Original Preface (slightly edited)

The purpose of this book is to provide a comprehensive introduction to the étale topology, sheaf theory, and cohomology. When a variety is defined over the complex numbers, the complex topology may be used to define cohomology groups that reflect the structure of the variety much more strongly than do those defined, for example, by the Zariski topology. For an arbitrary scheme the complex topology is not available, but the étale topology, whose definition is purely algebraic, may be regarded as a replacement. It gives a sheaf theory and cohomology theory with properties very close to those arising from the complex topology. When both are defined for a variety over the complex numbers, the étale and complex cohomology groups are closely related. On the other hand, when the scheme is the spectrum of a field and hence has only one point, the étale cohomology need not be trivial; in fact it is precisely equivalent to the Galois cohomology of the field. Étale cohomology has achieved an importance for the study of schemes comparable to that of complex cohomology for the study of the geometry of complex manifolds or of Galois cohomology for the study of the arithmetic of fields.

The étale topology was initially defined by A. Grothendieck and developed by him with the aid of M. Artin and J.-L. Verdier in order to explain Weil's insight (Weil [1]) that, for polynomial equations with integer coefficients, the complex topology of the set of complex solutions of the equations should profoundly influence the number of solutions of the equations modulo a prime number. In this, the étale topology has been brilliantly successful. We give a sketch of the explanation it provides. It must be assumed that the equations define a scheme proper and smooth over some ring of integers. The complex topology on the complex points of the scheme determines the complex cohomology groups. The comparison theorem says that these groups are essentially the same as the étale cohomology groups of the scheme regarded as a variety over the complex numbers. The proper and smooth base change theorems now show that these last groups are canonically isomorphic to the étale cohomology groups of the scheme regarded as a variety over the algebraic closure of a finite residue field. But the points of the scheme with coordinates in a finite field are the fixed points of the Frobenius operator acting on the set of points of the scheme with coordinates in the algebraic closure of the finite field. The Lefschetz trace formula now shows that the number of points in the finite field may be computed from the trace of the Frobenius operator acting on étale cohomology groups that are essentially equal to the original complex cohomology groups. A large part of this book may be regarded as a justification of this sketch.

To give the reader some idea of the similarities and differences to be expected between the étale and complex theories, we consider the case of a projective nonsingular curve X of genus g over an algebraically closed field k. If k is the complex numbers, then X may be regarded as a one-dimensional compact complex manifold $X(\mathbb{C})$, and its fundamental group $\pi_1(X,x)$ has $2g$ generators and a single, well-known relation. The most interesting cohomology group is H^1, and $H^1(X(\mathbb{C}),\Lambda) = \mathrm{Hom}(\pi_1(X,x),\Lambda)$ for a constant abelian sheaf Λ; for example, $H^1(X(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}^{2g}$. If k is arbitrary, then it is possible to define in a purely algebraic way a fundamental group $\pi_1^{\mathrm{alg}}(X,x)$ that, when k is of characteristic zero, is the pro-finite completion of $\pi_1(X,\mathrm{x})$. The étale cohomology group $H^1(X_{\mathrm{et}},\Lambda) = \mathrm{Hom}(\pi_1^{\mathrm{alg}}(X,x),\Lambda)$ for any constant abelian sheaf Λ, but now Hom refers to continuous

This is a revised corrected version of Chapter I of Étale Cohomology, J.S. Milne, Princeton University Press, 1980. Rings are no longer required to be noetherian. The numbering is unchanged (at present). Please send comments and corrections to me at jmilne at umich.edu. Dated April 25, 2017.

homomorphisms. Thus $H^1(X_{\mathrm{et}}, \Lambda) = \Lambda^{2g}$, if Λ is finite or is the l-adic integers $\mathbb{Z}_l$. But $H^1(X_{\mathrm{et}}, \mathbb{Z}) = 0$, for $\mathbb{Z}$ must be given the discrete topology, and the image of any continuous map $\pi_1^{\mathrm{alg}}(X,x) \to \mathbb{Z}$ is finite. Therefore the étale cohomology is as expected in the first two cases but is anomalous in the last.

It may seem that the étale topology should be superfluous when k is the complex numbers, but this is not so: the étale groups have one important advantage over the complex groups, namely, that if X is defined over a subfield k_0 of k, then every automorphism of k/k_0 acts on $H^r(X_{\mathrm{et}}, \Lambda)$.

The book's first chapter is concerned with the properties of étale morphisms, Henselian rings, and the algebraic fundamental group. It had been my original intention to state these without proof, but this would have been unsatisfactory since one of the essential differences between étale sheaf theory and the usual sheaf theory is that trivial facts from point set topology must frequently be replaced by subtle facts from algebraic geometry. On the other hand to give a complete treatment of these topics would require a book in itself. Thus Chapter I is a compromise: almost everything about étale morphisms and Henselian rings is proved and almost nothing about the fundamental group. The prerequisites for this chapter are a solid knowledge of basic commutative algebra, for example, the contents of Atiyah and Macdonald [1], plus a reasonable understanding of the language of schemes.

The next two chapters are concerned with the basic theory of étale sheaves and with elementary étale cohomology. The prerequisite for these chapters is some knowledge of homological algebra and Galois cohomology.

The fourth chapter treats Azumaya algebras over schemes and the Brauer groups of schemes. Here it is assumed that the reader is familiar with the corresponding objects over fields. This chapter may be skipped.

The fifth chapter contains a detailed analysis of the cohomology of curves and of surfaces. The section on curves assumes a knowledge of the representation theory of finite groups and that on surfaces assumes a more detailed knowledge of algebraic geometry than required earlier in the book.

The sixth chapter proves the fundamental theorems in étale cohomology and applies them to show the rationality of some very general classes of zeta functions and L-series.

The appendixes list definitions and results concerning limits, spectral sequences, and hypercohomology that the reader may find useful.

The most striking application of étale cohomology, that of Deligne to proving the Weil-Riemann hypothesis, is not included, but anyone who reads this book will find little difficulty with Deligne's original paper. Essentially the only results he uses that are not included here concern Lefschetz pencils of odd fiber dimension. However, we do treat Lefschetz pencils of fiber dimension one, and the general case is very similar and only slightly more difficult.

I have tried to keep things as concrete as possible. Only enough foundational material is included to treat the étale site and similar sites, such as the flat and Zariski sites. In particular, the word *topos* does not occur. Derived categories are not used although their spirit pervades the last part of Chapter VI.

For an account of the origins of étale cohomology and its results up to the mid 1960s, I recommend Artin's talk at the International Congress in Moscow, 1966 [3]; for a "popular" account of the history of the Weil conjectures (which is intimately related to the history of étale cohomology) and of Deligne's solution, I recommend Katz's article [2], and for a survey of the main ideas and results in étale cohomology and their relations to their classical analogues, I recommend Deligne's Arcata lectures [SGA. $4\frac{1}{2}$, Arcata]. The best introduction to the material from algebraic geometry required for reading this book is provided by Hartshorne [2].

It is a pleasure to thank M. Artin for explaining a number of points to me, R. Hoobler for his comments on Chapter IV, and the Institute for Advanced Study and l'Institut des Hautes Études Scientifiques where parts of the book were written.

Terminology and Conventions

We no longer require that all rings are noetherian and all schemes are locally noetherian (however, we sometimes insert noetherian hypotheses when they are not strictly needed).

A variety is a geometrically reduced and irreducible scheme of finite type over a field, and a curve or surface is a variety of dimension one or two.

For a field k, k_s or k_{sep} is the separable algebraic closure of k and k_{al} the algebraic closure. If K is Galois over k, then $\mathrm{Gal}(K/k)$ or $G(K/k)$ is the corresponding Galois group; G_k denotes $\mathrm{Gal}(k_s/k)$.

For a ring A, A^* denotes the group of units of A and $\kappa(\mathfrak{p})$ the field of fractions of $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in A.

For a scheme X, $R(X)$ is the ring of rational functions on X, X_i the set of points x of codimension i (that is, such that $\dim O_{X,x} = i$), and X^i the set of points x of dimension i (that is, such that $\overline{\{\mathrm{x}\}}$ has dimension i). A geometric point of X is a map $z \to X$ where z is the spectrum of a separably closed field.

Set is the category of sets, Ab the category of abelian groups, Gp the category of groups, G-sets the category of finite sets on which G acts (continuously on the left), G-mod the category of (discrete) G-modules, Sch/X the category of schemes over X, FEt/X the category of schemes finite and étale over X, LFT/X the category of schemes locally of finite type over X, and $\mathsf{Fun}(\mathrm{C}, \mathrm{A})$ the category of functors from C to A.

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q$ denote respectively, the natural numbers, the ring of integers, the field of rational numbers, the field of real numbers, the field of complex numbers, and the finite field of q elements.

The symbols $\alpha_p, \mu_n, \mathbb{G}_m, \mathbb{G}_a$ denote certain group schemes (II, 2.18).

An injection is denoted by $\hookrightarrow$, a surjection by $\twoheadrightarrow$ an isomorphism by $\approx$, a quasi-isomorphism (or homotopy) by $\sim$, and a canonical (or given) isomorphism by $\simeq$. The symbol $X \stackrel{\text{def}}{=} Y$ means that X is defined to be Y, or that X equals Y by definition.

The kernel and cokernel of multiplication by n, $M \xrightarrow{n} M$, are denoted respectively by M_n and $M^{(n)}$.

The empty set and empty scheme are both denoted by $\emptyset$.

The symbol $b \gg a$ means b is sufficiently greater than a.

Added 2012

These are my notes for a revised updated version of the book. Although the book was not published until 1980, the manuscript was completed, and submitted, in 1977.[1] At the time, apart from the semi-published notes Mumford [3], the only way to learn scheme theory was by reading EGA. When Hartshorne [2] became available, I was able to insert only a few references to it.

[1] At the time, Princeton University Press was notoriously slow. In addition, a certain editor left the manuscript and a referee's report buried in the material on his desk for many months. They only re-emerged when the editor changed offices.

In the 1970s, derived categories were still quite new, and known to only a few algebraic geometers, and so I avoided using them. In some places this worked out quite well, for example, contrary to statements in the literature they are not really needed for the Lefschetz trace formula with coefficients in $\mathbb{Z}/m\mathbb{Z}$, but in others it led to complications. Anyone who doubts the need for derived categories should try studying the Künneth formula (VI, 8) without them. In the new version, I shall use them.

I also regret treating Lefschetz pencils only in the case of fiber dimension 1. Apart from using derived categories and including Lefschetz pencils with arbitrary fiber dimension, I plan to keep the book much as before, but with the statements of the main theorems updated to take account of later work. Whether the new version will ever be completed, only time will tell.

Chapter I

Étale Morphisms

A flat morphism is the algebraic analogue of a map whose fibers form a continuously varying family. For example, a surjective morphism of smooth varieties is flat if and only if all fibers have the same dimension. A finite morphism to a reduced scheme is flat if and only if, over every connected component, all fibers have the same number of points (counting multiplicities). A flat morphism of finite type of noetherian schemes is open, and flat morphisms that are surjective on the underlying spaces are epimorphisms in a very strong sense.

An étale morphism is a flat quasi-finite morphism $Y \to X$ with no ramification (that is, branch) points. Locally Y is then defined by an equation $T^m + a_1T^{m-1} + \cdots + a_m = 0$, where $a_1, \ldots, a_m$ are functions on an open subset U of X and all roots of the equation over a point of U are simple. An étale morphism induces isomorphisms on the tangent spaces and so might be expected to be a local isomorphism. This is true over the complex numbers if local is meant in the sense of the complex topology, but the Zariski topology is too coarse for this to hold algebraically. However, an étale morphism induces an isomorphism on the completions of the local rings at a point where there is no residue field extension. Moreover, it has all the uniqueness properties of a local isomorphism.

A local scheme is Henselian if, for any scheme étale over it, every section of the closed fiber extends to a section of the scheme. It is strictly Henselian, or strictly local, if every scheme étale and faithfully flat over it has a section. The strictly local rings play the same role for the étale topology that the local rings play for the Zariski topology.

The fundamental group of a scheme classifies the finite étale coverings of it. For a smooth variety over the complex numbers, the algebraic fundamental group is simply the profinite completion of the topological fundamental group. There are algebraic analogues for many of the results on the topological fundamental group.

1 Finite and Quasi-Finite Morphisms

Recall that a morphism of schemes $f: Y \to X$ is *affine* if the inverse image of every open affine subset U of X is an open affine subset of Y. If, moreover, $\Gamma(f^{-1}(U), \mathcal{O}_Y)$ is a finite $\Gamma(U, \mathcal{O}_X)$-algebra for every such U, then f is said to be *finite*. These conditions need only be checked for all U in some open affine covering of X (Mumford [3, III, 1, Prop. 5]).

Examples of finite morphisms abound. Let X be an integral scheme with field of rational functions $R(X)$, and let L be a finite field extension of $R(X)$. The *normalization* of X in L is a pair (X', f) where X' is an integral scheme with $R(X') = L$ and $f: X' \to X$ is an affine morphism such that, for all open affines U of X, $\Gamma(f^{-1}(U), \mathcal{O}_{X'})$ is the integral closure of $\Gamma(U, \mathcal{O}_X)$ in L.

PROPOSITION 1.1 *If X is normal and locally noetherian, then the normalization $f: X' \to X$ of X in any finite separable extension of $R(X)$ is finite.*

PROOF. One has only to show that $\Gamma(f^{-1}(U), \mathcal{O}_{X'})$ is a finite $\Gamma(U, \mathcal{O}_X)$-algebra for U an open affine in X, but this is proved in Atiyah-Macdonald [1, 5.17]. □

REMARK 1.2 The above proposition holds for many schemes X without the separability assumption, for example, for reduced excellent schemes and so for varieties ([EGA IV, 7.8] and Bourbaki [2, V, 3.2]). (A field is excellent; a Dedekind domain A with field of fraction K is excellent if, for every maximal ideal $\mathfrak{m}$ of A, the field of fractions of the completion of $A_{\mathfrak{m}}$ is separable over K; every scheme locally of finite type over an excellent scheme is excellent.)

PROPOSITION 1.3 *(a) A closed immersion is finite.*

(b) The composite of two finite morphisms is finite.

(c) Every base change of a finite morphism is finite, that is, if $f: Y \to X$ is finite, then so also is $f_{(X')}: Y_{(X')} \to X'$ for any morphism $X' \to X$.

PROOF. These come down to statements about rings, all of which are obvious. □

The "going up" theorem of Cohen-Seidenberg has the following geometric interpretation.

PROPOSITION 1.4 *Every finite morphism $f: Y \to X$ is proper, that is, it is separated, of finite type, and universally closed.*

PROOF. For any open affine covering (U_i) of X, f restricted to $f^{-1}(U_i) \to U_i$ is separated for all i, and so f is separated (Hartshorne [2, II, 4.6]). To show that finite morphisms are universally closed it suffices, according to (1.3c), to show that they are closed, and for this it suffices, according to (1.3a,b), to show that they map the whole space onto a closed set. Thus it suffices to show that $f(Y)$ is closed. This reduces easily to the affine case with, for example, $f = {}^a g$ where $g: A \to B$ is finite. Let $\mathfrak{I} = \ker(g)$. Then f factors into $\operatorname{Spec} B \to \operatorname{Spec} A/\mathfrak{I} \to \operatorname{Spec} A$. The first map is surjective (Atiyah-Macdonald [1, 5.10]), and the second is a closed immersion. □

For morphisms $X \to \operatorname{Spec} k$, with k a field, there is a topological characterization of finiteness.

PROPOSITION 1.5 *Let $f: X \to \operatorname{Spec} k$ be a morphism of finite type with k a field. The following are equivalent:*

(a) *X is affine and $\Gamma(X, \mathcal{O}_X)$ is an Artin ring;*

(b) *X is finite and discrete (as a topological space);*

(c) *X is discrete;*

(d) *f is finite.*

PROOF. See Atiyah-Macdonald [1, Chapter VIII, especially the exercises]. □

A morphism $f\colon Y \to X$ is *quasi-finite* if it is of finite type and has finite fibers, that is, $f^{-1}(x)$ is discrete (and hence finite) for all $x \in X$ (EGA I, 6.11.3). Similarly an A-algebra B is *quasi-finite* if it is finitely generated and $B \otimes_A \kappa(\mathfrak{p})$ is a finite $\kappa(\mathfrak{p})$-algebra for all prime ideals $\mathfrak{p} \subset A$.

EXERCISE 1.6 (a) Let A be a discrete valuation ring. Show that $A[T]/(P(T))$ is a quasi-finite A-algebra if and only if some coefficient of $P(T)$ is a unit, and that it is finite if and only if the leading coefficient of $P(T)$ is a unit.

(b) Let A be a Dedekind domain with field of fractions K. Show that $\operatorname{Spec} K \to \operatorname{Spec} A$ is never finite, that it is quasi-finite if it is of finite type, and that it is of finite type if and only if A has only finitely many prime ideals.

PROPOSITION 1.7 *(a) Every quasi-compact immersion is quasi-finite.*

(b) The composite of two quasi-finite morphisms is quasi-finite.

(c) Every base change of a quasi-finite morphism is quasi-finite.

PROOF. (a) Let $f\colon Y \to X$ be an immersion. Clearly f has finite fibers, and to show that it is of finite type it suffices to show that $f^{-1}(U)$ is quasi-compact for every open affine U in X, but this is true by hypothesis.

(b) This is obvious.

(c) Let $f\colon Y \to X$ be quasi-finite and $X' \to X$ arbitrary. If $x' \mapsto x$ under $X' \to X$, then the fiber

$$f_{(X')}^{-1}(x') = f^{-1}(x) \otimes_{\kappa(x)} \kappa(x')$$

and hence is discrete. □

One way of constructing quasi-finite morphisms is to take a finite morphism $f\colon Y \to X$ and consider its restriction to an open subscheme U of Y. Remarkably, essentially every quasi-finite morphism comes in this way.

THEOREM 1.8 (ZARISKI'S MAIN THEOREM) *Let X be a noetherian scheme. Every separated quasi-finite morphism $f\colon Y \to X$ factors into the composite*

$$Y \xrightarrow{f'} Y' \xrightarrow{g} X$$

of an open immersion f' and a finite morphism g.

PROOF. Under the additional assumption that f is quasi-projective, there is a cohomological proof of the theorem in [EGA III, 4.4.3]. The above statement is proved in [EGA IV, 8.12.6].

When X and Y are both affine, the theorem follows from the slightly more precise statement:

> let B be a quasi-finite A-algebra, and let A' be the integral closure of A in B; then the map $\operatorname{Spec} B \to \operatorname{Spec} A'$ is an open immersion; moreover, there exists a subalgebra A_1 of A', finite over A, such that $\operatorname{Spec} B \to \operatorname{Spec} A_1$ is also an open immersion.

