Shimura varieties: modular interpretation and construction techniques for canonical models

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Abstract

This is a translation of the classic article
Deligne, Pierre. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 247–289, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.
It is available at `www.jmilne.org/math/Documents`. Corrections should be sent to the email address on that page. All footnotes have been added by the translator. Obvious misprints have been silently corrected.

Contents

Introduction

This article is a sequel to [**5**], whose essential results (those of paragraphs 4 and 5) we shall make use of. In the first part we try to motivate the axioms imposed on the systems (G,X) (2.1.1) starting from which Shimura varieties are defined. Roughly speaking, we show that they correspond to the moduli spaces X^+ of Hodge structures of the following type.

(a) X^+ is a connected component of the space of all Hodge structures on a fixed vector space V relative to which certain tensors $t_1,\cdots,t_n$ are of type $(0,0)$. The algebraic group G is the subgroup of $\mathrm{GL}(V)$ fixing the t_i, and X is the orbit[1] $G(\mathbb{R})\cdot X^+$ of X^+ under $G(\mathbb{R})$.

(b) The family of Hodge structures on V parametrized by X^+ satisfies certain conditions that are satisfied by the families of Hodge structures appearing naturally in algebraic geometry: for a suitable (uniquely determined) complex structure on X^+, it is a variation of polarizable Hodge structures.

The space X^+ is automatically a *symmetric* hermitian domain (= hermitian symmetric domain of curvature < 0). The hermitian symmetric domains can all be described as moduli spaces of Hodge structures (1.1.17), and I believe this description to be very useful. For example, the embedding of a hermitian symmetric domain D into its dual $D^\vee$ (a flag variety) corresponds to the map sending a Hodge structure to the corresponding Hodge filtration. Their description as "Siegel domains of the 3rd kind" can be interpreted as saying that, under certain hypotheses, if one superimposes on a Hodge structure a filtration by weights, one obtains a mixed Hodge structure, and hence a map from D into the moduli space of mixed Hodge structures (cf. the constructions of [**1**, III, 4.1]). This last point will not be mentioned, or used, in the article.

This point of view and the description of certain Shimura varieties as moduli spaces of abelian varieties, are related by a dictionary: it is the same (equivalence of categories $A\mapsto H_1(A,\mathbb{Z})$) to give an abelian variety or a polarizable Hodge structure of type $\{(-1,0),(0,-1)\}$ (it is a question here of $\mathbb{Z}$ Hodge structures without torsion; by passage to the dual ($A\mapsto H^1(A,\mathbb{Z})$), one can replace $\{(-1,0),(0,-1)\}$ by $\{(1,0),(0,1)\}$). To polarize the abelian variety is the same as polarizing its H_1. With parameters, similarly, it is the same to give a polarized abelian scheme over a smooth complex variety S or a variation of polarized Hodge structures of type $\{(-1,0),(0,-1)\}$ over the analytic space S^{an}. An analytic family of abelian varieties, parametrized by S^{an}, is automatically algebraic (this follows from [**3**]). In order to interpret more complicated Hodge structures, we would like to replace abelian varieties with suitable "motives", but this is still only a dream.[2]

In number 1.2, we give a convenient description, based on the preceding formalism of the classification of hermitian symmetric domains in terms of Dynkin diagrams and their special nodes. In number 1.3, we classify a certain type of embedding of hermitian symmetric domains into a Siegel half-space. The results are parallel to

[1] In the original typewritten manuscript, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{A}$ are in blackboard bold with instructions to the printer to be set in boldface. This instruction has been ignored by the translator.

[2] A necessary condition for a Shimura variety to be a moduli variety for motives is that its weight be rational. The Shimura varieties with rational weight proved in this article to have canonical models (nowadays called *Shimura varieties of abelian type*) are shown to be moduli varieties for abelian motives in Milne, Shimura Varieties and Motives, 1994. Together with the "maladroit" method [**5**, 5.11], this gives an alternative proof of the main result (2.7.21) of this article. Realizing Shimura varieties not of abelian type as moduli of motives remains a dream.

those of Satake 1965.[3] An application of the unitary trick of Weyl, for which we refer to [**7**], reduces the classification to the knowledge of a fragment of the table, given for example in Bourbaki [**4**], expressing the fundamental weights as a linear combination of simple roots.

The reader wishing to know more about variations of Hodge structures and the way they appear in algebraic geometry can consult [**6**] (whose sign conventions we do not follow); certain facts, announced in [**6**], are proved in [**7**] .

In numbers 2.1 and 2.2 we define, in an adèlic language, the *Shimura varieties* ${}_KM_{\mathbb{C}}(G,X)$ (denoted ${}_KM_{\mathbb{C}}(G,h)$ in [**5**], for h an arbitrary element of X), their projective limit $M_{\mathbb{C}}(G,X)$, and the notion of a *canonical model*. I refer to the text for the definitions, and will say only that a canonical model of $M_{\mathbb{C}}(G,X)$ is a model of $M_{\mathbb{C}}(G,X)$ over the *dual field* (2.2.1) $E(G,X)$, i.e., a scheme $M(G,X)$ over $E(G,X)$ equipped with an isomorphism $M(G,X)\otimes_{E(G,X)}\mathbb{C}\xrightarrow{\sim}M_{\mathbb{C}}(G,X)$, having certain properties ($G(\mathbb{A}^f)$-equivariance, and Galois properties at the *special points* (2.2.4))[4]. We also define the notion of a weakly canonical model (same definition as canonical models with $E(G,X)$ replaced by a finite extension $E\subset\mathbb{C}$). They play a technical role in rhe construction of canonical models. The apparent difference in the definitions 2.1, 2.2 and those of [**5**] come from a different choice of sign conventions (right action instead of left action, reciprocity law in the global class field theory . . .).[5]

For a heuristic description, I refer to the introduction of [**5**] . For a brief description, with examples, of how to pass from the adèlic language to the a more classical language, I refer to [**5**, 1.6–1.11, 3.14–3.16, 4.11–4.16].

In [**5**], we systematized the methods introduced by Shimura to construct canonical models. In the second part of the present article, we improve the results of [**5**]. In number 2.6, we determine the action of a Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the set of geometric connected components of a weakly canonical model (supposed to exist) of $M_{\mathbb{C}}(G,X)$ over E without assuming, as in [**5**], that the derived group of G is simply connected. The essential point is the construction, given in number 2.4, of a morphism of the following type. Let G be a (connected) reductive group over $\mathbb{Q}$, $\rho\colon\widetilde{G}\to G$ the universal covering of its derived group G^{der}, and M a conjugacy class, defined over a number field E, of morphisms from $\mathbb{G}_m$ into G. We construct a morphism q_M from the group of idèle classes of E into the abelian quotient $G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A})\cdot G(\mathbb{Q})$ of $G(\mathbb{A})$. This morphism is functorial in (G,M), and, if F is an extension of E, the diagram

$$\begin{array}{ccc} C(F) & & \\ & \searrow & \\ \Big\downarrow N_{F/E} & & G(\mathbb{A})/\rho(\widetilde{G}(\mathbb{A})\cdot G(\mathbb{Q}) \\ & \nearrow & \\ C(E) & & \end{array}$$

is commutative. If $\widetilde{G}$ has no factor G' over $\mathbb{Q}$ such that $G'(\mathbb{R})$ is compact, we can

[3] Citations “Name [**n**]” in the original have been replaced with “Name year”.

[4] that determine it uniquely up to a unique isomorphism.

[5] And also a sign error in this article — see footnote 44, p. 28.

deduce from the strong approximation theorem that

$$\pi_0(G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A})G(\mathbb{Q}) = \pi_0(G(\mathbb{A})/G(\mathbb{Q})),$$

and q_M defines an action on $\pi_0(G(\mathbb{Q})/G(\mathbb{Q}))$ of $\pi_0 C(E)$, which according to global class field theory is the Galois group made abelian $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$.

The second new idea — in fact, a return to the point of view of Shimura — is the following observation: the results of 2.6 allow us to reconstruct the weakly canonical model $M_E(G,X)$ of $M_{\mathbb{C}}(G,X)$ starting from its neutral component[6] $M^{\circ}_{\overline{\mathbb{Q}}}(G,X^+)$ (a geometrically connected component, depending on the choice of a connected component X^+ of X), equipped with a (semi-linear) action of the subgroup H of $G(\mathbb{A}^f)\times G(\overline{\mathbb{Q}}/E)$ that stabilizes it. Let Z be the centre of G, G^{ad} the adjoint group, $G^{\mathrm{ad}}(\mathbb{R})^+$ the neutral topological component of $G^{\mathrm{ad}}(\mathbb{R})$, and $G^{\mathrm{ad}}(\mathbb{Q})^+ = G^{\mathrm{ad}}(\mathbb{Q})\cap G^{\mathrm{ad}}(\mathbb{R})^+$. The closure $Z(\mathbb{Q})^-$ of $Z(\mathbb{Q})$ in $G(\mathbb{A}^f)$ acts trivially on $M_{\mathbb{C}}(G,X)$, and so the action of H on $M^{\circ}_{\overline{\mathbb{Q}}}(G,X^+)$ factors through $H/Z(\mathbb{Q})^-$. It is possible to make somewhat a larger group act, namely, an extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ by the completion of $G^{\mathrm{ad}}(\mathbb{Q})^+$ for the topology defined by the images of congruence subgroups of $G^{\mathrm{der}}(\mathbb{Q})$.

Up to a unique isomorphism, this extension depends only on G^{ad}, G^{der} and the projection $X^{+\mathrm{ad}}$ of X^+ into G^{ad} (2.5). We denote it by $\mathcal{E}_E(G^{\mathrm{ad}},G^{\mathrm{der}},X^{+\mathrm{ad}})$. The neutral component $M^{\circ}_{\mathbb{C}}(G,X^+)$ is the projective limit of the quotients of $X^{+\mathrm{ad}}$ by the arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ that are images of congruence subgroups of $G^{\mathrm{der}}(\mathbb{Q})$. Finally, one checks that the conditions that a model $M^{\circ}_{\overline{\mathbb{Q}}}(G,X^+)$ of $M^{\circ}_{\mathbb{C}}(G,X^+)$ over $\overline{\mathbb{Q}}$ equipped with an action of $\mathcal{E}_E(G^{\mathrm{ad}},G^{\mathrm{der}},X^{+\mathrm{ad}})$ must satisfy in order to correspond to a weakly canonical model depend only on the adjoint group G^{ad}, on $X^{+\mathrm{ad}}$, on the covering G^{der} of G^{ad}, and on the finite extension E (contained in $\mathbb{C}$) of $E(G^{\mathrm{ad}},X^{+\mathrm{ad}})$. These conditions define the *weakly canonical connected* models (resp. *canonical* for $E=E(G^{\mathrm{ad}},X^{+\mathrm{ad}})$) (2.7.10).

The problem of the existence of a canonical model therefore depends, roughly speaking, on the derived group. This reduction to the derived group is a much more convenient version of the maladroit method of making a central modification of h in [**5**, 5.11].

In 2.3, we construct a supply of canonical models with the help of symplectic embeddings, by invoking [**5**, 4.21 and 5.7]. The results of 1.3 allow us to obtain the desired symplectic embeddings with very little calculation. In 2.7, we explain the reduction to the derived group sketched above, and we deduce in 2.3 an existence criterion for canonical models that covers all the known cases (Shimura, Miyake, and Shih).[7]

In the article, we use the equivalence between weakly canonical models and connected weakly canonical models to transfer to these last the results of [**5**] (uniqueness, construction of a canonical model starting from a family of weakly canonical models). It would have been more natural to transpose the proofs, and to transpose even the

[6] The original uses ^{0} instead of $^\circ$ for a connected component.

[7] For a comparison of the canonical models in the sense of Deligne to the canonical models in the sense of Shimura, and a proof that the existence of the former implies that of the latter, see Milne and Shih, Amer. J. Math. 103 (1981), 1159-1175.

functoriality [**5**, 5.4] and the passage to a subgroup [**5**, 5.7] (avoiding the syballine proposition [**5**, 1.15]). Lack of time and weariness have prevented me.

I have recently proved[8] that one can give a purely algebraic sense to the notion of a rational cycle of type (p, p) on an abelian variety over a field of characteristic 0. Starting from this, one can recover the existence criterion 2.3.1 for canonical models, and give a modular description of the models obtained with its help (cf. [**9**]). This description does not lend itself unfortunately to reduction modulo p. This method avoids recours to [**5**, 5.7], (and from there to [**5**, 1.15]) and furnishes partial information on the conjugation of Shimura varieties.

0 Review, terminology, notation

0.1. We will make use of the strong approximation theorem, the real approximation theorem, the Hasse principle, and the vanishing of $H^1(K,G)$ for G a simply connected semisimple group over a nonarchimedean local field. Some bibliographic indications on the theorems are given in [**5**, (0.1) to (0.4)]. We note in addition the article of G. Prasad (*Strong approximation for semi-simple groups over function fields,* Ann. of Math. (2) **105** (1977), 553-572) proving the strong approximation theorem over any global field. Let G be a simply connected semisimple group, with centre Z, over a global field K. We will use the Hasse principle for $H^1(K,G)$ only for the classes in the image of $H^1(K,Z)$. In particular, the factors of E_8 do not affect this.[9]

0.2. *Reductive group* always means *connected reductive group*. A *finite covering*[10] of a reductive group is a *connected* finite covering. *Adjoint* group means *adjoint reductive* group. If G is a reductive group, we denote by G^{ad} its adjoint group, by G^{der} its derived group, and by $\rho\colon \widetilde{G} \to G^{\mathrm{der}}$ the universal finite covering of G^{der}. We sometimes let Z (or $Z(G)$) denote the centre of G, and (conflict of notation) $\widetilde{Z}$ that of $\widetilde{G}$.

0.3. We use the exponent $^\circ$ for an *algebraic connected component* (for example, Z° is the neutral component of the centre of G). The exponent $^+$ will denote a *topological connected component* (for example, $G(\mathbb{R})^+$ is the neutral component of the topological group of real points of the group G). We we will also write $G(\mathbb{Q})^+$ for the trace $G(\mathbb{R})^+$ on $G(\mathbb{Q})$. For a real reductive group G, we will use the subscript $_+$ to denote the inverse image of $G^{\mathrm{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$. The same notation $_+$ will be used for the trace on the group of rationals.[11]

For a topological space X, we let $\pi_0(X)$ denote the set of its connected components equipped with the quotient topology of that of X. In the article, the space $\pi_0(X)$ will always be discrete or compact and totally disconnected.

[8] See, P. Deligne (notes by J. Milne), Hodge cycles on abelian varieties, in Hodge cycles, motives, and Shimura varieties, Lecture Notes in Math. 900, Springer-Verlag, 1982,

[9] At the time the article was written, the Hasse principle had not been proved for groups of type E_8 — they have trivial centre.

[10] "revetment" in original = "finite covering" in translation.

[11] So $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$ and $G(\mathbb{Q})_+ = G(\mathbb{R})_+ \cap G(\mathbb{Q})$.

0.4. A *hermitian symmetric domain* is a symmetric hermitian space of curvature <0 (i.e., without euclidean or compact factors).

0.5. Unless expressly mentioned otherwise, a *vector space* is assumed to be of finite dimension, and a *number field* is assumed to be of finite degree over $\mathbb{Q}$. The number fields that we will consider will be most often contained in $\mathbb{C}$; $\overline{\mathbb{Q}}$ denotes the algebraic closure $\mathbb{Q}$ in $\mathbb{C}$.

0.6. We put $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$, $\mathbb{A}^f = \mathbb{Q} \otimes \widehat{\mathbb{Z}} = \prod_p \mathbb{Q}_p$ (restricted product), and we let $\mathbb{A} = \mathbb{R} \times \mathbb{A}^f$ denote the ring of adèles of $\mathbb{Q}$. We also sometimes denote by $\mathbb{A}$ the ring of adèles of an arbitrary global field.

0.7. $G(K)$, $G \otimes_F K$, G_K: for G a scheme over F (for example, an algebraic group over F), and K an F-algebra, we denote by $G(K)$ the set of points of G with values in K and by G_K or $G \otimes_F K$ the scheme over K deduced from G by extension of scalars.

0.8. We normalize the reciprocity isomorphism in global class field theory (= choice of one or its inverse)[12]

$$\pi_0 \mathbb{A}_E^\times / E^\times \xrightarrow{\sim} \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$$

so that the class of the idèle equal to a uniformizer at v and 1 elsewhere corresponds to the geometric Frobenius (inverse of the Frobenius map) (cf. 1.1.6 and the justification loc. cit. 3.6, 8.12).[13]

1 Hermitian symmetric domains

1.1 Moduli spaces of Hodge structures

1.1.1. Recall that a Hodge structure on a real vector space V is a bigradation $V_{\mathbb{C}} = \bigoplus V^{pq}$ of the complexification of V such that V^{pq} is the complex conjugate of V^{qp}.

Define an action h of $\mathbb{C}^\times$ on $V_{\mathbb{C}}$ by the formula

$$h(z)v = z^{-p}\overline{z}^{-q}v \text{ for } v \in V^{pq}. \tag{1.1.1.1}$$

The $h(z)$ commute with complex conjugation on $V_{\mathbb{C}}$, and therefore arise by extension of scalars from an action, again denoted h, of $\mathbb{C}^\times$ on V. Regard $\mathbb{C}$ as an extension of $\mathbb{R}$, and $\mathbb{S}$ as its multiplicative group considered as a real algebraic group (in other words, $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, Weil restriction of scalars); then $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, and h is an action of the algebraic group $\mathbb{S}$. One can check that this construction defines an equivalence of categories: (real vector spaces endowed with a Hodge structure)[14] $\rightsquigarrow$ (real vector spaces endowed with an action of the real algebraic group $\mathbb{S}$).

[12] The original writes R^* for the units in a ring R; this is translated as $R^\times$.

[13] The parenthetical statement should be deleted — it is not relevant.

[14] Here, and elsewhere, we use $\rightsquigarrow$ for functors instead of $\rightarrow$.

The inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times$ corresponds to an inclusion of real algebraic groups $\mathbb{G}_m \subset \mathbb{S}$. We write w_h (or simply w) for the restriction of h^{-1} to $\mathbb{G}_m$, and call it the *weight* $w\colon \mathbb{G}_m \to \mathrm{GL}_V$. We say that V is homogeneous of weight n if $V^{pq} = 0$ for $p+q \neq n$, i.e., if $w(\lambda)$ is the homothety with factor λ^n.

We write μ_h (or simply μ) for the action of $\mathbb{G}_m$ on $V_\mathbb{C}$ defined by $\mu(z)v = z^{-p}v$ for $v \in V^{pq}$. It is the composite

$$\mathbb{G}_m \to \mathbb{S}_\mathbb{C} \xrightarrow{h} \mathrm{GL}_{V_\mathbb{C}}.$$

The *Hodge filtration* F_h (or simply F) is defined by $F^p = \bigoplus_{r \geq p} V^{rs}$. We say that V is of type $\mathcal{E} \subset \mathbb{Z}\times\mathbb{Z}$ if $V^{pq} = 0$ for $(p,q) \notin \mathcal{E}$.

More generally, if A is a subring of $\mathbb{R}$ such that $A \otimes \mathbb{Q}$ is a field (in practice, $A = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$), a Hodge A-structure is an A-module of finite type V endowed with a Hodge structure on $V \otimes_A \mathbb{R}$.[15]

EXAMPLE 1.1.2. The fundamental example is that where $V = H^n(X,\mathbb{R})$ for X a compact kählerian manifold, and where $V^{pq} \subset H^n(X,\mathbb{C})$ is the space of cohomology classes represented by a closed form of type (p,q). Other useful examples can be constructed from this one by the operations of tensor products, passage to a direct factor of Hodge structures, or passage to the dual. Thus, the dual $H_n(X,\mathbb{R})$ of $H^n(X,\mathbb{R})$ is endowed with a Hodge structure of weight $-n$. Integral homology, or cohomology, furnishes Hodge $\mathbb{Z}$-structures.

EXAMPLE 1.1.3. The Hodge structures of type $\{(-1,0),(0,-1)\}$ are those for which the action h of $\mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ is induced by a complex structure on V; for V of this type, the projection pr of V onto $V^{-1,0} \subset V_\mathbb{C}$ is bijective, and satisfies $\mathrm{pr}(h(z)\cdot v) = z \cdot \mathrm{pr}(v)$.

EXAMPLE 1.1.4. Let A be a complex torus; it is the quotient L/Γ of its Lie algebra L by a lattice Γ. Then $\Gamma \otimes \mathbb{R} \xrightarrow{\sim} L$, from which we obtain a complex structure on $\Gamma \otimes \mathbb{R}$. Regard this as a Hodge structure (1.1.3). Via the isomorphism[16] $\Gamma \simeq H_1(A,\mathbb{Z})$, this structure is that of 1.1.2.

EXAMPLE 1.1.5. The Tate Hodge structure $\mathbb{Z}(1)$ is the Hodge $\mathbb{Z}$-structure of type $(-1,-1)$ with integral lattice $2\pi i\mathbb{Z} \subset \mathbb{C}$. The exponential identifies $\mathbb{C}^\times$ with $\mathbb{C}/\mathbb{Z}(1)$, from which we get an isomorphism $\mathbb{Z}(1) \simeq H_1(\mathbb{C}^\times)$. The Hodge structure $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$ $(n \in \mathbb{Z})$ is the Hodge $\mathbb{Z}$-structure of type $(-n,-n)$ with integral lattice $(2\pi i)^n\mathbb{Z}$. We denote by $\ldots(n)$ the tensor product of $\ldots$ with $\mathbb{Z}(n)$ (*Tate twist*).

REMARK 1.1.6. The rule $h(z)v = z^{-p}\bar{z}^{-q}v$ for $v \in V^{pq}$ is that which I used in *Les constants des équations fonctionnelles des fonctions L*, Antwerp II, Lecture Notes in Math., vol. 349, pp. 501–597) and is inverse to that in [**6**]. It is partly justified by the example 1.1.4 above and also by the desire to have $\mathbb{C}^\times$ acting on $\mathbb{R}(1)$ by multiplication by the norm (cf. loc. cit., end of 8.12).

[15] Usually one also requires the weight gradation to be defined over $A \otimes \mathbb{Q}$.

[16] Occasionally, "$=$" has been translated to "$\simeq$"

1.1.7. A *variation of Hodge structures* on a complex analytic variety S consists of

(a) a local system V of real vector spaces,

(b) at each point of S, a Hodge structure on the fibre of V at s, varying continuously with s.

The Hodge filtration is required to vary holomorphically with s, and satisfy the *transversality* axiom: the derivative of a section of F^p is in F^{p-1}.

Often there will be give a local system $V_{\mathbb{Z}} \subset V$ of $\mathbb{Z}$-modules of finite type such that $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. We then speak of a *variation of Hodge* $\mathbb{Z}$*-structures*, and similarly with $\mathbb{Z}$ replaced by a ring A as in 1.1.1.

