Bures-sur-Yvette, April 2, 1973

Dear Пятецкий-Шапиро,

I am ashamed the enclosed letter is so badly written, and written more for my sake than for yours. In it, I claim (except at 2) to prove for the supercuspidal representations what in your notes you prove for the principal (unramified serie. The idea is that

a) room is left for it in your notes only thanks to the supersingular elliptic curves;

b) supersingular elliptic curves correspond to ideals classes in the quaternion algebra ramified at p and ∞ ;

c) this, by a global argument using Jacquet Langlands § 14, forces the outcome

Corollary 1 : Let E be an elliptic curve $/\mathbb{Q}$, which is a direct factor (up to isogeny) of a jacobian of a modular curve, corresponding to a new form w_E . Then, except perhaps at 2, the conductors of E and w_E are the same (At 2, I still can prove $2 | f_E \Leftrightarrow 2 | f_{w_E}$)

Corollary 2 : (Casselman-Morita) : Let H be a quaternion algebra over $\mathbb{Q}$ split at ∞ , p be a prime at which H ramifies, $\mathcal{O}$ be an order of H maximal at p and $M = X/\mathcal{O}^*$ [X = Poincaré upper half plane]. Then, the irreductible components of the reduction of M mod p are rational. [M is defined over a cyclotomic field unramified at p].

Mumford has nice ideas about the compactification of $K\backslash G/\Gamma$ ($K\backslash G$ hermitian symmetric). For the moduli of abelian varieties, I hope it will eventually give nice compactification over $\mathbb{Z}\left[\frac{1}{\text{obvious primes}}\right]$.

I am looking forward meeting you again.

Yours most sincerely,

P. DELIGNE

Princeton, march 25, 1973

Dear Пятецкий-Шапиро,

Except at 2, I have now a good understanding of the bad primes of the ℓ-adic representations attached to modular forms (for $GL(2,\mathbb{Q})$)

A. Isogenies

Dfn The category of elliptic curves up to isogeny is obtained from that of elliptic curves by inverting isogenies

ie α) an elliptic curve E defines an elliptic curve up to isogeny $E \otimes \mathbb{Q}$

β) $\mathrm{Hom}(E \otimes \mathbb{Q}, F \otimes \mathbb{Q}) = \mathrm{Hom}(E, F) \otimes \mathbb{Q}$.

Lemma if F is a functor ~~[illegible]~~ (ell. curves) $\longrightarrow$ (- - -), and F(any isogeny) is an isomorphism, F makes sense for elliptic curves up to isogeny.

Notations: $T_\ell(E) = \varprojlim E_{\ell^n}$ (for ℓ prime to p, E/k alg clos of char p, it is a free module of rank 2 on $\mathbb{Z}_\ell$)

$V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ make sense for ell. curves up to isogeny

$\hat{T}_{p'}(E) = \varprojlim_{(p,n)=1} E_n$

$\hat{V}_{p'}(E) = \hat{T}_{p'}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ a $\mathbb{A}^{f,p'} = \prod_{\ell \neq p,\infty} \mathbb{Q}_\ell$ -module of rank 2 makes sense for ell. curves up to isogeny

$\mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}$

$\mathbb{Q}_\ell(1) = \cancel{\lim}\ \mathbb{Z}_\ell(1) \otimes \mathbb{Q}_\ell$

(1) : $\otimes \mathbb{Q}_\ell(1)$ or $\otimes \mathbb{Z}_\ell(1)$

$$d : \begin{cases} d(E) = \mathbb{Z} \\ f : E \longrightarrow F \rightsquigarrow d(f)\ d(E) \longrightarrow d(F) \text{ is } \deg(f) \quad (0 \text{ if } f = 0) \end{cases}$$?

Thm a) $d(E)\otimes\mathbb{Q}$ make sense for elliptic curve up to isogeny

notation $d(E\otimes\mathbb{Q}) = d(E)\otimes\mathbb{Q}$.

b) for E_0 an elliptic curve up to isogeny, the e_n-pairings, and their behaviour under isogeny, enables one to define

$$\overset{2}{\Lambda} V_\ell(E_0) \simeq d(E_0)\otimes\mathbb{Q}_\ell(1)$$

At p: Let us look at the case of supersingular curves of char p; then, a substitute for $T_p(E)$ is the formal group of E (a height 2 dim 1 formal group). The formalism of d has the following analogue

a) a height 2 dim 1 formal group defines $d_p(F)$, a rank one module over $\mathbb{Z}_p$: $d_p(F)$ = set of $F \xrightarrow{\sim} F^*$ (F^* = Cartier-dual), with ${}^t u = -u$.

b) if $\tilde{F}/S$ is a deformation of F over S (local complete), and if $T_p(\tilde{F})$ is the corresponding local system on $S\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, then
$$\overset{2}{\Lambda} T_p(\tilde{F}) \simeq d_p(F)(1)$$

c) for E a supersingular elliptic curve, with corresponding formal group $\hat{E}$, $\quad d(E)\otimes\mathbb{Z}_p \simeq d_p(\hat{E})$

Recovering E Let E_0 be $\underbrace{\text{an}}_{\text{supersingular}}$ elliptic curve up to isogeny. An elliptic curve E with an isomorphism $E\otimes\mathbb{Q} \xrightarrow[\rho]{\sim} E_0$ defines

a) a "lattice" $\hat{T}_{p'}(E)$ in $\hat{V}_{p'}(E_0)$

b) a "lattice" $d(E)$ in $d(E_0)$ (the p'-aspect of it is determined by a): $d(E)\otimes\mathbb{Z}_\ell(1) \simeq \overset{2}{\Lambda} T_\ell(E) \subset \overset{2}{\Lambda} V_\ell(E) = d(E_0)\otimes\mathbb{Q}_\ell(1)$)

Lemma 1 It amounts to the same to give either [E_0 supersingular]

α) (E,ρ)

β) the lattices $\hat{T}_{p'}(E) \subset \hat{V}_{p'}(E_0)$ and $d(E)\otimes\mathbb{Z}_p \subset d(E_0)\otimes\mathbb{Q}_p$

The reason why no more information is required at p is that, for supersingular E, the only degree p^k-isogeny of source E is $E \to E^{(p^k)}$.

