Addendum/Erratum for Arithmetic Duality Theorems 2nd edition 2006.

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p2. In the definition of a complete resolution of G, I should require that e has infinite order, so that in the factorization $d_0 = \iota \circ \epsilon$, ι is injective — otherwise the zero complex satisfies the definition (Kevin Buzzard).

p3. It should be noted that the formula (0.6.1) only only applies when x, y have degree ≥ 1.

Question p48–49: Let K be a number field, let S be a set of primes of K, and let K_S be the largest extension of K unramified outside S. Let P be the set of prime numbers ℓ such that $\ell^\infty | [K_S\colon K]$. It is now known that P contains *all* prime numbers provided that S contains all the prime ideals of K lying over at least two primes in $\mathbb{Q}$. See Chenevier and Clozel, JAMS 22 (2009), 467–519, Cor. 5.2. This strengthens the main results of §4.

Footnote p48. Brian Conrad has pointed out to me that Haberland's errors (e.g., his "proof" that a certain global $H^2(G_S, -)$ is the whole of $\mathbb{Q}/\mathbb{Z}$ only gives that it is a subgroup of $\mathbb{Q}/\mathbb{Z}$ which contains $\mathbb{Q}_p/\mathbb{Z}_p$ for all $p \in S$) don't affect his proofs of the duality theorems for finite discrete G_S-modules M whose order is an S-unit.

I, Lemma 6.1 holds more generally: for any isogeny $f\colon A \to B$ of abelian schemes over a normal scheme X such that $\deg f$ is a unit in $\mathcal{O}_X$, $A_f(k(x)_{sep}) = A_f(k(x)_{un})$. (notations as in Example I 5.2b of my book on Etale Cohomology).

From Timo Keller

p.8 l.-2: There is a P missing.

p.18 Question 1.4: In Neukirch's Algebraic Number Theory, group cohomology is somewhat avoided.

p.25: "equal to the set primes" should be "equal to the set of primes"

p.40 l.-8: f should be F

p.74 l.-5: g_v should be G_v

p.146 Proposition 0.9: $Z \to g^*F$ does not induce $g_*Z \to F*$ (the adjunction is the other way round)

p.149 l-13, should read $H^r(X, j_!F) = 0$, not $H^r(X, F) = 0$ (which true, but uninteresting).

p.150 Lemma 1.4: The stalks are at $\bar{x}$ and $\bar{u}$, not $\bar{x}$ and $\bar{x}$

p. 151 The lower row in the commutative diagram should read:

$$0 \to R^\times \to K^\times \xrightarrow{\text{ord}} \mathbb{Z} \to 0$$

p.172 2.11(b) $Z- > \mathbb{Z}$

p.197: "always an open subscheme or X" should be "always an open subscheme of X" p.318 Proposition C.2: There is a (missing

From Keenan Kidwell

I, Lemma 1.2. Replace M with M^U twice (G/U doesn't act on M).

I, Proposition 6.2. The proof should read: "this gives an injection $A(K)^{(n)} \hookrightarrow H^1(G_S, A_n)$".

Nguyen Quoc Thang has pointed out to me that I often misspell "Bégueri".

From David Harari

On page 117 (Example 9.1.), an integer r is defined as the g.c.d. of the local degrees. It should be the lowest common multiple.

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pvii. Here's an explanation of the footnote.

Let M be a p-primary abelian group (i.e., every element is killed by a power of p).

Let H be the subgroup of infinitely p-divisible elements, i.e.,

$$H = \{x \in M \mid \text{for all } n \geq 1, \text{ there exists a } y \in M \text{ such that } p^n y = x\}.$$

Then H need not be divisible, i.e., we need not have $pH = H$. We only know H is contained in pG. Moreover, for x in H, there need not exist an infinite sequence $y_1, y_2, y_3 ....$ such that $py_2 = y_1$, $py_3 = y_2, ...$ We only know that there exist arbitrarily long finite such sequences.

There does exist a unique maximal divisible subgroup D of M, and D is a subgroup of H. Moreover, $M = D \oplus N$ where N is a subgroup with no divisible subgroups and D is isomorphic to a direct sum of copies of $\mathbb{Q}_p/\mathbb{Z}_p$.

Thus, unless there is some finiteness condition on M, you do have to worry about H being different from D. For example, M could have infinitely divisible elements but no divisible subgroup.

For all this, see Kaplansky, Infinite Abelian Groups, University of Michigan Press, 1954.

In Milne 1988, 3.3,[1] the following is proved:

> Let N be an abelian group, and let $N^1 \stackrel{\text{def}}{=} \bigcap mN$ be its first Ulm subgroup.
>
> (a) If N/mN is finite for all integers m, then N^1 is divisible (and hence is the unique maximal divisible subgroup N_{div} of N); if, in addition, $\hat{N} \stackrel{\text{def}}{=} \varprojlim N/mN$ is finite, then N/N_{div} is finite and equals $\hat{N}$.
>
> (b) Assume N is torsion, and let M be the Pontryagin dual of N (regarding N as a discrete group). Then the pairing $N/N_{\text{div}} \times M_{\text{tors}} \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate, and so the Pontryagin dual of N/N_{div} is the closure of M_{tors} in M.

