CHAPTER **13**

Coherent Sheaves; Invertible Sheaves

In this chapter, k is an arbitrary field.

a Coherent sheaves

Let $V = \operatorname{Spm} A$ be an affine variety over k, and let M be a finitely generated A-module. There is a unique sheaf of $\mathcal{O}_V$-modules $\mathcal{M}$ on V such that, for all $f \in A$,

$$\Gamma(D(f), \mathcal{M}) = M_f \quad (= A_f \otimes_A M).$$

Such an $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***coherent***. A homomorphism $M \to N$ of A-modules defines a homomorphism $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}_V$-modules, and $M \mapsto \mathcal{M}$ is a fully faithful functor from the category of finitely generated A-modules to the category of coherent $\mathcal{O}_V$-modules, with quasi-inverse $\mathcal{M} \mapsto \Gamma(V, \mathcal{M})$.

Now consider a variety V. An $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***coherent*** if, for every open affine subset U of V, $\mathcal{M}|U$ is coherent. It suffices to check this condition for the sets in an open affine covering of V.

For example, $\mathcal{O}_V^n$ is a coherent $\mathcal{O}_V$-module. An $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***locally free of rank*** n if it is locally isomorphic to $\mathcal{O}_V^n$, i.e., if every point $P \in V$ has an open neighbourhood such that $\mathcal{M}|U \approx \mathcal{O}_V^n$. A locally free $\mathcal{O}_V$-module of rank n is coherent.

Let $v \in V$, and let $\mathcal{M}$ be a coherent $\mathcal{O}_V$-module. We define a $\kappa(v)$-module $\mathcal{M}(v)$ as follows: after replacing V with an open neighbourhood of v, we can assume that it is affine; hence we may suppose that $V = \operatorname{Spm}(A)$, that v corresponds to a maximal ideal $\mathfrak{m}$ in A (so that $\kappa(v) = A/\mathfrak{m}$), and $\mathcal{M}$ corresponds to the A-module M; we then define

$$\mathcal{M}(v) = M \otimes_A \kappa(v) = M/\mathfrak{m}M.$$

It is a finitely generated vector space over $\kappa(v)$. Don't confuse $\mathcal{M}(v)$ with the stalk $\mathcal{M}_v$ of $\mathcal{M}$ which, with the above notations, is $M_\mathfrak{m} = M \otimes_A A_\mathfrak{m}$. Thus

$$\mathcal{M}(v) = \mathcal{M}_v/\mathfrak{m}\mathcal{M}_v = \kappa(v) \otimes_{A_\mathfrak{m}} \mathcal{M}_\mathfrak{m}.$$

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Nakayama's lemma (AG 1.3) shows that

$$\mathcal{M}(v) = 0 \Rightarrow \mathcal{M}_v = 0.$$

The ***support*** of a coherent sheaf $\mathcal{M}$ is

$$\mathrm{Supp}(\mathcal{M}) = \{v \in V \mid \mathcal{M}(v) \neq 0\} = \{v \in V \mid \mathcal{M}_v \neq 0\}.$$

Suppose V is affine, and that $\mathcal{M}$ corresponds to the A-module M. Let $\mathfrak{a}$ be the annihilator of M:

$$\mathfrak{a} = \{f \in A \mid fM = 0\}.$$

Then $M/\mathfrak{m}M \neq 0 \iff \mathfrak{m} \supset \mathfrak{a}$ (for otherwise $A/\mathfrak{m}A$ contains a nonzero element annihilating $M/\mathfrak{m}M$), and so

$$\mathrm{Supp}(\mathcal{M}) = V(\mathfrak{a}).$$

Thus the support of a coherent module is a closed subset of V.

Note that if $\mathcal{M}$ is locally free of rank n, then $\mathcal{M}(v)$ is a vector space of dimension n for all v. There is a converse of this.

