A \) are regular if \( A \) is normal.

Now assume \( n=1 \). This says that every nonzero prime ideal of \( A \) is maximal or, geometrically, that the only proper subvarieties of \( V \) are points.

By the above discussion, for any \( x \in K \) which is transcendental over \( k \), \( [K:k(x)] \) is finite if the integral closure \( A \) of \( k(x) \) in \( K \) is finite over \( k[x] \); thus we may choose \( A = k[x_1, \ldots, x_r] \)

\( n = x_1, \ldots, x_r \) to be integrally closed.

Then
\[
\begin{align*}
\text{then} \quad & A \text{ is noetherian} \\
& \text{Integrally closed} \\
& \text{Has dim } 1
\end{align*}
\]

So \( A \) is a Dedekind ring.

\[
\begin{array}{c}
\mathbb{Q} \\
\text{finite} \\
\downarrow \\
\mathbb{Z}
\end{array}
\quad
\begin{array}{c}
\text{finite} \\
\downarrow \\
A \text{ alg integers}
\end{array}
\quad
\begin{array}{c}
\uparrow \\
\text{finite} \\
\downarrow \\
k(x)
\end{array}
\quad
\begin{array}{c}
\text{finite} \\
\downarrow \\
k[x]
\end{array}
\cdot
\begin{array}{c}
\text{finite} \\
\downarrow \\
k
\end{array}
\]
N.b. It is not true that every Dedekind ring $D$ admits in this way:

$$\begin{array}{c}
K \\
\downarrow \quad \uparrow \\
D \\
\downarrow \\
\text{not always} \\
\text{principal ring}
\end{array}$$

Fact on Dedekind Rings $A$

1. Every nonzero ideal $I \subset A$ has a unique expression as a product of prime ideals

$$I = \prod_{i} \mathfrak{p}_i^{m_i}$$

where $\mathfrak{p}_i$ are prime ideals.

2. $A/I$ is of finite length as an $A$-module; by the Chinese Remainder Theorem,

$$A/I \cong \bigoplus A/\mathfrak{p}_i^{m_i}$$

and $A/\mathfrak{p}_i^{m_i}$ has a composition series

$$A/\mathfrak{p}_i \supseteq A/\mathfrak{p}_i^{m_i} \supseteq \cdots \supseteq A/\mathfrak{p}_i^{m_i-1} \supseteq \mathfrak{p}_i^{m_i-1}/\mathfrak{p}_i$$

with $A/\mathfrak{p}_i \cong \prod \mathfrak{p}_i^{m_i-1}/\mathfrak{p}_i$ (as $A$-modules)

via $a \mapsto \overline{a}$, where $m_i$ is any nonzero elt of $\mathfrak{p}_i^{m_i-1}/\mathfrak{p}_i$.

In our case $A/\mathfrak{p}_i$ is a finite (= finitely generated algebraic) extension of $k = \dim_k(A/\mathfrak{p}_i) = \sum a_i [A/\mathfrak{p}_i : k]$.

E.g. $A = k[x,y]$; then $\mathfrak{p}_i \leftrightarrow$ irreducible polynomials $P_i$ of $k[x,y]$,

and $A/\mathfrak{p}_i \cong \prod P_i$.

3. The local ring $A_x$ of $A$ at a nonzero prime ideal $x$ is a Dedekind ring with only one nonzero prime ideal — hence it is a principal ring.

Let $x A_x = t A_x$.

Then any $f \neq 0 \in K$ can be written uniquely

$$f = u t^n, \quad u \text{ unit in } A_x, \quad n \in \mathbb{Z}$$

Let $V_x(f) = a$, $V_x(0) = \infty$; then $V_x$ is a
discrete valuation on \( K \), i.e. a \( \text{fn} \)
\[ v: K \rightarrow \mathbb{Z} \cup \{0\} \text{ s.t. } v(fg) = v(f) + v(g) \]
\[ v(f+g) \leq \min(v(f), v(g)) \]
\[ v(f) = 0 \iff f = 0 \]

Moreover, the corresponding valuation ring is \( \{ f \in K \mid v_x(f) \geq 0 \} = A_x \).

4) The nonzero ideals \( I \neq A \) are exactly the sets of the form
\[ I = \{ f \in K \mid v_x(f) \geq a \}, \quad i=1, \ldots, n \to 0 \}
\[ v_x(f) \geq 0 \iff f \not\equiv 0 \}
\[ A = \{ f \in K \mid v_x(f) \geq 0 \text{ all } x \} = \bigcap A_x \]

5) The \( A_x \) are exactly the nontrivial valuation rings of \( K \) containing \( A \) (cf. \( \mathbb{P}^1 \)).

3. The Associated Scheme

\( K/k \) a field and \( A \) a chosen set \( A \)
any ultrar ring ring containing \( K[x] \) contains \( A \) so such rings are exactly those of the form \( A_x \), \( x \) a maximal ideal of \( A \). Any other
ultrar ring of \( K/k \) contains \( k[x] \), hence the integral closure \( A_i \) of \( k[x] \), \( x \) belongs to its maximal ideal. But \( x \) belongs to only a finite number of nontrivial ideals of \( A_i \), so there can only be finitely many ultrar rings of \( K/k \) not containing \( A \).

**Def.** A prime divisor of \( K/k \) is a ultrar ring \( R_p, k \subset R_p \subset K \)
they are in fact d.v.r.'s i.e. regular local rings of \( k \).

**Def.** The canonical scheme \( (V, O_V) \) associated to \( K/k \) is as follows:
\[ V = \text{the set of prime divisors } P \text{ of } K/k \text{ with } \# P \]
the open sets of \( V \) are the cofinite subsets containing \( P_0 \) and \( \emptyset \), for \( U \) then \( \emptyset \), \( V(U, O_V) = \bigcap R_p \)
where \( R_{\sigma} = K \); the restriction maps are the obvious inclusions.

Remark: \((V, \mathcal{O}_V)\) is a normal (= nonsingular), complete normal scheme of finite type over \( k \) with \( R(A) = K \). Conversely, any such scheme is isomorphic to \((V, \mathcal{O}_V)\) (see EGA II 7.4.18 or C.A. C. II.1).