Variant Let F_0 be a height 2 dim 1 formal group law, up to isogeny. Then, to give F defining F_0 amounts to give $d_p(F) \subset d_p(F_0)$
$(\sim \mathbb{Z}_p)$ $(\sim \mathbb{Q}_p)$

B. The fundamental local construction

Let:

$\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$

$\overline{\mathbb{F}}_p$ be the alg closure of $\mathbb{F}_p$, residue field of $\overline{\mathbb{Q}}_p$

k be an alg closure of $\mathbb{F}_p$, provided with a class modulo integral powers of Frobenius of allowed isomorphism $k \xrightarrow{\sim} \overline{\mathbb{F}}_p$. The class is denoted $\mathrm{hom}(k, \overline{\mathbb{F}}_p)$

V_0 be a 2-dimensional vector space over $\mathbb{Q}_p$

E_0 be a supersingular elliptic curve up to isogeny over k

F_0 be the formal group law up to isogeny $/k$ defined by E_0.

To be able to use "transport de structure", I prefer not to take $k = \overline{\mathbb{F}}_p$ nor $V_0 = \mathbb{Q}_p^2$

(1) First construction

Let

$\sigma \in \mathrm{hom}(k, \overline{\mathbb{F}}_p)$ and $\beta \in \mathrm{hom}(d(E_0) \otimes \mathbb{Q}_p(1), \overset{2}{\Lambda} V_0)$.

Here, $\mathbb{Q}_p(1)$ is relative to $\overline{\mathbb{Q}}_p$:

$$\mathbb{Q}_p(1) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \left(\varprojlim_n \text{ (group of } p^n\text{-roots of unity of } \overline{\mathbb{Q}}_p) \right)$$

To σ and β we define $E_0(\sigma, \beta) = (\sigma(E_0), \sigma(\beta))$ where

$\sigma(E_0)$ is an elliptic curve up to isogeny on $\overline{\mathbb{F}}_p$

$\sigma(\beta)$ is an isomorphism of one-dimensional vector spaces over $\mathbb{Q}_p$

$$d(\sigma(E_0)) \otimes \mathbb{Q}_p(1) \underset{\sigma}{=} d(E_0) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \overset{2}{\Lambda} V_0$$

Let φ be the Frobenius ~~auto~~ substitution $\varphi\ x \mapsto x^p$. $\bar{k} \to \bar{k}$.

Then, ~~lies~~ for any elliptic curve $E/\bar{k}$, $\varphi(E)$ is $E^{(p)}$, and one disposes of the Frobenius isogeny $F \cdot E \longrightarrow E^{(p)}$.

The diagramm

$$\begin{array}{ccc} d(E) & \xleftrightarrow{\ \varphi\ } & d(E^{(p)}) \\ \| & & \| \\ \mathbb{Z} & = & \mathbb{Z} \\ \| & & \uparrow p \\ d(E) & \xrightarrow{\ F\ } & d(E^{(p)}) \end{array}$$

is commutative.

In particular, F is an isomorphism $F\ E_0 \xrightarrow{\sim} \varphi(E_0)$, and the diagramm

$$\begin{array}{ccc} d(E_0) & \xrightarrow{\ \varphi\ } & d(\varphi(E_0)) \\ \| & & \downarrow p \\ d(E_0) & \xrightarrow{\ F\ } & d(\varphi(E_0)) \end{array}$$

is commutative.

F hence induces an isomorphism

(1) $\quad F$ or $\sigma F \quad (\sigma(E_0), \sigma(\beta)) \xrightarrow{\sim} (\sigma\varphi(E_0), \sigma(p^{-1}\beta))$

Definition D_p is the one dimensional vector space over $\mathbb{Q}_p$, quotient of

$$\mathrm{hom}(k, \bar{\mathbb{F}}_p) \times \mathrm{hom}(d(E_0) \otimes \mathbb{Q}_p(1), \textstyle\bigwedge^2 V_0)$$

by the equivalence relation $(\sigma, \beta) \sim (\sigma\varphi^k, p^{-k}\beta)$ for $k \in \mathbb{Z}$.

(1) defines an isomorphism between $E_0(\sigma,\beta)$ and $E_0(\sigma',\beta')$ for $(\sigma,\beta) \sim (\sigma',\beta')$, and those isomorphisms form a transitive system of isomorphisms. They hence allow us to define $(E_0(\delta), \beta(\delta))$ for $\delta \in D_p$, where

$\begin{cases} E_0(\delta) \text{ is a (supersingular) elliptic curve up to isogeny on } \bar{\mathbb{F}}_p, \text{ and} \\ \beta(\delta) \text{ is an isomorphism } d(E_0(\delta)) \otimes \mathbb{Q}_p(1) \Rightarrow \bigwedge^2 V_0 \end{cases}$

Remark The following groups are acting on D_p (by "transport de structure")

~~1) [illegible]~~

2) $W(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, via its actions both on $\mathrm{hom}(k, \bar{\mathbb{F}}_p)$ and $\mathbb{Q}_p(1)$. If the isomorphism of $W(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab} = \mathbb{Q}_p^*$ is normalised $(\pm)$ so that Frobeniuses φ correspond to inverse of uniformizing parameter, then

$$\sigma\delta = d(\sigma)^{-1}\cdot\delta$$

γ) $GL(V_0)$, via its action on $\overset{2}{\Lambda} V_0$. One has

$$g.\delta = \det(g).\delta$$

δ) $\mathrm{Aut}(E_0)$, via its action on $d(E_0)$. We define

$$H = \mathrm{Aut}(E_0),$$

it is the multiplicative group of a quaternion algebra ramified at p and at ∞. One has $g.\delta = \mathrm{Nrd}(g)^{-1}.\delta$ (reduced norm)

<u>Remark 2</u> Assume that a lattice $\Lambda \simeq \mathbb{Z}_p$ has been chosen in $\overset{2}{\Lambda} V_0$. Then, $(E_0, \beta)/\overline{\mathbb{F}}_p$ as above define an elliptic curve up to p'-isogeny on $\overline{\mathbb{F}}_p$, corresponding to the lattice $\beta^{-1}(\Lambda)(-1) \subset d(E_0) \otimes \mathbb{Q}_p$. p'-isogeny means isogeny of degree prime to p. (cf Lemma A1).

