Automorphic Forms, Shimura Varieties, and L-functions

Volume II

Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6–16, 1988

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FOREWORD

In 1977, the AMS Summer Research Institute was devoted to "Automorphic Forms, Representations, and L-Functions". One of its central topics was the relation between automorphic forms (in their modern guise as automorphic forms on adèle groups) and various objects arising from algebraic geometry, most notably the Hasse-Weil zeta functions of varieties, Galois representations, and Grothendieck's motives. These conjectural relations had been explored by Shimura and others, but Langlands had formulated a systematic program to study them for Shimura varieties. At the time of the conference, Deligne and Langlands stated several fundamental conjectures concerning Shimura varieties, Galois representations, and L-functions.

The decade following the conference saw substantial progress on many of these problems, and the conference was organized in Ann Arbor in 1988 to review this progress and to explore new avenues of research and new questions. In the theory of automorphic forms, advances have been made in the study of the Arthur-Selberg trace formula, the analytic properties of automorphic L-functions including in some cases their analytic continuation, Langlands's functoriality principle including its proof in some important instances, the structure and properties of the discrete spectrum for classical groups, and the p-adic interpolation of certain L-functions. Moreover the baffling problems raised by L-indistinguishability are better understood. As regards Shimura varieties, the basic conjecture of Shimura and Deligne about their canonical models has been proved in the strenthened form conjectured by Langlands at the Corvallis conference, and a combinatorial conjecture of Langlands, allowing one to express their local zeta functions in automorphic terms, proved in some cases. Certain questions that arise in the comparison of the ℓ-adic representations associated with Shimura varieties and automorphic forms (Zucker's conjecture, formulas for the traces of Hecke operators in L^2- cohomology spaces) have been solved. Important arithmetic consequences of the theory of automorphic forms and the functoriality principle have been obtained or seem more accessible, for example, the construction of the Galois representations associated with Maass forms and proofs of the Tate conjecture for certain arithmetic varieties. There has also been progress in the study of the local zeta function of a Shimura variety at a prime of bad reduction. Finally, starting with Drinfeld, the analogues of these problems have been studied for function fields.

The articles in these Proceedings, which are expansions of the lectures given at the conference, are intended to reflect these advances. They are divided, in a somewhat arbitrary manner, between two volumes. The first volume contains expository articles on the trace formula (Labesse) and on the progress since Corvallis in understanding the analytic properties of L-functions (Shahidi). The articles of Milne and Clozel develop two different aspects of Langlands's paper at Corvallis: while Milne's article explains results on Langlands's conjecture on conjugates of Shimura varieties and how they should extend to holomorphic automorphic forms and mixed Shimura varieties, that of Clozel takes up the more speculative question of defining a category of automorphic representations that has the structure of a Tannakian category. The article of Laumon is concerned with finding a geometric interpretation for certain Eisenstein series in the function field case. The papers of Arthur and Kottwitz concern, *inter alia,* the conjectural Hecke-Galois relations for Shimura varieties in the most general case; Kottwitz's paper also includes a conjectural description of the number of points on a Shimura variety over a finite field.

The second volume contains papers on Galois representations associated with automorphic forms (Blasius, Carayol, Taylor); bad reduction of Shimura varieties (Rapoport); higher L-functions (Jacquet-Shalika); coherent cohomology and automorphic forms (Harris); a Lefschetz trace formula, conjectured by Deligne, of importance for zeta functions of Shimura varieties (Zink); the conjectures of Tate and Beilinson in the context of Shimura varieties and a review of the progress that has been made on them and related questions (Ramakrishnan); the p-adic L-functions associated with Shimura convolutions (Hida); and finally, the proof of the Zucker conjecture (Zucker).

The conference was supported by generous grants from the National Science Foundation through the Presidential Young Investigator and Special Projects programs. We are indebted to the Mathematics Department of the University of Michigan, and especially Don Lewis, for its assistance, and to the School of Business Administration for providing us with an air-conditioned auditorium during one of the hottest spells of the century in Ann Arbor. The manuscripts not submitted in TEX were TEX-ed by Steve Tinney and Chris Weider, and Steve Tinney had the difficult task of preparing a uniform manuscript for the publisher from files in submitted in every known dialect of TEX. Finally, the nonmathematical organization of the conference would not have been possible without the exceptional efforts of Lee Zukowski.

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CONTENTS OF VOLUME I

Automorphic Forms and Galois Representations: Some Examples

DON BLASIUS

1.1. Introduction. Let K be a number field. According to a general philosophy, one hopes to identify the L-functions of automorphic forms on $GL_n(\mathbb{A}_k)$ which are *arithmetic* (c.f. 1.7 below) with those of n-dimensional *motivic* Galois representations, i.e. which occur in the etale cohomology of varieties defined over K. Stated in this generality, the study of this correspondence is still in its infancy. Except in the case $n = 1$ where a complete result, due to Shimura and Taniyama, as recently complemented by Faltings, exists, the problem is not fully settled for any n or any K. The purpose of this paper is to sketch some recent developments concerning the problem of finding a motivic representation having a given L-function in the cases $n = 2$, 3, and where K is totally real.

Over the last 35 years, steady progress has been made towards providing such a motivic interpretation for the L-functions of holomorphic Hilbert modular forms which are discrete series at each infinite place. Indeed, after the foundational results of Eichler and Shimura, the well-known paper of Deligne ([**De2**]) constructed representations for all such holomorphic forms, matching L-factors at the unramified places. Later work of Langlands, Deligne, and Carayol ([**Ca**]) settled the problem at the ramified places. If K is totally real, then Rogaski and Tunnel ([**RT**]) found representations for many such forms, again matching Euler factors at the unramified places. Their method excluded exactly those forms not of CM type which are not supercuspidal at some finite place when $[K : \mathbb{Q}]$ is even. Carayol's work ([**Ca**]) also completed the problem for the remaining places. Recently, Taylor constructed the remaining representations by a sophisticated congruence argument. These representations have the sought L-factors at all finite places but do not come equipped with a motivic realization. Later, Rogawski and the author used the theory of $U(3)$ and its Shimura varieties to provide a motivic realization. That this theory is available is due principally to Rogawski, who established the stable trace formula for $U(3)$ and derived its global consequences, and

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to Kottwitz, who gave the description of the points modulo p on the associated Shimura varieties. Thus, altogether, the problem is solved for such holomorphic Hilbert modular forms.

The case of holomorphic forms which are limit discrete series (i.e. weight 1) at each infinte place has also attracted attention. For these forms, the ℓ-adic representation should arise from a complex Galois representation with finite image. If $K = \mathbf{Q}$, such representations were constructed by Deligne and Serre ([**DS**]). Their result was generalized to totally real K by Rogawski and Tunnell ([**RT**]), again excluding the case of certain insufficiently ramified forms in the case where $[K : \mathbf{Q}]$ is even. Later, Wiles found a p-adic method to construct all the sought representations, although now the result of [**Tay**] enables one to proceed also as in [**RT**].

Recently, Ramakrishnan, Clozel and the author found a simple construction using new cases of the principle of functoriality to address the case of forms which are non-holomorphic at an infinite place. This method at present yields only algebraicity statements ([**BHR**]) about the Hecke eigenvalues of the forms, but we hope that further progress in the theory of Shimura varieties and the trace formula will yield the Galois representations. These representations should have finite image only if the form is not discrete series at any infinite place. Thus, there should be many examples, although none has been constructed, where the image is infinite. For these forms, the directions sketched here may eventually yield representations with the correct Euler factors at unramified places, if not a motivic realization. The construction is summarized in Section 3.

In the fourth section, we again invoke the recent progress in the theory of $U(3)$ to obtain some easy consequences of this type for forms on GL_3. For a further contribution of this type, for general n, see Clozel's article in this volume.

The first and second sections of the paper review definitions and summarize the case $n = 1$, respectively.

1.2. Representations. Let $K \subseteq \bar{\mathbf{Q}} \subseteq \mathbf{C}$ be a number field. Let $G_K = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$. Let ℓ be a rational prime. Recall that a *λ-adic representation* of G_K is a continuous homomorphism $\rho : G_K \longrightarrow GL(V)$ where V is a finite dimensional vector space over a finite extension L of $\mathbf{Q}_\ell$ contained in $\bar{\mathbf{Q}}_\ell$. We always assume that ρ is *unramified* at almost all finite places v of K, i.e. for such v, if $\bar{v}$ is an extension of v to $\bar{\mathbf{Q}}$, with inertia group $I_{\bar{v}} \subseteq G_K$, then ρ is trivial on $I_{\bar{v}}$. We let

$F_{\bar{v}}$ denote a Frobenius element for $\bar{v}$. Let E be a number field and let $\eta : E \hookrightarrow \bar{\mathbb{Q}}_\ell$ be an embedding. We say that ρ is *E-rational at η* if the characteristic polynomial of $\rho(F_{\bar{v}})$ has coefficients in $\eta(E)$, for almost all unramified v.

1.3. Compatible systems. ([T]). Suppose we are given, for each prime ℓ and each $\eta : E \hookrightarrow \bar{\mathbb{Q}}_\ell$, a λ-adic representation (ρ_η, V_η) defined over L_η which is E-rational relative to η. Such a system is denoted $\mathbf{V}$. Let $W(\bar{v}/v) \subseteq G_K$ be a Weil group for the finite place v with associated Weil-Deligne group $WD(\bar{v}/v)$. Then, for v not dividing l, the restriction of ρ_η to $W(\bar{v}/v)$ defines an isomorphism class of representation $\sigma_v(\rho_\eta)$ of $WD(\bar{v}/v)$ on V_η. Let S be a finite set of finite places of K. Then $\mathbf{V}$ is *strictly compatible* (outside S) if, for $v \notin S$, and for each extension of η^{-1} to an isomorphism $\bar{\eta}^{-1} : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, the class of the complex representation $\bar{\eta}^{-1}(\sigma_v(\rho_\eta))$ is independent of ℓ and η. Denote by $\sigma_v(\mathbf{V})$ the class of complex representation of $WD(\bar{v}/v)$ so defined. We refer the reader to [T] for the definitions of conductor and epsilon factors for the $\sigma_v(\mathbf{V})$. For L-factors, we recall that, for $v \notin S$ and v not dividing ℓ,

$$L_v(\mathbf{V}, s) = P_v(X)^{-1} \mid_{X=N_v^{-s}}$$

where N_v is the number of elements in the residue field of K at v, and

$$P_v(X) = \eta^{-1}(\det(1 - \rho_\eta(F_{\bar{v}})X \mid_{V_\eta^{I_{\bar{v}}}})).$$

Put

$$L_S(\mathbf{V}, s) = \prod_{\substack{v \notin S \\ v \text{ finite}}} L_v(\mathbf{V}, s).$$

1.4. Local Langlands correspondence. Let $\chi_1, \dots, \chi_n$ be unramified quasicharacters of $K_v^* = W(\bar{v}/v)^{ab}$, such that $|\chi_j(\tilde{\omega}_v)|$ is independent of j, and let $\pi(\chi_1, \dots, \chi_n)$ be the irreducible unitarily induced principal series representation of $GL_n(K_v)$ defined by the χ_j. The local Langlands correspondence, proven for $n = 1, 2, 3$, and in many other cases, parametrizes all isomorphism classes of irreducible admissible representations π of $GL_n(K_v)$ by all classes $\sigma_v(\pi)$ of $F_{\bar{v}}$-semisimple complex representations of $WD(\bar{v}/v)$. We denote the inverse correspondence by $\sigma \mapsto \pi_v(\sigma)$. For example, for $\pi(\chi_1, \dots, \chi_n)$ above, we have $\sigma_v(\pi(\chi_1, \dots, \chi_n)) \cong \chi_1 \oplus \cdots \oplus \chi_n$, and

$$L_v(\pi(\chi_1, \dots, \chi_n), s) = \prod_{j=1}^{n} L_v(\chi_j, s)$$

where, as usual,

$$L(\chi_j, s) = (1 - \chi_j(\tilde{\omega}_v)N_v^{-s})^{-1}.$$

Hence, if $v \notin S$, and we put $\chi_j = \bar{\eta}^{-1}(\alpha_j)$ where $\alpha_1, \dots, \alpha_n : K_v^* \longrightarrow \overline{\eta(E)}$ are the characters occuring in the diagonalization of (the semisimplification of) the action of $F_{\bar{v}}$ on V_η, we have

$$L_v(\mathbf{V}, s) = L_v(\pi(\chi_1, \dots, \chi_n), s),$$

or equivalently,

$$\bar{\eta}^{-1}(\sigma_v(\mathbf{V})) \cong \sigma_v(\pi(\chi_1, \dots, \chi_n)).$$

Put, in general,

$$\pi_v(\mathbf{V}) = \pi_v(\bar{\eta}^{-1}(\sigma_v(\mathbf{V})),$$

and

$$\pi_S(\mathbf{V}) = \bigotimes_{\substack{v \notin S \\ v \text{ finite}}} \pi_v(\mathbf{V})$$

1.5. Motivic V.. We say that the E-rational strictly compatible system $\mathbf{V}$ is *motivic* if there exists a proper smooth variety X over K and, for each $\eta : E \hookrightarrow \bar{\mathbb{Q}}_\ell$, an L_η-linear G_K-embedding

$$\xi_\eta : V_\eta \longrightarrow H^w(\bar{X}, \bar{\mathbb{Q}}_\ell)$$

where $\bar{X} = X \times \operatorname{Spec}(\bar{\mathbb{Q}})$. If $\mathbf{V}$ is motivic, the component representations acquire many good properties: Riemann Hypothesis, Hodge-Tate, etc. Nevertheless, it is of interest to know whether $\mathbf{V}$ arises from a Grothendieck motive $M \subseteq H^w(X)$. This means that there exists a finite extension T of E and an embedding $T \hookrightarrow \operatorname{End}(M)$ such that V_η is isomorphic to the $\bar{\eta}$-*component* of the $T \otimes \bar{\mathbb{Q}}_\ell$-module $M_\ell \otimes \bar{\mathbb{Q}}_\ell$, where M_ℓ is the l-adic realization of M and $\bar{\eta}$ is any extension of η to T. One hopes that every motivic $\mathbf{V}$ arises from a Grothendieck motive.

1.6. Global Langlands Correspondence. Let $\mathbf{V}$ be an E-rational motivic system, strictly compatible outside S. One conjectures that there is an isobaric automorphic representation $\pi' = \pi'(\mathbf{V})$ of $GL_n(\mathbb{A}_K)$, $n = \dim \mathbf{V}$, such that

$$\pi'(\mathbf{V}) \xrightarrow{\sim} \Big(\bigotimes_{\substack{v \in S \text{ or} \\ \text{infinite}}} \pi_v \Big) \otimes \pi_S(\mathbf{V}).$$

Furthermore, $\pi'(\mathbf{V})$ should be a cusp form exactly when any (and hence, conjecturally, every) V_η is irreducible. Of course, one conjectures that $S = \emptyset$, and so $\pi'(\mathbf{V})$ should be simply the completion of $\pi_\phi(\mathbf{V})$ with factors at the archimedean places.

1.7. Archimedean places. If $\mathbf{V}$ arises from a Grothendieck motive M, then the Hodge theory of its topological realizations $M_{B,\tau}(\tau : K \hookrightarrow \mathbb{C})$ provides a rule (See [**Ta**]) for computing suitable factors $L_v(\mathbf{V}, s)$ where v is the infinite place defined by τ. In this case one attaches to each archimedian v a class $\sigma_v(\mathbf{V})$ of representations of the Weil group W_v, and hence, by the local Langlands correspondence, a class of representations $\pi_v = \pi(\sigma_v)$ of $GL_n(K_v)$. It is this π_v which should occur above (1.6) in $\pi'(\mathbf{V})$. However, we neglect this question in this paper. Conversely, given a representation π_v of $GL_n(K_v)$, we say that π_v is *arithmetic* if the restriction of $\sigma_v(\pi_v)$ to $\mathbb{C}^* \subseteq W_v$ diagonalizes as a sum of characters $z \mapsto z^a \bar{z}^b$ with $a, b \in \mathbb{Z}$. We say that an automorphic representation π is *arithmetic* if π_v is arithmetic for each archimedian v. Of course, if $\pi_v = \pi(\sigma_v(\mathbf{V}))$ as above, then π_v is arithmetic.

1.8. Converse to 1.6. If π is an isobaric arithmetic automorphic representation of $GL_n(\mathbb{A}_K)$, one conjectures that there exists a number field E and a motivic E-rational strictly compatible ($S = \phi$) system $\mathbf{V} = \mathbf{V}(\pi)$ such that

$$\sigma_v(\mathbf{V}(\pi)) = \sigma_v(\pi) \tag{1.8.1}$$

for all finite v. The purpose of this paper is to review some examples of this correspondence.

2.1. The case $n = 1$.

THEOREM. *(a) Let S be a finite set of finite primes. Let $\mathbf{V}$ be a system of E-rational compatible outside S one-dimensional λ-adic representations of G_K. Then $\mathbf{V}$ is strictly compatible with $S = \emptyset$ and there exists a unique quasicharacter $\chi = \chi(\mathbf{V}) : \mathbb{A}_K^*/K^* \longrightarrow \mathbb{C}^*$ such that $\sigma_v(\chi) = \chi_v = \sigma_v(\mathbf{V})$ for all finite v.* (b) *Conversely, let χ be a quasicharacter of $\mathbb{A}_K^*/K^*$ which is arithmetic (i.e. is of type A_0, in the sense of Weil). Let E be the field generated by the values of χ on the finite ideles of K. Then there exists a strictly compatible motivic E-rational system $\mathbf{V}(\chi)$ (with $S = \emptyset$) such that $\Pi'(\mathbf{V}(\chi)) = \chi$. We have $\sigma_v(\mathbf{V}(\chi)) = \sigma_v(\chi)$ at all finite places. The system $\mathbf{V}(\chi)$ is unique* *up to possible enlargements of the L_η, and there is a natural minimal choice where L_η is the closure of $\eta(E)$ for each η.* (c) *A multiple of* $\mathbf{V}(\chi)$ *arises from a Grothendieck motive in the category of such motives generated by abelian varieties defined over K and of CM type over K. The system* $\mathbf{V}(\chi)$ *itself arises from the etale realization of a motive for absolute Hodge cycles in the category of such motives generated by abelian varieties over K.* (d) *Let $V \subset H_\ell^w(X) \otimes \bar{\mathbb{Q}}_\ell$ be any 1-dimensional λ-adic representation. Then there exists an arithmetic quasicharacter χ and an embedding $\bar{\eta}$ of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_\ell$ such that $\bar{\eta}(\chi(\tilde{\omega}_v)) = \rho(F_v)$ for all v which are unramified for ρ.*

2.2. Theorem 2.1 is a summary of well-known results. For example, part (a) is a theorem of Serre, Waldschmidt and Henniart ([**He**]). Similarly, part (d) follows from a theorem of Tate ([**Se**]) since Faltings has shown that the $H_\ell^w(X)$ are Hodge-Tate. Part (b) follows from the theory of Shimura-Taniyama and a theorem of Casselman ([**S**]). Part (c) follows from [**DMOS**].

3.1. The case $n = 2$. We assume, for the rest of the paper, that K is totally real. Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_K)$ for which, at archimedean v, π_v belongs to the discrete series, i.e.

$$\sigma_v(\pi) = \mathrm{Ind}_{\mathbb{C}^*}^{W_v}(z^{1-k_v}) \otimes \omega_v^{(k_v-1-w)/2} \tag{3.1.1}$$

with integers $k_v \geq 2$ and w, $k_v \equiv w \bmod(2)$, and where $\omega_v : W_v \longrightarrow \mathbb{R}^*$ is the norm. These π correspond to all holomorphic forms of weight $\{k_v\}$ where all k_v are congruent modulo 2 and greater than one. Let $E(\pi) \subseteq \mathbb{C}$ be the field generated by the coefficients of the Hecke polynomials of the π_v for the finite, unramified v.

THEOREM. *([**BRo1**], [**Tay**]). For π as above, there exists a motivic $E(\pi)$-rational strictly compatible ($S = \emptyset$) system $\mathbf{V}(\pi)$ such that*

$$\sigma_v(\mathbf{V}(\pi)) = \sigma_v(\pi) \tag{3.1.2}$$

for all finite v. Each $V_\eta \in \mathbf{V}(\pi)$ can be realized over the closure in $\bar{\mathbb{Q}}_\ell$ of $\eta(E(\pi))$ and is irreducible.

3.2. Remarks on the Proof. In [**BRo1**] it is shown that there exists a motivic $\mathbf{V}(\pi)$ in the endoscopic part of the cohomology of a fiber system over a Shimura surface defined by a unitary group in 3 variables such that (3.1.2) holds at the unramified places. On the other hand, [**Tay**] constructs a $\mathbf{V}(\pi)$ which satisfies (3.1.2) at all places v where π_v is principal series. From here, one proceeds as in [**Ca**], part 12, using base change and the local Langlands classification to obtain (3.1.2) at all finite places. The irreducibility is well known.

3.3. There are many arithmetic Hilbert modular forms which are not treated by Theorem 3.1. However, at least if π_v is holomorphic for each infinite v, we can provide a partial solution.

THEOREM. *Suppose π is a cuspidal automorphic representation of $GL_2(\mathbb{A}_k)$ which satisfies* (3.1.1) *with integers $k_v \geq 1$ and $w, k_v \not\equiv w$* mod (2) *(so that π_v is now allowed to be limit discrete series as well). Then there exists an $E(\pi)$-rational system $\mathbf{V}(\pi)$, strictly compatible at all finite v which are unramified for π so that at such v* (3.1.2) *holds. Each V_η is irreducible and can be realized over the closure in $\bar{\mathbb{Q}}_\ell$ of $\eta(E(\pi))$.*

3.4. The proof of Theorem 3.3 follows from Wiles' method of pseudo-representations ([**Tay**]). If all $k_v = 1$, these representations are systems associated to totally odd Galois representations $G_K \longrightarrow GL_2(\mathbb{C})$ with finite image and hence $\mathbf{V}(\pi)$ is motivic. These were already constructed by [**DS**], [**RT**], and [**Wi**]. However, if some, but not all, $k_v \geq 1$, then $\mathbf{V}(\pi)$ is not known to be motivic unless π is of CM type.

3.5. The general case. If π is an arithmetic cusp form on $GL_2(\mathbb{A}_k)$, then, if $\sigma_v(\pi)$ is not of the form (3.1.1), we must have

$$\sigma_v(\pi) = (\text{triv.} \oplus \text{triv.}) \otimes \omega_v^{-\omega/2} \mathrm{sgn}_v^a$$

where $\omega \in 2\mathbb{Z}$ and $a \in \{0, 1\}$. Thus, π_v belongs to the principal series at such a place. For these π we cannot, at present, construct $\mathbf{V}(\pi)$. In [**BCR1**], [**BCR2**], [**BHR**], and [**BR**], a study of this case was begun when $K = \mathbb{Q}$. We now sketch how these methods extend to treat the remaining arithmetic Hilbert modular forms.

3.6. Fix an arithmetic π. Let L be a totally imaginary quadratic extension of K and let $\chi : \mathbb{A}_L^*/L^* \longrightarrow \mathbb{C}^*$ be an arithmetic quasicharacter such that $\chi_v(z_v) = z_v^{-a}$, with a (large) positive integer a, independent of v, having fixed an identification of K_v^* with $\mathbb{C}^*$ for each infinite v. Let π_L be the base change of π to $GL_2(\mathbb{A}_L)$; assume, without loss, that π_L is cuspidal. Let $I_{L/K}(\pi_L \otimes \chi)$ be the automorphic induction of $\pi_L \otimes \chi$ to $GL_4(\mathbb{A}_K)$. It is cuspidal. Let

$\Lambda^2 : GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ be the exterior square representation. Exactly as in [**BR**], §6.2, we find

$$\begin{aligned}(3.6.1)\quad & L_S(I_{L/K}(\pi_L \otimes \chi),\ \Lambda^2; s) \\ & = L_S(\pi, \mathrm{Sym}^2 \otimes \chi_0 \varepsilon_L; s) L_S(I_{L/K}(\chi^2) \otimes \omega_\pi, s) L_S(\omega_\pi \chi_0, s)\end{aligned}$$

where $\mathrm{Sym}^2 : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$ is the symmetric square, χ_0 denotes the restriction of χ to $\mathbb{A}_K^*$, ω_π is the central character of π and ε_L is the quadratic idele class character of $\mathbb{A}_K^*$ associated to L/K by classfield theory. Thus, a twist of the exterior square L-function (3.6.1) has a pole with non-zero residue, and so, by the theorem of Jacquet, Piatetski-Shapiro, and Shalika ([**BR**]), there exists a generic cusp form on $\Pi(\pi, \chi)$ on $GS_p(4, \mathbb{A}_K)$, where $GS_p(4, -)$ is the group of symplectic similitudes in dimension 4, such that

$$L_v(\Pi(\pi, \chi), s) = L_v(I_{L/K}(\pi_L \otimes \chi), s)$$

at all finite places where π and $I_{L/K}(\chi)$ are unramified, and

$$\sigma_v(\Pi(\pi, \chi)) = \sigma_v(\pi) \otimes \mathrm{Ind}_{\mathbb{C}^*}^{W_v}(z^{-a})$$

at infinite v. Thus,

$$\sigma_v(\Pi(\pi,\chi)) = \begin{cases} (\mathrm{Ind}_{\mathbb{C}^*}^{W_v}(z^{1-k_v-a}) \oplus \mathrm{Ind}_{\mathbb{C}^*}^{W_v}(z^{1-k_v}\bar{z}^{-a})) & \\ \qquad\qquad \otimes \omega_v^{(k_v-\omega-1)/2}, & (k_v \geq 1) \\ 2\ \mathrm{Ind}\ (\bar{z}^{-a}) \otimes \omega_v^{\omega/2}, & (\text{otherwise}). \end{cases}$$

Hence, if $k_v \geq 2$, $\Pi(\pi, \chi)_v$ belongs to a discrete series L-packet on $GS_p(4, \mathbb{R})$, and, otherwise, the packet of $\Pi(\Pi, \chi)_v$ is a limit of discrete series ([**BHR**]). In each case, $\Pi(\pi, \chi)$ is $\bar{\partial}$-cohomological in the sense of [**BHR**], and hence, as explained there, the unramified Hecke eigenvalues of $\Pi(\pi, \chi)$ occur in the action of the Hecke algebra on a higher coherent cohomology group $H^j(S, \mathcal{V})$, where $\mathcal{V}$ is an automorphic vector bundle, determined by the $\Pi(\pi, \chi)_v$ for v infinite, defined on the Shimura variety associated to $GS_p(4, K)$. Since $\mathcal{V}$ is defined over the Galois closure of K, this proves:

THEOREM [**BHR**]. *The coefficients of the unramified Hecke eigenvalues of π are algebraic numbers.*

3.7. According to Prop. 6.6 of [**BR**], the L-packet with parameter $\sigma_v(\Pi(\pi,\chi))$ contains a holomorphic element Π_v^h. Further, Hypothesis 1 (§7.1 of [**BR**]) extends naturally to this case: we expect a suitably refined trace formula to show that if

$$\Pi(\pi,\chi) = \otimes_{v|\infty}\Pi(\pi,\chi)_v \otimes \Pi(\pi,\chi)_f$$

then

$$\Pi^h(\pi,\chi) = \otimes_{v|\infty}\Pi(\pi,\chi)_v^h \otimes \Pi(\pi,\chi)_f$$

is automorphic. If each π_v is either principal series or limit of discrete series, so that $\mathbf{V}(\pi)$, if it exists, is the system associated to a complex Galois representation, then the program of [**BR**] extends. Accordingly, one needs to settle the further problem of finding systems $\mathbf{V}(\Pi)$ for Π a cusp form on $GS_p(4,\mathbb{A}_K)$ which is everywhere at infinity of discrete series type. However, if $K = \mathbb{Q}$, then R. Taylor has found a method, using the congruence relations of Shimura for the Siegel modular three-folds, to evade this last condition, i.e. it is enough to show that the $\Pi(\pi,\chi)^h$ exist. On the other hand, for more general π, one must hope to extend the interpolation methods of Wiles and Taylor; it is likely that such an extension can be found. The $\mathbf{V}(\Pi)$ for $GS_p(4,\mathbb{A}_K)$ will themselves be difficult to construct but at least a program exists: generalizing [**BR**] and [**BRo1**], one may hope to (a) transfer Π to a Π' on $GL_4(\mathbb{A}_K)$, (b) base change Π' to a Π'_L for a CM quadratic extension L of K; (c) descend a twist of Π'_L to a Π' on quasisplit $U(4)$, (d) realize Π'' as an endoscopic L-packet on a form of $U(5)$ which is compact at all but one infinite place, where we have $U(5)_v \xrightarrow{\sim} U(4,1)$, and quasisplit at all finite places, and (e) compute the zeta function of an associated Shimura variety. This procedure will construct representations $\mathbf{V}$ as in (1.8.1) for Π'_L, and varying L, we obtain $\mathbf{V}$ for Π. Of course, these steps are well out of reach at present.