REMARK 1.1.8. Regard S as a real manifold whose tangent space at each point is endowed with a complex structure, i.e., a Hodge structure of type $\{(-1,0),(0,-1)\}$. The integrability of the almost-complex structure on S can be expressed by saying that the bracket of vector fields is compatible with the Hodge filtration on the complexified tangent bundle: $[T^{0,-1}, T^{0,-1}] \subset T^{0,-1}$. Similarly, the axiom of the variations of Hodge structures expresses that the derivation

$$\text{(tangent bundle)} \otimes_{\mathbb{R}} (C^{\infty} \text{ sections of } V) \to (C^{\infty} \text{ sections of } V)$$

(or rather the complexification of this map) is compatible with the Hodge filtrations: $\partial_D F^p \subset F^p$ for D in $T^{0,-1}$ (holomorphy) and $\partial_D F^p \subset F^{p-1}$ for D arbitrary (transversality).

1.1.9. PRINCIPLE. In algebraic geometry, each time there appears a Hodge structure depending on complex parameters, it is a variation of Hodge structures on the space of parameters. The fundamental example is 1.1.2 with parameters: if $f\colon X \to S$ is a proper smooth morphism with kählerian fibre X_s, then the $H^n(X_s,\mathbb{Z})$ form a local system on S and the Hodge filtration on the complexification $H^n(X_s,\mathbb{C})$ varies holomorphically with s and satisfies the axiom of transversality.

1.1.10. A *polarization* of a real Hodge structure V of weight n is a morphism[17] $\Psi\colon V \otimes V \to \mathbb{R}(-n)$ such that the form $(2\pi i)^n \Psi(x, h(i)y)$ is symmetric and positive definite. For Hodge $\mathbb{Z}$-structures, we replace $\mathbb{R}(-n)$ with $\mathbb{Z}(-n)$, ... Since $\Psi(h(i)x, y) = \Psi(x, h(-i)y)$ ($h(i)$ is trivial on $\mathbb{R}(-n)$) and $h(-i)y = (-1)^n h(i)y$, the symmetry condition comes down to: Ψ is symmetric for n even and alternating for n odd.

The Hodge structures that arise in algebraic geometry are polarizable homogeneous Hodge $\mathbb{Z}$-structures. Fundamental example: the Hodge positivity theorems ensure that $H^n(X,\mathbb{Z})$ for X a smooth projective variety is polarizable (note that $h(i)$ is the operator denoted by C by Weil in his book on kählerian manifolds).

1.1.11. Let $(V_i)_{i\in I}$ be a family of real vector spaces and $(s_j)_{j\in J}$ a family of tensors in the tensor products of the V_i and their duals. We are interested in the families of Hodge structures on the V_i for which the s_j are of type $(0,0)$. In order to interpret the

[17] of Hodge structures

condition "type $(0,0)$" in particular cases, note that $f: V \to W$ is a morphism if and only if, as an element of $\mathrm{Hom}(V,W) \simeq V^\vee \otimes W$, it is of type $(0,0)$.

Let G be the algebraic subgroup of[18] $\prod \mathrm{GL}_{V_i}$ fixing the s_j. After 1.1.1, a family of Hodge structures on the V_i can be identified with a morphism $h: \mathbb{S} \to \prod \mathrm{GL}_{V_i}$. The s_j are of type $(0,0)$ if and only if h factors through G: it becomes a question of considering algebraic morphisms $h: \mathbb{S} \to G$.

We can regard G rather than the system of V_i and s_j as being the primitive object: if G is a real linear algebraic group, to give an $h: \mathbb{S} \to G$ is the same as giving on each representation V of G a Hodge structure that is functorial for G-morphisms and compatible with tensor products (cf. Saavedra 1982, VI, §2). The morphisms w_h and μ_h of 1.1.1 become morphisms of $\mathbb{G}_m$ into G and of $\mathbb{G}_m$ into $G_\mathbb{C}$ respectively.

1.1.12. The construction 1.1.11 suggests considering moduli spaces of Hodge structures of the following type: fix a real linear algebraic group G, and consider a (topological) connected component X of the space of morphisms (= homomorphisms of algebraic groups over $\mathbb{R}$) from $\mathbb{S}$ into G.

Let G_1 be the smallest algebraic subgroup of G through which all the $h \in X$ factor: X is again a connected component of the space of morphisms of $\mathbb{S}$ into G_1. Since $\mathbb{S}$ is of multiplicative type, any two elements of X are conjugate: the space X is a $G_1(\mathbb{R})^+$-conjugacy class of morphisms from $\mathbb{S}$ into G. It is also a $G(\mathbb{R})^+$-conjugacy class, and G_1 is a normal subgroup of the neutral component of G.

1.1.13. In view of 1.1.9 and 1.1.10, we consider only the X such that, for some faithful family V_i of representations of G, we have

- (α) For all i, the weight gradation on the V_i (whose complexification is the gradation of $V_{i\mathbb{C}}$ by the subspaces $V_{i\mathbb{C}}^n = \bigoplus_{p+q=n} V_i^{pq}$) is independent of $h \in X$. Equivalent conditions: $h(\mathbb{R}^\times)$ is central in $G(\mathbb{R})^+$;[19] the adjoint representation is of weight 0.
- (β) For a suitable complex structure on X, and all i, the family of Hodge structures defined by the $h \in X$ is a variation of Hodge structures on X.
- (γ) If V is a homogeneous component of weight n of a V_i, there exists $\Psi: V \otimes V \to \mathbb{R}(-n)$ which, for all $h \in X$, is a polarization of V.

PROPOSITION 1.1.14. *Suppose (α) is satisfied.*

(i) *There exists one and only one complex structure on X such that the Hodge filtrations on the V_i vary holomorphically with $h \in X$.*

(ii) *The condition 1.1.13(β) is satisfied if and only if the adjoint representation is of type*

$$\{(-1,1),(0,0),(1,-1)\}.$$

(iii) *The condition 1.1.13(γ) is satisfied if and only if G_1 (defined in 1.1.12) is reductive and, for all $h \in X$, the inner automorphism* $\mathrm{inn}\, h(i)$ *induces a Cartan involution of its adjoint group.*

[18]The original has $\mathrm{GL}(V)$, but GL_V has become the standard notation for the algebraic group and $\mathrm{GL}(V)$ for the abstract group.

[19]The original has $G(\mathbb{R})^\circ$.

(i) Let V be the sum of the V_i. It is a faithful representation of G. A Hodge structure is determined by the corresponding Hodge filtration (plus the weight gradation if we are not in the homogeneous case): in weight $n = p+q$, we have $V^{pq} = F^p \cap \overline{F^q}$. The map φ from X into the grassmannian of $V_{\mathbb{C}}$,

$$h \mapsto \text{(corresponding Hodge filtration)},$$

is therefore injective. We shall show that it identifies X with a complex subvariety of the the grassmannian; this will prove (i): the complex structure on X induced by that of the grassmanian is the only one for which φ is holomorphic.

Let L be the Lie algebra of G and $p\colon L \to \mathrm{End}(V)$ its action on V. The action p is a morphism of G-modules, injective by hypothesis. For all $h \in X$, it is also a morphism of Hodge structures.[20] The tangent space to X at h is the quotient of L by the Lie algebra of the stabilizer of h, namely the subspace L^{00} of L for the Hodge structure on L defined by h. The tangent space to the grassmanian at $\varphi(h)$ is $\mathrm{End}(V_{\mathbb{C}})/F^0(\mathrm{End}(V_{\mathbb{C}}))$. Finally, $d\varphi$ is the composite

$$\begin{array}{ccc} L/L^{00} & \xrightarrow{p} & \mathrm{End}(V)/\mathrm{End}(V)^{00} \\ \downarrow\wr & & \downarrow\wr \\ L_{\mathbb{C}}/F^0 L_{\mathbb{C}} & \xrightarrow{p} & \mathrm{End}(V_{\mathbb{C}})/F^0\,\mathrm{End}(V_{\mathbb{C}}). \end{array}$$

As p is an injective morphism of Hodge structures, $d\varphi$ is injective; its image is that of $L_{\mathbb{C}}/F^0 L_{\mathbb{C}}$, which is a complex subspace, whence the assertion.

(ii) The transversality axiom asserts that the image of $d\varphi$ is in

$$F^{-1}\,\mathrm{End}(V_{\mathbb{C}})/F^0\,\mathrm{End}(V_{\mathbb{C}}),$$

i.e., that $L_{\mathbb{C}} = F^{-1} L_{\mathbb{C}}$.

In order to prove (iii), we make use of [**7**, 2.8], recalled below. Recall that a *Cartan involution* of a real linear algebraic group G (not necessarily connected) is an involution σ of G such that the real form G^{σ} of G (with complex conjugation $g \mapsto \sigma(\overline{g})$) is *compact*: $G^{\sigma}(\mathbb{R})$ is compact and meets every connected component of $G^{\sigma}(\mathbb{C}) = G(\mathbb{C})$. For $C \in G(\mathbb{R})$ with central square, a *C-polarization* of a representation V of G is a G-invariant bilinear form Ψ such that $\Psi(x, Cy)$ is symmetric and positive definite. For all $g \in G(\mathbb{R})$, we then have that $\Psi(x, gCg^{-1}y) = \Psi(g^{-1}x, Cg^{-1}y)$: the notion of a C-polarization depends only on the $G(\mathbb{R})$-conjugacy class of C.

1.1.15. *Review [**7**, 2.8]. Let G be a real algebraic group and C an element of $G(\mathbb{R})$ whose square is central. The following conditions are equivalent:*

(a) $\mathrm{inn}\, C$ *is a Cartan involution of G;*

(b) *every real representation of G is C-polarizable;*

(c) *G admits a faithful real C-polarizable representation.*

[20] Because it is a morphism of G-modules.

Note that the condition 1.1.15(i) implies that G° is reductive, because it has a compact form. The condition depends only on the conjugacy class of C.

PROOF OF 1.1.14(iii). Let G_2 be the smallest algebraic subgroup of G through which the restrictions of the $h \in X$ to $U^1 \subset \mathbb{C}^\times$ factor. A bilinear form $\Psi \colon V \otimes V \to \mathbb{R}(-n)$ satisfies 1.1.13(γ) if and only if $(2\pi i)^n \Psi \colon V \otimes V \to \mathbb{R}$ is invariant under the $h(U^1)$ — therefore under G_2 — (this expresses that Ψ is a morphism) and an $h(i)$-polarization. After 1.1.15, 1.1.13(γ) is equivalent to: $\operatorname{inn} h(i)$ is a Cartan involution of G_2.

We deduce first that G_1 is reductive: G_2 is, because it has a compact form, and G_1 is a quotient of the product $G_2 \times \mathbb{G}_m$. Since G_2 is generated by compact subgroups, its connected centre is compact: it[21] is isogenous to the quotient of G_2 by its derived group. The involution $\theta = \operatorname{inn} h(i)$ is therefore a Cartan involution of G_2 if and only if it is of the adjoint group of G_2. We conclude by noting that G_1 and G_2 have the same adjoint group.

As the conditions in 1.1.14 depend only on (G, X), we have the following:

COROLLARY 1.1.16. *The conditions 1.1.13(α), (β), (γ) do not depend on the faithful family of representations V_i chosen.*

COROLLARY 1.1.17. *The spaces X in 1.1.13 are the hermitian symmetric domains.*[22]

A. *Proof that X is of this type.*

We reduce successively to supposing:

(1) That $G = G_1$: replacing G with G_1 changes neither X nor the conditions 1.1.13.

(2) That G is adjoint: by (1) , G is reductive, and its quotient by a finite central subgroup is the product of a torus T with its adjoint group G^{ad}. The space X can be identified with a connected component of the space of morphisms of $\mathbb{S}/\mathbb{G}_m$ into G^{ad}: if such a morphism lifts to a morphism from $\mathbb{S}$ into G with given projection into T, then the lifting is unique. The conditions stated in 1.1.14 remain true.

(3) That G is simple: decompose G into a product of simple groups G_i; this decomposes X into a product of spaces X_i relative to the G_i.

Therefore, let G be a simple adjoint group, and let X be a $G(\mathbb{R})^+$-conjugacy class of nontrivial morphisms $h \colon \mathbb{S}/\mathbb{G}_m \to G$ satisfying the conditions of 1.1.14(ii), (iii). The group G is noncompact: otherwise, $\operatorname{inn} h(i)$ would be trivial (by (iii)), $\operatorname{Lie} G$ would be of type $(0,0)$ (by (ii)), and h would be trivial. Let $h \in X$. After (iii), its centralizer is compact; there therefore exists on X a $G(\mathbb{R})^+$-equivariant riemannian structure. After (ii), $h(i)$ acts on the tangent space $\operatorname{Lie}(G)/\operatorname{Lie}(G)^{00}$ of X at h by -1: the riemannian space X is symmetric. One checks finally that it is hermitian symmetric for the complex structure 1.1.14(i). It is of noncompact type (curvature < 0) because G is not compact.

B. Conversely, if X is a symmetric hermitian space and $x \in X$, we know that multiplication by u ($|u| = 1$) on the tangent space T_x to X at x extends to an automorphism

[21] I.e., the connected centre.

[22] The original has "domaines hermitiens symétriques"="symmetric hermitian domains", but "hermitian symmetric domains" is more common in English.

$m_x(u)$ of X. Let A be the group of automorphisms of X, and $h(z) = m(z/\overline{z})$ for $z \in \mathbb{C}^\times$. The centralizer A_x of x commutes with h, and the condition 1.1.14(ii) is satisfied: $\mathrm{Lie}(A_x)$ is of type $(0,0)$, and $T_x = \mathrm{Lie}(A)/\mathrm{Lie}(A_x)$ of type $\{(-1,1),(1,-1)\}$. Finally, we know that A is the neutral component of $G(\mathbb{R})$ with G adjoint, and that the symmetric riemannian space X is of curvature < 0 if and only if the symmetry $h(i)$ furnishes a Cartan involution of G (see Helgason 1962).

1.1.18. We indicate two variants of 1.1.15 (cf. [**7**, 2.11]).

(a) Suppose given a reductive real algebraic group (0.2) G and a $G(\mathbb{R})$-conjugacy class of morphisms $h\colon \mathbb{S} \to G$. Assume that w_h — denoted w — is central, and therefore independent of h (condition 1.1.13(α)), and that $\mathrm{inn}\, h(i)$ is a Cartan involution of $G/w(\mathbb{G}_m)$.

Since G is reductive, $w(\mathbb{G}_m)$ admits a supplement G_2: a connected normal subgroup such that G is the quotient $w(\mathbb{G}_m) \times G_2$ by a finite central subgroup. It is unique: generated by the derived group and the largest compact subtorus of the centre. It contains the $h(U^1)$ ($h \in X$), and $\mathrm{inn}\, h(i)$ is a Cartan involution. If V is a representation of G, its restriction to G_2 therefore admits an $h(i)$-polarization Φ. If V is of weight n, then $w(\mathbb{G}_m)$ acts by similitudes, therefore G likewise: for a suitable representation of G on $\mathbb{R}$, Φ becomes covariant. For this representation, $\mathbb{R}$ is of type (n,n); this allows us to make G act on $\mathbb{R}(n)$ in a way that is compatible with its Hodge structure, and to see $\Psi = (2\pi i)^{-n}\Phi$ as a *G-invariant polarization form* $V \otimes V \to \mathbb{R}(-n)$.

(b) Suppose that G is deduced by extension of scalars to $\mathbb{R}$ from $G_\mathbb{Q}$ over $\mathbb{Q}$ and that w is defined over $\mathbb{Q}$. The group G_2 is then defined over $\mathbb{Q}$, because it is the unique supplement to $w(\mathbb{G}_m)$, and every character of G/G_2 is defined over $\mathbb{Q}$, because this group is trivial or isomorphic over $\mathbb{Q}$ to $\mathbb{G}_m$. If a (rational) representation V of $G_\mathbb{Q}$ is of weight n, the G-invariant bilinear forms $V \otimes V \to \mathbb{Q}(-n)$ form a vector space F over $\mathbb{Q}$. The set of such forms that are a polarization (relative to $h \in X$) is the intersection with F of an open subset of $F_\mathbb{R}$, and this open set is nonempty by (a). There therefore exist G-invariant polarization forms $\Psi\colon V \otimes V \to \mathbb{Q}(-n)$.[23]

Take care that the forms in (a) and (b) are not always polarizations for all $h' \in X$: if $h' = \mathrm{inn}(g)(h)$, the formula $\Psi(x, h'(i)y) = g\Psi(g^{-1}x, h(i)g^{-1}y)$ shows that the form $(2\pi i)^n \Psi(x, h'(i)y)$ is symmetric and definite — but positive or negative definite according to the action of g on $\mathbb{R}(-n)$.

1.2 Classification

In the rest of this paragraph, we use the relation 1.1.17 between hermitian symmetric domains and moduli spaces of Hodge structures to reformulate certain results of [**1**] and [**8**], and to give some complements.

1.2.1. Consider the systems (G, X) formed of an adjoint simple real algebraic group G and of a $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to G$ satisfying (the notation is that of 1.1.1, 1.1.11).

[23] The original has $\mathbb{Q}(n)$.

(i) The adjoint representation $\mathrm{Lie}(G)$ is of type $\{(-1,1),(0,0),(1,-1)\}$ (in particular, h is trivial on $\mathbb{G}_m \subset S$);

(ii) $\mathrm{inn}\, h(i)$ is a Cartan involution;

(iii) h is nontrivial or, what amounts to the same thing (cf. 1.1.17), G is noncompact.

After 1.1.17, the connected components of the spaces X thus obtained are irreducible hermitian symmetric domains.

The hypothesis (ii) ensures that the Cartan involutions of G are inner automorphisms, therefore that G is an inner form of its compact form (cf. 1.2.3). In particular G, being simple, is absolutely simple.

The $G(\mathbb{C})$-conjugacy class of $\mu_h\colon \mathbb{G}_m \to G_{\mathbb{C}}$ doesn't depend on the choice of $h \in X$. We denote it by M_X.

PROPOSITION 1.2.2. *Let $G_{\mathbb{C}}$ be an adjoint simple complex algebraic group. To each system (G,X) consisting of a real form G of $G_{\mathbb{C}}$ and an X satisfying 1.2.1(i), (ii), (iii), attach M_X. We obtain in this way a bijection between the $G(\mathbb{C})$-conjugacy classes of systems (G,X) and the $G(\mathbb{C})$-conjugacy classes of nontrivial morphisms $\mu\colon \mathbb{G}_m \to G_{\mathbb{C}}$ satisfying the following condition:*

> *(*) In the representation* $\mathrm{inn}\,\mu$ *of* $\mathbb{G}_m$ *on* $\mathrm{Lie}(G_{\mathbb{C}})$, *only the characters* z, 1, *and* z^{-1} *appear.*

In proving 1.2.2, we shall use the duality between hermitian symmetric domains and compact hermitian symmetric spaces.

1.2.3. Let G be a real form of $G_{\mathbb{C}}$, let X be a $G(\mathbb{R})$-conjugacy class of morphisms $\mathbb{S}/\mathbb{G}_m \to G$, and let $h \in X$. The real form G corresponds to a complex conjugation σ on $G_{\mathbb{C}}$; define G^* to be the real form with the complex conjugation $\mathrm{inn}(h(i))\sigma$:

$$G^*(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = \mathrm{inn}(h(i))\sigma(g)\}.$$

The morphism h is still defined over $\mathbb{R}$, from $\mathbb{S}/\mathbb{G}_m$ into G^*: we have $h(\mathbb{C}^\times/\mathbb{R}^\times) \subset G^*(\mathbb{R})$; define X^* to be the $G^*(\mathbb{R})$-conjugacy class of h. The construction[24] $(G,X) \mapsto (G^*,X^*)$ is an involution on the set of $G_{\mathbb{C}}(\mathbb{C})$-conjugacy classes of systems (G,X) consisting of a real form G of $G_{\mathbb{C}}$ and a $G(\mathbb{R})$-conjugacy class of nontrivial morphism from $\mathbb{S}/\mathbb{G}_m$ to G. It exchanges the (G,X) as in 1.2.2 and the (G,X) such that G is compact and X satisfies 1.2.1(i).

We know that the compact real forms of $G_{\mathbb{C}}$ are all conjugate. Since, if $g \in G_{\mathbb{C}}$ normalizes a real form G, then $g \in G(\mathbb{R})$ (because G is adjoint), the duality reduces 1.2.2 to the following statement:

LEMMA 1.2.4. *Let G be a compact form of $G_{\mathbb{C}}$. The construction $h \mapsto \mu_h$ induces a bijection between*

(a) *$G(\mathbb{R})$-conjugacy classes of morphisms $h\colon \mathbb{S}/\mathbb{G}_m \to G$ satisfying 1.2.1(i), and*

(b) *$G_{\mathbb{C}}(\mathbb{C})$-conjugacy classes of morphisms $\mu\colon \mathbb{G}_m \to G_{\mathbb{C}}$ satisfying 1.2.2(*).*

[24] Here and elsewhere, we have sometimes replaced $\to$ with $\mapsto$ when this is what is meant.

PROOF. Let T be a maximal torus of G. One checks first that the map

$$h \mapsto \mu_h : \mathrm{Hom}(\mathbb{S}/\mathbb{G}_m, T) \to \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})$$

is bijective. If W is the Weyl group of T, we know that

$$\mathrm{Hom}(U^1, T)/W \xrightarrow{\sim} \mathrm{Hom}(U^1, G)/G(\mathbb{R})$$

and

$$\mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})/W \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_m, G_{\mathbb{C}})/G_{\mathbb{C}}(\mathbb{C}).$$

The map $h \mapsto \mu_h$ therefore induces a bijection

$$\mathrm{Hom}(\mathbb{S}/\mathbb{G}_m, G)/G(\mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_m, G)/G_{\mathbb{C}}(\mathbb{C}),$$

and, h satisfies 1.2.1(i) if and only if μ_h satisfies 1.2.2(*).

1.2.5. Let G be a simple adjoint complex algebraic group. We shall enumerate the conjugacy classes of nontrivial homomorphisms $\mu: \mathbb{G}_m \to G$ satisfying 1.2.2(*) in terms of the Dynkin diagrams of G. Recall that this last is canonically attached to G — in particular, the automorphisms of G act on D — we can identify the nodes with the conjugacy classes of maximal parabolic subgroups.

Let T be a maximal torus, $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ (the dual of $X(T)$ for the pairing $X(T) \times Y(T) \xrightarrow{\circ} \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$), $R \subset X(T)$ the set of roots, B a system of simple roots, α_0 the opposite of the longest root and $B^+ = B \cup \{\alpha_0\}$. The nodes of D are parametrized by B, and those of the extended Dynkin diagram D^+ by B^+.

A conjugacy class of morphisms of $\mathbb{G}_m$ to G has a unique representative $\mu \in Y(T)$ in the fundamental chamber $\langle \alpha, \mu \rangle \geq 0$ for $\alpha \in B$. It is uniquely determined by the positive integers $\langle \alpha, \mu \rangle$ ($\alpha \in B$) and, G being adjoint, these can be prescribed arbitrarily. The condition 1.2.2(*) for μ nontrivial can be rewritten

$$(*)' \quad \langle \alpha_0, \mu \rangle = -1.$$

Write the longest root as a linear combination of simple roots, $\sum_{\alpha \in B^+} n(\alpha)\alpha = 0$, with $n(\alpha_0) = 1$, and call *special* the nodes of D^+ such that, for the corresponding root $\alpha \in B^+$, we have $n(\alpha) = 1$. We know that the quotient of the group of coweights by the subgroup of coroots acts on D^+, and the action is simply transitive on the set of special nodes. The special nodes are therefore conjugate under $\mathrm{Aut}(D^+)$ to the node corresponding to α_0, and their number is the index of connection $|\pi_1(G)|$ of G (cf. Bourbaki [**4**, VI, 2 ex 2 and 5a, p. 227]).

