Addendum/Erratum for Arithmetic Duality Theorems 2nd edition 2006.

J.S. Milne

Last revised March 10, 2024.

As noted in the original Preface, I wrote the book (before departing[1] the field), in order that there should be a written record of what was known, or believed to be known, at the time.

Addenda/errata

From Jungin Lee

Page 168, line 5 : Change i^* and i_*.

Page 210, line 7 : Change n to m (2 times).

Page 210, line 9 : Change $H_c^3(Y, \mu_m)$ to $H_c^3(U, \mu_m)$.

Page 211, Cor.2.7.7 : By the recent work of Geisser and Schmidt [GS18], this theorem can be generalized to regular, flat, separated morphism of finite type of relative dimension d (which is not necessarily smooth).

[GS18] T. Geisser and A. Schmidt, Poitou-Tate duality for arithmetic schemes. Compositio Math. 154 (2018), 2020-2044.

pvii. explanation of the footnote.

Let M be a p-primary abelian group (i.e., every element is killed by a power of p).

Let H be the subgroup of infinitely p-divisible elements, i.e.,

$$H = \{x \in M \mid \text{for all } n \geq 1, \text{ there exists a } y \in M \text{ such that } p^n y = x\}.$$

Then H need not be divisible, i.e., we need not have $pH = H$. We only know H is contained in pG. Moreover, for x in H, there need not exist an infinite sequence $y_1, y_2, y_3....$ such that $py_2 = y_1, py_3 = y_2,...$ We only know that there exist arbitrarily long finite such sequences.

There does exist a unique maximal divisible subgroup D of M, and D is a subgroup of H. Moreover, $M = D \oplus N$ where N is a subgroup with no divisible subgroups and D is isomorphic to a direct sum of copies of $\mathbb{Q}_p/\mathbb{Z}_p$.

Thus, unless there is some finiteness condition on M, you do have to worry about H being different from D. For example, M could have infinitely divisible elements but no divisible subgroup.

For all this, see Kaplansky, Infinite Abelian Groups, University of Michigan Press, 1954.

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[1] Not entirely successfully!

In Milne 1988, 3.3,[2] the following is proved:

> Let N be an abelian group, and let $N^1 \stackrel{\text{def}}{=} \bigcap mN$ be its first Ulm subgroup.
>
> (a) If N/mN is finite for all integers m, then N^1 is divisible (and hence is the unique maximal divisible subgroup N_{div} of N); if, in addition, $\hat{N} \stackrel{\text{def}}{=} \varprojlim N/mN$ is finite, then N/N_{div} is finite and equals $\hat{N}$.
>
> (b) Assume N is torsion, and let M be the Pontryagin dual of N (regarding N as a discrete group). Then the pairing $N/N_{\text{div}} \times M_{\text{tors}} \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate, and so the Pontryagin dual of N/N_{div} is the closure of M_{tors} in M.

Here is an example of a group M such that $H \neq 0$ but $D = 0$. Let C_n be the cyclic group of order p^{n+1} with generator e_n, and let $N = \bigoplus_{n \geq 0} C_n$. Then N_p is a vector space over $\mathbb{F}_p$ with basis e_0, pe_1, $p^2 e_2$, ... Let M be the quotient of N by the subspace of codimension 1 of N_p generated by $e_0 - pe_1$, $pe_1 - p^2 e_2$, ... In M the element e_0 becomes infinitely p-divisible, because $e_0 = pe_1 = p^2 e_2 = \dots$ in M. However, it is not in any p-divisible subgroup of M: otherwise we could find an infinite chain $e_0, a_1, a_2, \dots$ in M with $pa_1 = e_0$ and $pa_i = a_{i-1}$, which is clearly impossible. In fact, no such infinite chains exist in the quotient group. Note that M has countably infinitely many elements of order p.