A proof of this affine statement can be found, in Raynaud [3, p. 42] and in my notes on Commutative Algebra. Deducing the global statement from the affine statement requires an additional argument. Specifically, it requires the statement:

Let $f: Y \to X$ be a separated morphism of finite type such that $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism. Let V be the open subset of Y of points y such that the fiber over $f(y)$ has dimension 0. Then the restriction of f to V is an open immersion $V \to X$ and $f^{-1}(f(V)) = V$.

This is Theorem 12.83 of Görtz and Wedhorn, Algebraic Geometry I, 2010. To obtain (1.8), apply this statement to the map obtained from $f: Y \to X$ by replacing X with its normalization in Y. □

REMARK 1.9 Zariski's main theorem is, more correctly, the main theorem of Zariski [2]. There he was interested in the behavior of a singularity on a normal variety under a birational map. The original statement is essentially that if $f: Y \to X$ is a birational morphism of varieties and $\mathcal{O}_{X,x}$ is integrally closed, then either $f^{-1}(x)$ consists of one point and the inverse morphism f^{-1} is defined in a neighborhood of x or all components of $f^{-1}(x)$ have dimension ≥ 1. To relate this to Grothendieck's version, note that if in (1.8) X and Y are varieties, f is birational and X is normal, then g is an isomorphism. For a more complete discussion of the theorem, see Mumford [3, III.9]; see also my notes on Algebraic Geometry.

COROLLARY 1.10 *Let X be a noetherian scheme. Every proper, quasi-finite morphism $f: Y \to X$ is finite.*

PROOF. Let $f = gf'$ be the factorization as in (1.8). As g is separated and f is proper, f' is proper. (Use the factorization

$$f' = f_{(Y')} \circ \Gamma_{f'}: Y \to Y \times_X Y' \to Y'.)$$

Thus f' is an immersion with closed image, that is, a closed immersion. Now both f' and g are finite. □

REMARK 1.11 The separatedness is necessary in both of the above results: for if X is the affine line with the "origin doubled" (Hartshorne [2, II, 2.3.6]), and $f: X \to \mathbb{A}^1$ is the natural map, then f is universally closed and quasi-finite, but not finite. (It is even flat and étale; see the next two sections.)

EXERCISE 1.12 Let $f: Y \to X$ be separated and of finite type with X irreducible. Show that if the fiber over the generic point η is finite, then there exists an open neighborhood U of η in X such that $f^{-1}(U) \to U$ is finite. Cf. Hartshorne [2, II, Exercise 3.7].

2 Flat Morphisms

A homomorphism $f: A \to B$ of rings is *flat* if B is flat when regarded as an A-module by means of f. Thus, f is flat if and only if the functor $- \otimes_A B$ from A-modules to B-modules is exact. In particular, if $\mathfrak{I}$ is an ideal of A and f is flat, then $\mathfrak{I} \otimes_A B \to A \otimes_A B \simeq B$ is injective. The converse to this statement is also true.

PROPOSITION 2.1 *A homomorphism $f: A \to B$ is flat if the map*

$$a \otimes b \mapsto f(a)b: \mathfrak{I} \otimes_A B \to B$$

is injective for all ideals $\mathfrak{I}$ in A.

PROOF. Let $g: M' \to M$ be an injective map of A-modules where, following Atiyah-Macdonald [1, 2.19], we may assume M to be finitely generated.

Case (a) M is free. We prove this case by induction on the rank r of M. If $r = 1$, then we may identify M with A and M' with an ideal in A; then the statement to be proved is the statement given. If $r > 1$, then $M = M_1 \oplus M_2$ with M_1 and M_2 free of rank $< r$. Consider the exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M & \xrightarrow{p} & M_2 & \longrightarrow & 0 \\ & & \uparrow g_1 & & \uparrow g & & \uparrow g_2 & & \\ 0 & \longrightarrow & g^{-1}(M_1) & \longrightarrow & M' & \longrightarrow & pg(M') & \longrightarrow & 0 \end{array}$$

When tensored with B, the top row remains exact, and g_1 and g_2 remain injective. This implies that $g \otimes 1$ is injective.

Case (b) M arbitrary (finitely generated). Let $x_1, \ldots, x_r$ generate M, let M^* be the free A-module on $x_1, \ldots, x_r$, and consider the exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{j} & M^* & \xrightarrow{h} & M & \longrightarrow & 0 \\ & & \| & & \uparrow i & & \uparrow g & & \\ 0 & \longrightarrow & N & \longrightarrow & h^{-1}g(M') & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

By case (a), $i \otimes 1$ is injective, and it follows that $g \otimes 1$ is injective. □

PROPOSITION 2.2 *If $f: A \to B$ is flat, then so also is $S^{-1}A \to T^{-1}B$ for any multiplicative subsets $S \subset A$ and $T \subset B$ such that $f(S) \subset T$. Conversely, if $A_{f^{-1}(\mathfrak{n})} \to B_{\mathfrak{n}}$ is flat for all maximal ideals $\mathfrak{n}$ of B, then $A \to B$ is flat.*

PROOF. The map $S^{-1}A \to S^{-1}B$ is flat according to Atiyah-Macdonald [1, 2.20], and $S^{-1}B \to T^{-1}B$ is flat according to Atiyah-Macdonald [1, 3.6]. For the converse statement, let $M' \to M$ be an injective map of A-modules. To show that $B \otimes_A M' \to B \otimes_A M$ is injective, it suffices to show that

$$B_{\mathfrak{n}} \otimes_B (B \otimes_A M') \to B_{\mathfrak{n}} \otimes_B (B \otimes_A M)$$

is injective for all $\mathfrak{n}$, but this follows from the flatness of $A_{\mathfrak{p}} \to B_{\mathfrak{n}}$ ($\mathfrak{p} = f^{-1}(\mathfrak{n})$) and the existence of a canonical isomorphism

$$B_{\mathfrak{n}} \otimes_B (B \otimes_A N) \simeq B_{\mathfrak{n}} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A N),$$

natural in the A-module N. □

REMARK 2.3 If $a \in A$ is not a zero-divisor and $f: A \to B$ is flat, then $f(a)$ is not a zero-divisor in B because the injectivity of $x \mapsto ax: A \to A$ implies that of

$$x \mapsto f(a)x: B \to B \simeq A \otimes_A B.$$

Thus, if A is an integral domain and $B \neq 0$, then f is injective. Conversely, every injective homomorphism $f: A \to B$ of integral domains with A Dedekind is flat. In proving this, we may localize and hence assume that A is principal. According to (2.1), it suffices to prove that every nonzero ideal $\mathfrak{I}$ of A, the map $\mathfrak{I} \otimes_A B \to B$ is injective. But $\mathfrak{I} \otimes_A B$ is a free B-module of rank one, and the generator of $\mathfrak{I}$ is not mapped to zero in B.

A morphism $f: Y \to X$ of schemes is *flat* if, for all points y of Y, the induced map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat. Equivalently, f is flat if for every pair V and U of open affines of Y and X such that $f(V) \subset U$, the map $\Gamma(U,\mathcal{O}_X) \to \Gamma(V,\mathcal{O}_Y)$ is flat. From (2.2) it follows that the first condition needs only to be checked for closed points y of Y.

PROPOSITION 2.4 *(a) An open immersion is flat.*

(b) The composite of two flat morphisms is flat.

(c) Every base extension of a flat morphism is flat.

PROOF. (a) and (b) are obvious from the definition.

(c) If $f: A \to B$ is flat and $A \to A'$ is arbitrary, then to see that $A' \to B \otimes_A A'$ is flat, one may use the canonical isomorphism $(B \otimes_A A') \otimes_{A'} M \simeq B \otimes_A M$, which exists for any A'-module M. □

In order to get less trivial examples of flat morphisms we shall need the following criterion.

PROPOSITION 2.5 *Let $\alpha: A \to B$ be a flat homomorphism of noetherian rings, and let $b \in B$.*

(a) *If the image of b in $B/\alpha^{-1}(\mathfrak{n})B$ is not a zero-divisor for any maximal ideal $\mathfrak{n}$ in B, then $B/(b)$ is a flat A-algebra.*

(b) *Assume that A is Jacobson*[1] *and that B is a finitely generated A-algebra. If the image of b in $B/\mathfrak{m}B$ is not a zero-divisor for any maximal ideal $\mathfrak{m}$ of A, then $B/(b)$ is a flat A-algebra.* [2]

PROOF. Under the hypotheses of (b), the ideal $\alpha^{-1}(\mathfrak{n})$ is maximal, and so it suffices to prove (a). After applying (2.2), we may assume that $A \to B$ is a local homomorphism of local rings. Let $\mathfrak{m}$ be the maximal ideal of A. By assumption, if $c \in B$ and $bc = 0$, then $c \in \mathfrak{m}B$. We shall show by induction that in fact $c \in \mathfrak{m}^r B$ for all r, and hence $c \in \bigcap_{r \geq 1} \mathfrak{m}^r B = (0)$ (Krull intersection theorem). Assume that $c \in \mathfrak{m}^r B$, and write

$$c = \sum a_i b_i$$

where the a_i form a minimal generating set for $\mathfrak{m}^r$ and the $b_i \in B$. Then

$$0 = bc = \sum_i a_i b_i b,$$

and so, by one of the standard flatness criteria (see the note following 2.10 below), there are equations

$$b_i b = \sum_j a_{ij} b'_j$$

[1] A ring A is Jacobson if every prime ideal in A is an intersection of maximal ideals. For example, a Dedekind domain is Jacobson if and only if it has infinitely many maximal ideals. A local ring is Jacobson if and only if its maximal ideal is its only prime ideal. A general form of Hilbert's Nullstellensatz states that if A is a Jacobson ring, then so is any finitely generated A-algebra B. Moreover the pullback of any maximal ideal $\mathfrak{n}$ of B is a maximal ideal $\mathfrak{m}$ of A, and $B/\mathfrak{n}$ is a finite extension of the field $A/\mathfrak{m}$ (q.v. Wikipedia).

[2] The following example shows that (b) fails without the hypotheses on A and B. Let $A = k[[x,y]]$, let $B = A_{\mathfrak{p}}$ with $\mathfrak{p} = (x)$, and let $b = x$. The only maximal ideal in A is $\mathfrak{m} = (x,y)$, and $B/\mathfrak{m}B = 0$, and so b is not a zero-divisor in $B/\mathfrak{m}B$. However, the injective map $a \mapsto ab: A \to A$ doesn't stay injective when tensored with $B/(b)$, which therefore is not flat over A.

with $b'_j \in B$, $a_{ij} \in A$, such that

$$\sum_i a_i a_{ij} = 0$$

for all j. From the choice of the a_i, all $a_{ij} \in \mathfrak{m}$. Thus $b_i b \in \mathfrak{m}B$, and since b is not a zero-divisor in $B/\mathfrak{m}B$, this implies that $b_i \in \mathfrak{m}B$. Thus $c \in \mathfrak{m}^{r+1}B$, which completes the induction. We have shown that b is not a zero-divisor in B, and the same argument, with A replaced by $A/\mathfrak{I}$ and B by $B/\mathfrak{I}B$, shows that b is not a zero-divisor in $B/\mathfrak{I}$ for any ideal $\mathfrak{I}$ of A.

Fix such an ideal, and consider the exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathfrak{I}\otimes B & \longrightarrow & \mathfrak{I}\otimes B & \longrightarrow & \mathfrak{I}\otimes(B/(b)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{b} & B & \longrightarrow & B/(b) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B/\mathfrak{I} & \xrightarrow{b} & B/\mathfrak{I}B & \longrightarrow & (B/(b))/\mathfrak{I}(B/(b)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which b means multiplication by b. An application of the snake lemma shows that $\mathfrak{I}\otimes B/(b) \to B/(b)$ is injective, which shows that $B/(b)$ is flat over A, according to (2.1). □

REMARK 2.6 Let $\varphi: A \to B$ be a homomorphism of noetherian rings, and let $\mathfrak{q}$ be a prime ideal of B. If φ is flat, then

$$\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{p}) + \dim(B \otimes_A \kappa(\mathfrak{p})), \quad \mathfrak{p} = \varphi^{-1}(\mathfrak{q})$$

(see, for example, §23 of my notes on commutative algebra).

(a) Let A be a noetherian ring, and consider B/fB where $B = A[X_1,\ldots,X_n]$ and f is a nonzero element of B without constant term. Then B is a free A-module, and so $A \to B$ is flat. For any prime ideal $\mathfrak{p}$ in A, $B/\mathfrak{p}B = (A/\mathfrak{p})[X_1,\ldots,X_n]$ is an integral domain. Therefore (2.5) shows that B/fB is flat over A if no maximal ideal of A contains all the coefficients of f. In other words, B/fB is a flat k-algebra if the ideal generated by the coefficients of f is A.

Let $Z = \operatorname{Spec}(B/(f))$ be the hypersurface in $\mathbb{A}^n_A$ defined by f. The above discussion shows that Z is flat over $\operatorname{Spec}(A)$ if the fiber of Z over every closed point of $\operatorname{Spec}(A)$ has dimension $n-1$.

Now assume that A is Cohen-Macaulay. Then A is catenary, i.e., for any prime ideals $\mathfrak{p} \subset \mathfrak{q}$ in A, the maximal chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ all have the same length. Moreover, B is also Cohen-Macaulay. Let $\mathfrak{q}$ be a prime ideal of B/fB and let $\mathfrak{p} = \mathfrak{q} \cap A$. If B/fB is flat over A, then (see above),

$$\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{p}) + \dim\left(\kappa(\mathfrak{p})[X_1,\ldots,X_n]/(\bar{f})\right).$$

Let $\tilde{\mathfrak{q}}$ be the inverse image of $\mathfrak{q}$ in B. Using Krull's principal ideal theorem and that the rings are catenary, we find that

$$\operatorname{ht}(\tilde{\mathfrak{q}}) = \operatorname{ht}(\mathfrak{q}) + 1$$
$$\operatorname{ht}(\tilde{\mathfrak{q}}) = \operatorname{ht}(\mathfrak{p}) + n.$$

Therefore, if B/fB is flat over A, then $\dim\big(\kappa(\mathfrak{p})[X_1,\ldots,X_n]/(\bar{f})\big) = n-1$, and so the coefficients of f don't all lie in $\mathfrak{p}$. We have shown that, when A is Cohen-Macaulay, B/fB is flat over A if and only if the ideal generated by the coefficients of f is A.

(b) We may restate (a) as follows: a hypersurface Z is flat if and only if its closed fibers over $\operatorname{Spec} A$ all have the same dimension. This generalizes. Firstly, if $f: Y \to X$ is a flat morphism of noetherian schemes, then

$$\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{Y_x,y}) \qquad (x = f(y)). \qquad (*)$$

For varieties, (*) becomes

$$\dim(Y) = \dim(X) + \dim(Y_x).$$

The proof, which is quite elementary, may be found in [EGA IV, 6.1] or Hartshorne [2, III, 9.5] (the affine case was recalled above). Secondly, let $f: Y \to X$ be a morphism of noetherian schemes with X regular and Y Cohen-Macaulay; if (*) holds for all $y \in Y$, then f is flat. The proof again may be found in [EGA IV, 6.1]. (See also Hartshome [2, III, Ex. 10.9].)

(c) There is another criterion for flatness that is frequently very useful. It is easy to construct examples of morphisms of noetherian schemes $Z \xrightarrow{f} Y \xrightarrow{g} X$ in which g and gf are flat, but f is not flat. However, if one also knows that the maps on fibers $f_x: Z_x \to Y_x$ are flat for all closed $x \in X$, then f is flat ([SGA 1, IV, 5.9], or Bourbaki [2, III, 5.4 Prop. 2,3]).

(d) If B is flat over A and $b_1,\ldots,b_n$ is a sequence of elements of B whose image in $B/\mathfrak{m}B$ is regular for each maximal ideal $\mathfrak{m}$ of B, that is, b_i is not a zero-divisor in $B/(\mathfrak{m}+(b_1,b_2,\ldots.b_{i-1}))$ for any i, then $B/(b_1,\ldots,b_n)$ is flat over A. This follows by induction from (2.5).

(e) There is a second generalization of (a). Let X be an integral noetherian scheme and Z a closed subscheme of $\mathbb{P}^n_X$; for each $x \in X$, let $p_x \in \mathbb{Q}[T]$ be the Hilbert polynomial of the fiber $Z_x \subset \mathbb{P}^n_{\kappa(x)}$; then Z is flat over X if and only if p_x is independent of x (Hartshorne [2, III, 9.9]).

A flat morphism $f: A \to B$ is *faithfully flat* if $B \otimes_A M$ is nonzero whenever M is nonzero. On taking M to be a principal ideal in A, we see that such a morphism is injective.

PROPOSITION 2.7 *Let $f: A \to B$ be a flat morphism with $A \neq 0$. The following are equivalent:*

(a) *f is faithfully flat;*

(b) *a sequence $M' \to M \to M''$ of A-modules is exact whenever $B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$ is exact;*

(c) *${}^a f: \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective;*

(d) *for every maximal ideal $\mathfrak{m}$ of A, $f(\mathfrak{m})B \neq B$. In particular, a flat local homomorphism of local rings is automatically faithfully flat.*

PROOF. (a)⇒(b). Suppose that $M' \xrightarrow{g_1} M \xrightarrow{g_2} M''$ becomes exact after tensoring with B. Then $\mathrm{im}(g_2 g_1) = 0$ because

$$B \otimes_A \mathrm{im}(g_2 g_1) = \mathrm{im}((1 \otimes g_2)(1 \otimes g_1)) = 0,$$

and $\mathrm{im}(g_1) = \ker(g_2)$ because

$$B \otimes (\ker g_2 / \mathrm{im}\, g_1) = \ker(1 \otimes g_2)/\mathrm{im}(1 \otimes g_1) = 0.$$

(b)⇒(a). The sequence $M \xrightarrow{0} M \longrightarrow 0$ is exact if and only if $M = 0$.

(a)⇒(c). For any prime ideal $\mathfrak{p}$ of A, $B \otimes_A \kappa(\mathfrak{p}) \neq 0$, and so ${}^a f^{-1}(\mathfrak{p}) = \mathrm{spec}(B \otimes_A \kappa(\mathfrak{p}))$ is nonempty.

(c)⇒(d). This is trivial.