(1') <u>Variant</u> Let us start with F_0 instead of E_0. D_p is defined as before, using $d_p(F_0)$ instead of $d(E_0) \otimes \mathbb{Q}_p$, and is the same as before, via the isomorphism $d(E_0) \otimes \mathbb{Q}_p \simeq d_p(F_0)$ ($F_0 = \widehat{E}_0$). This time, D_p is acted upon by ~~[illegible]~~ $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $GL(V_0)$ and $\mathrm{Aut}(F_0) = H(\mathbb{Q}_p)$. The formulae are the same as before.

$\delta \in D_p$ defines

$$\begin{cases} F_0(\delta) & \text{, a (supersingular) formal group law up to isogeny on } \overline{\mathbb{F}}_p \\ \beta(\delta) & \text{, } d(F_0(\delta)) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \overset{2}{\Lambda} V_0 \end{cases}$$

If Λ has been chosen in $\overset{2}{\Lambda} V_0$, one get

$$\begin{cases} F(\delta) & \text{, a (supersingular) formal group law on } \overline{\mathbb{F}}_p \\ \beta(\delta) & \text{, } d(F(\delta)) \otimes \mathbb{Z}_p(1) \xrightarrow{\sim} \Lambda \end{cases}$$

(2) Second construction

I have now to define "vanishing cycles varieties" and vanishing cycles groups. Eventually, for K° an open compact subgroup of ~~GL(V)~~ $SL(V_0)$, and for $\delta \in D_p$, a scheme $V(K^\circ, \delta)$ over $\overline{\mathbb{Q}}_p$ will be defined. For $K^\#$ an open compact subgroup of $GL(V_0)$ such that $K^\circ = K^\# \cap SL(V_0)$, isomorphisms

$$V(K^\circ, \delta) \xleftrightarrow{(K^\#)} V(K^\circ, \alpha\delta) \qquad (\text{for } \alpha \in \det K^\circ \subset \mathbb{Z}_p^*)$$

are defined. For variable δ, the $V(K^\circ, \delta)$'s will thus be a "local system" ~~of D_p~~ of schemes on D_p. The $V(K^\circ, \delta)$ are not of finite type, but of the type usual in the vanishing cycle theory. However we won't need it, here is a description of the ~~pts~~ $\overline{\mathbb{Q}}_p$-valued points of $V(K^\circ, \delta)$.

— A $\overline{\mathbb{Q}}_p$-point ~~is an isomorphism class of systems consisting in~~:

~~*) an elliptic curve up to isogeny~~

a) δ provides us with $(E_0(\delta), \beta(\delta))$ on $\overline{\mathbb{F}}_p$

b) a point of $V(K^\circ, \delta)$ ~~consist~~ is an isomorphism class of systems consisting in:

α) an elliptic curve up to isogeny on $\overline{\mathbb{Q}}_p$: E

β) an isomorphism $\phi : V_p(E) \xrightarrow{\sim} V_0$, given mod K°

γ) an isomorphism of the reduction of E with $E_0(\delta)$. $\Psi\ E|_{\overline{\mathbb{F}}_p} = E_0(\delta)$

δ) required: ~~det(β)~~ the diagramm

$$(*)\qquad \begin{array}{ccc} \overset{2}{\Lambda} V_p(E) & \xrightarrow{\overset{2}{\Lambda}\phi} & \overset{2}{\Lambda} V_0 \\ \| & & \uparrow{\scriptstyle s}\ \beta(\delta) \\ \mathcal{L}(E)(1) & \xrightarrow[\Psi]{=} & \mathcal{L}(E_0(\delta))(1) \end{array}$$

is commutative.

If $K^\#$ is as above, it amounts to the same to give ~~ϕ~~ only mod $K^\#$, $(*)$ being required only mod $\det(K)$.

To actually construct $V(K,\delta)$, let us start with

K open compact ~~of~~ subgroup of $GL(V_0)$

E_0 a supersingular curve up to isogeny on $\overline{\mathbb{F}}_p$ — in practice, $E_0(\delta)$, $\rho(\delta)$

$\rho : d(E_0)\otimes\mathbb{Q}_p(1) \xrightarrow{\sim} \overset{2}{\Lambda} V_0$

$\bar\rho$ ρ mod $\det(K) \subset \mathbb{Z}_p^*$ (only $\bar\rho$, not ρ, will be used).

The construction will involve the auxiliary data of

L a K-stable lattice in V_0.