PROPOSITION 13.1. *If $\mathcal{M}$ is a coherent $\mathcal{O}_V$-module such that $\mathcal{M}(v)$ has constant dimension n for all $v \in V$, then $\mathcal{M}$ is a locally free of rank n.*

PROOF. We may assume that V is affine, and that $\mathcal{M}$ corresponds to the finitely generated A-module M. Fix a maximal ideal $\mathfrak{m}$ of A, and let $x_1, \ldots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a basis for it over $\kappa(v)$. Consider the map

$$\gamma\colon A^n \to M, \quad (a_1, \ldots, a_n) \mapsto \sum a_i x_i.$$

Its cokernel is a finitely generated A-module whose support does not contain v. Therefore there is an element $f \in A$, $f \notin \mathfrak{m}$, such that γ defines a surjection $A_f^n \to M_f$. After replacing A with A_f we may assume that γ itself is surjective. For every maximal ideal $\mathfrak{n}$ of A, the map $(A/\mathfrak{n})^n \to M/\mathfrak{n}M$ is surjective, and hence (because of the condition on the dimension of $\mathcal{M}(v)$) bijective. Therefore, the kernel of γ is contained in $\mathfrak{n}^n$ (meaning $\mathfrak{n} \times \cdots \times \mathfrak{n}$) for all maximal ideals $\mathfrak{n}$ in A, and the next lemma shows that this implies that the kernel is zero. □

LEMMA 13.2. *Let A be an affine k-algebra. Then*

$$\bigcap \mathfrak{m} = 0 \text{ (intersection of all maximal ideals in } A\text{).}$$

PROOF. When k is algebraically closed, we showed (AG, 2.18) that this follows from the strong Nullstellensatz. In the general case, consider a maximal ideal $\mathfrak{m}$ of $A \otimes_k k^{\mathrm{al}}$. Then

$$A/(\mathfrak{m} \cap A) \hookrightarrow (A \otimes_k k^{\mathrm{al}})/\mathfrak{m} = k^{\mathrm{al}},$$

and so $A/\mathfrak{m} \cap A$ is an integral domain. Since it is finite-dimensional over k, it is a field, and so $\mathfrak{m} \cap A$ is a maximal ideal in A. Thus if $f \in A$ is in all maximal ideals of A, then its image in $A \otimes k^{\mathrm{al}}$ is in all maximal ideals of A, and so is zero. □

For two coherent $\mathcal{O}_V$-modules $\mathcal{M}$ and $\mathcal{N}$, there is a unique coherent $\mathcal{O}_V$-module $\mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}$ such that

$$\Gamma(U, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}) = \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U, \mathcal{O}_V)} \Gamma(U, \mathcal{N})$$

for all open affines $U \subset V$. The reader should be careful not to assume that this formula holds for nonaffine open subsets U (see example 13.4 below). For a such a U, one writes $U = \bigcup U_i$ with the U_i open affines, and defines $\Gamma(U, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N})$ to be the kernel of

$$\prod_i \Gamma(U_i, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}) \rightrightarrows \prod_{i,j} \Gamma(U_{ij}, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}).$$

Define $\mathcal{H}om(\mathcal{M}, \mathcal{N})$ to be the sheaf on V such that

$$\Gamma(U, \mathcal{H}om(\mathcal{M}, \mathcal{N})) = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{M}, \mathcal{N})$$

(homomorphisms of $\mathcal{O}_U$-modules) for all open U in V. It is easy to see that this is a sheaf. If the restrictions of $\mathcal{M}$ and $\mathcal{N}$ to some open affine U correspond to A-modules M and N, then

$$\Gamma(U, \mathcal{H}om(\mathcal{M}, \mathcal{N})) = \mathrm{Hom}_A(M, N),$$

and so $\mathcal{H}om(\mathcal{M}, \mathcal{N})$ is again a coherent $\mathcal{O}_V$-module.

b Invertible sheaves.

An ***invertible sheaf*** on V is a locally free $\mathcal{O}_V$-module $\mathcal{L}$ of rank 1. The tensor product of two invertible sheaves is again an invertible sheaf. In this way, we get a product structure on the set of isomorphism classes of invertible sheaves:

$$[\mathcal{L}] \cdot [\mathcal{L}'] \stackrel{\text{def}}{=} [\mathcal{L} \otimes \mathcal{L}'].$$

The product structure is associative and commutative (because tensor products are associative and commutative, up to isomorphism), and $[\mathcal{O}_V]$ is an identity element. Define

$$\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_V).$$

Clearly, $\mathcal{L}^\vee$ is free of rank 1 over any open set where $\mathcal{L}$ is free of rank 1, and so $\mathcal{L}^\vee$ is again an invertible sheaf. Moreover, the canonical map

$$\mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_V, \quad (f, x) \mapsto f(x)$$

is an isomorphism (because it is an isomorphism over any open subset where $\mathcal{L}$ is free). Thus

$$[\mathcal{L}^\vee][\mathcal{L}] = [\mathcal{O}_V].$$

For this reason, we often write $\mathcal{L}^{-1}$ for $\mathcal{L}^\vee$.