(d)⇒(a). Let $x \in M$, $x \neq 0$. Because f is flat, it suffices to show that $B \otimes_A N \neq 0$ where $N = Ax \subset M$. But $N \approx A/\mathfrak{I}$ for some ideal $\mathfrak{I}$ of A, and hence $B \otimes N \approx B/\mathfrak{I}B$. If $\mathfrak{m}$ is a maximal ideal of A containing $\mathfrak{I}$, then $\mathfrak{I}B \subset f(\mathfrak{m})B \neq B$, and so $B/\mathfrak{I}B \neq 0$. □

COROLLARY 2.8 *Let $f: Y \to X$ be flat; let $y \in Y$, and let x' be such that $x = f(y)$ is in the closure $\overline{\{x'\}}$ of $\{x'\}$. Then there exists a y' such that $y \in \overline{\{y'\}}$ and $f(y') = x'$.*

PROOF. The x' such that $x \in \overline{\{x'\}}$ are exactly the points in the image of the canonical map $\mathrm{Spec}\, \mathcal{O}_x \to X$. The corollary therefore follows from the fact that the map $\mathrm{Spec}\, \mathcal{O}_y \to \mathrm{Spec}\, \mathcal{O}_x$ induced by f is surjective. □

A morphism $f: Y \to X$ is *faithfully flat* if it is flat and surjective. According to (2.7c), this agrees with the previous definition for rings.

We now consider the question of flatness for finite morphisms. The next theorem shows that, for such a morphism $f: Y \to X$, flatness has a very explicit interpretation in terms of the properties of $f_* \mathcal{O}_Y$ as an $\mathcal{O}_X$-module.

THEOREM 2.9 *The following conditions on an A-module are equivalent:*

(a) *M is finitely generated and projective;*

(b) *M is finitely presented and $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of A;*

(c) *$\tilde{M}$ is a locally free sheaf on $\mathrm{Spec}\, A$, i.e., there exists a finite family $(f_i)_{i \in I}$ of elements of A generating the ideal A and such that, for all $i \in I$, the A_{f_i}-module M_{f_i} is free of finite rank;*

(d) *M is finitely presented and flat.*

Moreover, when A is an integral domain and M is finitely presented, they are equivalent to:

(e) *$\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p}))$ is the same for all prime ideals $\mathfrak{p}$ of A (here $\kappa(\mathfrak{p})$ denotes the field of fractions of $A/\mathfrak{p}$).*

PROOF. (a)⇒(d). As tensor products commute with direct sums, every free module is flat and every direct summand of a flat module is flat. As projective modules are exactly the direct summands of free modules, they are flat. It remains to show that every finitely generated projective module M is finitely presented. The kernel of any surjective homomorphism $A^r \to M$ is a direct summand (hence quotient) of A^r, and so is finitely generated.

(b)⇒(c). Let $\mathfrak{m}$ be a maximal ideal of A, and let $x_1, \ldots, x_r$ be elements of M whose images in $M_\mathfrak{m}$ form a basis for $M_\mathfrak{m}$ over $A_\mathfrak{m}$. The kernel N' and cokernel N of the homomorphism

$$\alpha\colon A^r \to M, \quad \alpha(a_1, \ldots, a_r) = \sum a_i x_i,$$

are both finitely generated, and $N'_{\mathfrak{m}} = 0 = N_{\mathfrak{m}}$. Therefore, there exists[3] an $f \in A \smallsetminus \mathfrak{m}$ such that $N'_f = 0 = N_f$. Now α becomes an isomorphism when tensored with A_f.

The set T of elements f arising in this way is contained in no maximal ideal, and so generates the ideal A. Therefore, $1 = \sum_{i \in I} a_i f_i$ for certain $a_i \in A$ and $f_i \in T$.

(c)⇒(d). Let $B = \prod_{i \in I} A_{f_i}$. Then B is faithfully flat over A, and $B \otimes_A M = \prod M_{f_i}$, which is clearly a flat B-module. It follows that M is a flat A-module.

(c)⇒(e). This is obvious.

(e)⇒(c). Fix a prime ideal $\mathfrak{p}$ of A. For some $f \notin \mathfrak{p}$, there exist elements $x_1, \ldots, x_r$ of M_f whose images in $M \otimes_A \kappa(\mathfrak{p})$ form a basis. Then the map

$$\alpha\colon A_f^r \to M_f,\ \alpha(a_1, \ldots, a_r) = \sum a_i x_i,$$

defines a surjection $A_{\mathfrak{p}}^r \to M_{\mathfrak{p}}$ (Nakayama's lemma; note that $\kappa(\mathfrak{p}) \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$). Because the cokernel of α is finitely generated, the map α itself will be surjective once f has been replaced by a multiple. For any prime ideal $\mathfrak{q}$ of A_f, the map $\kappa(\mathfrak{q})^r \to M \otimes_A \kappa(\mathfrak{q})$ defined by α is surjective, and hence is an isomorphism because $\dim(M \otimes_A \kappa(\mathfrak{q})) = r$. Thus $\ker(\alpha) \subset \mathfrak{q}A_f^r$ for every $\mathfrak{q}$, which implies that it is zero as A_f is reduced. Therefore M_f is free. As in the proof of (b), a finite set of such f's will generate A. □

To prove the remaining implications, (d)⇒(a),(b) we shall need the following lemma.

LEMMA 2.10 *Let*

$$0 \to N \to F \to M \to 0 \tag{1}$$

be an exact sequence of A-modules with N a submodule of F.

(a) *If M and F are flat over A, then $N \cap \mathfrak{a}F = \mathfrak{a}N$ (inside F) for all ideals $\mathfrak{a}$ of A.*

(b) *Assume that F is free with basis $(y_i)_{i \in I}$ and that M is flat. If the element $n = \sum_{i \in I} a_i y_i$ of F lies in N, then there exist $n_i \in N$ such that $n = \sum_{i \in I} a_i n_i$.*

(c) *Assume that M is flat and F is free. For every finite set $\{n_1, \ldots, n_r\}$ of elements of N, there exists an A-linear map $f\colon F \to N$ with $f(n_j) = n_j$, $j = 1, \ldots, r$.*

PROOF. (a) Consider

$$\begin{array}{ccccccc} & & \mathfrak{a} \otimes N & \longrightarrow & \mathfrak{a} \otimes F & \longrightarrow & \mathfrak{a} \otimes M \\ & & & & \downarrow\simeq & & \downarrow\simeq \\ 0 & \longrightarrow & N \cap \mathfrak{a}F & \longrightarrow & \mathfrak{a}F & \longrightarrow & \mathfrak{a}M \end{array}$$

The first row is obtained from (1) by tensoring with $\mathfrak{a}$, and the second row is a subsequence of (1). Both rows are exact. On tensoring $\mathfrak{a} \to A$ with F we get a map $\mathfrak{a} \otimes F \to F$, which is injective because F is flat. Therefore $\mathfrak{a} \otimes F \to \mathfrak{a}F$ is an isomorphism. Similarly, $\mathfrak{a} \otimes M \to \mathfrak{a}M$ is an isomorphism. From the diagram we get a surjective map $\mathfrak{a} \otimes N \to N \cap \mathfrak{a}F$, and so the image of $\mathfrak{a} \otimes N$ in $\mathfrak{a}F$ is $N \cap \mathfrak{a}F$. But this image is $\mathfrak{a}N$.

(b) Let $\mathfrak{a}$ be the ideal generated by the a_i. Then $n \in N \cap \mathfrak{a}F = \mathfrak{a}N$, and so there are $n_i \in N$ such that $n = \sum a_i n_i$.

(c) We use induction on r. Assume first that $r = 1$, and write

$$n_1 = \textstyle\sum_{i \in I_0} a_i y_i$$

[3]To say that $S^{-1}N = 0$ means that, for each $x \in N$, there exists an $s_x \in S$ such that $s_x x = 0$. If $x_1, \ldots, x_n$ generate N, then $s \stackrel{\text{def}}{=} s_{x_1} \cdots s_{x_n}$ lies in S and has the property that $sN = 0$. Therefore, $N_s = 0$.

where $(y_i)_{i\in I}$ is a basis for F and I_0 is a finite subset of I. Then

$$n_1 = \sum\nolimits_{i\in I_0} a_i n_i'$$

for some $n_i' \in N$ (by (b)), and f may be taken to be the map such that $f(y_i) = n_i'$ for $i \in I_0$ and $f(y_i) = 0$ otherwise. Now suppose that $r > 1$, and that there are maps $f_1,\ f_2 : F \to N$ such that $f_1(n_1) = n_1$ and

$$f_2(n_i - f_1(n_i)) = n_i - f_1(n_i), \quad i = 2,\ \ldots\ r.$$

Then

$$f\colon F \to N, \quad f = f_1 + f_2 - f_2 \circ f_1$$

has the required property. □

We now complete the proof of the Theorem 2.9.

(d)⇒(a). Because M is finitely presented, there is an exact sequence

$$0 \to N \to F \to M \to 0$$

in which F is free and N and F are both finitely generated. Because M is flat, (c) of the lemma shows that this sequence splits, and so M is projective.

(d)⇒(b). We may suppose that A itself is local, with maximal ideal $\mathfrak{m}$. Let $x_1,\ \ldots,\ x_r \in M$ be such that their images in $M/\mathfrak{m}M$ form a basis for this over the field $A/\mathfrak{m}$. Then the x_i generate M (by Nakayama's lemma), and so there exists an exact

$$0 \to N \to F \xrightarrow{g} M \to 0$$

in which F is free with basis $\{y_1,\ \ldots,\ y_r\}$ and $g(y_i) = x_i$. According to (a) of the lemma, $\mathfrak{m}N = N \cap (\mathfrak{m}F)$, which equals N because $N \subset \mathfrak{m}F$. Therefore N is zero by Nakayama's lemma. □

NOTES Using (2.10), we prove the following statement:

> Let M be a flat A-module, and suppose that $\sum_{i=1}^{r} a_i x_i = 0$, $a_i \in A$, $x_i \in M$. Then there exist $a_{ij} \in A$ and $y_j \in M$ such that $\sum_i a_i a_{ij} = 0$ and $x_i = \sum_j a_{ij} y_j$. (In other words, every linear relation in M arises from a family of linear relations in A.)

Note that there exists a surjection $g\colon F \to M$ from a free module F onto M and a basis (y_i) for F such that $g(y_i) = x_i$ for $i = 1,\ldots,r$. Now $n = \sum a_i y_i \in \ker(g)$, and so n may be written $n = \sum a_i n_i$ with $n_i \in \ker(g)$ (apply 2.10(b)). Write $n_i = y_i - \sum_j a_{ij} y_j$, $a_{ij} \in A$. Then $x_i = \sum_j a_{ij} g(y_j)$, and

$$n = \sum\nolimits_i a_i n_i = n - \sum\nolimits_j \left(\sum\nolimits_i a_i a_{ij}\right) y_j,$$

which implies that $\sum_i a_i a_{ij} = 0$.

REMARK 2.11 Let $f\colon Y \to X$ be finite and flat. I claim that f is open, that is, maps open sets to open sets. Following (2.9), we may assume that $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $B \approx A^r$ as an A-module. Let $T^r + a_1 T^{r-1} + \cdots + a_r$ be the characteristic polynomial over A of an element $b \in B$. A prime ideal $\mathfrak{p}$ of A is in the image of $\operatorname{spec}(B_b) \to \operatorname{spec}(A)$ exactly when $B_b/\mathfrak{p}B_b$ is nonzero. But $B_b/\mathfrak{p}B_b \simeq (B/\mathfrak{p}B)_{\bar{b}}$ and so this ring is nonzero exactly when $\bar{b}$ is not nilpotent in $B/\mathfrak{p}B$ or, equivalently, when some coefficient of $T^r + a_1 T^{r-1} + \cdots + a_r$ is nonzero in $A/\mathfrak{p}$. Thus the image of $\operatorname{spec} B_b$ in $\operatorname{spec} A$ is $\bigcup \operatorname{spec} A_{a_i}$, which is open. A much more general statement holds.

THEOREM 2.12 *Every flat morphism locally of finite type of noetherian schemes is open.*

We first prove a lemma.

LEMMA 2.13 *Let $f\colon Y \to X$ be of finite type. For all pairs (Z,U) where Z is a closed irreducible subset of Y and U is an open subset such that $U \cap Z \neq \emptyset$, there exists an open subset V of X such that $f(U \cap Z) \supset V \cap \overline{f(Z)} \neq \emptyset$. (Here, $\overline{f(Z)}$ denotes the closure of the set $f(Z)$).*

PROOF. First note the following statements.

(a) The lemma is true for closed immersions.

(b) The lemma is true for f if it is true for $f_{\mathrm{red}}: Y_{\mathrm{red}} \to X_{\mathrm{red}}$.

(c) The lemma is true for gf if it is true for f and g. (For, if V' satisfies the conclusion of the lemma for the pair $(\overline{f(Z)}, V)$ and the map g, then it also satisfies the conclusion for the pair (Z,U) and the map gf.)

(d) It suffices to check the lemma locally on Y and X.

(e) In checking the lemma for a given Z, we note that X may be replaced by $\overline{f(Z)}$, and hence may be assumed to be irreducible.

Using (a), (c), and (d), we may reduce the question to the case that f is the projection $\mathbb{A}^n \times X \to X$ where X is affine. Using (b) and (e), we reduce the question further to the case that $X = \operatorname{Spec} A$, A an integral domain. Finally, using (c) again, we reduce the question to the case that f is the projection $\mathbb{A}^1 \times X \to X$.

Let Z be a closed irreducible subset of $\mathbb{A}^1_X$, say $Z = \operatorname{Spec} B$ where $B = A[T]/\mathfrak{q}$. We may assume that $\mathfrak{q} \neq 0$, for otherwise the lemma is easy. We may also assume, according to (e), that $\mathfrak{q} \cap A = (0)$, that is, that $f(Z) = X$. Let K be the field of fractions of A, and let $t = T \pmod{\mathfrak{q}}$. Since $\mathfrak{q}$ contains a nonconstant polynomial, t is algebraic over K, and so there is an $a \in A$, $a \neq 0$, such that at is integral over A. Then B_a is finite over A_a, and so $\operatorname{spec}(B_a) \to \operatorname{spec}(A_a)$ is surjective (Atiyah-Macdonald [1, 5.10]). Thus we are reduced to showing that the image of a nonempty open subset U of $\operatorname{spec}(B_a)$ contains a nonempty open subset of $\operatorname{spec}(A)$. But if U contains $(\operatorname{spec}(B_a))_b$, and b satisfies the polynomial $T^m + a_1T^{m-1} + \cdots + a_m = 0$, $a_i \in A_a$, then $f(U) \supset \bigcup (\operatorname{spec} A_a)_{a_i}$. □

PROOF (OF (2.12)) Let $f\colon Y \to X$ be as in the theorem. It suffices to show that $f(Y)$ is open. Let $W = X \smallsetminus f(Y)$ and let $Z_1, \ldots, Z_n$ be the irreducible components of $\bar{W}$. Let z_j be the generic point of Z_j. If $z_j \in f(Y)$, say $z_j = f(y)$, then (2.13) applied to $(\overline{\{y\}}, Y)$ shows that there exists an open U in X such that $f(Y) \supset U \cap Z_j \supset \{z_j\}$. But then

$$f(Y) \supset U \cap \Big(X \smallsetminus \bigcup\nolimits_{i \neq j} Z_i\Big) \supset \{z_j\},$$

and, as U and $(X \smallsetminus \bigcup_{i \neq j} Z_i)$ are open, this implies that $z_j \notin \bar{W}$, which is a contradiction. Thus $z_j \in W$, and, according to (2.8), all specializations of z_j belong to W. Thus $W \supset Z_j$, $W \supset \bigcup Z_j = \bar{W}$, and $f(Y)$ is open. □

REMARK 2.14 If $f\colon Y \to X$ is finite and flat, then it is both open and closed. Thus, if X is connected, then f is surjective and hence faithfully flat (provided Y is nonempty).

REMARK 2.15 In fact, every flat morphism locally of finite presentation is open [EGA IV, 2.4.6]. However, a flat morphism $f: Y \to X$ of noetherian schemes, even faithfully flat, need not be open. For example, let $X = \operatorname{Spec} A$ with A a Dedekind domain having infinitely many prime ideals, and let $Y = X \sqcup \operatorname{Spec} K$ with K the field of fractions of A. Under the natural map $Y \to X$, the image of $\operatorname{Spec} K$ is not open in X (cf. 1.6b).

If $f: Y \to X$ is finite, and for some $y \in Y$, $\mathcal{O}_y$ is free as an $\mathcal{O}_{f(y)}$-module, then clearly $\Gamma(f^{-1}(U), \mathcal{O}_Y)$ is free over $\Gamma(U, \mathcal{O}_X)$ for some open affine U in X containing $f(y)$. (See the proof of (b)⇒(c) in 2.9.) Thus the set of points $y \in Y$ such that $\mathcal{O}_y$ is flat over $\mathcal{O}_x$ is open in Y and is even nonempty if X is integral and $\overline{f(Y)} = X$. Again this holds more generally.

THEOREM 2.16 *Let $f: Y \to X$ be locally of finite type with X locally noetherian. The set of points $y \in Y$ such that $\mathcal{O}_y$ is flat over $\mathcal{O}_{f(y)}$ is open in Y; it is nonempty if f is dominant and X is integral.*

PROOF. See [EGA IV, 11.1.1]. A reasonably self-contained proof of the affine case can be found in Matsumura [1, Chapter VIII]; see also Mumford [2, p.57]. □

Recall that, in any category with fiber products, a morphism $Y \to X$ is a *strict epimorphism* if the sequence

$$Y \times_X Y \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y \longrightarrow X$$

is exact, that is, if the sequence of sets

$$\operatorname{Hom}(X, Z) \longrightarrow \operatorname{Hom}(Y, Z) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \operatorname{Hom}(Y \times_X Y, Z)$$

is exact for all objects Z, that is, the first arrow maps $\operatorname{Hom}(X, Z)$ bijectively onto the subset of $\operatorname{Hom}(Y, Z)$ on which p_1^* and p_2^* agree.

Clearly the condition that a morphism of schemes be surjective is not sufficient to imply that it is a strict epimorphism—consider the morphism $\operatorname{Spec} k \to \operatorname{Spec} A$ where A is a local Artin ring with residue field k—but for flat morphisms it is, almost.

THEOREM 2.17 *Every faithfully flat morphism $f: Y \to X$ of finite type is a strict epimorphism.*

It is convenient to prove the following result first.