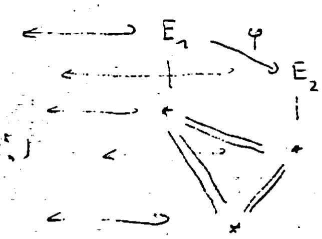
a) $\rho^{-1}(\overset{2}{\Lambda} L)(-1)$ is a lattice in $d(E_0)\otimes\mathbb{Q}_p$ and defines an elliptic curve up to p'-isogeny E', with $E'\otimes\mathbb{Q} \simeq E_0$. Let me choose an elliptic curve E with level n structure α_n ($n\geq 3$, $(n,p)=1$) with $E\otimes\mathbb{Z}_{(p)} = E'$.
[the construction will be shown to be independent of L, E', E, α_n].
Let M_n be the modular scheme for elliptic curves with level n structure and $e\in M_n(\overline{\mathbb{F}}_p)$ be the point of M_n defined by (E,α_n).

b) $M(E,\alpha_n)$ is the spectrum of the henselization of the local ring at e of $M_n\otimes_{\mathbb{Z}} W(\overline{\mathbb{F}}_p)$.

c) the completion of $M(E_*,\alpha_n)$ is isomorphic to $W(\overline{\mathbb{F}}_p)[[t]]$.

d) Assume that (E_1,α_n^1) (E_2,α_n^2) are two systems as above, and that $\varphi : E_1 \to E_2$ is a p'-isogeny. There is then one and only one isomorphism $\bar\varphi : M(E_1,\alpha_n^1) \to M(E_2,\alpha_n^2)$ fitting in a commutative diagramm

$$\begin{tikzcd}
\tilde E_1 \arrow[r, "\tilde\varphi"] \arrow[d, dash] & \tilde E_2 \arrow[d] \\
M(E_1,\alpha_n^1) \arrow[r, "\bar\varphi"] \arrow[dr] & M(E_2,\alpha_n^2) \arrow[d] \\
& W(\overline{\mathbb{F}}_p)
\end{tikzcd}$$



In this diagramm, $*$ means $\mathrm{Spec}(\overline{\mathbb{F}}_p)$ (= point), and $\tilde{E}_1, \tilde{E}_2$ are the pull-back over $M(E_1, \alpha_n^1)$, $M(E_2, \alpha_n^2)$ of the universal curve over M_n.

This can be better expressed by saying that $M(E_1, \alpha_n^1)$ is the parameter space of the universal deformation of the elliptic curve up to p'-isogeny $E'/\overline{\mathbb{F}}_p$ (universal is with respect to deformation over henselian local ~~ws~~ $W(\overline{\mathbb{F}}_p)$-algebras). This allows us to write simply $M(E')$ for $M(E_1, \alpha_n^1)$, and $\tilde{E}'$ for the elliptic curve up to p'-isogeny over $M(E')$ defined by (any) E_1.

For n large enough, K is the subgroup of $GL(L)$ inverse image of a suitable subgroup $\overline{K}$ of $GL(L/p^n L)$.

Let $K(\overline{\mathbb{F}}_p)$ be the field of fractions of $W(\overline{\mathbb{F}}_p)$, and let $\overline{\mathbb{Z}}_p$ be the ring of integers in $\overline{\mathbb{Q}}_p$. The group scheme E'_{p^n} over $M(E')$ is finite étale over $M(E') \otimes K(\overline{\mathbb{F}}_p)$, hence

$$\underline{\mathrm{Isom}}\,(E'_{p^n}, L/p^n L)$$

is a finite étale Galois covering of $M(E') \otimes K(\overline{\mathbb{F}}_p)$, with Galois group $GL(L/p^n L)$ Let us rather consider

$$\underline{\mathrm{Isom}}_{M(E') \otimes \overline{\mathbb{Q}}_p}(E'_{p^n}, L/p^n L).$$

This time, we get a <u>disconnected</u> covering of $M(E') \otimes \overline{\mathbb{Q}}_p$. A piece of it can be picked as follow: via β, one has $\stackrel{2}{\wedge} E'_{p^n} = d(E')_p \otimes \mathbb{Z}/p^n(1)$ $= \stackrel{2}{\wedge} L/p^n L$, and one consider only isomorphismes of "determinant 1".

Similarly, $\overline{\beta}$ enable one to pick a component of

$$\overline{K} \backslash \underline{\mathrm{Isom}}_{M(E') \otimes \overline{\mathbb{Q}}_p}(E'_{p^n}, L/p^n L)$$

We call this component $V_L(K, E_0^*, \beta)$. It is also the quotient by $\bar{K} \cap SL(L/p^n L)$ of the picked component of

$$\underline{\mathrm{hom}}_{\Gamma(E')}(E'_{p^n}, L/p^nL) \otimes_{W(\bar{\mathbb{F}}_p)} \bar{\mathbb{Q}}_p .$$

Summary ~~Remark~~ $V_L(K, E_0^*, \beta)$ depends on $K \subset GL(V_0)$, compact open, E_0^* up to ~~p~~ isogeny on $\bar{\mathbb{F}}_p$, β $d(E') \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \wedge^2 V_0$, and of a lattice L in V_0, stable by K. ~~Fr~~ It does not depend on the whole of β, but only on $\bar{\beta} = \beta$ mod $\det(K)$. For $K' \subset K$, one has a map

$$V_L(K', E', \beta) \longrightarrow V_L(K, E', \beta)$$

<u>Construction</u> <u>$V_L(K, E_0^*, \beta)$ is independent of L</u>

More precisely, the system consisting in

$$\left\{\begin{array}{l} V_L(K, E', \beta) \\ \text{the elliptic curve up to isogeny } \tilde{E}_0' \text{ over } V_L(K, E', \beta), \\ \text{the universal isomorphism } \cancel{\alpha}\ \tilde{E}_3 \text{ given mod } K \quad V_p(\tilde{E}_0') \longrightarrow V_0 \end{array}\right.$$

(deduced from $\bar{\alpha}$: $\cancel{T_p(E')} \; \tilde{E}_{p^n} \xrightarrow{\sim} L/p^nL$, mod $\bar{K}$

or $\bar{\alpha}$ $\quad T_p(\tilde{E}') \xrightarrow{\sim} L$ mod K)

is independent of L.