From these remarks, we see that the set of isomorphism classes of invertible sheaves on V is a group — it is called the ***Picard group***, $\mathrm{Pic}(V)$, of V.

We say that an invertible sheaf $\mathcal{L}$ is ***trivial*** if it is isomorphic to $\mathcal{O}_V$ — then $\mathcal{L}$ represents the zero element in $\mathrm{Pic}(V)$.

PROPOSITION 13.3. *An invertible sheaf $\mathcal{L}$ on a complete variety V is trivial if and only if both it and its dual have nonzero global sections, i.e.,*

$$\Gamma(V,\mathcal{L}) \neq 0 \neq \Gamma(V,\mathcal{L}^{\vee}).$$

PROOF. We may assume that V is irreducible. Note first that, for any $\mathcal{O}_V$-module $\mathcal{M}$ on any variety V, the map

$$\mathrm{Hom}(\mathcal{O}_V,\mathcal{M}) \to \Gamma(V,\mathcal{M}), \quad \alpha \mapsto \alpha(1)$$

is an isomorphism.

Next recall that the only regular functions on a complete variety are the constant functions (see AG, 7.9, in the case that k is algebraically closed), i.e., $\Gamma(V,\mathcal{O}_V) = k'$ where k' is the algebraic closure of k in $k(V)$. Hence $\mathcal{H}om(\mathcal{O}_V,\mathcal{O}_V) = k'$, and so a homomorphism $\mathcal{O}_V \to \mathcal{O}_V$ is either 0 or an isomorphism.

We now prove the proposition. The sections define nonzero homomorphisms

$$s_1\colon \mathcal{O}_V \to \mathcal{L}, \quad s_2\colon \mathcal{O}_V \to \mathcal{L}^{\vee}.$$

We can take the dual of the second homomorphism, and so obtain nonzero homomorphisms

$$\mathcal{O}_V \xrightarrow{s_1} \mathcal{L} \xrightarrow{s_2^{\vee}} \mathcal{O}_V.$$

The composite is nonzero, and hence an isomorphism, which shows that $s_2^{\vee}$ is surjective, and this implies that it is an isomorphism (for any ring A, a surjective homomorphism of A-modules $A \to A$ is bijective because 1 must map to a unit). □

c Invertible sheaves and divisors.

Now assume that V is nonsingular and irreducible. For a divisor D on V, the vector space $L(D)$ is defined to be

$$L(D) = \{f \in k(V)^{\times} \mid \mathrm{div}(f) + D \geq 0\}.$$

We make this definition local: define $\mathcal{L}(D)$ to be the sheaf on V such that, for any open set U,

$$\Gamma(U,\mathcal{L}(D)) = \{f \in k(V)^{\times} \mid \mathrm{div}(f) + D \geq 0 \text{ on } U\} \cup \{0\}.$$

The condition "$\mathrm{div}(f)+D \geq 0$ on U" means that, if $D = \sum n_Z Z$, then $\mathrm{ord}_Z(f) + n_Z \geq 0$ for all Z with $Z \cap U \neq \emptyset$. Thus, $\Gamma(U,\mathcal{L}(D))$ is a $\Gamma(U,\mathcal{O}_V)$-module, and if $U \subset U'$, then $\Gamma(U',\mathcal{L}(D)) \subset \Gamma(U,\mathcal{L}(D))$. We define the restriction map to be this inclusion. In this way, $\mathcal{L}(D)$ becomes a sheaf of $\mathcal{O}_V$-modules.

