CHAPTER **12**

Divisors and Intersection Theory

In this chapter, k is an arbitrary field.

a Divisors

Recall that a normal ring is an integral domain that is integrally closed in its field of fractions, and that a variety V is ***normal*** if $\mathcal{O}_v$ is a normal ring for all $v \in V$. Equivalent condition: for every open connected affine subset U of V, $\Gamma(U, \mathcal{O}_V)$ is a normal ring.

REMARK 12.1. Let V be a projective variety, say, defined by a homogeneous ring R. When R is normal, V is said to be ***projectively normal***. If V is projectively normal, then it is normal, but the converse statement is false.

Assume now that V is normal and irreducible.

A ***prime divisor*** on V is an irreducible subvariety of V of codimension 1. A ***divisor*** on V is an element of the free abelian group $\mathrm{Div}(V)$ generated by the prime divisors. Thus a divisor D can be written uniquely as a finite (formal) sum

$$D = \sum n_i Z_i, \quad n_i \in \mathbb{Z}, \quad Z_i \text{ a prime divisor on } V.$$

The ***support*** $|D|$ of D is the union of the Z_i corresponding to nonzero n_i's. A divisor is said to be ***effective*** (or ***positive***) if $n_i \geq 0$ for all i. We get a partial ordering on the divisors by defining $D \geq D'$ to mean $D - D' \geq 0$.

Because V is normal, there is associated with every prime divisor Z on V a discrete valuation ring $\mathcal{O}_Z$. This can be defined, for example, by choosing an open affine subvariety U of V such that $U \cap Z \neq \emptyset$; then $U \cap Z$ is a maximal proper closed subset of U, and so the ideal $\mathfrak{p}$ corresponding to it is minimal among the nonzero ideals of $R = \Gamma(U, \mathcal{O})$; so $R_{\mathfrak{p}}$ is a normal ring with exactly one nonzero prime ideal $\mathfrak{p}R$ — it is therefore a discrete valuation ring (CA 20.2), which is defined to be $\mathcal{O}_Z$. More intrinsically we can define $\mathcal{O}_Z$ to be the set of rational functions on V that are defined an open subset U of V with $U \cap Z \neq \emptyset$.

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Let ord_Z be the valuation of $k(V)^\times \twoheadrightarrow \mathbb{Z}$ with valuation ring $\mathcal{O}_Z$. The divisor of a nonzero element f of $k(V)$ is defined to be

$$\mathrm{div}(f) = \sum \mathrm{ord}_Z(f) \cdot Z.$$

The sum is over all the prime divisors of V, but in fact $\mathrm{ord}_Z(f) = 0$ for all but finitely many Z's. In proving this, we can assume that V is affine (because it is a finite union of affines), say $V = \mathrm{Spm}(R)$. Then $k(V)$ is the field of fractions of R, and so we can write $f = g/h$ with $g, h \in R$, and $\mathrm{div}(f) = \mathrm{div}(g) - \mathrm{div}(h)$. Therefore, we can assume $f \in R$. The zero set of f, $V(f)$ either is empty or is a finite union of prime divisors, $V = \bigcup Z_i$ (AG 3.42) and $\mathrm{ord}_Z(f) = 0$ unless Z is one of the Z_i.

The map

$$f \mapsto \mathrm{div}(f) \colon k(V)^\times \to \mathrm{Div}(V)$$

is a homomorphism. A divisor of the form $\mathrm{div}(f)$ is said to be ***principal***, and two divisors are said to be ***linearly equivalent***, denoted $D \sim D'$, if they differ by a principal divisor.

When V is nonsingular, the ***Picard group*** $\mathrm{Pic}(V)$ of V is defined to be the group of divisors on V modulo principal divisors. (Later, we shall define $\mathrm{Pic}(V)$ for an arbitrary variety; when V is singular it will differ from the group of divisors modulo principal divisors, even when V is normal.)

EXAMPLE 12.2. Let C be a nonsingular affine curve corresponding to the affine k-algebra R. Because C is nonsingular, R is a Dedekind domain. A prime divisor on C can be identified with a nonzero prime divisor in R, a divisor on C with a fractional ideal, and $Pic(C)$ with the ideal class group of R.