The condition $(*)'$ can be rewritten:

> $(*)''$ For one simple root $\alpha \in B$ corresponding to a special node of D, we have $\langle \alpha, \mu \rangle = 1$. For the other simple roots, $\langle \alpha, \mu \rangle = 0$.

1.2.6. In sum, the $G_{\mathbb{C}}(\mathbb{C})$-conjugacy classes of systems (G, X) as in (1.2.2) are parametrized by the special nodes of the Dynkin diagram D of $G_{\mathbb{C}}$. In particular, for a given real form G of $G_{\mathbb{C}}$, X is determined by the corresponding special node $s(X)$ ($G(\mathbb{R}) \subset G_{\mathbb{C}}(\mathbb{C})$ is indeed its own normalizer). The node corresponding to $X^{-1} = \{h^{-1} \mid h \in X\}$ is the transform of $s(X)$ by the opposition involution.

In 1.2.3, G and G^* are inner forms of each other. If there exists an X satisfying 1.2.1(i), (ii), (iii), then G is an inner form of its compact form. In other words, complex conjugation acts on the Dynkin diagram of $G_{\mathbb{C}}$ by the opposition involution.

PROPOSITION 1.2.7. *Let G be an adjoint simple real algebraic group, and suppose that there exist morphisms $h: \mathbb{C}^{\times}/\mathbb{R}^{\times} \to G$ satisfying 1.2.1(i), (ii), (iii). The set of such morphisms has two connected components, interchanged by $h \mapsto h^{-1}$. Each has for stabilizer the identity component $G(\mathbb{R})^{+}$ of G.*

The hypothesis (ii) ensures that the centralizer K of $h(i)$ is a maximal compact subgroup of $G(\mathbb{R})$. In particular, $\pi_0(K) \xrightarrow{\sim} \pi_0 G(\mathbb{R})$. It[25] has the same Lie algebra as the centralizer of h. This last is a connected algebraic group — as the centralizer of a torus — and compact — as a subgroup of the centralizer of $h(i)$. It is therefore topologically connected and $\mathrm{Cent}(h) = K^{+} = K \cap G(\mathbb{R})^{+}$. The centre of K^{+} is of dimension 1: the complexification of K^{+} is the centralizer of μ_h, therefore after (*)$''$, a Levi subgroup of a maximal parabolic subgroup. One can also deduce this from knowing that the representation of K^{+} on $\mathrm{Lie}(G)/\mathrm{Lie}(K^{+})$ is irreducible (cf. [**8**, proof of V, 1.1]). The morphism h is therefore an isomorphism of $\mathbb{S}/\mathbb{G}_m$ with the centre of K^{+}, and K^{+} determines h up to sign. A fortiori, $h(i)$ determines h up to sign. From this we deduce that

(a) the map $h \mapsto h(i)$ is 2 : 1;

(b) it maps the orbit $G(\mathbb{R})^{+}/K^{+}$ of h under $G(\mathbb{R})^{+}$ isomorphically onto the set $G(\mathbb{R})/K$ of all Cartan involutions in $G(\mathbb{R})$.

The proposition follows.

COROLLARY 1.2.8. *Let (G, X) be as in 1.2.1, and let s be the corresponding node of the Dynkin diagram of $G_{\mathbb{C}}$.*

(i) If s is not fixed by the opposition involution, $G(\mathbb{R})$ and X are connected.

(ii) If s is fixed by the opposition involution, $G(\mathbb{R})$ and X have two connected components; the components of X are interchanged by $h \mapsto h^{-1}$ and by the $g \in G(\mathbb{R}) \smallsetminus G(\mathbb{R})^{+}$.

Note that the case (i) is also characterized by the equivalent conditions:
(i$'$) the system of relative roots of G is of type C (rather than BC);
(i$''$) X is a tube domain.

1.3 Symplectic embeddings

1.3.1. Let V be a real vector space together with a nondegenerate alternating form Ψ. The corresponding *Siegel half-space* S^{+} has the following description: it is the

[25] i.e., K

space of complex structures h on V such that Ψ is of type $(1,1)$ (see 1.1.3 for the identification between complex structures and Hodge structures of type $\{(-1,0),(0,-1)\}$) and the form $\Psi(x,h(i)x)$ is symmetric and positive definite.

If we replace "positive definite" with "definite", then the *Siegel double half-space* $S^{\pm}$ obtained is a conjugacy class of morphisms $h\colon \mathbb{S}\to \mathrm{CSp}(V)$ (CSp = symplectic similitudes; in [**5**], this group is denoted by Gp).[26]

1.3.2. Let G be an adjoint real algebraic group (0.2) and X a conjugacy class of morphisms $h\colon \mathbb{S}\to G$. Assume that (G,X) satisfies the conditions (i), (ii) of 1.2.1, and replace (iii) by

(iii′) G has no compact factor.

The system (G,X) is therefore a product of systems (G_i,X_i) as in 1.2.1, and X_i corresponds to a special node of the Dynkin diagram of $G_{i\mathbb{C}}$ (1.2.6).

Consider the diagrams

$$(G,X)\leftarrow (G_1,X_1)\rightarrow (\mathrm{CSp}(V),S^{\pm}),$$

where G is the adjoint group of the reductive group G_1, and where X_1 is a $G_1(\mathbb{R})$-conjugacy class of morphisms from $\mathbb{S}$ into G_1. We have a section $\widetilde{G}\to G_1$, and so V is a representation of $\widetilde{G}$. Our goal is the determination 1.3.8 of the nontrivial irreducible complex representations W occurring in the complexification of a representation thus obtained.[27] This problem was resolved by Satake 1965.

LEMMA 1.3.3. *It suffices that there exist $(G_1,X_1)\to(G,X)$, as above, and a linear representation (V,ρ) of type $\{(-1,0),(0,-1)\}$ of G_1 such that W occurs in $V_{\mathbb{C}}$.*

After replacing G_1 with the subgroup generated by its derived group G_1' and the image of h, we may suppose that $\mathrm{inn}\,h(i)$ is a Cartan involution of $G_1/w(\mathbb{G}_m)$. There then exists on V a polarization Ψ (1.1.18(a))[28] such that ρ is a morphism of (G_1,X_1) into $(\mathrm{CSp}(V),S^{\pm})$.

1.3.4. Consider the following projective system[29] $(H_n)_{n\in N}$: N is ordered by divisibility, $H_n=\mathbb{G}_m$, and the transition morphism from H_{nd} to H_n is $x\mapsto x^d$ ($\varprojlim H_n$ is the universal covering — in the algebraic sense — of $\mathbb{G}_m$). A *fractional morphism of* $\mathbb{G}_m$ into an algebraic group H is an element of $\varinjlim \mathrm{Hom}(H_n,H)$. Similarly, for the group $\mathbb{S}$. For a fractional $\mu\colon\mathbb{G}_m\to H$, defined by $\mu^n\colon H_n=\mathbb{G}_m\to H$, and a linear representation V of H, V is a sum of subspaces V_a ($a\in\frac{1}{n}\mathbb{Z}$) such that, via μ^n, $\mathbb{G}_m$ acts on V_a as multiplication by x^{na}. The a such that $V_a\neq 0$ are the *weights* of μ in V. Similarly, a fractional morphism $h\colon\mathbb{S}\to H$ determines a fractional Hodge decomposition $V^{r,s}$ of V ($r,s\in\mathbb{Q}$).

LEMMA 1.3.5. *For $h\in X$, let $\widetilde{\mu}_h$ be the fractional lifting of μ_h to $\widetilde{G}_{\mathbb{C}}$. The representations W of 1.3.2 are those such that $\widetilde{\mu}_h$ has only two weights a and $a+1$.*

[26] It is usually denoted by GSp.

[27] For more details, see §10 of Milne, J., *Shimura varieties and moduli*, Handbook of Moduli, International Press of Boston, 2013, Vol II, 462–544.

[28] The original has (1.18(a))

[29] Here N is $\mathbb{N}\smallsetminus\{0\}$.

The condition is necessary: Lifting h to $h_1 \in X_1$, we have $\mu_{h_1} = \widetilde{\mu}_h \cdot \nu$ with ν central. On V, μ_{h_1} has weights 0 and 1. If $-a$ is the unique weight of ν on an irreducible subspace W of $V_{\mathbb{C}}$, the only weights of $\widetilde{\mu}_h$ on W are a and $a+1$. For nontrivial W, the action of $\mathbb{G}_m$ via $\widetilde{\mu}_h^n$ (n sufficiently divisible) is nontrivial (because $G_{\mathbb{C}}$ is simple[30]), therefore noncentral, and the two weights a and $a+1$ appear.

The condition is sufficient: Take for V the real vector space underlying W, and for G_1 the group generated by the image of $\widetilde{G}$ and by the group of homotheties. For $h \in X$, with fractional lifting $\widetilde{h}$ to $\widetilde{G}$, let $h_1(z) = h(z)z^{-a}\overline{z}^{1-a}$.[31] If W_a and W_{a+1} are subspaces of weight a and $a+1$ of W, $\overline{h}$ acts on W_a (resp. W_{a+1}) by $(z/\overline{z})^a$ (resp. $(z/\overline{z})^{1+a}$), and h_1 by $\overline{z}$ (resp. z): h_1 is a true morphism from $\mathbb{S}$ into G_1, with projection h onto G, and V is of type $\{(-1,0),(0,-1)\}$ relative to h_1. It remains only to apply 1.3.3.

1.3.6. We translate the condition 1.3.5 in terms of roots. Let T be a maximal torus in $G_{\mathbb{C}}$, $\widetilde{T}$ its inverse image in $\widetilde{G}_{\mathbb{C}}$, B a system of simple roots of T, and $\mu \in Y(T)$ the representative in the fundamental chamber of the conjugacy class of μ_h ($h \in X$). If α is the dominant weight of W, the smallest weight is $-\tau(\alpha)$ for τ the opposition involution. It is a question of expressing that $\langle \mu, \beta \rangle$ takes only two values a and $a+1$ for β a root of W. These weights are all of the form

$$\alpha + \text{a } \mathbb{Z}\text{-linear combination of roots,}$$

and the[32] $\langle \mu, r \rangle$, for r a root, are integers, and so the condition becomes

$$\langle \mu, -\tau(\alpha) \rangle = \langle \mu, \alpha \rangle - 1,$$

or

$$\langle \mu, \alpha + \tau(\alpha) \rangle = 1. \tag{1.3.6.1}$$

We determine the solutions of (1.3.6.1). For every dominant weight α, $\langle \mu, \alpha + \tau(\alpha) \rangle$ is an integer, because $\alpha + \tau(\alpha)$ is a $\mathbb{Z}$-linear combination of roots. If $\alpha \neq 0$, it is > 0, otherwise μ annihilates all the roots of the corresponding representation. A dominant weight α satisfying (1.3.6.1) cannot be the sum of two weights:

LEMMA 1.3.7. *Only the fundamental weights can satisfy 1.3.6.1.*

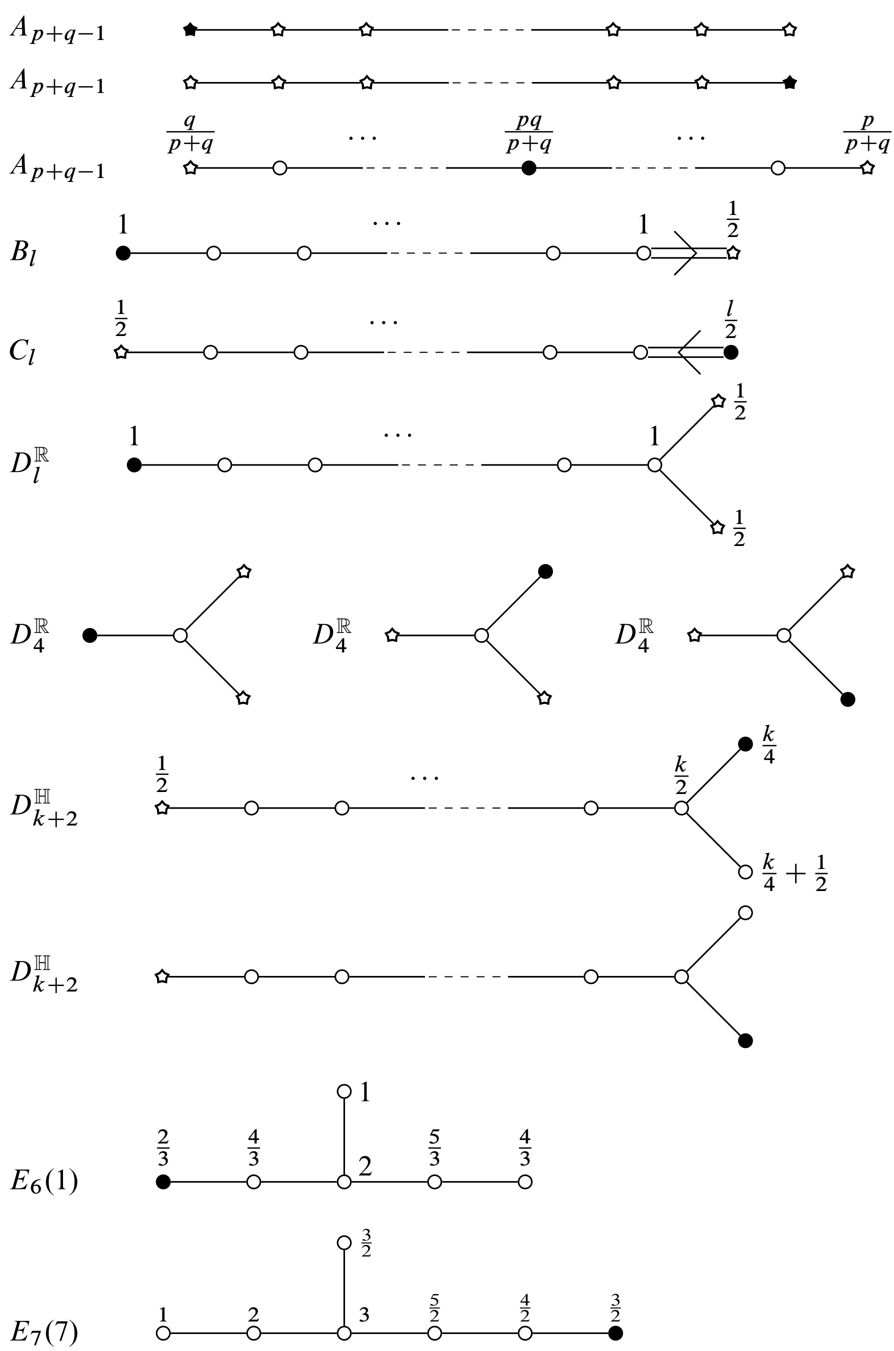
1.3.8. After 1.3.7, every representation sought factors through a simple factor G_i of G, and its dominant weight is a fundamental weight; it corresponds to a node of the Dynkin diagram D_i of $G_{i\mathbb{C}}$. The necessary and sufficient condition (1.3.6.1) depends only on the projection of μ into $G_{i\mathbb{C}}$; the representation corresponds to a special node s of D_i (1.2.6), and s to a simple root α_s. The number $\langle \mu, \omega \rangle$ for ω a weight is the coefficient of α_s in the expression of ω as a $\mathbb{Q}$-linear combination of simple roots. For ω fundamental, these coefficients are given in the tables in Bourbaki. They are given in the following table, which ennumerates the Dynkin diagrams equipped with a special node (filled). Each node corresponds to a fundamental weight ω, and we have given the number $\langle \mu, \omega \rangle$. The nodes corresponding to weights satisfying (1.3.6.1) are starred.

[30] Deligne forgot to require G to be simple.

[31] Should be $h_1(z) = h(z)z^{-a}\overline{z}^{-1-a}$.

[32] The original has $\langle \mu, z \rangle$.

1.3.9. Table:[33],[34] In the table, "$\cdots$" indicates an arithmetic progression. In the third row, $p \neq 1$, $p \neq p+q-1$, and the special node is in the pth position. In $D^{\mathbb{H}}_{k+2}$, $k+2 \geq 5$.



[33]As Deligne has pointed out, the table in the original is incorrect: all vertices in the first row should be underlined when $p=1$. The table included here has been enlarged and corrected.

[34]The filled nodes correspond to hermitian symmetric domains, and the starred nodes correspond to symplectic embeddings.

REMARK 1.3.10. (i) For G simple and exceptional, no representation W satisfies 1.3.2.

(ii) For a classical simple G, except for the case $D_l^{\mathbb{H}}$ ($l \geq 5$), the representations W of 1.3.2 form a faithful system of representations of $\widetilde{G}$. For $D^{\mathbb{H}}$, we obtain only a faithful representation of a double covering of G (namely, of the algebraic connected component of the group of automorphisms of a vector space over $\mathbb{H}$ equipped with a nondegenerate antihermitian form — an inner form of $\mathrm{SO}(2n)$).

2 Shimura varieties

2.0 Preliminaries

2.0.1. Let G be a group, Γ a subgroup, and $\varphi\colon \Gamma \to \Delta$ a morphism. Suppose given an action r of Δ on G, which stabilizes G, and is such that

(a) $r(\varphi(\gamma))$ is the inner automorphism inn_γ of G;

(b) φ is compatible with the actions of Δ on Γ by r and on itself by inner automorphisms: $\varphi(r(\delta)(\gamma)) = \mathrm{inn}_\delta(\varphi(\gamma))$.

Form the semi-direct product $G \rtimes \Delta$. The conditions (a) and (b) amount to saying that the set of $\gamma \cdot \varphi(\gamma)^{-1}$ is a normal subgroup, and we define $G *_\Gamma \Delta$ to be the quotient of $G \rtimes \Delta$ by this subgroup.

Note that the hypotheses imply that $Z = \mathrm{Ker}(\varphi)$ is central in G, and that $\mathrm{Im}(\varphi)$ is a normal subgroup of Δ. The rows of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & \Delta/\Gamma & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & G *_\Gamma \Delta & \longrightarrow & \Delta/\Gamma & \longrightarrow & 0 \end{array} \qquad (2.0.1.1)$$

are exact, and so we get an isomorphism,

$$\Gamma\backslash G \xrightarrow{\sim} \Delta\backslash G *_\Gamma \Delta, \qquad (2.0.1.2)$$

which makes obvious the right action of $G *_\Gamma \Delta$ on $\Gamma\backslash G$. For this action, G acts by right translation and Δ by the right action r^{-1}.

If G is a topological group, Δ is discrete, and the action r is continuous, then the group $G *_\Gamma \Delta$ equipped with the quotient topology of that of $G \rtimes \Delta$ is a topological group, $G/\mathrm{Ker}(\varphi)$ is an open subgroup, and the map (2.0.1.2) is a homeomorphism.

The construction (2.0.1) makes sense also in the category of algebraic groups over a field. If G is a reductive group over k, we have a canonical isomorphism

$G \simeq \widetilde{G} *_{Z(\widetilde{G})} Z(G)$ (for the trivial action of $Z(G)$ on $\widetilde{G}$):[35]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & Z(\widetilde{G}) & \longrightarrow & Z(G) & \longrightarrow & Z(G)/Z(\widetilde{G}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Z & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & Z(G)/Z(\widetilde{G}) & \longrightarrow & 0. \end{array}$$

2.0.2. Let G be an algebraic group over a field k and G^{ad} the quotient of G by its centre Z. The action of G on itself by inner automorphisms

$$(x,y) \mapsto xyx^{-1}: G \times G \to G$$

is invariant under $Z \times \{e\}$ acting by translation, and so factors through an action of G^{ad} on G. Caution: the action of $\gamma \in G^{\mathrm{ad}}(k)$ on $G(k)$ is not necessarily an inner automorphism of $G(k)$ (the projection of $G(k)$ onto $G^{\mathrm{ad}}(k)$ is not always surjective). A typical example is the action of $\mathrm{PGL}(n,k)$ on $\mathrm{SL}(n,k)$.

Similarly, the "commutator" map

$$(x,y) \mapsto xyx^{-1}y^{-1}: G \times G \to G$$

is invariant under $Z \times Z$ acting by translation, and so factors through a "commutator" map $(\,,\,): G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to G$.

All of this, and the fact that "commutators" and "inner automorphisms" satisfy the usual identities is best seen by descent, i.e., by interpreting G as a sheaf of groups on a suitable site and G^{ad} as the quotient of this sheaf of groups by its centre. In characteristic 0, if one is interested only in the points of G over the extensions of k, it suffices to use Galois descent — cf. 2.4.1, 2.4.2.

Variant. For G reductive over k, the groups G and $\widetilde{G}$ have the same adjoint group, and the preceding constructions for G and $\widetilde{G}$ are compatible. In particular, the commutator map $(\,,\,): G \times G \to G$ factors canonically

$$(\,,\,): G \times G \to G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to \widetilde{G} \to G.$$

We deduce that the quotient of $G(k)$ by the normal subgroup $\rho\widetilde{G}(k)$ is abelian.

2.0.3. Let k be a global field of characteristic zero, $\mathbb{A}$ the ring of adèles, G a semisimple group over k, and $N = \mathrm{Ker}(\rho: \widetilde{G} \to G)$. Let S be a finite set of places of k, $\mathbb{A}_S$ the ring of S-adèles (restricted product running over the $v \notin S$), and put $\Gamma_S = \rho\widetilde{G}(\mathbb{A}_S) \cap G(k)$ (intersection in $G(\mathbb{A}_S)$). This is the group of elements of $G(k)$ which, at every place $v \notin S$, can be lifted to $\widetilde{G}(k_v)$ (recall that $\rho: \widetilde{G}(\mathbb{A}) \to G(\mathbb{A})$ is proper).

The long exact cohomology sequence identifies $G(k)/\rho\widetilde{G}(k)$ with a subgroup of $H^1(\mathrm{Gal}(\bar{k}/k), N(\bar{k}))$ and $\Gamma_S/\rho\widetilde{G}(k)$ to the elements that are locally zero, at the places $v \notin S$, of this subroup. In particular, $\Gamma_S/\rho\widetilde{G}(k)$ is contained in the subgroup

[35] Diagram added by the translator. Recall that $\widetilde{G}$ is the universal covering group of G^{der}. For more on this construction, see Section 19d of Milne, J., *Algebraic Groups*, CUP, 2017.

$H^1_{\cdot}(\mathrm{Gal}(\overline{k}/k), N(\overline{k}))$ of classes whose restriction to any monogenic subgroup is trivial (argument and notations of [**12**]). If $\mathrm{Im}(\mathrm{Gal}(\overline{k}/k))$ is the image of Galois in $\mathrm{Aut}(N(\overline{k}))$, we have

$$H^1_{\cdot}(\mathrm{Gal}(\overline{k}/k), N(\overline{k})) = H^1_{\cdot}(\mathrm{Im}(\mathrm{Gal}(\overline{k}/k), N(\overline{k}))$$

(loc. cit.); in particular, $\Gamma_S/\rho\widetilde{G}(k)$ is finite.

