

11/30/2011

Dear Vasile,

Let  $\mathcal{C}$  be a tannabrian category over an algebraically closed field  $k$ . Your question amounts to: does  $\mathcal{C}$  always have a fiber functor over  $k$ . I believe it is true. The following argument is presumably too pedestrian, hiding which "compactness arguments" are really relevant.

Let  $A$  be the set of strictly full subcategories of  $\mathcal{C}$ , stable by  $\otimes$ , subquotients and duals, ~~also~~ let  $A_f \subset A$  be the set of those generated by finitely many objects, using the same operations. For  $\alpha \in A$ , I note  $\mathcal{C}_\alpha$  the corresponding subcategory (as defined:  $\mathcal{C}_\alpha = \alpha$ ). The set  $A$  is ordered by inclusion. I assume we already know the existence and unicity up to isomorphism of fiber functors for the  $\mathcal{C}_\alpha, \alpha \in A$ .

If  $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$ , it makes sense to say that a fiber functor  $\omega_\beta$  on  $\mathcal{C}_\beta$  extends (on the nose) a fiber functor  $\omega_\alpha$  on  $\mathcal{C}_\alpha$ . If an extension up to isomorphism exists, an actual extension exists too.

Let us order the set of  $(\alpha, \omega_\alpha) : \alpha \in A, \omega_\alpha$  fiber functor on  $\mathcal{C}_\alpha$ , by " $\omega_\beta$  extends  $\omega_\alpha$ ". To avoid set theoretical difficulties, one could consider only the fiber functors  $\omega_\alpha$  such that the  $\omega_\alpha(x)$  take value in the set of vector spaces  $k^n$ ,  $n \in \mathbb{N}$ .

The ordered set of  $(\alpha, \omega_\alpha)$  is inductive (Bourbaki  
 Ens Ch 3 §2, 4): If  $I$  is a totally ordered subset,  
 the "union" of the  $(\alpha, \omega_\alpha)$  is a  $\mathcal{C}_0$  in  $A$  ( $= \bigcup_{(\alpha, \omega_\alpha) \in I} \mathcal{C}_\alpha$ )  
 with the fiber functor  $\omega_0$  characterized by  
 $\omega_0|_{\mathcal{C}_\alpha} = \omega_\alpha$ . This "union" majorizes  $I$ . By Bourb  
 loc cit Th 2, the ordered set of the  $(\alpha, \omega_\alpha)$  has a  
 maximal element  $(\mathcal{C}_i, \omega_i)$ . To prove that  $\mathcal{C}_i = \mathcal{C}$ ,  
 it suffices to prove the following

Lemma 1 Let  $\mathcal{C}'$  be in  $A$  and  $\mathcal{C}''$  be in  $A_f$ . Let  
 $\langle \mathcal{C}', \mathcal{C}'' \rangle$  be generated by  $\mathcal{C}'$  and  $\mathcal{C}''$ : it is in  $A$ .

Then, any fiber functor  $\omega'$  on  $\mathcal{C}'$  can be extended (or the  
 next, or up to isomorphism, this amounts to the same)  
 to  $\langle \mathcal{C}', \mathcal{C}'' \rangle$

I first prove

Lemma 2 Suppose  $\mathcal{C}'$  is in  $\mathbb{Z}A_f$  too. "Restriction" is ~~an~~ then a  
 equivalence of categories

(fiber functors on  $\langle \mathcal{C}', \mathcal{C}'' \rangle$ )  $\longrightarrow$  (~~the~~ triples of  
 a fiber functor  $\omega'$  on  $\mathcal{C}'$ ,  $\omega''$  on  $\mathcal{C}''$  and an isomorphism  $\tau$   
 of the restrictions of  $\omega'$  and  $\omega''$  to  $\mathcal{C}' \cap \mathcal{C}''$  ( $\mathcal{C}' \cap \mathcal{C}''$  is  
 also in  $A_f$ ))

We may assume that  $\langle \mathcal{C}', \mathcal{C}'' \rangle$  is the category of  
 representations of a group  $G$ , and that for invariant subgroups  
 $A$  and  $B$ ,  $\mathcal{C}'$  (resp  $\mathcal{C}''$ ) is the subcategory of representations  
 where  $A$  (resp  $B$ ) acts trivially. That they generate  $\langle \mathcal{C}', \mathcal{C}'' \rangle$   
 means that  $A \cap B = \{e\}$ . The intersection  $\mathcal{C}' \cap \mathcal{C}''$  is the category