Let U be an open subset of V, and let Z be a prime divisor of V. Then $Z \cap U$ is either empty or is a prime divisor of U. We define the ***restriction*** of a divisor $D = \sum n_Z Z$ on V to U to be

$$D|_U = \sum_{Z \cap U \neq \emptyset} n_Z \cdot Z \cap U.$$

When V is nonsingular, every divisor D is ***locally principal***, i.e., every point P has an open neighbourhood U such that the restriction of D to U is principal. It suffices to prove this for a prime divisor Z. If P is not in the support of D, we can take $f = 1$. The prime divisors passing through P are in one-to-one correspondence with the prime ideals $\mathfrak{p}$ of height 1 in $\mathcal{O}_P$, i.e., the minimal nonzero prime ideals. Our assumption implies that $\mathcal{O}_P$ is a regular local ring. It is a (fairly hard) theorem in commutative algebra that a regular local ring is a unique factorization domain. It is a (fairly easy) theorem that a noetherian integral domain is a unique factorization domain if every prime ideal of height 1 is principal (CA 21.4). Thus $\mathfrak{p}$ is principal in $\mathcal{O}_\mathfrak{p}$, and this implies that it is principal in $\Gamma(U, \mathcal{O}_V)$ for some open affine set U containing P.

If $D|_U = \mathrm{div}(f)$, then we call f a ***local equation*** for D on U.

b Intersection theory.

Fix a nonsingular variety V of dimension n over a field k, ***assumed to be perfect***. Let W_1 and W_2 be irreducible closed subsets of V, and let Z be an irreducible component of $W_1 \cap W_2$. Then intersection theory attaches a multiplicity to Z. We shall only do this in an easy case.

DIVISORS.

Let V be a nonsingular variety of dimension n, and let $D_1,\ldots,D_n$ be effective divisors on V. We say that $D_1,\ldots,D_n$ ***intersect properly*** at $P \in |D_1|\cap\ldots\cap|D_n|$ if P is an isolated point of the intersection. In this case, we define

$$(D_1\cdot\ldots\cdot D_n)_P = \dim_k \mathcal{O}_P/(f_1,\ldots,f_n)$$

where f_i is a local equation for D_i near P. The hypothesis on P implies that this is finite.

EXAMPLE 12.3. In all the examples, the ambient variety is a surface.

(a) Let Z_1 be the affine plane curve Y^2-X^3, let Z_2 be the curve $Y = X^2$, and let $P = (0,0)$. Then

$$(Z_1\cdot Z_2)_P = \dim k[X,Y]_{(X,Y)}/(Y-X^2,Y^2-X^3) = \dim k[X]/(X^4-X^3) = 3.$$

(b) If Z_1 and Z_2 are prime divisors, then $(Z_1\cdot Z_2)_P = 1$ if and only if f_1, f_2 are local uniformizing parameters at P. Equivalently, $(Z_1\cdot Z_2)_P = 1$ if and only if Z_1 and Z_2 are *transversal* at P, that is, $T_{Z_1}(P)\cap T_{Z_2}(P) = \{0\}$.

(c) Let D_1 be the x-axis, and let D_2 be the cuspidal cubic Y^2-X^3. For $P = (0,0)$, $(D_1\cdot D_2)_P = 3$.

(d) In general, $(Z_1\cdot Z_2)_P$ is the "order of contact" of the curves Z_1 and Z_2.

We say that $D_1,\ldots,D_n$ ***intersect properly*** if they do so at every point of intersection of their supports; equivalently, if $|D_1|\cap\ldots\cap|D_n|$ is a finite set. We then define the intersection number

$$(D_1\cdot\ldots\cdot D_n) = \sum_{P\in|D_1|\cap\ldots\cap|D_n|}(D_1\cdot\ldots\cdot D_n)_P.$$

EXAMPLE 12.4. Let C be a curve. If $D = \sum n_i P_i$, then the intersection number

$$(D) = \sum n_i[k(P_i):k].$$

By definition, this is the degree of D.

Consider a regular map $\alpha\colon W \to V$ of connected nonsingular varieties, and let D be a divisor on V whose support does not contain the image of W. There is then a unique divisor α^*D on W with the following property: if D has local equation f on the open subset U of V, then α^*D has local equation $f\circ\alpha$ on $\alpha^{-1}U$. (Use AG, 3.42, to see that this does define a divisor on W; if the image of α is disjoint from $|D|$, then $\alpha^*D = 0$.)