PROPOSITION 2.18 *If $f: A \to B$ is faithfully flat, then the sequence*[4]

$$0 \to A \xrightarrow{f} B \xrightarrow{d^0} B^{\otimes 2} \longrightarrow \cdots \longrightarrow B^{\otimes r} \xrightarrow{d^{r-1}} B^{\otimes r+1} \longrightarrow \cdots$$

is exact, where

$$B^{\otimes r} = B \otimes_A B \otimes_A \cdots \otimes_A B \quad (r \text{ times})$$
$$d^{r-1} = \sum (-1)^i e_i$$
$$e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}.$$

[4] Sometimes called the Amitsur complex (Amitsur, S.A., Trans. Amer. Math. Soc. 90 1959 73–112).

PROOF. The usual argument shows that $d^r \circ d^{r-1} = 0$. We assume first that f admits a section, that is, that there exists a homomorphism $g\colon B \to A$ such that $gf = 1$, and we construct a contracting homotopy $k_r : B^{\otimes r+2} \to B^{\otimes r+1}$. Define

$$k_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0)b_1 \otimes b_2 \otimes \cdots \otimes b_{r+1},\ r \geq -1.$$

It is easily checked that $k_{r+1}d^{r+1} + d^r k_r = 1$, $r \geq -1$, and this implies that the sequence is exact.

Now let A' be an A-algebra, let $B' = A' \otimes_A B$, and let $f' = 1 \otimes f\colon A' \to B'$. The sequence corresponding to f' is obtained from the sequence for f by tensoring with A' (because $B^{\otimes r} \otimes_A A' \simeq (B')^{\otimes r}$). Thus, if A' is a faithfully flat A-algebra, it suffices to prove the theorem for f'. Take $A' = B$, and then $f' = (b \mapsto b \otimes 1)\colon B \to B \otimes_A B$ has a section, namely, $g(b \otimes b') = bb'$, and so the sequence is exact. □

REMARK 2.19 A similar argument to the above shows that if $f\colon A \to B$ is faithfully flat and M is an A-module, then the sequence

$$0 \to M \to M \otimes_A B \xrightarrow{1\otimes d^0} M \otimes_A B^{\otimes 2} \to \cdots \to M \otimes B^{\otimes r} \xrightarrow{1\otimes d^{r-1}} M \otimes B^{\otimes r+1} \to \cdots$$

is exact. Indeed, one may assume again that f has a section and construct a contracting homotopy as before.

PROOF (OF 2.17) We have to show that for every scheme Z and morphism $h\colon Y \to Z$ such that $hp_1 = hp_2$, there exists a unique morphism $g\colon X \to Z$ such that $gf = h$.

Case (a) $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$ are all affine. In this case the theorem follows from the exactness of

$$0 \to A \to B \xrightarrow{e_0 - e_1} B \otimes_A B$$

(since ${}^a e_0 = p_2$, ${}^a e_1 = p_1$).

Case (b) $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ affine, Z arbitrary. We first show the uniqueness of g. If $g_1, g_2\colon X \to Z$ are such that $g_1 f = g_2 f$, then g_1 and g_2 must agree on the underlying topological space of X because f is surjective. Let $x \in X$; let U be an open affine neighborhood of $g_1(x)(= g_2(x))$ in Z, and let $a \in A$ be such that $x \in X_a$ and $g_1(X_a) = g_2(X_a) \subset U$. Then B_b, where b is the image of a in B is faithfully flat over A_a, and it therefore follows from case (a) that $g_1|X_a = g_2|X_a$.

Now let $h\colon Y \to Z$ have $hp_1 = hp_2$. Because of the uniqueness just proved, it suffices to define g locally. Let $x \in X$, $y \in f^{-1}(x)$, and let U be an open affine neighborhood of $h(y)$ in Z. Then $f(h^{-1}(U))$ is open in X (apply 2.12), and so it is possible to find an $a \in A$ such that $x \in X_a \subset f(h^{-1}(U))$. I claim that $f^{-1}(X_a)$ is contained in $h^{-1}(U)$. Indeed, if $f(y_1) = f(y_2)$, there is a $y' \in Y \times Y$ such that $p_1(y') = y_1$ and $p_2(y') = y_2$; if $y_2 \in h^{-1}(U)$, then

$$h(y_1) = hp_1(y') = hp_2(y') = h(y_2) \in U,$$

which proves the claim. If now b is the image of a in B, then $h(Y_b) = h(f^{-1}(X_a)) \subset U$, and B_b is faithfully flat over A_a. Thus the problem is reduced to case (a).

Case (c) General case. It is easy to reduce to the case where X is affine. Since f is quasi-compact, Y is a finite union, $Y = Y_1 \cup \cdots \cup Y_n$, of open affines. Let Y^* be the disjoint union $Y_1 \sqcup \cdots \sqcup Y_n$. Then Y^* is affine and the obvious map $Y^* \to X$ is faithfully flat. In the commutative diagram,

$$\begin{array}{ccccc} \mathrm{Hom}(X,Z) & \longrightarrow & \mathrm{Hom}(Y,Z) & \rightrightarrows & \mathrm{Hom}(Y\times_X Y, Z) \\ \| & & \downarrow & & \downarrow \\ \mathrm{Hom}(X,Z) & \longrightarrow & \mathrm{Hom}(Y^*,Z) & \rightrightarrows & \mathrm{Hom}(Y^*\times_X Y^*, Z), \end{array}$$

the lower row is exact by case (b) and the middle vertical arrow is obviously injective. An easy diagram chase now shows that the top row is exact. □

EXERCISE 2.20 Show that $\mathrm{Spec}\, k[T] \to \mathrm{Spec}\, k[T^3, T^5]$ is an epimorphism, but not a strict epimorphism.

REMARK 2.21 Let $f\colon A \to B$ be a faithfully flat homomorphism, and let M be an A-module. Write M' for the B-module $f_*M = B \otimes_A M$. The module $e_{0*}M' = (B \otimes_A B) \otimes_B M'$ may be identified with $B \otimes_A M'$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(b \otimes m) = b_1 b \otimes b_2 m$, and $e_{1*}M'$ may be identified with $M' \otimes_A B$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(m \otimes b) = b_1 m \otimes b_2 b$. There is a canonical isomorphism $\phi\colon e_{1*}M' \to e_{0*}M'$ arising from

$$e_{1*}M' = (e_1 f)_* M = (e_0 f)_* M = e_{0*}M';$$

explicitly it is the map

$$\begin{array}{ccc} M' \otimes_A B & \to & B \otimes_A M' \\ (b \otimes m) \otimes b' & \mapsto & b \otimes (b' \otimes m), \quad m \in M. \end{array}$$

Moreover, M can be recovered from the pair (M', ϕ) because

$$M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}$$

according to (2.19).

Conversely, every pair (M', ϕ) satisfying certain conditions does arise in this way from an A-module. Given $\phi\colon M' \otimes_A B \to B \otimes_A M'$ define

$$\begin{aligned} \phi_1 &\colon B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M', \\ \phi_2 &\colon M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M', \\ \phi_3 &\colon M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B \end{aligned}$$

by tensoring ϕ with id_B in the first, second, and third positions respectively. Then a pair (M', ϕ) arises from an A-module M as above if and only if $\phi_2 = \phi_1 \phi_3$. The necessity is easy to check. For the sufficiency, define

$$M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}.$$

There is a canonical map $b \otimes m \mapsto bm\colon B \otimes_A M \to M'$, and it suffices to show that this is an isomorphism (and that the map arising from M is ϕ). Consider the diagram

$$\begin{array}{ccc} M' \otimes_A B & \overset{\alpha\otimes 1}{\underset{\beta\otimes 1}{\rightrightarrows}} & B \otimes_A M' \otimes_A B \\ \downarrow \phi & & \downarrow \phi_1 \\ B \otimes_A M' & \overset{e_0\otimes 1}{\underset{e_1\otimes 1}{\rightrightarrows}} & B \otimes_A B \otimes_A M' \end{array}$$

in which $\alpha(m) = 1 \otimes m$ and $\beta(m) = \phi(m \otimes 1)$. As the diagram commutes with either the upper or the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1\phi_3$), ϕ induces an isomorphism on the kernels. But, by definition of M, the kernel of the pair $(\alpha \otimes 1, \beta \otimes 1)$ is $M \otimes_A B$, and, according to (2.19), the kernel of the pair $(e_0 \otimes 1, e_1 \otimes 1)$ is M'. This essentially completes the proof.

More details on this, and the following two results may be found in Murre [1, Chapter VII] and Knus-Ojanguren [1, Chapter II].

PROPOSITION 2.22 *Let $f: Y \to X$ be faithfully flat and quasi-compact. To give a quasi-coherent $\mathcal{O}_X$-module M is the same as giving a quasi-coherent $\mathcal{O}_Y$-module M' plus an isomorphism $\phi: p_1^* M' \to p_2^* M'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi) p_{21}^*(\phi).$$

(Here the p_{ij} are the various projections $Y \times Y \times Y \to Y \times Y$, that is, $p_{ji}(y_1, y_2, y_3) = (y_j, y_i)$, $j > i$).

PROOF. In the case that Y and X are affine, this is a restatement of (2.21). □

By using the relation between schemes affine over a scheme and quasi-coherent sheaves of algebras (Hartshorne [2, II, Ex. 5.17]), one can deduce from (2.22) the following result.

THEOREM 2.23 *Let $f: Y \to X$ be faithfully flat and quasi-compact. To give a scheme Z affine over X is the same as giving a scheme Z' affine over Y plus an isomorphism $\phi: p_1^* Z' \to p_2^* Z'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi) p_{21}^*(\phi).$$

REMARK 2.24 The above is a sketch of part of descent theory. Another part describes which properties of morphisms descend. Consider a Cartesian square

$$\begin{array}{ccc} Y & \longleftarrow & Y' \\ \downarrow f & & \downarrow f' \\ X & \longleftarrow & X' \end{array}$$

in which the map $X' \to X$ is faithfully flat and quasi-compact. If f' is quasi-compact (respectively separated, of finite type, proper, an open immersion, affine, finite, quasi-finite, flat, smooth, étale), then f is also [EGA IV, 2.6, 2.7]. The reader may check that this statement implies the same statement for faithfully flat morphisms $X' \to X$ that are locally of finite type. (Use (2.12)).

Of a similar nature is the result that if $f: Y \to X$ is faithfully flat and Y is integral (respectively normal, regular), then so also is X [EGA 0_{IV}, 17.3.3].

Finally, we quote a result that may be regarded as a vast generalization of the Hilbert Nullstellensatz. Recall that the Nullstellensatz says that every morphism of finite type $f: X \to \operatorname{Spec}(k)$ with k a field has a quasi-section, that is, that there exists a k-morphism $g: \operatorname{Spec}(k') \to X$ with k' a finite field extension of k.

PROPOSITION 2.25 *Let $f: Y \to X$ be faithfully flat and locally of finite presentation, and assume that X is quasi-compact and quasi-separated. Then there exists an affine scheme X', a faithfully flat quasi-finite morphism $h: X' \to X$, and an X-morphism $g: X' \to Y$.*

PROOF. One has to show that, locally, there exist sequences satisfying the conditions of (2.6d) and of length equal to the relative dimension of Y/X. See [EGA IV, 17.16.2] for the details. □

3 Étale Morphisms

Let k be a field and $\bar{k}$ its algebraic closure. A k-algebra A is *separable* if $\bar{A} = A \otimes_k k$ has zero Jacobson radical, that is, if the intersection of the maximal ideals of $\bar{A}$ is zero.[5]

PROPOSITION 3.1 *Let A be a finite algebra over a field k. The following are equivalent:*

(a) *A is separable over k;*

(b) *$\bar{A}$ is isomorphic to a finite product of copies $\bar{k}$,*

(c) *A is isomorphic to a finite product of separable field extensions of k;*

(d) *the discriminant of any basis of A over k is nonzero (that is, the trace pairing $A \times A \to k$ is nondegenerate).*

PROOF. (a)⇒(b). From (1.5) we know that $\bar{A}$ has only finitely many prime ideals and that they are all maximal. Now (a) implies that their intersection is zero and (b) follows from the Chinese remainder theorem (Atiyah-Macdonald [1, 1.10]).

(b)⇒(c). The Chinese remainder theorem implies that A/I_r, where l_r is the Jacobson radical of A, is isomorphic to a finite product $\prod k_i$ of finite field extensions of k. Write $[K{:}k]_s$ for the separable degree of a field extension K/k. Then $\mathrm{Hom}_{k\text{-alg}}(A, \bar{k})$ has $\sum [k_i{:}k]_s$ elements. But

$$\mathrm{Hom}_{k\text{-alg}}(A, \bar{k}) \simeq \mathrm{Hom}_{\bar{k}\text{-alg}}(\bar{A}, \bar{k}),$$

and this set has $[\bar{A}{:}\bar{k}]$ elements by (b). Thus

$$[\bar{A}{:}\bar{k}] = \sum [k_i{:}k]_s \leq \sum [k_i{:}k] = [A/I_r{:}k] \leq [A{:}k].$$

Since $[\bar{A}{:}\bar{k}] = [A{:}k]$, equality must hold throughout and we have (c).

(c)⇒(d). If $A = \prod k_i$, where the k_i are separable field extensions of k, then $\mathrm{disc}(A) = \prod \mathrm{disc}(k_i)$, and this is nonzero by one of the standard criteria for a field extension to be separable.

(d)⇒(a). The discriminants of A and $\bar{A}$ are the same. If x is in the radical of $\bar{A}$, then xa is nilpotent for all $a \in \bar{A}$, and so $\mathrm{Tr}_{\bar{A}/\bar{k}}(xa) = 0$ all a. Thus $x = 0$. □

A morphism $f{:}Y \to X$ that is locally of finite presentation is said to be *unramified* at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable field extension of $\kappa(x)$, where $x = f(y)$. In terms of rings, this says that a homomorphism $f{:}A \to B$ of finite presentation is unramified at $\mathfrak{q} \in \operatorname{spec} B$ if and only if $\mathfrak{p} = f^{-1}(\mathfrak{q})$ generates the maximal ideal in $B_{\mathfrak{q}}$ and $\kappa(\mathfrak{q})$ is a finite separable field extension of $\kappa(\mathfrak{p})$. Thus this terminology agrees with that in number theory.

A morphism $f{:}Y \to X$ is *unramified* if it is unramified at all $y \in Y$.

[5]Bourbaki's terminology (A, V, §6) is that an algebra A over a field k is *diagonalizable* if it is isomorphic to a product algebra k^n for some n, and it is *étale* if $L \otimes_k A$ is diagonalizable for some field L containing k. Thus a finite k-algebra is étale if and only if it satisfies the equivalent conditions of (3.1).

PROPOSITION 3.2 *Let $f:Y \to X$ be locally of finite presentation. The following are equivalent:*

(a) *f is unramified;*

(b) *for all $x \in X$, the fiber $Y_x \to \operatorname{Spec}\kappa(x)$ over X is unramified;*

(c) *all geometric fibers of f are unramified (that is, for all morphisms $\operatorname{Spec}k \to X$, with k separably closed, $Y \times_{\operatorname{Spec}k} X \to \operatorname{Spec}k$ is unramified);*

(d) *for all $x \in X$, Y_x has an open covering by spectra of finite separable $\kappa(x)$-algebras;*

(e) *for all $x \in X$, Y_x is a disjoint union $\bigsqcup \operatorname{Spec}k_i$, where the k_i are finite separable field extensions of $\kappa(x)$.*

(If f is of finite presentation, then Y_x itself is the spectrum of a finite separable $\kappa(x)$-algebra in (d), and Y_x is a finite sum in (e); in particular f is quasi-finite.)

PROOF. (a)⇔(b). This follows from the isomorphism $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y} \simeq \mathcal{O}_{Y_x,y}$.

(b)⇒(d). Let U be an open affine subset of Y_x, and let $\mathfrak{q}$ be a prime ideal in $B = \Gamma(U, \mathcal{O}_{Y_x})$. According to (b), $B_\mathfrak{q}$ is a finite separable field extension of $\kappa(x)$. Also

$$\kappa(x) \subset B/\mathfrak{q} \subset B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} = B_\mathfrak{q},$$

and so $B/\mathfrak{q}$ is also a field. Thus $\mathfrak{q}$ is maximal, B is an Artin ring (Atiyah-Macdonald [1, 8.5]), and $B = \prod B_\mathfrak{q}$, where $\mathfrak{q}$ runs through the finite set $\operatorname{Spec}B$. This proves (d).

A similar argument shows that (c)⇒(d), and (d)⇒(e)⇒(c) and (d)⇒(b) are trivial consequences of (3.1). □

Notice that according to the above definition, every closed immersion $Z \hookrightarrow X$ is unramified. Since this does not agree with our intuitive idea of an unramified covering, for example, of Riemann surfaces, we need a more restricted notion. A morphism of schemes (or rings) is defined to be *étale* if it is flat and unramified (hence also locally of finite presentation).

PROPOSITION 3.3 *(a) Every open immersion is étale.*

(b) The composite of two étale morphisms is étale.

(c) Every base change of an étale morphism is étale.

PROOF. After applying (2.4), we only have to check that the three statements hold for unramified morphisms. Both (a) and (b) are obvious (every immersion is unramified). Also, (c) is obviously true according to (3.1) if the base change is of the form $k \to k'$, where k and k' are fields but, according to (3.2), this is all that has to be checked. □

EXAMPLE 3.4 . Let k be a field and $P(T)$ a monic polynomial over k. Then the monogenic extension $k[T]/(P)$ is separable (equivalently, unramified or étale) if and only if P is separable, that is, has no multiple roots in $\bar{k}$.

This generalizes to rings. A monic polynomial $P(T) \in A[T]$ is separable if $(P, P') = A[T]$, that is, if $P'(T)$ is a unit in $A[T]/(P)$ where $P'(T)$ is the formal derivative of $P(T)$. It is easy to see that P is separable if and only if its image in $\kappa(\mathfrak{p})[T]$ is separable for all prime ideals $\mathfrak{p}$ in A.

Let $B = A[T]/(P)$, where P is any monic polynomial in $A[T]$. As an A-module, B is free of finite rank equal to the degree of P. Moreover, $B \otimes_A \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[T]/(\bar{P})$ where $\bar{P}$ is the image of P in $\kappa(\mathfrak{p})[T]$. It follows from (3.2b) that B is unramified and so étale over

A if and only if P is separable. More generally, for any $b \in B$, B_b is étale over A if and only if P' is a unit in B_b.

For example, $B = A[T]/(T^r - a)$ is étale over A if and only if ra is invertible in A (for $ra \in A^* \iff r\bar{a} \in \kappa(\mathfrak{p})^*$, all $\mathfrak{p} \iff T^r - \bar{a}$ is separable in $\kappa(\mathfrak{p})[T]$ all $\mathfrak{p}$).

