# Addendum/Erratum for Etale Cohomology, Princeton U. Press, 1980

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**px, yt, zb** means page x, line y from top, line z from bottom.

**pxiii.** The only reason I assumed all the schemes are locally Noetherian was that I didn't want to keep saying finite presentation etc. for finite type. I can't believe it really matters anywhere. The fibre product schemes on p. 70, line 2, and p80 are not automatically Noetherian. If I remember correctly, Grothendieck's example of naturally arising nonnoetherian rings is the tensor product of two completions of ring. Products of Henselizations are probably only about as Noetherian as completions.

**p6, 18b.** It does not "follow easily". In fact, passing from the affine result to the global result is not easy.

**p7, 1.12.** Cf. Hartshorne, Algebraic Geometry, Ex. 3.7. (Hartshorne's book appeared just before I sent the final version of the manuscript to the publisher, in time only for me to change some of the references in the manuscript.)

**p7, 12b.** Atiyah-Macdonald [1, 2.19]

**p9.** Correct statement of 2.5: Let $\alpha\, A \to B$ be a flat $A$-algebra, and consider $b \in B$. If the image of $b$ in $B/\alpha^{-1}(\mathfrak{n})$ is not a zero-divisor for any maximal ideal $\mathfrak{n}$ of $B$, then $B/(b)$ is a flat $A$-algebra. This necessitates some changes on p10.

**p16.** In the statement of Theorem 2.16, one needs to assume that $f$ is dominant in order for the set to be nonempty (Suchandan Pal). For a proof of Theorem 2.16, see also Mumford [2, p57] or (9.11) of my Commutative Algebra notes.

**p31.** In Exercise 3.27, the $Y$ should be $T$.

**p40.** Remark 5.1c should read: ...states that the functor that...

**p42.** Chinberg criticizes the statement of Abhyankar's theorem (first paragraph) as being too global to be true.

**p47, 5t.** Drop the condition:"closed under fiber products". It is not actually needed, and it fails for the small flat site: $U, V, W$ can all be flat over $X$ without $U \times_V W$ being flat over $X$ (the point is that $U \to V$ need not be flat).

**p51, 11b.** Should be (I.2.19).

**p57.** The first line should read $\mathbf{A} = \mathbf{Ab}$ and $p = \pi^{\cdot}$.

**p67.** In Exercise 2.19, it is not true in general that (b) $\Rightarrow$ (c). In fact, all the statements in (2.19) should be treated with caution.

Lorenzini has pointed out to me that, contrary to what is asserted, (c′) *does imply* (b′), and that this can be proved as follows. Let $\phi\colon G \to G''$ be a morphism of group schemes (not necessarily commutative) flat and locally of finite presentation over a scheme $S$, and assume that $\phi$ is surjective as a map of sheaves for the flat (fppf) topology on $S$. We have to show that the group-scheme kernel $G'$ of $\phi$ is flat over $S$.

The surjectivity means that, for any $s' \in G''(U)$ with $U$ flat and locally of finite presentation over $S$, there exists a $W$ faithfully flat and locally of finite presentation over $U$ and an $s \in G(W)$ mapping to $s'|W$. When we apply this remark to the identity morphism $G'' \to G''$, we see that there is a faithfully flat morphism $f: W \to G''$ locally of finite presentation and a morphism $s: W \to G$ such that $\phi \circ s = f$.

There is an isomorphism of $W$-schemes

$$(g, w) \mapsto (g \cdot s(w)^{-1}, w): G \times_{G''} W \to G' \times_S W.$$

Because $G \times_{G''} W$ is flat over $G$ and $G$ is flat over $S$, $G \times_{G''} W$ is flat over $S$. Thus $G' \times_S W$ is flat over $S$. Because $G' \times_S W \to G'$ is faithfully flat, this implies that $G' \to S$ is flat.

Holger Deppe has pointed out to me that (a$'$) $\Longrightarrow$ (c$'$) doesn't hold either, i.e., a flat homomorphism $f: G \to G''$ commutative flat group schemes locally of finite type over $X$ might not give a surjective map of sheaves on $X_{\mathrm{fl}}$. Consider, for example, the constant group scheme $G$ defined by a finite commutative group $A \neq 1$ and let $f: G \to G$ be the zero map; this is flat (even a local isomorphism) but it is not surjective as a map of sheaves. I don't know whether the implication becomes correct when the group schemes are connected.

**p90, l7.** In "As any sheaf on $(\mathrm{spec}\, k)_{\mathrm{et}}$ is flabby", replace $k$ with $\bar{k}$ (Tiago J. Fonseca).

**p93.** In the proof of (1.27), we need to show that the two restrictions of $\gamma$ to $X' \times_X X'$ are equal, but this can be seen on the stalks.

**p97.** Matthieu Romagny points out to me that "universal monomorphism" is redundant: in a category with fibre products, monomorphisms are automatically universal monomorphisms.