**Classical Case**: \( k = \mathbb{C} \). Then \( \exists \) a compact connected complex manifold \( V \) of dim 1 (Riemann surface) \( \cong k \), \( K \) is the field of meromorphic functions on \( V \).

For each \( P \in V \), \( R_P \) = functions which are holomorphic at \( P \) \( \supset \mathcal{O}_V(P) \cong \) order of the zero of \( f \) at \( P \) (so the \( \mathcal{O}_V \) of the canonical scheme are in correspondence with the \( \mathcal{O}_V \) of the Riemann surface). The classical theory on the Riemann surface comes from \( \mathbb{C} \), \( \mathbb{C} \) has no counterpart in the general situation.

4. **Nonsingular Points on an Affine Curve**

We revert to the old notation. Let \( k \) be a field, \( F(X, Y) \) an irreducible in \( k[X, Y], A = k[X, Y](F) = k[x, y] \), \( K = k(x, y) \) so \( K \) is a field in one variable [Clearly \( \deg K \leq 1\); if \( x \) is any over \( k \) then its mini height \( f(X) \) say, \( \in k[X] \) divides \( F(X, Y) \). Hence \( F(X, Y) \) involves only \( X, Y \) as transcendental over \( k \)].

**Then**: let \((x_0, y_0) \in A_k(k) \), be o.k. \( F(x_0, y_0) = 0 \). T.F.A.E.

1. Not both \( \frac{\partial F}{\partial X}(x_0, y_0) \) and \( \frac{\partial F}{\partial Y}(x_0, y_0) \) vanish.

(\( e, 3 \), a unique tft)

2. \( \text{dim}_k \left( \mathcal{O}_k / \mathcal{O}_k \right) = 1 \), where \( \mathcal{O} = \) some ideal of \( A \), \( \mathcal{O} \) vanishing at \((x_0, y_0)\).

3. Ay is a d. u. r.

4. \( A_y \) is integrally closed.

If \( F \) we may assume \( x_0 = y_0 = 0 \), otherwise trans
late 

\[ X \rightarrow X-x_0, \quad Y \rightarrow Y-y_0. \]

\[ \iff 2: \quad y = \{ (x, y) \mid (x, y) = 0 \} 
\]

\[ = X_1 + Y_1 (x, y) / (F) \]

\[ = (X, y) / (F) \]

\[ y^2 = (x^2, x, y, y^2, F) / (F) \]

\[ \dim_k \left( \frac{(X, y)(x, y, y^2)}{(X^2, x, y, y^2)} \right) = 2 \quad (\text{case } X, y) \]

so \[ \dim_k (y^2) = \dim_k \left( \frac{(X, y, F)(x, x, y, y^2, F)}{(x^2, x, y, y^2, x, y)} + (F) \right) \]

\[ = \dim_k (X, y) / (x^2, y^2, x, y) + (F) \]

\[ = 1 \quad (\not= 0 \text{ for } \dim_k (F) = 1) \]

\[ \implies F \notin (x^2, y^2, x, y) \]

\[ \implies (1) \; \text{holds.} \]

\[ 3 \iff 4: \quad A_k \; \text{integ. closed} \quad (\implies) \quad A_k \; \text{a Dedekind ring} \quad (\iff) \quad A_k \; \text{a d.v.r.} \]

\[ 2 \implies 3: \quad \text{Let } m = y A_k; \quad m/m^2 \cong y/y^2 \; \forall \; \text{any } t \not= 0 \in k \]

\[ \text{generates } y/y^2 \; (t \not= 0 \in m/m^2). \; \text{Then} \]

\[ m = t A_k + m/m^2 \; \text{so, by Nakayama, } m = t A_k. \]

\[ \text{Claim every ideal of } A_k \; \text{is of the form } t A_k. \]

\[ \text{Let } n \; \text{be a max. non-zero ideal.} \; \text{Then } n \subset m = t A_k \]

\[ \implies \text{if } tn = n \; \text{then } n = 0 \; \text{by Nakayama.} \; \text{Hence } n = t_0. \; \text{A some } t_0 \; \text{we have are done.} \; \text{It is now clear that } A_k \; \text{is a d.v.r.} \]

\[ 3 \implies 2: \quad \text{obvious.} \]

When the point \((x_0, y_0) \in A \) \( (x, y) \) (instead \( A_2 (\mathbb{R}) \) (instead of \( A_2 (\mathbb{K}) \)) then \n
\[ 1 \implies (2'): \quad \dim_k \left( \frac{x^2}{(x_0, y_0)} \right) = 1 \equiv (3) \equiv (4) \]

\[ \not\in \text{translate to origin.} \]

\[ \text{If } F \; \text{satisfies } (1) \; \text{then } F \not\in y^2 \; \text{for otherwise} \]

\[ F = \sum \frac{\partial G}{\partial y} \]

\[ \frac{\partial F}{\partial x} (x_0, y_0) = \sum \left( \frac{\partial G}{\partial y} H_i + G_i \frac{\partial H_i}{\partial x} (x_0, y_0) \right) = 0 \]

Thus, as before, we get
\( \dim_K (Y^2 - f(x)) = \dim_k ((X-x_0, (Y-y_0), F)/(X-y_0)^2, \ldots) = 1 \) where \( k' = k(x_0, y_0) = A_k = A_{k'/k} \)

The remaining implications are exactly as before.

Example: \( F(X, Y) = Y^2 - f(X) \) where \( f(X) = X^n + \cdots + a_n, n > 0 \) has no double roots in \( k \).

Then \( F \) is separable, for \( F \neq (Y - f_1(X))(Y + f_1(X)) \) (assuming char \( k \neq 2 \))

\[
\frac{2F}{2Y} = 2Y, \quad \frac{\partial F}{\partial x} = -f'(x)
\]

which cannot have a common zero \( (a_{1,1}, k) \) with \( F(X, Y) \). Hence all points in \( A_k/k \) satisfy the condition of the theorem. If we assume \( k \) is algebraically closed, or \( f \) has no double roots in \( k \), then all points in \( A_k/k \) satisfy these conditions, so all \( A_k \) are dense, i.e. \( A_k \) is integrally closed.

2. \( F(X, Y) = X^2 - Y^2 \) if \( Y^2 = X^3 \)

Both partials vanish at the origin — hence have a singularity. The integral closure of \( k[t] \) is \( k[t^2, t^3] \) \( \cong k[X, Y]/(X^2 - Y^2) \).