For algebras generated by more than one element, there is the following Jacobian criterion: let $C = A[T_1, \ldots, T_n]$, let $P_1, \ldots, P_n \in C$, and let $B = C/(P_1, \ldots, P_n)$; then B is étale over A if and only if the image of $\det(\partial P_i/\partial T_j)$ in B is a unit. That B is unramified over A if and only if the condition holds follows directly from (3.5b) below. (The B-module $\Omega^1_{B/A}$ has generators $dT_1, \ldots, dT_n$ and relations $\sum(\partial P_i/\partial T_j)dT_j = 0$.) That B is flat over A may be proved by repeated applications of (2.5). (See Mumford [3, III, §10. Thm. 3′] for the details.)

Note that if $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ were analytic manifolds, then this criterion would say that the induced maps on the tangent spaces were all isomorphisms, and hence $Y \to X$ would be a local isomorphism at every point of Y by the inverse function theorem. It is clearly not true in the geometric case that $\operatorname{Spec} B \to \operatorname{Spec} A$ is a local isomorphism (unless local is meant in the sense of the étale topology—see later). For example, consider $\operatorname{Spec} \mathbb{Z}[T]/(T^2 - 2) \to \operatorname{Spec} \mathbb{Z}$, which is étale on the complement of $\{(2)\}$ but is not a local isomorphism.

PROPOSITION 3.5 *Let $f: Y \to X$ be locally of finite presentation. The following are equivalent:*

(a) *f is unramified;*

(b) *the sheaf $\Omega^1_{Y/X}$ is zero;*

(c) *the diagonal morphism $\Delta_{Y/X}: Y \to Y \times_X Y$ is an open immersion.*

PROOF. (a)⇒(b). Since $\Omega^1_{Y/X}$ behaves well with respect to base change, it suffices to consider the case that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine, then the case that $A \to B$ is a local homomorphism of local rings, and finally, using Nakayama's lemma, the case that A and B are fields. Then B is a separable field extension of A, and it is a standard fact that this implies that $\Omega^1_{B/A} = 0$.

(b)⇒(c). Since the diagonal is always at least locally closed, we may choose an open subscheme U of $Y \times_X Y$ such that $\Delta_{Y/X}: Y \to U$ is a closed immersion and regard Y as a subscheme of U. Let $\mathcal{I}$ be the sheaf of ideals on U defining Y. Then $\mathcal{I}/\mathcal{I}^2$, regarded as a sheaf on Y, is isomorphic to $\Omega^1_{Y/X}$ and hence is zero. Using Nakayama's lemma, one sees that this implies that $\mathcal{I}_y = 0$ for all $y \in Y$, and it follows that $\mathcal{I} = 0$ on some open subset V of U containing Y. Then $(Y, \mathcal{O}_Y) = (V, \mathcal{O}_V)$ is an open subscheme of $Y \times Y$.

(c)⇒(a). According to (3.2), it suffices to show that each geometric fiber of f is unramified. Thus we need only consider the case of a morphism $f: Y \to \operatorname{Spec} k$ where k is an algebraically closed field. Let y be a closed point of Y. Because k is algebraically closed, there exists a section $g: \operatorname{Spec} k \to Y$ whose image is $\{y\}$. The following square is Cartesian:

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \times_X Y \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle (gf,1)} \\ \{y\} & \xrightarrow{g} & Y. \end{array}$$

Since Δ is an open immersion, this implies that $\{y\}$ is open in Y. Moreover, the map

$$\operatorname{Spec} \mathcal{O}_y = \{y\} \to \operatorname{Spec} k$$

still has the property that $\operatorname{Spec}\mathcal{O}_y \xrightarrow{\Delta} \operatorname{Spec}(\mathcal{O}_y \otimes_k \mathcal{O}_y)$ is an open immersion. But $\mathcal{O}_y$ is a local Artin ring with residue field k, and so $\operatorname{Spec}\mathcal{O}_y \otimes_k \mathcal{O}_y$ has only one point, and $\mathcal{O}_y \otimes_k \mathcal{O}_y \to \mathcal{O}_y$ must be an isomorphism. By counting dimensions over k, one sees then that $\mathcal{O}_y = k$. Thus, on applying (3.1) and (3.2), we obtain (a). □

COROLLARY 3.6 *Consider morphisms* $Y \xrightarrow{g} X \xrightarrow{f} S$. *If* $f \circ g$ *is étale and* f *is unramified, then* g *is étale.*

PROOF. Write $g = p_2\Gamma_g$ where $\Gamma_g\colon Y \to Y \times_S X$ is the graph of g and $p_2\colon Y \times_S X \to X$ is the projection on the second factor. Now Γ_g is the pull-back of the open immersion $\Delta_{X/S}\colon X \to X \times_S X$ by $g \times 1\colon Y \times_S X \to X \times_S X$, and p_2 is the pull-back of the étale map $fg\colon Y \to S$ by $f\colon X \to S$. Thus, by using (3.3), we see that g is étale. □

REMARK 3.7 Let $f\colon Y \to X$ be locally of finite presentation. The annihilator of $\Omega^1_{Y/X}$ (an ideal in $\mathcal{O}_Y$) is called the *different* $\mathfrak{d}_{Y/X}$ of Y over X. That this definition agrees with the one in number theory is proved in Serre [7, III, §7].

The closed subscheme of Y defined by $\mathfrak{d}_{Y/X}$ is called the *branch locus* of Y over X. The open complement of the branch locus is precisely the set on which $\Omega^1_{Y/X} = 0$, that is, on which $f\colon Y \to X$ is unramified. Assume X is locally noetherian. The theorem of the purity of branch locus states that the branch locus (if nonempty) has pure codimension one in Y in each of the two cases: (a) when f is faithfully flat and finite over X; or (b) when f is quasi-finite and dominating, Y is regular and X is normal. (See Altman and Kleiman [1, VI, 6.8], [SGA 1, X, 3.1], and [SGA 2, X, 3.4].)

PROPOSITION 3.8 *If* $f\colon Y \to X$ *is locally of finite presentation, then the set of points* y *of* Y, *such that* $\mathcal{O}_{Y,y}$ *is flat over* $\mathcal{O}_{X,f(y)}$ *and* $\Omega^1_{Y/X,y} = 0$, *is open in Y. Thus there is a unique largest open set* U *in* Y *on which* f *is étale.*

PROOF. This follows immediately from (2.16). □

EXERCISE 3.9 Let $f\colon Y \to X$ be finite and flat, and assume that X is connected. Then $f_*\mathcal{O}_Y$ is locally free, of constant rank r say. Show that there is a sheaf of ideals $\mathfrak{D}_{Y/X}$ on X, called the *discriminant* of Y over X, with the property that if U is an open affine in X such that $B = \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is free with basis $\{b_1, \ldots, b_r\}$ over $A = \Gamma(U, \mathcal{O}_X)$, then $\Gamma(U, \mathfrak{D}_{Y/X})$ is the principal ideal generated by $\det(\operatorname{Tr}_{B/A}(b_i b_j))$. Show that f is unramified, hence étale, at all $y \in f^{-1}(x)$ if and only if $(\mathfrak{D}_{Y/X})_x = \mathcal{O}_{X,x}$ (use (3.1d)). Use this to show that if f is unramified at all $y \in f^{-1}(x)$ for some $x \in X$, then there exists an open subset $U \subset X$ containing x such that $f\colon f^{-1}(U) \to U$ is étale. Show that if $B = A[T]/(P(T))$ with P monic, then the discriminant $\mathfrak{D}_{B/A} = (D(P))$, where $D(P)$ is the discriminant of P, that is, the resultant, $\operatorname{res}(P, P')$, of P and P'. Show also that the different $\mathfrak{d}_{B/A} = (P'(t))$ where $t = T \pmod P$. (See Serre [7, III, §6].)

The next proposition and its corollaries show that étale morphisms have the uniqueness properties of local isomorphisms.

PROPOSITION 3.10 *Let* $f\colon Y \to X$ *be a closed immersion of noetherian schemes. If* f *is flat (hence étale), then it is an open immersion.*

PROOF. According to (2.12), $f(Y)$ is open in X and so, after replacing X with $f(Y)$, we may assume f to be surjective. As f is finite, $f_*\mathcal{O}_Y$ is locally free as an $\mathcal{O}_X$-module (2.9). Since f is a closed immersion, this implies that $\mathcal{O}_X \simeq f_*\mathcal{O}_Y$, that is, that f is an isomorphism. □

REMARK 3.11 By using Zariski's main theorem, we may prove a stronger result, namely, that every étale, universally injective, separated morphism $f: Y \to X$ of locally noetherian schemes is an open immersion. (An injective morphism is *universally injective* if and only if the maps $\kappa(f(y)) \to \kappa(y)$ are radicial for all $y \in Y$ [EGA I, 3.7.1].) In fact, by proceeding as above, we can assume that f is universally bijective, hence a homeomorphism (2.12), hence proper, and hence finite (1.10). Now f being étale and radicial implies that $f_*\mathcal{O}_Y$ must be free of rank one.

COROLLARY 3.12 *Let X be a connected noetherian scheme. If $f: Y \to X$ is étale (resp. étale and separated), then every section s of f is an open immersion (resp. an isomorphism onto an open connected component). Thus there is a one-to-one correspondence between the set of such sections and the set of those open (resp. open and closed) subschemes Y_i of Y such that f induces an isomorphism $Y_i \to X$. In particular, when f is separated, a section is determined by its value at a single point.*

PROOF. Only the first assertion requires proof. Assume first that f is separated. Then s is a closed immersion because $fs = 1$ is a closed immersion, and f is separated (compare the proof of (3.6)). According to (3.6) s is étale, and hence it is an open immersion. Thus s is an isomorphism onto its image, which is both open and closed in Y. If f is only assumed to be étale, then it is separated in a neighborhood of y and $x = f(y)$, and hence the above argument shows that s is a local isomorphism at x. □

COROLLARY 3.13 *Let $f, g: Y' \to Y$ be X-morphisms where X is locally noetherian, Y' is connected, and Y is étale and separated over X. If there exists a point $y' \in Y'$ such that $f(y') = g(y') = y$ and the maps $\kappa(y) \to \kappa(y')$ induced by f and g coincide, then $f = g$.*

PROOF. The graphs $\Gamma_f, \Gamma_g: Y' \to Y' \times_X Y$ of f and g are sections to the projection $p_1: Y' \times_X Y \to Y'$. The conditions imply that Γ_f and Γ_g agree at a point, and so Γ_f and Γ_g are equal (3.12). Thus $f = p_2\Gamma_f = p_2\Gamma_g = g$. □

We saw in (3.4) above that given a monic polynomial $P(T)$ over A, it is possible to construct an étale morphism $\operatorname{Spec} C \to \operatorname{Spec} A$ by taking $C = B_b$ where $B = A[T]/(P)$ and b is such that $P'(T)$ is a unit in B_b. We shall call such an étale morphism *standard*. The interesting fact is that locally every étale morphism $Y \to X$ is standard. Geometrically this means that in a neighborhood of any point x of X, there are functions $a_1, \dots, a_r$ on X such that Y is locally described by the equation

$$T^r + a_1 T^{r-1} + \cdots + a_r = 0,$$

and the roots of the equation are all simple (at any geometric point).

THEOREM 3.14 *Let X be a locally noetherian scheme. If $f: Y \to X$ is étale in some open neighborhood of $y \in Y$, then there are open affine neighborhoods V and U of y and $f(y)$, respectively, such that $f|V: V \to U$ is a standard étale morphism.*

PROOF. Clearly, we may assume that $Y = \operatorname{Spec} C$ and $X = \operatorname{Spec} A$ are affine. Also, by Zariski's main theorem (1.8), we may assume that C is a finite A-algebra. Let $\mathfrak{q}$ be the prime ideal of C corresponding to y. We have to show that there is a standard étale A-algebra B_b such that $B_b \approx C_c$ for some $c \notin \mathfrak{q}$. It is easy to see (because everything is finite over A) that it suffices to do this with A replaced by $A_\mathfrak{p}$, where $\mathfrak{p} = f^{-1}(\mathfrak{q})$, that is, that we may assume that A is local and that $\mathfrak{q}$ lies over the maximal ideal $\mathfrak{p}$ of A.

Choose an element $t \in C$ whose image $\bar{t}$ in $C/\mathfrak{p}C$ generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$, that is, $\bar{t}$ is such that $\kappa(\mathfrak{p})[\bar{t}] = \kappa(\mathfrak{q}) \subset C/\mathfrak{p}C$. Such an element exists because $C/\mathfrak{p}C$ is a product $\kappa(\mathfrak{q}) \times C'$, and $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is separable. Let $\mathfrak{q}' = \mathfrak{q} \cap A[t]$. I claim that $A[t]_{\mathfrak{q}'} \to C_{\mathfrak{q}}$ is an isomorphism. Note first that $\mathfrak{q}$ is the only prime ideal of C lying over $\mathfrak{q}'$ (in checking this, one may tensor with $\kappa(\mathfrak{p})$). Thus the semilocal ring $C \otimes_{A[t]} A[t]_{\mathfrak{q}}$, is actually local and so equals $C_{\mathfrak{q}}$. As $A[t] \to C$ is injective and finite, it follows that

$$A[t]_{\mathfrak{q}'} \to C \otimes_{A[t]} A[t]_{\mathfrak{q}'} = C_{\mathfrak{q}}$$

is injective and finite. It is surjective because $\kappa(\mathfrak{q}') \to \kappa(\mathfrak{q})$ is surjective, and Nakayama's lemma may be applied.

The A-algebra $A[t]$ is finite (it is a submodule of a noetherian A-module), and the isomorphism $A[t]_{\mathfrak{q}'} \to C_{\mathfrak{q}}$ extends to an isomorphism $A[t]_{c'} \to C$ for some $c \notin \mathfrak{q}$, $c' \notin \mathfrak{q}'$. Thus C may be replaced by $A[t]$, that is, we may assume that t generates C over A.

Let $n = [\kappa(\mathfrak{q}){:}\kappa(\mathfrak{p})]$, so that $1, \bar{t}, \ldots, \bar{t}^{n-1}$ generate $\kappa(\mathfrak{q})$ as a vector space over $\kappa(\mathfrak{p})$. Then $1, t, \ldots, t^{n-1}$ generate $C = A[t]$ over A (according to Nakayama's lemma), and so there is a monic polynomial $P(T)$ of degree n and a surjection $h{:}\, B = A[T]/(P) \to C$. Clearly $\bar{P}(T)$ is the characteristic polynomial of $\bar{t}$ in $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$ and so is separable. Thus B_b is a standard étale A-algebra for some $b \notin h^{-1}(\mathfrak{q})$. With a suitable choice of b and c we get a surjection $h'{:}\, B_b \to C_c$ with both B_b and C_c étale A-algebras. According to (3.6), h' is étale, and ${}^a h'{:}\operatorname{Spec} C_c \to \operatorname{Spec} B_b$ is a closed immersion. Hence, according to (3.10), ${}^a h'$ is an open immersion, which completes the proof. □

REMARK 3.15 The fact that f was flat was used only in the last step of the above proof. Thus the argument shows that locally every unramified morphism is a composite of a closed immersion with a standard étale morphism.

COROLLARY 3.16 *A morphism $f{:}Y \to X$ is étale if and only if for every $y \in Y$, there exist open affine neighborhoods $V = \operatorname{Spec} C$ of y and $U = \operatorname{Spec} A$ of $x = f(y)$ such that*

$$C = A[T_1, \ldots, T_n]/(P_1, \ldots, P_n)$$

and $\det(\partial P_i/\partial T_j)$ is a unit in C.

PROOF. Because of (3.4), we only have to prove the necessity. From the theorem, we may assume that $Y \to X$ is standard étale, say, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} C$, $C = B_b$, $B = A[T]/(P)$. Then $C \simeq A[T,U]/(P(T), bU - 1)$, and the determinant corresponding to this is $P'(T)b$. Since the image of $P'(T)b$ is a unit in C, this proves the corollary with the added information that n may be taken to be two. □

With this structure theorem, it is relatively easy to prove that if $Y \to X$ is étale, then Y inherits many of the good properties of X. (For the opposite inheritance, see (2.24).)

PROPOSITION 3.17 *Let $f{:}Y \to X$ be étale.*

(a) *For all $y \in Y$, $\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,f(y)})$*

(b) *If X is normal, then Y is normal.*

(c) *If X is regular, then Y is regular.*

PROOF. (a) We may assume that $X = \operatorname{Spec} A$ where $A(= \mathcal{O}_x)$ is local and that $Y = \operatorname{Spec} B$. The proof uses only the assumption that B is quasi-finite and flat over A. Let $\mathfrak{q}$ be the prime ideal of B corresponding to y (so $\mathfrak{q}$ lies over $\mathfrak{p}$, the maximal ideal of A). Then $\operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A$ is surjective (2.7), so $\dim(B_{\mathfrak{q}}) \geq \dim(A)$. Conversely we may assume $B = B'_b$, where B' is finite over A (see 1.8). Then $\dim(A) \geq \dim(B')(\geq \dim B_{\mathfrak{q}})$ (Atiyah-Macdonald [1, 5.9]).

(b) We may assume that $X = \operatorname{Spec} A$ where A is local (hence normal) and that $T = \operatorname{Spec} C$ where $C = B_b$ is a standard étale A-algebra with $B = A[T]/(P(T))$. Let K be the field of fractions of A, let $L = C \otimes_A K = K[T]/(P(T))$, and let A' be the integral closure of A in L. Note that L is a product of separable field extensions of K. Then we have the inclusions

$$\begin{array}{ccccc} & & C & \subset & A'_b & \subset & L \\ & & \cup & & \cup & & \\ A & \subset & B & \subset & A' & & \end{array}$$

Write $t = T \pmod{P(T)}$. Choose an $a \in A'$. We have to show that a/b^s, or equivalently, a itself, is in C.

Let $\bar{K}$ be the algebraic closure of K, and let $\phi_1, \ldots, \phi_r$ be the homomorphisms $L \to \bar{K}$ over K such that $\phi_1(t), \ldots, \phi_r(t)$ are the roots of $P(T)$ (so $r =$ degree P). Write

$$a = a_0 + a_1 t + \cdots + a_{r-1} t^{r-1}, \; a_i \in K.$$

Then we have r equations,

$$\phi_j(a) = a_0 + a_1 t_j + \cdots + a_{r-1} t_j^{r-1}$$

where $t_j = \phi_j(t)$. Let D be the determinant of these equations, regarding the a_i as unknowns, so that $D = \pm \prod_{i<j}(t_i - t_j)$, that is, $D^2 =$ discriminant of $P(T) = \mathfrak{D}_{B/A}$ (compare (3.9)). Since the $\phi_j(a)$ and t_j^i are integral over A, it follows from Cramer's rule that the $Da_i, i = 1, \ldots, r$, are also integral over A. Since the $Da_i \in K$ and A is normal, they belong to A, and this implies that $Da \in B \subset C$. Since D is a unit in C, it follows that $a \in C$.