Suppose D is principal on an open subset U, say $D|U = \mathrm{div}(g)$, $g \in k(V)^{\times}$. Then

$$\Gamma(U,\mathcal{L}(D)) = \{f \in k(V)^{\times} \mid \mathrm{div}(fg) \geq 0 \text{ on } U\} \cup \{0\}.$$

Therefore,

$$\Gamma(U,\mathcal{L}(D)) \to \Gamma(U,\mathcal{O}_V), \quad f \mapsto fg,$$

is an isomorphism. These isomorphisms clearly commute with the restriction maps for $U' \subset U$, and so we obtain an isomorphism $\mathcal{L}(D)|U \to \mathcal{O}_U$. Since every D is locally principal, this shows that $\mathcal{L}(D)$ is locally isomorphic to $\mathcal{O}_V$, i.e., that it is an invertible sheaf. If D itself is principal, then $\mathcal{L}(D)$ is trivial.

Next we note that the canonical map

$$\mathcal{L}(D) \otimes \mathcal{L}(D') \to \mathcal{L}(D+D'), \quad f \otimes g \mapsto fg$$

is an isomorphism on any open set where D and $D'$are principal, and hence it is an isomorphism globally. Therefore, we have a homomorphism

$$\mathrm{Div}(V) \to \mathrm{Pic}(V), \quad D \mapsto [\mathcal{L}(D)],$$

which is zero on the principal divisors.

EXAMPLE 13.4. Let V be an elliptic curve, and let P be the point at infinity. Let D be the divisor $D = P$. Then $\Gamma(V, \mathcal{L}(D)) = k$, the ring of constant functions, but $\Gamma(V, \mathcal{L}(2D))$ contains a nonconstant function x. Therefore,

$$\Gamma(V, \mathcal{L}(2D)) \neq \Gamma(V, \mathcal{L}(D)) \otimes \Gamma(V, \mathcal{L}(D)),$$

— in other words, $\Gamma(V, \mathcal{L}(D) \otimes \mathcal{L}(D)) \neq \Gamma(V, \mathcal{L}(D)) \otimes \Gamma(V, \mathcal{L}(D))$.

PROPOSITION 13.5. *For an irreducible nonsingular variety, the map $D \mapsto [\mathcal{L}(D)]$ defines an isomorphism*

$$\mathrm{Div}(V)/\mathrm{PrinDiv}(V) \to \mathrm{Pic}(V).$$

PROOF. (Injectivity). If s is an isomorphism $\mathcal{O}_V \to \mathcal{L}(D)$, then $g = s(1)$ is an element of $k(V)^\times$ such that

(a) $\mathrm{div}(g) + D \geq 0$ (on the whole of V);

(b) if $\mathrm{div}(f) + D \geq 0$ on U, that is, if $f \in \Gamma(U, \mathcal{L}(D))$, then $f = h(g|U)$ for some $h \in \Gamma(U, \mathcal{O}_V)$.

Statement (a) says that $D \geq \mathrm{div}(-g)$ (on the whole of V). Suppose U is such that $D|U$ admits a local equation $f = 0$. When we apply (b) to $-f$, then we see that $\mathrm{div}(-f) \leq \mathrm{div}(g)$ on U, so that $D|U + \mathrm{div}(g) \geq 0$. Since the U's cover V, together with (a) this implies that $D = \mathrm{div}(-g)$.

(Surjectivity). Define

$$\Gamma(U, \mathcal{K}) = \begin{cases} k(V)^\times \text{ if } U \text{ is open and nonempty} \\ 0 \text{ if } U \text{ is empty.} \end{cases}$$

Because V is irreducible, $\mathcal{K}$ becomes a sheaf with the obvious restriction maps. On any open subset U where $\mathcal{L}|U \approx \mathcal{O}_U$, we have $\mathcal{L}|U \otimes \mathcal{K} \approx \mathcal{K}$. Since these open sets form a covering of V, V is irreducible, and the restriction maps are all the identity map, this implies that $\mathcal{L} \otimes \mathcal{K} \approx \mathcal{K}$ on the whole of V. Choose such an isomorphism, and identify $\mathcal{L}$ with a subsheaf of $\mathcal{K}$. On any U where $\mathcal{L} \approx \mathcal{O}_U$, $\mathcal{L}|U = g\mathcal{O}_U$ as a subsheaf of $\mathcal{K}$, where g is the image of $1 \in \Gamma(U, \mathcal{O}_V)$. Define D to be the divisor such that, on a U, g^{-1} is a local equation for D.□

EXAMPLE 13.6. Suppose V is affine, say $V = \mathrm{Spm}\, A$. We know that coherent $\mathcal{O}_V$-modules correspond to finitely generated A-modules, but what do the locally free sheaves of rank n correspond to? They correspond to finitely generated *projective* A-modules (CA 12.5). The invertible sheaves correspond to finitely generated projective A-modules of rank 1.