of representations on which  $AB$  acts trivially

The triples  $(\omega', \omega'', \tau)$  are all isomorphic: as all  $\omega'$  (resp all  $\omega''$ ) are isomorphic, it suffices to see that  $(\omega', \omega'', \tau_1)$  and  $(\omega', \omega'', \tau_2)$  are isomorphic. Indeed  $\tau_1$  and  $\tau_2$  differ by an automorphism of  $\omega' | \tau_1 \cap \tau_2$ , and such an automorphism lifts to an automorphism of  $\omega'$ :

$$G/A(b) \rightarrow G/AB(b) \text{ is onto.}$$

We hence have here categories with just one isomorphism class of objects, and the question is to compare automorphism groups. We need to check

$$G \xrightarrow{\sim} \left\{ (g', g'') \in G/A, G/B \mid g' \text{ and } g'' \text{ have same image in } G/AB \right\}$$

$$\begin{array}{ccc} G & \longrightarrow & G/A \\ \downarrow & \square & \downarrow \\ G/B & \longrightarrow & G/AB \end{array}$$

Injectivity of this morphism of groups amounts to  $A \cap B = \{e\}$ .

Surjectivity: if  $g' \in G/A, g'' \in G/B$ ,  $\tau_1, \tau_2$  lift  $g', g''$ ,  $\tilde{g}'' = \tilde{g}' \cdot a \in B$  and  $\tilde{g}'' \cdot B = \tilde{g}' \cdot a \in G$  maps to  $(g', g'')$ .

Lemma 3 Same as lemma 2, but  $\tau'$  only assumed to be in  $A$

Let  $B$  be the set of  $\tau_a$  ( $a \in A_f$ ) contained in  $\tau'$ . One has  $\# \langle \tau', \tau'' \rangle = \bigcup_{B \in B} \langle \tau_B, \tau'' \rangle$ . One has equivalences

(fiber functors on  $\tau$ )  $\xrightarrow{\sim}$  (fiber functors<sup>exp</sup> on the  $\tau_p$ 's, plus a compatible system of isomorphisms  $\omega_p | \tau_\gamma \xrightarrow{\sim} \omega_\gamma$  for  $\tau_\gamma \subset \tau_p$ ; compatible: condition for  $\tau_\delta \subset \tau_\gamma \subset \tau_p$ )

Same for fiber functors on the  $\langle \mathcal{C}', \mathcal{C}'' \rangle = \bigcup \langle \mathcal{C}_\beta, \mathcal{C}'' \rangle$ .

The  $\mathcal{C}_\beta \cap \mathcal{C}''$  have a ~~smallest~~<sup>largest</sup> element: they if  $\mathcal{C}'' = \text{Rep}(G'')$ , they correspond to invariant subgroups of  $G''$ , subgroups are closed subschemes, and one uses the noetherian property. If  $\beta_0$  is such that  $\mathcal{C}_{\beta_0} \cap \mathcal{C}''$  is the largest  $\mathcal{C}_\beta \cap \mathcal{C}''$ , for any  $\beta_\beta > \beta_0$ , extending  $\omega_\beta = \omega|_{\mathcal{C}_\beta}$  to  $\langle \mathcal{C}_\beta, \mathcal{C}'' \rangle$  amounts to extending  $\omega_\beta|_{\mathcal{C}_\beta \cap \mathcal{C}''}$  to  $\mathcal{C}''$  (lemma 2), ~~or~~ that is to extend  $\omega_{\beta_0}$  from  $\mathcal{C}_{\beta_0} \cap \mathcal{C}''$  to  $\mathcal{C}''$ . If we choose one such extension, we get up to unique isomorphism a system of extensions of the  $\omega_\beta$  to  $\langle \mathcal{C}_\beta, \mathcal{C}'' \rangle$ , and by gluing them an extension of  $\omega$  to  $\langle \mathcal{C}', \mathcal{C}'' \rangle$ .

This is all we need to conclude that  $\mathcal{C}' = \mathcal{C}$ .

This proof should be cleaned up. After all, we are proving that some projective system of gerbs (of the fiber functors on the  $\mathcal{C}_\alpha, \alpha \in A_f$ ), where "projective system" is taken in a 2-categorical sense, has a non empty projective limit (again limit in a 2-categorical sense). Maybe such a translation would make the "compactness arguments" used clearer. They were two of them: "fiber functor is a property of finite type (cf Bourb. Ex Ch 3 §4.5)", and the noetherian property for subgroups.

The same arguments give unicity up to isomorphism of  $\omega$ , over  $k$ : we get a maximal  $\mathcal{O}_\alpha$  ( $\alpha \in A$ ) over which we have an isomorphism, and extend further if  $\mathcal{O}_\alpha \neq \mathcal{O}$ .

Bert

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