4.1. The case $n = 3$. If $n = 3$, we must impose strong conditions on the archimedean type of our representation in order to obtain any result. However, for this restricted class, we obtain the sought $\mathbf{V}$'s.

Definition. An automorphic representation π of $GL_n(\mathbb{A}_K)$ is *essentially self-dual* if $\pi^v \xrightarrow{\sim} \pi\otimes\psi$ for some quasicharacter ψ, where π^v is the contragredient of π.

If π is arithmetic, we see that it is also essentially self-dual, for $n = 3$ and K totally real, only if, for infinite v,

$$\sigma_v(\pi)\mid_{\mathbb{C}^*} \xrightarrow{\sim} \mathrm{Diag}(z^a\bar{z}^b, \bar{z}^a z^b, (z\bar{z})^{\frac{a+b}{2}})$$

with $a + b \in 2\mathbb{Z}$. Hence, if $a \neq b$,

$$\sigma_v(\pi) = \mathrm{Ind}_{\mathbb{C}^*}^{W_v}(z^a\bar{z}^b) \oplus \omega_v^{\frac{a+b}{2}}\mathrm{sgn}_v^m\ ,$$

with $m \in \{0, 1\}$, or, if $a = b$,

$$\sigma_v(\pi) = \omega_v^a\mathrm{sgn}^\alpha \oplus \omega_v^a\mathrm{sgn}_v^\beta \oplus \omega_v^a\mathrm{sgn}_v^\gamma$$

with $\alpha, \beta, \gamma \in \{0, 1\}$. Following Clozel, we say that Π is *regular* if $a \neq b$ for each $\sigma_v(\pi)$.

4.2. THEOREM. *Let π be a regular, essentially self-dual cuspidal automorphic representation of $GL_3(\mathbb{A}_K)$. Let S be the set of finite places where π ramifies. Then there exists a motivic strictly compatible outside S system of $E(\pi)$-rational λ-adic representations $\mathbf{V}(\pi)$ which is unramified outside S such that* (1.8.1) *holds for all finite $v \notin S$. Each $V_\eta \in \mathbf{V}(\pi)$ can be realized over the closure of $\eta(E(\pi))$ in $\bar{\mathbb{Q}}_\ell$; a multiple of $\mathbf{V}(\pi)$ arises from the etale realization of a Grothendieck motive. Each V_η is absolutely irreducible.*

4.3. PROOF: The argument which follows is similar to that of [**BRo1**]. Let L be a quadratic CM extension of K. Since the theorem is true for π if and only if it is true for $\pi \otimes \mu$ with a character μ of finite order, we may assume that $\pi^v \xrightarrow{\sim} \pi \otimes \omega^w\psi$ with $w \in \mathbb{Z}$ and a totally even ψ of finite order. Such ψ are trivial on the kernel of the morphism $\pi_0(\mathbb{A}_K^*/K^*) \longrightarrow \pi_0(\mathbb{A}_L^*/L^*)$ defined by the inclusion $K \hookrightarrow L$. Thus, ψ extends to a character $\bar{\psi}$ of $\mathbb{A}_L^*/L^*$, also of finite order. For a cusp form η on $GL_3(\mathbb{A}_L)$, let $\varepsilon\eta$ be defined by $(\varepsilon)(g) = \eta^v(\tau g)$ where τ is the nontrivial automorphism of L over K. Then $\varepsilon(\pi_L \otimes \bar{\psi}) = \varepsilon(\pi_L) \otimes \bar{\psi}^{-1} \circ \tau = (\pi_L \otimes \psi \circ N_{L/K}) \otimes \bar{\psi}^{-1} \circ \tau = \pi_L \otimes \bar{\psi}$. Thus, by [**R**], $\pi_L \otimes \bar{\psi}$ is the base change to $GL_3(\mathbb{A}_L)$ of a cuspidal L-packet Π' on the quasisplit unitary group V' in 3 variables, defined relative to L/K. By [**R**] again, there exists Π'' on a group U'' which is quasisplit at the finite place defined by the given embedding $K \hookrightarrow \mathbb{C}$ and at all finite places, and is compact at all remaining infinite places. From [**M**], we conclude that there is a compatible system

$\mathbf{V}(\Pi'')$ of motivic $E(\pi)E(\bar{\psi})$-rational λ-adic representations each that $\sigma_v(\pi_L \otimes \bar{\psi}) = \sigma_v(\mathbf{V}(\Pi''))$ for all finite v at which π_L and $\bar{\psi}$ are unramified, as well as infinite v. Further, by [**BRo2**], each constituent $V_\eta \in \mathbf{V}(\Pi'')$ is absolutely irreducible. Put $\mathbf{V}(\pi_L) = \mathbf{V}(\Pi'') \otimes \bar{\psi}^{-1}$, where $\bar{\psi}^{-1}$ is identified with the system defined by $\bar{\psi}$, viewed as a Galois character. (Here, we enlarge each $L = L_\eta$ as needed.) Suppose that the finite place v of K splits in L and is unramified for π. Let w/v be one of the places of L which lie over v. Then we can choose $\bar{\psi}$ to be unramified at w: suppose $\bar{\psi}_\omega$ is ramified. By the Grunwald-Hasse-Wang theorem, there is a character of finite order φ of $\mathbb{A}_L^*/L^*$ such that $\varphi_w = \bar{\psi}_w^{-1}$ and $\varphi_{\bar{w}} = 1$. Put $\lambda = \varphi/\varphi \circ \tau$ and $\bar{\psi}' = \bar{\psi}\lambda$. Then $\varepsilon(\pi \otimes \bar{\psi}') = \pi \otimes \bar{\psi}'$ since $\varepsilon(\lambda) = \lambda$ and the claim is proved.

4.4. Now let $\{L_j | j \geq 1\}$ be an infinite family of distinct quadratic CM extensions of K such that each finite prime of K splits in at least one L_j. For each j, let $\mathbf{V}_j$ be the system constructed above j it is E_j-rational. Then the isomorphism class of $V_{j,\eta}$ depends only upon j and η. In particular,

$$(V_{j,\eta})_{L_j L_m} \xrightarrow{\sim} (V_{m,\eta})_{L_j L_m}$$

where, as usual, the subscript $L_j L_m$ denotes the restriction to $\mathrm{Gal}(\bar{\mathbb{Q}}/L_j L_m)$. Exactly as in [**BR**], §4, this means that we can descend each family $\{V_{j,\eta} \mid j \geq 1\}$ to a representation V_η of G_K. In view of our result at the end of 4.3, we know that

$$\sigma_v(V_\eta) = \sigma_v(\pi)$$

for all $v \notin S$.

4.5. Let $E = E(\pi)$ and for each $\eta : E \hookrightarrow \bar{\mathbb{Q}}_\ell$, let E_η denote the closure of $\eta(E)$. We now show that each V_η is of the form $W_\eta \otimes_{E_\eta} L_\eta$ with an E_η-vector space W_η on which G_K acts. The claim that $\mathbf{V}(\pi)$, rather than just $\mathbf{V}(\pi)_{L_j}$ is motivic follows if we consider that $\mathbf{V}(\pi)_{L_1} \otimes \bar{\psi}_1$ occurs in the cohomology of $R_{L_1/K}(A)$ where A is a suitable fiber system of abelian varieties over the Shimura variety used above, and $R_{L_1/K}$ is the restriction of scalars functor. Let $c_v \in G_K$ be a complex conjugation defined by the place $v : K \longrightarrow \mathbb{R}$. As in [**BRo2**], an analysis of the Hodge types of the motivic realization of the V_η shows that c_v cannot act on V_η as a scalar. If V_η is not of the sought form, we can only have $3V_\eta$ of the form $W_\eta \otimes_{E_\eta} L_\eta$ with a representation

W_η, irreducible over E_η. Then Im $(E_\eta[G_K])$ is a division algebra of degree 9 over E_η. Since c_v has distinct rational eigenvalues, $E_\eta[c_v]$ does not have image a field, which is impossible. Thus, V_η descends to E_η, and the claim is proved. Let $\mathbf{V}(\pi) = \{W_\eta\}$.

4.6. This proves all claims of the theorem except that pertaining to Grothen-dieck motives. To show that a multiple of $\mathbf{V}(\pi)$ arises as the etale realization of a Grothendieck motive, one must use the observation that the Shimura varieties (and fiber systems) defined by unitary groups descend canonically to the maximal totally real subfield of their canonical field of definition. Although the action of the Hecke operators does not descend, they satisfy the reciprocity law $[g]^\tau = [g^\tau]$ for $g \in GU(\mathbb{A}_{K,f})$ where $[g]$ denotes the (K-rational) automorphism of the Shimura variety defined by g. The claim follows from these facts but we omit details.

4.7. We conclude by noting that this theorem, excepting the claim about irreducibility, extends to the π treated by Clozel in the last section of his article.

REFERENCES

[**Ca**] Carayol, H., *Sur les représentations ℓ-adique associés aux formes moddaires de Hilbert*, Ann. Sci. Ec. Norm. Sup. (4) **19** (1986), 409-468.

[**BCR1**] Blasius, D., Clozel, L., and Ramakrishnan, D., *Algébricité de l'action des opérateur de Hecke sur certaines formes de Maass,* C.R. Acad. Sci. Paris, **t. 305**, Série I, p. 705–708.

[**BCR2**] ———, *Opérateurs de Hecke et formes de Maass: application de la formule des traces,* C. R. Acad. Sci. Paris, **t, 306**, Série I' p. 59–62.

[**BHR**] Blasius, D., Harris, M., and Ramakrishnan, D., *Coherent cohomology, limits of discrete series and Maass forms of Galois type*, in preparation.

[**BR**] Blasius, D., and Ramakrishnan, D., Maass forms and Galois representations, to appear: Proc. of Conf. on Galois groups and related topics, MSRI, 1987.

[**BRo1**] Blasius, D., and Rogawski, J., *Galois representations for Hilbert modular forms,* to appear: Bull. Am. Math. Soc., July 1989.

[**BRo2**] ———and ———, *Tate classes and arithmetic quotients of the unit ball,* to appear in: Proc. of a Seminar on Shimura varieties, ed. J. Arthur, R. Langlands, D. Ramakrishnan.

[**Cl**] Clozel, L., Article in this volume.

[**De**] Deligne, P., *Valeurs de fonctions L et périodes d' integrales, in Automorphic Forms, Representations, and L-functions,* Proc. Symp. Pure Math., Vol. 33, Part 2, p. 313-346.

[**DMOS**] Deligne, P., Milne, J., Ogus, A., and Shih, K. *Hodge Cycles, Motives, and Shimura varieties,* Springer Lecture Notes 900.

[DS] Deligne, P. and Serre, J.P., *Formes Modulaire de Poids 1,* Ann. Sci. Ec. Norm. Sup., (4) **7** (1974) p. 507-530.

[He] Henniart, G., *Représentations ℓ-adique abéliennes,* Sem. Delange-Pisot-Poitou, p. 91-106, 1981, Birkhauser, Boston.

[L] Langlands, R., *Base change for GL_2* Ann. of Math. Studies, **96**, 1980.

[Ro] Rogawski, J., *Automorphic representations of unitary groups of three variables,* to appear: Ann. of Math. Studies, Princeton.

[RT] Rogawski, J. and Tunnell, J., *On Artin L-functions associated to Hilbert modular forms of weight one,* Inv. Math. **74** (1983), 1-42.

[Se] Serre, J.P., *Abelian ℓ-adic Representations and Elliptic Curves,* Benjamin, New York, 1968.

[S] Shimura, G., *On the zeta function of an abelian variety with complex multiplication,* Ann. of Math. **94** (1971) p. 504–533.

[Ta] Tate, J., *Number theoretic background,* in Proc. Symp. Pure Math., Vol. 33, Part 2, p. 3-26.

[Tay] Taylor, R., article in this volume.

[Tu] Tunnell, J., *Artin's conjecture for representations of octahedral type,* Bull. AMS, **5**, No. 2 (1981), p. 173–175.

[Wi] Wiles, A. *On ordinary λ-adic representations associated to modular forms,* preprint.

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Non-abelian Lubin-Tate Theory

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0. INTRODUCTION

0.1. Let $f = \sum a_n q^n \in S_k(N, \varepsilon)$ be a primitive normalized cuspidal eigenform of weight $k \geq 2$, level N and character ε. Then a classical construction, due to Eichler and Shimura in the weight 2 case and to Deligne in the general situation, associates to f a system (ρ_λ) of two-dimensional λ-adic Galois representations :

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(E_\lambda),$$

where λ ranges over the set of primes of the number field E, which is generated by the coefficients a_n and the values of ε. In classical terms, the relationship between f and ρ_λ is as follows : if we denote by l the residual characteristic of λ, then ρ_λ is unramified outside Nl; and for p any prime number not dividing Nl, the trace and determinant of ρ_λ on the arithmetic Frobenius ϕ_p are given by :

$$\begin{cases} \mathrm{tr}\ \rho_\lambda(\phi_p) & = a_p \\ \det\ \rho_\lambda(\phi_p) & = \varepsilon(p)p^{k-1}. \end{cases}$$

The classical theory of modular forms, however, was not suitable to formulate a precise conjecture describing the behaviour of the representations ρ_λ at *bad primes* : that means, for $p \neq l$ a divisor of N, giving a recipe to compute the restriction $\rho_{\lambda,p}$ of ρ_λ to the local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. It was only after Jacquet-Langlands' work that such a precise recipe was elaborated : a modular form f as above gives rise to an automorphic representation $\pi = \otimes \pi_v$ of the group $\mathrm{GL}_2(\mathbb{A})$, and the conjecture was that the local restriction $\rho_{\lambda,p}$ should correspond to the local factor π_p via the *local Langlands correspondence* (suitably normalized).

0.2. At the time when the Antwerp conference was held, the existence of the local Langlands correspondence had been checked for $p \neq 2$, via an explicit "dictionary" : principal series were associated to decomposed Galois representations, special GL(2) representations to

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special Galois representations; and finally, cuspidal (Weil) representations to irreducible (induced) Galois representations. And it was during this Antwerp conference that Langlands gave a proof of the above conjecture *whenever* π_p *is either a principal or a special series representation.*

As soon as π_p is not a spherical principal series representation, the Galois representation ρ_λ occurs in the cohomology of a modular curve which has *bad reduction* at p, and the whole question amounts to computing that cohomology group : it sits in an exact sequence, with on the left side the cohomology group of the special fiber and on the right the cohomology of vanishing cycles. One can explicitely describe the set of points of the special fiber, together with the Hecke and Galois actions. Using such a description, Langlands was able to compute the cohomology of the special fiber and (comparing the Selberg and Lefschetz trace formulas) to prove the above conjecture in the case of principal or special representations; more precisely, it turned out that principal series representations occured only in the special fiber cohomology and cuspidal ones only in the vanishing cycles cohomology, while special representations contributed to both. The case of cuspidal series remained open, as there was no way to compute explicitely the vanishing cycles cohomology.

0.3. However, this last case was solved some time later by Deligne [De 1], at least for $p \neq 2$. His method consisted in constructing a local representation of the product group $\mathrm{GL}_2(\mathbb{Q}_p) \times B_p^* \times W_{\mathbb{Q}_p}$, where B_p denotes the quaternion division algebra over $\mathbb{Q}_p$, and $W_{\mathbb{Q}_p}$ the Weil group. Using this local representation and its interplay with the global representation (on the vanishing cycles group of the modular curve), he was able to prove that the local restriction $\rho_{\lambda,p}$ was expressible in terms of the local component π_p alone. As a consequence of this, when the (cuspidal) factor π_p is a Weil representation (which is always the case if $p \neq 2$), one can reduce oneself to the situation where the automorphic representation π itself is obtained from the (global) Weil construction, and then the conjecture is easy to prove.

But the case of so-called "extraordinary" cuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_2)$ was still unsettled, and the very existence of the local Langlands correspondence had not yet been shown in that case. Some ten years after the works of Langlands and Deligne, I studied the same question for Hilbert modular forms. I found that a theory of bad reduction for Shimura curves existed, similar to the one for modular curves, and, using this theory, I was able to generalize the results of

Deligne and Langlands to the Hilbert case. In this more general context, it became possible to use base change arguments and finally to prove the above conjecture even in the case of extraordinary cuspidal representations (Kutzko had proven in the interval the existence in all cases of the local Langlands correspondence for GL(2)).

0.4. In the sequel, we are going to focus our attention on the "local fundamental representation". In the framework of my paper [C], it was defined, for F a finite extension of $\mathbb{Q}_p$, as an l-adic ($l \neq p$) representation $\mathcal{U}_F$ of the product $\mathrm{GL}_2(F) \times B_F^* \times W_F$, where B_F is the quaternion division algebra with center F and W_F the Weil group. One purpose of what follows is to explain how to generalize that construction to arbitrary dimensions, thus obtaining for every local field F (i.e. a finite extension of $\mathbb{Q}_p$ as above, or some $\mathbb{F}_q(T)$) and for every integer $h \geq 1$, a representation $\mathcal{U}_{h,F}^v$ of the product $\mathrm{GL}(h, F) \times B_{h,F}^* \times W_F$, where $B_{h,F}$ denotes the skew field with center F and invariant $1/h$.

In the notation just introduced, the upper-script "v" stands for "vanishing", as our representation is defined from the cohomology of a vanishing cycle variety. I also want to show that another construction can be obtained by considering rigid-analytic coverings (defined by Drinfeld) of the so-called "p-adic generalized upper half-plane" : the cohomologies of these coverings give rise to another representation, which I call $\mathcal{U}_{h,F}^r$, of the same product group.

Then I state a conjecture which predicts the decomposition of the representation $\mathcal{U}_{h,F}^v$, in terms of both the Jacquet-Langlands correspondence (between $\mathrm{GL}(h, F)$ and $B_{h,F}^*$) and the (conjectural) Langlands correspondence (between $\mathrm{GL}(h, F)$ and W_F). I also state a conjecture for $\mathcal{U}_{h,F}^r$: it is essentially the same, except for some dualities. Thus those two representations should be closely related, but I do not know exactly how, nor why! Note finally that those conjectures are not specially mine. They seem to have been known for a certain time by some people, mostly by Deligne and Drinfeld : for instance, the conjecture for $\mathcal{U}_{h,F}^r$ is implicitly suggested in the introduction of [Dr 3].

0.5. For $h = 1$, both conjectures constitute an easy exercise : in this case, it is easy to see that $\mathcal{U}_{1,F}^v$ and $\mathcal{U}_{1,F}^r$ coincide up to a sign, and that our conjectures are exactly equivalent to Lubin-Tate theory. In case $h = 2$, it was proved in [C] for $\mathcal{U}_{2,F}^v$ when F is a p-adic field, and an analogous proof (although never explicitely written) works for F a local field of positive characteristic. Below, I will sketch a proof of the conjecture for $\mathcal{U}^r_{2,F}$, when F is p-adic : this constitutes an alternative route, maybe easier in some sense, to reach the results of [C].

In contrast to the $h = 1$ case, the proofs for $h = 2$ are of global nature, using a comparison between the local fundamental representation and the cohomology of some Shimura curves (resp. Drinfeld modular curves in the geometric case). My last aim is to convince you that for $h \geq 3$ things should probably work in a similar way, now using the cohomology of Shimura varieties associated to some unitary groups (resp. Drinfeld modular varieties). The expected proof of the conjecture for $\mathcal{U}^r_{h,F}$ ($h \geq 3$, F p-adic) should use a higher-dimensional generalization, due to Rapoport, of Čerednik's theorem.

Notation. We denote by F a local non-archimedean field, and we choose a separable closure $\overline{F}$; we write F^{nr} for the maximal unramified extension of F (inside $\overline{F}$) and $\widehat{F}^{nr}$ (resp. $\widehat{\overline{F}}$) for the respective completions of F^{nr} and $\overline{F}$. The respective rings of integers are written :

$$\begin{array}{ccccc} \mathcal{O} \subset & \mathcal{O}^{nr} & \subset & \overline{\mathcal{O}} \\ & \cap & & \cap \\ & \widehat{\mathcal{O}}^{nr} & \subset & \widehat{\overline{\mathcal{O}}} \end{array}$$

We call k the residue field of F, p its characteristic and q its cardinality. The residue field $\overline{k}$ of F^{nr} (or $\widehat{F}^{nr}$) is an algebraic closure of k.

Finally, we choose an uniformizing parameter $\tilde{\omega} \in \mathcal{O}$ (this choice will only play an auxiliary role).

1. Definition of the local representations : the vanishing cycle side

1.1. Our representation $\mathcal{U}^v_{h,F}$ is constructed by considering *deformations of formal $\mathcal{O}$-modules*, and I first recall (after Drinfeld [Dr 1]) some definitions and results.

Let A be any $\mathcal{O}$-algebra. Then a (one dimensional) *formal $\mathcal{O}$-module over A* is a couple constituted of a (one dimensional) formal group over A together with an action of $\mathcal{O}$ on it (1 acting as the identity), such that the derived action on the Lie algebra coincides with the structural morphism $\mathcal{O} \to A$.

The theory of those objects is a straighforward generalization of the theory of formal groups (which we recover when $F = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$). To begin with, one looks at formal $\mathcal{O}$-modules over $\overline{k}$. The action of the uniformizer $\tilde{\omega}$ is expressed by a formal power series with coefficients in $\overline{k}$, and it is easy to see that the order of this series is a power q^h of q, or else infinite. The integer h so defined (we shall always suppose in the sequel that this order is finite) is by definition the *height* of our formal module (this does not coincide with the height of the underlying formal group). It turns out that formal $\mathcal{O}$-modules over $\overline{k}$ are classified by their height : for any fixed $h \geq 1$, there exists exactly one (up to isomorphism) $\mathcal{O}$-module of height h. We will write Ψ_h for such a module.

Moreover, it can be shown that the ring of endomorphisms of Ψ_h (as a formal $\mathcal{O}$-module) is isomorphic to the maximal order $\mathcal{O}(B_{h,F})$ of the skew-field $B_{h,F}$ over F with invariant $1/h$. We fix for the sequel such an isomorphism. (Note : while F and h are fixed, we shall often drop these indices, in order to simplify notations).

1.2. Deformation theory. It was also proved by Drinfeld that the functor of deformations of $\Psi = \Psi_h$ (over complete local $\widehat{\mathcal{O}}^{nr}$-algebras with residue field $\overline{k}$) was *representable* by a ring D_0, isomorphic to the ring of formal power series in $h-1$ variables over $\widehat{\mathcal{O}}^{nr}$. Call X_0 the spectrum of D_0 : this scheme is equipped with a formal $\mathcal{O}$-module $\widetilde{\Psi}$ (universal deformation). The group $\mathcal{O}(B)^*$ of automorphisms of Ψ acts on X_0 and $\widetilde{\Psi}$.

Now consider, for any integer $n \geq 1$, the group $\widetilde{\Psi}_n$ of $\tilde{\omega}^n$-torsion points in $\widetilde{\Psi}$. Over the generic fiber $X_{0,\eta} = X_0 \otimes_{\widehat{\mathcal{O}}^{nr}} \widehat{F}^{nr}$, this is an étale $\mathcal{O}$-module locally isomorphic to $(\tilde{\omega}^{-n}\mathcal{O}/\mathcal{O})^h$. Then we define an *étale Galois covering* $X_{n,\eta}$ of $X_{0,\eta}$, as the classifying space for isomorphisms :

$$(\tilde{\omega}^{-n}\mathcal{O}/\mathcal{O})^h \xrightarrow[\alpha]{\sim} \widetilde{\Psi}_{n,\eta}$$

(such an isomorphism is usually called a "level n structure".)

This covering has Galois group $\mathrm{GL}(h, \mathcal{O}/\tilde{\omega}^n\mathcal{O})$; it can be extended, using the concept of "Drinfeld basis" (cf. [Dr 1]), to a flat covering X_n of the whole X_0. As n increases, the various X_n constitute a projective system in an obvious way.

1.3. Vanishing cycles. We are interested in the vanishing cycles cohomology groups :

$$H^{h-1}(X_{n,\eta} \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_l) \quad (\text{with } l \neq p)$$

They constitute, for increasing n, an injective system with injective transition maps. We denote by $\mathcal{H}^v$ the injective limit. On each of these cohomology groups, and hence on the limit, we have obvious actions of the three groups $\mathrm{GL}(h,\mathcal{O})$, $\mathcal{O}(B)^*$ and $\mathrm{Gal}(\overline{F}/F^{nr})$ (the inertia group). These actions commute with each other, and thus we get an action on $\mathcal{H}^v$ of the product group $\mathrm{GL}(h,\mathcal{O}) \times \mathcal{O}(B)^* \times \mathrm{Gal}(\overline{F}/F^{nr})$.

1.4. My next objective is to extend the above action to a bigger subgroup P of the product $\mathrm{GL}(h,F) \times B^* \times W_F$. This subgroup is defined as the kernel of the homomorphism :

$$\begin{aligned} \mathrm{GL}(h,F) \times B^* \times W_F &\longrightarrow \mathbb{Z} \\ (g,b,w) &\longrightarrow \mathrm{val}[\det(g)^{-1}\nu(b)cl(w)] \end{aligned}$$

(where ν denotes the reduced norm and $cl : W_F \to F^*$ the local class field homomorphism, normalized in such a way that geometric Frobeniuses go to uniformizing parameters; as usual, val is the normalized valuation of F).

The extended action of the group P on $\mathcal{H}^v$ will be trivial on central elements of the form $(z,z,1)$ with $z \in F^*$. I will confine myself to a description of the action of the following elements, which generate the group P (together with those of the form $(z,z,1)$) :

$$(a) : (g^{-1}, b^{-1}, 1) \text{ with : } g \in M_h(\mathcal{O}).$$
$$[\text{then : val det } g = \text{val } \nu(b) = m \geq 0]$$
$$(b) : (g^{-1}, 1, \sigma^{-1}), \text{ again with : } g \in M_h(\mathcal{O}).$$
$$[\text{val det } g = \text{val } cl\ (\sigma) = m \geq 0]$$

case (a) : The (left) action of the given element on $\mathcal{H}^v$ will result from a (right) action on the projective system of the $X_{n,\eta}$. To simplify notations, I shall describe this action at the projective limit level $X_{\infty,\eta}$: on the projective limit X_∞, we still have our universal deformation $\widetilde{\Psi}$, now endowed with an infinite level structure above the generic fiber : $\alpha : (F/\mathcal{O})^h \xrightarrow{\sim} \widetilde{\Psi}_{\infty,\eta}$ (where $\widetilde{\Psi}_\infty$ denotes the torsion group of $\widetilde{\Psi}$).

Our element $g \in M_h(\mathcal{O})$ defines a surjective homomorphism :

$$(F/\mathcal{O})^h \xrightarrow{g} (F/\mathcal{O})^h.$$

Let N be its kernel (it has cardinality q^m) and put : $L = \alpha(N) \subset \widetilde{\Psi}_{\infty,\eta}$. We consider the closure $\bar{L}$ of L in $\widetilde{\Psi}_{\infty}$, and form the quotient $E = \widetilde{\Psi}/\bar{L}$. This quotient is a formal $\mathcal{O}$-module over X_∞, endowed with the obvious isogeny $\widetilde{\Psi} \xrightarrow{i} E$ of degree q^m. Moreover, an (infinite) level structure β on E_η is naturally defined in such a way that the following diagram commutes :

$$\begin{array}{ccc} \alpha : (F/\mathcal{O})^h & \xrightarrow{\sim} & \widetilde{\Psi}_{\infty,\eta} \\ g \downarrow & & i \downarrow \\ \beta : (F/\mathcal{O})^h & \xrightarrow{\sim} & E_{\infty,\eta} \end{array}$$

Finally, b defines the following isomorphism between the special fiber E_s of E and that $\widetilde{\Psi}_s = \Psi$ of $\widetilde{\Psi}$:

$$\Psi \xrightarrow{b^{-1}} \Psi \xrightarrow{i} E_s.$$

(this quasi-isogeny is an isomorphism for degree reasons).