PROPOSITION 2.0.4. *(i) The group Γ_S depends only on the set of places $v \in S$ where the decomposition group $D_v \subset \mathrm{Im}\,\mathrm{Gal}(\overline{k}/k)$ is noncyclic. In particular, it does not change when we add to S the places at infinity.*

(ii) The quotient $\Gamma_S/\rho\widetilde{G}(k)$ can be identified with the finite group

$$H^1_{\cdot}(\mathrm{Im}\,\mathrm{Gal}(\overline{k}/k), N(\overline{k}))$$

consisting of the classes with nul restriction to every decomposition subgroup D_v, $v \notin S$. In particular, for S large, we have

$$\Gamma_S/\rho\widetilde{G}(k) = H^1_{\cdot}(\mathrm{Im}\,\mathrm{Gal}(\widetilde{k}/k), N(\overline{k})).$$

The restriction of an element of $H^1_{\cdot}(\mathrm{Im}\,\mathrm{Gal}(\widetilde{k}/k), N(\overline{k}))$ to a cyclic decomposition group is automatically nul, whence (i). For (ii), we may suppose that S contains the places at infinity. The Hasse principle for $\widetilde{G}$ (for classes coming from the centre) then ensures that all the elements of the group (ii) are effectively realized as obstruction classes.

COROLLARY 2.0.5. *Every sufficiently small S-congruence subgroup of $G(k)$ is in Γ_S.*

If U is an S-congruence subgroup, $U/U \cap \rho\widetilde{G}(k)$ is finite: the obstruction to lifting to $\widetilde{G}(k)$ dies in a Galois extension of bounded degree and ramification, and therefore in $H^1(\mathrm{Gal}, N(\overline{k}))$ for Gal a finite quotient of $\mathrm{Gal}(\overline{k}/k)$. The S-congruence conditions then allow us to pass from this H^1 to $\Gamma_S/\rho\widetilde{G}(k)$, cf. [**12**].

REMARK 2.0.6. We note moreover that if $\widetilde{G}$ satisfies the strong approximation theorem relative to S, every S-congruence subgroup $U \subset \Gamma_S$ of $G(k)$ maps onto $\Gamma_S/\rho\widetilde{G}(k)$.

COROLLARY 2.0.7. *For every archimedean place v, a sufficiently small S-congruence subgroup U of $G(k)$ is in the topological connected component $G(k_v)^+$ of $G(k_v)$.*

Since $\widetilde{G}(k_v)$ is connected, we have $G(k_v)^+ = \rho\widetilde{G}(k_v)$ and $U \subset \Gamma_S = \Gamma_{S\cup\{v\}} \subset G(k_v)^+$ (2.0.4 and 2.0.5).

COROLLARY 2.0.8. *The subgroup $G(k)\rho\widetilde{G}(\mathbb{A}_S)$ of $G(\mathbb{A}_S)$ is closed and is topologically isomorphic to $\rho\widetilde{G}(\mathbb{A}_S) *_{\Gamma_S} G(k)$ (i.e., $\rho\widetilde{G}(\mathbb{A}_S)$ is an open subgroup).*

It is a subgroup because, in view of 2.0.2, $\rho\widetilde{G}(\mathbb{A}_S)$ is normal in $G(\mathbb{A}_S)$ with commutative quotient. Let $T \supset S$ be sufficiently large that $\widetilde{G}(k)$ is dense in $\widetilde{G}(\mathbb{A}_T)$ (strong approximation). Let $k_{T \smallsetminus S}$ denote the product of the k_v for $v \in T \smallsetminus S$. For K a compact open subgroup of $G(\mathbb{A}_T)$, we have

$$G(k)\rho\widetilde{G}(\mathbb{A}_S) = G(k)\rho\left(\widetilde{G}(k)\cdot\widetilde{G}(k_{T\smallsetminus S})\times\rho^{-1}K\right) \subset G(k)(\rho\widetilde{G}(k_{T\smallsetminus S})\times K).$$

After 2.0.5, for K sufficiently small,

$$G(k)\cap K \subset \Gamma_T, \quad (\text{in } G(\mathbb{A}_T)),$$

whence,

$$G(k)\cap(\rho\widetilde{G}(k_{T\smallsetminus S})\times K) \subset \Gamma_S \subset \rho\widetilde{G}(\mathbb{A}_S) \quad (\text{in } G(\mathbb{A}_S)).$$

The intersection of $G(k)\rho\widetilde{G}(\mathbb{A}_S)$ with the open subgroup $\rho\widetilde{G}(k_{T\smallsetminus S})\times K$ is therefore contained in $\rho\widetilde{G}(\mathbb{A}_S)$, and the corollary follows.

COROLLARY 2.0.9. *If $\widetilde{G}$ satisfies the strong approximation theorem relative to S, the closure of $G(k)$ in $G(\mathbb{A}_S)$ is $G(k)\cdot\rho\widetilde{G}(\mathbb{A}_S)$.*

2.0.10. Let T be a torus over k and S a finite set of places containing the archimedean places. Let $U \subset T(k)$ be the group of S-units. After a theorem of Chevalley, every subgroup of finite index in U is a congruence subgroup (see [**12**] for an elegant proof). It follows that if $T' \to T$ is an isogeny, the image of a congruence subgroup of T' is a congruence subgroup of T.

2.0.11. Let G be a reductive group of k, $\rho\colon\widetilde{G} \to G^{\mathrm{der}}$ the universal covering of its derived group, and Z° the neutral component of its centre Z. Here are some corollaries of 2.0.10 (we suppose that the finite set S contains the archimedean places).

COROLLARY 2.0.12. *For U of finite index in the group of S-units of $Z(k)$, there exists an open compact subgroup K of $G(\mathbb{A}_S)$ such that*

$$G(k)\cap(K\cdot G^{\mathrm{der}}(\mathbb{A}_S)) \subset G^{\mathrm{der}}(k)\cdot U.$$

We apply 2.0.10 to the isogeny $Z^\circ \to G/G^{\mathrm{der}}$: for K small, an element γ of $G(k)$ in $K\cdot G^{\mathrm{der}}(\mathbb{A}_S)$ has in $(G/G^{\mathrm{der}})(k)$ an image that is small for the topology of S-congruence subgroups, and therefore can be lifted to a small element z of $Z(k)$, and $\gamma = (\gamma z^{-1})\cdot z$.

COROLLARY 2.0.13. *The product of a congruence subgroup of G^{der} and of a subgroup of finite index of the group of S-units of $Z^\circ(k)$ is an S-congruence subgroup of $G(k)$.*

COROLLARY 2.0.14. *Every sufficiently small S-congruence subgroup of $G(k)$ is contained in the neutral topological component $G(\mathbb{R})^+$ of $G(\mathbb{R})$.*

Apply 2.0.13, 2.0.7 to G^{der} and 2.0.10 to Z°.

2.0.15. We know that $G^{\mathrm{der}}(k)\rho\widetilde{G}(\mathbb{A})$ is open in $G(k)\rho\widetilde{G}(\mathbb{A})$ (because it is the the inverse image trivial subgroup $\{e\}$ of the discrete subgroup $\left(G/G^{\mathrm{der}}\right)(k)$ of $(G/G^{\mathrm{der}})(\mathbb{A})$). After 2.0.8, $G(k)\rho\widetilde{G}(\mathbb{A})$ is therefore a closed subgroup of $G(\mathbb{A})$. We put

$$\pi(G) = G(\mathbb{A})/G(k)\rho\widetilde{G}(\mathbb{A}) \tag{2.0.15.1}$$

The existence of commutators 2.0.2 shows that the action of $G^{\mathrm{ad}}(k)$ on $\pi(G)$, deduced from the action 2.0.2 of G^{ad} on G, is trivial.

2.1 Shimura varieties

2.1.1. Let G be a reductive group, defined over $\mathbb{Q}$, and X a $G(\mathbb{R})$-conjugacy class of morphisms of real algebraic groups from $\mathbb{S}$ into $G_{\mathbb{R}}$. We suppose satisfied the following axioms[36] (the notations are those of 1.1.1 and 1.1.11):

2.1.1.1. For $h \in X$, $\mathrm{Lie}(G_{\mathbb{R}})$ is of type $\{(-1,1),(0,0),(1,-1)\}$.

2.1.1.2. The involution $\mathrm{inn}\, h(i)$ is a Cartan involution of the adjoint group $G^{\mathrm{ad}}_{\mathbb{R}}$.

2.1.1.3. The adjoint group admits no factor G' defined over $\mathbb{Q}$ on which the projection of h is trivial.

Axiom[37] 2.1.1.1 ensures that the morphism w_h $(h \in X)$ takes values in the centre of G, and therefore is independent of h. We denote it by w_X, or simply w. Some simplification appear when we suppose that,

2.1.1.4. The morphism $w\colon \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$.

2.1.1.5. $\mathrm{inn}\, h(i)$ is a Cartan involution on the group $(G/w(\mathbb{G}_m))_{\mathbb{R}}$.

After 1.1.14(i), X admits a unique complex structure such that, for every representation V of $G_{\mathbb{R}}$, the Hodge filtration F_h of V varies holomorphically with h. For this complex structure. the connected components of X are hermitian symmetric domains. The proof of 1.1.17 shows also that, when we decompose $G^{\mathrm{ad}}_{\mathbb{R}}$ into simple factors, h projects trivially onto the compact factors, and that each connected component of X is a product of hermitian symmetric spaces corresponding to the noncompact factors. Axiom 2.1.1.3 can be expressed by saying that G^{ad} (equivalently $\widetilde{G}$) has no factor G' (defined over $\mathbb{Q}$) such that $G'(\mathbb{R})$ is compact, and the strong approximation theorem ensures that $\widetilde{G}(\mathbb{Q})$ is dense in $\widetilde{G}(\mathbb{A}^f)$.

2.1.2. The *Shimura varieties* ${}_K M_{\mathbb{C}}(G,X)$ — or simply ${}_K M_{\mathbb{C}}$ — are the quotients

$${}_K M_{\mathbb{C}}(G,X) = G(\mathbb{Q})\backslash X \times \left(G(\mathbb{A}^f)/K\right)$$

for K a compact open subgroup of $G(\mathbb{A}^f)$. After 1.2.7, and with the notation of 0.3, the action of $G(\mathbb{R})$ on X makes $\pi_0(X)$ into a principal homogeneous space under

[36]These are the famous axioms 2.1.1.1–2.1.1.5. A less barbaric numeration would be S1–S5. Note that, when referring to the axioms, Deligne himself often loses track of the number of decimal points.
[37]The original has 2.1.1. Similar errors will sometimes be silently corrected.

$G(\mathbb{R})/G(\mathbb{R})_+$. Since $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ (real approximation theorem), we have $G(\mathbb{Q})/G(\mathbb{Q})_+ \xrightarrow{\sim} G(\mathbb{R})/G(\mathbb{R})_+$, and, if X^+ is a connected component of X, we have

$$_K M_{\mathbb{C}}(G,X) = G(\mathbb{Q})_+ \backslash X^+ \times (G(\mathbb{A}^f)/K).$$

This quotient is a disjoint sum, indexed by the finite set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}^f)/K$ of double coset classes, of the quotients $\Gamma_g \backslash X^+$ of the hermitian symmetric domain X^+ by the images $\Gamma_g \subset G^{\mathrm{ad}}(\mathbb{R})^+$ of the subgroups $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ of $G(\mathbb{Q})_+$. The Γ_g are arithmetic groups, and so we have the structure of an analytic space on $\Gamma_g \backslash X^+$. The article [**2**] provides a natural structure of a quasi-projective algebraic variety on the quotients, and therefore on $_K MG, X)$. If Γ_g is torsion-free (this is the case for K sufficiently small), it follows from [**3**] that this structure is unique. More precisely, for any reduced scheme Z, an analytic morphism from Z into $\Gamma_g \backslash X^+$ is automatically algebraic.

2.1.3. We have

$$\begin{aligned}\pi_0(_K M_{\mathbb{C}}) &= G(\mathbb{Q})\backslash \pi_0(X) \times (G(\mathbb{A}^f)/K)\\ &= G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})_+ \times K\\ &= G(\mathbb{Q})_+ \backslash G(\mathbb{A}^f)/K.\end{aligned}$$

As $G(\mathbb{A}^f)/K$ is discrete, we can replace $G(\mathbb{Q})_+$ by its closure in $G(\mathbb{A}^f)$. The connectedness of $\widetilde{G}(\mathbb{R})$ ensures that $\rho\widetilde{G}(\mathbb{Q}) \subset G(\mathbb{Q})_+$. By the strong approximation theorem for $\widetilde{G}$, $\rho\widetilde{G}(\mathbb{Q})$ is dense in $\rho\widetilde{G}(\mathbb{A}^f)$, and $G(\mathbb{Q})_+^- \supset \rho\widetilde{G}(\mathbb{A}^f)$. From this,

$$\begin{aligned}\pi_0(_K M_{\mathbb{C}}) &= G(\mathbb{Q})_+ \rho\widetilde{G}(\mathbb{A}^f)\backslash G(\mathbb{A}^f)/K\\ &= G(\mathbb{A}^f)/\rho\widetilde{G}(\mathbb{A}^f)\cdot G(\mathbb{Q})_+ \cdot K \qquad (2.1.3.1)\\ &= G(\mathbb{A}^f)/G(\mathbb{Q})_+ \cdot K,\end{aligned}$$

as $\rho\widetilde{G}(\mathbb{A}^f)$ is a normal subgroup with abelian quotient. Similarly, on putting $\overline{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0 G(\mathbb{R})_+$, we have

$$\begin{aligned}\pi_0(_K M_{\mathbb{C}}) &= G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A})\cdot G(\mathbb{Q})\cdot G(\mathbb{R})_+ \times K\\ &= \pi(G)/G(\mathbb{R})_+ \times K \qquad (2.1.3.2)\\ &= \overline{\pi}_0\pi(G)/K.\end{aligned}$$

In particular, $\pi_0(_K M_{\mathbb{C}})$ depends only on the image of K in $G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A})$.

2.1.4. For K variable (smaller and smaller), the $_K M_{\mathbb{C}}$ form a projective system. It is equipped with a right action of $G(\mathbb{A}^f)$: a system of isomorphisms

$$g\colon {}_K M_{\mathbb{C}} \xrightarrow{\sim} {}_{g^{-1}Kg} M_{\mathbb{C}}.$$

It is convenient to consider rather the scheme $M_{\mathbb{C}}((G,X)$ — or simply $M_{\mathbb{C}}$ — equal to the projective limit of the $_K M_{\mathbb{C}}$. The projective limit exists because the transition morphisms are finite. This scheme is equipped with a right action of $G(\mathbb{A}^f)$, and it gives back the ${}_K M_{\mathbb{C}}$:

$$ {}_K M_{\mathbb{C}} = M_{\mathbb{C}}/K. $$

We propose to determine $M_{\mathbb{C}}$ and its decomposition into connected components.

DEFINITION 2.1.5. Fix a connected component X^+ of X. The *neutral component* $M_{\mathbb{C}}^{\circ}$ of $M_{\mathbb{C}}$ is the connected component containing the image of $X^+ \times \{e\} \subset X \times G(\mathbb{A}^f)$.

DEFINITION 2.1.6. Let G_0 be an adjoint group over $\mathbb{Q}$, with no factor G_0' defined over $\mathbb{Q}$ such that $G_0'(\mathbb{R})$ is compact, and let G_1 be a finite covering of G_0. The *topology* $\tau(G_1)$ on $G_0(\mathbb{Q})$ is that admitting as a fundamental system of neighbourhoods of the origin the images of the congruence subgroups of $G_1(\mathbb{Q})$.

We denote by ^(rel. G_1), or simply ^, the completion for this topology. Let $\rho\colon \widetilde{G}_0 \to G_1$ be the natural map, let $^-$ denote closure in $G_1(\mathbb{A}^f)$, and put $\Gamma = \rho \widetilde{G}_0(\mathbb{A}) \cap G_1(\mathbb{Q})$. Since $\widetilde{G}_0(\mathbb{R})$ is connected, $\Gamma \subset G_1(\mathbb{Q})^+$. We have (2.0.9, 2.0.14)

$$ G_0(\mathbb{Q})\hat{}\,(\text{rel.}\, G_1) = G_1(\mathbb{Q})^- *_{G_1(\mathbb{Q})} G_0(\mathbb{Q}) = \rho \widetilde{G}_0(\mathbb{A}^f) *_{\Gamma} G_0(\mathbb{Q}), \quad (2.1.6.1) $$

$$ G_0(\mathbb{Q})^{+}\hat{}\,(\text{rel.}\, G_1) = G_1(\mathbb{Q})_+^- *_{G_1(\mathbb{Q})_+} G_0(\mathbb{Q})^+ = \rho \widetilde{G}_0(\mathbb{A}^f) *_{\Gamma} G_0(\mathbb{Q})^+. \quad (2.1.6.2) $$

PROPOSITION 2.1.7. *The neutral component $M_{\mathbb{C}}^{\circ}$ is the projective limit of the quotients $\Gamma \backslash X^+$ for Γ running over the arithmetic subgroups of $G^{\text{ad}}(\mathbb{Q})^+$ open for the topology $\tau(G^{\text{der}})$.*

After 2.1.2, it is the limit of the $\Gamma \backslash X^{\circ}$ for Γ the image of a congruence subgroup of $G(\mathbb{Q})_+$. Corollary 2.0.13 allows us to replace G with G^{der}.

2.1.8. The projection of G into G^{ad} induces an isomorphism of X^+ with a $G(\mathbb{R})^+$-conjugacy class of morphisms of $\mathbb{S}$ into $G_{\mathbb{R}}^{\text{ad}}$ and, after 2.1.7, $M_{\mathbb{C}}^{\circ}(G,X)$ depends only on G^{ad}, G^{der}, and this class. We formalize this remark. Let G be an adjoint group, X^+ a $G(\mathbb{R})^+$-conjugacy class of morphisms of $\mathbb{S}$ into $G(\mathbb{R})$ satisfying[38] (2.1.1.1), (2.1.1.2), (2.1.1.3) and G_1 a finite covering of G. The *connected Shimura varieties* (rel. G, G_1, X^+) are the quotients $\Gamma \backslash X^+$ for Γ an arithmetic subgroup of $G(\mathbb{Q})^+$ open for the topology $\tau(G_1)$. We let $M_{\mathbb{C}}^{\circ}(G, G_1, X^+)$ denote their projective limit for Γ getting smaller and smaller. Note that the action by transport of structure of $G(\mathbb{Q})^+$ on $M_{\mathbb{C}}^{\circ}(G, G_1, X^+)$ extends by continuity to an action of the completion $G(\mathbb{Q})^{+}\hat{}\,(\text{rel.}\, G_1)$.

With the notation of 2.1.7 and the above identification of X^+ with a $G(\mathbb{R})^+$-conjugacy class of morphisms from $\mathbb{S}$ into $G_{\mathbb{R}}^{\text{ad}}$, we have

$$ M_{\mathbb{C}}^{\circ}(G,X) = M_{\mathbb{C}}^{\circ}(G^{\text{ad}}, G^{\text{der}}, X^+). $$

[38] The original has (2.1.1), (2.1.2), (2.1.3).

2.1.9. Let Z be the centre of G and $Z(\mathbb{Q})^-$ the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}^f)$. After Chevalley (2.0.10), it is the completion of $Z(\mathbb{Q})$ for the topology of subgroups of finite index in the group of units; it receives isomorphically the closure of $Z(\mathbb{Q})$ in $\pi_0 Z(\mathbb{R}) \times Z(\mathbb{A}^f)$.

For $K \subset G(\mathbb{A}^f)$ compact and open, we have $Z(\mathbb{Q}) \cdot K = Z(\mathbb{Q})^- \cdot K$ (in[39] $G(\mathbb{A}^f)$), and

$$\begin{aligned} {}_K M_{\mathbb{C}} &= G(\mathbb{Q}) \backslash X \times \left(G(\mathbb{A}^f / K) \right) \\ &= \frac{G(\mathbb{Q})}{Z(\mathbb{Q})} \Big\backslash X \times \left(G(\mathbb{A}^f)/Z(\mathbb{Q}) \cdot K \right) \\ &= \frac{G(\mathbb{Q})}{Z(\mathbb{Q})} \Big\backslash X \times \left(G(\mathbb{A}^f)/Z(\mathbb{Q})^- \cdot K \right). \end{aligned}$$

The action of $G(\mathbb{Q})/Z(\mathbb{Q})$ on $X \times (G(\mathbb{A}^f)/Z(\mathbb{Q})^-)$ is proper. This allows us to pass to the limit over K:

PROPOSITION 2.1.10. *We have*

$$M_{\mathbb{C}}(G, X) = \frac{G(\mathbb{Q})}{Z(\mathbb{Q})} \Big\backslash X \times \left(G(\mathbb{A}^f)/Z(\mathbb{Q})^- \cdot K \right).$$

COROLLARY 2.1.11. *If the conditions*[40] *2.1.1.4 and 2.1.1.5 are satisfied, then*

$$M_{\mathbb{C}}(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^f).$$

In this case, $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}^f)$ and $Z(\mathbb{Q})^- = Z(\mathbb{Q})$.

COROLLARY 2.1.12. *The right action of $G(\mathbb{A}^f)$ factors through $G(\mathbb{A}^f)/Z(\mathbb{Q})^-$.*

2.1.13. Let $G^{\mathrm{ad}}(\mathbb{R})_1$ denote the image of $G(\mathbb{R})$ in $G^{\mathrm{ad}}(\mathbb{R})$, and let

$$G^{\mathrm{ad}}(\mathbb{Q})_1 = G^{\mathrm{ad}}(\mathbb{Q}) \cap G^{\mathrm{ad}}(\mathbb{R})_1.$$

The action 2.0.2 of G^{ad} on G induces a (left) action of $G^{\mathrm{ad}}(\mathbb{Q})_1$ on the system of the ${}_K M_{\mathbb{C}}$,

$$\mathrm{inn}(\gamma) \colon {}_K M_{\mathbb{C}} \xrightarrow{\sim} {}_{\gamma K \gamma^{-1}} M_{\mathbb{C}},$$

and, in the limit, on $M_{\mathbb{C}}$. For $\gamma \in G^{\mathrm{ad}}(\mathbb{Q})^+$, this action stabilizes the neutral component (therefore all components, cf. later) and it induces the action 2.1.6.

We convert this action into an action on the right, denoted $\cdot\gamma$. If γ is the image of $\delta \in G(\mathbb{Q})$, the action $\cdot\gamma$ coincides with the action of δ viewed as an element of $G(\mathbb{A}^f)$: for $u \in M_{\mathbb{C}}$, equal to the image of $(x, g) \in X \times G(\mathbb{A}^f)$, $u \cdot \gamma$ is the image of

$$(\gamma^{-1}(x), \mathrm{inn}_\gamma^{-1}(g)) = (\delta^{-1}(x), \delta^{-1} g \delta) \sim (x, g\delta) \bmod G(\mathbb{Q}) \text{ at left.}$$

In sum, we obtain thus a right action on $M_{\mathbb{C}}$ of the group

$$\frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})_1 = \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+. \tag{2.1.13.1}$$

[39] The original has $Z(\mathbb{A}^f)$.

[40] The original has 2.1.4…

PROPOSITION 2.1.14. *The right action of $G(\mathbb{A}^f)$ on $\pi_0 M_{\mathbb{C}}$ makes $\pi_0 M_{\mathbb{C}}$ into a principal homogeneous space under its abelian quotient $G(\mathbb{A}^f)/G(\mathbb{Q}_+^-) = \pi_0\pi(G)$.*

This results immediately by passage to the limit in the formulas 2.1.3.