EXAMPLE 12.5. Let C be a curve on a surface V, and let $\alpha\colon C'\to C$ be the normalization of C. For any divisor D on V,

$$(C\cdot D) = \deg\alpha^*D.$$

LEMMA 12.6 (ADDITIVITY). *Let $D_1,\ldots,D_n,D$ be divisors on V. If $(D_1\cdot\ldots\cdot D_n)_P$ and $(D_1\cdot\ldots\cdot D)_P$ are both defined, then so also is $(D_1\cdot\ldots\cdot D_n+D)_P$, and*

$$(D_1\cdot\ldots\cdot D_n+D)_P = (D_1\cdot\ldots\cdot D_n)_P + (D_1\cdot\ldots\cdot D)_P.$$

PROOF. One writes some exact sequences. See Shafarevich 1994, IV.1.2. □

Note that in intersection theory, unlike every other branch of mathematics, we add first, and then multiply.

Since every divisor is the difference of two effective divisors, Lemma 12.1 allows us to extend the definition of $(D_1 \cdot \ldots \cdot D_n)$ to all divisors intersecting properly (not just effective divisors).

LEMMA 12.7 (INVARIANCE UNDER LINEAR EQUIVALENCE). *Assume V is complete. If $D_n \sim D_n'$, then*

$$(D_1 \cdot \ldots \cdot D_n) = (D_1 \cdot \ldots \cdot D_n').$$

PROOF. By additivity, it suffices to show that $(D_1 \cdot \ldots \cdot D_n) = 0$ if D_n is a principal divisor. For $n = 1$, this is just the statement that a function has as many poles as zeros (counted with multiplicities). Suppose $n = 2$. By additivity, we may assume that D_1 is a curve, and then the assertion follows from Example 12.5 because

$$D \text{ principal } \Rightarrow \alpha^* D \text{ principal.}$$

The general case may be reduced to this last case (with some difficulty). See Shafarevich 1994, IV.1.3. □

LEMMA 12.8. *For any n divisors $D_1, \ldots, D_n$ on an n-dimensional variety, there exists n divisors $D_1', \ldots, D_n'$ intersect properly.*

PROOF. See Shafarevich 1994, IV.1.4. □

We can use the last two lemmas to define $(D_1 \cdot \ldots \cdot D_n)$ for any divisors on a complete nonsingular variety V: choose $D_1', \ldots, D_n'$ as in the lemma, and set

$$(D_1 \cdot \ldots \cdot D_n) = (D_1' \cdot \ldots \cdot D_n').$$

EXAMPLE 12.9. Let C be a smooth complete curve over $\mathbb{C}$, and let $\alpha: C \to C$ be a regular map. Then the Lefschetz trace formula states that

$$(\Delta \cdot \Gamma_\alpha) = \mathrm{Tr}(\alpha | H^0(C,\mathbb{Q}) - \mathrm{Tr}(\alpha | H^1(C,\mathbb{Q}) + \mathrm{Tr}(\alpha | H^2(C,\mathbb{Q}).$$

In particular, we see that $(\Delta \cdot \Delta) = 2 - 2g$, which may be negative, even though Δ is an effective divisor.

Let $\alpha: W \to V$ be a finite map of irreducible varieties. Then $k(W)$ is a finite extension of $k(V)$, and the degree of this extension is called the ***degree*** of α. If $k(W)$ is separable over $k(V)$ and k is algebraically closed, then there is an open subset U of V such that $\alpha^{-1}(u)$ consists exactly $d = \deg \alpha$ points for all $u \in U$. In fact, $\alpha^{-1}(u)$ always consists of exactly $\deg \alpha$ points if one counts multiplicities. Number theorists will recognize this as the formula $\sum e_i f_i = d$. Here the f_i are 1 (if we take k to be algebraically closed), and e_i is the multiplicity of the ith point lying over the given point.

A finite map $\alpha: W \to V$ is ***flat*** if every point P of V has an open neighbourhood U such that $\Gamma(\alpha^{-1}U, \mathcal{O}_W)$ is a free $\Gamma(U, \mathcal{O}_V)$-module — it is then free of rank $\deg \alpha$.