In this way, E appears as a deformation of Ψ over X_∞, endowed with an infinite level structure. By universality, we get a map $X_{\infty,\eta} \longrightarrow X_{\infty,\eta}$ which is the required one.

Case (b). We use g exactly as above to obtain E. Then E_s appears as a deformation of the special fiber of $\widetilde{\Psi}^\sigma$, via :

$$\widetilde{\Psi}_s^\sigma \xrightarrow{F_q^{-m}} \widetilde{\Psi}_s \xrightarrow{i} E_s.$$

(here F_q stands for the $\log_p(q)$- power of the usual Frobenius isogeny). So we get a map $X_{\infty,\eta} \longrightarrow X^\sigma_{\infty,\eta}$ which induces on cohomology the required action.

1.5. I leave it as an unpleasant exercise to check that the above constructions define an action of the group P on the space $\mathcal{H}^v$. *Then we define the local fundamental representation $\mathcal{U}^v = \mathcal{U}^v_{h,F}$ to be the induced representation, from P to* $\mathrm{GL}(h, F) \times B^*_{h,F} \times W_F$, *of* $\mathcal{H}^v$.

In [C], we constructed this representation (for $h = 2$ and F p-adic) in a slightly different way : using a theory of polarizations for formal modules, we constructed a formalism to define deformations of polarized formal modules that are only given "up to isogeny". This more abstract approach had the advantage of being more functorial, and consequently to allow a more natural definition of the group actions. The way we have just followed now, on the contrary, gives a quick and "down to earth" description of the representation space, but it requires some brute force to define the actions.

It is possible to proceed in general exactly as in [C]. The generalization is easy and formal, except for the existence of polarizations, which was established in [C] by global methods : those can be generalized, or else, one can use a purely local construction due to Lubin [Lu].

2. The rigid side

2.1. This second approach relies on Drinfeld's construction of a system of coverings of the p-adic upper half plane (and its generalizations), and we first recall (very briefly) some of the main features of Drinfeld's paper [Dr 3].

Ω_F^h denotes the rigid analytic space obtained by removing all rational hyperplanes from the projective space $\mathbb{P}_F^{h-1}$; the rigid structure is explained in [Dr 1]. According to Raynaud's theory, this rigid space can also be defined from some formal scheme over $\mathcal{O}$, and such a formal scheme $\widehat{\Omega}_F^h$ was effectively supplied by Deligne [De 2]. Drinfeld's basic discovery consisted in an interpretation of the formal scheme $\widehat{\Omega}_F^h \otimes_{\mathcal{O}} \widehat{\mathcal{O}}^{nr}$ as a moduli space for certain formal groups endowed with an action of the ring $\mathcal{O}(B_{h,F})$. This gives on the formal scheme a universal family Φ of such formal groups (X in Drinfeld's terminology), and we denote by Φ_n the group of $\tilde{\omega}^n$-torsion points in Φ ($= \Gamma_n$ in Drinfeld). Returning to the rigid category, those torsion groups correspond to some étale rigid coverings $\Phi_n \otimes_{\widehat{\mathcal{O}}^{nr}} \widehat{F}^{nr}$ of the rigid space $\Omega_F^h \otimes_F \widehat{F}^{nr}$ deduced from Ω_F^h by scalar extension. Finally Drinfeld considers the difference (points exactly killed by $\tilde{\omega}^n$) :

$$\Sigma_F^{h,n} \underset{\text{def}}{=} (\Phi_n \otimes_{\widehat{\mathcal{O}}^{nr}} \widehat{F}^{nr}) - (\Phi_{n-1} \otimes_{\widehat{\mathcal{O}}^{nr}} \widehat{F}^{nr}).$$

This is an étale rigid Galois covering of $\Omega_F^h \otimes_F \widehat{F}^{nr}$ with Galois group $[\mathcal{O}(B)/\tilde{\omega}^n \mathcal{O}(B)]^*$. When n increases, these coverings constitute a projective system, with transition maps given by the action of $\tilde{\omega}$.

[As in the 1^{st} paragraph, we will often drop the indices h and F.]

2.2. Group actions. There exists on the space Ω an obvious (at least from the set-theoretic point of view) action of the projective linear group $\mathrm{PGL}(h, F)$. Drinfeld considers the semilinear action on $\Omega \otimes_F \widehat{F}^{nr}$ of the product group $\mathrm{GL}(h, F) \times B^*$ obtained as the twist of the above action (factorized via $\mathrm{PGL}(h, F)$) by the following action on $\widehat{F}^{nr}$:

$$(g,b) \in \mathrm{GL}(h,F) \times B^* \longrightarrow \varphi_q^{\mathrm{val}[\det(g).\nu(b)]}$$

where φ_q denotes the "geometric" Frobenius element in the Galois group $\mathrm{Gal}(F^{nr}/F)$. It turns out that the action thus defined admits a natural modular interpretation, and consequently lifts to an action on the universal formal group Φ. It results from this fact that *the action of the group* $\mathrm{GL}(h,F) \times B^*$ *lifts to the system of coverings* Σ^n. It will be more convenient for our purposes in the sequel to convert Drinfeld's left action into a right action. In any case, one sees easily that central elements of the form (z, z^{-1}), with $z \in F^*$, act trivially.

We now introduce a subgroup P' of the product $\mathrm{GL}(h,F) \times B^* \times W_F$, analogous to the group P of the first paragraph. Now P' is defined as the kernel of the homomorphism :

$$\mathrm{GL}(h,F) \times B^* \times W_F \longrightarrow \mathbb{Z}$$

defined by

$$(g,b,w) \longrightarrow \mathrm{val}(\det(g)\nu(b)\mathrm{cl}(w)^{-1}).$$

Let $(g,b,w) \in P'$. The semi-linearity of the above action means that (g,b), acting on the right, induces an isomorphism :

$$\left[\Sigma^n \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}\right] \xrightarrow{\sim} \left[\Sigma^n \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}\right]^w.$$

2.3. Rigid étale cohomology. The construction we want to make in this paragraph relies on the existence of a good theory of étale l-adic cohomology for p-adic rigid analytic spaces ($l \neq p$), satisfying usual GAGA-type comparison theorems. Although it is easy to construct such a cohomological functor (cf. for instance [FVDP]), it seems much more difficult to prove comparison theorems (cf. (4.2) below). For the moment, I only assume the existence of a cohomology theory, with simply the minimal requirement that it should be invariant under base change on the ground field (supposed to be separably closed).

Then we consider the space :

$$\mathcal{H}^r = \varinjlim_n H^{h-1}(\Sigma^n \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_l).$$

One sees, from the discussion above, that this is a representation space for the group P'. *We define, in analogy with the first paragraph,* *the local fundamental representation* $\mathcal{U}^r = \mathcal{U}^r_{h,F}$ *to be the induced representation, from* P' *to* $\mathrm{GL}(h,F) \times B^*_{h,F} \times W_F$, *of* $\mathcal{H}^r$.

It will be convenient to use a slight variant of this definition, by considering the restriction of scalars from $\widehat{F}^{nr}$ to F of the space Σ^n. If we re-extend the scalars to $\widehat{F}^{nr}$, then we get an $\widehat{F}^{nr}$-rigid space which is the disjoint union of the $\Sigma^n \otimes_{\widehat{F}^{nr},\sigma} \widehat{F}^{nr}$, where σ varies inside the Galois group $\mathrm{Gal}(F^{nr}/F)$. This space is too big, so we rather consider the subspace of it obtained as σ only varies inside the group generated by the Frobenius φ_q : we write $\mathrm{Res}'_{\widehat{F}^{nr}/F}(\Sigma^n)$ for this subspace. It is an $\widehat{F}^{nr}$-rigid space, endowed with a descent datum to F defined only on the Weil group. It is obvious how to define "extension of scalars" to $\widehat{\overline{F}}$ for such a structure, and to obtain on the cohomology an action of W_F. Thus we can now define the local fundamental representation as follows :

$$\mathcal{U}^r = \varinjlim_n H^{h-1}(\mathrm{Res}'_{\widehat{F}^{nr}/F}(\Sigma^n) \otimes_F \widehat{\overline{F}}, \overline{\mathbb{Q}}_l).$$

3. Statement of the conjectures

3.1. Jacquet-Langlands correspondence. Let us write $\mathfrak{A}(h,F)$ for the set of equivalence classes of admissible irreducible representations of the group $\mathrm{GL}(h,F)$, and $\mathfrak{A}^d(h,F)$ for the subset of those which are *essentially square integrable (or discrete)* : that means that the coefficients are square integrable modulo center. Inside $\mathfrak{A}^d(h,F)$ lies the smaller subset $\mathfrak{A}^0(h,F)$ of *cuspidal* representations (it is in some sense the main and more interesting part of $\mathfrak{A}^d$). The difference $\mathfrak{A}^d - \mathfrak{A}^0$ contains the *special representations* (which constitute a single class modulo twisting), and nothing else if h is prime. On the contrary, if h is composite, we also have "generalized special representations", associated to cuspidal representations of $\mathrm{GL}(h_1,F)$ for h_1 a proper divisor of h.

On the other hand, let $\widehat{B}^*_{h,F}$ denote the set of all irreducible admissible representations of the group $B^*_{h,F}$ (those representations are finite dimensional). The following theorem is due to Jacquet-Langlands when $h=2$, and to Bernstein-Deligne-Kazhdan-Vigneras ([BDKV]) and Rogawski ([R]) in general :

THEOREM. *There exists a bijection*

$$\begin{aligned}\mathfrak{A}^d(h,F) &\longrightarrow \widehat{B}^*_{h,F}\\ \pi &\longmapsto j(\pi)\end{aligned}$$

characterized by the property that, on regular elliptic elements, the characters of π *and* $j(\pi)$ *coincide up to the* $(-1)^{h-1}$ *sign.*

In fact, this theorem makes sense not only for $\mathbb{C}$-representations, but also *over any algebraically closed field of zero characteristic* (and we shall use it in the sequel for $\overline{\mathbb{Q}}_l$-representations). To see that, it is enough to prove that the notion of an essentially square integrable representation is "algebraic", i.e. invariant under $\mathbb{C}$-automorphisms : this is obvious for cuspidal representations; for the other ones, this results from the cuspidal case and Zelevinski's classification of discrete series for GL_h. Clozel knows another (unpublished) proof, which is valid for any reductive p-adic group.

3.2. Langlands local conjecture. We now write $\mathfrak{G}(h,F)$ for the set of equivalence classes of φ-semisimple h-dimensional representations of the Weil-Deligne group WD_F (cf. [T]). When the field of definition is an l-adic field ($l \neq p$), it is well-known how to interpret those representations as continuous representations of the Weil group. Let $\mathfrak{G}^0$ (resp. $\mathfrak{G}^{\mathrm{in}}$) be the subset of irreductible (resp. indecomposable) representations.

Langlands conjectures *the existence of a bijection* (we are assuming at the moment that our base field is $\mathbb{C}$) :

$$\begin{aligned}\mathfrak{A}(h,F) &\longrightarrow \mathfrak{G}(h,F)\\ \pi &\longmapsto \mathcal{L}(\pi).\end{aligned}$$

This bijection should be characterized by equalities of L and ε factors on both sides, for all possible twists, or even (if $h > 3$) for pairs of representations. It should also restrict to bijections :

$$\begin{aligned}\mathfrak{A}^d &\longrightarrow \mathfrak{G}^{\mathrm{in}},\\ \mathfrak{A}^0 &\longrightarrow \mathfrak{G}^0\end{aligned}$$

And in fact the existence of Langlands correspondence is essentially equivalent to the existence (for all h) of this last bijection between $\mathfrak{A}^0$ and $\mathfrak{G}^0$. This conjecture has been proved for $h = 2$ (Kutzko [K]), $h = 3$ (Henniart [H1]), and numerically in all cases (Henniart [H2]).

We now choose a different normalization for the Langlands correspondence (assumed to exist) : the "Hecke" correspondence differs from Langlands' by a contragredient, followed by a twist :

$$\mathfrak{H}(\pi) \underset{\text{def}}{=} \mathcal{L}(\pi^\vee \otimes |\ |^{\frac{1-h}{2}}).$$

(where $|\ |$ denotes the normalized absolute value on F).

The advantage of that new correspondence should be its invariance under automorphisms of the field of definition. As a consequence, it should make sense over any algebraically closed field of zero characteristic, in particular over $\overline{\mathbb{Q}}_l$. This property can be proved at least for $h = 1, 2, 3$.

3.3. Conjectures. These conjectures predict a decomposition of the representations $\mathcal{U}^v$ (resp. $\mathcal{U}^r$). It is convenient to begin with a fixed (arbitrary) smooth quasi-character χ (with values in $\overline{\mathbb{Q}}_l^*$) of F^*. Then we write $\mathcal{U}^v(\chi)$ (resp. $\mathcal{U}^r(\chi)$) for the subspace of $\mathcal{U}^v$ (resp. $\mathcal{U}^r$) where the center F^* of $\mathrm{GL}(h, F)$ acts as χ.

Conjecture (vanishing cycle side). The representation $\mathcal{U}^v(\chi)$ of the group $\mathrm{GL}(h, F) \times B_{h,F}^* \times W_F$ decomposes as the direct sum :

$$\mathcal{U}^v(\chi) = \bigoplus_{\pi \in \mathfrak{A}^d(\chi)} \pi \otimes j(\pi)^\vee \otimes \mathfrak{H}(\pi)',$$

where π varies through the set $\mathfrak{A}^d(\chi)$ of discrete representations of $\mathrm{GL}(h, F)$ with central character χ. The representation $\mathfrak{H}(\pi)'$ is the unique irreducible quotient of $\mathfrak{H}(\pi)$, which is $\mathfrak{H}(\pi)$ itself if π is cuspidal.

The conjecture for $\mathcal{U}^r$ is almost the same, except for the contragredient on the second factor.

CONJECTURE (RIGID SIDE). *The representation $\mathcal{U}^r(\chi)$ decomposes as the direct sum :*

$$\mathcal{U}^r(\chi) = \bigoplus_{\pi \in \mathfrak{A}^d(\chi)} \pi \otimes j(\pi) \otimes \mathfrak{H}(\pi)',$$

with the same notations as before.

Remark. Maybe I have been a little bit rash in the above rule predicting $\mathfrak{H}(\pi)'$ for $\pi \in \mathfrak{A}^d - \mathfrak{A}^0$. That could be specially the case, when h is composite, for generalized special series. Thus there is a possibility that the given rule should have to be modified for discrete non-cuspidal series. In this case, the required modification would not necessarily be the same in both conjectures.

3.4. The $h = 1$ case : Lubin-Tate theory. In this case, both conjectures are true, and easily reduced to Lubin-Tate theory; this is only a matter of unravelling definitions, with the only true difficulties lying in sign questions.

Let us look first at the vanishing cycle side : with the notations of the first paragraph, the ring D_0 is $\widehat{\mathcal{O}}^{nr}$. Our universal deformation $\widetilde{\Psi}$ is nothing else but the restriction to $\widehat{\mathcal{O}}^{nr}$ of any Lubin-Tate group. Applying Lubin-Tate theory, one sees that the covering X_n defined in §1 coincides with the spectrum of the completion $\widehat{\mathcal{O}}^n$ of the abelian extension of F^{nr} corresponding to the subgroup $(1 + \tilde{\omega}^n\mathcal{O})^* \subset \mathcal{O}^*$. Extending the scalars to $\widehat{\overline{F}}$, then one gets a finite set isomorphic to $(\mathcal{O}/\tilde{\omega}^n\mathcal{O})^*$, where the three "right" actions of $\mathrm{GL}(1,\mathcal{O}) = \mathcal{O}^*$, $\mathcal{O}(B)^* = \mathcal{O}^*$ and $\mathrm{Gal}(\overline{F}/F^{nr})$ are respectively :

$$
\begin{array}{lll}
(1) & g \longmapsto & \text{multiplication by } g. \\
(2) & b \longmapsto & \text{multiplication by } b^{-1}. \\
(3) & w \longmapsto & \text{multiplication by } \mathrm{cl}(w)^{-1}.
\end{array}
$$

The cohomology group $H^0(X_{n,\eta} \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_l)$ is then isomorphic to the set of functions from $(\mathcal{O}/\tilde{\omega}^n\mathcal{O})^*$ to $\overline{\mathbb{Q}}_l$. Going to the limit, our representation $\mathcal{H}^v$ of the group P is then isomorphic to the space of locally constant functions $\mathcal{O}^* \longrightarrow \overline{\mathbb{Q}}_l$, with the above actions. Further, it is easy to check that elements of the form $(\tilde{\omega}, \tilde{\omega}, 1)$ or $(\tilde{\omega}, 1, \tilde{\omega})$ act trivially, and consequently P acts through the homomorphism :

$$
\begin{array}{ccc}
P & \longrightarrow & \mathcal{O}^* \\
(g, b, w) & \longmapsto & gb^{-1}\mathrm{cl}(w)^{-1}
\end{array}
$$

by multiplication on the variable. Then it is an immediate exercise to check that the induced representation $\mathcal{U}^v$ is isomorphic to the space of locally constant functions $F^* \longrightarrow \overline{\mathbb{Q}}_l$, with the product group $F^* \times F^* \times W_F$ acting (by multiplication on the variable) through the homomorphism :

$$
\begin{array}{ccc}
F^* \times F^* \times W_F & \longrightarrow & F^* \\
(g, b, w) & \longmapsto & gb^{-1}\mathrm{cl}(w)^{-1}.
\end{array}
$$

The conjecture follows.

On the rigid side, it turns out that $\widehat{\Omega}^1$ is isomorphic to the formal spectrum of $\widehat{\mathcal{O}}^{nr}$, with again the Lubin-Tate group as the universal formal group Φ. The systems of coverings $\Sigma^n \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}$ thus coincides with the system $X_{n,\eta} \otimes_{\widehat{F}^{nr}} \widehat{\overline{F}}$ and so we get the same representation as before, with some sign changes in the group actions.

3.5. For $h = 2$ and F a p-adic field, the conjecture relative to $\mathcal{U}^v_{2,F}$ was proved in $[C]$. It seems clear to me that the same proof (although never written) should also work for F a local field of equal characteristics, with Drinfeld modular curves replacing Shimura curves. This proof should be even shorter, as it suffices to show how the global correspondence (between automorphic forms on function fields and Galois representations) can be computed, at bad (discrete) places, by means of the local fundamental representation. No further (base change) arguments are required, since the behaviour at bad primes of compatible systems of Galois representations over function fields is a priori controlled by Grothendieck's theory of L-functions. On the other hand, this approach does not produce any new result of a global nature, as it does in the arithmetic case, but merely a proof of the local conjecture.

Consider now the rigid side. When F is a p-adic field, the conjecture for $\mathcal{U}^r_{2,F}$ can be proved, using Čerednik's theorem for Shimura curves. In the next paragraph, I will sketch this proof for $F = \mathbb{Q}_p$. This restriction is essentially for notational convenience, except for the fact that Čerednik's theorem in Drinfeld's style has never been written for number fields but $\mathbb{Q}$: however, it is known to specialists how to deduce it from [Dr 3]. Finally, when F is of equal characteristics, there also exists a Čerednik-Drinfeld type theorem, relating the (global) coverings constructed in [Dr 2] to the (local) coverings of the p-adic upper half plane (this results from a letter Drinfeld wrote to me some years ago). Using this fact, one can probably also prove the conjecture for $\mathcal{U}^r_{2,F}$ when F is of equal characteristics.

4. Proof of the conjecture for $\mathcal{U}^r_{2,\mathbb{Q}_p}$ (sketch)

4.1. We begin with slightly modifying our notations : we will now write B for a global quaternion algebra over $\mathbb{Q}$ that splits at ∞ but not at p; thus its completion $B_p = B \otimes \mathbb{Q}_p$ at p is "the" quaternion skew field over $\mathbb{Q}_p$, which was denoted $B_{2,\mathbb{Q}_p}$ in the preceeding paragraphs. We call $\overline{B}$ the quaternion algebra obtained by interchanging the local invariants at p and ∞ : so $\overline{B}_p$ is isomorphic to $M_2(\mathbb{Q}_p)$, and $\overline{B}_\infty$ to the skew field of Hamilton quaternions; everywhere else $\overline{B}$ has the same (unspecified) invariants as B.

Let G (resp. $\overline{G}$) be the reductive group over $\mathbb{Q}$ defined by the multiplicative group of B (resp. $\overline{B}$). Inside the group $G(\mathbb{A}_f) = (B \otimes \mathbb{A}_f)^*$ of points with values in the finite adèles, we will consider open-compact subgroups K of the following form :

$$K = K_p^n K^p$$

where $K_p^n \subset B_p^* = G(\mathbb{Q}_p)$ is the group of those units in $\mathcal{O}(B_p)^*$ which are congruent to 1 modulo $\tilde{\omega}^n$, and K^p is any (open compact) subgroup of $G(\mathbb{A}_f^p)$; here we denote, as usual, by $\mathbb{A}_f^p$ the ring of finite adèles without the p-component.

The corresponding Shimura curve is a complete curve S_K, defined over $\mathbb{Q}$, whose set of complex points is given by :

$$S_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (H^{\pm} \times G(\mathbb{A}_f)/K),$$

where $H^{\pm}$ denotes the "double" Poincaré half plane $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. The quotient above is nothing else but a finite union of quotients $\Gamma\backslash H$ of the Poincaré upper half-plane by arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$.

We fix an isomorphism $G(\mathbb{A}_f^p) \simeq \overline{G}(\mathbb{A}_f^p)$, coming from an algebra *anti-isomorphism*: $B \otimes \mathbb{A}_f^p \simeq \overline{B} \otimes \mathbb{A}_f^p$. We shall write $\overline{K}^p$ for the image of K^p under this isomorphism. On the other hand, we also fix an isomorphism $\overline{G}(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$ obtained from an algebra *isomorphism* $\overline{B} \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$.

Then the Čerednik-Drinfeld theorem gives the following rigid-analytic description of the curve $S_K \otimes \mathbb{Q}_p$:

$$(S_K \otimes \mathbb{Q}_p)_{an} \simeq (\Sigma^n \times X_{\overline{K}^p})/\mathrm{GL}_2(\mathbb{Q}_p),$$

where Σ^n is the covering of the p-adic upper half plane, as defined in the second paragraph, and :

$$X_{\overline{K}^p} = \overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}_f)/\overline{K}^p.$$

In the above formula, Σ^n (a priori defined over $\widehat{F}^{nr}$) is viewed as defined over F by restriction of scalars. This can also be easily written, using the notations of the second paragraph :

$$(S_K \otimes \mathbb{Q}_p)_{an} \simeq (\mathrm{Res}'\Sigma^n \times X_{\overline{K}^p})/\mathrm{GL}_2(\mathbb{Q}_p).$$

(The "difference" between $Res'\Sigma^n$ and $Res\Sigma^n$ vanishes in the quotient). This quotient is nothing over F^{nr} (there are some Galois twists over F!) but a finite union of quotients Σ^n/Γ for some Schotty groups $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$. For $n = 0$, we thus get Mumford quotients, and a generalization of them when $n \geq 1$. The meaning of those quotients is easy when K^p (and so Γ) is small enough : when Γ acts sufficiently freely on the Bruhat-Tits tree, then taking the quotient amounts to glueing together affinoid pieces.

Group actions : The group $G(\mathbb{A}_f)$ (resp. $\overline{G}(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$) acts on the projective system of the S_K (resp. of the analytic quotients written above). The Drinfeld-Čerednik isomorphisms are compatible (when K varies) with the transition maps of both projective systems. They are also compatible with the $G(\mathbb{A}_f)$-actions, via the "outer" automorphism which was fixed above.

4.2. Rigid étale cohomology and algebraic étale cohomology. . Very little reference is known to me on this subject. It would be reasonable to expect, for any proper and smooth algebraic variety over $\overline{\mathbb{Q}}_p$, a canonical isomorphism between its étale l-adic ($l \neq p$) cohomology, and the cohomology of the underlying rigid analytic space. On the other hand, the cohomology of a quotient X/Γ of a rigid space by a discrete group should be computable by means of a Cartan-Leray spectral sequence :

$$H^p(\Gamma, H^q(X, \overline{\mathbb{Q}}_l)) \Longrightarrow H^{p+q}(X/\Gamma, \overline{\mathbb{Q}}_l)).$$

According to a letter that Berthelot wrote to me, those questions were studied by Gabber, who should in principle be able to prove everything we need. This, however, requires a lot of work, and no proof has been written by now.

For curves, everything is more explicit : at least the comparison theorem is known (cf. [Dr 1]), and the above spectral sequence should materialize as an exact sequence :

(∗)
$$0 \longrightarrow H^1(\Gamma, H^0(X, \overline{\mathbb{Q}}_l)) \longrightarrow H^1(X/\Gamma, \overline{\mathbb{Q}}_l) \longrightarrow H^1(X, \overline{\mathbb{Q}}_l)^\Gamma \longrightarrow 0$$

beyond of course the obvious relation $H^0(X/\Gamma, \mathbb{Q}_l) = H^0(X, \mathbb{Q}_l)^\Gamma$. The existence of this exact sequence was established in [Dr1] (cf. also [G] and [De-Hu]) in the case when $X = \Omega^2$ (i.e. $n = 0$), using explicit

Čech coverings. It can be checked-if no other reference is available-that an analogous method also works for Drinfeld's coverings Σ^n. In any case, I will assume from now on the existence and exactness of the above sequence.

4.3. Connected components. Our aim is the determination of the representation $\mathcal{U} = \mathcal{U}^r = \varinjlim H^1(\mathrm{Res}'\Sigma^n \otimes_F \widehat{\overline{F}}, \overline{\mathbb{Q}}_l)$ of the group $\mathrm{GL}_2(\mathbb{Q}_p) \times B_p^* \times W_{\mathbb{Q}_p}$. We will begin by the computation, by using a local-global comparison method, of the representation, which I call $\mathcal{U}^0$, obtained when the H^1 above is replaced by H^0. This method is the exact prefiguration of the one we will use later for the representation $\mathcal{U}$ itself.

If we apply the rule $H^0(X/\Gamma) = H^0(X)^\Gamma$ (I have dropped $\overline{\mathbb{Q}}_l$ to simplify notations) to the Čerednik-Drinfeld formula (after a trivial translation in adèlic terms), then we get the following :

$$H^0(S_K \otimes \overline{\mathbb{Q}}_p) = [H^0(\mathrm{Res}'\Sigma^n \otimes_F \widehat{\overline{F}}) \otimes H^0(X_{\overline{K}^p})]^{\mathrm{GL}_2(\mathbb{Q}_p)}$$

Here, $H^0(X_{\overline{K}^p})$ simply denotes the space of $\overline{\mathbb{Q}}_l$-valued functions on the set $X_{\overline{K}^p} = \overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}_f)/\overline{K}^p$. This set can be replaced (without changing the above formula) by the space $\overline{\mathcal{A}}_{\overline{K}^p}$ of $\overline{\mathbb{Q}}_l$-valued smooth functions on $\overline{G}(\mathbb{A}_f)$, left invariant under $\overline{G}(\mathbb{Q})$ and right invariant under $\overline{K}^p$; the limit of these spaces when $\overline{K}^p$ decreases is the space $\overline{\mathcal{A}}$ of $\overline{\mathbb{Q}}_l$-valued smooth automorphic functions on $\overline{G}(\mathbb{A})$ which are $\overline{G}(\mathbb{R})$-invariant. $\overline{\mathcal{A}}$ decomposes into the sum of all automorphic representations with trivial infinite component of the group $\overline{G}(\mathbb{A})$. Those automorphic representations are of two types :

(i) Those which factor through the norm $\overline{G}(\mathbb{A}) \longrightarrow \mathbb{A}^*$.