2.1.15. As $G^{\text{ad}}(\mathbb{Q})$ acts trivially on $\pi(G)$ (2.0.15) and $G^{\text{ad}}(\mathbb{Q})^+$stabilizes at least one connected component (2.1.13), the group $G^{\text{ad}}(\mathbb{Q})^+$ stabilizes them all. For the action 2.1.13 of the group (2.1.13.1) on $M_{\mathbb{C}}$, the stabilizer of each connected component is therefore

$$\frac{G(\mathbb{Q})_+^-}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\text{ad}}(\mathbb{Q})^+ \overset{2.0.13}{=} G^{\text{ad}}(\mathbb{Q})^{+\wedge}(\text{rel. } G^{\text{der}}). \tag{2.1.15.1}$$

SUMMARY 2.1.16. *The group*

$$\frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})}{Z(\mathbb{Q})}} G^{\text{ad}}(\mathbb{Q})_1$$

acts on the right on $M_{\mathbb{C}}$. The profinite set $\pi_0 M_{\mathbb{C}}$ is a principal homogeneous space under the action of the abelian quotient

$$\frac{G(\mathbb{A}^f)}{G(\mathbb{Q})_+^-} = \bar{\pi}_0\pi(G)$$

of this group by the closure of $G^{\text{ad}}(\mathbb{Q})^+$. This closure is the completion of $G^{\text{ad}}(\mathbb{Q})^+$ for the topology of the images of the congruence subgroups of $G^{\text{der}}(\mathbb{Q})$. The action of this completion on the neutral component, once converted into a left action, is the action 2.1.8.

2.2 Canonical models

2.2.1. Let G and X be as in 2.1.1.[41] For $h \in X$, the morphism μ_h (1.1.1, completed by 1.1.11) is a morphism over $\mathbb{C}$ of algebraic groups defined over $\mathbb{Q}$: $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$. The *dual field* (= reflex field)[42] $E(G,X) \subset \mathbb{C}$ of (G,X) is the field of definition of its conjugacy class. If X^+ is a connected component of X, we sometimes denote it by $E(G,X^+)$.

Let (G',X') and (G'',X'') be as in 2.1.1. If a morphism $f : G' \to G''$ sends X' into G'', then $E(G',X') \supset E(G'',X'')$.

2.2.2. Let T be a torus, E a number field, and μ a morphism defined over E from $\mathbb{G}_m$ into T_E. The group $E^\times$, viewed as an algebraic group over $\mathbb{Q}$, is the Weil restriction of scalars $R_{E/\mathbb{Q}}(\mathbb{G}_m)$. On applying $R_{E/\mathbb{Q}}$, we obtain

$$R_{E/\mathbb{Q}}(\mu) : E^\times \to R_{E/\mathbb{Q}} T_E.$$

[41] Throughout, Deligne writes "let G and X be as in 2.1.1" to mean that G and X are as in 2.1.1 and satisfy the Axioms 2.1.1.1, 2.1.1.2, and 2.1.1.3 (but not necessarily 2.1.1.4 or 2.1.1.5), i.e., in modern terminology, that (G,X) is a Shimura datum.

[42] Today, "reflex field" is much more common that "dual field".

We have also a norm morphism $N_{E/\mathbb{Q}}: R_{E/\mathbb{Q}} T_E \to T$ (on rational points, it is the norm $T(E) \to T(\mathbb{Q})$), whence by composition a morphism $N_{E/\mathbb{Q}} \circ R_{E/\mathbb{Q}}(\mu): E^\times \to T$, which we denote simply by $NR_E(\mu)$, or even $NR(\mu)$. If E' is an extension of E, then μ is again defined over E', and

$$NR_{E'}(\mu) = NR_E(\mu) \circ N_{E'/E}. \tag{2.2.2.1}$$

2.2.3. In particular, let T be a torus, $h: \mathbb{S} \to T_\mathbb{R}$, and $X = \{h\}$. If $E \subset \mathbb{C}$ contains $E(T, X)$, the morphism μ_h is defined over E, and so we have a morphism $NR(\mu_h): E^\times \to T$. On passing to the adèlic points modulo the rational points, we get a homomorphism of the idèle class group $C(E)$ of E into[43] $T(\mathbb{Q}) \backslash T(\mathbb{A})$, and, by passage to the set of connected components, a morphism

$$\pi_0 NR(\mu_h): \pi_0 C(E) \to \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})).$$

Global class field theory identifies $\pi_0 C(E)$ to the Galois group of E made abelian.

The group $\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}))$ is a profinite group, projective limit of the finite groups $T(\mathbb{Q}) \backslash T(\mathbb{A}) / T(\mathbb{R})^+ \times K$ for K compact open in $T(\mathbb{A}^f)$. It equals

$$\pi_0 T(\mathbb{R}) \times T(\mathbb{A}^f) / T(\mathbb{Q})^-.$$

The Shimura varieties ${}_K M_\mathbb{C}(T, X)$ are the finite sets

$$T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}^f) / K = T(\mathbb{Q}) \backslash T(\mathbb{A}^f) / K.$$

Their projective limit $T(\mathbb{A}^f) / T(\mathbb{Q})^-$, calculated in 2.1.10, is the quotient of $\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}))$ by $\pi_0 T(\mathbb{R})$.

We will call the *reciprocity morphism* the morphism

$$r_E(T, X): \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \to T(\mathbb{A}^f) / T(\mathbb{Q})^-$$

inverse to the composite[44] of the isomorphism of global class field theory (0.8), the morphism of $\pi_0 NR(\mu_h)$, and the projection of $\pi_0 T(\mathbb{A}) / T(\mathbb{Q})$ onto $T(\mathbb{A}^f) / T(\mathbb{Q})^-$. It defines an action r_E of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$ on the ${}_K M_\mathbb{C}(T, X)$:

$$\sigma \mapsto \text{right translation by } r_E(T, X)(\sigma).$$

The universal case (in E) is that where $E = E(T, X)$: it follows from (2.2.2.1) that the action of r_E on $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ is the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(T, X))$ of $r_{E(T,X)}$.

2.2.4. Let G and X be as in 2.1.1. A point $h \in X$ is said to be *special*, or of *CM-type*, if $h: \mathbb{S} \to G(\mathbb{R})$ factors through a torus $T \subset G$ defined over $\mathbb{Q}$.[45] Note that, if T is

[43] Here and elsewhere, the original has $T(\mathbb{Q}) / T(\mathbb{A})$.

[44] Here "inverse" should be "equal". For an explanation of why "inverse" is wrong, see the letter from Milne to Deligne 28.03.90, available on Milne's website (1990b, under articles).

[45] It would be better to preserve the second appellation for the points x that give CM-motives for any finite-dimensional representation of G, or, equivalently, such that there exists a $\mathbb{Q}$-rational homomorphism $\rho: S \to G$ such that $\mu_{\mathrm{can}} \circ \rho_\mathbb{C} = \mu_x$. Here S is the Serre group. Every special point is CM when (G, X) satisfies the following two conditions: (a) the weight w_X is defined over $\mathbb{Q}$; and (b) the centre of G is split by a CM-field. See p. 127 of Milne, Inventiones math., 92 (1988), 91-128.

such a torus, the Cartan involution $\operatorname{inn} h(i)$ is trivial on the image of $T(\mathbb{R})$ in the adjoint group, and so this image is compact. The field $E(T,\{h\})$ depends only on h. It is the *dual field*[46] $E(h)$ of h.

We transport this terminology to the points of ${}_K M_{\mathbb{C}}(G,X)$ and $M_{\mathbb{C}}(G,X)$: for $x \in {}_K M_{\mathbb{C}}(G,X)$ (resp. $M_{\mathbb{C}}(G,X)$), the class of $(h,g) \in X \times G(\mathbb{A}^f)$, the $G(\mathbb{Q})$-conjugacy class of h depends only on x. We will say that x is *special* if h is, that $E(h)$ is the *dual field* $E(x)$ of x, and that the $G(\mathbb{Q})$-conjugacy class of h is the *type* of x.

On the set of special points of ${}_K M_{\mathbb{C}}(G,X)$ (resp. of $M_{\mathbb{C}}(G,X)$) of given type, corresponding to a dual field E, we shall define an action r of $\operatorname{Gal}(\overline{\mathbb{Q}}/E)$. Therefore, let $x \in {}_K M_{\mathbb{C}}(G,X)$ (resp. $M_{\mathbb{C}}(G,X)$) be the class of $(h,g) \in X \times G(\mathbb{A}^f)$, $T \subset G$ a torus defined over $\mathbb{Q}$ through which h factors, $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/E)$, and $\tilde{r}(\sigma)$ a representative in $T(\mathbb{A}^f)$ of $r_E(T,\{h\})(\sigma) \in T(\mathbb{A}^f)/T(\mathbb{Q})^-$. We put $r(\sigma)x$ equal to the class of $(h, \tilde{r}(\sigma)g)$. The reader will check that this class depends only on x and σ. The action thus defined commutes with the right action of $G(\mathbb{A}^f)$ on $M_{\mathbb{C}}(G,X)$.

2.2.5. A *canonical model* $M(G,X)$ of $M_{\mathbb{C}}(G,X)$ is a form over $E(G,X)$ of $M_{\mathbb{C}}(G,X)$, equipped with a right action of $G(\mathbb{A}^f)$, such that

(a) the special ponts are algebraic;

(b) on the set of special points of a given type τ, corresponding to the dual field $E(\tau)$, the Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}/E(\tau)) \subset \operatorname{Gal}(\overline{\mathbb{Q}}/E(G,X))$ acts through the action 2.2.4.

By "form" we mean a scheme M over $E(G,X)$ equipped with a right action of $G(\mathbb{A}^f)$ and an equivariant isomorphism $M \otimes_{E(G,X)} \mathbb{C} \xrightarrow{\sim} M_{\mathbb{C}}(G,X)$.

Let $E \subset \mathbb{C}$ be a number field that contains $E(G,X)$. A *weakly canonical model* of $M_{\mathbb{C}}(G,X)$ over E is a form over E of $M_{\mathbb{C}}(G,X)$ equipped with a right action of $G(\mathbb{A}^f)$ that satisfies (a) and

(b*) same condition as (b) with $\operatorname{Gal}(\overline{\mathbb{Q}}/E(\tau))$ replace by $\operatorname{Gal}(\overline{\mathbb{Q}}/E(\tau)) \cap \operatorname{Gal}(\overline{\mathbb{Q}}/E)$.

2.2.6. In [**5**, 5.4, 5.5], inspired by the methods of Shimura, we have shown that $M_{\mathbb{C}}(G,X)$ admits at most one weakly canonical model over E (for $E(G,X) \subset E \subset \mathbb{C}$), and that, when it exists, it is functorial in (G,X).

2.3 Construction of canonical models

In this number, we determine the cases where the following criterion, which was proved in [**5**, 4.2, 5.7], applies to construct canonical models.

CRITERION 2.3.1. Let (G,X) be as in (2.1.1), let V be a rational vector space endowed with a nondegenerate alternating form Ψ, and let $S^{\pm}$ be the corresponding Siegel doublespace (cf. 1.3.1). If there exists an embedding $G \hookrightarrow \operatorname{CSp}(V)$ sending X into $S^{\pm}$, then $M_{\mathbb{C}}(G,X)$ admits a canonical model $M(G,X)$.[47]

[46] Better, reflex field.

[47] Shimura varieties satisfying this criterion are now said to be of Hodge type. As Deligne notes (end of the Introduction above), they are moduli varieties for abelian varieties equipped with a collection of absolute Hodge classes. As the moduli problem is defined over the reflex field, this gives a more natural proof of the existence of canonical models in this case than that in the text.

PROPOSITION 2.3.2. *Let (G,X) be as in 2.1.1, let $w = w_h$ ($h \in X$), and let (V,ρ) be a faithful representation of type $\{(-1,0),(0,-1)\}$ of G. If $\operatorname{inn} h(i)$ is a Cartan involution of $G_{\mathbb{R}}/w(\mathbb{G}_m)$, then there exists an alternating form Ψ on V such that ρ induces $(G,X) \hookrightarrow (\mathrm{CSp}(V), S^{\pm})$.*

By hypothesis, the faithful representation V is homogeneous of weight -1. The weight w is defined over $\mathbb{Q}$, and one takes for Ψ the polarization form as in 1.1.18(b).

COROLLARY 2.3.3. *Let (G,X) be as in 2.1.1, let $w = w_h$ ($h \in X$), and let (V,ρ) be a faithful representation of type $\{(-1,0),(0,-1)\}$ of G. If the centre Z° of G is split by a CM-field, then there exists an algebraic subgroup G_2 of G with the same derived group through which X factors, and an alternating form Ψ on V such that ρ induces $(G_2,X) \hookrightarrow (\mathrm{CSp}(V), S^{\pm})$*

The hypothesis on Z° amounts to saying that the largest compact torus of $Z^\circ_{\mathbb{R}}$ is defined over $\mathbb{Q}$. Take G_2 to be the algebraic subgroup of G generated by its derived group, this torus, and the image of w, and apply 2.3.2.

2.3.4. Let (G,X) be as in 2.1.1 with G adjoint and $\mathbb{Q}$-simple. Axiom (2.1.1.2) ensures that $G_{\mathbb{R}}$ is *an inner form of its compact form*. We exploit this fact.

(a) The simple components of $G_{\mathbb{R}}$ are absolutely simple. If we write G as a Weil restriction of scalars, $G = R_{F/\mathbb{Q}} G^s$ with G^s absolutely simple over F, then this implies that F is totally real. Introduce the notation,

$$I = \text{the set of real embeddings of } F,$$

and, for $v \in I$,

$$\begin{cases} G_v = G^s \otimes_{F,v} \mathbb{R} \\ D_v = \text{Dynkin diagram of } G_{v\mathbb{C}}. \end{cases}$$

Then $G_{\mathbb{R}} = \prod_{v \in I} G_v$, $G_{\mathbb{C}} = \prod_{v \in I} G_{v\mathbb{C}}$, and the Dynkin diagram D of $G_{\mathbb{C}}$ is the disjoint union of the D_v. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on D and I, and these actions are compatible with the projection of D onto I.

(b) Complex conjugation acts on D by the opposition involution. This is central in $\mathrm{Aut}(D)$. It follows that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on D via a faithful action of $\mathrm{Gal}(K_D/\mathbb{Q})$ with K_D totally real if the opposition involution is trivial and totally imaginary quadratic over a totally real field otherwise.

2.3.5. We have $X = \prod_{v \in I} X_v$ with X_v a $G_v(\mathbb{R})$-conjugacy class of morphisms from $\mathbb{S} \to G_v$. When G_v is compact, X_v is trivial. When G_v is noncompact, X_v is described by a node s_v of the Dynkin diagram D_v of $G_{v\mathbb{C}}$ (1.2.6).

Some notation: I_c = the set of $v \in I$ such that G_v is compact, $I_{nc} = I \smallsetminus I_c$, D_c (resp. D_{nc}) = the union of the D_v for $v \in I_c$ (resp. $v \in I_{nc}$), G_c (resp. G_{nc}) = the product of the G_v for $v \in I_c$ (resp. $v \in I_{nc}$); similarly for the universal coverings; finally, $\Sigma(X)$ = the set of s_v for $v \in I_{nc}$. The definition 2.2.1 gives:

PROPOSITION 2.3.6. *The dual field of (G,X) is the subfield of K_D fixed by the subgroup of $\mathrm{Gal}(K_D/\mathbb{Q})$ which stabilizes $\Sigma(X)$.*

2.3.7. Suppose that there exists a diagram

$$(G,X) \leftarrow (G_1,X_1) \hookrightarrow (\mathrm{CSp}(V),S^{\pm}). \tag{2.3.7.1}$$

The universal covering $\widetilde{G} \to G$ lifts to a map $\widetilde{G} \to G_1$, and so the representation of G_1 on V defines a representation of $\widetilde{G}$ on V. The quotient of $\widetilde{G}$ that acts faithfully is by hypotheses the derived group of G_1. On applying 1.3.2, 1.3.8 to the diagram

$$(G_{nc},X) \leftarrow (\mathrm{Ker}(G_{1\mathbb{R}} \to G_c)^{\circ},X_1) \hookrightarrow (\mathrm{CSp}(V),S^{\pm}),$$

we find that the nontrivial irreducible components of the representation $V_{\mathbb{C}}$ of $\widetilde{G}_{n\mathbb{C}}$ factor through one of the $\widetilde{G}_{v\mathbb{C}}$ ($v \in I_{nc}$), and that their dominant weights are fundamental and of one of the types permitted by Table 1.3.9. The set of dominant weights of the irreducible components of the representation $V_{\mathbb{C}}$ of $\widetilde{G}_{\mathbb{C}}$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on I, and $I_{nc} \neq \emptyset$, we find that,

(a) Every irreducible component W of $V_{\mathbb{C}}$ is of the form $\bigotimes_{v\in T} W_v$ with W_v a fundamental representation of $G_{v\mathbb{C}}$ ($v \in T \subset I$) corresponding to a node $\tau(v)$ of D_v. We denote by $\mathcal{S}(V)$ the set of $\tau(T) \subset D$ for $W \subset V_{\mathbb{C}}$ irreducible.

(b) If $S \in \mathcal{S}(V)$, then $S \cap D_{nc}$ is empty or reduced to a single point $s_S \in D_v$ ($v \in I_{nc}$) and, in the Table 1.3.9 for (D_v, s_v), s_S is one of the starred nodes.

(c) $\mathcal{S}$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathcal{S} \not\subset \{\emptyset\}$.

If one of the subsets $\mathcal{S}$ of D satisfies (b) and (c), we let $\widetilde{G}(\mathcal{S})_{\mathbb{C}}$ denote the quotient of $\widetilde{G}_{\mathbb{C}}$ which acts faithfully in the corresponding representation of $\widetilde{G}_{\mathbb{C}}$. The condition (c) ensures that it is defined over $\mathbb{Q}$. The most interesting case is where

(d) $\mathcal{S}$ is formed of one-element sets.

If $\mathcal{S}$ satisfies (b), (c), the set $\mathcal{S}'$ of $\{s\}$ for $s \in S \in \mathcal{S}$ satisfies (b), (c), (d) and $\widetilde{G}(\mathcal{S}')$ dominates $\widetilde{G}(\mathcal{S})$.

In the table below — deduced from 1.3.9 — we give,

A list of the cases where there exists an $\mathcal{S}$ satisfying (b), (c). After 1.3.10, this can happen only if G is one of the types A,B,C,D, and we successively review each of these types.

The maximal set $\mathcal{S}$ satisfying (b), (c), (d), and the corresponding group $\widetilde{G}(\mathcal{S})$ (it dominates all the $\widetilde{G}(\mathcal{S})$ for $\mathcal{S}$ satisfying (b), (c)).

2.3.8. TABLE

Types A,B,C. The only $\mathcal{S}$ satisfying (b), (c), (d) is the set of $\{s\}$ for s an endpoint — corresponding to a short root for the types B,C — of a diagram D_v ($v \in I$). The covering $\widetilde{G}(\mathcal{S})$ is the universal covering.

Type D_l ($l \geq 5$). In order for there to exist an $\mathcal{S}$ satisfying (b), (c), it is necessary and sufficient that (G_v, X_v) ($v \in I_{nc}$) all be of type $D_l^{\mathbb{R}}$ or all of type $D_l^{\mathbb{H}}$. Distinguish these cases:

Subcase $D_l^{\mathbb{R}}$. The maximal $\mathcal{S}$ satisfying (b), (c), (d) maximal is the set of $\{s\}$ for s a "right" endpoint of a D_v. The covering $\widetilde{G}(\mathcal{S})$ is the universal covering.

Subcase $D_l^{\mathbb{H}}$. The unique $\mathcal{S}$ satisfying (b), (c), (d) is the set of $\{s\}$ for s the "left" endpoint of a D_v. The covering $\widetilde{G}(\mathcal{S})$ of G is of the form $R_{F/\mathbb{Q}}\widetilde{G}^*$ for $\widetilde{G}^*$ the double covering of G which is a form $\mathrm{SO}(2l)$, cf. 1.3.10.

Type D_4. Replace $\mathcal{S}$ satisfying (d) with $S = \{s \mid \{s\} \in \mathcal{S}\}$. The condition (b) on $\mathcal{S}$ becomes: S is contained in the set E of endpoints of D, and $S \cap \Sigma(X) = \emptyset$. For the definition of $\Sigma(X)$, see 2.3.5. The maximal subset of E stable under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ for this property is the complement of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \Sigma(X)$. It meets each D_v in 0, 1, or 2 points. In the first case, there does not exist an $\mathcal{S}$ satisfying (b), (c). In the second (resp. third), it (resp. its complement) is the image of a section τ of $X \to I$, invariant under the Galois action; $\tau(I)$ is disjoint (resp. contains) $\Sigma(X)$. Calling $\tau(v)$ the "left" node of D_v, one refinds the situation of D_l $(l \leq 5)$

Subcase $D_4^{\mathbb{R}}$. There exists a section τ to $X \to I$ with $\tau(I) \supset \Sigma(X)$. This section is then unique, and the situation is the same as that of $D_l^{\mathbb{R}}$, $l \geq 5$.

Subcase $D_4^{\mathbb{H}}$. There is a section τ to $X \to I$ with $\tau(I) \cap \Sigma(X) = \emptyset$. If we are not in the case $D_4^{\mathbb{R}}$, this section is unique, and the unique $\mathcal{S}$ satisfying (b), (c), (d) is the set of $\{s\}$ for s the "left" endpoint of a D_v. The covering $\widetilde{G}(\mathcal{S})$ of G is of the form $R_{F/\mathbb{Q}}\widetilde{G}^s$ for $\widetilde{G}^s$ a double covering of G^s which can be described in terms of τ.

For the rest of this article, it will be convenient to redefine the case $D_4^{\mathbb{H}}$ to exclude $D_4^{\mathbb{R}}$. With this terminology, there exists an $\mathcal{S}$ satisfying (b), (c) if and only if (G,X) is one of the types A, B, C, $D^{\mathbb{R}}$, $D^{\mathbb{H}}$ and, except for type $D^{\mathbb{H}}$, there exists an $\mathcal{S}$ satisfying (b), (c), (d) such that $\widetilde{G}(\mathcal{S})$ is the universal covering of G.

2.3.9. We wish to consider totally imaginary quadratic extensions K of F endowed with a set T of complex embedding, one above each real embedding $v \in I_c$. Such a T defines a Hodge structure $h_T : \mathbb{S} \to K_{\mathbb{R}}^{\times}$ on K (considered as $\mathbb{Q}$-vector space, on which $K^{\times}$ acts by multiplication): if J is the set of complex embeddings of K, we have $K \otimes \mathbb{C} \simeq \mathbb{C}^J$, and we define h_T by requiring that the factor with index $\sigma \in J$ be of type $(-1,0)$ for $\sigma \in T$, $(0,-1)$ for $\overline{\sigma} \in T$, and $(0,0)$ if σ is above I_{nc}. The main result of this number is the following.