Here is an example of a group M such that $H \neq 0$ but $D = 0$. Let C_n be the cyclic group of order p^{n+1} with generator e_n, and let $N = \bigoplus_{n \geq 0} C_n$. Then N_p is a vector space over $\mathbb{F}_p$ with basis e_0, pe_1, $p^2e_2, \ldots$. Let M be the quotient of N by the subspace of codimension 1 of N_p generated by $e_0 - pe_1$, $pe_1 - p^2e_2, \ldots$. In M the element e_0 becomes infinitely p-divisible, because $e_0 = pe_1 = p^2e_2 = \ldots$ in M. However, it is not in any p-divisible subgroup of M: otherwise we could find an infinite chain $e_0, a_1, a_2, \ldots$ in M with $pa_1 = e_0$ and $pa_i = a_{i-1}$, which is clearly impossible. In fact, no such infinite chains exist in the quotient group. Note that M has countably infinitely many elements of order p.

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[1] Milne, J. S. Motivic cohomology and values of zeta functions. Compositio Math. 68 (1988), no. 1, 59–100, 101–102.

I 3, 3.8. Ahmed Matar has pointed out to me that, in the last paragraph of the proof, I can't apply the lemma in Serre to show that $H^2(G/I, \mathcal{A}^\circ(R^{\mathrm{un}})) = 0$ because $R^{\mathrm{un}} \neq \varprojlim R^{\mathrm{un}}/\mathfrak{m}^n R^{\mathrm{un}}$ and $\mathcal{A}^\circ(R^{\mathrm{un}}) \neq \varprojlim \mathcal{A}^\circ(R^{\mathrm{un}}/\mathfrak{m}^n R^{\mathrm{un}})$ (in fact, $\mathcal{A}^\circ(R^{\mathrm{un}})$ is countable but $\varprojlim \mathcal{A}^\circ(R^{\mathrm{un}}/\mathfrak{m}^n R^{\mathrm{un}})$ is uncountable). To fix this, one can argue as follows.

Let K' be a finite unramified extension of K, let R' be the ring of integers in K', let k' be the residue field, and let $\Gamma = \mathrm{Gal}(K'/K) \simeq \mathrm{Gal}(k'/k)$. For each $n \geq 1$, there is an exact sequence

$$0 \to \omega_{\mathcal{A}} \otimes_R k' \to \mathcal{A}^\circ(R'/\mathfrak{m}^{n+1}R') \to \mathcal{A}^\circ(R'/\mathfrak{m}^n R') \to 0$$

(cf. III 4.3). Now $H^i(\Gamma, \omega_{\mathcal{A}} \otimes_R k') = 0$ for $i > 0$ (by Serre, Local Fields, X §1 Prop 1, for example), and so

$$H^i(\Gamma, \mathrm{Ker}(\mathcal{A}^\circ(R'/\mathfrak{m}^{n+1}R') \to \mathcal{A}^\circ(k')) = 0$$

for $i > 0$ and all $n \geq 0$. When K'/K is finite, we can apply the lemma in Serre to deduce that

$$H^i(\Gamma, \mathrm{Ker}(\mathcal{A}^\circ(R') \to \mathcal{A}^\circ(k')) = 0$$

for $i > 0$. On passing to the limit over increasing K', we get the same result for $K' = K^{\mathrm{un}}$, and so

$$H^i(\Gamma, \mathcal{A}^\circ(R^{\mathrm{un}})) \simeq H^i(\Gamma, \mathcal{A}^\circ(k^{\mathrm{un}})),$$

which is zero for $i > 1$ because $\mathcal{A}^\circ(k^{\mathrm{un}})$ is torsion and Γ has cohomological dimension 1.

I 6, 6.24. As Peter Jossen pointed out to me, the statement is proved only for the m-components where m is a unit in $R_{K,S}$, for example, any integer prime to characteristic of K when S contains all finite primes. Probably the statement is still true without this condition — cf. the note on II 5.6 below.

I 6, p82, footnote 17. In his ICM 1982 talk, (3.3), Tate correctly states that the pairing on the Tate-Shafarevich group defined by a principal polarization is alternating when the polarization is defined by a divisor rational over the base field k. In his Bourbaki talk (1966, p306-06) Tate omits the condition, and incorrectly states that the order of the Tate-Shafarevich group of a Jacobian is a square. In the original version of the book (6.12, p100), I followed Tate in incorrectly stating that the pairing on the Tate-Shafarevich group of a Jacobian is alternating.