(c) Let $y \in Y$ and let $x = f(y)$. Then $\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,x})$ and $\mathfrak{m}_y = \mathfrak{m}_x \mathcal{O}_{Y,y}$, which can be generated by $\dim(\mathcal{O}_{X,x})$ elements. □

REMARK 3.18 An argument, similar to that in (b), shows that if X is reduced, then Y is reduced (Raynaud [3, p. 74]).

We now determine the structure of étale morphisms $Y \to X$ when X is normal.

PROPOSITION 3.19 *Let $f: Y \to X$ be étale, where X is normal and noetherian. Then locally f is a standard étale morphism of the form* $\operatorname{Spec} C \to \operatorname{Spec} A$ *where A is an integral domain, $C = B_b$, $B = A[T]/(P(T))$, and $P(T)$ is irreducible over the field of fractions of A.*

PROOF. The only new fact to be shown is that $P(T)$ may be chosen to be irreducible over the field of fractions K of A. It suffices to consider the case that $X = \operatorname{Spec} A$, where A is a local ring, and $Y = \operatorname{Spec} C$, where C is a standard étale A-algebra, say $C = B_b, B = A[T]/(P(T))$ with $P(T)$ possibly reducible. Fix a prime ideal $\mathfrak{q}$ in C such that $\mathfrak{p} = \mathfrak{q} \cap A$ is the maximal ideal of A.

Note that every monic factor $Q(T)$ of $P(T)$ in $K[T]$ automatically has coefficients in A. (Let K' be a splitting field for $Q(T)$; the roots of $Q(T)$ in K' are also roots of $P(T)$ and hence are integral over A; it follows that the coefficients of $Q(T)$ are also integral over A since they can be expressed in terms of the roots.) Choose $P_1(T)$ to be a monic irreducible factor of $P(T)$ whose image in $\kappa(\mathfrak{q})$ is zero, and write $P(T) = P_1(T)Q(T)$ with $P_1, Q \in A[T]$. Then the images $\bar{P}_1$ and $\bar{Q}$ of P_1 and Q in $\kappa(\mathfrak{p})[T]$ are coprime since $\bar{P}(T)$ is separable and so has no multiple roots. It follows that $(P_1, Q) = A[T]$ (compare (4.1a) below), and the Chinese remainder theorem shows that $B \simeq A[T]/(P_1) \times A[T]/(Q)$. Let b_1 be the image of b in $B_1 = A[T]/(P_1)$. Obviously $C_1 = (B_1)_{b_1}$ is the standard A-algebra sought. □

THEOREM 3.20 *Let X be a normal noetherian scheme and $f: Y \to X$ an unramified morphism. Then f is étale if and only if, for all $y \in Y$, $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is injective.*

PROOF. If f is flat, then $\mathcal{O}_{f(y)} \to \mathcal{O}_y$ is injective according to (2.3). For the converse, note that locally f factors into $Y \xrightarrow{f'} Y' \xrightarrow{g} X$ with f' a closed immersion and g étale (3.15). Write $A = \mathcal{O}_{X,f(y)}$; following (3.19), we may write $\mathcal{O}_{Y',f'(y)} = C_\mathfrak{q}$ where $C = A[T]/(P(T))$ with $P(T)$ irreducible over the field of fractions K of A. We have $A \to C_\mathfrak{q} \to \mathcal{O}_{Y,y}$, which, when tensored with K, becomes $K \to C_\mathfrak{q} \otimes_A K \to \mathcal{O}_{Y,y} \otimes_A K$. As $A \to \mathcal{O}_{Y,y}$ is injective, $K \to \mathcal{O}_{Y,y} \otimes_A K$ is injective, which shows that $C_\mathfrak{q} \otimes_A K \to \mathcal{O}_{Y,y} \otimes_A K$ is not the zero map. But $C_\mathfrak{q} \otimes_A K = K[T]/(P)$ is a field, and so this last map is injective. Hence $C_\mathfrak{q} \to \mathcal{O}_{Y,y}$ is injective, and we already know that it is surjective because f' is a closed immersion. Thus $\mathcal{O}_{Y,y} = C_\mathfrak{q}$ is flat over A. □

THEOREM 3.21 *Let X be a connected normal noetherian scheme, and let $K = R(X)$. Let L be a finite separable field extension of K, let X' be the normalization of X in L, and let U be any open subscheme of X' that is disjoint from the support of $\Omega^1_{X'/X}$. Then $U \to X$ is étale, and conversely every separated étale morphism $Y \to X$ of finite presentation can be written $Y = \coprod U_i \to X$ where each $U_i \to X$ is of this form.*

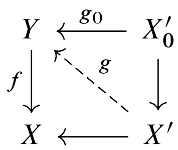
PROOF. The sheaf $\Omega^1_{U/X} = \Omega^1_{X'/X}|U = 0$, and so $U \to X$ is unramified according to (3.5). It is étale according to (3.20).

Conversely, let $Y \to X$ be separated, étale, and of finite presentation. The connected components Y_i of Y are irreducible (because the irreducible components of Y containing y are in one-to-one correspondence with the minimal prime ideals of $\mathcal{O}_{Y,y}$ and Y is normal). If $\operatorname{Spec} L_i \to \operatorname{Spec} K$ is the generic fiber of $Y_i \to X$ and X_i is the normalization of X in L_i, then Zariski's main theorem implies that $Y_i \to X_i$ is an open immersion (see (1.8)). □

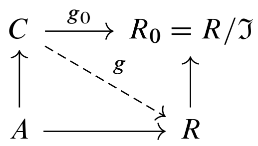
REMARK 3.22 In [EGA IV, 17] the following functorial definitions are made. Let X be a scheme and F a contravariant functor $\mathsf{Sch}/X \to \mathsf{Set}$. Then F is said to be *formally smooth* (*lisse*) (respectively, *formally unramified* (*net*), *formally étale*) if for every affine X-scheme X' and every subscheme X_0 of X' defined by a nilpotent ideal $\mathfrak{I}$, $F(X') \to F(X'_0)$ is surjective (respectively, injective, bijective).

A scheme Y over X is said to be *formally smooth*, *formally unramified*, or *formally étale* over X when the functor $h_Y = \operatorname{Hom}_X(\text{-}, Y)$ it defines has the corresponding property. If, in addition, Y is locally of finite presentation over X, then one says simply that Y is *smooth*, *unramified*, or *étale* over X.

Let X be noetherian. We show that a morphism $f: Y \to X$ that is étale in our sense is also étale in the above sense. (The converse, which is more difficult, may be found, for example, in Artin [9, I, 1.1].) Thus, given an X-morphism $g_0 \colon X'_0 \to Y$, we must show that there is a unique X-morphism $g \colon X' \to Y$ lifting it:



The uniqueness implies that it sufficies to do this locally. Thus we may assume that f is standard, for example, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} C$, $C = B_b$, $B = A[T]/(P) = A[t]$. Let $X' = \operatorname{Spec} R$, $X'_0 = \operatorname{Spec} R_0$ and $R_0 = R/\mathfrak{I}$. Then we are given an A-homomorphism $g_0 \colon C \to R_0$, and we want to find a unique $g \colon C \to R$ lifting it:



Induction on the length of $\mathfrak{I}$ shows that it suffices to treat the case that $\mathfrak{I}^2 = 0$. Let $r \in R$ be such that $g_0(t) = r \pmod{\mathfrak{I}}$. We have to find an $r' \in R$ such that $r' \equiv r \pmod{\mathfrak{I}}$ and $P(r') = 0$. Write $r' = r + h$, $h \in \mathfrak{I}$. Then h must satisfy the equation $P(r + h) = 0$. But $P(r + h) = P(r) + hP'(r)$, where $P(r) \in \mathfrak{I}$ and $P'(r)$ is a unit (since $P'(t) \in C^* \Rightarrow P'(r) \in R_0^*$), and so there is a unique h.

Alternatively, this may be proved by applying (3.12) to $Y \times_X X'/X'$.

THEOREM 3.23 *(Topological invariance of étale morphisms.) Let X_0 be the closed subscheme of a noetherian scheme X defined by a nilpotent ideal. The functor $Y \rightsquigarrow Y_0 = Y \times_X X_0$ is an equivalence from the category of étale X-schemes to the category of étale X_0-schemes.*

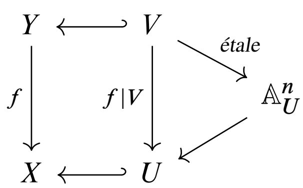
PROOF. To give an X-morphism $Y \to Z$ of étale X-schemes is the same as giving its graph, that is, a section to $Y \times_X Z \to Y$. According to (3.12), such sections are in one-to-one correspondence with the open subschemes of $Y \times_X Z$ that map isomorphically onto Y. Since the same is true for X_0-morphisms $Y_0 \to Z_0$, it is easy to see using (3.10) or (3.11) that our functor is faithfully full. Thus it remains to show that it is essentially surjective on objects. Because of the uniqueness assertion for morphisms, it suffices to locally lift an étale X_0-scheme Y_0 to an X-scheme Y. But then we may assume that $Y_0 \to X_0$ is standard, and the assertion is obvious. □

For completeness, we list some conditions equivalent to smoothness.

PROPOSITION 3.24 *Let $f \colon Y \to X$ be locally of finite presentation. The following are equivalent:*

(a) *f is smooth in the sense of (3.22);*

(b) *for any $y \in Y$, there exist open affine neighborhoods V of y and U of $f(y)$ such that $f|V$ factors into $V \to \mathbb{A}^n_U \to U \hookrightarrow X$ where $V \to \mathbb{A}^n_U$ is étale and $\mathbb{A}^n_U$ is affine n-space over U,*



(c) *for any $y \in Y$, there exist open affine neighborhoods $V = \operatorname{Spec} C$ of y and $U = \operatorname{Spec} A$ of $f(y)$ such that*

$$C = A[T_1, \ldots . T_n]/(P_1, \ldots, P_m),\ m \leq n,$$

and the ideal generated by the $m \times m$ minors of $(\partial P_i / \partial T_j)$ is C;

(d) *f is flat and for every algebraically closed geometric point $\bar{x}$ of X, the fiber $Y_{\bar{x}} \to \bar{x}$ is smooth;*

(e) *f is flat and for every algebraically closed geometric point $\bar{x}$ of X, $Y_{\bar{x}}$ is regular;*

(f) *f is flat and $\Omega^1_{Y/X}$ is locally free of rank equal to the relative dimension of Y/X.*[6]

PROOF. See [SGA 1, II] or Demazure-Gabriel [1, I, §4.4]. □

REMARK 3.25 (a) In the case that f is of finite presentation, conditions (d) and (e) may be paraphrased by saying that Y is a flat family of nonsingular varieties over X.

(b) Condition (b) shows that for a morphism of finite presentation, "étale" is equivalent to "smooth and quasi-finite".

Finally we note that, when the morphism f in (2.25) is smooth, the covering h can be taken to be étale.

PROPOSITION 3.26 *Let $f: Y \to X$ be smooth and surjective, and assume that X is quasi-compact. Then there exists an affine scheme X', a surjective étale morphism $h: X' \to X$, and an X-morphism $g: X' \to Y$.*

PROOF. See [EGA IV, 17.16.3]. □

EXERCISE 3.27 (Hochster). Let A be the ring $k[T^2, T^3]$ localized at its maximal ideal (T^2, T^3) (that is, A is the local ring at a cusp on a curve); let $B = A[S]/(S^3T^2 + S + T^2)$, and let C be the integral closure of A in B. Show that B is étale over A, but that C is not flat over A. (Hint: show that TS and T^2S are in C; hence $TS \in (T^2 : T^3)_C$. If C were flat over A, then

$$(T^2 : T^3)_C = (T^2 ; T^3)_A C = (T^2, T^3);$$

but $TS \in (T^2, T^3)$ would imply $S \in C$.)

EXERCISE 3.28 Let Y and X be smooth varieties over a field k. Show that a morphism $Y \to X$ is étale if and only if it induces an isomorphism on tangent spaces for every closed point of Y.

EXERCISE 3.29 (Hartshorne [2, III, Ex. 10.6]). Let C be the plane nodal cubic curve $Y^2 = X^2(X+1)$. Show that C has a finite étale covering C' of degree 2, where X is a union of two irreducible components, each isomorphic to the normalization of C.

[6]In more detail, f is flat and $\Omega^1_{Y/X}$ is locally free with rank $\dim_y f$ at each point y of Y, where $\dim_y f$ is the dimension of the topological space $f^{-1}(f(y))$.

4 Henselian Rings

Throughout this section, A will be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. The homomorphisms $A \to k$ and $A[T] \to k[T]$ will be written as $a \mapsto \bar{a}$ and $f \mapsto \bar{f}$.

Two polynomials $f(T)$, $g(T)$ with coefficients in a ring B are *strictly coprime* if the ideals (f) and (g) are coprime in $B[T]$, that is, if $(f,g) = B[T]$. For example, $f(T)$ and $T-a$ are coprime if and only if $f(a) \neq 0$ and are strictly coprime if and only if $f(a)$ is a unit in B.

If A is a complete discrete valuation ring, then Hensel's lemma (in number theory) states the following: if f is a monic polynomial with coefficients in A such that $\bar{f}$ factors as $\bar{f} = g_0h_0$ with g_0 and h_0 monic and coprime, then f itself factors as $f = gh$ with g and h monic and such that $\bar{g} = g_0$, $\bar{h} = h_0$. In general, any local ring A for which the conclusion of Hensel's lemma holds is said to be Henselian.

REMARK 4.1 (a) The g and h in the above factorization are strictly coprime. More generally, if $f, g \in A[T]$ are such that $\bar{f}, \bar{g}$ are coprime in $k[T]$ and f is monic, then f and g are strictly coprime in $A[T]$. Indeed, let $M = A[T]/(f,g)$. As f is monic, this is a finitely generated A-module; as $(\bar{f},\bar{g}) = k[T]$, $(f,g) + \mathfrak{m}A[T] = A[T]$ and $\mathfrak{m}M = M$, and so Nakayama's Iemma implies that $M = 0$.

(b) The factorization $f = gh$ is unique, for let $f = gh = g'h'$ with g, h, g', h' all monic, $\bar{g} = \bar{g}'$, $\bar{h} = \bar{h}'$, and $\bar{g}$ and $\bar{h}$ coprime. Then g and h' are strictly coprime in $A[T]$, and so there exist $r, s \in A[T]$ such that $gr + h's = 1$. Now

$$g' = g'gr + g'h's = g'gr + ghs,$$

and so g divides g'. As they are monic and have the same degree, they must be equal.

THEOREM 4.2 *Let x be the closed point of $X = \operatorname{Spec} A$. The following are equivalent:*

(a) *A is Henselian;*

(b) *every finite A-algebra B is a direct product of local rings $B = \prod B_i$ (the B_i are then necessarily isomorphic to the rings $B_{\mathfrak{m}_i}$, where the $\mathfrak{m}_i$ are the maximal ideals of B);*

(c) *if $f\colon Y \to X$ is quasi-finite and separated, then $Y = Y_0 \sqcup Y_1 \sqcup \cdots \sqcup Y_n$ where $f(Y_0)$ does not contain x and Y_i is finite over X and is the spectrum of a local ring, $i \geq 1$;*

(d) *if $f\colon Y \to X$ is étale and there is a point $y \in Y$ such that $f(y) = x$ and $\kappa(y) = \kappa(x)$, then f has a section $s\colon X \to Y$;*

(d$'$) *let $f_1, \ldots f_n \in A[T_1, \ldots, T_n]$; if there exists an $a = (a_1, \ldots a_n) \in k^n$ such that $\bar{f_i}(a) = 0$, $i = 1, 2, \ldots, n$, and $\det((\partial \bar{f_i}/\partial T_j)(a)) \neq 0$, then there exists a $b \in A^n$ such that $\bar{b} = a$ and $f_i(b) = 0$, $i = 1, \ldots, n$;*

(e) *let $f(T) \in A[T]$; if $\bar{f}$ factors as $\bar{f} = g_0h_0$ with g_0 monic and g_0 and h_0 coprime, then f factors as $f = gh$ with g monic and $\bar{g} = g_0$, $\bar{h} = h_0$.*

PROOF. (a)$\Rightarrow$(b). According to the going-up theorem, every maximal ideal of B lies over $\mathfrak{m}$. Thus B is local if and only if $B/\mathfrak{m}B$ is local.

Assume first that B is of the form $B = A[T]/(f)$ with $f(T)$ monic. If f is a power of an irreducible polynomial, then $B/\mathfrak{m}B = k[T]/(\bar{f})$ is local and B is local. If not, then (a) implies that $f = gh$ where g and h are monic, strictly coprime, and of degree ≥ 1. Then $B \approx A[T]/(g) \times A[T]/(h)$ (Atiyah-Macdonald [1, 1.10]), and this process may be continued to get the required splitting.

Now let B be an arbitrary finite A-algebra. If B is not local, then there is a $b \in B$ such that b is a nontrivial idempotent in $B/\mathfrak{m}B$. Let f be a monic polynomial such that $f(b) = 0$; let $C = A[T]/(f)$, and let $\phi: C \to B$ be the map that sends T to b. Since C is monogenic over A, the first part implies that there is an idempotent $c \in C$ such that $\overline{\phi(c)} = \bar{b}$. Now $\phi(c) = e$ is a nontrivial idempotent in B; $B = Be \times B(1-e)$ is a nontrivial splitting, and the process may be continued.

(b)⇒(c). According to (1.8), f factors into $Y \xrightarrow{f'} Y' \xrightarrow{g} X$ with f' an open immersion and g finite. Then (b) implies that $Y' = \coprod \mathrm{Spec}(\mathcal{O}_{Y',y})$ where the y run through the (finitely many) closed points of Y. Let $Y_* = \coprod \mathrm{Spec}(\mathcal{O}_{Y',y})$, where the y runs through the closed points of Y' that are in Y. Then Y_* is contained in Y and is both open and closed in Y because it is so in Y'. Let $Y = Y_* \sqcup Y_0$. Then clearly $f(Y_0)$ does not contain x.