p2. In the definition of a complete resolution of G, I should require that e has infinite order, so that in the factorization $d_0 = \iota \circ \epsilon$, ι is injective — otherwise the zero complex satisfies the definition (Kevin Buzzard).

p3. It should be noted that the formula (0.6.1) only only applies when x, y have degree ≥ 1.

p46. 3.8. Ahmed Matar has pointed out to me that, in the last paragraph of the proof, I can't apply the lemma in Serre to show that $H^2(G/I, \mathcal{A}^\circ(R^{\text{un}})) = 0$ because $R^{\text{un}} \neq \varprojlim R^{\text{un}}/\mathfrak{m}^n R^{\text{un}}$ and $\mathcal{A}^\circ(R^{\text{un}}) \neq \varprojlim \mathcal{A}^\circ(R^{\text{un}}/\mathfrak{m}^n R^{\text{un}})$ (in fact, $\mathcal{A}^\circ(R^{\text{un}})$ is countable but $\varprojlim \mathcal{A}^\circ(R^{\text{un}}/\mathfrak{m}^n R^{\text{un}})$ is uncountable). To fix this, one can argue as follows.

Let K' be a finite unramified extension of K, let R' be the ring of integers in K', let k' be the residue field, and let $\Gamma = \text{Gal}(K'/K) \simeq \text{Gal}(k'/k)$. For each $n \geq 1$, there is an exact sequence

$$0 \to \omega_A \otimes_R k' \to \mathcal{A}^\circ(R'/\mathfrak{m}^{n+1} R') \to \mathcal{A}^\circ(R'/\mathfrak{m}^n R') \to 0$$

(cf. III 4.3). Now $H^i(\Gamma, \omega_A \otimes_R k') = 0$ for $i > 0$ (by Serre, Local Fields, X §1 Prop 1, for example), and so

$$H^i(\Gamma, \text{Ker}(\mathcal{A}^\circ(R'/\mathfrak{m}^{n+1} R') \to \mathcal{A}^\circ(k'))) = 0$$

for $i > 0$ and all $n \geq 0$. When K'/K is finite, we can apply the lemma in Serre to deduce that

$$H^i(\Gamma, \text{Ker}(\mathcal{A}^\circ(R') \to \mathcal{A}^\circ(k'))) = 0$$

for $i > 0$. On passing to the limit over increasing K', we get the same result for $K' = K^{\text{un}}$, and so

$$H^i(\Gamma, \mathcal{A}^\circ(R^{\text{un}})) \simeq H^i(\Gamma, \mathcal{A}^\circ(k^{\text{un}})),$$

which is zero for $i > 1$ because $\mathcal{A}^\circ(k^{\text{un}})$ is torsion and Γ has cohomological dimension 1.

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[2]Milne, J. S. Motivic cohomology and values of zeta functions. Compositio Math. 68 (1988), no. 1, 59–100, 101–102.

p48 Parenthetical question at bottom of page: Let K be a number field, let S be a set of primes of K, and let K_S be the largest extension of K unramified outside S. Let P be the set of prime numbers ℓ such that $\ell^\infty | [K_S : K]$. It is now known that P contains *all* prime numbers provided that S contains all the prime ideals of K lying over at least two primes in $\mathbb{Q}$. See Chenevier and Clozel, JAMS 22 (2009), 467–519, Cor. 5.2. This strengthens the main results of §4.

p48 Footnote . Haberland's errors, for example, his "proof" that a certain global $H^2(G_S, -)$ is the whole of $\mathbb{Q}/\mathbb{Z}$ in fact only shows that it is a subgroup of $\mathbb{Q}/\mathbb{Z}$ containing $\mathbb{Q}_p/\mathbb{Z}_p$ for all $p \in S$, don't affect his proofs of the duality theorems for finite discrete G_S-modules M whose order is an S-unit (Brian Conrad).

p49. It would be better to *define* $J_{F,S}$ to be the restricted topological product of the $F_w^\times$ (so that the S_F-units embeds into it).