THEOREM 12.10. *Let $\alpha: W \to V$ be a finite map between nonsingular varieties. For any divisors $D_1, \ldots, D_n$ on V intersecting properly at a point P of V,*

$$\sum_{\alpha(Q)=P} (\alpha^* D_1 \cdot \ldots \cdot \alpha^* D_n) = \deg \alpha \cdot (D_1 \cdot \ldots \cdot D_n)_P.$$

PROOF. After replacing V by a sufficiently small open affine neighbourhood of P, we may assume that α corresponds to a map of rings $A \to B$ and that B is free of rank $d = \deg\alpha$ as an A-module. Moreover, we may assume that $D_1, \ldots, D_n$ are principal with equations $f_1, \ldots, f_n$ on V, and that P is the only point in $|D_1| \cap \ldots \cap |D_n|$. Then $\mathfrak{m}_P$ is the only ideal of A containing $\mathfrak{a} = (f_1, \ldots, f_n)$. Set $S = A \smallsetminus \mathfrak{m}_P$; then

$$S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a}) = A/\mathfrak{a}$$

because $A/\mathfrak{a}$ is already local. Hence

$$(D_1 \cdot \ldots \cdot D_n)_P = \dim A/(f_1, \ldots, f_n).$$

Similarly,

$$(\alpha^* D_1 \cdot \ldots \cdot \alpha^* D_n)_P = \dim B/(f_1 \circ \alpha, \ldots, f_n \circ \alpha).$$

But B is a free A-module of rank d, and

$$A/(f_1, \ldots, f_n) \otimes_A B = B/(f_1 \circ \alpha, \ldots, f_n \circ \alpha).$$

Therefore, as A-modules, and hence as k-vector spaces,

$$B/(f_1 \circ \alpha, \ldots, f_n \circ \alpha) \approx (A/(f_1, \ldots, f_n))^d$$

which proves the formula. □

EXAMPLE 12.11. Assume k is algebraically closed of characteristic $p \neq 0$. Let $\alpha\colon \mathbb{A}^1 \to \mathbb{A}^1$ be the Frobenius map $c \mapsto c^p$. It corresponds to the map $k[X] \to k[X]$, $X \mapsto X^p$, on rings. Let D be the divisor c. It has equation $X - c$ on $\mathbb{A}^1$, and $\alpha^* D$ has the equation $X^p - c = (X - \gamma)^p$. Thus $\alpha^* D = p(\gamma)$, and so

$$\deg(\alpha^* D) = p = p \cdot \deg(D).$$

THE GENERAL CASE.

Let V be a nonsingular connected variety. A ***cycle of codimension*** r on V is an element of the free abelian group $C^r(V)$ generated by the prime cycles of codimension r.

Let Z_1 and Z_2 be prime cycles on any nonsingular variety V, and let W be an irreducible component of $Z_1 \cap Z_2$. Then

$$\dim\ Z_1 + \dim\ Z_2 \leq \dim\ V + \dim\ W,$$

and we say Z_1 and Z_2 ***intersect properly*** at W if equality holds.

Define $\mathcal{O}_{V,W}$ to be the set of rational functions on V that are defined on some open subset U of V with $U \cap W \neq \emptyset$ — it is a local ring. Assume that Z_1 and Z_2 intersect properly at W, and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the ideals in $\mathcal{O}_{V,W}$ corresponding to Z_1 and Z_2 (so $\mathfrak{p}_i = (f_1, f_2, ..., f_r)$ if the f_j define Z_i in some open subset of V meeting W). The example of divisors on a surface suggests that we should set

$$(Z_1 \cdot Z_2)_W = \dim_k \mathcal{O}_{V,W}/(\mathfrak{p}_1, \mathfrak{p}_2),$$

but examples show this is not a good definition. Note that

$$\mathcal{O}_{V,W}/(\mathfrak{p}_1, \mathfrak{p}_2) = \mathcal{O}_{V,W}/\mathfrak{p}_1 \otimes_{\mathcal{O}_{V,W}} \mathcal{O}_{V,W}/\mathfrak{p}_2.$$

