Addendum/Erratum for Etale Cohomology, Princeton U. Press, 1980

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px, yt, zb means page x, line y from top, line z from bottom.

pxiii. The only reason I assumed all the schemes are locally Noetherian was that I didn't want to keep saying finite presentation etc. for finite type. I can't believe it really matters anywhere. The fibre product schemes on p. 70, line 2, and p80 are not automatically Noetherian. If I remember correctly, Grothendieck's example of naturally arising nonnoetherian rings is the tensor product of two completions of ring. Products of Henselizations are probably only about as Noetherian as completions.

p6, 18b. It does not "follow easily". In fact, passing from the affine result to the global result is not easy.

p7, 1.12. Cf. Hartshorne, Algebraic Geometry, Ex. 3.7. (Hartshorne's book appeared just before I sent the final version of the manuscript to the publisher, in time only for me to change some of the references in the manuscript.)

p7, 12b. Atiyah-Macdonald [1, 2.19]

p9. Correct statement of 2.5: Let $\alpha\ A \to B$ be a flat A-algebra, and consider $b \in B$. If the image of b in $B/\alpha^{-1}(\mathfrak{n})$ is not a zero-divisor for any maximal ideal $\mathfrak{n}$ of B, then $B/(b)$ is a flat A-algebra. This necessitates some changes on p10.

p16. For the proof of Theorem 2.16, see also Mumford [2, p57].

p31. In Exercise 3.27, the Y should be T.

p40. Remark 5.1c should read: ...states that the functor that...

p42. Chinberg criticizes the statement of Abhyankar's theorem (first paragraph) as being too global to be true.

p47, 5t. Drop the condition:"closed under fiber products". It is not actually needed, and it fails for the small flat site: U, V, W can all be flat over X without $U \times_V W$ being flat over X (the point is that $U \to V$ need not be flat).

p51, 11b. Should be (I.2.19).

p57. The first line should read $\mathbf{A} = \mathbf{Ab}$ and $p = \pi^{\cdot}$.

p67. In Exercise 2.19, it is not true in general that (b) $\Rightarrow$ (c). In fact, all the statements in (2.19) should be treated with caution.

Lorenzini has pointed out to me that, contrary to what is asserted, (c$'$) *does imply* (b$'$), and that this can be proved as follows. Let $\phi\colon G \to G''$ be a morphism of group schemes (not necessarily commutative) flat and locally of finite presentation over a scheme S, and assume that ϕ is surjective as a map of sheaves for the flat (fppf) topology on S. We have to show that the group-scheme kernel G' of ϕ is flat over S.

The surjectivity means that, for any $s' \in G''(U)$ with U flat and locally of finite presentation over S, there exists a W faithfully flat and locally of finite presentation over U and an $s \in G(W)$ mapping to $s'|W$. When we apply this remark to the identity morphism $G'' \to G''$, we see that there is a faithfully flat morphism $f: W \to G''$ locally of finite presentation and a morphism $s: W \to G$ such that $\phi \circ s = f$.

There is an isomorphism of W-schemes

$$(g, w) \mapsto (g \cdot s(w)^{-1}, w): G \times_{G''} W \to G' \times_S W.$$

Because $G \times_{G''} W$ is flat over G and G is flat over S, $G \times_{G''} W$ is flat over S. Thus $G' \times_S W$ is flat over S. Because $G' \times_S W \to G'$ is faithfully flat, this implies that $G' \to S$ is flat.