p57. The exact sequence in Theorem 4.10 has become known as the Poitou-Tate exact sequence.

p72. II §6, Abelian varieties over global fields Tate outlines a proof of his duality theorem in a letter to Serre, 28 July 1962 (Correspondance Serre–Tate, Vol. I, p.82). (When I was writing this chapter, I had trouble proving the theorem, and wrote to Tate, but received no reply.)

p73. Lemma 6.1. holds more generally: for any isogeny $f : A \to B$ of abelian schemes over a normal scheme X such that $\deg f$ is a unit in $\mathcal{O}_X$, $A_f(k(x)_{sep}) = A_f(k(x)_{un})$. (notations as in Example I 5.2b of my book on Etale Cohomology).

p82, footnote 17. In his 1962 ICM talk, Theorem 3.3, Tate correctly states that the pairing on the Tate-Shafarevich group defined by a principal polarization is alternating when the polarization is defined by a divisor rational over the base field k. In his Bourbaki talk (1966, p306-06) Tate omits the condition, and incorrectly states that the order of the Tate-Shafarevich group of a Jacobian is a square. In the original version of the book (6.12, p100), I incorrectly stated that the pairing on the Tate-Shafarevich group of a Jacobian is alternating.

p91, 6.24. As Peter Jossen pointed out to me, the statement is proved only for the m-components where m is a unit in $R_{K,S}$, for example, any integer prime to characteristic of K when S contains all finite primes. Probably the statement is still true without this condition — cf. the note on II 5.6 below.

p93. Footnote 21. This has been published as: Harari, David; Szamuely, Tamás. Arithmetic duality theorems for 1-motives. J. Reine Angew. Math. 578 (2005), 93–128. Correction ibid. 632 (2009), 233-236.

p111. Jiu-Kang Yu has pointed out to me that there is a gap in the argument in the paragraph at the bottom of the page, namely, where I claim that, with $A = C_F \otimes M$, the canonical map $\mathrm{Hom}(A, \mathbb{C}^\times)_G \to \mathrm{Hom}(A^G, \mathbb{C}^\times)$ is *obviously* an isomorphism. Here the Hom is as abstract groups (i.e., disregarding the topology). As he writes:

If we replace $\mathrm{Hom}(A, \mathbb{C}^\times)$ with the Pontrijagin dual of A (assuming A locally compact), then this is obvious, by Pontrijagin duality. Lacking a duality theory, I see no reason that the quotient $\mathrm{Hom}(A, \mathbb{C}^\times)_G$ of $\mathrm{Hom}(A, \mathbb{C}^\times)$ should be of the form $\mathrm{Hom}(B, \mathbb{C}^\times)$ for a subgroup B of A. In Labesse's paper (1984) on Langlands correspondence for tori, he argued by splitting $\mathbb{C}^\times$ to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, and argued the two parts separately (the part for $\mathbb{R}/\mathbb{Z}$ is Pontrijagin duality as above). I think that that is OK.

p120. Proof of Corollary 9.6 In order to be able to apply Theorem 9.2, one should first reduce to the case that the Galois action factors through an ℓ-group, as in the "elementary" proof of Cassels, cited on the same page (Ari Shnidman).

p194. Theorem 4.11. This is true with the big étale site replaced by the smooth site, as predicted in 4.12(b) — see Suzuki arXiv:1410.3046v1, Remark 5.2 (this became 5.2.1 in later versions).

pp197-205, II 5. Throughout this section, in the function field case I'm always working prime to the characteristic p. Sometimes I forget to say this, for example, in the statement of Theorem 5.6(a) (but not in its proof).

p200, Corollary 5.3. From David Harari: I think that the proof of Corollary II.5.3. in your book "Arithmetic duality Theorems" (page 201 in the new edition) is incomplete. Indeed the second line of the diagram on top of page 201 doesn't make sense ($H_c^1(U, A)$ and $H^0(K_v, A)$ are not torsion groups). One could try to remove the m in this second line, but then one runs into the problem that the kernel of the first vertical map is not necessarily divisible (it is just a subgroup of a divisible group).