Let L_1 and L_2 be two K-invariant lattices, and let E_1 and E_2 be the corresponding elliptic curves up to p-isogeny on $\bar{\mathbb{F}}_p$. We are to define an isomorphism

~~$\underline{\mathrm{hom}}_{M(E_1)}(V(E_1),$~~

$$\begin{tikzcd} K \backslash \underline{\mathrm{hom}}_{M(E_1) \otimes K(\bar{\mathbb{F}}_p)}\left(\begin{array}{c} V_p(\check{E}_1) \\ \cup \\ T_p(\check{E}_1) \end{array}, \begin{array}{c} V_0 \\ \cup \\ L_1 \end{array} \right) \arrow[r, leftrightarrow] \arrow[d] & K \backslash \underline{\mathrm{hom}} \text{ same with 2 instead of 1.} \arrow[d] \\ M(E_1) & M(E_2) \end{tikzcd}$$

For simplicity, we assume $L_1 \subset L_2$. Then, over $K \backslash \underline{\mathrm{Isom}}_{M(E_1)} \cdots$, $\tilde{E}_1$ is provided with a subgroup isomorphic to L_2/L_1, call it H. Let $\overline{W}_1(K, E_0, \beta)$ be the normalization of $M(E_1)$ in $K \backslash \underline{\mathrm{Isom}} \cdots$. Due to the normality of $\overline{W}$ and the fact that an elliptic curve has only finitely many subgroupscheme of given order, the subgroup-scheme of H of $\tilde{E}_1$ on $\overline{W}_1(K, E_0, \beta) \otimes K(\overline{\mathbb{F}}_p)$ extends as a subgroup-scheme of H of $\tilde{E}_1$ on $\overline{W}_1(K, E_0, \beta)$. The quotient $\tilde{E}_1/H$ is a deformation of E_2, hence a map

$$\overline{W}_1(K, E_0, \beta) \longrightarrow M(E_2).$$

Further, $V_p(\tilde{E}_1/H) = V_p(\tilde{E}_1)$, and, over $K \backslash \underline{\mathrm{Isom}}\left(\begin{matrix} V_p(\tilde{E}_1) \\ \cup \\ T_p(\tilde{E}_1)\end{matrix}, \begin{matrix} V_0 \\ \cup \\ L_1\end{matrix}\right)$, isomorphism $\alpha \cdot V_p(E_1) \to V_0$ carrying $T_p(\tilde{E}_1)$ to L_1 do carry $T_p(\tilde{E}_1/H)$ to L_2: if W_2 is defined as W_1, one has

$$\begin{tikzcd}
K \backslash \underline{\mathrm{Isom}}_{M(E_1) \otimes K(\overline{\mathbb{F}}_p)}\left(\begin{matrix} V_p \\ \cup \\ T_p\end{matrix}(\tilde{E}_1), \begin{matrix} V_0 \\ \cup \\ L_1\end{matrix}\right) \arrow[r, dashed] \arrow[d, hook] \arrow[rd] \arrow[rdd] & K \backslash \underline{\mathrm{Isom}}_{M(E_2) \otimes} \cdots \arrow[d, hook] \\
\overline{W}_1 \arrow[r, dashed] \arrow[d] \arrow[rd, "\text{by } \tilde{E}_1/H"] & \overline{W}_2 \arrow[d] \\
M(E_1) & M(E_2)
\end{tikzcd}$$

This defines the dotted maps; they are the seeked for isomorphism.

By extending the scalars to $\overline{\mathbb{Q}}_p$ and taking one component, one get the isomorphisms expressing that $V_L(K, E_0, \beta)$ is independant of L. For $\delta \in D_p$, we note

$$V(K, \delta) = V_L(K, E_0(\delta), \beta(\delta))$$

<u>Summary</u> $V_{\xi}(K,\delta)$ is a scheme over $\bar{\mathbb{Q}}_p$, it depends on K compact open in $GL(V_0)$ and on $\delta \in D_p$, given modulo multiplication by elements of $\det(K) \subset \mathbb{Z}_p^+$. For K smaller and smaller, the $V(K,\delta)$ form a projective system. Over $V(K,\delta)$ is given an elliptic curve up to isogeny $\tilde{E}$, provided with an isomorphism given mod K $\alpha : V_p(\tilde{E}) \xrightarrow{\sim} V_0$. In a sense, $\tilde{E}$ is a deformation of $E_0(\delta)$, in particular $d(E_0(\delta)) = d(\tilde{E})$. The isomorphism α is compatible with $\rho(\delta)$.

$$\overset{2}{\Lambda} V_p(\tilde{E}) = d(E_0(\delta)) \otimes \mathbb{Q}_p(1) \underset{\beta(\delta)}{\overset{\det\alpha}{\underset{\parallel}{\longrightarrow}}} \overset{2}{\Lambda} V_0 \qquad (\mathrm{mod}\ \det K).$$

In fact, the statement that $\tilde{E}$ is a deformation of E_0 can be made more precise by introducing a suitable $\tilde{E} / \bar{V}(K,\delta) / \bar{\mathbb{Z}}_p$.

<u>(2') Variant</u> Starting with F_0 instead of E_0, one can construct analogues of the $V(K,\delta)$, called $\hat{V}(K,\delta)$, with complete local rings replacing henselian local rings.

<u>(3) Third construction</u>

We are interested in the l-adic cohomology groups

$$H^{n}(V(K,\delta), \mathbb{Q}_{\ell}).$$

Those groups are finite dimensional, and locally constant as a function of δ (as $V(K,\delta)$ itself is). If K' is a distinguished subgroup of K, one clearly has

$$H^{n}(V(K,\delta), \mathbb{Q}_{\ell}) = H^{\cdot}(V(K',\delta), \mathbb{Q}_{\ell})^{K\cap SL(V_0)/K'\cap SL(V_0)} \qquad (\text{invariants})$$

($V(K,\delta)$ depends also only on $K \cap SL(V_0)$ and δ)

The <u>local fundamental object</u> is the "bundle" $\mathcal{X}/D_p$, with

$$\mathcal{X}_\delta = \varinjlim_K H^n(V(K,\delta), \mathbb{Q}_\ell) .$$

There is a notion of "locally constant section of $\mathcal{X}_\delta / D_p$" : it is a function $\varphi(\delta)$ $(\delta \in D_p, \varphi(\delta) \in \mathcal{X}_\delta)$ with locally $\varphi(\delta)$ in a $H^n(V(K,\delta), \mathbb{Q}_\ell)$ and locally constant. The space of locally constant sections is noted $\Gamma(D_p, \mathcal{X})$

On the local fundamental object are acting, by "transport de structure":

~~[illegible]~~

a) $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

b) $GL(V_0)$

c) $\mathrm{Aut}(E_0) = H$

The actions over D_p being those already described.

