CHAPTER **18**

Algebraic Schemes and Algebraic Spaces

In this course, we have attached an affine algebraic variety to any algebra finitely generated over a field k. For many reasons, for example, in order to be able to study the reduction of varieties to characteristic $p \neq 0$, Grothendieck realized that it is important to attach a geometric object to *every* commutative ring. Unfortunately, $A \mapsto \operatorname{spm} A$ is not functorial in this generality: if $\varphi\colon A \to B$ is a homomorphism of rings, then $\varphi^{-1}(\mathfrak{m})$ for $\mathfrak{m}$ maximal need not be maximal — consider for example the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Thus he was forced to replace $\operatorname{spm}(A)$ with $\operatorname{spec}(A)$, the set of all prime ideals in A. He then attaches an ***affine scheme*** $\operatorname{Spec}(A)$ to each ring A, and defines a ***scheme*** to be a locally ringed space that admits an open covering by affine schemes.

There is a natural functor $V \mapsto V^*$ from the category of algebraic spaces over k to the category of schemes of finite-type over k, which is an equivalence of categories. The algebraic varieties correspond to geometrically reduced schemes. To construct V^* from V, one only has to add one point p_Z for each irreducible closed subvariety Z of V of dimension > 0; in other words, V^* is the set of irreducible closed subsets of V (and V is the subset of V^* of zero-dimensional irreducible closed subsets of V, i.e., points). For any open subset U of V, let U^* be the subset of V^* containing the points of U together with the points p_Z such that $U \cap Z$ is nonempty. Thus, $U \mapsto U^*$ is a bijection from the set of open subsets of V to the set of open subsets of V^*. Moreover, $\Gamma(U^*, \mathcal{O}_{V^*}) = \Gamma(U, \mathcal{O}_V)$ for each open subset U of V. Therefore the topologies and sheaves on V and V^* are the same — only the underlying sets differ. For a closed irreducible subset Z of V, the local ring $\mathcal{O}_{V^*, p_Z} = \varinjlim_{U \cap Z \neq \emptyset} \Gamma(U, \mathcal{O}_U)$. The reverse functor is even easier: simply omit the nonclosed points from the base space.[1]

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[1]Some authors call a geometrically reduced scheme of finite-type over a field a variety. Despite their similarity, it is important to distinguish such schemes from varieties (in the sense of these notes). For example, if W and W' are subvarieties of a variety, their intersection in the sense of schemes need not be reduced, and so may differ from their intersection in the sense of varieties. For example, if $W = V(\mathfrak{a}) \subset \mathbb{A}^n$ and $W' = V(\mathfrak{a}') \subset \mathbb{A}^{n'}$ with $\mathfrak{a}$ and $\mathfrak{a}'$ radical, then the intersection W and W' in the sense of schemes is $\operatorname{Spec} k[X_1, \ldots, X_{n+n'}]/(\mathfrak{a}, \mathfrak{a}')$ while their intersection in the sense of varieties is $\operatorname{Spec} k[X_1, \ldots, X_{n+n'}]/\operatorname{rad}(\mathfrak{a}, \mathfrak{a}')$ (and their intersection in the sense of algebraic spaces is $\operatorname{Spm} k[X_1, \ldots, X_{n+n'}]/(\mathfrak{a}, \mathfrak{a}')$).

Every aspiring algebraic and (especially) arithmetic geometer needs to learn the basic theory of schemes, and for this I recommend reading Chapters II and III of Hartshorne 1997.