Therefore, I believe that one has to use the analogue of your lemmas I.6.15. and I.6.17 (pp 86-88) to complete the proof. With Tamas Szamuely, we wrote this recently for 1-motives (see http://www.math-inst.hu/~szamuely/erratcrelle.pdf) because we went wrong at this point as well in our Crelle paper (proof of Cor 3.5.).

p202, Theorem 5.6b. On March 11, 1991, a student of Tate's (Ki-Seng Tan) wrote to me asking two questions, the first of which was "What one can say about the global duality for the Galois cohomologies, when p equals the characteristic of K". Following is my response:

> I am not sure I understand your notation, but it seems to me that the answer to your first question is already in my book "Arithmetic Duality Theorems".
>
> Specifically, let X be a complete smooth curve over a finite field k and let $K = k(X)$. Let A be an abelian variety over K, and let U be an open subset of X such that A extends to a smooth scheme (also denoted A) over U. Then (see p204) there is an exact sequence of étale cohomology groups
>
> $$H^1(U, A) \to \bigoplus_{v \notin U} H^1(K_v, A) \to H_c^2(U, A).$$
>
> When we pass to the limit over shrinking U, this becomes
>
> $$H^1(K, A) \to \bigoplus_{v \in X} H^1(K_v, A) \to \varinjlim H_c^2(U, A).$$

According to the theorem on p370, we can replace $H_c^2(U, A)$ with $H^0(U, A^t)^\wedge = A^t(K)^\wedge$. Here A^t is the dual abelian variety, and the hat means profinite completion. In fact, it seems to me that by using the results in Chapter III, one can prove II, Theorem 5.6b (p247) with m replaced by p.

Unfortunately, there is nothing I can say about your second question.

J.S. Milne, 19th March, 1991.

See González-Avilés, Cristian D.; Tan, Ki-Seng. A generalization of the Cassels-Tate dual exact sequence. Math. Res. Lett. 14 (2007), no. 2, 295–302. Also On the Poitou-Tate exact sequence for 1-motives, Cristian D. Gonzalez-Aviles, arXiv:0806.0772.

p.223 In Remark 0.6(b) I state (without proof) that the sequence (0.4c) remains exact after certain rings and fields have been replaced by their completions. Using (I, 3.10), Takashi Suzuki has written a proof of this for the sheaf defined by a smooth group scheme, and hence for finite group schemes (and suggests that the statement may be false in general). Probably this is all that is needed when we apply the remark in §8 of Chapter III.

p.224/225. I misstate the hypotheses of the Tate-Oort theorem:

In Arithmetic Duality Theorems (2nd edition) on p. 225 you recall the Oort-Tate classification of group schemes of order p. It seems to me that there zeta should be assumed to be a primitive $(p-1)$-st root of unity instead of a p-th root of unity. The discussion before Proposition III.1.20 appears to be correct: K automatically contains the $(p-1)$-st roots of unity because it is a fraction field of a Henselian discrete valuation ring of residue characteristic p.

On the last line of p.244 the Euler characteristic should be $1/(R:aR)$, as would be given by Theorem III.1.14. (Kęstutis Česnavičius).

III, §0, p.218 For detailed proofs that the cohomology groups $H_c^r(U, F)$ defined in 0.6(b) do have the properties claimed there, see Cyril Demarche, David Harari, *Artin-Mazur-Milne duality Theorem for fppf cohomology*, ANT, 13:10 (2019), 2323–2357, arXiv:1804.03941.