The actions of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $GL(V_0)$ are "continuous" (the latter, with respect with the notion of locally constant section).

<u>Proposition</u> The action of $\mathrm{Aut}(E_0)$ extends as a continuous action of $H(\mathbb{Q}_p)$ on $\mathcal{X}/D_p$

I don't have a satisfactory proof. My idea of proof would be to go back to the $\overline{W}$ introduced earlier and to express $\mathcal{X}$ in terms of special fiber of stable models of $\overline{W}$ over suitable ramified extensions of $W(\overline{\mathbb{F}}_p)$.

Intuitively, one may argue that $V(K,\delta)$ and $\hat{V}(K,\delta)$ could have the same cohomology, that $\mathrm{Aut}(F_0) = H(\mathbb{Q}_p)$ acts on $\hat{V}(K,\delta)$, hence on $H^n(\hat{V}(K,\delta), \mathbb{Q}_\ell)$, and that it would be the hell if the action were not continuous

C Statement of the local results

(The proofs will be of a global nature)

It will be easier to work not with $\mathbb{Q}_\ell$ cohomology, but with $\overline{\mathbb{Q}}_\ell$-cohomology, obtained by extending $\mathbb{Q}_\ell$ to an algebraic closure $\overline{\mathbb{Q}}_\ell$. By abuse of language, we will again denote ~~by~~ $\mathcal{H}/D_p$ the "admissible" bundle over D_p, with fibres

$$\mathcal{H}_\delta = \varinjlim_K H^1(V(K,\delta), \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell .$$

The groups $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $GL(V_0)$, $H(\mathbb{Q}_p)$ act admissibly on $\mathcal{H}/D_p$. The actions on D_p have been computed. Of course, the four actions commute the one with the other. If $a \in \mathbb{Q}_p^*$, the actions of the elements $a \in GL(V_0)$ and $a \in H(\mathbb{Q}_p)$ are <u>inverse</u> the one of the other. In the term of a fixed $\delta_0 \in D_p$, this could as well be expressed by saying

A/ the subgroup of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times GL(V_0) \times H(\mathbb{Q}_p)$

formed by the elements such that $\ell(\sigma)^{-1} \cdot \det(g) \cdot \mathrm{Nrd}(\ell)^{-1} = 1$ acts on $\mathcal{H}_{\delta_0}$

B/ the subgroup $(1, 1, a, a)$ acts trivially.

<u>Mnemotechnik way</u> to check the action on the $V(K,\delta)/D_p$, it is good to view a point of $\varprojlim V(K,\delta)/D_p$ as consisting of

- $\delta \in D_p$
- $\tilde{E}$ elliptic curve up to isogeny on $\overline{\mathbb{O}}_p$
- α $V_p(\tilde{E}) \xrightarrow{\sim} V_0$
- Ψ "specialisation map" $\tilde{E} \overset{\Psi}{\dashrightarrow} \delta(E_0)$

with a compatibility between α, Ψ et $\rho(\delta)$

Let $\chi : \mathbb{Q}_p^* \to \overline{\mathbb{Q}}_\ell^*$ be a quasi-character (with open kernel)

We denote by

$\Gamma(D_p, \mathcal{X})$ the $\overline{\mathbb{Q}}_\ell$-vector space of locally constant sections of $\mathcal{X} / D_p$

$\Gamma_\chi(D_p, \mathcal{X})$ the subspace of those sections ~~f~~ for which, for any $a \in \mathbb{Q}_p^*$, with image $z(a)$ in ~~$GL(V_0)$~~ the centrum of $GL(V_0)$,

$$~~f~~ \quad z(a) \cdot f = \chi(a) \cdot f \quad , \text{ ie}$$

$$z(a) f(\delta) = \chi(a) f(a^2 \delta)$$

<u>Theorem</u> (i) $\Gamma_\chi(D_p, \mathcal{X})$ is a direct sum of triple tensor products

$$\Gamma_\chi(D_p, \mathcal{X}) = \bigoplus_{\varphi \in \Phi} V_\varphi \otimes V'_\varphi \otimes W_\varphi \qquad \text{where}$$

α) V_φ is an admissible ~~representation~~ irreducible of $GL(V_0)$, the centrum of $GL(V_0)$ acting by the character χ, and V_φ being of the ~~discrete serie~~ (= special or supercuspidal)

β) V'_φ is an admissible irreducible (finite dimentional) representation of $H(\mathbb{Q}_p)$, with the centrum acting by χ^{-1}

γ) W_φ is a 2-dimentional (or 1-dimentional) continuous $\overline{\mathbb{Q}}_\ell$-adic irreducible representation of $W(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$. If it is 2-dimentional, then $W(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$ acts on $\overset{2}{\Lambda} W_\varphi$ by the character . If 1-dimentional, defined by a character ν, then $\nu =$ ($\overset{2}{\Lambda}(W_\varphi \oplus W_\varphi(1))$ corresponds again to)

(ii) V'_φ runs (once and only once) through the representations said in (i)

(iii) The same holds for V_φ, and V_φ and V'_φ correspond by the Weil representation construction (suitably normalized)

(iv) W_φ 1-dimentional $\Leftrightarrow$ V'_φ is $\Leftrightarrow$ V_φ special, and ~~#~~ if V'_φ is $\mu(\mathrm{Nrd})$, then W_φ is the character of $W(\overline{\mathbb{Q}}_p / \mathbb{Q}_p) \xrightarrow[\mathcal{C}]{} \mathbb{Q}_p^*$

(v) If V_φ is defined by a ~~character of a~~ quasi-character of a quadratic extension of $\mathbb{Q}_p$ (with values in $\overline{\mathbb{Q}}_\ell^*$), then (with a suitable normalization) W_φ is induced by that same character

Except for $p=2$, this gives a complete description of $\Gamma_\chi(D_p, \mathcal{X})$.