(ii) The "true" ones, which are infinite dimensional.

Going to the limit in the above formula, one gets :

$$\varinjlim_K H^0(S_K \otimes \overline{\mathbb{Q}}_p) = [\mathcal{U}^0 \otimes \overline{\mathcal{A}}]^{\mathrm{GL}_2(\mathbb{Q}_p)}.$$

Or else, if we decompose $\overline{\mathcal{A}}$ into the sum $\bigoplus_{\overline{\pi}} \overline{\pi}$ of all automorphic representations with trivial infinite component, and if we notice that $[\mathcal{U}^0 \otimes \overline{\pi}_p]^{\mathrm{GL}_2(\mathbb{Q}_p)}$ is isomorphic to the isotypic component $\mathcal{U}^0(\overline{\pi}_p^\vee)$ for the contragredient $\overline{\pi}_p^\vee$, we get :

$$\varinjlim H^0(S_K \otimes \overline{\mathbb{Q}}_p) = \bigoplus_{\overline{\pi}} \Big[\mathcal{U}^0(\overline{\pi}_p^\vee) \otimes \otimes_{q \neq p} \overline{\pi}_q\Big].$$

But now, the left-hand term above is given by the "reciprocity law", which describes the set of connected components of any Shimura variety. In our very simple situation, this rule says that the representation of the group $G(\mathbb{A}_f) \times W_{\mathbb{Q}_p}$ on $\varinjlim H^0(S_K \otimes \overline{\mathbb{Q}}_p)$ decomposes as the sum :

$$\bigoplus_{\rho} \rho_f \otimes \mathfrak{H}(\rho_p),$$

where ρ ranges over the set of those automorphic representations of $G(\mathbb{A})$ which factor through the norm and with trivial infinite component, and $\mathfrak{H}$ denotes the Hecke correspondence for GL(1). Comparing with the preceeding expression, one gets :

$$\mathcal{U}^0(\overline{\pi}_p^\vee) = \begin{cases} 0 \quad \text{if } \overline{\pi}_p \text{ is infinite dimensional.} \\ \rho_p \otimes \mathfrak{H}(\rho_p) \text{ if } \overline{\pi}_p = \rho_p \text{ is one-dimensional.} \end{cases}$$

[There is a change of sign due to the fact that the chosen isomorphism between $G(\mathbb{A}_f^p)$ and $\overline{G}(\mathbb{A}_f^p)$ induces the inverse map at the norm level.]

At this stage, one has to be a little bit careful because the formula just written is a priori only valid when $\overline{\pi}_p$ is the local component of an automorphic representation with trivial infinite component : that is, when its central character is of finite order. But it is easy to prove that the representation $\mathcal{U}^0$ is invariant under twisting by those characters of the group $\mathrm{GL}_2(\mathbb{Q}_p) \times B_p^* \times W_{\mathbb{Q}_p}$ which can be factored via : $(g, b, w) \longrightarrow \mathrm{val}(\det(g).\nu(b).\mathrm{cl}(w)^{-1})$; that results from the fact $\mathcal{U}^0$ (like $\mathcal{U}$ itself) is induced from the subgroup P'. As a result, the above formula is valid in all cases. If we denote (as for $\mathcal{U}$) $\mathcal{U}^0(\chi)$ the subspace of $\mathcal{U}^0$ where the center of GL_2 acts via χ, then we get :

$$\mathcal{U}^0 = \bigoplus \mu \otimes \mu \otimes \mathfrak{H}(\mu),$$

where the sum is extended to the characters μ of $\mathbb{Q}_p^*$ such that $\mu^2 = \chi$ (such a character is viewed via the determinant -resp. the norm- as a character of GL_2 - resp. B_p^*).

This representation-theoretic formula can easily be translated in more practical terms : it means that the set $\varprojlim \pi_0[\mathrm{Res}'\ \Sigma^n \otimes_F \widehat{\overline{F}}]$

is isomorphic to $\mathbb{Q}_p^*$, with $\mathrm{GL}_2(\mathbb{Q}_p)$ and B_p^* acting through the norm and W_F through the inverse of the class-field homomorphism.

4.4. We now want to use the exact sequence $(*)$ in order to compare the local and global H^1. Our first task is to evaluate the left hand side. For that, one works essentially along the same lines as in [Dr 1]. Writing $\mathcal{T}$ for the tree associated to $\mathrm{PGL}_2(\mathbb{Q}_p)$, one has, for any Schotty group Γ :

$$H^1(\Gamma, \overline{\mathbb{Q}}_l) = H^1(\mathcal{T}/\Gamma, \overline{\mathbb{Q}}_l).$$

We apply this to the Čerednik-Drinfeld formula, and we get the (projective limit of the) left hand side :

$$\mathrm{LHS} = \varinjlim H^1(\mathcal{T} \times \mathbb{Q}_p^* \times X_{\overline{K}_p}/\mathrm{GL}_2(\mathbb{Q}_p))$$

This is essentially the same formula as in [Dr 1], except for the term $\mathbb{Q}_p^*$, coming from the fact that our coverings are not absolutely connected, contrarily to Ω^2 itself. Reasoning like in [Dr 1], one sees that LHS identifies to the space of $\mathrm{GL}_2(\mathbb{Q}_p)$-invariants inside the tensor product $Z \otimes \mathcal{U}^0 \otimes \overline{\mathcal{A}}$, where Z denotes the space of harmonic 1-cochains on $\mathcal{T}$: this space Z is a realization of the (dual of) the special representation Sp of the group $\mathrm{GL}_2(\mathbb{Q}_p)$.

Using the decomposition of $\overline{\mathcal{A}}$ into automorphic representations and the decomposition of $\mathcal{U}^0$, one finds immediately :

$$\mathrm{LHS} = \bigoplus_{\substack{\overline{\pi} \\ \overline{\pi}_\infty = 1; \overline{\pi}_p = \mu^{-1} Sp}} (\otimes_{q \neq p} \overline{\pi}_q) \otimes \mu \otimes \mathfrak{H}(\mu),$$

where the sum is extended to the set of automorphic representations of $\overline{G}(\mathbb{A})$ with trivial infinite component and p-component isomorphic to a twist $\mu^{-1}Sp$ of the special representation. Via the Jacquet-Langlands global correspondence, such an automorphic representation corresponds to an automorphic representation π of $G(\mathbb{A})$ with p-component μ^{-1} and infinite component the discrete series $D_2 = D_{2,0}$ (with the notation of [C]) of $\mathrm{GL}_2(\mathbb{R})$; everywhere else π has the same factors as $\overline{\pi}$. Using this, the LHS can be rewritten as :

$$\mathrm{LHS} = \bigoplus_{\substack{\pi \\ \pi_\infty = D_2; \pi_p = \mu}} \pi_f \otimes \mathfrak{H}(\mu).$$

(Remember that the chosen isomorphism between $G(\mathbb{A}_f^p)$ and $\overline{G}(\mathbb{A}_f^p)$ is "outer" : this transforms the correspondence above, changing π_q $(q \neq p)$ into its contragredient).

4.5. We are now ready to compare the local and global H^1. First, the following decomposition of the cohomology of Shimura curves is well-known (cf. [C]) :

$$\varinjlim_K H^1(S_K \otimes \overline{\mathbf{Q}}) = \bigoplus_{\substack{\pi \\ \pi_\infty = D_2}} \pi_f \otimes \sigma(\pi),$$

where the sum is extended to the set of all automorphic representations of $G(\mathbb{A})$, with infinite component isomorphic to D_2, and $\sigma(\pi)$ stands for some two-dimensional l-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. If we write $\sigma(\pi)_q$ for the restriction of $\sigma(\pi)$ to the local Weil group $W_{\mathbf{Q}_q}$, then the Eichler-Shimura theory proves that $\sigma(\pi)_q = \mathfrak{H}(\pi_q)$ for q a "good" prime (i.e. B_q unramified and π_q spherical). The global conjecture we want to prove says that this is true for every q (the assertion at a prime q where B_q is ramified must be understood as : $\sigma(\pi)_q = \mathfrak{H}(j^{-1}(\pi_q))$). Using the result at almost all places, it is not hard to prove that at least the determinant $\det \sigma(\pi)$ is everywhere as expected.

Using the above result giving the LHS of the Cartan-Leray exact sequence, and the fact that the determinant of $\sigma(\pi)$ is known, are sees immediately that *in the case when π_p is one-dimensional, then at least the semi-simplification of $\sigma(\pi)_p$ is as predicted.* One would like to prove in fact that $\sigma(\pi)_p$ is a special Galois representation, i.e. that the extension defined by restricting the exact sequence $(*)$ to $\sigma(\pi)_p$ is *not split.* A similar phenomenon occurs on the vanishing cycle side, and in this case the non-splitting was proved by Langlands ([La]) (using the Picard-Lefschetz formula). In our present situation, we can either use directly Langland's result (because both questions are equivalent from the global point of view), or else give a similar argument on the rigid-analytic side : the required expression for the variation can be extracted from [M-D].

Taking the "difference" between the above formula and the one for the LHS, one gets the following expression for the right hand side :

$$\mathrm{RHS} = \bigoplus_{\substack{\pi \\ \pi_\infty = D_2}} \pi_f \otimes \sigma(\pi)'_p,$$

with

$$\sigma(\pi)'_p = \begin{cases} \sigma(\pi)_p \text{ if } \pi_p \text{ is not 1-dimensional.} \\ \mathfrak{H}(\mu||^{-1}) \text{ if } \pi_p = \mu. \end{cases}$$

On the other hand, this RHS is also the space of $\mathrm{GL}_2(\mathbb{Q}_p)$-invariants in the tensor product $\mathcal{U} \otimes \overline{\mathcal{A}}$ (cf. 4.3) and we get, by decomposing $\overline{\mathcal{A}}$:

$$\mathrm{RHS} = \bigoplus_{\substack{\overline{\pi} \\ \overline{\pi}_\infty = 1}} \Big[\bigotimes_{q \neq p} \overline{\pi}_q \otimes \mathcal{U}(\overline{\pi}_p^\vee) \Big],$$

where the sum is extended to the set of all automorphic representations of $\overline{G}(\mathbb{A})$ with trivial infinite component. A comparison -using the global Jacquet-Langlands correspondence- between both expressions giving the RHS proves that our local restriction $\sigma(\pi)'_p$ only depends on the local component π_p (thus we are allowed to write $\sigma(\pi_p)'$ instead of $\sigma(\pi)'_p$), and this gives moreover :

$$\mathcal{U}(\overline{\pi}_p) = \begin{cases} 0 & \text{if } \overline{\pi}_p \text{ is a } 1-\text{dimensional or a} \\ & \text{principal series representation,} \\ j(\overline{\pi}_p) \otimes \sigma(j(\overline{\pi}_p))' & \text{otherwise,} \end{cases}$$

where $\overline{\pi}_p$ varies through the set of all admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ whose central character is of finite order.

It remains to prove the equality : $\sigma(j(\overline{\pi}_p))' = \mathfrak{H}(\overline{\pi}_p)'$. This results from the above if π_p is special. If π_p is ordinary cuspidal (i.e. comes from a character of the multiplicative group of a quadratic extension of $\mathbb{Q}_p$) then it is possible to compute $\sigma(j(\overline{\pi}_p))'$ from the global theory because we can then assume that π itself comes from an idèle class character of a quadratic extension of $\mathbb{Q}$. This gives $\sigma(j(\overline{\pi}_p))'$ whenever $\overline{\pi}_p$ is ordinary cuspidal, for instance for $p \neq 2$. Returning to the formula above, one obtains the predicted decomposition of $\mathcal{U}(\chi)$:

$$\mathcal{U}(\chi) = \bigoplus_{\overline{\pi}_p} \overline{\pi}_p \otimes j(\overline{\pi}_p) \otimes \mathfrak{H}(\overline{\pi}_p)'$$

where $\overline{\pi}_p$ ranges over the set of discrete admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character χ. [Note that this is a priori only valid for χ of finite order, but we can use the same argument as in 4.3.].

The case $p = 2$ requires the use of base change arguments exactly similar to those of [C], and so it is first necessary to make the above local-global comparison arguments work for all totally real fields, and not only for $\mathbb{Q}$. Another possibility is to use the global results obtained from the vanishing cycle side : these results directly imply that $\sigma(\pi)_p = \mathfrak{H}(\pi_p)$ in all cases, including $p = 2$ and π_p extraordinary cuspidal.

5. The higher dimensional case

5.1 For $h \geq 3$, similar approaches should work, by using comparisons between our local representations and global ones. In the geometric case, one should simply use moduli spaces for Drinfeld modules of rank ≥ 3. While in the arithmetic case, one should look at unitary groups over totally real fields, with archimedean type $U(h-1,1) \times U(h) \times \ldots \times U(h)$. More precisely (assume for simplicity that the base field is $\mathbb{Q}$) :

(a) The vanishing representation $\mathcal{U}^v_{h,\mathbb{Q}_p}$ should be studied in comparison with the cohomology of a Shimura variety associated to some form G of the unitary group such that $G_{\mathbb{R}} \simeq U(h-1,1)$ and $G_{\mathbb{Q}_p} \simeq \mathrm{GL}(h, \mathbb{Q}_p)$. For instance, one can take some "true" unitary group associated with a hermitian form over a quadratic field E which splits at p. It is easy to check that a one dimensional formal group is defined on the special fiber at p of such a variety, and that the height $h' \leq h$ at geometric points of this formal group "controls" the singularity type. This height h' defines a stratification of the special fiber : the "worst" singularity occurs on the stratum $h' = h$, which is a discrete set. The contribution of this stratum to the vanishing cycles is related to the local representation $\mathcal{U}^v_{h,\mathbb{Q}_p}$. For $h' < h$, the corresponding stratum is of dimension $h - h'$ and is related to $\mathcal{U}^v_{h',\mathbb{Q}_p}$.

The projective system of Shimura varieties associated to G is defined over E. If we consider the cohomology (in degree $h-1$) of those varieties, and decompose it under the action of Hecke operators, we get systems of l-adic representations of $Gal(\overline{E}/E)$. In this way, we obtain a (global) correspondence between automorphic representations of $G(\mathbb{A})$ and representations of the Galois group of E. It is expected that this correspondence is expressible, locally at p, by means of the representations $\mathcal{U}^v_{h',\mathbb{Q}_p}(h' \leq h)$ in a way similar to the GL_2-case (cf. [C]).

(b) For the rigid representation $\mathcal{U}^r_{h,\mathbb{Q}_p}$, one should rather use some form G with the same infinite type as before, but now with $G_{\mathbb{Q}_p} \simeq B^*_{h,\mathbb{Q}_p}$. Indeed, it is known in this case that the corresponding Shimura variety admits a Čerednik-Drinfeld type uniformization (by the generalized p-adic upper half plane Ω^h and its coverings $\Sigma^{h,n}$). This generalization of Čerednik's theorem was discovered by Rapoport (as far as I know, he has not written the proof by now). It has the same expression as in [4.1], where $\overline{G}$ now denotes the inner form of G which satisfies :

$$\overline{G}_{\mathbb{Q}_p} \simeq GL(h, \mathbb{Q}_p), \quad \overline{G}_{\mathbb{Q}_q} \simeq G_{\mathbb{Q}_q} \text{ (for } q \neq p), \text{ and } \overline{G}_{\mathbb{R}} \simeq U(h).$$

In both cases, similar methods as in the GL_2-case should in principle work. That should be easier on the rigid side, because the consideration of a group which is anisotropic at p cuts out the whole non-discrete spectrum : while in the first case, on the contrary, all representations occur at p, and that means that a difficult generalization of Langlands' trace formula methods [La] is needed : in the spectral sequence of vanishing cycles, we must indeed take care of the contribution of all the strata in the special fiber, and we must find the corresponding parts in the Selberg trace formula.

5.2. In conclusion of this report, let me ask some questions that must be solved in order to be able to work out the above program.

(a) *Algebro-geometric questions.* From the point of view of the rigid local representation, it is needed to establish properties of GAGA-type for the rigid étale cohomology. On the other hand, working on the vanishing cycle side would require results, in some sense analogous, allowing to replace "henselian" vanishing cycles by "formal" ones. Those results were proved for curves (using resolution of singularities) by Brylinski, in an appendix to my thesis ([C]).

(b) *"Usual" questions on Shimura varieties.* That means computing at good places the Galois representations associated to automorphic forms. For the varieties considered here, that could be now within reach, thanks to Kottwitz' results on the structure of the set of mod p points; note that we can always choose our groups in such a way that the variety should be proper, and the problems of L-indistinguishability should be empty.

(c) *Automorphic questions.* One essential tool for the local-global comparison in case $h \geq 3$ (in the arithmetic case) would be the Jacquet-Langlands correspondence between two inner forms of the unitary group, together with base change over the quadratic extension where this unitary group becomes an inner form of GL_n. I do not know the exact state of these questions. For unitary groups in three variables, they should be solved in Rogawski's forthcoming book.

(d) The last problem might be the most difficult. The local-global comparison is expected to prove the local conjecture whenever π_p is ordinary cuspidal. In the case $h = 2$, a base-change argument then gives the answer in general (cf. [C] : in fact a non-normal cubic base change is sometimes needed). I do not know what to do in general, for instance if $h = 3$, $p = 3$.

REFERENCES

[B.D.K.V] J.N. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vigneras, *Représentations des groupes réductifs sur un corps local*, "Travaux en cours", Hermann 1984.

[C] H. Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. E.N.S. **19**, (1986), 409-468.

[De 1] P. Deligne, *Letter to Piatetskii-Shapiro* (1973).

[De 2] P. Deligne, *Description de Ω^d comme fibre générale d'un schéma formel*, unpublished manuscript.

[De - Hu] P. Deligne and D. Husemöller, *Survey of Drinfeld modules, in : Current Trends in Arithmetical Algebraic Geometry*, (K. Ribet, Editor), Contemporary Mathematics, Vol. **67** (1987), 25-91.

[Dr 1] V. G. Drinfeld, *Elliptic modules*, Math. USSR-Sb. **23**, (1974), 561-592 (English translation).

[Dr 2] V. G. Drinfeld, *Elliptic modules II*, Math. USSR-Sb. **31** (1977), 159-170 (English transl.)

[Dr 3] V. G. Drinfeld, *Coverings of p-adic symmetric domains*, Funct. Anal. Appl. **10** (1976), 107-115 (English transl.).

[F.V.D.P] J. Fresnel and M. Van Der Put, *Géométrie Analytique Rigide et Applications*, Progress in Mathematics **18**, Birkhäuser 1981.

[G] E. U. Gekeler, *Drinfeld modular curves*, Springer Lecture Notes in Math., Vol. **1231** (1986).

[H1] G. Henniart, *La conjecture de Langlands locale pour* GL(3), Mém. Soc. Math. Fr. Nouv. Sér. **11/12** (1983).

[H2] G. Henniart, *La conjecture de Langlands locale numérique pour* GL(n), Ann. Sci. E.N.S., **21** (1988), 497-544.

[K] Ph. Kutzko, *The local Langlands conjecture for* GL(2), Ann. of Math. **112**, (1980), 381-412.

[La] R. P. Langlands, *Modular forms and l-adic representations, in Modular Functions of one Variable II*, Springer Lecture Notes in Math., Vol. **349**, (1973), 361-500.

[Lu] J. Lubin, *Determinants for one dimensional formal modules*, unpublished manuscript (Brown University).

[M-D] Yu. Manin and V.G. Drinfeld, *Periods of p-adic Schotty groups*, J. für die reine und angewandte Mathematik, Band 262/263 (1973), 239-247.

[R] J. Rogawski, *Representations of* GL(n) *and division algebras over a p-adic field*, Duke Math. J. Vol. **50** (1983), 161-196.

[T] J. Tate, *Number theoretic background*, in : *Automorphic forms, Representations, and L-functions*, Proc. Symp. Pure Math., Vol. XXXIII, Part 2, (1979), 3-26.

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Automorphic Forms and the Cohomology of Vector Bundles on Shimura Varieties

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INTRODUCTION

The systematic study of the cohomology with twisted coefficients of locally symmetric varieties, and its relation with Lie algebra cohomology, was initiated by Matsushima, Murakami, and Kuga in the 1960s [**38**]. The subject was taken up again by Borel, Casselman, Kumaresan, Vogan, Wallach, Zuckerman and others in the 1970s, using methods of infinite-dimensional representation theory; the standard reference for this work is [**11**]. In the cocompact case, Hodge theory provides a ready expression for these cohomology groups in terms of automorphic forms; in general, Borel's theory of cohomology with growth conditions [**7**],[**8**] permits harmonic cusp forms, at least, to be interpreted cohomologically.

When the locally symmetric variety in question is a Shimura variety, automorphic forms can also define classes in the cohomology of certain coherent sheaves—the automorphic vector bundles discussed in Milne's talk—computed in the Zariski topology. The existence of canonical models of automorphic vector bundles introduces a new rationality principle for automorphic forms, with applications to arithmetic. This may be seen as the natural generalization of the classical theory of elliptic modular forms with algebraic Fourier coefficients.

The following abstract considerations may shed some light on the significance of this rationality principle. If M is a motive over $\mathbb{Q}$, then, following Greg Anderson, we define its *arithmetic Hodge structure* to be the triple $(H_{DR}(M), H_B(M), I : H_B(M)\mathbb{C} \tilde{\rightarrow} H_{DR}(M)\mathbb{C})$, where H_{DR} (resp. H_B) is the algebraic de Rham (resp. topological) cohomology of M with coefficients in $\mathbb{Q}$, and I is the comparison isomorphism (cf. [**13**], §0); $H_{DR}(\mathsf{M})$ is assumed to be endowed with its $\mathbb{Q}$-rational Hodge filtration. Now let Sh_i, $i = 1, 2$, be Shimura varieties, V_i^{∇} a flat automorphic vector bundle (cf. §1) over Sh_i and M_i a motive over $\mathbb{Q}$ occuring in the cuspidal cohomology of Sh_i with

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coefficients in V_i^∇, $i = 1, 2$. We assume furthermore (this may be unnecessary) that M_i is defined as an eigenspace for the Hecke operators of the group attached to Sh_i at almost all places. Then standard conjectures, most importantly the Tate conjecture (cf. Ramakrishnan's article in this volume) suggest the following

PRINCIPLE. *Suppose the ℓ-adic cohomology groups $H_\ell(M_1) \tilde{\rightarrow} H_\ell(M_2)$ as $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-modules for all ℓ. Then the arithmetic Hodge structures attached to M_1 and M_2 are isomorphic.*

The rationality principle discussed in these notes is (as we indicate in 4.3.2) related to the natural rational structure on $H_{DR}(M)$. The results described in 7.1 and 7.2.4 may be viewed as a attempts to verify the above principle in some specific cases.

It may reasonably be asked to what extent the converse of the above principle also is valid: i.e., to what extent the periods of a motive determine its ℓ-adic representations.

This survey is primarily an exposition of recent results on the coherent cohomology classes defined by automorphic forms. These results are contained in the author's article [**28**] and in joint work with Phong [**31**], Blasius-Ramakrishnan [**6**], and Kudla [**30**]. In order to emphasize the parallel with the earlier work on cohomology with twisted coefficients, we have included a review of this theory. The relation between the two cohomology theories is worked out in §4, using Faltings' idea of the B-G-G resolution of local systems on Shimura varieties [**18**]. The results in the last part of §4 are new, as are the results on Eisenstein cohomology and the periods of Hilbert modular forms, described in §6 and 7.1, respectively.

Discussions with Arthur, Blasius, Garrett, Kudla, and Ramakrishnan were helpful in the preparation of this manuscript. I also thank Borel for comments which led to clearing up some confusing points in §1.

NOTATION AND CONVENTIONS

By $\mathbf{A}$ (resp. $\mathbf{A}^f$) we mean the ring of rational adèles (resp. of rational finite adèles). The group schemes $GL(n)$ and $\mathbb{G}_m$ are denoted as usual. By $\bar{\mathbb{Q}}$ we always mean the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

If V and T are schemes over the scheme S, then $V(T)$ denotes the set of T-valued points of V; $V_T = V \times_S T$. If T is $\mathrm{Spec}(A)$ for some ring A, we often write $V(A)$ and V_A in place of $V(T)$ and V_T. If $S = \mathrm{Spec}\, k'$, where k' is a finite field extension of the field k, then

$R_{k'/k}V$ is the scheme over k obtained by Weil's restriction of scalars functor. The structure sheaf of V is denoted $\mathcal{O}_V$.

If G is an algebraic group, then G^{ad}, G^{der}, G^{ab}, and Z_G are the adjoint group, the derived subgroup, the abelianization G/G^{der}, and the center, respectively, of G. The Lie algebra of G is denoted $\mathfrak{g}$ or Lie(G); the enveloping algebra of $\mathfrak{g}$ is $U(G)$, and the center of $U(\mathfrak{g})$ is written $Z(\mathfrak{g})$. The unipotent radical of G is denoted $R_u(G)$. If G is a topological group, then G^0 is its connected component containing the identity.

If X is a C^∞-manifold and V is a complex vector space, then $C^\infty(X, V)$ is the space of C^∞ functions on X with values in V. If X is an adelic group then $C^\infty(X, V)$ is the space of V-valued functions on X which are C^∞ (resp. locally constant) in the archimedean (resp. non-archimedean) variables.

If $\mathcal{E}$ is a vector bundle over the (algebraic or analytic) variety X, then $\Gamma(X, \mathcal{E})$ is the space of global sections of $\mathcal{E}$ over X. The same notation is used for C^∞ vector bundles. We make no notational distinction between $\mathcal{E}$ and its associated locally free sheaf; in particular, if X is an algebraic variety, then $H^*(X, \mathcal{E})$ denotes cohomology of the sheaf of sections of $\mathcal{E}$ in the Zariski topology.

For Hodge structures, we use the notation of Milne's article in this volume, except that we write $\underline{S}$ instead of $\mathbb{S}$ for $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. If G is an algebraic group, and $\rho : G \to GL(V)$ is an algebraic representation, we often denote the representation (ρ, V), and use ρ and V interchangeably. If G is a topological group and V is a topological vector space, we use the same convention. If G is a reductive Lie group, $K_\infty \subset G$ an algebraic subgroup containing a maximal compact subgroup, and (π, V) is a unitary representation of G, we denote again by π or V the associated $(\mathfrak{g}, K_\infty)$ module. Here $(\mathfrak{g}, K_\infty)$ modules are defined as in [8], with the following modification: since K_∞ typically contains the center of G and is thus not compact, we require that the K_∞ -types occurring in the restriction of π to K_∞ be finite-dimensional *algebraic* representations of K_∞.

§1. De Rham cohomology of local systems with growth conditions

In this section we review some of Borel's work [4],[5],[8] on the cohomology of Shimura varieties with twisted coefficients. We remark that Borel studies local systems over general locally symmetric spaces, and that the existence of a complex structure plays no role at this stage of the theory.

1.1. Let (G,X) be the datum defining the Shimura variety $Sh = Sh(G,X)$ (see Milne's article in this volume for notation and hypotheses). In order to avoid technical complications, we assume, except when otherwise indicated, that $Z_G(\mathbb{R})/Z'_G(\mathbb{R})$ is compact, where Z_G is the center of G and Z'_G is its maximal $\mathbb{Q}$-split subtorus. For any compact open subgroup $K \subset G(\mathbf{A}^f)$, let ${}_KSh$ be the Shimura variety of level K, denoted $Sh_K(G,X)$ in Milne's article:

$$(1.1.1) \qquad {}_KSh = G(\mathbb{Q})\backslash X \times G(\mathbf{A}^f)/K, Sh = \varprojlim_K {}_KSh.$$

For $h \in X$, let K_h denote its stabilizer in $G(\mathbb{R})$. The adjoint action of $h(\underline{S})$ on the Lie algebra $\mathfrak{g}$ of G defines a Hodge structure on $\mathfrak{g}$ such that

$$(1.1.2) \qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}, \text{ where } \mathfrak{g}^{0,0} = \mathfrak{k}_{h,\mathbb{C}} = \ [\mathbf{12}].$$

We let $\mathfrak{p}^+ = \mathfrak{g}^{-1,1}, \mathfrak{p}^- = \mathfrak{g}^{1,-1}$. Then $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are invariant under the adjoint action of $\mathfrak{k}_{h,\mathbb{C}}$. We let $\mathcal{P}_h$ be the parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{P}_h = \mathfrak{k}_{h,\mathbb{C}} \oplus \mathfrak{p}^-$, and let $\check{X}$ be the flag variety of parabolic subgroups of G conjugate to $\mathcal{P}_h$. The association $h \mapsto \mathcal{P}_h$ defines a holomorphic, $G(\mathbb{R})$-equivariant imbedding $X \to \check{X}(\mathbb{C})$, the Borel imbedding (cf. Milne's article, III).