III, §3, Finite sheaves, p.252 For a detailed analysis of the commutativity of the diagrams in this section, see Demarche and Harari 2019.[3]

III, §8, The duality theorem, p.290. For a detailed exposition of the proof of **Theorem 8.2**, see Demarche and Harari 2019. In particular, they check that the cohomology with compact support defined using completions does have the properties I claim in III, 0.6(b), and they check that all the diagrams commute. For a slightly more general result with a different proof, see T. Suzuki, *Duality for cohomology of curves with coefficients in abelian varieties*, arXiv:1803.09291.

p.273. Kęstutis Česnavičius has questioned my computation of the flat cohomology of the Weil restriction. He writes:

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[3]For the "prime to the characteristics" case, they appeal to the Artin-Verdier theorem. For this they refer to the first edition of ADT, which contains only a flawed proof, and to Geisser and Schmidt 2018, which contains no proof at all, but does correctly refer to the second edition of my book.

In the proof of Proposition III.6.1, more precisely, in the proof of Lemma III.6.3, you claim that $H^r(K, G') = H^r(L, G)$ (flat cohomology), where $G' = \operatorname{Res}_{L/K} G_L$. I was wondering: what argument did you have in mind for this? [I have no idea, jsm] It seems that generalities on the exactness of the push-forward along a finite map a priori do not apply because we are dealing with fppf cohomology rather than étale. On the one hand, the full strength of your claim is not needed for the argument at hand: every locally of finite type commutative K-group scheme G embeds into a commutative smooth K-group H (which can be taken to be quasi-compact if so is G), so for the purposes of the proof of III.6.3 one may first embed into H and then use the étale topology, for which all is clear. On the other hand, I would be curious to know your general argument because your claim seems useful beyond the proof of III.6.3. (Although the quotient H/G is smooth, the map $H \to H/G$ need not be, so I do not see why $\operatorname{Res}_{L/K}$ would preserve the exact sequence (if it did, one could argue by the 5-lemma).)

p.272, III.6 Local fields of characteristic p.

In this section, I endow certain cohomology groups with a topology by identifying them with Cech groups. I then claim that it is "obvious", for example, that the maps in the cohomology sequence arising from a short exact sequence of group schemes are continuous (Lemma 6.5). It was pointed out to me by P. Gille and Nguyễñ Quốc Thăńg that this is far from obvious. Fortunately the questions on the topology have been successfully resolved. See:

Băć, Dào Phuong; Thăńg, Nguyễñ Quốc. On the topology of relative and geometric orbits for actions of algebraic groups over complete fields. J. Algebra 390 (2013), 181–198.

Băć, Dào Phuong; Thăńg, Nguyễñ Quốc. On the topology on group cohomology of algebraic groups over complete valued fields. J. Algebra 399 (2014), 561–580.

Česnavičius, Kęstutis. Topology on cohomology of local fields, (2014), arXiv:1405.2009.

Fibrés principaux sur les corps valués henséliens. Ofer Gabber, Philippe Gille, Laurent Moret-Bailly. arXiv:1309.6430.

As Gille writes (26.09.2013): In conclusion, the topology on cohomology groups of abelian k-groups defined in ADT is nice in a broader setting, that of admissible henselian valued fields.

p294, Notes

Milne 1977 should be Milne 1967 (my thesis).

p300. Section III.10.

For a criticism/correction of Bester's proof of his Theorem 3.1, see Remark 5.7 (p.48) of *Grothendieck's pairing on Neron component groups: Galois descent from the semistable case.* Takashi Suzuki. arXiv:1410.3046.

p322. Paragraph preceding C.11.