For μ a quasi-character of $\mathbb{Q}_p^*$ and $\delta_0 \in D_p$, multiplication by the function $\mu(\delta\delta_0^{-1})$ on D_p provides an isomorphism

$$\Gamma_\chi(D_p, \mathcal{X}) \longrightarrow \Gamma_{\chi\mu^{-2}}(D_p, \mathcal{X}) \quad ;$$

if $V_\varphi \otimes V'_\varphi \otimes W_\varphi$ occurs in $\Gamma_\chi(D_p, \mathcal{X})$, then

$$(V_\varphi \otimes \mu^{-1}\det(g)) \otimes (V'_\varphi \otimes \mu\, \mathrm{Nrd}) \otimes (W_\varphi \otimes \mu\, \mathrm{cl})$$

occurs in $\Gamma_{\chi\mu^{-2}}(D_p, \mathcal{X})$.

D. Global theory

We consider

K open compact subgroup of $GL(2, \mathbb{A}^f)$

$X^\pm$ Poincaré upper and lower half-plane.

$$X^\pm = \mathrm{Hom}_{\mathbb{R}}(\mathbb{Z}^2 \otimes \mathbb{R}, \mathbb{C}) / \mathbb{C}^* \subset \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) / \mathbb{C}^*$$

($GL(2,\mathbb{R})$ acts on the right on $\mathbb{Z}^2 \otimes \mathbb{R}$ via its action on $\mathbb{Z}^2 \otimes \mathbb{R} = \mathbb{R}^2$

$M^\circ_K(\mathbb{C}) = K \backslash X^\pm \times GL(2, \mathbb{A}^f) / GL(2, \mathbb{Q})$

k an integer ($k \geq 0$)

μ the representation Sym^k (dual of obvious representation) of $GL(2, \mathbb{Q})$

$F_\mu^{\mathbb{Q}}$ the corresponding local system on $M^\circ_K(\mathbb{C})$

$M_K(\mathbb{C})$ = Satake compactification of $M^\circ_K(\mathbb{C})$; $j : M^\circ_K(\mathbb{C}) \hookrightarrow M_K(\mathbb{C})$

$F_\mu^{\mathbb{Q}}$ the sheaf $j_* F_\mu^{\mathbb{Q}}$ on $M_K(\mathbb{C})$

$\mathcal{H}^{\mathbb{Q}}(\mu) = \varinjlim_K H^1(M_K(\mathbb{C}), F_\mu^{\mathbb{Q}})$

(an admissible representation of $GL(2, \mathbb{A}^f)$ defined $/\mathbb{Q}$)

$\mathcal{X}^{\mathbb{C}}(\mu) = \mathcal{X}(\mu) \otimes \mathbb{C}$, $\mathcal{X}(\mu)^{\mathbb{Q}_\ell} = \mathcal{X}(\mu) \otimes \mathbb{Q}_\ell$, $F_\mu^{\mathbb{Q}_\ell} = F_\mu \otimes \mathbb{Q}_\ell$.

One has a decomposition

$\mathcal{X}^{\mathbb{C}}(\mu) = \mathcal{X}^{k+1,0} \oplus \mathcal{X}^{0,k+1}$ of $\mathcal{X}^{\mathbb{C}}$ into 2 complex conjugate subspaces; $\mathcal{X}^{k+1,0}$ is a complex admissible representation of $GL(2, \mathbb{A}^f)$, which can be defined over $\mathbb{Q}$; it is hence isomorphic to the complex conjugate representation $\mathcal{X}^{0,k+1}$. It correspond to holomorphic modular cusp forms of weight $k+2$ (note that $k+2 \geqslant 2$!). For some explicit admissible representation D_{k+2} of $GL(2,\mathbb{R})$, of the discrete series,

$$\mathcal{X}^{k+1,0} = \mathrm{Hom}_{GL(2,\mathbb{R})}\left(D_{k+2},\ L_0\left(GL(2,\mathbb{A})/GL(2,\mathbb{Q})\right)\right).$$

Further, $\{M_K(\mathbb{C})\}$ ~~...~~ F_μ is naturally defined $/\mathbb{Q}$ (with its $GL(2,\mathbb{A}^f)$-action: $M_K \xrightarrow[g]{\sim} M_{gKg^{-1}}$), and $F_\mu^{\mathbb{Q}_\ell}$ is an ℓ-adic sheaf, defined $/\mathbb{Q}$. Hence,

$\mathcal{X}^{\mathbb{Q}_\ell}(\mu)$ carries a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action, commuting with $GL(2,\mathbb{A}^f)$, acting by "transport de structure". After extension of $\mathbb{Q}_\ell$ to $\overline{\mathbb{Q}}_\ell$, one has a decomposition

$$\mathcal{X}^{\mathbb{Q}_\ell}(\mu) = \bigoplus_{f\in F} \left(\bigotimes_p V_{f,p}\right) \otimes W_f$$

$V_{f,p}$: irreducible admissible representation of $GL(2,\mathbb{Q}_p)$

W_f : 2-dimentional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

(F = spectrum of $GL(2)$)

By Eichler – Shimura – Kuga – Deligne – Ihara – Пятецкий-Шапиро – Langlands, if $\check{V}_{f,p}$ is of the principal serie, then $W_f \mid Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (restriction to the decomposition group) is sum of 2 corresponding characters. If $V_{f,p}$ is special, then $W_f \mid Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the corresponding special ~~representation~~ ℓ-adic representation of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

I can now prove

(A) If $V_{f,p}$ is supercuspidal, then ~~W_f is irreducible~~ $W_f \mid Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is irreducible, and, with the notations of C Theorem, if $V_{f,p} \sim V_\varphi$, then $W_f \sim W_\varphi$.