For \( t \) is integral over \( k[t] \) \( (t^2 - x = 0) \) or \( t \) is integrally closed (it is principal).

5 Sections

\( K/k \) for \( k \) in \( \bar{K} \) with \( z \in K \) is called if it is algebraic over \( k \); otherwise it is variable. We call it a variable of \( K \), which is finite over \( k \) or by the normalization theorem; or let \( \alpha \) be transcendental of \( k \) over \( k \) and \( k(\alpha) \) are linearly disjoint, i.e.

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In future we will assume \( k = k_1 \), i.e. that \( k \) is algebraically closed in \( K \).
\((V, O_V)\) will denote the canonical associated scheme. \(V = \{P\} \cup \{P_0\}\) where \(P_0 = \text{general}\) \(\mathfrak{m}\). \(\mathfrak{m}_P = \text{local ring at } P\) (it is a d. u. r.), \(\mathfrak{m}_P = \mathfrak{m}_P \mathfrak{m}_P\) is the max. ideal, \(+ \mathfrak{m}(P)\) the residue field.

**Defn:** A (Weil) prime divisor of \(V\) is an irreducible subscheme of codim 1, in an irreducible subscheme of dim 1, — we may identify it to a pt \(P \in V\).

**Defn:** The degree of (the divisor) \(P\) is \(\deg(P) = [k(P) : k]\) (\(< \infty\)).

The degree of \(P\) is the number of its \(V\) with values in \(\overline{k}\). Let \(r\) be the number of \(k\)-points \(k(P) \to \overline{k}\) of the number of conjugates of \((x_0, y_0)\) in \(\overline{k} \times \overline{k}\) where \(k(P) \cong k(x_0, y_0).

**Defn:** A (Weil) divisor is formal linear combination \(D = \sum n_P P\) , \(P \in V - P_0\) , \(n_P = 0\) almost all \(P\).

We write \(D \geq E \iff V_p(D) \geq V_p(E)\) all \(P\)
(where \(D = \sum V_p(D)\), \(\cdots\)

\(D \geq 0 \iff V_p(D) \geq 0\) all \(P\)

— such a \(D\) is effective.

The support of \(D\),

\(\text{supp}(D) = \{P \mid V_P(D) \neq 0\}\)

Every divisor can be written uniquely as the difference of two effective divisors,

\(D = D^+ - D^-\), \(\text{supp}(D^+) \cap \text{supp}(D^-) = \emptyset\)

The divisor of \(Z \in R(V) = K\) is

\(Z = \sum V_P(Z) P = (Z)_0 - (Z)_0^\vee\)

divisor of zeros of \(Z\) divisor of poles of \(Z\)
The degree of a divisor \( D \) is
\[
\deg(D) = \sum v_p(D) \deg P
\]

**Note:**
\[
\deg(D + E) = \deg(D) + \deg(E)
\]
for \( D \geq E \) \( \iff \) \( \deg(D) \geq \deg(E) \).

**Thm.** For any \( z \in K \),
\[
\deg(z) = \deg(z)_0 = \begin{cases} 0 & \text{if } z \in k \\ [K:k(z)] & \text{if } z \notin k \\ \end{cases}
\]

If the first part is obvious, we shall \( z \) as a variable. Let \( A \) be the integral closure of \( k[z] \) in \( K \). \( k[z] \) is fractional, \( A \) as free over \( k[z] \) with basis \( w_1, \ldots, w_n \), say.

Since \( K = k(z) A \), \( n = [K:k(z)] \)
\[
\begin{align*}
\text{Let } z \ A = \prod p_i^{e_i}, \quad p_i \leftrightarrow p_i, \quad z_0 = \sum e_i p_i; \\
\text{Then } \deg(z) = \sum e_i \deg p_i, \quad p_i = \dim_k(A/z_i)
\end{align*}
\]

But \( A/z_1 \cong \bigoplus_{i=1}^n k(z_{i})/k(z) \)

so \( \dim_k(A/z) = n \)

Finally, \( (z)_0 = (z^{-1})_0 = [K:k(z)] = [K:k(z)] = n \).

**Defn.**
**The two divisors** \( D + E \) **are linearly equivalent** \( D \sim E, \ M : D - E = (z) \) **for some** \( z \).

**Defn.** The divisors linearly equivalent to zero are those of the form \( (z) \), are called the **principal divisors**. They form a subring of the ring of all divisors.

**Defn.** Two divisors \( D + E \) are algebraically equivalent if \( \deg D = \deg E \).
6. Locally Free Sheaves

Let $L$ be a sheaf of $O_V$-modules and $L$ be locally free of rank $r$. Then $\Gamma(V, L_{|U_i}) \cong O_V^{r|U_i}$ for an open covering $(U_i)$ of $V$. Let $\mathcal{L}$ be the stalks of $L$ at $p \in V$, then it is easy to check that

1. $L$ is a vector space of dimension $r$ over $K = \mathbb{R}$.
2. $L_p$ is an $O_p$-module of $L_{|U_i}$.
4. $L_p$ is free over $O_p$.
5. Any $K$-basis of $L$ is an $O_p$-basis for $L_p$, almost all $p$.

Conversely, any such system $(L, L_p)$ defines a locally free sheaf of rank $r$ over $O_V$ by

$$L(U) = \bigcup_{p \in U} L_p$$

Note 1: $(\alpha) \Rightarrow L_p$ is free of rank $r$ over $O_p$ (since $R_p$ is principal).

Note 2: It suffices to check $(b)$ for any basis $\{w_i\}$. For let $(w_i)$ be another; then

$$w_i = \sum a_{ij} w_j \quad a_{ij} \in K, \quad t \in R_p \text{ for almost } p$$

is a basis for $L_p$ at all those points $p$ such that $a_{ij}, b_{ij}$ are all in $R_p$.