It turns out that we also need to consider the higher Tor terms. Set

$$\chi^{\mathcal{O}}(\mathcal{O}/\mathfrak{p}_1, \mathcal{O}/\mathfrak{p}_2) = \sum_{i=0}^{\dim V} (-1)^i \dim_k(\mathrm{Tor}_i^{\mathcal{O}}(\mathcal{O}/\mathfrak{p}_1, \mathcal{O}/\mathfrak{p}_2))$$

where $\mathcal{O} = \mathcal{O}_{V,W}$. It is an integer ≥ 0, and $= 0$ if Z_1 and Z_2 do not intersect properly at W. When they do intersect properly, we define

$$(Z_1 \cdot Z_2)_W = mW, \quad m = \chi^{\mathcal{O}}(\mathcal{O}/\mathfrak{p}_1, \mathcal{O}/\mathfrak{p}_2).$$

When Z_1 and Z_2 are divisors on a surface, the higher Tor's vanish, and so this definition agrees with the previous one.

Now assume that V is projective. It is possible to define a notion of rational equivalence for cycles of codimension r: let W be an irreducible subvariety of codimension $r-1$, and let $f \in k(W)^{\times}$; then $\mathrm{div}(f)$ is a cycle of codimension r on V (since W may not be normal, the definition of $\mathrm{div}(f)$ requires care), and we let $C^r(V)'$ be the subgroup of $C^r(V)$ generated by such cycles as W ranges over all irreducible subvarieties of codimension $r-1$ and f ranges over all elements of $k(W)^{\times}$. Two cycles are said to be ***rationally equivalent*** if they differ by an element of $C^r(V)'$, and the quotient of $C^r(V)$ by $C^r(V)'$ is called the ***Chow group*** $CH^r(V)$. A discussion similar to that in the case of a surface leads to well-defined pairings

$$CH^r(V) \times CH^s(V) \to CH^{r+s}(V).$$

In general, we know very little about the Chow groups of varieties — for example, there has been little success at finding algebraic cycles on varieties other than the obvious ones (divisors, intersections of divisors,...). However, there are many deep conjectures concerning them, due to Beilinson, Bloch, Murre, and others.

We can restate our definition of the degree of a variety in $\mathbb{P}^n$ as follows: a closed subvariety V of $\mathbb{P}^n$ of dimension d has degree $(V \cdot H)$ for any linear subspace of $\mathbb{P}^n$ of codimension d. (All linear subspaces of $\mathbb{P}^n$ of codimension r are rationally equivalent, and so $(V \cdot H)$ is independent of the choice of H.)

REMARK 12.12. (Bezout's theorem). A divisor D on $\mathbb{P}^n$ is linearly equivalent of δH, where δ is the degree of D and H is any hyperplane. Therefore

$$(D_1 \cdot \cdots \cdot D_n) = \delta_1 \cdots \delta_n$$

where δ_j is the degree of D_j. For example, if C_1 and C_2 are curves in $\mathbb{P}^2$ defined by irreducible polynomials F_1 and F_2 of degrees δ_1 and δ_2 respectively, then C_1 and C_2 intersect in $\delta_1\delta_2$ points (counting multiplicities).

c Exercises

You may assume the characteristic is zero if you wish.

12-1. Let $V = V(F) \subset \mathbb{P}^n$, where F is a homogeneous polynomial of degree δ without multiple factors. Show that V has degree δ according to the definition in the notes.

12-2. Let C be a curve in $\mathbb{A}^2$ defined by an irreducible polynomial $F(X,Y)$, and assume C passes through the origin. Then $F = F_m + F_{m+1} + \cdots$, $m \geq 1$, with F_m the homogeneous part of F of degree m. Let $\sigma: W \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at $(0,0)$, and let C' be the closure of $\sigma^{-1}(C \smallsetminus (0,0))$. Let $Z = \sigma^{-1}(0,0)$. Write $F_m = \prod_{i=1}^{s}(a_i X + b_i Y)^{r_i}$, with the $(a_i : b_i)$ being distinct points of $\mathbb{P}^1$, and show that $C' \cap Z$ consists of exactly s distinct points.

12-3. Find the intersection number of $D_1 : Y^2 = X^r$ and $D_2 : Y^2 = X^s$, $r > s > 2$, at the origin.

12-4. Find $\mathrm{Pic}(V)$ when V is the curve $Y^2 = X^3$.

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