(Hence $V_{f,p}$ determines, by a local rule, $W_{f,p} = W_f \mid Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, and the rule is the obvious one (up to a normalization) if $V_{f,p}$ corresponds by the Weil construction to a character of a quadratic extension)

The method is to use the theory of vanishing cycles to prove that ~~the~~ a space to be described below is a quotient of $\mathcal{X}(\mu) \otimes \mathbb{Q}_\ell$ (this is accurate only for $k \neq 0$; I will not bother much about $k=0$ and the phenomena related to the special representation)

We keep the notation of B., except that now $V_0 = \mathbb{Q}_p^2$. Let us consider

$$\mathrm{hom}(V_p(E_0), (\mathbb{A}^{f,p})^2) \times D_p$$

and the right action of ~~H~~ H on it (by composition for the first factor, and the inverse of the already defined action on D_p).
On this space, we have the following H-equivariant local system:

$$\mathrm{Sym}^k(V_\ell(E)^*) \otimes \varinjlim_K V(K;s)$$

(on the second factor, a right action is required, one takes the inverse of the one already constructed). ~~This local system is right equivariant (trivial action on the first factor, defined as on the~~ The space + local system is acted by $GL(2, \mathbb{A}^f)$

$\begin{cases} \text{space} & \text{composition on } 1^{\text{st}} \text{ factor, already described on } D_p \\ \ell\text{ system} & \text{trivial on first factor, described on } 2^{\text{d}} \end{cases}$

We ~~may~~ now take as announced representation of $GL(2, \mathbb{A}^f)$:

$$H^0\left(\begin{matrix}\text{Local system } \mathrm{Sym}^k(V_\ell(E_\bullet)^*) \otimes \varinjlim_K V(K,s) \\ \text{on} \\ \mathrm{Isom}(V_{p'}(E_0), \mathbb{A}^{f,p\,2}) \times D_p \end{matrix} \Big/ H\right)$$

The action of $W(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is via its action on $V(K,s)/D_p$

I now wish to relate that H^0 with the spectrum of $H(\mathbb{A})/H(\mathbb{Q})$ and $\Gamma(D_p, \chi)$.

For simplicity, let me in any way extend the scalars from $\mathbb{Q}_\ell$ to $\mathbb{C}$ (not to loose the Gal action, I have to use here that it is continuous, ~~...~~ for the discrete topology of $\mathbb{Q}_\ell$ - which I can do directly). Let me also choose ~~isomorphisms~~ an isomorphism

$$\begin{cases} V_{p'}(E_0) = \mathbb{A}^{f p'\,2} & , \text{ hence} \\ H(\mathbb{Q}_\ell) = GL(2, \mathbb{Q}_\ell) & (\ell \neq p) \end{cases}$$

Then

$$H^0 = \{\text{functions } H(\mathbb{A}^{f,p}) \longrightarrow \mathrm{Sym}^k(\ \ ^*) \otimes \Gamma(D_p, \chi) \quad \text{with} \quad f(x\gamma) = f(x)\gamma\}$$

$$= \{\text{functions } H(\mathbb{A}) \longrightarrow \mathrm{Sym}^k(\ \ ^*) \otimes \Gamma(D_p, \chi) \quad \text{with} \quad f(x\gamma) = f(x) \text{ and } f(g_\infty g_p x) = g_\infty g_p f(x)\}$$

$$= \bigoplus_{(\mathbb{A}^*/\mathbb{Q}^*)^\wedge} \{\ \}_\chi$$

$$= \bigoplus_{\substack{\chi \\ (\chi_\infty = \cdots)}} \mathrm{Hom}_{H(\mathbb{R}) \times H(\mathbb{Q}_p)}\left(\mathrm{Sym}^k(V) \otimes \Gamma_{\chi_p^{-1}}(D_p, \chi)^\vee, \ L_0^\lambda(H(\mathbb{A})/H(\mathbb{Q}))\right)$$

This can be expressed as follows: In $L_0(H(\mathbb{A})/H(\mathbb{Q}))$, one takes only those elements which at ∞ transforms ~~according to~~

as a given vector of a given representation of $H(\mathbb{R})$. The representation of $H(\mathbb{A}^f)$ so obtained being written

$$\bigoplus_{f_0 \in F_0} \Big(\bigotimes_{p} V_{f_0, p} \Big),$$

the following representation occurs in $\mathcal{X}(,\mu)$:

$$\boxed{\bigoplus_{f_0 \in F} \Big(\bigotimes_{\ell \neq p} V_{f_0,\ell} \Big) \otimes \bigoplus_{V_\varphi' \sim V_{f_0,p}} \big(V_\varphi \otimes W_\varphi \big)}$$

Now, by comparison with Jacquet-Langlands §16, plus the fact that supercuspidal representation cannot occur outside $\mathcal{X}(\mu)$ (this is given by [illegible] – [illegible] or Langlands + vanishing cycle theory), one get (i) (α) (β) (ii) (iii), 2-dimensionality in (i) γ) of Theorem [illegible] of C. One also get statement (A). By checking this general rule against modular forms attached to L functions with grossencharacter of ~~quadratic~~ imaginary quadratic fields (+ the end remark of C), one gets statement (v). The proof of (i) γ) in characteristic 2, in cases not covered by (v), uses entirely different ideas, which I cannot explain here.

Yours sincerely

P. Deligne