Suzuki (arXiv:1410.3046v1, 5.2) writes:

[in this paragraph] it is stated and used that

$$\mathcal{E}xt_X^r(\mathcal{A}, i_*\mathbb{Z}) = i_*\mathcal{E}xt_x^r(i^*\mathcal{A}, \mathbb{Z}).$$

This is incorrect: I only state the equality for $r = 1$, which is a much more benign statement. In any case, the statement is true for all r (Suzuki, ibid.). Suzuki fixed this in later versions of the manuscript, 5.2.1.

p313. Lemma III.B.1: I think it should read "and let $0 \to N' \to N \to N'' \to 0$" be an exact sequence of finite group schemes over *K* (not: R). (Timo Keller).

p323. Conjecture C.13 (Grothendieck's). I should also have mentioned (in the footnote) that Werner proved the conjecture for abelian varieties with semistable reduction (Werner, Annette On Grothendieck's pairing of component groups in the semistable reduction case. J. Reine Angew. Math. 486 (1997), 205–215.). More recently, Suzuki (arXiv:1410.3046 2014) has proved the conjecture in general (perfect residue fields) by deducing it from the semistable case.

Miscellaneous errata

From Kęstutis Česnavičius.

III.0.3 (a)–(b): Z should be $\mathbf{Z}$ in three instances.

pp. 220–221: in addition to being open, U and V are, presumably, meant to be nonempty.

p. 220, line 5: $H_c^r(X_\mathrm{fl}, F)$ should be $H_c^r(U_\mathrm{fl}, F)$.

III.0.4 (e): the statement and the proof seem to use different notation; one possible correction is to change G to F' and the displayed formula to

$$\mathrm{Ext}_U^s(F, F') \times H_c^r(U, F) \to H_c^{r+s}(U, F')$$

in the statement, change Ext^r to Ext^s on the first line of p. 223, and modify the line 6 on that page to read "... this morphism with c'.".

From Timo Keller

p.8 l.-2: There is a P missing.

p.18 Question 1.4: In Neukirch's Algebraic Number Theory, group cohomology is somewhat avoided.

p.25: "equal to the set primes" should be "equal to the set of primes"

p.40 l.-8: f should be F

p.74 l.-5: g_v should be G_v

p.146 Proposition 0.9: $Z \to g^*F$ does not induce $g_*Z \to F*$ (the adjunction is the other way round)

p.149 l-13, should read $H^r(X, j_!F) = 0$, not $H^r(X, F) = 0$ (which true, but uninteresting).

p.150 Lemma 1.4: The stalks are at $\bar{x}$ and $\bar{u}$, not $\bar{x}$ and $\bar{x}$

p. 151 The lower row in the commutative diagram should read:

$$0 \to R^\times \to K^\times \xrightarrow{\mathrm{ord}} \mathbb{Z} \to 0$$

p.172 2.11(b) $Z {-}{>} \mathbb{Z}$

p.197: "always an open subscheme or X" should be "always an open subscheme of X"

p.200 In line 1, I 0.20e should be I 0.20b.

p.205. "for any sheaf F on an open subscheme U of K" should be "for any sheaf F on an open subscheme U of X"

p.318 Proposition C.2: There is a (missing

From Keenan Kidwell

I, Lemma 1.2. Replace M with M^U twice (G/U doesn't act on M).

I, Proposition 6.2. The proof should read: "this gives an injection $A(K)^{(n)} \hookrightarrow H^1(G_S, A_n)$". **Nguyen Quoc Thang** has pointed out to me that I often misspell "Bégueri".

From David Harari

On page 117 (Example 9.1.), an integer r is defined as the g.c.d. of the local degrees. It should be the lowest common multiple.

From Matthieu Romagny

p.313. In the statement of Lemma B.1, the first exact sequence should be defined over K, not R.

Work since 2006.

There has been much progress in the field since 2006. Below I list some of the papers (in addition to those mentioned above).

Silverman, Joseph H. A survey of local and global pairings on elliptic curves and abelian varieties. Pairing-based cryptography—Pairing 2010, 377–396, Lecture Notes in Comput. Sci., 6487, Springer, Berlin, 2010.

"I wrote a survey article whose title could have been 'Where do pairings really come from, anyway?' It was for a cryptography conference on pairings. I tried to explain, from a functorial point of view, the origins and relationships of the various pairings on abelian varieties associated with the names of Weil, Tate, Lichtenbaum, Néron, Cassels, ... It's just a survey, so lacks many details, but may be useful in providing an overview." (Silverman).