(c)⇒(d). Using (c), we may reduce the question to the case of a finite étale local homomorphism $A \to B$ such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{n})$ where $\mathfrak{n}$ is the maximal ideal of B. According to (2.9b), B is a free A-module, and since

$$\kappa(\mathfrak{n}) \simeq B \otimes_A \kappa(\mathfrak{m}) \simeq \kappa(\mathfrak{m})$$

it must have rank 1, that is, $A \simeq B$.

(d)⇒(d′). Let

$$B = A[T_1, \ldots . T_n]/(f_1, \ldots, f_n) = A[t_1, \ldots, t_n],$$

and let $J(T_1, \ldots, T_n) = \det(\partial f_i / \partial T_j)$. The conditions imply that there exists a prime ideal $\mathfrak{q}$ in B lying over $\mathfrak{m}$ such that $J(t_1, \ldots, t_n)$ is a unit in $B_{\mathfrak{q}}$. It follows that $J(t_1, \ldots, t_n)$ is a unit in B_b for some $b \in B$, $b \notin \mathfrak{q}$ and thus that B_b is étale over A (compare (3.4); to convert B_b to an algebra of the form considered there, use the trick of the proof of (3.16)). Now apply (d) to lift the solution in k^n to one in A^n.

(d′)⇒(e). Write

$$f(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0,$$

and consider the equations,

$$\begin{aligned} X_0 Y_0 &= a_0, \\ X_0 Y_1 + X_1 Y_0 &= a_1, \\ X_0 Y_2 + X_1 Y_1 + X_2 Y_0 &= a_2, \\ &\cdots \\ X_{r-1} Y_s + Y_{s-1} &= a_{n-1}, \\ Y_s &= a_n \end{aligned}$$

where $r = \deg(g_0)$ and $s = n - r$. Clearly $(b_0, \ldots, b_{r-1}; c_0, \ldots, c_s)$ is a solution to this system of equations if and only if

$$f(T) = (T^r + b_{r-1} T^{r-1} + \cdots + b_0)(c_s T^s + \cdots + c_0).$$

The Jacobian of the equations is

$$\det \begin{pmatrix} Y_0 & & X_0 & & \\ Y_1 & Y_0 & X_1 & X_0 & \\ \vdots & Y_1 & X_2 & X_1 & X_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_s & & & & \\ & & 1 & & \end{pmatrix} = \mathrm{res}(g, h),$$

the resultant of g and h, where $g = T^r + X_{r-1}T^{r-1} + \cdots + X_0$ and $h = Y_sT^s + \cdots + Y_0$. To prove (e) we have only to show that $\operatorname{res}(g_0, h_0) \neq 0$. But $\operatorname{res}(g_0, h_0)$ can be zero only if both $\deg(g_0) < r$ and $\deg(h_0) < s$, or g_0 and h_0 have a common factor, and neither of these occurs.

(e)⇒(a). This is trivial. □

COROLLARY 4.3 *If A is Henselian, then so also is every finite local A-algebra B and every quotient ring $A/\mathfrak{I}$.*

PROOF. Obviously B satisfies condition (b) of the theorem, and $A/\mathfrak{I}$ satisfies the definition of a Henselian ring. □

PROPOSITION 4.4 *If A is Henselian, then the functor $B \rightsquigarrow B \otimes_A k$ induces an equivalence between the category of finite étale A-algebras and the category of finite étale k-algebras.*

PROOF. After applying (4.2b), we need only consider local A-algebras B. The canonical map

$$\operatorname{Hom}_A(B, B') \to \operatorname{Hom}_k(B \otimes k, B' \otimes k)$$

is injective according to (3.13). To see the surjectivity, note that a k-homomorphism $B \otimes k \to B' \otimes k$ induces an A-homomorphism $g: B \to B' \otimes_A k$ by composition with $B \to B \otimes k$ and hence an A-homomorphism

$$b' \otimes b \mapsto b'g(b): B' \otimes_A B \to B' \otimes_A k.$$

Now apply (4.2d) to the map $\operatorname{Spec}(B' \otimes B) \to \operatorname{Spec}(B')$ to get an A-homomorphism $B' \otimes_A B \to B'$ that induces the required map $B \to B'$. Thus the functor is fully faithful. To complete the proof, one only has to observe that every local étale k-algebra k' can be written in the form $k[T]/(f_0(T))$, where $f_0(T)$ is monic and irreducible and then that $B = A[T]/(f(T))$, where $f(T) = f_0(T)$ and f is monic, has the property that $B \otimes_A k = k'$. □

So far, we have had no examples of Henselian rings. The following is a generalization of Hensel's lemma in number theory.

PROPOSITION 4.5 *Every complete local ring A is Henselian.*

PROOF. Let B be an étale A-algebra, and suppose that there exists a section $s_0: B \to k$. We have to show (4.2d) that this lifts to a section $s: B \to A$. Write $A_r = A/\mathfrak{m}^{r+1}$; if we can prove that there exist compatible sections $s_r: B \to A_r$, then these maps will induce a section $s: B \to \varprojlim A_r = A$. For $r = 0$ the existence of s_r is given. For $r > 0$ the existence of s_r follows from that of s_{r-1} because of the property of the functor defined by an étale morphism (3.22). □

REMARK 4.6 (a) The last two propositions show that the functor $B \rightsquigarrow B \otimes_A \hat{A}$ gives an equivalence between the categories of finite étale algebras over A and over its completion when A is Henselian. Under certain circumstances, notably when X is proper over a Henselian ring A, this result extends to the categories of schemes finite and étale over X and over $\hat{X} = X \otimes_A \hat{A}$. See Artin [2] and [5].

(b) The result in (4.4) has the following generalization. Let X be a scheme proper over a Henselian local ring A, and let X_0 be the closed fiber of X. The functor $Y \rightsquigarrow Y \times_X X_0$ induces an equivalence between the category of schemes Y finite and étale over X and those over X_0.

When A is complete, proofs of this may be found in Artin [9, VII, 11.7] and Murre [1, 8.1.3]. The Henselian case is deduced from the complete case by means of the approximation theorem (Artin [9, II]; see also Artin [5, Theorem 3.1]).

(c) Part (c) of (4.2) also has a generalization. Let $f: Y \to X$ be separated and of finite presentation, where $X = \operatorname{Spec} A$ with A Henselian local. If y is an isolated point in the closed fiber Y_0 of f, so that $Y_0 = \{y\} \sqcup Y_0'$ (as schemes), then $Y = Y'' \sqcup Y'$ with Y'' finite over X and Y'' and Y' having closed fibers $\{y\}$ and Y_0' respectively. For a proof, see Artin [9, I, 1.10].

(d) If X is an analytic manifold over $\mathbb{C}$, then the local ring at a point x of X is Henselian. (For this, and similar examples, see Raynaud [3, VII, 4].)

REMARK 4.7 Let $f: Y \to X$ be étale, and suppose $\kappa(x) = \kappa(y)$ for some $y \in Y$, $x = f(y)$. Then the map on the completions $\hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{Y,y}$ is étale and, according to (4.5) and (4.2), has a section, which implies (3.12) it is an isomorphism $\hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{Y,y}$. (See Hartshorne [2, III. Ex. 10.4] for a converse statement.)

This may be used to give an example of an injective unramified map of rings that is not étale. Let X be a curve over a field having a node at x_0, and let $f: Y \to X$ be the normalization of X in $k(X)$. It is obvious that this map is unramified, and $\mathcal{O}_{X,f(y)} \hookrightarrow \mathcal{O}_{Y,y}$ is injective for all y, but if y lies over x_0, then $\hat{\mathcal{O}}_{X,x_0} \to \hat{\mathcal{O}}_{Y,y}$ is not an isomorphism because $\hat{\mathcal{O}}_{X,x_0}$ is not an integral domain. (It has two minimal prime ideals; see Hartshorne [2, I.5.6.3].)

If A is noetherian, then it is a subring of its completion $\hat{A}$, and so A is a subring of a Henselian ring; the smallest such ring is called the Henselization of A. More precisely, let $i: A \to A^{\mathrm{h}}$ be a local homomorphism of local rings; A^{h} is the *Henselization* of A if it is Henselian and if every other local homomorphism from A into a Henselian local ring factors uniquely through i. Clearly (A^{h}, i) is unique, up to a unique isomorphism, if it exists.

Before proving the existence of A^{h}, we introduce the notion of an *étale neighborhood* of a local ring A. It is a pair $(B, \mathfrak{q})$ where B is an étale A-algebra and $\mathfrak{q}$ is a prime ideal of B lying over $\mathfrak{m}$ such that the induced map $k \to \kappa(\mathfrak{q})$ is an isomorphism.

LEMMA 4.8 *(a) If $(B, \mathfrak{q})$ and $(B', \mathfrak{q}')$ are étale neighborhoods of A such that $\operatorname{Spec} B'$ is connected, then there is at most one A-homomorphism $f: B \to B'$ such that $f^{-1}(\mathfrak{q}') = \mathfrak{q}$.*

(b) Let $(B, \mathfrak{q})$ and $(B', \mathfrak{q}')$ be étale neighborhoods of A; there is an étale neighborhood $(B'', \mathfrak{q}')$ of A with $\operatorname{Spec} B''$ connected and A-homomorphisms $f: B \to B''$, $f': B' \to B''$ such that $f^{-1}(\mathfrak{q}'') = \mathfrak{q}$, $f'^{-1}(\mathfrak{q}'') = \mathfrak{q}'$.

PROOF. (a) This is an immediate consequence of (3.13).

(b) Let $C = B \otimes_A B'$. The maps $B \to \kappa(\mathfrak{q}) = k$, $B' \to \kappa(\mathfrak{q}') = k$ induce a map $C \to k$. Let $\mathfrak{q}'$ be the kernel of this map. Then $(B'', \mathfrak{q}' B'')$, where $B'' = C_c$ with some $c \notin \mathfrak{q}''$ such that $\operatorname{Spec} B''$ is connected, is the required étale neighborhood. □

It follows from the lemma that the étale neighborhoods of A with connected spectra form a filtered direct system. Define $(A^{\mathrm{h}}, \mathfrak{m}^{\mathrm{h}})$ to be its direct limit, $(A^{\mathrm{h}}, \mathfrak{m}^{\mathrm{h}}) = \varinjlim (B, \mathfrak{q})$. It is easy to check that A^{h} is a local A-algebra with maximal ideal $\mathfrak{m}^{\mathrm{h}}$, $A^{\mathrm{h}}/\mathfrak{m}^{\mathrm{h}} = k$ and A^{h} is the Henselization of A. Also A^{h} is obviously flat over A. Slightly less trivial is the fact that A^{h} is noetherian if A is noetherian. This may be found in Artin [1, III.4.2].

EXERCISE 4.9 Instead of building A^{h} from below, it is possible to descend on it from above. Let A be a noetherian local ring, and let $\tilde{A}$ be the intersection of all local Henselian subrings H of $\hat{A}$, containing A, that have the property that $\hat{\mathfrak{m}} \cap H = \mathfrak{m}_H$. Show that $(\tilde{A}, i)$, where $i : A \hookrightarrow \tilde{A}$ is the inclusion map, is a Henselization of A. (Hint: to show that $\tilde{A}$ satisfies the definition of Henselian ring, note that the factorization $f = g_0 h_0$ lifts to a factorization $f = gh$ in any H; use the uniqueness of the factorization to show that g and h are in $\bigcap H = \tilde{A}$.)

EXAMPLE 4.10 (a) Let A be normal; let K be the field of fractions of A, and let K_s be a separable closure of K. The Galois group G of K_s over K acts on the integral closure B of A in K_s. Let $\mathfrak{n}$ be a maximal ideal of B lying over $\mathfrak{m}$, and let $D \subset G$ be the decomposition group of $\mathfrak{n}$, that is, $D = \{\sigma \in G | \sigma(\mathfrak{n}) = \mathfrak{n}\}$. Let A^{h} be the localization at $\mathfrak{n}^D$ of the integral closure B^D of A in K_s^D. (Here

$$B^D = \{\, b \in B \mid \sigma(b) = b \text{ all } \sigma \in D\}$$

etc.) I claim that A^{h} is the Henselization of A.

Indeed, if A^{h} were not Henselian, there would exist a monic polynomial $f(T)$ that is irreducible over A^{h} but whose reduction $f(T)$ factors into relatively prime factors. But from such an f one can construct a finite Galois extension L of K_s^D such that the integral closure A' of A^{h} in L is not local. This is a contradiction since the Galois group of L over K_s^D permutes the prime ideals of A' lying over $\mathfrak{n}^D$ and hence cannot be a quotient of D. To see that A^{h} is the Henselization, one only has to show that it is a union of étale neighborhoods of A, but this is easy using (3.21).

(b) Let k be a field, and let A be the localization of $k[T_1, \ldots, T_n]$ at $(T_1, \ldots, T_n)$. The Henselization of A is the set of power series $P \in k[[T_1, \ldots, T_n]]$ that are algebraic over A. (For a good discussion of why this should be so, see Artin [8]; for a proof, see Artin [9, II.2.9].)

(c) The Henselization of $A/\mathfrak{I}$ is $A^{\mathrm{h}}/\mathfrak{I}A^{\mathrm{h}}$. This is immediate from the definition of the Henselization and (4.3).

Every ring is a quotient of a normal ring, and so it would have sufficed to construct A^{h} for A normal. This is the approach adopted by Nagata [1].

REMARK 4.11 We have seen that if A is normal, then so also is A^{h}. It is also true that if A is reduced or regular, then A^{h} is reduced or regular and $\dim A^{\mathrm{h}} = \dim A$. These statements follow from (3.18) or (3.17).

Let X be a scheme and let $x \in X$. An *étale neighborhood* of x is a pair (Y, y) where Y is an étale X-scheme and y is a point of Y mapping to x such that $\kappa(x) = \kappa(y)$. The connected étale neighborhoods of x form a filtered system and clearly the limit $\varinjlim \Gamma(Y, \mathcal{O}_Y) = \mathcal{O}_{X,x}^{\mathrm{h}}$.

By definition, A being Henselian means that it has no finite étale extensions with trivial residue field extension (4.2d) except those of the form $A \to A^r$. Thus if the residue field of A is separably algebraically closed, then A has no finite étale extensions at all. Such a Henselian ring is called *strictly Henselian* or *strictly local*. Most of the above theory can be rewritten for strictly Henselian rings. In particular, the strict Henselization of A is a pair (A^{sh}, i) where A^{sh} is a strictly Henselian ring and $i : A \to A^{\mathrm{sh}}$ is a local homomorphism such that every other local homomorphism $f : A \to H$ with H strictly Henselian extends to a local homomorphism $f' : A^{\mathrm{sh}} \to H$; moreover, f' is to be uniquely determined once the induced map $A^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \to H/\mathfrak{m}_H$ on residue fields is given.

Fix a separable closure k_s of k. Then $A^{\mathrm{sh}} = \varinjlim B$, where the limit runs over all commutative diagrams

$$\begin{array}{ccc} B & \longrightarrow & k_s \\ \uparrow & \nearrow & \\ A & & \end{array}$$

in which $A \to B$ is étale. If $A = k$ is a field, then A^{sh} is any separable closure of k; if A is normal, then A^{sh} can be constructed the same way as A^{h} except that the decomposition group must be replaced by the inertia group; if A is normal and Henselian, then A^{sh} is the maximal unramified extension of A in the sense of the number theorists. Finally, $(A/\mathfrak{I})^{\mathrm{sh}} = A^{\mathrm{sh}}/\mathfrak{I}A^{\mathrm{sh}}$.

Let X be a scheme and $\bar{x} \to X$ a geometric point of X. An *étale neighborhood* of $\bar{x}$ is a commutative diagram:

$$\begin{array}{ccc} \bar{x} & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array}$$

with $U \to X$ étale. Clearly $\left(\mathcal{O}_{X,x}\right)^{\mathrm{sh}} = \varinjlim \Gamma(U, \mathcal{O}_U)$ where the limit is taken over all étale neighborhoods of $\bar{x}$. We write $\mathcal{O}^{\mathrm{sh}}_{X,\bar{x}}$, or simply $\mathcal{O}_{X,\bar{x}}$ for this limit. As we shall see, $\mathcal{O}_{X,\bar{x}}$ is the analogue for the étale topology of the local ring for the Zariski topology, that is, it is the local ring relative to a stronger notion of localization. Note that its definition is formally the same, for $\mathcal{O}_{X,x} = \varinjlim \Gamma(U, \mathcal{O}_U)$ where the limit is taken over all Zariski (open) neighborhoods of x.

EXERCISE 4.12 Study the properties of the ring $A^{\mathrm{qh}} = \varinjlim B$, where the limit is taken over all diagrams

$$\begin{array}{ccc} B & & \\ \uparrow & \searrow & \\ A & \longrightarrow & k \end{array}$$

in which $\operatorname{Spec} B$ is connected and $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite and étale or an open immersion or a composite of such morphisms.

EXERCISE 4.13 Let X be a smooth scheme over $\operatorname{Spec} A$, where A is Henselian with residue field k. Show that $X(A) \to X(k)$ is surjective. (Use 3.24b).

5 The Fundamental Group: Galois Coverings

In this section we summarize some of the basic properties of the fundamental group of a scheme. For simplicity, we require all schemes X to be locally noetherian. We let FEt/X denote the category of schemes finite and étale over X.

The fundamental group $\pi_1(X, x_0)$ of an arcwise connected, locally connected, and locally simply connected topological space with base point x_0 may be defined in two ways: either as the group of closed paths through x_0 modulo homotopy equivalence or as the automorphism group of the universal covering space of X. The first definition does not generalize well to schemes—there are simply too few algebraically defined closed paths—but the second does. Thus the important, defining property of the fundamental group of a scheme is that it classifies in a natural way the étale coverings of X, étale being the most natural analogue of local homeomorphism.

Thus let X be a connected scheme, and let $\bar{x} \to X$ be a geometric point of X. Let $F\colon \mathsf{FEt}/X \to \mathsf{Set}$ be the functor $Y \rightsquigarrow \mathrm{Hom}_X(\bar{x}, Y)$. Thus to give an element of $F(Y)$ is to give a point $y \in Y$ lying over x and a $\kappa(x)$-homomorphism $\kappa(y) \to \kappa(\bar{x})$. It may be shown that this functor is strictly prorepresentable, that is, that there exists a directed set I, a projective system $(X_j, \phi_{ij})_{i \in I}$ in FEt/X in which the transition morphisms $\phi_{ij}\colon X_j \to X_i$ $(i \leq j)$ are epimorphisms, and elements $f_i \in F(X_i)$ such that

(a) $f_i = \phi_{ij} \circ f_j$, and

(b) the natural map $\varinjlim \mathrm{Hom}(X_i, Z) \to F(Z)$ induced by the f_i is an isomorphism for any Z in FEt/X.