**p98.** From Changlong Zhong: "I think there is a typo in the book Etale Cohomology' page 98, Ch III, Proposition 2.3. $H^0(U, -)$ should be $\check{H}^0(U, -)$.

**p112.** I define a site to be noetherian if every covering has a finite subcovering, and then say "It is not clear to the author why this is called a Noetherian site rather than a compact site". As was pointed out to me by Thomas Wyler, the reason is the following: a topological space is said to be noetherian if it has the ascending chain condition on open sets, and one sees easily that a topological space is noetherian if and only if every open subset is quasi-compact (i.e., every open covering of an open subset has a finite subcovering).

**p121.** In Theorem III.4.3 (b) one needs an additional assumption that G is of finite presentation (i.e., under the omnipresent locally Noetherian assumption, one may need to assume in addition that G is of finite type). This assumption is present in the cited reference Raynaud [2, XI.3.1 1)]. Also, presumably, in the proof of the same theorem, the reference for part (e) to Raynaud [2, XII.2.3] should be replaced by Raynaud [2, XIII.2.3 (iii)]. (Kestutis Cesnavicius).

**p134. "**$\underline{\mathrm{Aut}}(Y)$ is the sheaf associated with the presheaf ..." The presheaf is actually a sheaf — there is no need to take the associated sheaf.

**p139, 1t.** The bar over the $R$ has been omitted.

**p143.** The last term in the display in Step 3 has a spurious \check on the $X$ (Timo Keller).

**p151.** From Tate 12/4/92: "It was not at all clear to me, especially for schemes as opposed to rings, that a maximal order is locally maximal. And I think the comment on line -7 (after possibly replacing...) is misleading, since (I hope) if $M_n(A)$ is Azumaya, then $A$ is also, and so is maximal."

**p153.** From Tate: The statement "if an Azumaya algebra..." should refer to classes in the Brauer group. For example, there exists a vector bundle $V$ on $\mathbb{A}^3 - \{(0,0,0)\}$ that doesn't extend, and $\mathrm{End}(V)$ is then an Azumaya algebra on the same variety that doesn't extend.

**p153.** G. Ofer should be Ofer Gabber.

**p155.** The heading should have a semicolon, not a colon.

**p174.** See Gamst and Hoechsmann, Tohoku 1968, for more on pairings.

**p190, 4b.** Drop the subscript on $\mathbb{Q}$.

**p197.** Treating only surfaces makes the theory more concrete, but the theory of vanishing cycles is not essentially more difficult in higher dimensions—probably I should have included the general case.

**p202.** In the diagram at the bottom of the page, the curve $X_s'$ should not cross itself (think of the picture in 3-dimensions).

**p221, 10b.** Should read: $X' = X \times_{\mathrm{spec}\, k} \mathbb{A}^d_k \cdots$

**p224.** Corollary 2.8 is incorrect. (Is it O.K. if the Galois group of $k$ is finitely generated?)

**p227.** Higher direct images with *proper* support is better than ... compact support.

**p229.** Remark 3.3e. For a modern proof of Nagata's theorem, see Lutkebohmert, W., On compactification of schemes, Manuscripta Math. 80 (1993), 95-111.

**p244.** Corollary 5.5 is incorrect. (Cf. 2.8 above.)

**p253.** In the statement of Theorem 7.1, $i : Z \to X$ is the inclusion of a hyper*surface* section.

**p253.** Should add the Lefschetz hypersurface-section theorem: Let $X$ be a projective variety of dimension $d$ over a separably closed field $k$, and let $Z$ be a hypersurface section of $X$ such that $U \stackrel{\mathrm{def}}{=} X \smallsetminus Z$ is smooth; for any locally constant sheaf $F$ on $X$ with finite stalks whose torsion is prime to the characteristic of $k$,

$$H^i(X, F) \to H^i(Z, F)$$

is injective for $i = d-1$ and an isomorphism for $i < d-1$.

To prove this, use the exact sequence

$$\cdots \to H^i_c(U, F) \to H^i(X, F) \to H^i(Z, F) \to \cdots$$

(III 1.30) noting that $H^i_c(U, F)$ is dual to $H^{2d-i}(U, F^\vee(d))$ (VI 11.2), which is zero for $2d - i > d$, i.e., $i < d$ (VI 7.1).

**p253, 1b.** In fact, it's only proved prime to the characteristic.

**p256, 9t.** $\Gamma(X_\alpha, \mathcal{O}_{X_\alpha})$ *not* $\Gamma(X_\alpha, \mathcal{O}_{X\alpha})$.

**p256.** The reader is invited to rewrite Section 8 using derived categories.

**p264.** In Lemma 8.17, assume that $\phi$ is a quasi-isomorphism.

**p268.** At the top of the page, assume that the torsion in $\Lambda$ is prime to the characteristic of $k$.

**p268, 15b.** Alas, $\sum H^r(X, \Lambda([\frac{r}{2}]))$ does not become a ring under cup-product. To get a ring, take $\sum H^{2r}(X, \Lambda([r]))$.