PROPOSITION 2.3.10. *Let (G,X) be as in 2.1.1 with G simple and adjoint, and one of the types A, B, C, $D^{\mathbb{R}}$, $D^{\mathbb{H}}$. For any totally imaginary quadratic extension K of F, endowed with a T as in 2.3.9, there exists a diagram*

$$(G,X) \leftarrow (G_1,X_1) \hookrightarrow (\mathrm{CSp}(V), S^{\pm})$$

for which

(i) $E(G_1,X_1)$ is the composite of $E(G,X)$ and of $E(K^{\times},h_T)$;

(ii) The derived group G_1' is simply connected for G of type A, B, C, $D^{\mathbb{R}}$, and the covering of G described in 2.3.8 for the type $D^{\mathbb{H}}$.

Let S be the largest set of nodes of the Dynkin diagram D of $G_{\mathbb{C}}$ such that $\{\{s\} \mid s \in S\}$ satisfies 2.3.7(b), (c). We have determined them in 2.3.8. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on S, and we can identify S with $\mathrm{Hom}(K_S,\mathbb{C})$ for K_S a suitable product of extensions of $\mathbb{Q}$, isomorphic to a subfield of K_D since $\mathrm{Gal}(\overline{\mathbb{Q}}/K_D)$ acts trivially on D, therefore on S. In particular, K_S is a product of totally real or CM fields. The projection $S \to I$ corresponds to a homomorphism $F \to K_S$.

For $s \in S$, let $V(s)$ be the complex representation of $\widetilde{G}_{\mathbb{C}}$ with dominant weight the fundamental weight corresponding to s. The isomorphism class of the representation

$\bigoplus_{s \in S} V(s)$ is defined over $\mathbb{Q}$. This does not suffice to show that it is defined over $\mathbb{Q}$; the obstruction lies in a Brauer group. However, a multiple of this representation is always defined over $\mathbb{Q}$. Therefore, let V be a representation of $\widetilde{G}$ over $\mathbb{Q}$ with $V_{\mathbb{C}} \sim \bigoplus_{s \in S} V(s)^n$ for some n. We let V_s denote the unique factor of $V_{\mathbb{C}}$ isomorphic to $V(s)^n$. These factors are permuted by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in a fashion compatible with the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on S, and the decomposition $V_{\mathbb{C}} \sim \bigoplus_{s \in S} V_s$ corresponds therefore to a structure of a K_S-module on V: on V_s, K_S acts as multiplication by the corresponding homomorphism from K_S into $\mathbb{C}$.

Let $\widetilde{G}'$ denote the quotient of $\overline{G}$ that acts faithfully on V. It is the covering of G considered in (ii).

Let $h \in X$, and lift h to a fractional morphism (1.3.4) of $\mathbb{S}$ into $\widetilde{G}'_{\mathbb{R}}$. We deduce a fractional Hodge structure on V, of weight 0. Let $s \in S$, and let v be its image in I. The type of the decomposition of V_s is given by the Table 1.3.9:

(a) if $v \in I_c$, V_s is of type $(0,0)$;

(b) if $v \in I_{nc}$, V_s is of type $\{(r,-r),(r-1,1-r)\}$ where r is given by 1.3.9: it is the number which marks the node s of D_v equipped with the special node defining X_v.

We define a Hodge structure h_2 on V, by keeping V_s of type $(0,0)$ for $v \in I_c$ and, for $v \in I_{nc}$, renaming the subset of type $(r,-r)$ (resp. $(r-1,1-r)$) of V_s as being of type $(0,-1)$ (resp. $(-1,0)$). If G_2 is the algebraic subgroup of GL_V generated by $\widetilde{G}'$ and $K_S^{\times}$, the Hodge structure h_2 is a morphism $\mathbb{S} \to G_{2\mathbb{R}}$. Denoting by X_2 its $G_2(\mathbb{R})$-conjugacy class, we have a map $(G_2, X_2) \to (G, X)$ and $E(G_2, X_2) = E(G, X)$.

Equip $K \otimes_F V$ with the Hodge structure h that is the tensor product of the Hodge structures of V and K (2.3.9). We have

$$(K \otimes_F V) \otimes \mathbb{R} = \bigoplus_{v \in I} (K \otimes_{F,v} \mathbb{R}) \otimes_{\mathbb{R}} (V \otimes_{F,v} \mathbb{R}).$$

This decomposition is compatible with the Hodge structure, and on the factor corresponding to $v \in I_c$ (resp. $v \in I_{nc}$), the Hodge structure is the tensor product of a structure of type $\{(-1,0),(0,-1)\}$ on $K \otimes_{F,v} \mathbb{R}$ (resp. $V \otimes_{F,v} \mathbb{R}$) by one of type $\{(0,0)\}$ on $V \otimes_{F,v} \mathbb{R}$ (resp. $K \otimes_{F,v} \mathbb{R}$). In sum, h_3 is of type $\{(-1,0),(0,-1)\}$. If G_3 is the algebraic subgroup of $\mathrm{GL}(K \otimes_F V)$ generated by $K^{\times}$and G_2, then the Hodge structure h_3 is a morphism $\mathbb{S} \to G_{3\mathbb{R}}$.

If X_3 is the conjugacy class of h_3, then we have a morphism $(G_3, X_3) \to (G, X)$. The derived group of G_3 is $\widetilde{G}'$, and $E(G_3, X_3)$ is the composite of $E(G_2, X_2) = E(G, X)$ and of $E(K^{\times}, h_T)$. In order to obtain the (G_1, X_1) sought, it remains only to apply 2.3.3 to (G_3, X_3) and its faithful linear representation on $K \otimes_F V$.

REMARK 2.3.11. The construction given generalizes to furnish a diagram (2.3.7.1) where $\mathcal{S}(V)$ is any set of subsets of D satisfying 2.3.7(b),(c). Here are the main steps:

(a) if $\mathcal{S}$ satisfies 2.3.7(b), (c), define $K_{\mathcal{S}}$ by $\mathrm{Hom}(K_{\mathcal{S}}, \mathbb{C}) = \mathcal{S}$; construct a representation V of $\widetilde{G}$ such that $\mathcal{S}(V) = \mathcal{S}$; the decomposition $V_{\mathbb{C}} = \bigoplus_{S \in \mathcal{S}} V_S$ provides V with the structure of a $K_{\mathcal{S}}$-module;

(b) the fractional Hodge structure of V_S is of type $(0,0)$ for S above I_c, and of type $\{(r,-r),(r-1,1-r)\}$ with r described — as above — by the point of S above I_{nc} otherwise;

(c) convert $\{(r,-r),(r-1,1-r)\}$ to $\{(0,-1),(-1,0)\}$ as above;

(d) in order to convert $(0,0)$ into $\{(-1,0),(0,-1)\}$, tensor with V over K_S with K'_S of CM-type endowed with a suitable Hodge structure h.

By this method, we obtain for (G_1,X_1) a derived group $\widetilde{G}(\mathcal{S})$ and a dual field, the composite of $E(G,X)$ and $E(K'^{\times}_S,h)$. Note that, even for $\mathcal{S}$ satisfying (b), (c), (d), the conversion indicated by $(0,0)$ is more general than that of 2.3.10.

REMARK 2.3.12. For the types A, with $\Sigma(X)$ fixed by the opposition involution, and B, C, and $D^{\mathbb{R}}$, the dual field $E(G,X)$ is the (totally real) subfield of K_D fixed by the subgroup of $\mathrm{Gal}(K_D/\mathbb{Q})$ stabilizing I_c. If $I_c=\emptyset$, it is $\mathbb{Q}$. If I_c (resp. I_{nc}) consists of a single element v, it is $v(F)$. The fields $E(K,h_T)$ are extensions of $E(G,X)$.

REMARK 2.3.13. For these types and $D^{\mathbb{H}}$, the V_s of 2.3.10 for $v\in I_{nc}$, are of type $\left\{(-\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right\}$. This allows us, in 2.3.10, to replace G_2 by the subgroup of GL_V generated by $F^{\times}$ and $\widetilde{G}'$. If $I_c=\emptyset$, we can even replace it by the subgroup of GL_V generated by $\mathbb{Q}^{\times}$ and $\widetilde{G}'$, and the criterion 2.3.2 applies directly to this group, whence we get a (G_1,X_1) with $E(G_1,X_1)=E(G,X)$ ($=\mathbb{Q}$ except in the case $D^{\mathbb{H}}$).

2.4 The reciprocity laws: preliminaries

The constructions of this number will allow us, in number 2.6, to calculate the reciprocity law for the canonical models, i.e., the action of the Galois group on the set of geometric connected components.

While they are best expressed in the language of *fppf* descent, we have expressed them here in terms of Galois descent, believing it to be more familiar to nongeometers. This gives rise to some repetitions and inconsistencies, and introduced some parasitic hypotheses on separability or of characteristic zero.

Let G be a reductive group over a global field k. With the notation of 2.0.15, our purpose is to construct canonical morphisms of the following two types.

(a) For k' a finite extension (which we will suppose separable) of k and G' deduced from G by extension of scalars to k', a norm map

$$N_{k'/k}\colon \pi(G')\to\pi(G). \tag{2.4.0.1}$$

(b) For a torus T and M a conjugacy class, defined over k, of morphisms from T into G, a morphism

$$q_M\colon \pi(T)\to\pi(G) \tag{2.4.0.2}$$

If $m\in M(k)$, q_M will be the morphism induced by m; the problem is to show that this morphism does not depend on the choice of m, and to construct q_M even if M has no representative defined over k.

The functoriality properties of these morphisms will be obvious from their definitions.

2.4.1. We will systematically use the language of torsors (which I prefer to that of cocycles), and of Galois descent, under the form given to it by Grothendieck (cf. SGA 1, of SGA $4\frac{1}{2}$[Arcata]).[48]

Galois descent. Let K be a finite separable extension of a field k. To construct an object X over k (for example a torsor), it suffices to construct

(a) for each separable extension k' of k such that there exists a morphism of the k-algebra K into k', an object $X_{k'}$ over k';

(b) for k'' an extension of k', an isomorphism $\chi_{k'',k'}\colon X_{k'} \otimes k'' \xrightarrow{\sim} X_{k''}$;

(c) and to check a compatibility $\chi_{k''',k''} \circ \chi_{k'',k'} = \chi_{k''',k'}$.[49]

In practice, this signifies that to construct X, we may suppose to exist auxiliary objects that exist only over a separable extension K of k — provided we show that the X constructed does not depend — up to a unique isomorphism — on the choice of the auxiliary object.

REMARK 1. Galois descent is a special case of localization in the étale topology; a construction as in (a), (b), (c) above will be sometimes introduced by the adverb "locally".

EXAMPLE 2.4.2. We explain the canonical lifting used in 2.0.2 of the commutator map. The usage of Galois descent — rather than *fppf* — requires us to assume that the projection of G onto G^{ad} is smooth, and to consider only $(\ ,\ )\colon G^{\mathrm{ad}}(k) \times G^{\mathrm{ad}}(k) \to G(k)$, rather than the morphism $G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to G$. If $x_1, x_2 \in G^{\mathrm{ad}}(k)$, we can, locally, write $x_i = \rho(\widetilde{x}_i)z_i$ with z_i in the centre of G. The element $\widetilde{x}_i$ is unique up to multiplication by an element of the centre of $\widetilde{G}$. The commutator of $\widetilde{x}_1$ and $\widetilde{x}_2$ does not depend on the choice of the $\widetilde{x}_i$, and we put $(x_1, x_2) = \widetilde{x}_1\widetilde{x}_2\widetilde{x}_1^{-1}\widetilde{x}_2^{-1}$.

2.4.3. For an algebraic group G over a field k, a G-*torsor* is a scheme P over k, equipped with a *right* action of G that makes it a principal homogeneous space. The *trivial* G-torsor G_r is G equipped with the action of G by right translation. The points $x \in P(k)$ can be identified with the *trivializations* of P (isomorphisms $\varphi\colon G_d \xrightarrow{\sim} P$) by $\varphi(g) = xg$.

If $f\colon G_1 \to G_2$ is a morphism and P a G_1-torsor, then there exists a G_2-torsor $f(P)$ equipped with $f\colon P \to f(P)$ satisfying $f(pg) = f(p)f(g)$, and it is unique up to a unique isomorphism. We are interested in the category $[G_1 \to G_2]$ of G_1-torsors P equipped with a trivilisation $f(P)$. For morphisms, we take isomorphisms of G_1-torsors compatible with the G_2-trivializations. We let $\mathbb{H}^0(G_1 \to G_2)$ denote the group of automorphisms of (G_{1r}, e) (it is $\mathrm{Ker}(G_1(k) \to G_2(k))$ and[50] $\mathbb{H}^1(G_1 \to G_2)$ the set (pointed by (G_{1r}, e)) of isomorphism classes of objects.

Each $x \in G_2(k)$ defines an object $[x]$ of $[G_1 \to G_2]$: the trivial G_1-torsor G_{1r} equipped with the trivilization $x \in f(G_{1r}) = G_{2r}$. When it does not cause confusion,

[48] The form given by Galois is sometimes easier: if k'/k is a Galois extension, then, roughly speaking, to give an object over k is the same as giving an object over k' together with an action of the Galois group. See 2.7.5 for a precise statement.

[49] Misprints fixed.

[50] The original typed manuscript has $\mathbb{H}$, which was typeset as boldface math.

we denote it simply by x. The set of morphisms of $[x]$ into $[y]$ can be identified with $\{g \in G_1(k) \mid f(g)x = y\}$: to g attach $u \mapsto gu: G_{1r} \to G_{1r}$. An object is of the form $[x]$ if and only if, as a G_1-torsor, it is trivial — from this we get an exact sequence

$$1 \to \mathbb{H}^0(G_1 \to G_2) \to G_1(k) \to G_2(k) \to \mathbb{H}^1(G_1 \to G_2) \to H^1(G_1) \to H^1(G_2) \tag{2.4.3.1}$$

(this sequence does not describe the inverse image of $p \in H^1(G_1)$; to describe it, it is necessary to proceed by twisting, as in [**13**]).

2.4.4. If f is an epimorphism, with kernel K, to give a G_1-torsor P G_2-trivialized by $x \in f(P)(k)$ is the same as giving the K-torsor $f^{-1}(x) \subset P$: the natural functor $[K \to \{e\}] \to [G_1 \to G_2]$ is an equivalence.

More generally, if $g: G_2 \to H$ induces an epimorphism from G_1 onto H, and $K_i = \mathrm{Ker}(G_i \to H)$, then the natural functor is an equivalence

$$[K_1 \to K_2] \to [G_1 \to G_2].$$

2.4.5. If G is commutative, the sum $s: G \times G \to G$ is a morphism, and we define the sum of two G-torsors by $P + Q = s(P \times Q)$. If G_1 and G_2 are commutative, we can add similarly the objects of $[G_1 \to G_2]$, which becomes a (strictly commutative) *Picard category* (SGA 4, XVIII, 1.4).

All of the above is valid for sheaves of groups on an arbitrary topos.

2.4.6. If k' is a finite extension of k (the case where k'/k is separable suffices for us) and G' is an algebraic group over k', Weil restriction of scalars $R_{k'/k}$ is an equivalence from the category of G'-torsors to that of $R_{k'/k}G'$-torsors. This corresponds to the Shapiro lemma $H^1(k', G') = H^1(k, R_{k'/k}G)$. If G' is deduced by extension of scalars from a commutative G — over k we have a trace morphism $R_{k'/k}G' \to G$ — whence a trace functor $\mathrm{Tr}_{k'/k}$ from G'-torsors to G-torsors. More generally, for $G_1 \to G_2$ a morphism commutative groups, we obtain an additive functor

$$\mathrm{Tr}_{k'/k}: [G_1' \to G_2'] \to [G_1 \to G_2]. \tag{2.4.6.1}$$

Such functors are described in great generality in [SGA 4, XVII, 6.3]. For k'/k separable, it is possible to give a simple definition by descent: locally k' is the sum of $[k':k]$ copies of k, $[G_1' \to G_2']$ can be identified with the category of $[k':k]$-tuples of objects in $[G_1 \to G_2]$, and $\mathrm{Tr}_{k'/k}$ is the sum.

When the groups are written multiplicatively, we speak rather of the *norm* functor $N_{k'/k}$.

2.4.7. Let G be a reductive group (0.2) over k and $\rho: \tilde{G} \to G$ the universal covering of its derived group. The particular case of 2.4.3 that interests us is $[\tilde{G} \to G]$. Let G^{ad} be the adjoint group of G, Z the centre of G, and $\tilde{Z}$ the centre of $\tilde{G}$. The morphism $\tilde{G} \to G^{\mathrm{ad}}$ is an epimorphism, from which we obtain an equivalence (2.4.4)

$$[\tilde{Z} \to Z] \to [\tilde{G} \to G]. \tag{2.4.7.1}$$

As Z and $\tilde{Z}$ are commutative, $[\tilde{Z} \to Z]$ is a strictly commutative Picard category (2.4.5). Using the equivalence (2.4.7.1), we can also make $[\tilde{G} \to G]$ into such a category. We shall calculate $[x_1]+[x_2]$ in $[\widetilde{G}\to G]$ and the associativity and commutativity laws. We suppose that $\rho\colon\widetilde{G}\to G^{\mathrm{der}}$ is separable in order to be able to proceed by Galois descent. Writing (locally) $x_i=\rho(g_i)z_i$ with z_i central, we have isomorphisms $g_i\colon[z_i]\to[x_i]$ and $g_1g_2\colon[z_1z_2]\to[x_1x_2]$, from which we get an isomorphism

$$[x_1]+[x_2]\xleftarrow{g_1+g_2}[z_1]+[z_2]=[z_1z_2]\xrightarrow{g_1g_2}[x_1x_2]. \tag{2.4.7.2}$$

When we change the decomposition, $x_i=\rho(g_i')z_i$ with $g_i=g_i'u_i$ ($u_i\in\widetilde{Z}$), the diagram

$$\begin{array}{ccccccc}
 & & [z_1]+[z_2] & = & [z_1z_2] & & \\
 & \swarrow^{g_1+g_2} & \downarrow{\scriptstyle u_1+u_2} & & \downarrow{\scriptstyle u_1u_2} & \searrow^{g_1g_2} & \\
[x_1]+[x_2] & & & & & & [x_1x_2] \\
 & \nwarrow_{g_1'+g_2'} & & & & \nearrow_{g_1'g_2'} & \\
 & & [z_1']+[z_2'] & = & [z_1'z_2'] & &
\end{array}$$

commutes. The isomorphism (2.4.7.2)

$$[x_1]+[x_2]=[x_1x_2] \tag{2.4.7.3}$$

is therefore independent of the choices made. The reader will check easily that, via this isomorphism, that the associativity law follows from the associativity of the product, and that the commutativity law is $(x_1,x_2)\colon[x_2x_1]\to[x_1x_2]$. He will also check that if $y_i=\rho(g_i)x_i$ ($i=1,2$), then the sum of the $g_i\colon[x_i]\to[y_i]$ is

$$g_1+g_2=g_1\,\mathrm{inn}_{x_1}(g_2)\colon[x_1x_2]\to[y_1y_2],$$

where inn denotes the action of G on $\widetilde{G}$ (defined by transport of structure, or via the action 2.0.2 of $G^{\mathrm{ad}}=\widetilde{G}^{\mathrm{ad}}$).

The category $[\widetilde{G}\to G]$ being Picard, the set $\mathbb{H}^1(\widetilde{G}\to G)$ of isomorphism classes of objects is an abelian group. The above formulas show that the injection

$$G(k)/\rho(\widetilde{G}(k))\to\mathbb{H}^1(\widetilde{G}\to G)$$

is a homomorphism. After (2.4.3.1), it is an isomorphism if $H^1(\widetilde{G})=0$.

Let k' be a finite extension of k, and denote by a $'$ extension of scalars to k'. The equivalence (2.4.7.1) allows us to deduce from 2.4.6.1 a trace functor (which we will baptise *norm*)

$$N_{k'/k}\colon[\widetilde{G}'\to G']\to[\widetilde{G}\to G]. \tag{2.4.7.4}$$

PROPOSITION 2.4.8. *If k is a local or global field, then the morphism deduced from (2.4.7.4) by passage to the set of isomorphism classes of objects induces a morphism from $G(k')/\rho\widetilde{G}'(k)$ into $G(k)/\rho\widetilde{G}(k)$,*

$$\begin{array}{ccc}
G(k')/\rho\widetilde{G}(k') & \longrightarrow & G(k)/\rho\widetilde{G}(k) \\
\big\downarrow & & \big\downarrow \\
\mathbb{H}^1(\widetilde{G}'\to G') & \longrightarrow & \mathbb{H}^1(\widetilde{G}\to G).
\end{array} \tag{2.4.8.1}$$

If $H^1(\widetilde{G}) = 0$, the vertical arrow at right is an isomorphism, and the assertion is obvious. This nullity is valid for k local and nonarchimedean. For k local archimedean the only interesting case is $k = \mathbb{R}$, $k' = \mathbb{C}$, and the commutative diagram

$$\begin{array}{ccc} Z(\mathbb{C}) & \twoheadrightarrow & \mathbb{H}^1(\widetilde{G}_{\mathbb{C}} \to G_{\mathbb{C}}) \\ \downarrow{\scriptstyle N_{\mathbb{C}/\mathbb{R}}} & & \downarrow \\ Z(\mathbb{R}) & \longrightarrow & \mathbb{H}^1(\widetilde{G} \to G) \end{array}$$

shows that (2.4.8.1) is again defined — with values in the image of $Z(\mathbb{R})$ (and even in its neutral component).

For k global and $x \in G(k')/\rho(\widetilde{G}(k'))$, the image of $N_{k'/k}(x) \in \mathbb{H}^1(\widetilde{G} \to G)$ in $H^1(\widetilde{G})$ is therefore locally nul. After the Hasse principle, it is nul. We are using here the Hasse principle only for the cohomology classes in the image of $H^1(\widetilde{Z})$, so that E_8 factors create no trouble. The image $N_{k'/k}(x)$ is therefore in $G(k)/\rho\widetilde{G}(k)$, as promised.

2.4.9. For k local nonarchimedean with ring of integers V, k' unramified over k with ring of integers V', and G reductive over V, the morphism 2.4.8 induces a morphism of[51] $G(V')/\rho\widetilde{G}(V)$: one sees this by repeating the preceding arguments over V, the Galois descent being replaced by étale localization (here, formally identical to a Galois descent over the residue field).

It is possible therefore to adèlize 2.4.8: for k global, the restricted product of the morphisms 2.4.8 for the completions of k is a morphism

$$N_{k'/k}: G(\mathbb{A}')/\rho\widetilde{G}(\mathbb{A}') \to G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A}).$$

On dividing by the global trace morphism, we obtain finally the morphism (2.4.0.1)

$$N_{k'/k}: \pi(G') \to \pi(G).$$

Just as the construction of the morphism (2.4.0.1) rests on that of the functor (2.4.7.1), that of (2.4.0.2) rests on the

CONSTRUCTION 2.4.10. *Let G be a connected reductive group of k, $\rho: \widetilde{G} \to G$ the universal covering of the derived group, T a torus over k, and M a conjugacy class, defined over k, of morphisms from T into G. We shall define an additive functor*

$$q_M: [\{e\} \to T] \to [\widetilde{G} \to G].$$

We will give two variants of the construction.