Any homomorphism $\varphi: L \to M$ of $L.F.S.(r)$ is $k,M$ defines

(1) a $K$-hom $\varphi: L \to M$

(2) $\varphi(L_p) \subseteq M_p$

Conversely, such a $K$-hom $\varphi: L \to M$ defines
a homomorphism \( \varphi \) of sheaves by

\[
\varphi(U) = \varphi|_{\mathcal{L}_p} : \mathcal{L}_p = \varphi(U) \to \bigwedge^p M_p
\]

Operations on L.F.S is

1. **Direct Sum**

\[
\mathcal{L} \oplus \mathcal{L}' = (\mathcal{L} \oplus \mathcal{L}', (\mathcal{L}_p \oplus \mathcal{L}'_p))
\]

+ the ranks add

2. **Tensor Product**

\[
\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' = (\mathcal{L} \otimes \mathcal{L}', (\mathcal{L}_p \otimes_{\mathcal{O}_{\mathcal{L}_p}} \mathcal{L}'_p))
\]

+ again the ranks add

3. **Hom**

\[
\text{Hom}(\mathcal{L}, \mathcal{L}')_p = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}')
\]

\[
\text{Hom}(\mathcal{L}, \mathcal{L}')_p = \text{Hom}_{\mathcal{L}_p}(\mathcal{L}_p, \mathcal{L}'_p)
\]

(cf. F.A.C. 14.4thm5)

4. **Tensor Products**

\[
\bigwedge^k \mathcal{L} = (\bigwedge^k \mathcal{L}, \bigwedge^k \mathcal{L}_p)
\]

If \( \omega_1, \ldots, \omega_p \) are bases for \( \mathcal{L}_p \), then

\[
\bigwedge^k \mathcal{L}_p = \sum_{i_1 < \cdots < i_k} \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}
\]

\[
\bigwedge^k \mathcal{L} = \bigwedge^k \mathcal{L}_p
\]

is locally free of rank \((k)\)

**Cohomology**: Let \( \mathcal{L} \) be a L.F.S of rank \( r \); consider

\[
0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_V \to K \to 0
\]

where \( K \) is the constant sheaf, \( \Gamma(U, K) = K \)

Let \( (U_i) \) be a covering of \( V \) by open affines \( U_i \), \( \cong \mathcal{O}_V \). Then on each \( U_i \), the above sequence is isomorphic to

\[
0 \to \mathcal{O}_i |_{U_i} \to K |_{U_i} \to K^r |_{U_i} \to 0
\]

\( K^r |_{U_i} \) is flabby, in fact \( K^r \) is flabby because any open subset of \( V \) is connected.

Thus,

\[
0 \to H^0(V, L) \to H^0(V, \mathcal{L} \otimes K) \to \bigoplus \Gamma(U) \to H^1(V, L) \to 0
\]

\( H^r(V, L) = 0, r \geq 2 \) — prove either by showing \( \mathcal{L} \) to be flabby, or apply F.A.C. 66 thm.1, or Exercise II 4.15.
But \( \mathcal{E} \) is a skyscraper sheaf, i.e., if \( s \in \mathcal{E}(U) \), \( U \) a nbhd of \( P \), then \( \exists \) a nbhd \( U' \subseteq U \) of \( P \) s.t. \( s = 0 \) on \( U' - P \). (Take \( U' = \{ q \in U | \exists \delta > 0 \text{ s.t. } q \in U \} \)) It follows that \( \mathcal{E}(V) = \bigoplus L/k \).

\[
\begin{align*}
H^0(V, L) &= \bigoplus L/k \\
H^1(V, L) &= \text{other} (L \rightarrow \bigoplus L/k) \\
H^p(V, L) &= 0 \quad p > 1.
\end{align*}
\]

2. **Divisors**

We have already defined Weil divisors and principal Weil divisors.

**Def.**

1. An invertible sheaf is a locally free sheaf of rank 1.

2. A Cartier divisor is an elt \( g \in \Gamma(V, K^*/\mathcal{O}_V^*) \); \( g \) is principal if it may be lifted to an elt of \( \Gamma(V, K^*) \).

3. A line bundle is a \( k \)-scheme \( E \) + a surjective morphism \( E \xrightarrow{\pi} V \) with the following property: \( E \) is an open cover \((U_i)\) of \( V \).

For each \( i \), \( E \) is an isomorphism \((\pi^{-1}(U_i), E \mid \pi^{-1}(U_i))\):

\[
\begin{array}{c}
\phi_i \uparrow \\
\downarrow \pi_{1U_i} \\
(U_i, \mathcal{O}_{V1U_i})
\end{array}
\]

**Theorem.**

a) There is a correspondence between Weil divisors + Cartier divisors under which principal divisors correspond to principal divisors.

b) The following groups are isomorphic:

1. Weil divisors / principal Weil divisors

2. Cartier -- / Cartier

3. Isomorphism classes of invertible sheaves (isomorphism being tensor product)

4. Isomorphism classes of line bundles

5. \( H^1(V, \mathcal{O}_V^*) \)

**Pf.** a) Consider the Weil divisor \( \sum P \); for each \( P \in V \) choose \( \mathcal{O}_P \in \mathcal{O}_V, p \in \mathcal{O}_P \). Let \( \mathcal{O}_P(\mathcal{O}_P) = \mathcal{O}_P \), \( \times \) then
choose an open nhbd \( U_0 \) of \( V \) s.t. \( \forall t_0 \in V \), \( \forall U, (t_0, U) \preceq V \) defines an elt of \( \Gamma(V, k^*/U) \).

To get the reverse mapping, use that \( k^*/U \) is a sheaves sheaf.

(b)\( \ast \) already proven

2\( \Rightarrow \) 5: \( \begin{array}{c}
0 \rightarrow O_V \rightarrow k^* \rightarrow k^*/O_V \rightarrow 0
\end{array} \)

is exact, so \( k^* \) is flabby (unreducible).

\( \begin{array}{c}
0 \rightarrow \Gamma(V, O^*_V) \rightarrow \Gamma(V, k^*) \rightarrow \Gamma(V, k^*/O^*_V)
\end{array} \)

is exact.

3\( \Rightarrow \) 5: (b) not yet that \( H^1(V, O^*_V) = \tilde{H}^1(V, O^*_V) \)

let \( F \) be an invertible sheaf s.t. \( U_1 \subseteq V \) a covering of \( F \) on \( U_1 \), to free, so we have

\[ F_{U_1} \cong O_{U_1} \]

Then we have

\[ q_{i,j} : q_{i,j} : O_{U_i \cap U_j} \rightarrow F_{U_i \cap U_j} \]

which must just be multiplication by an invertible elt, \( f_{i,j} \) say, \( f_{i,j} \in O_{U_i \cap U_j} \). \( (f_{i,j}) \)

now defines an elt of \( H^1(V, O^*_V) \)

The converse is clear

5\( \Rightarrow \) 3 same as 2\( \Rightarrow \) 5.