Demarche, Cyril, Suites de Poitou-Tate pour les complexes de tores à deux termes. [Poitou-Tate sequences for two-term torus complexes] Int. Math. Res. Not. IMRN 2011, no. 1, 135–174.

Proves local and global duality theorems for the hypercohomology of a complex of tori of length 2 defined over a number field, and deduces a the existence of a Poitou-Tate exact sequence for such a complex. The motivation for such results comes from the study of algebraic groups.

Suzuki, Takashi. Duality for local fields and sheaves on the category of fields. arXiv:1310.4941 (2013).

Introduces a new site (a local version of that in Artin and Milne 1976) and uses it to reformulate, reprove, and extend the results of Bégueri and Bester on the duality of finite group schemes over local fields with perfect residue fields (ADT III.4, III.10).

Pépin, Cédric. Dualité sur un corps local de caractéristique positive à corps résiduel algébriquement clos. 154pp. arXiv:1411.0742 (2014).

Let K be a local field with algebraically closed residue field, and let R be the ring of integers in K. When K has mixed characteristic, Bégueri (1980) proved duality theorems for finite group schemes over R and K, and when K has characteristic $p \neq 0$, Bester (1978) proved duality theorems for finite group schemes over R (ADT III.4, III.10). In the second situation, Pépin proves duality theorems for finite group schemes over K (cf. Suzuki 2013).

Suzuki, Takashi. Grothendieck's pairing on Neron component groups: Galois descent from the semistable case. arXiv:1410.3046 (2014).

Suzuki uses his new site (see above) to formulate a statement for an abelian variety over a local field with perfect residue field and proves:

1. the statement is equivalent to the conjunction of Grothendieck's conjecture (ADT, p. 323, Conjecture C.13) and the local duality theorem (ADT, III 4.15, III 10.7);

2. if the statement becomes true over a finite Galois extension of the field, then it is true over the field.

As every abelian variety acquires semistable reduction after a finite Galois extension, this allows him to deduce both Grothendieck's conjecture and the local duality theorem from the semistable case, thus finally settling Grothendieck's conjecture (it is true if the residue field is perfect, but not otherwise). He also proves the local duality theorem for one-motives.

Česnavičius, Kęstutis. Poitou-Tate without restrictions on the order. Math. Res. Lett. 22 (2015), no. 6, 1621–1666. arXiv:1410.2621 (2014)

Attaches an exact sequence to a finite flat group scheme over an open subscheme of a curve over a finite field or the spectrum of the ring of integers in a number field (cf. ADT III 3.1, III.8.2). When the group scheme is taken to be étale with étale dual, the sequence becomes the Poitou-Tate exact sequence (ADT, I 4.10).

Generalizations of Poitou-Tate duality (Nekovář et al.)

Nekovář, Jan. Selmer complexes. Astérisque No. 310 (2006), Chapter 5.
Lim, Meng Fai. Poitou-Tate duality over extensions of global fields. J. Number Theory 132 (2012), no. 11, 2636–2672.
Lim, Meng Fai; Sharifi, Romyar T. Nekovář duality over p-adic Lie extensions of global fields. Doc. Math. 18 (2013), 621–678.

Various papers of González-Avilés (and Tan).

Mostly noted earlier.

Tate Duality In Positive Dimension Over Function Fields. Zev Rosengarten. 220 pages, arXiv:1805.00522

Abstract: We extend the classical duality results of Poitou and Tate for finite discrete Galois modules over local and global fields (local duality, nine-term exact sequence, etc.) to all affine commutative group schemes of finite type, building on the recent work of Česnavičius extending these results to all finite commutative group schemes. We concentrate mainly on the more difficult function field setting, giving some remarks about the number field case along the way.