**p273.** In A4, replace $y$ and $Y$ with $z$ and $Z$.

**p274, 13t.** Replace $c$ with $c_t$.

**p275.** The proof of Proposition 10.6 is incorrect, because the map at the bottom of the page is not invertible. A correct proof is rather long.

**p276, 9t.** Replace $\mathbb{Q}_t$ with $\mathbb{Q}_\ell$.

**p292.** Add $\frac{t^n}{n}$ to the end of the top formula.

**p307.** In the 4th line, it should say $R^p g(f(I))$ rather than $R^p g(I)$ (Lucio Guerberoff).

**p311, 13t.** (f) not $(f)$.

**p322.** Godement, not Grodement.

**Chapter VI.** Should add the Lefschetz hypersurface-section theorem in the following form: for any hypersurface section $Z$ of a projective variety $X$ such that $U \overset{\text{def}}{=} X \smallsetminus Z$ is smooth, $H_l^i(X) \to H_l^i(Z)$ is injective if $i = \dim X - 1$ and an isomorphism if $i < \dim X - 1$. To prove this, use the exact sequence

$$\cdots \to H_c^i(U, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l) \to H^i(Z, \mathbb{Q}_l) \to \cdots$$

(III 1.30) noting that $H_c^i(U, \mathbb{Q}_l)$ is dual to $H^{2d-i}(U, \mathbb{Q}_l(d))$, $d = \dim X$, (VI 11.2), which is zero for $2d - i > d$, i.e., $i < d$ (VI 7.1).

**dust jacket** On the back of the first printing, replace 1971 with 1980.

**From Yang Cao 14/12/11:**

Page 170. At the end of your proof of Proposition 1.13, you say that "using the fact that $\pi_! F' \to \bar{\pi}_* J^\bullet$ is an injective resolution of $\pi_! F'$, we get the required diagram." But this can not prove that $j_! \pi^* F \to j_! \pi^* I^\bullet[r+s]$ is an injective resolution of $j_! \pi^* F$. So it can not prove that the homotopy class of $\mathrm{Hom}_{\bar{X}'}(J^\bullet[r], j_! \pi^* I^\bullet[r+s])$ is equal to $\mathrm{Ext}^s_{X'}(F', \pi^* F) = \mathrm{Ext}^s_{X'}(j_! F', j_! \pi^* F)$, and the homotopy class of $\mathrm{Hom}_{\bar{X}'}(\mathbb{Z}, j_! \pi^* I^\bullet[r+s])$ is equal to $H^{r+s}(\bar{X}', j_! \pi^* F)$.

Page 242. Line 7 from the top. I think

$$\underline{\mathrm{Ext}}^i_X(j_! j^* \mathbb{Z}, F) \approx j_* \underline{\mathrm{Ext}}^i_U(j^* \mathbb{Z}, j^* F) \approx R^i j_*(j^* F)$$

is wrong. It should be a spectral sequence

$$R^s j_* \underline{\mathrm{Ext}}^i_U(j^* \mathbb{Z}, j^* F) \implies \underline{\mathrm{Ext}}^{i+s}_X(j_! j^* \mathbb{Z}, F),$$

and $\underline{\mathrm{Ext}}^i_U(j^* \mathbb{Z}, j^* F)$ is $j^* F$ when $i = 0$ and 0 when $i > 0$. So we get

$$\underline{\mathrm{Ext}}^i_X(j_! j^* \mathbb{Z}, F) \approx R^i j_*(j^* F).$$

**Some typos (Bhupendra Nath Tiwari)**

p16. (Proposition 2.18) ... is exact where $B^{\otimes r} = B \otimes_A B \otimes_A \ldots \otimes_A B$

p177. Proof. where $\mathrm{tr}\colon \mathrm{Pic}(X') \to \mathrm{Pic}(X)$ is ...

p178. Step 3: Proof. Let ... and let $i\colon Z \hookrightarrow U$ be the complementary closed immersion.

p232. A morphism $g\colon Y \to X$ is $n$-acyclic ... is bijective for $0 \le i \le n$, not $0 \le 1 \le n$.

p233. Sublemma 4.7 ... and $(h_i'\colon X_i' \to X')$ is a covering of ...

p250. Proposition 6.5 (b) Let $Z \overset{i_1}{\hookrightarrow} Y \overset{i_2}{\hookrightarrow} X$ be a smooth $S$-triple;

p262. Remark 8.12... $\mathbf{R}h_\star(F \boxtimes G) = C^\bullet(F) \boxtimes C^\bullet(G)$ should be

$$\mathbf{R}h_\star(F \boxtimes G) = h_\star(C^\bullet(F) \boxtimes C^\bullet(G))$$

(seems so by the symmetry of the canonical pairing ... (you gave, just below it))

p268. The first display should be ... $F \boxtimes^{\mathbf{L}} G$).

p290. Lemma 13.2, replace "identify map" with "identity map".