### Definition

The growth in (b) of the above thin is the

\[ \text{Picard group} \text{ of } \text{Pic}(V) \text{ of } V. \]

We make more explicit the connection between
invertible sheaves and Weil divisors: if \( L \)

is an invertible sheaf we may take \( L = K \)

as a Weil divisor \( V \) is a basis for \( L \), \( V \) is a univalent \( K \) all \( P \), \( P \) for almost

all \( P \), \( L \) is a basis for \( L \), \( K \) all \( P \)

Thus \( L = \sum v_P P \) with \( v_P = 0 \) almost

all \( P \), \( L \) defines a Weil divisor \( D = \sum -v_P P \). Conversely, a Weil divisor \( D = \sum -v_P P \) defines

an invertible sheaf \( L = [D] \)

by

\[ [D]_P = \{ f \in K | v_P(f) \leq -v_P \} = \mathcal{O}^{-v_P}_P \]
\[ [\mathcal{O}] = K \]
\[ \{ \mathfrak{d} \in \mathcal{D}(U) \mid \mathfrak{d} \in K, V_p(f) \geq -V_p(f) \text{ all } p \in U \} \]

\[ [\mathcal{O}] \cong [\mathcal{E}] \iff \exists \text{ an iso } f : [\mathcal{O}] \to [\mathcal{E}] \]

\[ [\mathcal{O}] \otimes K \xrightarrow{\cong} [\mathcal{E}] \otimes K \]

\[ \text{must be of form, } g \mapsto gf, f \equiv 0 \in K \]

\[ [\mathcal{O}] \cong [\mathcal{E}] \iff \exists f \neq 0 \in K, \mathfrak{d} \longmapsto [\mathcal{E}] = [\mathcal{O}]f \]

\[ \mathfrak{d} - V_p(E) = -V_p(D) + V_p(f) \text{ all } p \]

\[ E = D - (f) \]

\[ E \equiv D \text{ are linearly equivalent} \]

\[ \text{Remark: Koch (First Form)} \]

We shall first consider the projective line \[ \mathbb{P}^1 \text{ we take } K = k(x) \]

\[ \text{If } K = k(x) \text{ then the Jacobian group is } 0 \text{ (i.e. linear equivalence = algebraic equivalence)} \]

\[ \text{If } P \in \mathbb{P}^1 \text{ corresponds to a non-zero \[ \mathbb{P}^1 \text{ polynomial } f_0(x) \text{ in } k(x) \text{ and } + deg f_0 \]

\[ \text{Let } D \text{ be any Weil divisor, choose } f = \prod f_0(x)^{v_p(P)}, \text{ choose } \]

\[ D - (f) = D - \sum_{P \neq P_0} V_p(D)P + \sum_{P \neq P_0} V_p deg P, P_0 \]

\[ = nP_0, n = deg D. \]

\[ \text{Cohomology of } [nP_0] = \mathbb{L} : \]

\[ H^0(V, \mathbb{L}) = \{ f \in k(x) \mid f \in k[x], v_p(f) = -deg f \geq -n \} \]

\[ = \{ f \in k[x] \mid deg(f) \leq +n \} \]

\[ \dim_k(H^0(V, [nP_0])) = n+1 \quad \forall \ n \geq 0 \]

\[ = 0 \quad \forall \ n < 0 \]

\[ \text{N.B. Karhunen, } D \geq 0 \text{ (ne effective) then} \]
\[ H^0(X, [D]) = \{ f \in K \mid (f)_\infty \leq 0 \} \]
\[ \text{for } (f) \geq 0 \iff (f)_\infty - (f)_0 \leq 0 \iff (f)_\infty \leq 0 \]
(\(\leq\) the places of \(f\) have at most order \(v_D(B)\)).

**Consider**
\[ k(X) \to \bigoplus_{P \neq P_0} K/(P) \to H^1(V, [D]) \to 0 \]
Any \( f \in k(X) \) can be written uniquely
\[ f(X) = \sum_{P \neq P_0} \frac{g_p(X)}{f_0(X), j(P)} + g(X) \]
where \( g_p, g \) are polynomials
\[ g_p = 0 \iff j(P) = 0 \]
\[ f_0 \neq 0 \iff g_p + \deg g_p \leq \deg f_0 \]
(see Lang Alg p125)

*The image of \( f \) in \( K/(P) \) \((P \neq P_0)\) is represented by \( \frac{g_p(X)}{f_0(X), j(P)} \).*

But any elt of \( K/(P) \) \((P \neq P_0)\) is represented by such a rational function.\(\)
\[ K \to \bigoplus_{P \neq P_0} K/(P) \text{ is surjective} \]
\[ L_0 = X^n P_0 = \{ f/g \mid \deg g + n \geq \deg f \} \]
If \( n < 0 \), then the image of \( f \) in \( K/(P) \) is \( g(t) \).
In this case \( K \to \bigoplus_{P \neq P_0} K/(P) \) is still surjective, but if
\[ n < 0, \text{ then } (\bigoplus_{P \neq P_0} K/(P)) / \text{im}(K) \text{ is free on the generators } x^n, \ldots, x^{n-1} \]
Hence \( \dim_k H^1(V, [nP_0]) = 0 \iff n \geq 0 \]
\[ = \dim_k H^1(V, [nP_0]) = 0 \iff n < 0 \]

**Define** (Arithmetically \( V \)). \( X((V, B)) = \dim_k H^0(V, B) - \dim_k H^1(V, B) \)
(\(\geq 0 \) provided the \( n \) \( B \) makes sense, i.e. \( h^0, h^1 \) are finite)
\[ X(V, B) = X(V, [B]) \]

Thus, if \( V \) is the projective line \((\cong K = k(X))\), then for any divisor \( D \),
\[ X(D) = \deg D + 1 \]
\(\because D = n P_0, n = \deg D, \text{ so } [D] = [nP_0] \)
\[
X(nP_0) = n + 1 + 0 \quad n \geq 0 \\
= 0 - (-n-1) \quad n < 0
\]

**Theorem (Riemann-Roch, 1st Form):** For all locally free sheaves \( F \) of rank \( r \) over \( V \), \( H^0(V, F) \to H^r(V, F) \) are flat, so

\[
X(V, F) = \text{deg}(\mathcal{L}^r F) + r(1-g)
\]

where \( g = \dim_k(H^1(V, \mathcal{O}_V)) = \text{genus of } K/k, r > 0 \).