In Milne's talk it is explained how to associate to each G-homogeneous vector bundle $\mathcal{E}$ over $\check{X}$ a $G(\mathbf{A}^f)$-homogeneous automorphic vector bundle $[\mathcal{E}]$ (denoted $\mathcal{V}(\mathcal{E})$ by Milne) over Sh. The association $\mathcal{E} \mapsto [\mathcal{E}]$ is functorial and respects the tensor operations on vector bundles, and $[\mathcal{E}]$ is endowed with a canonical model over the field of definition of $\mathcal{E}$ [32,cf. 19].

Let $\mathcal{E}$ be a G-homogeneous vector bundle over $\check{X}$. For each point $h \in X$, the fiber $\mathcal{E}_h$ of $\mathcal{E}$ is a representation space for the isotropy group $\mathcal{P}_h$. The association $\mathcal{E} \mapsto \mathcal{E}_h$ defines an equivalence of categories between the category of G-homogeneous vector bundles over $\check{X}$ of finite rank and the category of finite-dimensional representations of $\mathcal{P}_h$.

In particular, any finite-dimensional representation (ρ, V) of the algebraic group G defines, by the functor of the previous paragraph, an automorphic vector bundle $\tilde{V}$ over Sh. Similarly, to any finite-dimensional representation (σ, W_σ) of the algebraic group K_h we may associate a homogeneous vector bundle $\mathcal{E}_\sigma$ over $\check{X}$, and an automorphic vector bundle $E_\sigma = [\mathcal{E}_\sigma]$ over Sh. Automorphic vector bundles of the first type are called flat; they are naturally endowed with integrable $G(\mathbf{A}^f)$-invariant connections [**19**]. Automorphic vector bundles of the second type are called fully decomposed. Holomorphic automorphic forms may be viewed as sections of fully decomposed automorphic vector bundles [19,II]. We let $k(\sigma)$ be the field of definition of $\mathcal{E}_\sigma$ as a homogeneous vector bundle. Then E_σ, together with its canonical $G(\mathbf{A}^f)$-action, has a canonical model over $k(\sigma)$. We warn the reader that the subgroup K_h of G, and *a fortiori* its representation σ, are not generally defined over $k(\sigma)$.

Let $K \subset G(\mathbf{A}^f)$ be an open compact subgroup. The bundles $\tilde{V}$ and E_σ defined above descend to automorphic vector bundles, also denoted $\tilde{V}$ and E_σ over ${}_KSh$, as explained in Milne's talk. Occasionally we denote these bundles ${}_K\tilde{V}$ and ${}_KE_\sigma$.

1.1.3. Example. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in GL(2n)$, where I_n is the $n \times n$ identity matrix, and let $G_n = GSp(n)$ be the similitude group of the alternating form defined by J. The $G_n(\mathbb{R})$ orbit of the homomorphism $h : \underline{S} \to G_{n,\mathbb{R}}$, which takes $z = x + iy \in \mathbb{C}^\times \cong \underline{S}(\mathbb{R})$ to the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in G_n(\mathbb{R})$, is analytically isomorphic to the union $\mathfrak{S}_n^\pm$ of the Siegel upper and lower half-planes of genus n. The pair $(G_n, \mathfrak{S}_n^\pm)$ defines a Shimura variety $\mathcal{M}_n = Sh(G_n, \mathfrak{S}_n^\pm)$, isomorphic to the moduli space of principally polarized abelian varieties of dimension n with a consistent family of level N structures for all positive integers N. In this case, we have

$$\mathfrak{k}_{h,\mathbb{C}} = \left\{ k(z, x_1, x_2) = \begin{pmatrix} z \cdot x_1 & x_2 \\ -x_2 & z \cdot x_1 \end{pmatrix} \;\middle|\; {}^t x_1 = -x_1,\ {}^t x_2 = x_2,\ z \in \mathbb{C} \right\},$$

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} x & -ix \\ -ix & -x \end{pmatrix} \;\middle|\; {}^t x = x \right\},\ \mathfrak{p}^+ = \left\{ \begin{pmatrix} x & ix \\ ix & -x \end{pmatrix} \;\middle|\; {}^t x = x \right\},$$

where x_1, x_2 and x are all $n \times n$ complex matrices.

1.2. To each finite-dimensional representation (ρ, V) of G we have associated a flat vector bundle $\tilde{V}$ over Sh. The sheaf V^∇ of horizontal sections of $\tilde{V}$ is a $G(\mathbf{A}^f)$-invariant local system over Sh, whose cohomology can be computed using the de Rham resolution. Thus, for $p = 0, 1, \ldots, 2n$, let $\mathcal{A}^p(V)$ denote the space of $C^\infty p$-forms on Sh with coefficients in V^∇. Here and in what follows, a space of differential forms on Sh with coefficients in a vector bundle is the direct limit over open compact subgroups $K \subset G(\mathbf{A}^f)$ of the corresponding spaces of differential forms on ${}_KSh$. Cohomology is likewise defined as a direct limit. Letting d be exterior differentiation, we have the de Rham complex

$$0 \to \mathcal{A}^0(V) \xrightarrow{d} \mathcal{A}^1(V) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{A}^{2n}(V) \to 0, \tag{1.2.1}$$

and a canonical $G(\mathbf{A}^f)$-equivariant isomorphism $H^*(Sh, V^\nabla) \cong H^*(\mathcal{A}^\cdot(V))$.

Fix a point $h \in X$; let $K_h = \mathrm{Lie}(\mathfrak{k}_h)$, and let $\mathfrak{g} = \mathfrak{k}_h \oplus \mathfrak{p}_h$ be the Cartan decomposition. Note that $\mathfrak{p}_h$ is canonically isomorphic to the tangent space of X at h. There are canonical lifting maps (cf. [**8**], VII, §2)

$$\mathcal{A}^p(V) \tilde{\to} (C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})) \otimes \Lambda^p(\mathfrak{p})^* \otimes V)^{K_h}, \tag{1.2.2}$$

the fixed vectors under the diagonal action of K_h, whose action on $C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A}))$ is given by right translation. Note that $G(\mathbf{A}^f)$ acts on both sides of (1.2.2): the action on $\mathcal{A}^p(V)$ is induced from the $G(\mathbf{A}^f)$-homogeneity of V^∇, whereas the action on the right-hand side is induced from the right action of $G(\mathbf{A}^f)$ on $C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A}))$. The isomorphism (1.2.2) is clearly $G(\mathbf{A}^f)$-equivariant.

We recall briefly the standard construction of the relative Lie algebra cohomology of $(\mathfrak{g}, K_h)$-modules. Let W be a $(\mathfrak{g}, K_h)$-module, and let

(1.2.3)
$$\begin{aligned} C^q(G, K_h, W) &= \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}_h), W) \\ &= \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}), W), \quad 0 \le q \le \dim \mathfrak{p}. \end{aligned}$$

Define $d : C^q(\mathfrak{g}, K_h, W) \to C^{q+1}(\mathfrak{g}, K_h, W)$ by the formula

(1.2.4)
$$\begin{aligned} df(x_0, \ldots, x_q) &= \sum_i (-1)^i x_i \cdot f(x_0, \ldots, \hat{x}_i, \ldots, x_q) \\ &+ \sum_{i<j} (-1)^{i+1} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q), \end{aligned}$$

where $\{x_0, \ldots, x_q\} \subset \mathfrak{p}$ and the variables wearing $\hat{}$ are omitted from the summation. Then $d^2 = 0$ and $H^*(\mathfrak{g}, K_h, W) = H*(C^{\cdot}(\mathfrak{g}, K_h, W))$.

For each p, there is an obvious isomorphism

$$(1.2.5) \quad (C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes \Lambda^p(\mathfrak{p})^* \otimes V)^{K_h} \\ \tilde{\to} C^p(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V).$$

Combining (1.2.5) and (1.2.2), we obtain isomorphisms

$$(1.2.6) \qquad \mathcal{A}^p(V) \tilde{\to} C^p(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V).$$

The isomorphisms (1.2.6) commute with the differentials on both sides [**30**], [8,VII]. In other words,

1.2.7. PROPOSITION. *([8], VII, Corollary 2.7). There is a canonical isomorphism of graded complexes*

$$\mathcal{A}^{\cdot}(V) \tilde{\to} C^{\cdot}(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V),$$

which induces canonical isomorphisms of cohomology groups

$$H^*(Sh, V^\nabla) \tilde{\to} H^*(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V).$$

These isomorphisms commute with the natural $G(\mathbf{A}^f)$-actions on both sides.

1.2.8. Remark. Note that $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ is not strictly speaking a $(\mathfrak{g}, K_h)$-module, since it is not equal to its submodule $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))_0$ of K_h-finite vectors. However, the relative Lie algebra complex only notices K_h-finite vectors; one can thus replace $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ by $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))_0$ in all the formulas without changing the cohomology. We will not dwell on this point.

1.3. When G^{der} has $\mathbb{Q}$-rank 0, the varieties ${}_KSh$ are compact, and $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ becomes a unitarizable $G(\mathbf{A})$-module (modulo the action of $Z_G(\mathbf{A})$). One can then apply Hodge theory to the Lie algebra complex $C^{\cdot}(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V)$, as in [8,esp. II and VII, §6]. The cohomology $H^*(Sh, V^\nabla)$ is then entirely represented by automorphic forms. Computation of $H^*(Sh, V^\nabla)$ is carried out in two steps:

(1) Decomposition of $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))_0$ as a $(\mathfrak{g}, K_h) \times G(\mathbf{A}^f)$-module. Say

$$C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))_0 \cong \bigoplus_{\pi_\infty, \pi_f} m(\pi_\infty \otimes \pi_f) V_{\pi_\infty} \otimes V_{\pi_f}$$

where $(\pi_\infty, V_{\pi_\infty})$ (resp. (π_f, V_{π_f}) runs through the set of unitarizable $(\mathfrak{g}, K_h)$-modules (resp. $G(\mathbf{A}^f)$-modules). It is well known that the multiplicities $m(\pi_\infty \otimes \pi_f)$ are finite.

(2) Computation of $H^*(\mathfrak{g}, K_h, V_{\pi_\infty})$ for any unitarizable $(\mathfrak{g}, K_h)$-module $(\pi_\infty, V_{\pi_\infty})$. This step is carried out completely in **[54]**.

When G has $\mathbb{Q}$-rank > 0, this approach fails, and it is not known in general to what extent $H^*(Sh, V^\nabla)$ can be represented by automorphic forms. As a partial substitute, Borel was led to introduce de Rham cohomology with growth conditions. We review the most important elements of this theory here; the analogous theory for coherent cohomology will be described in §2.

Let $G_0 = G^{der}(\mathbb{R})^0$, and let $g \mapsto \tilde{g}$ be the Cartan involution $ad(h(i))$ on G_0 with respect to K_h, and define $\|g\|_{G_0} = tr(Ad(\tilde{g}^{-1} \cdot g)), g \in G_0$. If $(V, \|\|_V)$ is a normed complex vector space, $f \in C^\infty(G(\mathbf{A}), V)$ is called *slowly increasing* (resp. *rapidly decreasing*) if

(1.3.1) f is a finite sum of eigenfunctions for $Z_G(\mathbf{A})$; and

(1.3.2) $\|f(g_0\gamma)\|_V < C\|g_0\|^m_{G_0}, \forall g_0 \in G_0, \gamma \in G(\mathbf{A})$, for some (resp for all) $m \geq 0, C \in \mathbb{R}^+$(resp. $C \in \mathbb{R}^+$ depending on m and γ).

The condition (1.3.1) is included here for convenience; it is automatically satisfied in every case of interest to us.

We let ${}_KC_{si} = {}_KC_{si}(G)$ (resp. ${}_KC_{rd} = {}_KC_{rd}(G)$) denote the space of all C^∞ functions on $G(\mathbb{Q})\backslash G(\mathbf{A})/K$ which, together with all their right $U(\mathfrak{g})$-derivatives, are slowly increasing (resp. rapidly decreasing), in the above sense. Let $C_{si} = C_{si}(G) = \varinjlim_K C_{si}, C_{rd} = C_{rd}(G) = \varinjlim_K {}_KC_{rd}$.

1.3.3. THEOREM. *(Borel [4],[5],[6]). Let (ρ, V) be a finite dimensional representation of G. The inclusion of Lie algebra complexes*

$$C^\cdot(\mathfrak{g}, K_h, C_{si} \otimes V) \subset C^\cdot(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V)$$

defines an isomorphism on cohomology. In particular, there is a natural isomorphism

$$H^*(\mathfrak{g}, K_h, C_{si} \otimes V) \xrightarrow{\sim} H^*(Sh, V^\nabla)$$

of admissible $G(\mathbf{A}^f)$-modules.

1.4. The constructions in 1.2 above also apply to cohomology with compact support. Thus, let $H_c^*(Sh, V^\nabla)$ denote the cohomology with compact support of the local system V^{∇}, and let $C_c^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ denote the space of C^∞ functions on $G(\mathbb{Q})\backslash G(\mathbf{A})$ with compact support modulo $Z_G(\mathbf{A})$. Then the complex $C^{\cdot}(\mathfrak{g}, K_h, C_c^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V)$ computes $H_c^*(Sh, V^{\nabla})$ [5], and the isomorphism

$$C^{\cdot}(\mathfrak{g}, K_h, C_c^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V)) \tilde{\to} H_c^*(Sh, V^{\nabla})$$

is $G(\mathbf{A}^f)$-equivariant.

1.4.1. THEOREM. *(Borel, [5],[6]). The inclusion of Lie algebra complexes*

$$C^{\cdot}(\mathfrak{g}, K_h, C_c^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes V)) \subset C^{\cdot}(\mathfrak{g}, K_h, C_{rd} \otimes V)$$

defines an isomorphism on cohomology. In particular, there is a natural isomorphism $H^*(\mathfrak{g}, K_h, C_{rd} \otimes V) \tilde{\to} H_c^*(Sh, V^{\nabla})$ of admissible $G(\mathbf{A}^f)$-modules.

The product of a slowly increasing function by a rapidly decreasing function is rapidly decreasing. Contraction thus defines a morphism of complexes

(1.4.2)
$$C^{\cdot}(\mathfrak{g}, K_h, {}_KC_{si} \otimes V) \otimes C^{\cdot}(\mathfrak{g}, K_h, {}_KC_{rd} \otimes V^*) \to C^{\cdot}(\mathfrak{g}, K_h, {}_KC_{rd})$$

where the double complex on the left hand side is identified with the associated single complex. In particular, (1.4.2) defines, for each $i \in \{0, \ldots, 2n\}$, a bilinear pairing

$$H^i(\mathfrak{g}, K_h, {}_KC_{si} \otimes V) \otimes H^{2n-i}(\mathfrak{g}, K_h, {}_KC_{rd} \otimes V^*) \to H^{2n}(\mathfrak{g}, K_h, {}_KC_{rd})$$

and thus, by 1.3.3 and 1.4.1, a bilinear pairing

$$(1.4.3) \qquad H^i({}_KSh, V^{\nabla}) \otimes H_c^{2n-i}({}_KSh, (V^*)^{\nabla}) \to H_c^{2n}({}_KSh, \mathbb{C})$$

We denote this pairing $\smile$. Define

$$\widetilde{Tr} : C^{2n}(\mathfrak{g}, K_h, {}_KC_{rd}) \to \mathbb{C}; \widetilde{Tr}(\omega) = (2\pi i)^{-n} \int_{G(\mathbb{Q})\backslash G(\mathbf{A})/KK_h} .$$

Note that the integral is well-defined because $K_h \supset Z_G(\mathbb{R})$. Borel proves

1.4.4. PROPOSITION. *(Borel, [5], 5.6).*

(a) The map $\widetilde{Tr}$ *factors through* $H^{2n}(\mathfrak{g}, K_h, {}_KC_{rd})$ *and defines a surjective homomorphism* $Tr : H^{2n}_c({}_KSh, \mathbb{C}) \to \mathbb{C}$.

(b) For each connected component Sh^0 *of* ${}_KSh$*, the restriction of* Tr *defines an isomorphism* $H^{2n}_c(Sh^0, \mathbb{C}) \to \mathbb{C}$.

(c) The bilinear pairing

$$H^i({}_KSh, V^\nabla) \otimes H^{2n-i}_c({}_KSh, (V^*)^\nabla) \to \mathbb{C}, \omega \otimes \omega' \mapsto Tr(\omega \smile \omega')$$

coincides (up to a non-zero scalar multiple) with Poincare duality.

1.5. The symmetric space X possesses a $G(\mathbb{R})$-invariant Hermitian metric. In what follows we assume this metric fixed; it descends to a complete Hermitian metric on ${}_KSh$, for any $K \subset G(\mathbf{A}^f)$. If (ρ, V) is a finite-dimensional complex representation of G, we may endow V with a K_h-invariant Hermitian inner product with respect to which the elements of $\mathfrak{p}_h$ are self-adjoint. Likewise, if (σ, W_σ) is a finite-dimensional complex representation of K_h, W_σ has a K_h-invariant Hermitian inner product. In this way, the flat (resp. fully decomposed) automorphic vector bundle $\tilde{V}$ (resp. E_σ) becomes a Hermitian vector bundle over Sh. We fix these metrics in what follows.

Define the de Rham complex $\mathcal{A}^{\cdot}(V)$ as in 1.2; let $\Omega^{p,q}(E_\sigma)$ be the space of C^∞ (p,q) forms on Sh with values in E_σ, $0 \le p, q \le n$. With respect to the metrics defined above, the exterior derivative $d_V : \mathcal{A}^{\cdot}(V) \to \mathcal{A}^{\cdot}(V)$ (resp. the $\bar\partial$ operator $\bar\partial_\sigma : \Omega^{0,\cdot}(E_\sigma) \to \Omega^{0,\cdot}(E_\sigma)$) has a formal adjoint δ_V (resp. θ_σ), defined by the usual formulas (cf. [**10**]). Completeness of ${}_KSh$ (any K) implies that the formal adjoints coincide with the Hilbert space adjoints on the respective spaces of square integrable forms. Let $\Delta_V = d_V\delta_V + \delta_V d_V : \mathcal{A}^{\cdot}(V) \to \mathcal{A}^{\cdot}(V), \Box_\sigma = \bar\partial_\sigma\theta_\sigma + \theta_\sigma\bar\partial_\sigma : \Omega^{0,\cdot}(E_\sigma) \to \Omega^{0,\cdot}(E_\sigma)$ be the corresponding Laplacians. We save $\Box_\sigma$ for 2.6.

The operator Δ_V corresponds, under the lifting (1.2.2), to an operator, also denoted Δ_V, on the Lie algebra complex $C^{\cdot}(\mathfrak{g}, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))\otimes V)$. Let $C_\mathfrak{g}$ denote the Casimir operator in the center $Z(\mathfrak{g}_\mathbb{C})$ of the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$, and let R denote the right regular representation of $U(\mathfrak{g}_\mathbb{C})$ on $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$. Kuga's formula [**30**],[**8**] states that, in terms of the identification (1.2.4),

$$\Delta_V = R(C_\mathfrak{g}) \otimes 1 \otimes 1 - 1 \otimes 1 \otimes d\rho(C_\mathfrak{g}), \tag{1.5.1}$$

where 1 is the identity operator.

Let $\mathcal{A}(G)$ (resp. $\mathcal{A}_{(2)}(G)$, resp. $\mathcal{A}_0(G)$) denote the space of all automorphic forms (resp. automorphic forms which are square-integrable modulo $Z_G(\mathbf{A})$, resp. cusp forms) on $G(\mathbb{Q})\backslash G(\mathbf{A})$. Thus $\mathcal{A}(G)$ is the $(\mathfrak{g}, K_h)$-submodule of K_h-finite and $Z(\mathfrak{g}_{\mathbb{C}})$-finite vectors in C_{si}. Let $\mathcal{H}^p_{\text{cusp},V}$ (resp. $\mathcal{H}^p_{(2),V}$) denote the kernel of Δ_V on $C^p(\mathfrak{g}, K_h, \mathcal{A}_0(G) \otimes V)$ (resp. $C^p(\mathfrak{g}, K_h, \mathcal{A}_{(2)}(G) \otimes V)$; these are the harmonic cusp forms (resp. harmonic square integrable forms) with values in V. As for $\mathcal{H}^p_{\text{cusp},V}$, it follows from Theorem 1.4.1 (since cusp forms are rapidly decreasing) that there is a map $\mathcal{H}^*_{\text{cusp},V} \to H^*_c(Sh, V^\nabla)$. Borel proved the following theorem:

1.5.2 THEOREM. *(Borel, [4],[5]). Let $\bar{H}^*(Sh, V^\nabla)$ denote the image of $H^*_c(Sh, V^\nabla)$ in $H^*(Sh, V^\nabla)$. Then the canonical map $\mathcal{H}^*_{\text{cusp},V} \to \bar{H}^*(Sh, V^\nabla)$ is an injection of $G(\mathbf{A}^f)$-modules.*

§2. AUTOMORPHIC FORMS AS COHERENT COHOMOLOGY CLASSES

The arithmetic properties of the canonical models play no role in Borel's theory, which, as mentioned above, is valid for any locally symmetric space. We now want to study the coherent cohomology of certain automorphic vector bundles in terms of automorphic forms. Automorphic vector bundles, and therefore their cohomology groups, have canonical models over number fields. The automorphic forms which contribute to these cohomology groups have arithmetic properties connected with the existence of canonical models.

Borel's study of the cohomology of local systems exploits the fact that a locally symmetric space is homotopy equivalent to a well-behaved compact manifold with corners, the *Borel-Serre compactification.* The analogue for the holomorphic theory is provided by Mumford's theory of toroidal compactifications. The toroidal compactification of a Shimura variety is not unique, but, as we explain below, its coherent cohomology with coefficients in suitably extended automorphic vector bundles is independent of the choice of compactification. This is the starting point in the application of coherent cohomology to automorphic forms.

2.1. Let K be an open compact subgroup of $G(\mathbf{A}^f)$. We say K is *neat* if for each $k \in K$, there exists a prime p such that the p-component k_p of k has the following property: for any faithful finite-dimensional $\mathbb{Q}_p$-rational representation ρ of $G(\mathbb{Q}_p)$, the subgroup of $\bar{\mathbb{Q}}_p^\times$ generated by the eigenvalues of the semisimple part of $\rho(k_p)$ contains no roots of unity.

Assume K is neat, and let Sh^0 be a connected component of ${}_KSh$. In [**1**], Mumford and his collaborators construct a certain class of compactifications of Sh^0, called *toroidal compactifications.* The toroidal compactification Sh^0_Σ, associated to a rather complicated collection Σ of combinatorial data, is in general a complex algebraic space. Tai shows in [**1**] that Σ may be chosen in such a way as to assure that the compactifications are smooth projective varieties, and that the complements of Sh^0 in Sh^0_Σ is a divisor with normal crossings. A toroidal compactification with these properties will be called SNC. Such compactifications arise in Looijenga's proof of the Zucker conjecture; cf. Zucker's talk.

In [**22**], we construct toroidal compactifications in the adelic framework. In this way we can find combinatorial data Σ defining toroidal compactifications ${}_KSh_\Sigma$ of ${}_KSh$, some of which are projective varieties rational over $E(G,X)$ [**22**]. For K fixed, the Σ, and hence the ${}_KSh_\Sigma$, form an inverse system with respect to the relation of *refinement*; cf. [**1**],[**22**].

Let $\mathcal{V}$ be an automorphic vector bundle over ${}_KSh$. For any ${}_KSh_\Sigma$, there exist two functorial extensions of $\mathcal{V}$ to vector bundles over ${}_KSh_\Sigma$, rational over $k(\sigma)$ [21,§ 2]. The first one, called the *canonical extension* and denoted $\mathcal{V}^{\mathrm{can}}$, was constructed by Mumford in [**34**] and, more generally, in [**22**]. The functor $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{can}}$ commutes with tensor operations.

Let $Z_\Sigma = {}_KSh_\Sigma - {}_KSh$; let $\mathcal{I}(Z_\Sigma) \subset \mathcal{O} = \mathcal{O}_{{}_KSh_\Sigma}$ be the ideal sheaf defining the divisor Z_Σ. The *subcanonical extension* of $\mathcal{V}$ is the vector bundle $\mathcal{V}^{\mathrm{sub}} = \mathcal{V}^{\mathrm{can}} \otimes_{\mathcal{O}} \mathcal{I}(Z_\Sigma)$. Then $\mathcal{V}^{\mathrm{sub}}$ is naturally a subsheaf of $\mathcal{V}^{\mathrm{can}}$, and there is a short exact sequence

$$0 \to \mathcal{V}^{\mathrm{sub}} \to \mathcal{V}^{\mathrm{can}} \to \mathcal{V}^{\infty} \to 0, \tag{2.1.1}$$

where $\mathcal{V}^\infty$ is the restriction of $\mathcal{V}^{\mathrm{can}}$ to Z_Σ. When necessary, we write $\mathcal{V}^{\mathrm{sub},\Sigma}, \mathcal{V}^{\mathrm{can},\Sigma}, \mathcal{V}^{\infty,\Sigma}$. If Σ' is a refinement of Σ, let $\pi : {}_KSh_{\Sigma'} \to {}_KSh_\Sigma$ be the natural map; then $\pi^*(\mathcal{V}^{\mathrm{sub},\Sigma}) \cong \mathcal{V}^{\mathrm{sub},\Sigma'}, \pi^*(\mathcal{V}^{\mathrm{can},\Sigma}) \cong \mathcal{V}^{\mathrm{can},\Sigma'}$.

The following proposition is a summary of the contents of §2 of [**21**]:

2.2. PROPOSITION. *(a) Let ${}_KSh_\Sigma, {}_KSh_{\Sigma'}$ be two toroidal compactifications of ${}_KSh$. Then there are natural isomorphisms of sheaf co-*

homology

$$H^*({}_KSh_\Sigma, \mathcal{V}^{\mathrm{can},\Sigma}) \tilde{\to} H^*({}_KSh_{\Sigma'}, \mathcal{V}^{\mathrm{can},\Sigma'}),$$
$$H^*({}_KSh_\Sigma, \mathcal{V}^{\mathrm{sub},\Sigma}) \tilde{\to} H^*({}_KSh_{\Sigma'}, \mathcal{V}^{\mathrm{sub},\Sigma'}).$$

(b) We define

$$H^*_K(\mathcal{V}^{\mathrm{can}}) = \varinjlim_{\Sigma} H^*({}_KSh_\Sigma, \mathcal{V}^{\mathrm{can},\Sigma}),$$
$$H^*_K(\mathcal{V}^{\mathrm{sub}}) = \varinjlim_{\Sigma} H^*({}_KSh_\Sigma, \mathcal{V}^{\mathrm{sub},\Sigma}),$$

where the direct limits are taken with respect to refinement. Let

$$\tilde{H}^*(\mathcal{V}^{\mathrm{can}}) = \varinjlim_{K} H^*_K(\mathcal{V}^{\mathrm{can}}), \tilde{H}^*(\mathcal{V}^{\mathrm{sub}}) = \varinjlim_{K} H^*_K(\mathcal{V}^{\mathrm{sub}}),$$

Then $\tilde{H}^*(\mathcal{V}^{\mathrm{can}})$ and $\tilde{H}^*(\mathcal{V}^{\mathrm{sub}})$ are naturally admissible graded $G(\mathbf{A}^f)$-modules, and the natural homomorphism $\tilde{H}^*(\mathcal{V}^{\mathrm{sub}}) \to \tilde{H}^*(\mathcal{V}^{\mathrm{can}})$ is $G(\mathbf{A}^f)$-equivariant. Moreover, if $\mathcal{V} = E_\sigma$, (notation 1.1) then the action of $G(\mathbf{A}^f)$ preserves the natural $k(\sigma)$-rational structures on $\tilde{H}^*(E_\sigma^{\mathrm{can}})$ and $\tilde{H}^*(E_\sigma^{\mathrm{sub}})$.