First method. Locally, there exist m in M. Put $X(m) = Z \cdot m(T) \subset G$ and $Y(m) = \rho^{-1}X(m) = \widetilde{Z} \cdot (\rho^{-1}m(T))^{\circ}$. The groups $X(m)$ and $Y(m)$, extensions of tori by central subgroups of multiplicative type, are commutative. They give rise to a diagram

$$\begin{array}{ccccc} \widetilde{Z} & \longrightarrow & Y(m) & & \\ \downarrow & & \downarrow & & \\ Z & \longrightarrow & X(m) & \longleftarrow & T \end{array} \qquad (1m)$$

[51] The V should be V' (twice).

If g in G conjugates m into m', it conjugates $(1m)$ into $(1m')$ (we make it act on $\widetilde{G}$ through the action of $\widetilde{G}^{\mathrm{ad}} = G^{\mathrm{ad}}$). Moreover, the isomorphism $\mathrm{inn}(g)$ of $(1m)$ with $(1m')$ does not depend on g: if g centralizes m, then it centralizes Z, $m(T)$, $\widetilde{Z}$, as well as $(\rho^{-1}(T))^{\circ}$ (a torus isogenous to a subtorus of T), and therefore $X(m)$ and $Y(m)$. Two m in M being locally conjugate, this allows us to identify the diagrams $(1m)$, and deduce a unique diagram

$$\begin{array}{ccccc} \widetilde{Z} & \longrightarrow & Y & & \\ \downarrow & & \downarrow & & \\ Z & \longrightarrow & X & \longleftarrow & T \end{array} \tag{1}$$

defined over k.

We define q_M to be the composite functor

$$q_M\colon [\{e\} \to T] \longrightarrow [Y \to X] \xleftarrow[2.4.4]{\sim} [\widetilde{Z} \to Z] \xrightarrow{\sim} [\widetilde{G} \to G].$$

Second method. We suppose that $\rho\colon \widetilde{G} \to G$ is separable so that we can apply Galois descent. The objects of $[\{e\} \to T]$ having no automorphisms, a functor $[\{e\} \to T]$ into $[\widetilde{G} \to G]$ is simply a rule that to each $t \in T(k)$ assigns a G-trivialized $\widetilde{G}$-torsor $q_M([t])$. We proceed by Galois descent. Locally there exists $m \in M(k)$. Let q_m be the functor $[t] \to [m(t)]$. We shall define a transitive system of isomorphisms between the q_m. This done, we can define q_M to be any one of the q_m.

To define $\iota_{m',m}\colon q_m \xrightarrow{\sim} q_{m'}$, we choose a g such that $m' = gmg^{-1}$ (new application of the method of descent) and we put $\iota_{m'.m}([t])\colon [m'(t)] \to [m'(t)]$ equal to $(g, m(t)) \in \widetilde{G}(k)$. We certainly have $m'(t) = (\rho((g, m(t))m(t)$, and it remains to show that $(g, m(t))$ is independent of g.

The space C of g conjugating m into m' is connected and reduced, because it is a torsor under the centralizer of the torus $m(T)$. The function $(g, m(t))$ from C into $\widetilde{G}$ has a projection $\rho((m, m(t))) = m'(t)m(t)^{-1}$ into G which is constant. The fibre of ρ being discrete is also constant.

The construction is summarized by the commutative diagram

$$\begin{array}{ccc} & q_M([t]) & \\ \swarrow & & \searrow \\ q_m([t]) & \xrightarrow{(g, m(t))} & q_m([t]) \end{array} \qquad (q_m([t]) = [m(t)],\ m' = gmg^{-1}). \tag{2.4.10.1}$$

We construct the additivity datum for q_M. We have to give, for each $t_1, t_2 \in T(k)$, an isomorphism $q_M([t_1 t_2]) \to q_M([t_1]) + q_M([t_2])$, these isomorphisms being compatible with the associativity and commutativity rules for the sum in $[\widetilde{G} \to G]$. We define it by descent: for $m \in M$ and $m' = gmg^{-1}$, the following diagram is commutative

$$\begin{array}{ccccccc}
 & & [m(t_1t_2)] & \xrightarrow{2.4.7} & [m(t_1)]+[m(t_2)] & & \\
 & \nearrow & & & & \nwarrow & \\
q_m([t_1t_2]) & & \downarrow{\scriptstyle (g,m(t_1t_2))} & & \downarrow{\scriptstyle (g,m(t_1))+(g,m(t_2))} & & q_M([t_1])+q_M([t_2]) \\
 & \searrow & & & & \swarrow & \\
 & & [m'(t_1t_2)] & \xrightarrow{2.4.7} & [m'(t_1)]+[m'(t_2)] & &
\end{array}$$

(oblique arrows 2.4.10.1), and defines the sought isomorphism independently of m. Its commutativity expresses a certain identity, in $\widetilde{G}$, between the commutators (2.4.2), and the projection of this identity into G follows from the written arrows making sense. To prove it, note that locally (descent) it is the projection of an analogous identity for $\widetilde{G}\times Z^\circ$ — the projection is true in $\widetilde{G}\times Z^\circ$, therefore true in $\widetilde{G}$.

The compatibility with associativity and commutativity is seen by descent, by fixing m; we use that the commutator 2.4.2 is trivial on $m(T)$.

2.4.11. *Complements*

(i) The construction 2.4.10 is compatible with extension of scalars: denoting by $'$ the extension of scalars from k to an extension k' of k, we define in an obvious way an isomorphism of additive functors making commutative the diagram

$$\begin{array}{ccc}
[\{e\}\to T] & \xrightarrow{q_M} & [\widetilde{G}\to G] \\
\downarrow & & \downarrow \\
[\{e\}\to T'] & \xrightarrow{q_{M'}} & [\widetilde{G}'\to G'].
\end{array}$$

(ii) It is possible to make the construction 2.4.10 over any base; Galois descent is replaced by étale localization.

(iii) The construction 2.4.10 is compatible with the norm functors. To define the isomorphism of additive functors making commutative the diagram

$$\begin{array}{ccc}
[\{e\}\to T'] & \xrightarrow{q_{M'}} & [\widetilde{G}'\to G'] \\
\downarrow & & \downarrow{\scriptstyle 2.4.7.4} \\
[\{e\}\to T] & \xrightarrow{q_M} & [\widetilde{G}\to G].
\end{array}$$

(notation of (i) with k'/k finite and separable) and check its properties, the simplest approach is to proceed by descent: locally, k' becomes a sum[52] k^I of copies of k, $[\widetilde{G}'\to G']$ becomes[53] $[\widetilde{G}\to G]^I$, the norm (trace) becomes the sum, and everything is trivial.

[52] Actually, a product.

[53] The original has an errant $'$.

2.4.12. For k local and nonarchimedean, $H^1(\widetilde{G}) = 0$ and, passing to the sets of isomorphism classes of objects, we deduce from 2.4.10 a morphism

$$q_M : T(k) \to G(k)/\rho\widetilde{G}(k).$$

For G reductive over the valuation ring V of k, T a torus over V, and M over V, it induces a morphism from $T(V)$ into $G(V)/\rho\widetilde{G}(V)$ (2.4.11(ii)).

For k local and archimedean, there may be an obstruction in $\mathbb{H}^1(\widetilde{G})$, but it disappears for x in the neutral (topological) component $T(k)^+$ of $T(k)$: by 2.4.11(ii), it depends continuously on x, and it is nul for $x = e$. We therefore have again an isomorphism $T(k)^+ \to G(k)/\rho\widetilde{G}(k)$.

For k global, taking the restricted product of the morphisms for the completions of k, we obtain

$$q_M : T(\mathbb{A})^+ \to G(k)/\rho\widetilde{G}(k). \tag{2.4.12.1}$$

When we let

$$T(k)^+ = \{x \in T(k) \mid x \in T(k_v)^+ \text{ for } v \text{ real}\},$$

the Hasse principle, used as in 2.4.8, furnishes

$$q_M : T(k)^+ \to G(k)/\rho\widetilde{G}(k). \tag{2.4.12.2}$$

As $T(\mathbb{A})^+/T(k)^+ \xrightarrow{\sim} T(\mathbb{A})/T(k)$ (real approximation theorem for tori), we finally obtain by passage to the quotient the promised morphism (2.4.0.2)

$$q_M : \pi(T) \to \pi(G).$$

2.5 Application: a canonical extension

2.5.1. Let G be a reductive group over $\mathbb{Q}$. Suppose that $\widetilde{G}$ has no factor G' (defined over $\mathbb{Q}$) such that $G'(\mathbb{R})$ is compact. The strong approximation theorem then states that $\widetilde{G}(\mathbb{Q})$ is dense in $\widetilde{G}(\mathbb{A}^f)$. The case that is important for us is that of a group as in 2.1.1.

Return to the calculation of 2.1.3. For K compact open in $G(\mathbb{A}^f)$,

$$\begin{aligned} \pi_0(G(\mathbb{Q})\backslash G(\mathbb{A})/K) &= G(\mathbb{Q})\backslash\pi_0(G(\mathbb{R})) \times (G(\mathbb{A}^f)/K) \\ &= G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})^+ \times K \\ &= G(\mathbb{Q})^+\backslash G(\mathbb{A}^f)/K \end{aligned} \tag{2.5.1.1}$$

and we can replace $G(\mathbb{Q})$ (resp. $G(\mathbb{Q})^+$) by its closure in $G(\mathbb{A}^f)$. This contains $\rho\widetilde{G}(\mathbb{A}^f)$, a normal subgroup with abelian quotient in $G(\mathbb{A}^f)$, and, with the notation of 2.0.15,

$$\begin{aligned} \pi_0(G(\mathbb{Q})\backslash G(\mathbb{A})/K) &= \pi(G)/G(\mathbb{R})^+ \times K \\ &= \pi_0\pi(G)/K \\ &= G(\mathbb{A}^f)/G(\mathbb{Q})^+ \cdot K \end{aligned} \tag{2.5.1.2}$$

Passing to the limit over K, we deduce that

$$\pi_0(G(\mathbb{Q})\backslash G(\mathbb{A})) = \pi_0\pi(G) = G(\mathbb{A}^f)/G(\mathbb{Q})^{+-}.$$

Let $\overline{\pi}_0\pi(G)$ be the quotient $G(\mathbb{A}^f)/G(\mathbb{Q})^-_+$ of[54] $\pi_0\pi(G)$ by $\pi_0 G(\mathbb{R})_+$ (0.3). The sequence

$$0 \to G(\mathbb{Q})^-_+/Z(\mathbb{Q})^- \to G(\mathbb{A}^f)/Z(\mathbb{Q})^= \to \overline{\pi}_0\pi(G) \to 0 \qquad (2.5.1.3)$$

is exact. The action of G^{ad} on G induces an action of $G^{\mathrm{ad}}(\mathbb{Q})$ on this exact sequence. The existence of the commutator (2.0.2) shows that the action of $G^{\mathrm{ad}}(\mathbb{A})$ on $G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A})$ is trivial — a fortiori that of $G^{\mathrm{ad}}(\mathbb{Q})$ on $\overline{\pi}_0\pi(G)$. The map of the subgroup $G(\mathbb{Q})_+/Z(\mathbb{Q})$ of $G(\mathbb{Q})^-_+/Z(\mathbb{Q})^-$ into $G^{\mathrm{ad}}(\mathbb{Q})^+$ satisfies the conditions of 2.0.1. The group

$$G(\mathbb{Q})^-_+/Z(\mathbb{Q}) *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+$$

is nothing but the completion of $G^{\mathrm{ad}}(\mathbb{Q})^+$ for the topology $\tau(G^{\mathrm{der}})$ (2.1.6). Applying the same construction $*G^{\mathrm{ad}}(\mathbb{Q})^+$ to the central term in 2.5.1.3, we finally obtain an extension

$$0 \to G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\text{rel. } G^{\mathrm{der}}) \to \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+ \to \overline{\pi}_0\pi(G) \to 0 \quad (2.5.1.4)$$

2.5.2. Because G^{ad} is not functorial in G, the functoriality of this sequence is troublesome to make explicit. We content ourselves with the following two cases:

(a) When we consider only central extensions of groups with a given adjoint group, and morphisms compatible with the projection to the adjoint group, (2.5.1.4) is functorial in G in the obvious sense.

(b) Let $H \subset G$ be a torus and, contrary to the general conventions, *let H^{ad} denote its image in G^{ad}*. This being the case in the applications, we suppose that $H^{\mathrm{ad}}(\mathbb{R})$ is compact — therefore connected because H^{ad} is connected. This hypothesis ensures that $H^{\mathrm{ad}}(\mathbb{Q})$ is discrete in $H^{\mathrm{ad}}(\mathbb{A}^f)$, and that $H(\mathbb{Q}) \subset G(\mathbb{Q})_+$. Put $Z' = Z \cap H$. The commutative diagram

$$\begin{array}{ccccc} G(\mathbb{A}^f) & \longrightarrow & G(\mathbb{A}^f)/G(\mathbb{Q})^-_+ & = & \overline{\pi}_0\pi(G) \\ \uparrow & & \uparrow & & \\ H(\mathbb{A}^f) & \longrightarrow & H(\mathbb{A}^f)/H(\mathbb{Q})^- & = & \overline{\pi}_0\pi(H) \end{array}$$

furnishes a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\text{rel.}\, G^{\mathrm{der}}) & \longrightarrow & \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+ & \longrightarrow & \overline{\pi}_0\pi(G) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^{\mathrm{ad}}(\mathbb{Q}) & \longrightarrow & \frac{H(\mathbb{A}^f)}{Z'(\mathbb{Q})^-} *_{\frac{H(\mathbb{Q})}{Z'(\mathbb{Q})}} H^{\mathrm{ad}}(\mathbb{Q}) & \longrightarrow & \overline{\pi}_0\pi(H) & \longrightarrow & 0. \end{array}$$

(2.5.2.1)

[54] The original has $\overline{\pi}_0\pi(G)$.

2.5.3. Let G be as in 2.5.1, E a finite extension of $\mathbb{Q}$, T a torus over E, and M a conjugacy class defined over E of morphisms from T into G, $M: T \to G_E$. When we apply π_0 to the the composite of (2.4.0.1) and (2.4.0.2),

$$N_{E/\mathbb{Q}} q_M: \pi(T) \to \pi(G_E) \to \pi(G),$$

we get a morphism[55]

$$\pi_0\pi(T) \to \pi_0\pi(G) \to \overline{\pi}_0\pi(G).$$

We denote its inverse by[56] $r(G,M)$, and by $\mathcal{E}'_E(G,M)$ the inverse image of the extension (2.5.1.4) by $r(G,M)$,

$$\begin{array}{ccccccccc} 0 & \to & G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,G^{\mathrm{der}}) & \longrightarrow & \mathcal{E}'_E(G,M) & \longrightarrow & \pi_0\pi(T) & \to & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle r(G,M)} & & \\ 0 & \to & G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,G^{\mathrm{der}}) & \to & \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+ & \to & \overline{\pi}_0\pi(G) & \to & 0. \end{array}$$

Let $u: H \to G$ be a morphism as in 2.5.2(a) ($H^{\mathrm{ad}} = G^{\mathrm{ad}}$), and N a cohomology class, defined over E, of morphisms from T into H. We suppose that u sends N into M. By functoriality, u then defines an isomorphism of the quotient of $\mathcal{E}'_E(H,N)$ by

$$\mathrm{Ker}\left(G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}) \to G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\ H^{\mathrm{der}})\right)$$

with $\mathcal{E}'_E(G,M)$.

For $H \to G$ a torus as in 2.5.2(b), equipped with $m: T \to H_E$ in M, we find a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\mathrm{rel.}G^{\mathrm{der}}) & \to & \mathcal{E}'_E(G,M) & \to & \pi_0\pi(T) & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & H^{\mathrm{ad}}(\mathbb{Q}) & \longrightarrow & \dots & \longrightarrow & \pi_0\pi(T) & \to & 0. \end{array}$$

2.5.4. Let E, G, T, M be as above with G adjoint. Consider the systems (G_1, M_1, u) formed of a central extension $u: G_1 \to G$ of G (so $G_1^{\mathrm{ad}} = G$), defined over $\mathbb{Q}$, and a conjugacy class of morphisms M_1 of T into G_1, defined over E, which lifts M. Over $\overline{\mathbb{Q}}$, for m_1 in M_1 with image m in M, the centralizer of m_1 is the inverse image of the centralizer of m: this true for their Lie algebras, they are connected as the centralizers of tori, and the centralizer of m_1 contains the centre of G_1. We therefore have $M_1 \xrightarrow{\sim} M$.

LEMMA 2.5.5. *There exist systems (G_1, M_1, u) for which G_1^{der} is an arbitrary prescribed finite covering of G.*

[55] For the definition of $\overline{\pi}_0$ see 2.1.3.

[56] In the original, the symbol r is missing. Also, the "inverse" should be omitted as in 2.2.3. See footnote 44, p. 28.

It suffices to show that we can obtain the universal finite covering $\widetilde{G}$. Over $\overline{\mathbb{Q}}$, for m in M, the neutral component of the inverse image by m of the finite covering $\widetilde{G}$ of G is a finite covering $\pi\colon\widetilde{T}\to T$ of T. It does not depend on m, hence is defined over E, and M lifts to a conjugacy class $\widetilde{M}$ of morphisms from $\widetilde{T}$ into $\widetilde{G}_E$. By passing to the quotient by $\mathrm{Ker}(\pi)$, we deduce from $\widetilde{M}\times\mathrm{id}\colon\widetilde{T}\to\widetilde{G}_E\times\widetilde{T}$ a finite covering $M_1'\colon T\to(\widetilde{G}_E\times\widetilde{T})/\mathrm{Ker}(\pi)$. This finite covering takes values in a group G_E', defined over E, with adjoint group G_E. It remains to replace G_E' with a group defined over $\mathbb{Q}$. Write $G_E'=\widetilde{G}_E *_{\widetilde{Z}_E} Z_E'$. The idea is to replace Z_E' with the coproduct over $\widetilde{Z}_E$ of its conjugates: if we put

$$Z=R_{E/\mathbb{Q}}(Z_E')/\mathrm{Ker}(\mathrm{Tr}_{E/\mathbb{Q}}\colon R_{E/\mathbb{Q}}(\widetilde{Z}_E)\to\widetilde{Z})$$

and $G_1=\widetilde{G}*_{\widetilde{Z}}Z$, then we have $G_E'\subset G_{1E}$, and M_1' provides the lifting sought.

CONSTRUCTION 2.5.6. *Up to a unique isomorphism, the extension $\mathcal{E}'(G_1,M_1)$ depends only on M and G_1^{der}.*

Suppose that the two systems (G_1',M_1') and (G_1'',M_1'') have the same derived group. Consider the identity component G_1 of the fibred product of G_1' and G_1'' over G and the class $M_1=M_1'\times_M M_1''$. The diagram of extensions

$$\mathcal{E}'(G_1',M_1')\xleftarrow{\sim}\mathcal{E}'(G_1,M_1)\xrightarrow{\sim}\mathcal{E}'(G_1'',M_1'')$$

gives the isomorphism sought.

DEFINITION 2.5.7. Let G be an adjoint group, G' a finite covering of G, and M a conjugacy class, defined over E, of morphisms from T into G. The extension $\mathcal{E}_E(G,G',M)$ of $\pi_0\pi(T)$ by the completion $G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,G')$ is the extension $\mathcal{E}_E'(G_1,M_1)$ for any system (G_1,M_1) whatever as in 2.5.4 with $G_1^{\mathrm{der}}=G'$.

For F an extension of E, the class M furnishes by extension of scalars from E to F a conjugacy class M_F of morphisms of T_F into G_F; the corresponding extension $\mathcal{E}_F(G,G',M)$ is the inverse image of $\mathcal{E}_E(G,G',M)$ by the norm $N_{F/E}\colon\pi_0\pi(T_F)\to\pi_0\pi(T)$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,G') & \longrightarrow & \mathcal{E}_F(G,G',M_F) & \longrightarrow & \pi_0\pi(T_F) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow N_{F/E} & & \\
0 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,G') & \longrightarrow & \mathcal{E}_E(G,G',M) & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0.
\end{array}$$

The extensions $\mathcal{E}_E(G,G',M)$ can all be deduced from $\mathcal{E}_E(G,\widetilde{G},M)$ by passage to the quotient: replace $G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,\widetilde{G})$ with its quotient $G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\,G')$.

2.5.8. Let $H\to G$ be a torus with $H(\mathbb{R})$ compact and $m\in M$, defined over E, which factors through H. For any system $(G_1,M_1)\to(G,M)$ as above, let H_1 be the neutral component of the inverse image of H in G_1 and m_1 the element of

M_1 above m (2.5.4). Take the inverse image by $r(H_1, \{m_1\})$ of the morphism of extensions (2.5.2.1),

$$\begin{array}{ccccc} G(\mathbb{Q})^{+\wedge}(\mathrm{rel}.G') & \longrightarrow & \mathcal{E}_E(G,G',M) & \longrightarrow & \pi_0\pi(T) \\ \uparrow & & \uparrow & & \| \\ H(\mathbb{Q}) & \longrightarrow & \dots & \longrightarrow & \pi_0\pi(T). \end{array}$$

We see as in 2.5.6 that, up to a unique isomorphism, this diagram does not depend on the choice of (G_1, M_1). As in 2.5.7, this diagram, rel. a finite covering G' of G is deduced from the same diagram rel. $\widetilde{G}$ by passage to the quotient. We also have the same functoriality in E as in 2.5.7. In particular, for m in M, defined over an extension F of E, which factors through H, we find a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\mathrm{rel}.G') & \longrightarrow & \mathcal{E}_E(G,G',M) & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle N_{F/E}} & & \\ 0 & \longrightarrow & H(\mathbb{Q}) & \longrightarrow & \dots & \longrightarrow & \pi_0\pi(T_F) & \longrightarrow & 0 \end{array} \tag{2.5.8.1}$$

We use this diagram in the following way: if $\mathcal{E}_E(G,G',M)$ acts on a set V and a point $x \in V$ is fixed under $H(\mathbb{Q}) \subset G(\mathbb{Q})$, it makes sense to ask if it is fixed "by $\pi_0\pi(T_F)$" i.e., by the image subgroup of the extension on the second line.

2.5.9. We specialize the hypotheses to the case that interests us. We start with a system (G, X) as in 2.1.1, with G adjoint, and we fix a connected component X^+ of X. We take for E a finite extension, contained in $\mathbb{C}$, of $E(G,X)$, and we take $T = \mathbb{G}_m$, M =the conjugacy class of μ_h for $h \in X$. It is defined over E.

The group $\pi(T)$ is the idèle class group of E, and global class field theory identifies $\pi_0\pi(T)$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$. If G' is a finite covering of G, the inverse image by the morphism $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$ of the extension $\mathcal{E}_E(G,G',M)$ is an extension

$$0 \to G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\ G') \to \mathcal{E}_E(G,G',X) \to \mathrm{Gal}(\overline{\mathbb{Q}}/E) \to 0 \tag{2.5.9.1}$$

The universal case is that where $E = E(G,X)$ and $G' = \widetilde{G}$; after 2.5.7, $\mathcal{E}_E(G,G',M)$ is the inverse image of $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$ in $\mathcal{E}_{E(G,X)}(G,G',X)$ and $\mathcal{E}_E(G,G',X)$ is a quotient of $\mathcal{E}_E(G,\widetilde{G},X)$.