In particular,

\[
X(V, \mathcal{O}_V) = \text{deg } D + 1 - g
\]

**Note:** \( \mathcal{L}^r F \) is an invertible sheaf, \( r \) so corresponds to a divisor \( D_r \), \( r \) we define

\[
\text{deg}(\mathcal{L}^r F) = \text{deg } D
\]

**Lemma:** Suppose we are given two L.F.S \( L \to M \) with

\[
L = M, \quad L_p = M_p \quad \text{all } P \quad \text{(or } L_p = M_p \text{ almost all } P)
\]

If the cohomology for \( L \) or for \( M \) is 0, then it is likewise for the other, i.e.,

\[
\chi(M) = \chi(L) = \dim \left( \bigoplus M_p/L_p \right)
\]

**Proof:** Applying the snake lemma to

\[
\begin{array}{cccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \\
0 & \oplus M_p/L_p & \downarrow & \downarrow & \downarrow & \\
0 & H^0(M) & \rightarrow & M & \rightarrow & \oplus M_p/M_p & \rightarrow & H^1(M) & \rightarrow & 0 \\
0 & H^0(L) & \rightarrow & L & \rightarrow & \oplus L_p/L_p & \rightarrow & H^1(L) & \rightarrow & 0 \\
0 & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Yields an exact sequence

\[
0 \to H^0(L) \to H^0(M) \to \oplus M_p/L_p \to H^1(L) \to H^1(M) \to 0
\]

\( M_p + L_p \) are free \( R_p \)-modules with the same rank, hence we may choose a basis

\( u_1, \ldots, u_r \) of \( M_p \). If \( L_p \) has basis \( t_1, \ldots, t_r \) for \( R_p \)

Then \( M_p/L_p \cong R_p/m_p^{a_r(p)} \oplus \cdots \oplus R_p/m_p^{a_1(p)} \) which is finite-dimensional, so \( \oplus M_p/L_p \) is fed.
The rest follows from
\[ h^0(L) - h^0(M) \leq \dim_{k}(\oplus M_{p}/L_{p}) \]
\[ h^1(L) - h^1(M) \leq \dim_{k}(\oplus M_{p}/L_{p}) \]
\[ + h^0(L) - h^0(M) + \dim_{k}(\oplus M_{p}/L_{p}) - h^1(L) - h^1(M) = 0. \]

Remark: In particular, if \( D \neq E \) are two divisors
\( D \leq E \quad (= [D]_{p} \leq [E]_{p}) \) then
\[ \chi(E) - \chi(D) = \dim_{k}(\oplus [E]_{p}/[D]_{p}) \]
\[ = \dim_{k}(\oplus (M_{p}^{-v_{p}(E)})) \]
\[ = \sum \deg P (v_{p}(E) - v_{p}(D)) \]
\[ = \deg E - \deg D. \]

**Lemma 2:** \( L \subseteq M \) (i.e., \( L, M \) as in Lemma 1); then the
R.K. holds for \( L \iff \) it holds for \( M \)

By the finiteness statements have already
been shown equivalent.

Suffices to show \( \dim_{k}(\oplus M_{p}/L_{p}) = \deg(\Lambda^{r}M) - \deg(\Lambda^{r}L) \)

1. \( \text{H.S.} = \sum_{p} (\deg P) (a_{1}(P) + \ldots + a_{r}(P)) \)
   (in the notation of Lemma 1)

By the above remark
\[ \text{H.S.} = \dim_{k}(\oplus \left( \Lambda^{r}M\right)_{p}) \]
\[ = \dim_{k}(\oplus \left( \Lambda^{r}L\right)_{p}) \]
\[ = \sum_{p} (\deg P) (\sum a_{i}(P)) \]
\[ = \text{H.S.} \]

**Lemma 3:** If the R.K. holds for some \( L \) of rank \( r \), then
it holds for all L.F.S of rank \( r \).

Prove: We may assume all vector spaces \( L \) are the
same. If \( L \neq L' \) are both L.F.S of rank \( r \),
then \( L \wedge L' \) (as subspaces of the constant
sheaf of smooth \( L \)) is again L.F.S of
rank \( r \).
Lemma 4: If R.K. holds for \( L + L' \) then it holds for \( L'' = L \oplus L' \).

Proof: Since cohomology commutes with direct sums, \( H^r(L'') = H^r(L) + H^r(L') \)

Also \( r'' = r + r' \).

It remains to show that
\[
\deg (\Lambda^r L'') = \deg (\Lambda^r L) + \deg (\Lambda^r L')
\]

But,
\[
\Lambda^r (L'') \cong (\Lambda^r L) \oplus (\Lambda^r L')
\]

And
\[
\deg ([D] \oplus [E]) = \deg ([D+E]) = \deg D + \deg E
\]

Lemma 5: The R.K. then is true for \( L = \mathcal{O}_V (-507) \).

If \( \Lambda R^p = 1 \), then \( h^0 = 1 \).

\[
\deg (\mathcal{O}_V) = 0
\]

hence get,

\[
1 - h'(\mathcal{O}_V) = 0 + 1 - g
\]

and by defn \( h'(\mathcal{O}_V) = g \).

It remains to show that \( h'(\mathcal{O}_V) \) is finite.

Sublemma: since \( K_0 + K \) are function fields over \( k \),

\( + K \) is a finite degree over \( K_0 \). Let \( (V, \mathcal{O}_V) \) and \( (V_0, \mathcal{O}_{V_0}) \) be the corresponding schemes. Then \( \exists \) a canonical map
\[
V_0 \to V
\]

For any \( L.F.S \) of rank \( r \)

on \( V \) is a \( L.F.S \) of rank \( r \) \( \mathcal{O}_K; K_0 \) on \( K_o \).