2.2.1. Remark. The methods of Milne [**32**] imply a stronger assertion. For any automorphic vector bundle $\mathcal{V}$ over Sh, let $\tilde{H}^*(\mathcal{V})(\infty) = \varinjlim_K \varinjlim_\Sigma H^*({}_KSh_\Sigma, \mathcal{V}^{\infty,\Sigma})$. Let $\tau \in \mathrm{Aut}(\mathbf{C})$, choose a special point $h \in X$, and let $({}^{\tau,h}G, {}^{\tau,h}X)$ be the basic pair which appears in the Langlands conjecture, such that $Sh(G,X)^\tau \cong Sh({}^{\tau,h}G, {}^{\tau,h}X)$. Let $\mathcal{E}$ be a G-homogeneous vector bundle over $\check{X}$, ${}^{\tau,h}\mathcal{E}$ the corresponding ${}^{\tau,h}G$-homogeneous vector bundle over ${}^{\tau,h}\check{X}$. Then there is a canonical isomorphism of long exact sequences (depending on the choice of h):

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \tilde{H}^q([\mathcal{E}]^{\mathrm{sub}})^\tau & \longrightarrow & \tilde{H}^q([\mathcal{E}]^{\mathrm{can}})^\tau & \longrightarrow & \tilde{H}^q([\mathcal{E}])(\infty)^\tau & \longrightarrow \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
\ldots \longrightarrow & \tilde{H}^q([{}^{\tau,h}\mathcal{E}]^{\mathrm{sub}}) & \longrightarrow & \tilde{H}^q([{}^{\tau,h}\mathcal{E}]^{\mathrm{can}}) & \longrightarrow & \tilde{H}^q([{}^{\tau,h}\mathcal{E}])(\infty) & \longrightarrow \ldots.
\end{array}$$

This isomorphism commutes with the natural action of $G(\mathbf{A}^f) \cong {}^{\tau,h}G(\mathbf{A}^f)$ on both sides, and depends on the choice of h only up to a canonical isomorphism $\phi^{\mathcal{E}}(\tau, h', h) : [{}^{\tau,h'}\mathcal{E}] \tilde{\to} [{}^{\tau,h'}\mathcal{E}]$ (cf. Lemma 5.1

and Theorem 5.2 of [**32**] for the relevant definitions for connected Shimura varieties).

Let $\bar{H}^q_K(\mathcal{V})$ denote the image of $H^q_K(\mathcal{V}^{\mathrm{sub}})$ in $H^q_K(\mathcal{V}^{\mathrm{can}})$; let $\bar{H}^q(\mathcal{V}) = \varinjlim_K \bar{H}^q_K(\mathcal{V}) =$ the image of $\tilde{H}^q(\mathcal{V}^{\mathrm{sub}})$ in $\tilde{H}^q(\mathcal{V}^{\mathrm{can}})$.

2.3. Let Δ be the disc of radius $\frac{1}{2}$ in $\mathbb{C}$, and let $\Delta^* = \Delta - \{0\}$ be the punctured disc. Let z be the variable in $\Delta, r = |z|$. Let θ be the differential operator $z \cdot \frac{\partial}{\partial z}, \bar{\theta} = \bar{z} \cdot \frac{\partial}{\partial \bar{z}}$. If $N \in \mathbb{Z}$, we say a function $g \in C^\infty(\Delta^*)$ is *slowly increasing of exponent* N if it satisfies an inequality of the form

$$|g(z)| < C|\log r|^N. \tag{2.3.1}$$

We say g is *slowly increasing* (resp., *rapidly decreasing*) if g is slowly increasing of exponent N for some $N \in \mathbb{Z}$ (resp. for all $N \in \mathbb{Z}$). We say g is *slowly increasing to all orders* (resp. *rapidly decreasing to all orders*) if $\theta^i g$ and $\bar{\theta}^i g$ are slowly increasing (resp. rapidly decreasing) for all $i, j \geq 0$.

The relation between the cohomology groups $\tilde{H}^q(E^{\mathrm{can}}_\sigma)$ and $\tilde{H}^q(E^{\mathrm{sub}}_\sigma)$ and the theory of automorphic forms is based on the following lemma:

2.3.2. LEMMA. *(Harris-Phong). (a) Let $g \in C^\infty(\Delta^*)$ be a function which is slowly increasing of exponent N. Then the equation $\bar{\theta} f = g$ has a solution f which is slowly increasing of exponent $N + 2$.*

(b) In (a), if g is rapidly decreasing, then the equation $\bar{\theta} f = g$ has a solution f which is rapidly decreasing.

(c) In (a), suppose g is slowly increasing (resp. rapidly decreasing) to all orders. Then the equation $\bar{\theta} f = g$ has a solution f which is slowly increasing (resp. rapidly decreasing) to all orders.

Parts (a) and (b) are proved in [**24**],[**21**]. Suppose g is slowly increasing to all orders and, for $r \geq 0$, let f_r be the slowly increasing solution to the equation $\bar{\theta} f_r = \theta^r g$ constructed as in [**24**]. A simple modification of the arguments in [**24**],[**21**] (see Remark 2.3.4, below) shows:

(2.3.3) If $h \in C^\infty(\Delta^*)$ is such that h and θh are slowly increasing and ϕ is the slowly increasing solution to $\bar{\theta}\phi = h$ constructed in [**24**], then $\theta\phi$ is slowly increasing.

In particular θf_r is slowly increasing for all r. Since the operators θ and $\bar{\theta}$ commute, we see that, for all r, $\theta f_r - f_{r+1}$ is holomorphic and slowly increasing on Δ^*, hence is holomorphic on Δ. It follows by induction that, for all r, $\theta^r f - f_r$ is slowly increasing and holomorphic on Δ, hence f is slowly increasing to all orders. If now g is rapidly decreasing to all orders, the analogue of (2.3.3) holds, and the obvious argument (with f_r now rapidly decreasing) shows that, for all r, $\theta f_r - f_{r+1}$ is holomorphic and rapidly decreasing on Δ^*, hence has a zero at 0; thus $\theta^r f - f_r$ is rapidly decreasing and holomorphic on Δ. This implies (c).

2.3.3. Remark. The proof of (2.3.3) can be found in an unpublished manuscript of C. Soulé entitled "Dolbeault Hodge theory with logarithmic growth."

2.4. The growth conditions defined in 2.3 extend naturally to polydiscs. Thus, if V is a smooth complex algebraic variety and Z is a divisor with normal crossings on V, there is no difficulty in defining smooth functions on $V - Z$ which are slowly increasing, rapidly decreasing, or slowly increasing to all orders along Z. These rings of functions are denoted $C_{si,Z}(V), C_{rd,Z}(V)$, and $C_{sia,Z}(V)$, respectively.

More generally, let $X = X_{n,r} = (\Delta^*)^r \times \Delta^{n-r}$, with coordinates $z_1, \dots, z_n$. Let $A^{\cdot}_{si}(X)$ be the algebra generated over the ring of slowly increasing functions on X by the differentials $d\bar{z}_1/|z_1|, \dots, d\bar{z}_r/|z_r|, d\bar{z}_{r+1}, \dots, d\bar{z}_n$. An element of $A^{\cdot}_{si}(X)$ is called a *slowly increasing (antiholomorphic) differential form.* We similarly define $A^{\cdot}_{rd}(X)$ (resp. $A^{\cdot}_{sia}(X)$), whose elements are called *rapidly decreasing differential forms* (resp. differential forms *slowly increasing to all orders*, resp. differential forms *rapidly decreasing to all orders*).

Again, these notions globalize; with V and Z as above, we may define $A^{\cdot}_{si,Z}(V)$, $A^{\cdot}_{rd,Z}(V)$, $A^{\cdot}_{sia,Z}(V)$ and $A^{\cdot}_{rda,Z}(V)$ in the obvious way. The algebras $A^{\cdot}_{sia,Z}(V)$ and $A^{\cdot}_{rda,Z}(V)$ are already complexes under $\bar{\partial}$; we let $C^{\cdot}_{si,Z}(V)$ (resp. $C^{\cdot}_{rd,Z}(V)$) be the complex of forms ω such that both ω and $\bar{\partial}\omega$ are slowly increasing (resp. rapidly decreasing). If $\mathcal{E}$ is a holomorphic vector bundle on V, we let $C^{\cdot}_{si,Z}(V, \mathcal{E})$ be the differential graded sheaf $C^{\cdot}_{si,Z}(V) \otimes \mathcal{E}$, and define $C^{\cdot}_{rd,Z}(V, \mathcal{E})$, $C^{\cdot}_{sia,Z}(V, \mathcal{E})$ and $C^{\cdot}_{rda,Z}(V, \mathcal{E})$ similarly. It follows from Lemma 2.3.2 and the arguments in [**24**],[**21**] that the complexes $C^{\cdot}_{sia,Z}(V, \mathcal{E})$ and $C^{\cdot}_{si,Z}(V, \mathcal{E})$ (resp. $C^{\cdot}_{rda,Z}(V, \mathcal{E})$ and $C^{\cdot}_{rd,Z}(V, \mathcal{E})$ are fine resolutions of $\mathcal{E}$ (resp. of $\mathcal{E}(-Z)$).

Let $\mathcal{I}_Z \subset \mathcal{O}_V$ be the ideal sheaf defining Z. With $\mathcal{E}$ as above, let $\mathcal{E}(-Z) = \mathcal{E} \otimes_{\mathcal{O}_V} \mathcal{I}_Z$. Let $\Omega^{0,\cdot}(\mathcal{E})$ (resp. $\Omega^{0,\cdot}(\mathcal{E}(-Z))$) be the standard C^∞ Dolbeault complex of $\mathcal{E}$ (resp. $\mathcal{E}(-Z)$). It follows formally from the above remarks (cf. [**24**],[**21**]) that

2.4.1. THEOREM. *The inclusions*

$$\Omega^{0,\cdot}(\mathcal{E}) \subset C^{\cdot}_{sia,Z}(V,\mathcal{E}) \subset C^{\cdot}_{si,Z}(V,\mathcal{E})$$

and

$$\Omega^{0,\cdot}(\mathcal{E}(-Z)) \subset C^{\cdot}_{rd,Z}(V,\mathcal{E}) \subset C^{\cdot}_{rda,Z}(V,\mathcal{E})$$

induce isomorphisms on cohomology. In particular, there are natural isomorphisms:

$$H^{\cdot}(V,\mathcal{E}) \subset H^{\cdot}_{\bar\partial}(C^{\cdot}_{sia,Z}(V,\mathcal{E})) \subset H^{\cdot}_{\bar\partial}(C^{\cdot}_{si,Z}(V,\mathcal{E})),$$
$$H^{\cdot}(V,\mathcal{E}(-Z)) \subset H^{\cdot}_{\bar\partial}(C^{\cdot}_{rd,Z}(V,\mathcal{E})) \subset H^{\cdot}_{\bar\partial}(C^{\cdot}_{rda,Z}(V,\mathcal{E})).$$

where the left hand side is computed in the Zariski topology.

We now apply this to the spaces defined in 2.2. Let $j : {}_KSh \hookrightarrow {}_KSh_\Sigma$ be an SNC toroidal compactification. Fix a point $h \in X$, define K_h, $\mathcal{P}_h$ and $\mathfrak{P}_h$ as in 1.1, and let (σ, W_σ) be a finite-dimensional representation of $\mathcal{P}_h$. Let E_σ be the corresponding automorphic vector bundle over Sh. As in 1.2, there is a canonical lifting

$$(2.4.2)\quad \Gamma({}_KSh_\Sigma, j_*\Omega^{0,\cdot}(E_\sigma)) \cong \Gamma({}_KSh, \Omega^{0,\cdot}(E_\sigma)) \\ \qquad \tilde\to (C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K) \otimes \Lambda^{\cdot}(\mathfrak{p}^-)^* \otimes W_\sigma)^{K_h},$$

where $\Omega^{0,\cdot}(E_\sigma)$ is the standard C^∞ Dolbeault complex of E_σ over ${}_KSh$. As in [**21**], we may identify the right hand side with the Lie algebra complex $C^{\cdot}(\mathfrak{P}_h, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K) \otimes W_\sigma)$. Let d be the differential of this complex.

2.4.3. LEMMA. *Let* Lift *be the lifting (2.4.2). Then*

(a) Lift$(\Gamma({}_KSh_\Sigma, C^{\cdot}_{si,Z_\Sigma}({}_KSh_\Sigma, E_\sigma)))$ *(resp.* Lift$(\Gamma({}_KSh_\Sigma, C^{\cdot}_{rd,Z_\Sigma}({}_KSh_\Sigma, E_\sigma)))$ *is the subcomplex* ${}_KC^{\cdot}_{si,\sigma}$ *(resp.* ${}_KC^{\cdot}_{rd,\sigma}$ *of* $C^{\cdot}(\mathfrak{P}_h, K_h, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K)\otimes W_\sigma)$*, consisting of cochains* ω *such that both* ω *and* $d\omega$ *are slowly increasing (resp. rapidly decreasing)* W_σ *-valued forms, in the sense of 1.3.*

(b) Lift$(\Gamma({}_KSh_\Sigma, C^{\cdot}_{sia,Z_\Sigma}({}_KSh_\Sigma, E_\sigma))) = C^{\cdot}(\mathfrak{P}_h, K_h, C_{si} \otimes W_\sigma)^K$.

(c) Lift$(\Gamma({}_KSh_\Sigma, C^{\cdot}_{rda,Z_\Sigma}({}_KSh_\Sigma, E_\sigma))) = C^{\cdot}(\mathfrak{P}_h, K_h, C_{rd} \otimes W_\sigma)^K$.

2.4.3.1. Remark. Part (a) is essentially [**21**], 3.3.4. Parts (b) and (c) follow from standard estimates for the coefficients of left-invariant differential operators on $G(\mathbb{R})$ in symmetric space coordinates, as in [**6**], Correction and complement. In particular, the image under <u>Lift</u> of each of these complexes is independent of the choice of toroidal compactification. Let $C^{\cdot}_{si,\sigma} = \varinjlim_K {}_KC^{\cdot}_{si,\sigma}, C^{\cdot}_{rd,\sigma} = \varinjlim_K {}_KC^{\cdot}_{rd,\sigma}$. Combining 2.4.1 and 2.4.3, we obtain the following analogue of Borel's results.

2.4.4. THEOREM. *Let E_σ be the fully decomposed automorphic vector bundle on Sh, attached to the representation (σ, W_σ) of K_h. There is a natural commutative diagram of admissible graded $G(\mathbf{A}^f)$-modules*

$$(2.4.5)\qquad \begin{array}{ccccc} H^{\cdot}(\mathfrak{P}_h, K_h, C_{rd}\otimes W_\sigma) & \xrightarrow{\sim} & H^{\cdot}(C^{\cdot}_{rd,\sigma}) & \xrightarrow{\sim} & \tilde{H}^{\cdot}(E^{\mathrm{sub}}_\sigma) \\ \downarrow & & \downarrow & & \downarrow \\ H^{\cdot}(\mathfrak{P}_h, K_h, C_{si}\otimes W_\sigma) & \xrightarrow{\sim} & \tilde{H}^{\cdot}(C^{\cdot}_{si,\sigma}) & \xrightarrow{\sim} & \tilde{H}^{\cdot}(E^{\mathrm{can}}_\sigma) \end{array}$$

The horizontal arrows are isomorphisms.

Fix an open compact subgroup $K \subset G(\mathbf{A}^f)$. Let $\mathbb{K} = \mathbb{K}_{{}_KSh} = \Omega^n_{{}_KSh}$, and, when ${}_KSh_\Sigma$ is a toroidal compactification of ${}_KSh$, write $\mathbb{K}_\Sigma = \mathbb{K}_{{}_KSh_\Sigma}$ for the dualizing sheaf of ${}_KSh_\Sigma$, whether or not ${}_KSh_\Sigma$ is smooth. Then $\mathbb{K}^{\mathrm{sub}} \cong \mathbb{K}_\Sigma$ [**21**]. For any vector bundle $\mathcal{E}$ over ${}_KSh$, let $\mathcal{E}' = \mathbb{K}\otimes\mathcal{E}^*$.

Define E_σ as in Theorem 2.4.4. Let τ (resp. ω) be the representation of K_h corresponding to E'_σ (resp. to $\mathbb{K} = \Omega^n_{{}_KSh}$). The natural pairing $E'_\sigma \otimes E_\sigma \to \mathbb{K}$ defines a morphism of complexes

$$(2.4.6)\qquad C^{\cdot}_{rd,\tau}\otimes C^{\cdot}_{si,\sigma} \to C^{\cdot}_{rd,\omega}, f\otimes g \mapsto [f\wedge g]$$

where the double complex on the left hand side is identified with the associated single complex. The analogue of Borel's Proposition 1.4.4 is the following interpretation of Serre duality:

2.5. PROPOSITION [**21**]. *(a) For any automorphic vector bundle $\mathcal{V}$, and for any $q = 0, 1, \ldots, n$, the cup product*

$$(2.5.1)\qquad H^{n-q}_K(\mathcal{V}^{\prime,\mathrm{sub}})\otimes H^q_K(\mathcal{V}^{\mathrm{can}}) \to H^n_K(\mathbb{K}^{\mathrm{sub}}) \cong \mathbf{C}$$

is a nondegenerate pairing (Serre duality), rational over any base field k over which ${}_KSh$ and $\mathcal{V}$ are defined.

(b) Let $\phi \in H^{n-q}_K(E^{\mathrm{sub}}_\tau), \psi \in H^q_K(E^{\mathrm{can}}_\sigma)$. Let f (resp. g) be a $\bar\partial$-closed form in ${}_KC^{n-q}_{rd,\tau}$ (resp. ${}_KC^q_{si,\sigma}$) representing the cohomology

class ϕ (resp. ψ). The Serre duality pairing is given, up to a constant multiple, by

$$(2.5.2) \qquad <\phi, \psi> = (2\pi i)^{-n} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})/KK_h} [f \wedge g].$$

2.6. Define $\Box_\sigma : \Omega^{0,\cdot}(E_\sigma) \to \Omega^{0,\cdot}(E_\sigma)$ as in 1.5. Under the isomorphism Lift of (2.4.2), $\Box_\sigma$ corresponds to an operator, also denoted

$$\begin{aligned} \Box_\sigma : C^\cdot(\mathfrak{P}_h, K_h, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})) \otimes W_\sigma) \\ \to C^\cdot(\mathfrak{P}_h, K_h, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})) \otimes W_\sigma) \end{aligned}$$

As in 1.5, we may define spaces of harmonic cusp forms and harmonic square integrable forms with values in E_σ: let $\mathcal{H}^p_{\mathrm{cusp},\sigma}$ (resp. $\mathcal{H}^p_{(2),\sigma}$) denote kernel of $\Box_\sigma$ on $C^p(\mathfrak{P}_h, K_h, \mathcal{A}_0(G) \otimes W_\sigma)$ (resp. $C^p(\mathfrak{P}_h, K_h, \mathcal{A}_{(2)}(G) \otimes W_\sigma)$).

The following theorem is a slight strengthening (using Theorem 2.4.4) of the main result of [**21**]:

2.7. THEOREM. *Let (σ, W_σ) be a representation of K_h, and let E_σ be the corresponding fully decomposed automorphic vector bundle. Then*

(a) The canonical map $cl : \mathcal{H}^*_{\mathrm{cusp},\sigma} \to \bar{H}^*(E_\sigma)$, derived from (2.4.5) and the obvious inclusion $\mathcal{H}^*_{\mathrm{cusp},\sigma} \subset H^\cdot(\mathfrak{P}_h, K_h, C_{rd} \otimes W_\sigma)$, is an injection of $G(\mathbf{A}^f)$-modules.

(b) $\bar{H}^*(E_\sigma)$ is contained in the image of $\mathcal{H}^*_{(2),\sigma}$ in $\tilde{H}^*(E^{\mathrm{can}}_\sigma)$, under the homomorphism derived from (2.4.5) and the obvious inclusion $\mathcal{H}^*_{(2),\sigma} \subset C^\cdot(\mathfrak{P}_h, K_h, C_{si} \otimes W_\sigma)$.

Here part (a) is the analogue of Borel's theorem 1.5.2, and is proved in the same way. The analogue of part (b) for the cohomology of local systems is well known to the experts, but I have not seen it stated in print. This analogue is an immediate consequence of the theorem of Borel and Casselman [**7**] that, under certain hypotheses (verified by Shimura varieties) the L_2 cohomology of a locally symmetric space with coefficients in a local system is *finite-dimensional.*

2.7.1. Remark. We emphasize that we do not know at present whether or not the L_2 cohomology of a fully decomposed automorphic vector bundle is finite-dimensional. The methods of Borel and Casselman do not appear to apply. The proof of part (b) in [**21**] is based on the fact that the continuous spectrum in $L_2(G(\mathbf{Q})\backslash G(\mathbf{A})/K)$ contains no vectors which transform under a finite-dimensional representation of the Hecke algebra of $G(\mathbf{A}^f)$ relative to K.

In the holomorphic and anti-holomorphic cases we can improve on Theorem 2.7:

2.7.2. PROPOSITION [21,5.4.2]. *When $q = 0$ or $q = n$, The canonical map $cl : \mathcal{H}^q_{\text{cusp},\sigma} \to \bar{H}^q(E_\sigma)$ is an isomorphism, for any σ.*

The proof is an application of the results of Baily and Borel.

§3. $\bar{\partial}$-COHOMOLOGY OF IRREDUCIBLE UNITARY REPRESENTATIONS; APPLICATIONS

3.0 We retain the notation of the preceding chapters. It follows from 3.2.2, below, that there are natural isomorphisms

$$(3.0.1)\quad \mathcal{H}^*_{\text{cusp},\sigma} \cong H^*(\mathfrak{P}_h, K_h, \mathcal{A}_0(G) \otimes W_\sigma),$$
$$\mathcal{H}^*_{(2)},\sigma \cong H^*(\mathfrak{P}_h, K_h, \mathcal{A}_{(2)}(G) \otimes W_\sigma)$$

Let $\mathcal{A}_{res}(G)$ denote the orthogonal complement to $\mathcal{A}_0(G)$ in $\mathcal{A}_{(2)}(G)$. As in 1.3, we may write

$$(3.0.2)\qquad \mathcal{A}_*(G) \cong \bigoplus_{\pi_\infty,\pi_f} m_*(\pi_\infty \otimes \pi_f) V_{\pi_\infty} \otimes V_{\pi_{f'}},$$

where $* = (2)$, 0, or res, and $(\pi_\infty, V_{\pi_\infty})$ (resp. (π_f, V_{π_f}) runs through the set of unitarizable $(\mathfrak{g}, K_h)$-modules (resp. $G(\mathbf{A}^f)$-modules). A well-known theorem of Gelfand and Piatetski-Shapiro asserts that the multiplicities $m_0(\pi_\infty \otimes \pi_f)$ are finite; the analogous theorem for $m_{(2)}(\pi_\infty \otimes \pi_f)$ and $m_{res}(\pi_\infty \otimes \pi_f)$ is due to Langlands. As in 1.3, we are thus led to study the spaces $H^*(\mathfrak{P}_h, K_h, V_{\pi_\infty} \otimes W_\sigma)$ for general unitary $(\mathfrak{g}, K_h)$-modules $(\pi_\infty, V_{\pi_\infty})$.

In contrast to what is known for $(\mathfrak{g}, K_h)$-cohomology, the classification of unitary $(\mathfrak{g}, K_h)$-modules with non-trivial $(\mathfrak{P}_h, K_h)$-cohomology is not complete. There are certainly many unitary $(\mathfrak{g}, K_h)$-modules which do not have $(\mathfrak{g}, K_h)$-cohomology, and whose $(\mathfrak{P}_h, K_h)$-cohomology does not vanish; the nondegenerate limits of discrete series furnish an important class of examples (cf. Theorem 3.4, below). It is nevertheless possible to make a number of strong qualitative assertions about unitary $(\mathfrak{g}, K_h)$-modules with non-trivial $(\mathfrak{P}_h, K_h)$-cohomology; these assertions in turn have consequences for the arithmetic of the coherent cohomology spaces introduced in §2.

3.1. In what follows, all $(\mathfrak{g}, K_h)$-modules will be assumed to be complex vector spaces. If V is a $\mathfrak{g}$-module on which K_h acts, consistently with the adjoint action of K_h on $\mathfrak{g}$, we let V_0 denote the space of K_h-finite vectors in V. Choose a maximal torus $H \subset K_h$, and let $\mathfrak{h}$ be its Lie algebra. Then $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ as well as of $\mathfrak{k}_{h,\mathbf{C}}$. Choose a set R^+ of positive roots for $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$; let R_c^+ (resp. R_n^+) be the subset of R^+ of compact (resp. non-compact) roots. We assume henceforward that R_n^+ is the set of roots on $\mathfrak{p}^+$. Let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha, \rho_n = \frac{1}{2}\sum_{\alpha \in R_n^+} \alpha, \rho_c = \rho - \rho_n$.

3.1.1. Example: *1.1.3, continued.* In Example 1.1.3, let $d(t_1, \dots, t_n)$ denote the $n \times n$ diagonal matrix with entries $t_1, \dots, t_n$, and let $\mathfrak{h}_{\mathbf{C}}$ be the algebra of matrices $\{k(z, I_n, d(t_1, \dots, t_n))\} \subset \mathfrak{k}_{h,\mathbf{C}}$. For $j = 1, \dots, n$, let $\alpha_j(k(z, I_n, d(t_1, \dots, t_n))) = it_j, \nu(k(z, I_n, d(t_1, \dots, t_n))) = z$. Then for R_n^+ (resp. R_c^+) we may take the set of characters $\{\alpha_i + \alpha_j, i, j = 1, \dots, n\}$ (resp. $\{\alpha_i - \alpha_j, i, j = 1, \dots, n, i < j\}$. We write $(a_1, \dots, a_n; c)$ for the character $\sum_{j=1}^n a_j\alpha_j + c\nu \in \mathfrak{h}^*\mathbf{C}$; then $\rho = (n, n-1, \dots, 1; 0)$. The finite dimensional representations of $\mathfrak{g}_{\mathbf{C}}$ (resp. of $K_{h,\mathbf{C}}$) are parametrized by integer $n+1$-tuples $(a_1, \dots, a_n; c)$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ (resp. $a_1 \geq a_2 \geq \cdots \geq a_n$) such that $c \equiv \sum_{j=1}^n a_j (\mod 2)$.

3.1.2. DEFINITION. *Let (π, V) be a G-module on which K acts, consistently with the adjoint action of K_h on $\mathfrak{g}$, and let (σ, W_σ) be a finite dimensional representation of K_h. We say (π, V) has $\bar{\partial}$-cohomology with coefficients in σ (or in W_σ) if the relative Lie algebra cohomology space $H^*(\mathfrak{P}_h, K_h, V \otimes W_\sigma) \neq \{0\}$. If $H^q(\mathfrak{P}_h, K_h, V \otimes W_\sigma) \neq \{0\}$ for some degree q, we say (π, V) has $\bar{\partial}$-cohomology in degree q with coefficients in σ (or in W_σ). We say (π, V) is a representation with $\bar{\partial}$-cohomology if (π, V) has $\bar{\partial}$-cohomology with coefficients in σ for some (σ, W_σ).*

It is not difficult to verify [21,§4] that, with V and W_σ as above,

(3.1.3)
$$H^*(\mathfrak{P}_h, K_h, V \otimes W_\sigma) \cong (H^*(\mathfrak{p}^-, V) \otimes W_\sigma)^{K_h} \cong (H^*(\mathfrak{p}^-, V_0) \otimes W_\sigma)^{K_h}.$$

Suppose (π, V) is an irreducible admissible $(\mathfrak{g}, K_h)$-module. Let $Z(\mathfrak{g}_{\mathbf{C}})$ denote the center of the enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$. There exists a homomorphism $\chi_\pi : Z(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{C}$, the *infinitesimal character* of π, such that $\pi(z) = \chi_\pi(z), \forall z \in Z(\mathfrak{g}_{\mathbf{C}})$. Let $\theta : Z(\mathfrak{g}_{\mathbf{C}}) \tilde{\to} S(\mathfrak{h})W$ be the Harish-Chandra isomorphism, where $W = W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ is the Weyl group [**28**]. Any $\Lambda \in \mathfrak{h}^*$ naturally defines a homomorphism

$e_\Lambda : S(\mathfrak{h}) \to \mathbb{C}$, and thus an algebra homomorphism $\chi_\Lambda = e_\Lambda \circ \theta : Z(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$. Note that $\chi_\Lambda = \chi_{w\Lambda}, \forall w \in W$. For any irreducible admissible $(\mathfrak{g}, K_h)$-module π, it is known that $\chi_\pi = \chi_\Lambda$, for some $\Lambda \in \mathfrak{h}^*$, determined uniquely modulo the action of W.