The projective system $\tilde{X} = (X_i, \phi_{ij})$ will play the role of the universal covering space of a topological space, and we want to define π_1 to be its automorphism group.

For an X-scheme Y, we let $\mathrm{Aut}_X(Y)$ denote the group of X-automorphisms of Y acting on the right. For any $Y \in \mathsf{FEt}/X$, $\mathrm{Aut}_X(Y)$ acts on $F(Y)$ (on the right), and if Y is connected, then this action is faithful, that is, for any $g \in F(Y)$, the map

$$\sigma \mapsto \sigma \circ g\colon \mathrm{Aut}_X(Y) \to F(Y)$$

is injective (this follows from (3.13)). If Y is connected and $\mathrm{Aut}_X(Y)$ acts transitively on $F(Y)$, so that the above map $\mathrm{Aut}_X(Y) \to F(Y)$ is bijective, then Y is said to be *Galois* over X. For any $Y \in \mathsf{FEt}/X$ there is a $Y' \in \mathsf{FEt}/X$ that is Galois and an X-morphism $Y' \to Y$ (see 5.4 below). It follows that the objects X_i in $\tilde{X}$ may be assumed to be Galois over X. Now, given $j \geq i$, we can define a map $\psi_{ij}\colon \mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i)$ by requiring that $\psi_{ij}(\sigma) f_i = \phi_{ij} \circ \sigma \circ f_j$. We define $\pi_1(X, \bar{x})$ to be the profinite group $\varprojlim \mathrm{Aut}_X(X_i)$.

REMARK 5.1 (a) If $\bar{x}'$ is any other geometric point of X, then $\pi_1(X, \bar{x}')$ is isomorphic to $\pi_1(X, \bar{x})$ and the isomorphism is canonically determined up to an inner automorphism of $\pi_1(X, \bar{x})$.

(b) If the above process is carried through for an arcwise connected, locally connected, and locally simply connected topological space X, a point x on it, and the category of covering spaces of X, then one finds that F is representable, that is, not merely prorepresentable, by the universal covering space $\tilde{X}$ of X. Thus $\pi_1(X, x)$ can be directly defined as the automorphism group of $\tilde{X}$ over X.

(c) Let X be a smooth projective variety over $\mathbb{C}$, and let X^{an} be the associated analytic manifold. The Riemann existence theorem states that the functor that associates with any finite étale map $Y \to X$ the local isomorphism of analytic manifolds $Y^{\mathrm{an}} \to X^{\mathrm{an}}$, is an equivalence of categories. Thus the étale fundamental group $\pi_1(X, x)$, x a closed point of X, and the analytic fundamental group $\pi_1(X^{\mathrm{an}}, x)$ have the same finite quotients. It follows that their completions with respect to the topology defined by the subgroups of finite index are equal. But $\pi_1(X, x)$ by definition, is already complete, and so $\pi_1(X, x) \simeq \widehat{\pi_1(X^{\mathrm{an}}, x)}$.

The reason that no algebraically defined fundamental group can equal $\pi_1(X^{\mathrm{an}}, x)$ is that, in general, the covering space does not exist algebraically .

(d) A theorem of Grauert and Remmert implies that (c) also holds for nonprojective varieties. This fact is the basis of the proof of the theorem comparing étale and complex cohomology. (See III, 3.)

REMARK 5.2 (a) Let $X = \operatorname{Spec} k$, k a field. The X_i may be taken to be the spectra of all finite Galois extensions K_i of k contained in $\kappa(\bar{x})$. Thus $\pi_1(X,\bar{x})$ is the Galois group over k of the separable closure of $\kappa(x)$ in $\kappa(\bar{x})$. Changing $\bar{x}$ corresponds therefore to choosing a different separable algebraic closure.

(b) Let X be a normal scheme, and let $\bar{x} = \operatorname{Spec}\kappa(x)_{\mathrm{sep}}$ where x is the generic point of X. Then the X_i may be taken to be the normalizations of X in K_i, where the K_i run through the finite Galois extensions of $\kappa(x)$ contained in $\kappa(\bar{x})$ such that the normalization of X in K_i is unramified. Thus $\pi_1(X,\bar{x})$ is the Galois group of $\kappa(x)_{\mathrm{un}}$ over $\kappa(x)$, where $\kappa(x)_{\mathrm{un}} = \bigcup K_i$.

(c) Let $X = \operatorname{Spec} A$, where A is a strictly Henselian local ring. Then $\pi_1(X,\bar{x}) = \{1\}$ since FEt/X consists only of direct sums of copies of X. (If X is a scheme and $\bar{x}$ a geometric point of X, then $\operatorname{Spec}\mathcal{O}_{X,\bar{x}}$ is the algebraic analogue of a sufficiently small ball about a point x on a manifold, and so $\pi_1 = \{1\}$ agrees with the ball being contractible.)

(d) Let $X = \operatorname{Spec} A$, where A is Henselian. Let $\bar{x}$ be a geometric point over the closed point x of X. The equivalence of categories $\mathsf{FEt}/X \leftrightarrow \mathsf{FEt}/\operatorname{Spec}\kappa(x)$ (see 4.4) induces an isomorphism $\pi_1(X,\bar{x}) \simeq \pi_1(\operatorname{Spec}\kappa(x),\bar{x})$.

(e) Let $X = \operatorname{Spec} K$ where K is the field of fractions of a strictly Henselian discrete valuation ring A. Then X is the algebraic analogue of a punctured disc in the plane (compare with (c) above) and so one might hope that $\pi_1(X,\bar{x}) \approx \hat{\mathbb{Z}}$. This is true if the residue field $A/\mathfrak{m}$ has characteristic zero because (Serre [7, IV, Proposition 8]) the Galois extensions of K are exactly the Kummer extensions K_n/K, where $K_n = K[t^{1/n}]$ with t a uniformizing parameter. The map

$$\sigma \mapsto \sigma(t^{1/n})/t^{1/n} : \operatorname{Gal}(K_n/K) \to \mu_n(K)$$

is an isomorphism from the Galois group onto the nth roots of unity. Thus

$$\pi_1(X,\bar{x}) = \varprojlim \operatorname{Gal}(K_n/K) = \varprojlim \mu_n(K) \approx \hat{\mathbb{Z}}.$$

If the residue field has characteristic p, then this is no longer true because of wild ramification. However, every tamely ramified extension of K is still Kummer (Serre [7, IV]) and the tame fundamental group

$$\pi_1^t(X,\bar{x}) = \varprojlim_{p\nmid n} \mu_n(K) \approx \varprojlim_{p\nmid n} \mathbb{Z}/n\mathbb{Z}.$$

(In general, if K is a field and A is a discrete valuation ring with field of fractions K, then a finite separable field extension L of K is *tamely ramified*[7] with respect to A if, for each valuation ring B of L lying over A, the residue field extension $B/\mathfrak{n} \supset A/\mathfrak{m}$ is separable and the ramification index of B/A is not divisible by the characteristic p of $A/\mathfrak{m}$. Let X be a connected normal noetherian scheme; let D be a finite union $D = \bigcup D_i$ of irreducible divisors on X, and let x_i be the generic point of D_i. Then a map $f : Y \to X$ is a tamely ramified covering if it is finite and étale over $X \smallsetminus D$, Y is connected and normal, and $R(Y)/R(X)$ is tamely ramified with respect to the rings $\mathcal{O}_{X,x_i}$. The tame fundamental group π_1^t is defined so as to classify such coverings. A basic theorem of Abhyankar generalizes the statement that, in the above example, tame coverings are all Kummer. Assume that the D_i have only normal crossings, and that $f : Y \to X$ is a finite covering; if f is tame, then there exists a

[7] See also the article of Kerz and Schmidt http://arxiv.org/abs/0807.0979.

surjective family of étale maps $(U_i \to X)_i$ such that $Y \times_X U_i \to U_i$ is a Kummer covering for each i. See Grothendieck-Murre [1], especially 2.3.)

(f) Let X be the projective line $\mathbb{P}^1$ over a separably closed field k. If $k = \mathbb{C}$, then X is topologically a sphere, and so $\pi_1(X,\bar{x}) = \{1\}$. To see that this is true in general we have to show that FEt/X is trivial or, more precisely, that any finite étale map $Y \to X$ with Y connected is an isomorphism. But if ω is the differential dt on $\mathbb{P}^1$, then ω has a double pole at infinity and no other poles or zeros. Thus $f^*(\omega)$ has $2n$ poles, where n is the degree of f, and no zeros. But $-2n = 2g-2 \geq -2$, where g is the genus of Y, and so $n = 1$ and f is an isomorphism.

(The same argument shows that there is no nontrivial finite map $Y \to \mathbb{P}^1$ that is étale over $\mathbb{A}^1 \subset \mathbb{P}^1$ and tamely ramified at infinity. In this case $f^*(\omega)$ has at most $2n-(n-1) = n+1$ poles and no zeros, and the inequalities $-(n+1) \geq 2g-2 \geq -2$ show again that $n = 1$.)

(g) Let X be a proper scheme over a Henselian local ring A such that the closed fiber X_0 of X is geometrically connected. Let $\bar{x}_0 \to X_0$ be a geometric point of X_0 and let $\bar{x}$ be its image in X. It follows from (4.6b) that $\pi_1(X,\bar{x}) \simeq \pi_1(X_0,\bar{x}_0)$.

(h) Let U be an open subscheme of a regular scheme X whose closed complement $X \smallsetminus U$ has codimension ≥ 2. It follows immediately from the theorem of the purity of branch locus (3.7) and the description of the fundamental group of a normal scheme given in (b) above, that $\pi_1(U,\bar{x}) \simeq \pi_1(X,\bar{x})$ for any geometric point $\bar{x}$ of U.

It follows that the fundamental group is a birational invariant for varieties complete and regular over a field k (because any dominating rational map of such varieties is defined on the complement of a closed subset of codimension ≥ 2; see Hartshorne [2, V, 5.1]).

(i) Let X be a scheme proper and smooth over $\operatorname{Spec} A$ where A is a complete discrete valuation ring with algebraically closed residue field of characteristic p. Assume that $X_{\bar{K}}$ is connected, where $\bar{K}$ is the algebraic closure of the field of fractions of A, and that the special fiber of X/A is connected. Then the kernel of the surjective homomorphism $\pi_1(X_{\bar{K}},\bar{x}) \to \pi_1(X,\bar{x})$, for any geometric point $\bar{x}$ of $X_{\bar{K}}$, is contained in the kernel of every homomorphism of $\pi_1(X_{\bar{K}},\bar{x})$ into a finite group of order prime to p (see [SGA. 1, X.] or Murre [1]).

(j) Let X_0 be a smooth projective curve of genus g over an algebraically closed field k of characteristic p. Then there is a complete discrete valuation ring A with residue field k and a smooth projective curve X over A such that $X_0 = X \otimes_A k$. (The obstructions to lifting a smooth projective variety lie in second (Zariski) cohomology groups and hence vanish for a curve ([SGA. 1, III.7])). It follows from (g) above that $\pi_1(X_0,\bar{x}) \simeq \pi_1(X,\bar{x})$ and from (i) that there is a surjection $\pi_1(X_{\bar{K}},\bar{x}) \to \pi_1(X,\bar{x})$ with small kernel. The comparison theorem (5.1c) shows that $\pi_1(X_{\bar{K}},\bar{x})$ is the profinite completion of the topological fundamental group of a curve of genus g over $\mathbb{C}$ and hence is well-known. On putting these facts together, one finds that $\pi_1(X_0,\bar{x}_0)^{(p)} \approx G^{(p)}$ where G is the free group on $2g$ generators u_i, v_i $(i = 1,\ldots,g)$ with the single relation

$$(u_1 v_1 u_1^{-1} v_1^{-1}) \cdots (u_g v_g u_g^{-1} v_g^{-1}) = 1$$

and where the superscript (p) on a group H means replace H by $\varprojlim H_i$ with the H_i running through all finite quotients of H of order prime to p.

This computation of the prime-to-p part of the fundamental group of a curve in characteristic p was one of the first major successes of Grothendieck's approach to algebraic geometry.

Once $\pi_1(X,\bar{x})$ has been constructed, the important result is that it really does classify finite étale maps $Y \to X$.

THEOREM 5.3 *Let $\bar{x}$ be a geometric point of the connected scheme X. The functor F defines an equivalence between the category* FEt/X *and the category $\pi_1(X,\bar{x})$-sets of finite sets on which $\pi_1(X,\bar{x})$ acts continuously (on the left).*

PROOF. See Murre [1]. □

REMARK 5.4 Recall that if $Y \to X$ is finite and étale and both Y and X are connected, then we defined Y to be Galois over X if $\mathrm{Aut}_X(Y)$ has as many elements as the degree of Y over X. We wish to extend this notion slightly.

If G is a finite group, then G_X (for any scheme X) denotes the scheme $\coprod_{\sigma\in G} X_\sigma$, where $X_\sigma = X$ for each σ. Note that we may define an action of G on G_X (on the right) by requiring $\sigma|X_\tau$ to be the identity map $X_\tau \to X_{\tau\sigma}$.

Let G act on Y over X; then $Y \to X$ is *Galois*, with *Galois group* G, if it is faithfully flat and the map

$$\psi : G_Y \to Y \times Y,\ \psi|Y_\sigma = (y \mapsto (y, y\sigma))$$

is an isomorphism. Equivalently, $Y \to X$ is Galois with Galois group G if there is a faithfully flat morphism $U \to X$, locally of finite presentation, such that Y_U is isomorphic (with its G-action) to G_U. In terms of rings, a ring $B \supset A$ on which G acts by A-automorphisms (on the left) is a Galois extension of A with Galois group G if B is a finite flat A-algebra and $\mathrm{End}_A(B)$ has a basis $\{\sigma|\sigma \in G\}$ as a left B-module with multiplication table

$$(b\sigma)(c\tau) = b\sigma(c)\sigma\tau,\ b,\ c \in B,\ \sigma,\ \tau\ \in G.$$

We leave it to the reader to check that these are all equivalent and that if $Y \to X$ is Galois with Galois group G, then so also is $Y_{(X')} \to X'$ for any $X' \to X$. Also the reader may check that any finite étale morphism $Y \to X$ can be embedded in a Galois extension, that is, that there exists a Galois morphism $Y' \to X$ that factors through $Y \to X$. (For X normal, this is obvious from the Galois theory of fields; the general case is not much more difficult; see, for example, Murre [1, 4.4.1.8].)

Let X be connected, with geometric point $\bar{x}$. According to (5.3), to give an $\bar{x}$-pointed Galois morphism $Y \to X$ with a given action of G on Y over X is the same as giving a continuous morphism $\pi_1(X,\bar{x}) \to G$. We shall sometimes write $\pi^1(X,\bar{x};G) \stackrel{\text{def}}{=} \mathrm{Hom}_{\text{conts}}(\pi_1(X,\bar{x}),G) \approx$ set of $\bar{x}$-pointed Galois coverings of X with Galois group G (modulo isomorphism). As an application of étale cohomology we shall show how to compute $\pi^1(X,\bar{x};G)$ for some schemes X when G is commutative. Note that in this case it is not necessary to endow the Galois coverings with $\bar{x}$-points.

REMARK 5.5 Let A be a ring with no idempotents other than 0 and 1, that is, such that $X = \mathrm{Spec}\,A$ is connected, and let $X = (X_i, \phi_{ij})$ be the universal covering scheme (as above) relative to some geometric point $\bar{x}$ of X. Each X_i is an affine scheme $\mathrm{Spec}\,A_i$ and $A = \varinjlim A_i$ has the following properties: $\mathrm{Spec}\,\tilde{A} = \varprojlim X_i$; there is no nontrivial finite étale map $\tilde{A} \to B$; $\pi_1(X,\bar{x}) = \mathrm{Aut}_A(\tilde{A})$. Thus A may reasonably be called the étale closure of A. If A has only a finite number of idempotents, then $A = \prod A_j$ (finite product), $X = \coprod X_j$ ($X_j = \mathrm{Spec}\,A_j$), and $\mathsf{FEt}/X \approx \prod_j \mathsf{FEt}/X_j$. If A has an infinite number of idempotents, then $A = \bigcup A_j$ where each A_j has only a finite number of idempotents, $X = \varprojlim X_j$, and $\mathsf{FEt}/X = \varprojlim \mathsf{FEt}/X_j$ [EGA IV, 17]. Thus the study of FEt/X, X affine, essentially comes down to the case with X connected.

REMARK 5.6 We have seen that $\pi_1(X, \bar{x})$ prorepresents the functor that takes a finite group N to the set of isomorphism classes of $\bar{x}$-pointed Galois coverings of X with a given action of N. Clearly this property determines $\pi_1(X, \bar{x})$, and, since the category of finite groups is Artinian, a standard theorem (Grothendieck [3]) shows that the existence of $\pi_1(X, \bar{x})$ is equivalent to the left exactness of the functor.

It is natural to ask whether there exists a larger fundamental group that, in addition, classifies coverings whose structure groups are finite group schemes. More precisely, consider a variety X over a field k; fix a Spec(k_{al})-point $\bar{x}$ on X, and consider the functor that takes a finite group scheme N over k to the set of isomorphism classes of pairs $(Y, \bar{y})$ where Y is a principal homogeneous space for N over X (see III.4 below) and $\bar{y}$ is a Spec(k_{al})-point of Y lying over $\bar{x}$. Unless X is complete and k is algebraically closed, this functor will not be left exact, as one may see by looking at the cohomology sequence of a short exact sequence of commutative finite group schemes. However, when these conditions hold, then the functor is left exact and is represented by a profinite group scheme $\pi_1(X_{\text{fl}}, \bar{x})$, (Nori [1]). The fundamental group $\pi_1(X, \bar{x})$ is the maximal proétale quotient of this true fundamental group $\pi_1(X_{\text{fl}}, \bar{x})$. See also [SGA 1, p. 271, 289, 309].

COMMENTS ON THE LITERATURE

Proofs for this section may be found in Murre [1].

The first source for much of the material in this chapter is Grothendieck's Bourbaki talks [3] and [SGA 1], and the ultimate source is Chapter IV of [EGA]. Some of the same material can be found in Raynaud [3] and Iverson [2] in the affine case and in the notes of Altman-Kleiman [1]. The first chapter of Artin [9] also contains an elegant, if brief, treatment of étale maps and Henselian rings. See also Kurke-Pfister-Roczen [1]. The most useful introduction to the fundamental group is Murre [1].[8] The theory of the higher étale homotopy groups is developed in Artin-Mazur [1].

[8] Available on the TIFR website.