Moreover,
\[
H^m(V_0, \mathcal{O}_V) \cong H^m(V, \mathcal{L})
\]

(over \( k \)-sheaves) all \( m \).

Proof: any ultra mug \( R^p \) of \( K/k \) dominates a unique valuation ring \( R_0 \) of \( K_0/k \).

is the map on the underlying \( k \)-sheaves, on the sheaves we define
\[
\mathcal{O}_V(U) = \bigwedge_{p \in U} R_0 \longrightarrow \mathcal{O}_V(U') = \bigwedge_{p \in U'} R^p
\]
to be the inclusion map (where \( U' \) is any open subset of \( V \) and \( \pi(U') \subset U, u = U \supset U' \)).

(\( R_0 \supset R_0 \), some \( q \in U' \))

(Note that the rank is \( 1 \); in fact, it is directly closed, then \( [K : K_0] = 1 \) except at ramified \( S \).)

Thus, \( (\pi + L)(U) = \mathbb{L}(\pi^{-1}(U)) = \bigwedge_{\rho \in \pi^{-1}(U)} L_{\rho}, \)

is clearly a L.F.S. of rank \( r \) \( [K : K_0] \) on \( V_0 \).

\( H^0(X_0, \pi_+ L) = H^0(X, L) \) by defn.

One checks easily that

\[
(\pi + L)q = \bigwedge_{\pi(p) \neq q} L_p
\]

Consider,

\[
0 \rightarrow H^0(X_0, \pi_+ L) \rightarrow L \rightarrow \bigoplus \frac{L}{L_{\rho}} \rightarrow H'(V_0, \pi_+ L) \rightarrow 0
\]

\[
0 \rightarrow H^0(X, L) \rightarrow L \rightarrow \bigoplus \frac{L}{L_{\rho}} \rightarrow H'(V, L) \rightarrow 0
\]

is exact (Choose a basis of \( L \) which is contained in all \( R_0 \) with \( \pi(p) = q \), and apply the approximation there)

Thus \( q \) is an isomorphism \( \pi_0 \)

\[ H'(V_0, \pi_+ L) \cong H'(V, L) \]

This, in prove the lemma, it suffices to show that \( H'(V_0, \pi_+ O_V) \) is \( 0 \) where \( V_0 \) corresponds to \( K_0 = k(X) \subset K \); \( \pi_+(O_V) \)

is locally free of rank \( [K : k(Z)] \)

By a previous consideration, \( H'(O_V) = 0 \), so the genus of \( K_0 = 0 \), + the theorem on p17 says that the R.R. theorem holds for all L.F.S. of rank \( 1 \) (on \( V_0 \)). Now lemmas 3 and 4 easily that the R.R. holds for all L.F.S. on \( V_0 \), in particular \( H'(\pi_+ O_V) \) is finite.

The proof of the Riemann Roch theorem is now complete.
[Paragraphs from the document are translated into natural text]

**Remark:** Let \( K = k(z), A_1 = k[U_1], A_2 = k[U_2] \)
\[ U_1 = \text{the } A_1, \quad V = U_1 \cup U_2 \]
\[ U_2 = \text{the } A_2, \quad V = U_1 \cup U_2 \]
\[ \Gamma(U_1 \cup U_2, O_V) = k[2, z^{-1}] \]

Let \( L \) be locally free of rank \( r \) on \( V \). Then, since \( A_1 \) is smooth, \( \Gamma(U_1, L) = L(U_1) \) is a free \( A_1 \)-module of rank \( r \).
\[ L(U_1) = \sum_i k[z] \omega_i = M_1 \]
\[ L(U_2) = \sum_i k[z] \omega_i = M_2 \]

Clearly, it gives a locally free sheaf of rank \( r \) on \( V \) to give

(a) a free \( A_1 \)-module \( M_1 \) of rank \( r \)

(b) an isomorphism of \( k[2, z^{-1}] \)-modules
\[ k[z, z^{-1}] \otimes_{k[z]} M_1 \cong k[2, z^{-1}] \otimes_{k[z]} M_2 \]

It is to give an invertible \( r \times r \) matrix \( (a_{ij}) = a \) with entries in \( k[z, z^{-1}] \) (take \( \omega_i = \sum_j a_{ij} \omega_j \))

The sheaves corresponding to \( a, a' \in GL(r, k[2, z^{-1}]) \)
are isomorphic if \( a' = bac \) where
\[ b \in GL(r, k[z^{-1}]) \quad c \in GL(r, k[z]) \]

**Proof:** If \( V \) is the projective line over \( k \), then every locally free sheaf on \( V \) is a direct sum of invertible sheaves.

**Remark:** By the above remarks, need only prove

**Lemma:** For any \( a \in GL(r, k[z, z^{-1}]), \exists b \in GL(r, k[z]) \)
\[ + c \in GL(r, k[z^{-1}]) \text{ s.t. } cab \text{ is diagonal.} \]

**Proof:** Let \( a_{ij} \in k[z, z^{-1}], \) we define \( p(z) \in k[z, z^{-1}] \)
to be the gcd of \(a_1, \ldots, a_n\) (in the sense of \(k[z]\)) if \(p k[z] = (a_1, \ldots, a_n)\) (i.e., the ideal of \(k[z]\) generated by \(a_1, \ldots, a_n\)).

Now let \(p\) be the gcd (in the sense of \(k[z]\)) of the first row of \(A\); then \(p + a_1, \ldots, a_n\) also generate the same ideal in \(k[z, z^{-1}]\). But det \((a) \in p k[z, z^{-1}]\) since each term of the expansion of the determinant contains a factor from the first row. Hence \(p/2\) must be invertible in \(k[z, z^{-1}]\).

\(k[z, z^{-1}]\) is the ring of "Laurent polynomials" \(\sum_{m=0}^{\infty} c_m z^m + \cdots + c_n z^{-n}\). The invertible elements in this ring are clearly the monomials. Hence the gcd of each row \(m\) of the form \(2^m\).

Let \(2^m\) be the gcd of the \(i\)th row \(a_i\). The polynomials of the \(i\)th row are of the form \((2^m + \text{higher terms})\) with \(2^m\) appearing at least once.