Similarly, we let $Z(\mathfrak{k}_{h,\mathbb{C}})$ denote the center of the enveloping algebra of $\mathfrak{k}_{h,\mathbb{C}}, W_c = W(\mathfrak{k}_{h,\mathbb{C}}, \mathfrak{h}_\mathbb{C}) \subset W$. Let $\theta_\mathfrak{k} : Z(\mathfrak{k}_{h,\mathbb{C}}) \tilde{\to} S(\mathfrak{h})^{W_c}$ be the Harish-Chandra isomorphism for $K_{h,\mathbb{C}}$. As above, any $\Lambda \in \mathfrak{h}^*$ defines an algebra homomorphism $\chi'_\Lambda : Z(\mathfrak{k}_{h,\mathbb{C}}) \to \mathbb{C}$. Suppose Λ is integral and R_c^+-dominant, and let $(\sigma_\Lambda, W_\Lambda)$ be the irreducible finite-dimensional K_h-module with highest weight Λ. Let χ'_σ denote the infinitesimal character of σ; then $\chi'_\sigma = \chi'_{\Lambda+\rho_c}$. The inclusion $W_c \subset W$ defines a *surjective* restriction map

$$\xi : \mathrm{Hom}_{alg}(Z(\mathfrak{k}_{h,\mathbb{C}}), \mathbb{C}) \to \mathrm{Hom}_{alg}(Z(\mathfrak{g}_\mathbb{C}), \mathbb{C})$$

such that $\xi(\chi'_\Lambda) = \chi_{\Lambda+\rho_n}$. In our situation, a theorem of Casselman-Osborne [53,3.1.5] implies

3.1.4. PROPOSITION [**21**]. *Let (π, V) be an irreducible admissible $(\mathfrak{g}, K_h)$-module. Let $(\sigma_\Lambda, V_\Lambda)$ be the finite dimensional representation of K_h with highest weight Λ. Suppose (π, V) has $\bar\partial$-cohomology with coefficients in σ_Λ. Then $\chi_\pi = \xi(\chi'_{(\sigma_\Lambda)^*}) = \chi_{-\Lambda-\rho}$. In particular, for a given finite-dimensional representation (σ, V_σ) of K, the number of irreducible admissible $(\mathfrak{g}, K_h)$-modules with $\bar\partial$-cohomology with coefficients in σ is finite.*

The last assertion is a consequence of Harish-Chandra's well-known theorem that the number of irreducible admissible $(\mathfrak{g}, K_h)$-modules with given infinitesimal character is finite.

3.2. We henceforth assume that (π, V) is a *unitary* $(\mathfrak{g}, K_h)$-module; i.e. that there is a positive non-degenerate hermitian scalar product $(\cdot,\cdot)_\pi$ on V such that

$$(Xv, w)_\pi + (v, Xw)_\pi = 0, \forall X \in \mathfrak{g}^{der}(\mathbb{R}), v, w \in V. \tag{3.2.1}$$

The $(\mathfrak{g}, K_h)$-module of K_h-finite vectors of a unitary representation of the identity component G^0 of $G(\mathbb{R})$ is unitary in this sense.

Let $\Lambda \in \mathfrak{h}^*$ be R_c^+-dominant and integral. Choose K_h-invariant hermitian inner products on W_Λ and on $\mathfrak{p}^-$; together with the given inner product on V, these define K_h-invariant hermitian inner products on each of the terms of the complex $C^\cdot(\mathfrak{P}_h, K_h, V \otimes W_\Lambda)$. We let d^*_Λ denote the adjoint of d_Λ with respect to these inner products, and let $\Box_{\Lambda,\pi} = d_\Lambda d^*_\Lambda + d^*_\Lambda d_\Lambda$. The analogue of Kuga's formula (1.5.1) implies

3.2.2. PROPOSITION. *Let (π, V) be an irreducible unitary $(\mathfrak{g}, K_h)$-module, and let $(\sigma_\Lambda, V_\Lambda)$ be the irreducible representation of K_h with highest weight Λ. Let $c_\Lambda = <\Lambda+\rho, \Lambda+\rho> - <\rho,\rho>$.*

(a) If $\chi_\pi(C_\mathfrak{g}) \neq c_\Lambda$, then (π, V) has no $\bar{\partial}$-cohomology with coefficients in σ_Λ.

(b) If $\chi_\pi(C_\mathfrak{g}) = c_\Lambda$, then all cochains in the complex $C^\cdot(\mathfrak{P}_h, K_h, V \otimes W_\Lambda)$ are closed,

$$H^q(\mathfrak{P}_h, K_h, V\otimes W_\Lambda) = C^q(\mathfrak{P}_h, K_h, V\otimes W_\Lambda) \cong \mathrm{Hom}_{K_h}(\Lambda^q(\mathfrak{p}^-)\otimes W_\Lambda^*, V), \quad q = 0, \dots, n,$$

and every class in $H^q(\mathfrak{P}_h, K_h, V\otimes W_\Lambda)$ has a unique $\square_{\Lambda,\pi}$-harmonic representative.

3.3. Let $\mathcal{F} \subset \mathfrak{h}_\mathbb{C}^*$ denote the set of differentials of *algebraic* characters of the torus $H \subset K_h$. Let $\mathcal{F}+\rho = \{\Lambda+\rho | \Lambda \in \mathcal{F}\} \subset \mathfrak{h}_\mathbb{C}^*$. Let $<,>$ be the bilinear form on $\mathfrak{h}_\mathbb{C}^*$ induced by the Killing form. Choose a system ψ of positive roots ψ for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}), \psi \supset R_c^+$, such that λ is dominant relative to ψ, and suppose $\lambda \in \mathcal{F}+\rho$ satisfies

(3.3.1)
$$<\lambda, \alpha> \; > 0, \forall \alpha \in R_c^+ \text{ such that } \alpha \text{ is } \textit{simple} \text{ with respect to } \psi.$$

If λ is nonsingular for R, then $\psi = \psi_\lambda$ is uniquely determined, and we define the *discrete series* π_λ with Harish-Chandra parameter λ; in general, we may define the *limit of discrete series* $\pi(\lambda, \psi)$ as in [**28**], XII, §7. Let $V(\lambda, \psi)$ (resp. V_λ) be the $(\mathfrak{g}, K_h)$-module associated to $\pi(\lambda,\psi)$ (resp. to π_λ). The infinitesimal character of $\pi(\lambda,\psi)$, or of π_λ, is χ_λ.

3.3.2. Remark. Strictly speaking, π_λ is discrete series only if $\lambda|_{\mathrm{Lie}(Z_G)}$ corresponds to a *unitary* character; we drop this condition, which is unnatural from the arithmetic point of view. In any case, both π_λ and $\pi(\lambda,\psi)$ are irreducible representations of $G(\mathbb{R})^0$ which are unitary on $G_0 = G^{der}(\mathbb{R})^0$.

The discrete series representations are the representations of $G(\mathbb{R})^0$ whose matrix coefficients are square-integrable on G_0 or, equivalently, are the representations which occur with positive measure in the Plancherel formula for G_0. The limits of discrete series share with the discrete series the property of being *tempered*, a weaker growth condition for the matrix coefficients than square-integrability.

Tempered representations are the basic building blocks in the Langlands classification of irreducible $(\mathfrak{g}, K_h)$-modules. The following theorems completely describe the $\bar\partial$-cohomology of tempered $(\mathfrak{g}, K_h)$-modules:

3.4. THEOREM ([41],[57],[3]). *Let $\lambda \in \mathcal{F}+\rho$, and suppose $< \lambda, \alpha > \, > 0 \ \forall \alpha \in R_c^+$. Let ψ be as above, and let $q_{\lambda,\psi}$ be the cardinality of $\psi \cap R_n^+$. Let $\tau \in \mathfrak{h}_{\mathbf{C}}^*$ be an R_c^+-dominant integral weight and let (σ_τ, W_τ) be the finite-dimensional irreducible representation of K_h with highest weight τ. Then*

(i) $H^q(\mathfrak{P}_h, K_h, (\pi(\lambda,\psi))^* \otimes V_\tau) = 0$ unless $q = q_{\lambda,\psi}$ and *$\tau = \Lambda =_{\text{def.}} \lambda - \rho$.*

(ii) The character $\Lambda \in \mathfrak{h}_{\mathbf{C}}^*$ is R_c^+-dominant and integral, and

$$\dim H^{q_{\lambda,\psi}}(\mathfrak{P}_h, K_h, (\pi(\lambda,\psi))^* \otimes V_\Lambda) = 1.$$

When λ is regular, the analogous statements hold with $q = q_\lambda$ and $\pi(\lambda,\psi)$ replaced by π_λ.

3.5. THEOREM. *(Mirković, [33]) Suppose V is a tempered representation with $\bar\partial$-cohomology. Then V is a discrete series or nondegenerate limit of discrete series.*

Theorem 3.4, which is proved in [3], is a simple adaptation of the computation by Schmid [41] (generalized to limits of discrete series by Williams [57]) of the N-cohomology of discrete series, where N is the maximal unipotent subalgebra of $\mathfrak{g}_{\mathbf{C}}$ corresponding to $-R^+$. Theorem 3.5 was verified explicitly in [3] when G is the symplectic group of genus 2, here denoted $Sp(2)$. The general proof of Mirković uses the Beilinson-Bernstein technique of localization ($\mathcal{D}$-modules on the flag variety). It should be mentioned that the N-cohomology versions of these theorems have been proved whenever rank $G =$ rank K_h, and do not require that G/K_h have a complex structure.

3.6. It is obvious that the matrix coefficients of unitary representations are bounded; in fact, Howe has proved that the coefficients of a non-trivial unitary representation of G_0 vanish at infinity. This fact, together with the Langlands classification, places strong restrictions on the infinitesimal characters of non-tempered unitary representations ([8], IV, Theorem 5.2). A crude version of these restrictions is the following statement:

3.6.1. LEMMA. *There is a constant $b > 0$, depending only on G, with the following property: Let (π, V) be an irreducible unitary $(\mathfrak{g}, K_h)$-module, and suppose that $\chi_\pi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}_{\mathbf{C}}^*$. Suppose $| < \lambda, \alpha > | > b$ for all $\alpha \in R^+$. Then π is tempered.*

The importance to us of tempered representations lies in the following theorem of Wallach:

3.6.2. THEOREM (WALLACH, [**55**]). *Let $(\pi_\infty, V_{\pi_\infty})$ be a tempered $(\mathfrak{g}, K_h)$-module, and let (π_f, V_{π_f}) be any $G(\mathbf{A}^f)$-module. Then, in the notation of 3.0, $m_0(\pi_\infty \pi_f) = m_{(2)}(\pi_\infty \otimes \pi_f)$.*

The following existence theorem is an immediate consequence of Theorems 2.7 and 3.4, (3.0.1), and the last two results:

3.6.3 THEOREM ([**21**], COROLLARY 5.3.3). *There is a constant $b > 0$, depending only on G, with the following property. Assume the highest weight Λ of the irreducible representation σ of K_h satisfies $| < \Lambda + \rho, \alpha > | > b$ for every $\alpha \in R^+$. Then the inclusion $\mathcal{H}^*_{\mathrm{cusp},\sigma} \to \bar{H}^*(E_\sigma)$ of Theorem 2.7 is an isomorphism. In particular*

$$\bar{H}^q(E_\sigma) = 0, q \neq q_{\Lambda+\rho};$$

and there is a natural isomorphism of $G(\mathbf{A}^f)$-modules:

$$\bar{H}^q(E_\sigma) \cong \mathrm{Hom}_{(\mathfrak{g},K_h)}((\pi_{\Lambda+\rho})^*, \mathcal{A}_0(G)). \tag{3.6.4}$$

3.6.5. Remark. This theorem is extremely crude; for any given (G, X), one can prove much more precise statements. For example, using Mirković's Theorem
3.5, one can prove a version of the above theorem for limits of discrete series whose infinitesimal characters are "far from as many walls as possible" [**3**]. This fact has applications to Maass forms of Galois type [**3**].

Under the hypotheses of Theorem 3.6.3, we obtain a natural $k(\sigma)$-rational structure on $\mathrm{Hom}_{(\mathfrak{g},K_h)}((\pi_{\Lambda+\rho})^*, \mathcal{A}_0(G))$. Techniques for recognizing $k(\sigma)$-rational elements of $\mathrm{Hom}_{(\mathfrak{g},K_h)}((\pi_{\Lambda+\rho})^*, \mathcal{A}_0(G))$ will be discussed in §5. More generally, the methods involved in the proof of Theorem 3.6.3 provide a number of rationality theorems for representations with $\bar{\partial}$-cohomology, many of which cannot be proved using $(\mathfrak{g}, K_h)$-cohomology. A simple example is

3.7. PROPOSITION [**3**]. *Suppose π is an irreducible $(\mathfrak{g}, K_h)$-module with $\bar{\partial}$-cohomology with coefficients in σ, and suppose π^f is an irreducible admissible representation of $G(\mathbf{A}^f)$ such that the representation $\pi \otimes \pi^f$ occurs in $\mathcal{A}_0(G)$. Then π^f can be defined over a finite extension of $k(\sigma)$.*

§4. FALTINGS' $B-G-G$ SPECTRAL SEQUENCE

In [**15**], Faltings defines a spectral sequence which relates the cohomology groups studied in §2 to those in §1. In favorable cases this spectral sequence degenerates at E_1 and defines mixed Hodge structures on the cohomology of local systems over Shimura varieties [**14**],[**58**],[**10**]. In this section we recall Faltings' construction, which is based on the Bernstein-Gelfand-Gelfand resolution in category $\mathcal{O}$ of a finite dimensional representation of a semi-simple Lie algebra. We then interpret Faltings' spectral sequence in terms of Lie algebra cohomology.

4.1. We define R^+, W, and W_c as in §3. Let $W^1 = \{w \in W | w(R^+) \supset R_c^+\}$. Then every element $w \in W$ has a unique decomposition $w = w_c \cdot w^1$, with $w_c \in W_c, w^1 \in W^1$, and $\ell(w) = \ell(w_c) + \ell(w^1)$, where $\ell(w)$ is the length of w. For any integer $p \geq 0$, let $W^1(p) = \{w \in W^1, \ell(w) = p\}$.

Let $\mu \in \mathfrak{h}_{\mathbf{C}}^*$ be an R^+-dominant integral weight, and let (ρ, V_μ) be the finite dimensional representation of G with highest weight μ, relative to R^+. If $\Lambda \in \mathfrak{h}_{\mathbf{C}}^*$ is an R_c^+-dominant integral weight, let W_Λ be the finite dimensional K_h-module with highest weight Λ, as in §3; W_Λ extends trivially to a $\mathfrak{P}_h$-module. We let $\mathcal{E}_\Lambda$ be the corresponding homogeneous vector bundle on $\check{X}$. By adapting the method of Bernstein-Gelfand-Gelfand, Faltings constructs an exact sequence of $(\mathfrak{g}, K_h)$-modules [**15**]:

$$0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to V_\mu^* \to 0, \tag{4.1.1}$$

where

$$L_p = \bigoplus_{w \in W^1(p)} U(\mathfrak{g}_{\mathbf{C}}) \otimes_{U(\mathfrak{P}_h)} (W_{w(\mu+\rho)-\rho})^* \tag{4.1.2}$$

Let $D(\Lambda) = U(\mathfrak{g}_{\mathbf{C}}) \otimes_{U(\mathfrak{P}_h)} (W_\Lambda)^*$. It is easy to see [15,Theorem 2;26,§7] that for any two R_c^+-dominant integral weights Λ, Λ', there is a natural bijection

$$\{\text{homogeneous diff. ops. } \mathcal{E}_\Lambda \to \mathcal{E}_{\Lambda'} \text{ over } \check{X}\} \cong \mathrm{Hom}_{U(\mathfrak{g}_{\mathbf{C}})}(D(\Lambda'), D(\Lambda)). \tag{4.1.3}$$

The exact sequence (4.1.1) thus gives rise to a complex

$$0 \to \mathbf{K}^0_\mu \xrightarrow{\delta_0} \mathbf{K}^1_\mu \dots \xrightarrow{\delta_{n-1}} \mathbf{K}^n_\mu \to 0, \tag{4.1.4}$$

of homogeneous vector bundles $\mathbf{K}^p_\mu$, where

$$\mathbf{K}^p_\mu = \bigoplus_{w \in W^1(p)} \mathcal{E}_{w(\mu+\rho)-\rho}, \tag{4.1.5}$$

and where the δ_p are homogeneous differential operators. Moreover, the kernel of δ_0 is the constant sheaf $\check{X} \times V_\mu$.

Applying the functor $\mathcal{E} \to [\mathcal{E}]$ of 1.1, we obtain a $G(\mathbf{A}^f)$-homogeneous complex of automorphic vector bundles over $Sh = Sh(G, X)$:
(4.1.6)

$$0 \to K^0_\mu \xrightarrow{\delta_0} K^1_\mu \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{n-1}} K^n_\mu \to 0, K^p_\mu = \bigoplus_{w \in W^1(p)} E_{w(\mu+\rho)-\rho}.$$

Moreover, the sequence $K^\cdot_\mu$ is a resolution in the category of abelian sheaves of the local system V^∇_μ. Thus the hypercohomology $\mathbf{H}^*(Sh, K^\cdot_\mu)$ is isomorphic to $H^*(V^\nabla_\mu)$. However, it is more useful to work on the toroidal compactifications.

We use the same notation as in (4.1.6) to denote the corresponding sequence over ${}_KSh$, for any open compact $K \subset G(\mathbf{A}^f)$. Assume K is neat and fix a projective SNC toroidal compactification $j_\Sigma : {}_KSh \hookrightarrow {}_KSh_\Sigma$ defined over $E(G, X)$. We can extend (4.1.6) to a complex $(K^\cdot_\mu)^{\mathrm{can},\Sigma}$ of canonical extensions. Faltings verifies ([**15**], §7) that the inclusion $(K^\cdot_\mu)^{\mathrm{can},\Sigma} \hookrightarrow j_{\Sigma,*}(K^\cdot_\mu) \cong Rj_{\Sigma,*}(K^\cdot_\mu)$ is a quasi-isomorphism. In view of the above remarks, the hypercohomology $\mathbf{H}^*({}_KSh_\Sigma, (K^\cdot_\mu)^{\mathrm{can},\Sigma})$ is thus isomorphic to $H^*(V^\nabla_\mu)$. This can be seen in another way. Since $K^\cdot_\mu$ is a resolution of the local system V^∇_μ, it follows from [**22**], §4 that $(K^\cdot_\mu)^{\mathrm{can},\Sigma}$ is a resolution of the canonical extension, in the sense of Deligne [**11**], of V^∇_μ to a vector bundle $V^{\nabla,\mathrm{can}}_\mu$ with a regular connection. The result then follows from Deligne's comparison Theorem ([**11**], Theorem 6.2; cf. [9, IV, Theorem 6.2]).

As an immediate consequence, we obtain the following theorem:

4.2. THEOREM. *(Faltings, [**15**]) There is a spectral sequence*

$$E_1^{p,q} = H^q({}_KSh_\Sigma, (K^p_\mu)^{\mathrm{can},\Sigma}) \Rightarrow H^{p+q}({}_KSh, V^\nabla_\mu). \tag{4.2.1}$$

Passing to the limit over Σ and K, we obtain a $G(\mathbf{A}^f)$-equivariant spectral sequence

$$E_1^{p,q} = \tilde{H}^q((K_\mu^p)^{\mathrm{can}}) \Rightarrow H^{p+q}(Sh, V_\mu^\nabla). \tag{4.2.2}$$

4.3. THEOREM. *(a) (Faltings, [**15**]) If G^{der} has $\mathbf{Q}$-rank 0, then the spectral sequence (4.2.2) degenerates at E_1 and defines a Hodge structure on $H^{p+q}(Sh, V_\mu^\nabla)$.*

*(b) (Faltings, Chai-Faltings, [**14**],[**10**]) If $G = GSp(n)$, for some n, and Sh is the Siegel modular Shimura variety of genus n, then the spectral sequence (4.2.2) degenerates at E_1 and defines the F-filtration of a mixed Hodge structure on $H^{p+q}(Sh, V_\mu^\nabla)$.*

4.3.1. Remark. A result equivalent to Theorem 4.3 (a) is proved by Zucker in [**58**], following Deligne. Both proofs are based on the theory of harmonic forms. The proof of 4.3 (b) is based on Deligne's mixed Hodge theory for complete varieties, specifically for the universal abelian variety over the Siegel modular variety with level structure. It presumably extends to any situation in which V is obtained by tensor operations from the cohomology of a family of abelian varieties over Sh. Faltings has conjectured that the spectral sequence (4.2.2) always degenerates at E_1.

4.3.2. Suppose the complex (4.1.4) of homogeneous vector bundles on $\check{X}$ is defined over the number field $k'(\mu)$. The spectral sequence (4.2.2), and the existence of canonical models for automorphic vector bundles, determines a $k'(\mu)$-rational structure and F-filtration on $H^*(Sh, V_\mu^\nabla)$. On the other hand, if $k(\mu)$ is the field of definition of (ρ, V_μ), then $H^*(Sh, V_\mu^\nabla)$ comes equipped with a $k(\mu)$-rational structure. Following Deligne [**13**], we refer to the first structure as the *de Rham* rational structure, the second as the *Betti* rational structure. Even if $k(\mu) = k'(\mu)$, these two rational structures are expected in general to be quite different, reflecting the presence of transcendental *periods.*

4.4. We have already interpreted the E_1 and E_∞ terms in (4.2.2) in terms of Lie algebra cohomology. It is therefore not surprising that the spectral sequence itself has an interpretation in terms of Lie algebra cohomology. With $D(\Lambda)$ as in 4.1, let $D(\Lambda)\check{}$ denote the space of K_h-finite vectors in $\mathrm{Hom}_{\mathbf{C}}(D(\Lambda), \mathbf{C})$. Dualizing (4.1.2), we obtain an exact sequence of $(\mathfrak{g}, K_h)$-modules

$$0 \to V_\mu \to L^0 \to L^1 \to \cdots \to L^n \to 0, \tag{4.4.1}$$

where

$$L^p = \bigoplus_{w \in W^1(p)} D(w(\mu+\rho)-\rho)\check{} \tag{4.4.2}$$

The actions of $\mathfrak{g}$ and K_h on L^q are the usual contragredient actions: if $f \in \mathrm{Hom}_{\mathbf{C}}(D(\Lambda), \mathbf{C}), X \in \mathfrak{g}, k \in K_h$, and $v \in D(\Lambda)$, then

$$X \cdot f(v) = -f(Xv), \quad k \cdot f(v) = f(k^{-1} \cdot v). \tag{4.4.3}$$

If Z is any $(\mathfrak{P}_h, K_h)$-module we define, following Vogan [53,6.1.21]

$$\mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(Z) = \mathrm{Hom}_{U\mathfrak{P}_h}(U\mathfrak{g}_{\mathbf{C}}, Z)_0, \tag{4.4.4}$$

where the subscript 0 denotes K_h-finite vectors, as before. Here $U\mathfrak{g}_{\mathbf{C}}$ and Z are viewed as left $U\mathfrak{P}_h$-modules, and $U\mathfrak{g}_{\mathbf{C}}$ acts on $\mathrm{Hom}_{U\mathfrak{P}_h}(U\mathfrak{g}_{\mathbf{C}}, Z)$ by right multiplication. With the diagonal action of K_h, $\mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(Z)$ becomes in a natural way a $(\mathfrak{g}, K_h)$-module.

If $\mathcal{H}$ is any $(\mathfrak{g}, K_h)$-module, let $L^{\cdot}(\mathcal{H})$ be the complex $L^{\cdot} \otimes_{\mathbf{C}} \mathcal{W}$. The diagonal action of $\mathfrak{g}$ makes $L^{\cdot}(\mathcal{H})$ into a complex of $(\mathfrak{g}, K_h)$-modules, and we have an exact sequence

$$0 \to \mathcal{H} \otimes_{\mathbf{C}} V_\mu \to L^0(\mathcal{H}) \to L^1(\mathcal{H}) \to \cdots \to L^n(\mathcal{H}) \to 0. \tag{4.4.5}$$

For each q, there is a natural isomorphism of $(\mathfrak{g}, K_h)$-modules

$$L^q(\mathcal{H}) \cong \bigoplus_{w \in W^1(p)} \mathcal{H} \otimes_{\mathbf{C}} \mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(W_{w(\mu+\rho)-\rho}) \tag{4.4.6}$$

(Warning: the right and left actions of $\mathfrak{g}$ have been interchanged!) On the other hand, for any $(\mathfrak{P}_h, K_h)$-module Z, there is a natural isomorphism

$$\mathcal{H} \otimes_{\mathbf{C}} \mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(Z) \tilde{\to} \mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(\mathcal{H} \otimes_{\mathbf{C}} Z) \tag{4.4.7}$$

given by the map

(4.4.8)

$h \otimes \lambda \mapsto (Y \mapsto Y \cdot h \otimes \lambda(Y)), \lambda \in \mathrm{Hom}_{U\mathfrak{P}_h}(U\mathfrak{g}_{\mathbf{C}}, Z)_0, h \in \mathcal{H}, Y \in U\mathfrak{g}_{\mathbf{C}}$.

It is clear that (4.4.8) intertwines the $(\mathfrak{g}, K_h)$-actions on the two sides of (4.4.7). That (4.4.7) is an isomorphism follows from the fact that, as K_h-modules, $\mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(Z)$ is isomorphic to $\bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathfrak{p}^+)^* \otimes Z$.

By (4.4.6) and (4.4.7), (4.4.5) becomes

$$(4.4.9) \qquad 0 \to \mathcal{H} \otimes_{\mathbf{C}} V_\mu \to \mathcal{L}^0(\mathcal{H}) \to \mathcal{L}^1(\mathcal{H}) \to \cdots \to \mathcal{L}^n(\mathcal{H}) \to 0,$$

where

$$\mathcal{L}^q(\mathcal{H}) \cong \bigoplus_{w \in W^1(p)} \mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(\mathcal{H} \otimes_{\mathbf{C}} W_{w(\mu+\rho)-\rho}).$$

We obtain a spectral sequence in relative Lie algebra cohomology:

$$(4.4.10) \qquad E_1^{p,q} = H^q(\mathfrak{g}, K_h, \bigoplus_{w \in W^1(p)} \mathcal{H} \otimes_{\mathbf{C}} \mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(W_{w(\mu+\rho)-\rho})) \Rightarrow H^{p+q}(\mathfrak{g}, K_h, \mathcal{H} \otimes_{\mathbf{C}} V_\mu).$$

Now Shapiro's Lemma, in the form given in [**53**], Proposition 6.1.27, implies that, for any $(\mathfrak{P}_h, K_h)$-module Z,

$$(4.4.11) \qquad H^q(\mathfrak{g}, K_h, \mathrm{pro}_{\mathfrak{P}_h, K_h}^{\mathfrak{g}, K_h}(Z)) \cong H^q(\mathfrak{P}_h, K_h, Z).$$

Combining (4.4.10) and (4.4.11), we obtain

4.4.12. Proposition. *Let V_μ be the finite dimensional $(\mathfrak{g}, K_h)$-module with highest weight μ, and let $\mathcal{H}$ be any $(\mathfrak{g}, K_h)$-module. There is a spectral sequence (the B-G-G spectral sequence)*

$$E_1^{p,q} = H^q(\mathfrak{P}_h, K_h, \bigoplus_{w \in W^1(p)} \mathcal{H} \otimes_{\mathbf{C}} W_{w(\mu+\rho)-\rho}) \Rightarrow H^{p+q}(\mathfrak{g}, K_h, \mathcal{H} \otimes_{\mathbf{C}} V_\mu).$$

If $\mathcal{H}$ is a unitary $(\mathfrak{g}, K_h)$-module, then the B-G-G spectral sequence degenerates at E_1.