Holger Deppe has pointed out to me that (a′) $\implies$ (c′) doesn't hold either, i.e., a flat homomorphism $f: G \to G''$ commutative flat group schemes locally of finite type over X might not give a surjective map of sheaves on X_{fl}. Consider, for example, the constant group scheme G defined by a finite commutative group $A \neq 1$ and let $f: G \to G$ be the zero map; this is flat (even a local isomorphism) but it is not surjective as a map of sheaves. I don't know whether the implication becomes correct when the group schemes are connected.

p90, l7. In "As any sheaf on $(\operatorname{spec} k)_{\mathrm{et}}$ is flabby", replace k with $\bar{k}$ (Tiago J. Fonseca).

p93. In the proof of (1.27), we need to show that the two restrictions of γ to $X' \times_X X'$ are equal, but this can be seen on the stalks.

p97. Matthieu Romagny points out to me that "universal monomorphism" is redundant: in a category with fibre products, monomorphisms are automatically universal monomorphisms.

p98. From Changlong Zhong: "I think there is a typo in the book Etale Cohomology' page 98, Ch III, Proposition 2.3. $H^0(U, -)$ should be $\check{H}^0(U, -)$.

p112. I define a site to be noetherian if every covering has a finite subcovering, and then say "It is not clear to the author why this is called a Noetherian site rather than a compact site". As was pointed out to me by Thomas Wyler, the reason is the following: a topological space is said to be noetherian if it has the ascending chain condition on open sets, and one sees easily that a topological space is noetherian if and only if every open subset is quasi-compact (i.e., every open covering of an open subset has a finite subcovering).

p134. "$\underline{\mathrm{Aut}}(Y)$ is the sheaf associated with the presheaf ..." The presheaf is actually a sheaf — there is no need to take the associated sheaf.

p139, 1t. The bar over the R has been omitted.

p151. From Tate 12/4/92: "It was not at all clear to me, especially for schemes as opposed to rings, that a maximal order is locally maximal. And I think the comment on line -7 (after possibly replacing...) is misleading, since (I hope) if $M_n(A)$ is Azumaya, then A is also, and so is maximal."

p153. From Tate: The statement "if an Azumaya algebra..." should refer to classes in the Brauer group. For example, there exists a vector bundle V on $\mathbb{A}^3 - \{(0,0,0)\}$ that doesn't extend, and $\mathrm{End}(V)$ is then an Azumaya algebra on the same variety that doesn't extend.

p153. G. Ofer should be Ofer Gabber.

p155. The heading should have a semicolon, not a colon.

p174. See Gamst and Hoechsmann, Tohoku 1968, for more on pairings.

p190, 4b. Drop the subscript on $\mathbb{Q}$.

p197. Treating only surfaces makes the theory more concrete, but the theory of vanishing cycles is not essentially more difficult in higher dimensions—probably I should have included the general case.

p202. In the diagram at the bottom of the page, the curve X_s' should not cross itself (think of the picture in 3-dimensions).

p221, 10b. Should read: $X' = X \times_{\text{spec } k} \mathbb{A}_k^d \cdots$

p224. Corollary 2.8 is incorrect. (Is it O.K. if the Galois group of k is finitely generated?)

p227. Higher direct images with *proper* support is better than ... compact support.

p229. Remark 3.3e. For a modern proof of Nagata's theorem, see Lutkebohmert, W., On compactification of schemes, Manuscripta Math. 80 (1993), 95-111.

p244. Corollary 5.5 is incorrect. (Cf. 2.8 above.)

p253. In the statement of Theorem 7.1, $i : Z \to X$ is the inclusion of a hyper*surface* section.

p253. Should add the Lefschetz hypersurface-section theorem: Let X be a projective variety of dimension d over a separably closed field k, and let Z be a hypersurface section of X such that $U \overset{\text{def}}{=} X \smallsetminus Z$ is smooth; for any locally constant sheaf F on X with finite stalks whose torsion is prime to the characteristic of k,

$$H^i(X, F) \to H^i(Z, F)$$

is injective for $i = d - 1$ and an isomorphism for $i < d - 1$.