Similarly, compute the gcd of each \(c_i\) in the sense of \(k[z^{-1}]\), say \((z^{-1})^{\tau_i}\) for the \(i\)th col, so the els of the \(i\)th col are of the form \(z^{-\tau_i} + \text{lower terms}\) \((m_2)\), \(z^{-\tau_i}\) appears at least once.

Now consider the effect of multiplying on the right by \(b \in GL(V, k[z])\), \(a \rightarrow ab\).

\(2^m\) is a new gcd for the \(i\)th row, so gecs can only go up, but \(b\) is invertible \(\Rightarrow\) so they must stay the same.

Only \(a \rightarrow ca\) preserves the gcd of the columns. (It may change the row geds, but \(\text{gcd} \leq \text{max} (-\tau_i)\))

Step 1: replace a by \(ca\), where \(c\) is chosen so that the gcd of the first row is maximal among all \(ca\) (these are operations in the row).

Step 2: by an \(a \rightarrow ab\) can reduce \(a\) to the form

\[
\begin{pmatrix}
2^m & 0 & \cdots & 0 \\
\star & \star & \cdots & \star \\
\vdots & \vdots & \ddots & \vdots \\
\star & \star & \cdots & \star
\end{pmatrix}
\]
Step 3: now use $b, c, d$ of the form
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & * & \cdots & * \\
\vdots & \vdots & & \vdots \\
0 & * & \cdots & *
\end{pmatrix}
\] to reduce to the form
\[
\begin{pmatrix}
2^a & & & \\
& 2^b & & \\
& & \ddots & \\
& & & 2^c
\end{pmatrix}
\]
with $\rho_1 \geq \rho_2 \geq \rho_3 \cdots$

Step 4: by multiplication on the left, get elt of first col in form $+2^{\rho_1} + \text{higher}$

Step 5: by multiplication on right, get elt of first col (not in the first row) to be zero.

Thus, the diagonal form is unique if $\rho_1 \geq \cdots \geq \rho_r$.

**Exercise:** the diagonal form is unique if $\rho_1 \geq \cdots \geq \rho_r$.

Note that if $\Lambda$ is the sheaf corresponding to the above matrix, then

$$\text{deg}(\Lambda \otimes L) = \Sigma \rho_i$$

Indeed, by additivity, one need only check this for invertible sheaves, where it is clear.

(O or note that $\Lambda \otimes L$ corresponds to the $1 \times 1$ matrix $(2^{\rho_i})$)

**Example:** let $K = k[y, z]$ where char $k \neq 2$, $r$

$$y = c_{2n+2} z^{2n+1} + \cdots + c_0$$

with not both $c_{2n+2} = 0$ and $c_{2n+1} = 0$, $r$ with the $r$th square free.

Then $[K : k(2)] = 2$, $r$ the integral closure of $k[z]$ in $K$ is $k[z, y] = k[z, 1 + k[z] y]$ (see p. 9 ex. 1).

Inly, the integral closure of $k[z^{-1}]$ in $K$ is $k[z^{-1}, y/z^{n+1}] = k[z^{-1}, 1 + k[z^{-1}] y/z^{n+1}]
$ (for $(y/z^{n+1})^2 = (c_{2n+2} + \cdots + c_0 z^{-2n-2}$ so in the same form as the original equation).

We compute $\Pi x(0v)$ where $V \longrightarrow \mathbb{P}_v \ (\text{corresponding to} \ z)$

(cf. p. 20, sublemna). It clearly corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & z^{n+1} \end{pmatrix}$$

hence
\[ \dim_1 H'(V, O_V) = n + 1 = g \cdot (K/k) \]

(Note that \( \dim_1 H^0(O_V) = 1 + 0 \), for \( \mu_n O_V = (1) \otimes k \).

9. Duality + Differentials

\( K/k \) is a function field in 1 var, \( V \) the corresponding scheme, \( V \) the closure of \( V \).

If \( T \) is a vector space over \( k \) then \( T^* = \text{dual space}, \ Hom_k(T, k) \)

We wish to define an invertible sheaf \( \Delta \) on \( V \):

\[ \text{if } L' = \text{Hom}_k(L, \Delta), \text{ then } \]

\[ H^i(L')^* \cong H^{i+2}(L') \]

Want \[ \Delta = \Delta(V) = \lim_{\to -\infty} H'(\mathcal{D})^* \]

where the limits in the limit are as follows:

\[ D' \leq D \Rightarrow \mathcal{D}' \subset \mathcal{D} \]

\[ \Rightarrow H'(D') \rightarrow H'(D) \]

\[ \Rightarrow H'(D')^* \leftarrow H'(D)^* \]

From \( K \rightarrow \bigoplus_p K/\mathcal{D}_p \rightarrow H'(D) \rightarrow 0 \), we can make the following concrete def. for \( f \in \bigoplus \Delta \)

Def. a \textit{pseudo-diff} \( f \) is a family \( (\mathcal{D}_p)_{p \in \mathcal{V}} \) s.t.
1) \( \mathcal{D}_p : K \rightarrow k \) is \( k \)-linear
2) In degree \( D, \text{ s.t. } \mathcal{D}_p(\mathcal{D}_3) = 0, \forall P \in \mathcal{V} \)
3) \( \sum \mathcal{D}_p(f) = 0, \forall f \in K \)

(This is only a finite sum, for \( f \in \mathcal{D}_p \)
almost all \( p \))

The pseudo-diffs form a vector space \( \Delta \) over \( K \).

We define \( (f \mathcal{D}_p)(g) = \mathcal{D}_p(fg) \) for \( f, g \in K \)

Set \( \Delta^0 = \{ f \in \Delta | (2) \text{ holds for that } D \} = H'(D)^* \)

\[ \Delta(U) = \{ f \in \Delta | \mathcal{D}(\mathcal{D}_p) = 0 \text{ all } \forall P \in U \} \]
Equivalently, define $\Delta$ to be the sheaf given by $(\Delta, \Delta_p)$ where
$$\Delta_p = \{ s \in \Delta \mid s(R_p) = 0 \}$$

We will now show that $\Delta$ is an invertible sheaf. Clearly it will suffice to show that $\dim \Delta = 1$.

If $\deg D < -1$, then $0 < -g + h'(D)$, so $h'(D) > 0$, $\Delta^p \neq 0$. Hence $\Delta^p \neq 0 \iff \dim_x(\Delta) \geq 1$. 