Suppose for example that V is a curve, so that A is a Dedekind domain. This gives a new interpretation of the ideal class group: it is the group of isomorphism classes of finitely generated projective A-modules of rank one (i.e., such that $M \otimes_A K$ is a vector space of dimension one).

This can be proved directly. First show that every (fractional) ideal is a projective A-module — it is obviously finitely generated of rank one; then show that two ideals are isomorphic as A-modules if and only if they differ by a principal divisor; finally, show that every finitely generated projective A-module of rank 1 is isomorphic to a fractional ideal (by assumption $M \otimes_A K \approx K$; when we choose an identification $M \otimes_A K = K$, then $M \subset M \otimes_A K$ becomes identified with a fractional ideal). [Exercise: Prove the statements in this last paragraph.]

REMARK 13.7. Quite a lot is known about $\mathrm{Pic}(V)$, the group of divisors modulo linear equivalence, or of invertible sheaves up to isomorphism. For example, for any complete nonsingular variety V, there is an abelian variety P canonically attached to V, called the ***Picard variety*** of V, and an exact sequence

$$0 \to P(k) \to \mathrm{Pic}(V) \to \mathrm{NS}(V) \to 0$$

where $\mathrm{NS}(V)$ is a finitely generated group called the Néron-Severi group.

Much less is known about algebraic cycles of codimension > 1, and about locally free sheaves of rank > 1 (and the two don't correspond exactly, although the Chern classes of locally free sheaves are algebraic cycles).

d Direct images and inverse images of coherent sheaves.

Consider a homomorphism $A \to B$ of rings. From an A-module M, we get an B-module $B \otimes_A M$, which is finitely generated if M is finitely generated. Conversely, an B-module M can also be considered an A-module, but it usually won't be finitely generated (unless B is finitely generated as an A-module). Both these operations extend to maps of varieties.

Consider a regular map $\alpha\colon W \to V$, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_V$-modules. There is a unique coherent sheaf of $\mathcal{O}_W$-modules $\alpha^*\mathcal{F}$ with the following property: for any open affine subsets U' and U of W and V respectively such that $\alpha(U') \subset U$, $\alpha^*\mathcal{F}|U'$ is the sheaf corresponding to the $\Gamma(U', \mathcal{O}_W)$-module $\Gamma(U', \mathcal{O}_W) \otimes_{\Gamma(U, \mathcal{O}_V)} \Gamma(U, \mathcal{F})$.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_V$-modules. For any open subset U of V, we define $\Gamma(U, \alpha_*\mathcal{F}) = \Gamma(\alpha^{-1}U, \mathcal{F})$, regarded as a $\Gamma(U, \mathcal{O}_V)$-module via the map $\Gamma(U, \mathcal{O}_V) \to \Gamma(\alpha^{-1}U, \mathcal{O}_W)$. Then $U \mapsto \Gamma(U, \alpha_*\mathcal{F})$ is a sheaf of $\mathcal{O}_V$-modules. In general, $\alpha_*\mathcal{F}$ will not be coherent, even when $\mathcal{F}$ is.

LEMMA 13.8. (a) *For any regular maps $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ and coherent $\mathcal{O}_W$-module $\mathcal{F}$ on W, there is a canonical isomorphism*

$$(\beta\alpha)^*\mathcal{F} \xrightarrow{\approx} \alpha^*(\beta^*\mathcal{F}).$$

(b) *For any regular map $\alpha\colon V \to W$, α^* maps locally free sheaves of rank n to locally free sheaves of rank n (hence also invertible sheaves to invertible sheaves). It preserves tensor products, and, for an invertible sheaf $\mathcal{L}$, $\alpha^*(\mathcal{L}^{-1}) \simeq (\alpha^*\mathcal{L})^{-1}$.*

PROOF. (a) This follows from the fact that, given homomorphisms of rings $A \to B \to T$, $T \otimes_B (B \otimes_A M) = T \otimes_A M$.

(b) This again follows from well-known facts about tensor products of rings. □

See Kleiman.

e Principal bundles

To be added.