To prove this, use the exact sequence

$$\cdots \to H_c^i(U, F) \to H^i(X, F) \to H^i(Z, F) \to \cdots$$

(III 1.30) noting that $H_c^i(U, F)$ is dual to $H^{2d-i}(U, F^\vee(d))$ (VI 11.2), which is zero for $2d - i > d$, i.e., $i < d$ (VI 7.1).

p253, 1b. In fact, it's only proved prime to the characteristic.

p256, 9t. $\Gamma(X_\alpha, \mathcal{O}_{X_\alpha})$ *not* $\Gamma(X_\alpha, \mathcal{O}_{X\alpha})$.

p256. The reader is invited to rewrite Section 8 using derived categories.

p264. In Lemma 8.17, assume that ϕ is a quasi-isomorphism.

p268. At the top of the page, assume that the torsion in Λ is prime to the characteristic of k.

p268, 15b. Alas, $\sum H^r(X, \Lambda([\frac{r}{2}]))$ does not become a ring under cup-product. To get a ring, take $\sum H^{2r}(X, \Lambda([r]))$.

p273. In A4, replace y and Y with z and Z.

p274, 13t. Replace c with c_t.

p275. The proof of Proposition 10.6 is incorrect, because the map at the bottom of the page is not invertible. A correct proof is rather long.

p276, 9t. Replace $\mathbb{Q}_t$ with $\mathbb{Q}_\ell$.

p292. Add $\frac{t^n}{n}$ to the end of the top formula.

p307. In the 4th line, it should say $R^p g(f(I))$ rather than $R^p g(I)$ (Lucio Guerberoff).

p311, 13t. (f) not (f).

p322. Godement, not Grodement.

Chapter VI. Should add the Lefschetz hypersurface-section theorem in the following form: for any hypersurface section Z of a projective variety X such that $U \overset{\text{def}}{=} X \smallsetminus Z$ is smooth, $H_l^i(X) \to H_l^i(Z)$ is injective if $i = \dim X - 1$ and an isomorphism if $i < \dim X - 1$. To prove this, use the exact sequence

$$\cdots \to H_c^i(U, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l) \to H^i(Z, \mathbb{Q}_l) \to \cdots$$

(III 1.30) noting that $H_c^i(U, \mathbb{Q}_l)$ is dual to $H^{2d-i}(U, \mathbb{Q}_l(d))$, $d = \dim X$, (VI 11.2), which is zero for $2d - i > d$, i.e., $i < d$ (VI 7.1).

dust jacket On the back of the first printing, replace 1971 with 1980.

From Yang Cao 14/12/11:

Page 170. At the end of your proof of Proposition 1.13, you say that "using the fact that $\pi_! F' \to \bar{\pi}_* J^\bullet$ is an injective resolution of $\pi_! F'$, we get the required diagram." But this can not prove that $j_! \pi^* F \to j_! \pi^* I^\bullet[r+s]$ is an injective resolution of $j_! \pi^* F$. So it can not prove that the homotopy class of $\mathrm{Hom}_{\bar{X}'}(J^\bullet[r], j_! \pi^* I^\bullet[r+s])$ is equal to $\mathrm{Ext}^s_{X'}(F', \pi^* F) = \mathrm{Ext}^s_{X'}(j_! F', j_! \pi^* F)$, and the homotopy class of $\mathrm{Hom}_{\bar{X}'}(\mathbb{Z}, j_! \pi^* I^\bullet[r+s])$ is equal to $H^{r+s}(\bar{X}', j_! \pi^* F)$.

Page 242. Line 7 from the top. I think

$$\underline{\mathrm{Ext}}^i_X(j_! j^* \mathbb{Z}, F) \approx j_* \underline{\mathrm{Ext}}^i_U(j^* \mathbb{Z}, j^* F) \approx R^i j_*(j^* F)$$

is wrong. It should be a spectral sequence

$$R^s j_* \underline{\mathrm{Ext}}^i_U(j^* \mathbb{Z}, j^* F) \implies \underline{\mathrm{Ext}}^{i+s}_X(j_! j^* \mathbb{Z}, F),$$

and $\underline{\mathrm{Ext}}^i_U(j^* \mathbb{Z}, j^* F)$ is $j^* F$ when $i = 0$ and 0 when $i > 0$. So we get

$$\underline{\mathrm{Ext}}^i_X(j_! j^* \mathbb{Z}, F) \approx R^i j_*(j^* F).$$