2.5.10. Let $h \in X^+$ be a special point: h factors through a torus $H \subset G$ defined over $\mathbb{Q}$. As $\mathrm{inn}\, h(i)$ is a Cartan involution, $H(\mathbb{R})$ is compact. We can therefore apply 2.5.8 to H and to μ_h (defined over the extension $E(H,h)$ of $E(G,X)$. By inverse image, we deduce from (2.5.8.1) a morphism of extensions

$$\begin{array}{ccccc} G(\mathbb{Q})^{+\wedge}(\mathrm{rel}.G') & \longrightarrow & \mathcal{E}_E(G,G',M) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/E) \\ \uparrow & & \uparrow & & \uparrow \\ H(\mathbb{Q}) & \longrightarrow & \dots & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/E\cdot E(H,h)) \end{array} \tag{2.5.10.1}$$

2.6 The reciprocity law for canonical models

2.6.1. Let (G, X) be as in 2.1.1 and $E \subset \mathbb{C}$ a number field containing $E(G, X)$. Suppose that $M_{\mathbb{C}}(G, X)$ admits a weakly canonical model $M_E(G, X)$ over E. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ then acts on the profinite set $\pi_0(M_{\mathbb{C}}(G, X))$ of geometric connected components of $M_E(G, X)$. This action commutes with that of $G(\mathbb{A}^f)$ which, by hypothesis, is defined over E. After 2.1.14, the (right) action of $G(\mathbb{A}^f)$ makes $\pi_0 M_{\mathbb{C}}(G, X)$ into a principal homogeneous space under the abelian quotient $\overline{\pi}_0 \pi G = G(\mathbb{A}^f)/G(\mathbb{Q})^-_+$. The Galois action is therefore defined by a homomorphism $r_{G,X}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ into $\overline{\pi}_0\pi(G)$, said to be *reciprocity*. Sign convention: the (left) action of σ coincides with the (right) action of $r_{G,X}(\sigma)$. This morphism factors through the Galois group made abelian, identified by global class field theory with $\pi_0\pi(\mathbb{G}_{mE})$, whence

$$r_{G,X}: \pi_0\pi(\mathbb{G}_{mE}) \to \overline{\pi}_0\pi(G). \tag{2.6.1.1}$$

2.6.2. Let M be the conjugacy class of μ_h for $h \in X$. As $E \supset E(G, X)$, it is defined over E. On composing the morphisms (2.4.0.1) and (2.4.0.2),[57] we obtain

$$N_{E/\mathbb{Q}} q_m: \pi(\mathbb{G}_{mE}) \to \pi(G_E) \to \pi(G).$$

By applying π_0, we deduce

$$\pi_0 N_{E/\mathbb{Q}} q_m: \pi_0\pi(\mathbb{G}_{mE}) \to \pi_0\pi(G_E) \to \overline{\pi}_0\pi(G). \tag{2.6.2.1}$$

THEOREM 2.6.3. *The morphism (2.6.1.1) giving the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ *on the set of geometric connected components of a weakly canonical model* $M_E(G, X)$ *of* $M_{\mathbb{C}}(G, X)$ *over* E *is the inverse of*[58] *the morphism* $\pi_0 N_{E/\mathbb{Q}} q_M$ *of 2.6.2.*

The idea of the proof is that, for each type τ of special points (2.2.4), we know the action of a subgroup of finite index Gal_τ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the special points of this type (by definition of the weakly canonical models) — therefore on the set of connected components because the map that sends each point to its connected component is compatible with the Galois action. That the action of Gal_τ obtained is the restriction to Gal_τ of the action defined by the inverse[59] of $\pi_0(N_{E/\mathbb{Q}} q_W)$ is shown in 2.6.4 below, and it remains to show that the Gal_τ generate $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

A type τ of special points is defined by $h \in X$ factoring through a torus $\iota: T \to G$ defined over $\mathbb{Q}$. The corresponding subgroup Gal_τ is

$$\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h)) = \mathrm{Gal}(\overline{\mathbb{Q}}/E \cdot E(T, h)).$$

After [**5**, 5.1], for every finite extension F of $E(G, X)$, there exists a (T, h) such that the extension $E(T, h)$ of $E(G, X)$ is linearly disjoint from F. This is more than enough to ensure that the Gal_τ generate $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

[57] The original has "morphisms (2.4.0)"...

[58] Here "the inverse of" should be omitted — see footnote 44, p. 28.

[59] Here "the inverse of" should be omitted — see footnote 44, p. 28.

2.6.4. Let T and h be as above and $\mu = \mu_h$. The morphism $\mu\colon \mathbb{G}_m \to T$ is defined over $E(T,h)$ and the morphism $\pi_0 NR(\mu_h)$ of 2.2.3 is obtained from it by applying the functor π_0 to $N_{E(T,h)/\mathbb{Q}} \circ q_\mu\colon \pi(\mathbb{G}_{mE(T,h)}) \to \pi(T)$. We deduce that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E(T,h))$ on the special points of type τ is compatible with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T,h))^{\mathrm{ab}} = \pi_0\pi(\mathbb{G}_{mE(T,h)})$ on $\pi_0(M_{\mathbb{C}}(G,X))$ deduced, by application of the functor π_0, from the inverse of [60]

$$\iota \circ N_{E(T,h)/\mathbb{Q}} \circ q_\mu\colon \pi(\mathbb{G}_{mE(T,h)}) \to \pi(T) \to \pi(G).$$

From the functoriality of N and of q, it follows that the composite is $N_{E(T,h)/\mathbb{Q}} \circ q_M$,

$$\begin{array}{ccccc}
\pi(\mathbb{G}_{mE(T,h)}) & \xrightarrow{q_\mu} & \pi(T_{E(T,h)}) & \longrightarrow & \pi(T) \\
 & \searrow^{q_M} & \downarrow & & \downarrow \\
 & & \pi(G_{E(T,h)}) & \longrightarrow & \pi(G)
\end{array}$$

equal to $N_{E(G,X)/\mathbb{Q}} \circ N_{E(T,h)/E(G,X)}$

$$\begin{array}{ccccc}
\pi(\mathbb{G}_{mE(T,h)}) & \xrightarrow{q_M} & \pi(G_{E(T,h)}) & \longrightarrow & \pi(G) \\
\downarrow & & \downarrow & & \| \\
\pi(\mathbb{G}_{mE(G,X)}) & \xrightarrow{q_M} & \pi(G_{E(G,X)}) & \longrightarrow & \pi(G).
\end{array}$$

As the norm $N_{E(T,h)/E(G,X)}$ corresponds, via class field theory to the inclusion of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T,h))$ into $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$, we have the promised action.

2.7 *Reduction to the derived group, and the existence theorem*

In this number, scheme means "scheme admitting an ample invertible sheaf". This allows us to pass without scruples to the quotient by a finite group. The stability of this condition will be obvious in the applications, and I will not check it each time. All of this is moreover only a question of convenience.[61]

2.7.1. Let Γ be a locally compact totally disconnected group. We are interested in projective systems, equipped with a left action of Γ, of the following type.

(a) A projective system, indexed by the compact open subgroups K of Γ, of schemes S_K.

(b) An action ρ of Γ on the system (defined by isomorphisms $\rho_K(g)\colon S_K \xrightarrow{\sim} S_{gKg^{-1}}$).

(c) We assume that $\rho_K(k)$ is the identity for $k \in K$. For L normal in K, the $\rho_L(k)$ define an action on S_L of the finite quotient K/L, and we assume that $(K/L)\backslash S_L \xrightarrow{\sim} S_K$.

[60] Here "the inverse of" should be omitted — see footnote 44, p. 28.

[61] Deligne would rather be using stacks.

Such a system is determined by its projective limit $S = \varprojlim S_K$ equipped with an action of Γ: we have $S_K = K\backslash S$. We shall call S a *scheme equipped with a continuous left action of* Γ. We similarly define continuity of a right action by the condition $S = \varprojlim S/K$.

2.7.2. Let π be a profinite set equipped with a continuous action of Γ. We suppose that the action is transitive and that the orbits of a compact open subgroup are open: for $e \in \pi$ with stabilizer Δ, the bijection $\Gamma/\Delta \to \pi$ is a homeomorphism.

If Γ acts continuously on a scheme S equipped with a continuous equivariant map to π, the fibre S_e is equipped with a continuous action of Δ: for K compact open in Γ, $K \cap \Delta\backslash S_e$ is the fibre over the image of e in $K\backslash S \to K\backslash\pi$, and S_e is the limit of these quotients.

LEMMA 2.7.3. *The functor*[62] $S \rightsquigarrow S_e$ *is an equivalence of the category of schemes* S *equipped with a continuous action of* Γ *and an equivariant map to* π, *with the category of schemes equipped with a continuous action of* Δ.

The inverse functor is the *induction* functor from Δ to Γ: formally, $\mathrm{Ind}_\Delta^\Gamma(T)$ is the quotient of $\Gamma \times T$ by Δ acting by $\delta(\gamma,t) = (\gamma\delta^{-1}, \delta t)$; this makes sense because the action of Δ on Γ is proper; for K compact open in Γ, we have

$$\begin{aligned} K\backslash \mathrm{Ind}_\Delta^\Gamma(T) &= K\backslash\Gamma \times T \text{ divided by } \Delta \\ &= \coprod_{\gamma \in K\backslash\Delta = K\backslash\pi} (\gamma K \gamma^{-1} \cap \Delta)\backslash T. \end{aligned}$$

The detailed verification is left to the reader.[63]

2.7.4. Let E be a field and F a Galois extension of E. The Galois group $\mathrm{Gal}(F/E)$ acts continuously on $\mathrm{Spec}(F)$. More generally, if X is a scheme over E, it acts continuously (by transport of structure) on $X_F = X \times_{\mathrm{Spec}(E)} \mathrm{Spec}(F)$. We have (Galois descent)

LEMMA 2.7.5. *The functor* $X \rightsquigarrow X_F$ *is an equivalence of the category of schemes over* E, *with the category of schemes over* F *equipped with a continuous action of* $\mathrm{Gal}(F/E)$ *compatible with the action of this Galois group on* F.

2.7.6. Let $E \hookrightarrow \overline{\mathbb{Q}}$ be a number field, Γ a locally compact totally disconnected group, π a profinite set equipped with an action of Γ, as in 2.7.2, except that we take here a right action, and $e \in \pi$. We give also a left action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ commuting with the action of Γ. Let Γ_e denote the stabilizer of e. When we convert the right action of Γ into a left action, we obtain a left action of $\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$. The stabilizer of e for this action is an extension $\mathcal{E}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ by Γ_e,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_e & \longrightarrow & \mathcal{E} & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/E) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & \Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/E) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/E) & \longrightarrow & 0. \end{array}$$

[62] $S \to S_e$ in the original.

[63] Has any reader done this, and written it out? It would be useful to have a detailed account of the theory sketched by Deligne in 2.7.1–2.7.5.

2.7.7. When the action of Γ makes π into a principal homogeneous space under the abelian quotient $\pi(\Gamma)$ of Γ, the Galois action is defined by a morphism $r{:}\operatorname{Gal}(\overline{\mathbb{Q}}/E) \to \pi(\Gamma)$ such that $\sigma \cdot x = x \cdot r(\sigma)$, $\mathcal{E}$ does not depend on e: Γ_e is the kernel of the projection of Γ onto $\pi(\Gamma)$ and the extension $\mathcal{E}$ is the inverse image by r of the extension Γ of $\pi(\Gamma)$ by Γ_e.

2.7.8. Consider the schemes S over E equipped with a continuous right action of Γ and map $S_{\overline{\mathbb{Q}}} \to \pi$ that is both $\operatorname{Gal}(\overline{\mathbb{Q}}/E)$- and Γ-equivariant. We let S_e denote the fibre at e. It is a scheme over $\overline{\mathbb{Q}}$ equipped with a continuous (left) action of the extension $\mathcal{E}$, and the action of $\mathcal{E}$ on S_e is compatible with its action, via $\operatorname{Gal}(\overline{\mathbb{Q}}/E)$, on $\overline{\mathbb{Q}}$. Combining 2.7.3 and 2.7.5 we find

LEMMA 2.7.9. *The functor $S \rightsquigarrow S_e$ is an equivalence of categories.*

The case of interest to us is that where the S/K are of finite type over E for K compact open in Γ, and where the map from $S_{\overline{\mathbb{Q}}}$ onto π identifies π with $\pi_0(S_{\overline{\mathbb{Q}}})$. These conditions correspond to the following: the $K\backslash S_e$ for K compact open in Γ_e are connected and of finite type over $\overline{\mathbb{Q}}$.

2.7.10. Let G be an adjoint group, G' a finite covering of G, X^+ a $G(\mathbb{R})^+$-conjugacy class of morphisms from S into $G_{\mathbb{R}}$ satisfying the conditions 2.1.1, and $E \subset \overline{\mathbb{Q}}$ a finite extension of $E(G,X^+)$. A *(connected) weakly canonical model* of $M^\circ(G,G',X^+)$ over E consists of

(a) a model $M^\circ_{\overline{\mathbb{Q}}}$ of $M^\circ(G,G',X^+)$ over $\overline{\mathbb{Q}}$, i.e., a scheme $M^\circ_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$ equipped with an isomorphism from the scheme over $\mathbb{C}$ obtained from $M^\circ_{\overline{\mathbb{Q}}}$ by extension of scalars to $M^\circ(G,G',X^+)$;

(b) a continuous action of $\mathcal{E}_E(G,G',X^+)$ (2.5.9.1) on the scheme $M^\circ_{\overline{\mathbb{Q}}}$, compatible with the action of the quotient $\operatorname{Gal}(\overline{\mathbb{Q}}/E)$ of $\mathcal{E}_E$ on $\overline{\mathbb{Q}}$, and such that the action of the subgroup $G(\mathbb{Q})^{+\wedge}$(rel. G') (a $\overline{\mathbb{Q}}$-linear action this time) defines by extension of scalars to $\mathbb{C}$ the action 2.1.8;

(c) we require that for any special point $h \in X^+$, factoring through a torus $H \to G$ defined over $\mathbb{Q}$, the point of $M^\circ(G,G',X^+)$ defined by h — fixed by $H(\mathbb{Q})$ — is defined over $\overline{\mathbb{Q}}$ and (as a closed point of $M^\circ_{\overline{\mathbb{Q}}}$) fixed by the image of the extension in the second line of (2.5.10.1) (rel. μ_h).

When $E = E(G,X)$, we speak of a *(connected) canonical model.*[64]

2.7.11. The following functorial properties are immediate.

(a) Let (G_i,G_i',X_i^+) be a finite collection of systems as in 2.7.10, and $E \subset \overline{\mathbb{Q}}$ a number field containing the $E(G_i,X_i^+)$. If the $M^\circ_{\overline{\mathbb{Q}}i}$ are weakly canonical models over E of the $M^\circ_{\mathbb{C}}(G_i,G_i',X_i^+)$, their product is a weakly canonical model over E of $M^\circ_{\mathbb{C}}(\prod G_i,\prod G_i',\prod X_i^+)$.

[64]Nowadays, we can avoid speaking of *connected* canonical models by proving directly Langlands's conjugation conjecture, which has the existence of (nonconnected) canonical models as a corollary (by a nontrivial descent argument). The proof still passes through connected Shimura varieties, but now only over $\mathbb{C}$. See the articles of Milne (1983a, 1999c, on his website).

(b) Let (G, G', X^+) be as in 2.7.10, and G'' a finite covering of G which is a quotient of G'. If $M^\circ_{\overline{\mathbb{Q}}}$ is a weakly canonical model over E of $M^\circ_{\mathbb{C}}(G, G', X^+)$, its quotient by $\mathrm{Ker}(G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\, G') \to G(\mathbb{Q})^{+\wedge}(\mathrm{rel.}\, G''))$ is a weakly canonical model over E of $M^\circ_{\mathbb{C}}(G, G'', X^+)$.

2.7.12. Let G be a reductive group over $\mathbb{Q}$, X as in 2.1.1, X^+ a connected component of X, and $E \subset \overline{\mathbb{Q}}$ a finite extension of $\mathbb{Q}$ containing $E(G,X)$. If $M_{\mathbb{C}}(G,X)$ admits a weakly canonical model $M_E(G,X)$ over E, this last is unique up to a unique isomorphism [**5**, 3.5]. The action (2.0.2) of G^{ad} on G induces therefore an action by transport of structure of $G^{\mathrm{ad}}(\mathbb{Q})$ on $M_E(G,X)$. We convert this action into a right action. Combined with the action of $G(\mathbb{A}^f)$, it furnishes a right action of

$$\frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})_+ = \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+.$$

After extension of scalars to $\mathbb{C}$, this is the action (2.1.13).

Let π be the profinite set $\pi_0(M_{\overline{\mathbb{Q}}}(G,X)) = \pi_0(M_{\mathbb{C}}(G,X))$ and $e \in \pi$ the neutral component (2.1.7) rel. X^+. The functor 2.7.9 transforms $M_E(G,X)$ equipped with the natural projection $M_{\overline{\mathbb{Q}}}(G,X)$ to π into a scheme $M^\circ(G,X)$ over $\overline{\mathbb{Q}}$ equipped with a continuous action of the extension (2.5.9.1).

PROPOSITION 2.7.13. *The equivalence of categories 2.7.9 makes the weakly canonical models of $M(G,X)$ over E correspond to the weakly canonical models of*

$$M^\circ(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$$

over E.

In the definition 2.2.5 of weakly canonical models, we have imposed the action of a subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/E(\tau)) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the set of special points of type τ. These form a single orbit under $G(\mathbb{A}^f)$ and the prescribed action commutes with the action of $G(\mathbb{A}^f)$. In the definition 2.2.5, we can therefore be content to require that, for a special point of type τ, its conjugates by Galois are as prescribed. In particular, it suffices to consider the systems (H,h) consisting of a special point $h \in X^+$ factoring through a torus H defined over $\mathbb{Q}$, and, for each system of this type, to prescribe the conjugates under $\mathrm{Gal}(\overline{\mathbb{Q}}/E(H\{h\}) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ of image of $(h,e) \in X \times G(\mathbb{A}^f)$ in $M_{\mathbb{C}}(G,X)$. We recover thus the variant [**5**, 3.13] of the definition: $M_{\mathbb{C}}(H,\{h\})$ has trivially a canonical model (it is a profinite set, and we take the model over $E(H,\{h\})$ for which the Galois action on the points is the prescribed action), and we impose that the natural morphism $M_{\mathbb{C}}(H,\{h\}) \to M_{\mathbb{C}}(G,X)$ be defined over $E \cdot E(H,\{h\})$.

We leave to the reader the task of checking that the equivalence of the categories 2.7.9 transforms this condition of functoriality into that defining connected weakly canonical connected models.

2.7.14. Let G be a real adjoint algebraic group and X^+ a $G(\mathbb{R})^+$-conjugacy class of morphisms of $\mathbb{S}/\mathbb{G}_m$ into $G_{\mathbb{R}}$. Let M denote the conjugacy class of μ_h for $h \in X^+$: a conjugacy class of morphisms from $\mathbb{G}_m$ into $G_{\mathbb{C}}$. If G_1 is a reductive group with adjoint group G, a lifting of X^+ to a $G_1(\mathbb{R})^+$-conjugacy class X_1^+ of morphisms from $\mathbb{S}$ to $G_{\mathbb{R}}$ defines a lifting $M(X_1^+)$ of M: this is the conjugacy class of μ_h for $h \in X_1^+$ (cf. 2.5.4).

LEMMA 2.7.15. *The construction $X_1^+ \to M(X_1^+)$ puts in one-to-one correspondence the liftings of X^+ and those of M.*

The construction $h \to \mu_h$ is a bijection of the set of morphisms h of S into a real group G with the set of morphisms μ of $\mathbb{G}_m$ into $G_{\mathbb{C}}$ that commute with their complex conjugate: we have $h(z) = \mu(z)\overline{\mu}(z)$. Via this dictionary, the problem becomes that of checking that $\mu_1 \colon \mathbb{G}_m \to G_{\mathbb{C}}$ commutes with $\overline{\mu}_1$. This follows from the rigidity of tori: the morphism $\operatorname{inn} \overline{\mu}_1(z)(\mu_1)$ coincides with μ_1 for $z = 1$, and lifts μ_1 for every value of z. It is therefore constantly equal to μ_1.

This dictionary allows us to translate 2.5.5 into the

LEMMA 2.7.16. *Let G, G', and X^+ be as in 2.7.10. There exists a reductive group G_1 with adjoint group G and derived group G' and a*[65] *$G_1(\mathbb{R})^+$-conjugacy class X_1^+ of morphisms of $\mathbb{S}$ into G that lifts X^+ and is such that $E(G, X^+) = E(G_1, X_1^+)$.*

2.7.17. This lemma, and the equivalence 2.7.13, allow us to transfer to the weakly canonical models of connected Shimura varieties the results of [**5**] on the weakly canonical models of Shimura varieties, and to establish an equivalence between the corresponding construction problems.

COROLLARY 2.7.18. *Let (G, X) be as in 2.1.1, X^+ a connected component of X, and $E \subset \overline{\mathbb{Q}}$ a finite extension of $E(G, X)$. In order for $M(G, X)$ to admit a weakly canonical model over E, it is necessary and sufficient that $M^\circ(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ admit one. In particular, the existence of such a model depends only on $(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+, E)$.*

COROLLARY 2.7.19. *(Cf.* [**5**, 5.5, 5.10, 5.10.2]*.) Let G, G', X^+, and E be as in 2.7.10.*

(i) *$M^\circ(G, G', X^+)$ admits at most one weakly canonical model over E (uniqueness up to a unique isomorphism).*
(ii) *Suppose that, for every extension F of E, there exists a finite extension F' of E in $\overline{\mathbb{Q}}$, linearly disjoint from F, and a weakly canonical model of $M^\circ(G, G', X^+)$ over F'. Then there exists a weakly canonical model of $M^\circ(G, G', X^+)$ over E.*

The corollary 2.7.19 and 2.3.1, 2.3.10 provide numerous canonical models.

THEOREM 2.7.20. *Let G be a $\mathbb{Q}$-simple adjoint group, G' a finite covering of G, and X^+ a $G(\mathbb{R})^+$-conjugacy class of morphisms from $\mathbb{S}$ to $G_{\mathbb{R}}$ satisfying (2.1.1.1), (2.1.1.2), (2.1.1.3). In the following cases, $M^\circ(G, G', X^+)$ admits a canonical model*

(a) *G is of type A, B, C and G' is the universal finite covering of G.*
(b) *(G, X) is of type $D^{\mathbb{R}}$ and G' is the universal finite covering of G.*

[65] The original has $G_1(\mathbb{R})^\circ$.

(c) (G, X) *is of type* $D^{\mathbb{H}}$ *and* G' *is the covering* 2.3.8 *of* G.

Applying 2.7.11, 2.7.18, we deduce the

COROLLARY 2.7.21. *Let* G *be a reductive group,* X *a* $G(\mathbb{R})$*-conjugacy class of morphisms from* $\mathbb{S}$ *into* $G_{\mathbb{R}}$ *satisfying the conditions 2.1.1, and* X^+ *a connected component of* X*. In order for* $M(G, X)$ *to admit a canonical model it suffices that* (G^{ad}, X^+) *be a product of systems* (G_i, X_i^+) *of the type considered in 2.7.20, and that the finite covering* G^{der} *of* G^{ad} *be a quotient of a product of finite coverings of the* G_i *considered in 2.7.20.*

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