I 8, p111. Jiu-Kang Yu has pointed out to me that there is a gap in the argument in the paragraph at the bottom of the page, namely, where I claim that, with $A = C_F \otimes M$, the canonical map $\mathrm{Hom}(A, \mathbb{C}^\times)_G \to \mathrm{Hom}(A^G, \mathbb{C}^\times)$ is *obviously* an isomorphism. Here the Hom is as abstract groups (i.e., disregarding the topology). As he writes:

> If we replace $\mathrm{Hom}(A, \mathbb{C}^\times)$ with the Pontrijagin dual of A (assuming A locally compact), then this is obvious, by Pontrijagin duality. Lacking a duality theory, I see no reason that the quotient $\mathrm{Hom}(A, \mathbb{C}^\times)_G$ of $\mathrm{Hom}(A, \mathbb{C}^\times)$ should be of the form $\mathrm{Hom}(B, \mathbb{C}^\times)$ for a subgroup B of A. In Labesse's paper (1984) on Langlands correspondence for tori, he argued by splitting $\mathbb{C}^\times$ to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, and argued the two parts separately (the part for $\mathbb{R}/\mathbb{Z}$ is Pontrijagin duality as above). I think that that is OK.

II 5. Throughout this section, in the function field case I'm always working prime to the characteristic p. Sometimes I forget to say this, for example, in the statement of Theorem 5.6(a) (but not in its proof).

II 5.3. From David Harari: I think that the proof of Corollary II.5.3. in your book "Arithmetic duality Theorems" (page 201 in the new edition) is incomplete. Indeed the second line of the diagram on top of page 201 doesn't make sense ($H^1_c(U,A)$ and $H^0(K_v,A)$ are not torsion groups). One could try to remove the m in this second line, but then one runs into the problem that the kernel of the first vertical map is not necessarily divisible (it is just a subgroup of a divisible group).

Therefore, I believe that one has to use the analogue of your lemmas I.6.15. and I.6.17 (pp 86-88) to complete the proof. With Tamas Szamuely, we wrote this recently for 1-motives (see `http://www.math-inst.hu/~szamuely/erratcrelle.pdf`) because we went wrong at this point as well in our Crelle paper (proof of Cor 3.5.).

II 5.6b. On March 11, 1991, a student of Tate's (Ki-Seng Tan) wrote to me asking two questions, the first of which was "What one can say about the global duality for the Galois cohomologies, when p equals the characteristic of K". Following is my response:

> I am not sure I understand your notation, but it seems to me that the answer to your first question is already in my book "Arithmetic Duality Theorems".
>
> Specifically, let X be a complete smooth curve over a finite field k and let $K = k(X)$. Let A be an abelian variety over K, and let U be an open subset of X such that A extends to a smooth scheme (also denoted A) over U. Then (see p204) there is an exact sequence of étale cohomology groups
>
> $$H^1(U,A) \to \bigoplus_{v\notin U} H^1(K_v,A) \to H^2_c(U,A).$$
>
> When we pass to the limit over shrinking U, this becomes
>
> $$H^1(K,A) \to \bigoplus_{v\in X} H^1(K_v,A) \to \varinjlim H^2_c(U,A).$$
>
> According to the theorem on p370, we can replace $H^2_c(U,A)$ with $H^0(U,A^t)\hat{} = A^t(K)\hat{}$. Here A^t is the dual abelian variety, and the hat means profinite completion. In fact, it seems to me that by using the results in Chapter III, one can prove II, Theorem 5.6b (p247) with m replaced by p.
>
> Unfortunately, there is nothing I can say about your second question.
>
> J.S. Milne, 19th March, 1991.

See González-Avilés, Cristian D.; Tan, Ki-Seng. A generalization of the Cassels-Tate dual exact sequence. Math. Res. Lett. 14 (2007), no. 2, 295–302. Also On the Poitou-Tate exact sequence for 1-motives, Cristian D. Gonzalez-Aviles, arXiv:0806.0772.

III 6.5/6.6. P. Gille writes (16.03.2010):

> In the beginning of the proof of Lemma III.6.5 of "Arithmetic duality theorems", one uses that the image of B^r in C^{r-1} is closed. The characteristic zero case is clear to me since a group homomorphism is smooth. I do not see how it works in the positive characteristic case. More precisely, it seems that we need to know that the image is locally compact for applying topological sorites.

Shatz (1964, 1972) defines the (a nice) topology on $H^1(k,G)$ in the case of an abelian finite group scheme G/k (and on $H^i(k,G)$ for $G = \mathbb{G}_m$), so the first case this may be a problem is when G is not finite (abelian) and not smooth.

Nguyen Quoc Thang has also questioned why the maps in the cohomology sequence arising from short exact sequence are continuous,[2] which I claim to be "obvious" (Lemma III 6.5(b)). At this late date, I have no idea why I thought these things were obvious. Probably Lemma III 6.5 should be taken to be unproven.

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[2]While reading your book "Arithmetic duality theorems", in Chap. III, sections 6.5, 6.6 on topology on the cohomology groups, I am really stuck with the proof of part b), where you wrote "Obvious".

Could you please give me some more details on the proof that all the coboundary maps between flat cohomology groups induced from a short exact sequence $1 \to A \to B \to C \to 1$ of commutative group schemes over a local field k are continuous ?

For simplicity we assume A is a closed subgroup of B, $f : B \to C$ is the projection. I think, we may reduce the general case $d\colon H^r(k, C) \to H^{r+1}(k, A)$ perhaps by dimension shifting, to the case $d\colon C(k) \to H^1(k, A)$. The problem is how to find (if any) a continuous mapping $g : C(k) \to B$, such that $f(g(c)) = c$, for all $c \in C(k)$.