The last statement is an immediate consequence of the usual harmonic theory (Kuga's formula (1.5.1) and Proposition 3.2.2), and is a somewhat more efficient way of expressing the bigrading on $H^*(\mathfrak{g}, K_h, \mathcal{H} \otimes_{\mathbf{C}} V_\mu)$ in the Hermitian symmetric case ([**30**], [8, II, §4]).

4.4.13. Example: *The discrete series.* Fix an integer $p_0 \in \{0, \ldots, n\}$ and a $w \in W^1(p_0)$, and let $\mathcal{H}$ be the discrete series module $\pi^*_{w(\mu+\rho)}$. Let $q_0 = q_{w(\mu+\rho)}, \Lambda_0 = w(\mu+\rho)-\rho$. Evidently $p_0+q_0 = n$. It follows from Theorem 3.4 that every term $E_1^{p,q}$ in the B-G-G spectral sequence vanishes except the term $E_1^{p_0,q_0}$, which is one-dimensional and equal to $H^{q_0}(\mathfrak{P}_h, K_h, \mathcal{H} \otimes_{\mathbf{C}} V_{\Lambda_0})$. The B-G-G spectral sequence thus becomes a canonical isomorphism

$$H^{q_0}(\mathfrak{P}_h, K_h, \pi^*_{w(\mu+\rho)} \otimes_{\mathbf{C}} V_{w(\mu+\rho)-\rho}) \tilde{\to} H^n(\mathfrak{g}, K_h, \pi^*_{w(\mu+\rho)} \otimes_{\mathbf{C}} V_\mu)$$

of one-dimensional vector spaces; all $H^i(\mathfrak{g}, K_h, \pi^*_{w(\mu+\rho)} \otimes_{\mathbf{C}} V_\mu)$, with $i \neq n$, vanish. This computation of the cohomology $H^*(\mathfrak{g}, K_h, \pi^*_{w(\mu+\rho)} \otimes_{\mathbf{C}} V_\mu)$ is the same as the one given in [**8**].

We now return to the global situation of §1–2. Let V_μ and K^p_μ be as in 4.1, and take $\mathcal{H}$ in (4.4.12) to be the $(\mathfrak{g}, K_h)$-module $C_{si} = C_{si}(G)$. We obtain

4.5. THEOREM. *The B-G-G spectral sequence (4.4.12) for the $(\mathfrak{g}, K_h)$-module C_{si} gives, via the isomorphisms (1.3.3) and (2.4.5), a $G(\mathbf{A}^f)$-equivariant spectral sequence*

$$E_1^{p,q} = \tilde{H}^q((K^p_\mu)^{\mathrm{can}}) \Rightarrow H^{p+q}(Sh, V^\nabla_\mu),$$

which coincides with Faltings' spectral sequence (4.2.1).

That the spectral sequences 4.5 and (4.2.1) coincide follows immediately from the definitions. We note that Chai and Faltings also construct a spectral sequence

$$E_1^{p,q} = \tilde{H}^q((K^p_\mu)^{\mathrm{sub}}) \Rightarrow H_c^{p+q}(Sh, V^\nabla_\mu)$$

The above arguments show that this coincides with the B-G-G spectral sequence (4.4.12) for the $(\mathfrak{g}, K_h)$-module C_{rd}, via (1.3.3) and (2.4.5).

It does not seem that the methods of 4.4 provide a simple route to proving the degeneration of the spectral sequence (4.5), except in the case where G^{der} has $\mathbf{Q}$-rank 0, where the result is well-known. One might consider replacing C_{si} with the space C_{umg} of functions of uniform moderate growth, as in [**6**]. Borel shows that $H^*(Sh, V^\nabla_\mu) \cong H^*(\mathfrak{g}, K_h, C_{umg} \otimes V_\mu)$, and his methods probably work for $(\mathfrak{P}_h, K_h)$-cohomology as well. The advantage of C_{umg}, as Borel points out, is that Langlands' theory of Eisenstein series implies that C_{umg} decomposes as the direct sum of pieces corresponding to the standard rational parabolic subgroups of G. One might then hope that the degeneration can be proved by induction on the $\mathbf{Q}$-rank of G^{der}. The problem is that the Levi components of parabolic subgroups of G are in general not of Hermitian type; nevertheless, their cohomology contributes to the cohomology of Sh, as we see in §6.

4.6. Example: *1.1.3, continued.* We retain the coordinates of 3.1.1, and work out the parameters for the B-G-G spectral sequence for $GL(2) \cong GSp(1)$ and $GSp(2)$. For $G = GL(2)$, the highest weight μ of V_μ is given by a pair $(a;c) \in \mathfrak{h}^*_{\mathbf{C}}$, with $c \equiv a(mod 2)$, $a \geq 0$. Then $K^0_\mu = E_{(a;c)}$, $K^1_\mu = E_{(-a-2;c)}$. The discrete series representation π^*_λ, with $\lambda = (a+1;c)$ (resp. $\lambda = (-a-1;c)$) has $\bar\partial$-cohomology with coefficients in W_Λ, with $\Lambda = (a;c)$ (resp. $\Lambda = (-a-2;c)$). Hence $\pi^*_{(a+1;c)}$ (resp. $\pi^*_{(-a-1;c)}$) contributes to $\bar{H}^1(K^0_\mu)$ (resp. to $\bar{H}^0(K^0_\mu)$); if $a \geq 1$ (λ far from the walls), then there are isomorphisms

$$(4.6.1)\quad \bar{H}^1(K^0_\mu) \cong \mathrm{Hom}(\mathfrak{g}, K_h)(\pi^*_{(a+1;c)}, \mathcal{A}_0(G)), \bar{H}^0(K^1_\mu) \cong \mathrm{Hom}(\mathfrak{g}, K_h)(\pi^*_{(-a-1;c)}, \mathcal{A}_0(G)).$$

The latter group corresponds to the space of holomorphic cusp forms of all levels, of weight $a+2$ and central character $-c$ (at the archimedean prime).

Now assume $G = GSp(2)$. Then μ is given by $(a,b;c)$, with $a \geq b \geq 0, c \equiv a+b(mod 2)$. We have $K^0_\mu = E_{(a,b;c)}, K^1_\mu = E_{(a,-b-2;c)}, K^2_\mu = E_{(b-1,-a-3;c)}, K^3_\mu = E_{(-b-3,-a-3;c)}$. The corresponding discrete series representations, in the same order, are $\pi^*_{(a+2,b+1;c)}$, $\pi^*_{(a+2,-b-1;c)}$, $\pi^*_{(b+1,-a-2;c)}$, and $\pi^*_{(-b-1,-a-2;c)}$.

§5. Rationality criteria for harmonic cusp forms

5.1. If $\pi = \pi_{\Lambda+\rho}$ is a sufficiently regular discrete series $(\mathfrak{g}, K_h)$-module, then Theorem 3.6.3 defines a natural $k(\sigma_\Lambda)$-rational structure on $\mathrm{Hom}_{(\mathfrak{g},K_h)}(\pi^*, \mathcal{A}_0(G))$, or equivalently on the space $\mathcal{H}^q_{\mathrm{cusp},\Lambda}$ of harmonic cuspidal $(0,q)$-forms with values in W_Λ, where $q = q_{\Lambda+\rho}$. The question was raised in §3 of recognizing the $k(\sigma_\Lambda)$-rational elements in $\mathcal{H}^q_{\mathrm{cusp},\Lambda}$. An extended discussion of the motivation for this question can be found in [**20**].

When $q = 0$, $\mathcal{H}^0_{\mathrm{cusp},\Lambda}$ may be identified with a space of holomorphic functions on $X \times G(\mathbf{A}^f)$, and the question has been studied by numerous authors. The cases of elliptic, Hilbert, and Siegel modular forms are familiar: a $\bar{\mathbf{Q}}$-rational structure is defined by forms with algebraic Fourier coefficients. Generalizations of this criterion were studied by Shimura, Garrett, Brylinski, Milne, and the author [**46**],[**16**],[**19**], and are discussed in Milne's talk. We will see below that the obvious extension of this criterion fails when $q > 0$.

An alternative characterization of "arithmetic" holomorphic automorphic forms was introduced by Shimura in [**48**] and exploited in his subsequent papers. It is based on the values of automorphic forms at CM points: an arithmetic section of E_{σ_Λ} is one whose values at the CM point w is an algebraic multiple of a certain product of periods of CM abelian varieties, depending on w and Λ. This criterion has the advantage that it is available for all Shimura varieties, whereas the criterion based on Fourier coefficients (or, more generally, on Fourier-Jacobi expansions) only works for Shimura varieties with point boundary components. Suitably generalized, this criterion is at the heart of the author's construction over the reflex field of the functor $\mathcal{E} \to [\mathcal{E}]$ of 1.1, and of Milne's subsequent construction of canonical models of automorphic vector bundles [**19**],[**32**].

The next case to study is the case $q = n = \dim X$. Let $\mathcal{V} = E_{\sigma_\Lambda}$, and define $\mathcal{V}'$ as in 2.4; let $\Lambda' \in \mathfrak{h}_{\mathbf{C}}$ be the character such that $\mathcal{V}' = E_{\sigma_{\Lambda'}}$. Complex conjugation, followed by a certain character twist, defines an antilinear isomorphism

$$C : \mathcal{H}^0_{\mathrm{cusp},\Lambda} \tilde{\to} \mathcal{H}^n_{\mathrm{cusp},\Lambda'} (\text{cf. } [\mathbf{23}, \S 5]). \tag{5.1.1}$$

When $G = GL(2,\mathbf{Q}), C$ takes holomorphic forms with algebraic Fourier coefficients to anti-holomorphic forms with algebraic Fourier coefficients. More generally, C preserves the arithmetic properties of values of forms at CM points. But Serre duality shows that the map $C_{\mathcal{V}} : \bar{H}^0(\mathcal{V}) \to \bar{H}^n(\mathcal{V}') \cong \bar{H}^0(\mathcal{V})^*$ defined by C is not, in general, $\bar{\mathbf{Q}}$-rational. Indeed, suppose $f \in \mathcal{H}^0_{\mathrm{cusp},\Lambda}$ is such that $\phi = cl(f) \in \bar{H}^0(\mathcal{V})$ is arithmetic, in the above sense; here cl is as in 2.7. Suppose f belongs to an irreducible representation π^f of $G(\mathbf{A}^f)$. If $\psi =_{\mathrm{def}} cl(C(f)) \in \bar{H}^0(\mathcal{V})^*$ were arithmetic, then, by Proposition 2.5, we would have

$$< \phi, \psi > = (2\pi i)^{-n} \int_{G(\bar{\mathbf{Q}})\backslash G(\mathbf{A})/K_h} [f \wedge C(f)] \in \bar{\mathbf{Q}}. \tag{5.1.2}$$

In other words, the Petersson square norm of ϕ, normalized as in (5.1.2), would have to be an algebraic number, which is extremely unlikely.

We remark that when $G = GL(2,F)$, with F a totally real field, Shimura has shown [47, Prop. 4.14] that, if ϕ and ϕ' are two arithmetic holomorphic vectors belonging to the same irreducible $G(\mathbf{A}^f)$-submodule of $\mathcal{Q}_0(G)$, then

$$< \phi, C_{\mathcal{V}}(\phi)) > \sim_{\bar{\mathbf{Q}}} < \phi', C_{\mathcal{V}}(\phi')) >, \tag{5.1.3}$$

where if $a, b \in \mathbb{C}^\times$ and k is a subfield of $\mathbb{C}$, we write $a \sim_k b$ if $a/b \in k^\times$. The proof is based on the relation between the Petersson inner product and special values (more precisely, residues) of L-functions. For general G, a similar argument of Shimura [**51**] shows, in some cases, that (5.1.3) holds under the weaker assumption that ϕ and ϕ' generate *isomorphic* representations of $G(\mathbf{A}^f)$. When this is true, it follows easily that

$$(5.1.4) \qquad \frac{C_{\mathcal{V}}(\phi)}{< \phi, C_{\mathcal{V}}(\phi) >} \text{ is a } \bar{\mathbb{Q}}\text{-rational element of } \bar{H}^n(\mathcal{V}').$$

One way to distinguish arithmetic cohomology classes of intermediate dimension is provided, in some cases, by the following criterion, which may be viewed as a generalization of Shimura's criterion in [**48**] for arithmetic holomorphic automorphic forms. Let $(G^\#, X^\#) \hookrightarrow (G, X)$ be a morphism of data defining Shimura varieties; then $Sh^\# = Sh(G^\#, X^\#)$ is naturally a $G^\#(\mathbf{A}^f)$-homogeneous subvariety of Sh. Similarly, the flag variety $\check{X}^\#$ is a $G^\#(\mathbf{A}^f)$-homogeneous subvariety of $\check{X}$. We assume our point $h \in X$ is actually in $X^\#$, and let $K_h^\# \subset K_h$ be its stabilizer in $G^\#(\mathbb{R})$; define $\mathfrak{h}^\#$, $\mathfrak{p}^{\#,-}$ and $\mathfrak{P}_h^\#$ in the obvious way. The following is a special case of Theorem 7.6 of [**21**]:

5.2. THEOREM. *Let (π, V) (resp. $(\pi^\#, V^\#)$) be an irreducible unitary representation of $G(\mathbb{R})^0$ (resp. $G^\#(\mathbb{R})^0$) such that $\pi^\#$ is a closed direct factor of $\pi|_{G^\#(\mathbb{R})^0}$. Let (σ, W_σ) (resp. $(\sigma^\#, W_{\sigma^\#})$) be an irreducible unitary representation of K_h (resp. $K_h^\#$) such that $\sigma^\#$ is a direct factor of $\sigma|_{K_h^\#}$. Define the homogeneous vector bundles $\mathcal{E}_\sigma$, $\mathcal{E}_{\sigma^\#}$, over $\check{X}$ and $\check{X}^\#$, respectively, and let $\mathcal{V} = E_\sigma, \mathcal{V}^\# = E_{\sigma^\#}$ be the corresponding automorphic vector bundles. Assume that $\mathcal{E}_\sigma$, $\mathcal{E}_{\sigma^\#}$, and the natural homomorphism $\mathcal{E}_\sigma|_{\check{X}} \to \mathcal{E}_{\sigma^\#}$ are all defined over the extension $k^\#$ of $E(G^\#, X^\#)$. Assume*

(a) The representations π and $\pi^\#$ belong to the discrete series, and their parameters λ and $\lambda^\#$ are both sufficiently regular (cf. Remark 5.2.1, below);

(b) $\dim H^q(\mathfrak{P}_h, K_h, V_0 \otimes W_\sigma) = \dim H^q(\mathfrak{P}_h^\#, K_h^\#, V_0^\# \otimes W_{\sigma^\#}) = 1$, and the orthogonal projection $p \otimes p_{\sigma,\sigma^\#} : V \otimes W_\sigma \to V(\pi^\#) \otimes W_{\sigma^\#}$ induces a non-trivial homomorphism

$$H^q(\mathfrak{P}_h, K_h, V_0 \otimes W_\sigma) \to H^q(\mathfrak{P}_h^\#, K_h^\#, V(\pi^\#)_0 \otimes W_{\sigma^\#})$$

, where $V(\pi^{\#})$ is the $\pi^{\#}$-isotypic subspace of V.

Let $\phi \in \mathcal{H}^q_{\mathrm{cusp},\sigma}$, and let $F = cl(\phi) \in \bar{H}^q(\mathcal{V})$, in the notation of 2.7. For $\gamma \in G(\mathbf{A}^f)$, we let $\psi_\gamma : \bar{H}^q(\mathcal{V}) \to \bar{H}^q(\mathcal{V}^{\#})$ denote $\psi \circ t^*_\gamma$, where $\gamma \mapsto t^*_\gamma$ is the natural action of $G(\mathbf{A}^f)$ on $\bar{H}^q(\mathcal{V})$. Then F is rational over the extension L of $k^{\#}$ if and only if, for every $\gamma \in G(\mathbf{A}^f)$, the element $\psi_\gamma(F)$ is an L-rational element of $\bar{H}^q(\mathcal{V}^{\#})$.

5.2.1. Remark. (cf. [**21**], Remark 7.6.1): In (a) it is necessary to assume in particular that $\pi^{\#}$ belongs to the integrable discrete series, and that both λ and $\lambda^{\#}$ are sufficiently regular to satisfy Theorem 3.6.3. It is actually not necessary that π be a discrete series representation; it suffices that it satisfy the analogue of (3.6.4).

The remainder of this section will discuss some examples in which this criterion can be applied.

5.3. In this section $G = GL(2, F)$, where F is a real quadratic field (possibly $\mathbf{Q} \times \mathbf{Q}$), and $G^{\#} = GL(2, \mathbf{Q})$, with its natural imbedding in G. This case is studied in [21, §8]. As in 3.1.1, representations of $K_h^{\#}$, and hence automorphic vector bundles over $Sh^{\#}$, are parametrized by pairs of integers $(a; c)$, with $a \equiv c \pmod 2$. Similarly, $G(\mathbf{R}) \cong G^{\#}(\mathbf{R}) \times G^{\#}(\mathbf{R}), K_h \cong K_h^{\#} \times K_h^{\#}$; representations of K_h and automorphic vector bundles over Sh correspond to quadruples $((a_1; c_1), (a_2; c_2))$ satisfying $a_i \equiv c_i \pmod 2$; this congruence condition will henceforth be assumed. We note that Z_G does not satisfy the hypothesis imposed in 1.1, so that the automorphic vector bundles over Sh exist, in general, only in the sense of stacks, as in Milne's article in this volume. The reader who dislikes stacks may replace G by its subgroup of elements with determinant in $\mathbf{G}_{m,\mathbf{Q}} \subset R_{F/\mathbf{Q}} G_{m,F}$.

For a and c as above, $a > 0$, let $\pi(a; c)$ and $\pi(-a; c)$ be the discrete series representations of $GL(2, \mathbf{R})^0$, parametrized as in 4.6. A theorem of Repka [**37**] describes the restriction to the diagonal of completed tensor products of discrete series representations.

5.3.1. Theorem. *([**37**], Theorem 7.3). Let $k, \ell, c, c' \in \mathbf{Z}$, with $k \equiv c \pmod 2, \ell \equiv c' \pmod 2, k, \ell \geq 1$. The restriction to the diagonal $G^{\#}(\mathbf{R})^0 \subset G(\mathbf{R})^0$ of the completed tensor product $\pi_{(k;c)} \hat{\otimes} \pi_{(-\ell;c')}$ contains as a closed direct summand the representation $\pi_{(a-1,c+c')}$ with multiplicity one, for every $a \in \mathbf{Z}$ such that (i) $2 \leq |a| \leq |k - \ell|$; and (iii) a is of the same sign and parity as $k - \ell$.*

5.3.2. Corollary. *([**21**], Theorem 8.6). Let $\Lambda = ((a_1; c_1), (a_2; c_2)) \in \mathfrak{h}^*_{\mathbf{C}}$, and suppose $a_1 \leq -2, a_2 \geq 0$, and $a_1 + a_2 > 0$.*

Let $\Lambda^{\#} = (a_1 + a_2; c_1 + c_2) \in \mathfrak{h}_{\mathbf{C}}^{\#,*}$. Let E_Λ and $E_{\Lambda^\#}$ be the corresponding automorphic vector bundles over Sh and $M^{\#}$, respectively. Then $f \in \bar{H}^1(E_\Lambda)$ is rational over the extension L of F if and only if, for every $\gamma \in G(\mathbf{A}^f)$, the element $\psi_\gamma(f)$, defined as above, is an L-rational element of $\bar{H}^1(E_{\Lambda^\#})$.

An analogous result holds when G is replaced by the multiplicative group of certain totally indefinite quaternion algebras. A test for rationality of elements of $\bar{H}^1(E_{\Lambda^\#})$ in terms of integrals against holomorphic cusp forms on $M^{\#}$, along the lines described in 5.1, above, is discussed in [**20**] and [21,§8], where it is explained how such integrals arose in earlier work of Shimura [**50**].

5.4. In this section $G = GSp(2)$, $G^{\#} = \{(g_1, g_2) \in GL(2) \times GL(2) | \det(g_1) = \det(g_2)\}$, imbedded diagonally in G. In this case $\mathfrak{h} = \mathfrak{h}^{\#} = \{k(z, I_2, d(t_1, t_2))\}$, in the notation of 3.1.1. Automorphic vector bundles on Sh and $Sh^{\#}$, and discrete series representations of $G(\mathbf{R})^0$ and $G^{\#}(\mathbf{R})^0$, will all be parametrized by triples $(a, b; c)$, as in 3.1.1 and 4.6. The following results are joint work with S. Kudla.

5.4.1. THEOREM [**23**, §4]. *Let π be the discrete series representation $\pi^*_{(b+1,-a-2;-c)}$ of $G(\mathbf{R})^0$, with $a \geq b \geq 0$. The discrete spectrum of the restriction of π to $G^{\#}(\mathbf{R})^0$ is given by the the union of three sets of discrete series representations:*

(i) $\pi_{(k,\ell;c)}, |k-\ell| \geq a+b+4;$

(ii) $\pi_{(k,\ell;c)}, k+\ell \leq a-b;$

(iii) $\pi_{(k,-\ell;c)}$ *and* $\pi_{(-\ell,k;c)}, k-\ell \geq a-b+2, k+\ell \leq a+b+2.$

In each case, $k, \ell \geq 1$ are integers which satisfy $k + \ell \equiv a + b (\mathrm{mod}\, 2)$, and each of these representations occurs in π with multiplicity one.

Applying this theorem both to $\pi^*_{(b+1,-a-2;-c)}$, which has $\bar{\partial}$-cohomology in degree one, and to its contragredient, which has $\bar{\partial}$-cohomology in degree two, we can prove

5.4.2. THEOREM [**23**, §2]. *In the notation of 5.4.1, suppose $b > 1, a - b > 1$.*

(a) Let $\Lambda = \Lambda^{\#} = (b-1, -a-3, c)$. For every $\gamma \in G(\mathbf{A}^f)$, restriction defines a natural map $\psi_\gamma : \bar{H}^1(E_\Lambda) \to \bar{H}^1(E_{\Lambda^\#})$, as above, and $f \in \bar{H}^1(E_\Lambda)$ is rational over the extension L of $\mathbf{Q}$ if and only if $\psi_\gamma(f)$ is an L-rational element of $\bar{H}^1(E_{\Lambda^\#})$, for every $\gamma \in G(\mathbf{A}^f)$.

(b) Let $\Lambda = (a, -b-2, -c), \Lambda^{\#}(i) = (a-i, -b-2+i, -c), 0 \leq i \leq a+b+2$. *Then* $E_{\Lambda^{\#}(i)}$ *is a direct factor of* $E_{\Lambda|Sh^{\#}}$, *and for every* $\gamma \in G(\mathbf{A}^f)$, *restriction defines a natural map* $\psi_\gamma : \bar{H}^2(E_\Lambda) \to \bar{H}^2(E_{\Lambda^{\#}})$. *Suppose* $b+2 < i < a$. *Then* $f \in \bar{H}^2(E_\Lambda)$ *is rational over the extension* L *of* $\mathbb{Q}$ *if and only if* $\psi_\gamma(f)$ *is an* L*-rational element of* $\bar{H}^2(E_{\Lambda^{\#}(i)})$, *for every* $\gamma \in G(\mathbf{A}^f)$.

The inequalities above ensure that hypothesis (a) is verified in Theorem 5.2; however, they are probably unnecessarily restrictive. Some arithmetic applications only require *necessary* conditions for rationality, in which case the inequalities can be removed.

We note that (a) of 5.4.2 uses the set of representations in (iii) of 5.4.2, whereas (b) uses the representations in (ii) of 5.4.1. The arithmetic significance of the infinite family (i) of 5.4.1 is unknown.

5.5. Remark. In contrast to the situation with $q = 0$, nothing guarantees *a priori* that the restriction maps ψ_γ on coherent cohomology in higher degree are ever non-trivial, much less that they determine the rational structure on the coherent cohomology of the ambient group. Thus our criterion 5.4.2 may seem surprising. However, Weissauer has recently announced that the divisor class group of the Siegel modular variety ${}_K\mathcal{M}_2$ of genus 2 and level K is generated by imbedded products of modular curves. In particular, if ${}_K\mathcal{M}_2 \hookrightarrow {}_K\mathcal{M}_{2,\Sigma}$ is a projective toroidal compactification, then one can find a positive linear combination D of imbedded products of modular curves and divisors at infinity which is linearly equivalent to a hyperplane section on ${}_K\mathcal{M}_{2,\Sigma}$; restriction to this D thus defines (by Serre's FAC) an *injective* morphism $H^q({}_K\mathcal{M}_{2,\Sigma}, \mathcal{V}^{\mathrm{can}}) \to H^q(D, \mathcal{V}^{\mathrm{can}}|_D)$ for $q \leq 2$. Here D is viewed as an infinitesimal neighborhood of $\mathrm{Supp}(D)$. We note that, if $Sh^{\#}$ is an imbedded product of modular curves in ${}_K\mathcal{M}_2$, restricting cohomology to an infinitesimal neighborhood of $Sh^{\#}$ corresponds, on the level of representation theory, to projecting a discrete series representation of $G(\mathbb{R})^0$ simultaneously onto several of the factors in (ii) and (iii) of 5.4.1.

Weissauer's results raise the possibility that the restriction maps $H^q({}_K\mathcal{M}_{2,\Sigma}, \mathcal{V}^{\mathrm{can}}) \to H^q(D, \mathcal{V}^{\mathrm{can}}|_D)$ can be used to determine the $\mathbb{Z}$-structure of the former group, defined by Chai and Faltings [**10**].

§6. Coherent cohomology defined by Eisenstein series

Classical holomorphic Eisenstein series may be regarded, like cusp forms, as sections of automorphic vector bundles, or of their canonical extensions to toroidal compactifications. In general, it is well-known that, if $\mathcal{V}$ is a sufficiently regular automorphic vector bundle on $Sh = Sh(G, X), \tilde{H}^0(\mathcal{V}^{\mathrm{can}})$ decomposes as a direct sum of cusp forms and convergent holomorphic Eisenstein series, corresponding to the rational maximal parabolic subgroups of G. On the other hand, if $n = \dim X$, then $\tilde{H}^n(\mathcal{V}^{\mathrm{can}})$ always consists entirely of anti-holomorphic cusp forms.

For $\tilde{H}^q(\mathcal{V}^{\mathrm{can}})$, with $0 < q < n$, the situation is much less clear, even for highly regular $\mathcal{V}$. The spectral sequence (4.2.2) shows that this problem is related to the problem of representing elements of $H^*(Sh, V_\mu^\nabla)$ by automorphic forms. The work of Harder and Schwermer [**17**],[**42**],[**43**] has succeeded, in some cases, in representing $H^*(Sh, V_\mu^\nabla)$ as a direct sum of cuspidal cohomology (1.5.2) and either the values or residues of certain Eisenstein series. Their methods, which in general make no reference to the holomorphic structure of M, produce Eisenstein cohomology classes which are rational for the *Betti* rational structure on $H^*(Sh, V_\mu^\nabla)$.

It is desirable to have a similar theory of Eisenstein cohomology which, corresponds to the *de Rham* rational structure on $H^*(Sh, V_\mu^\nabla)$, or more precisely, which provides a complement to the cuspidal classes in $\tilde{H}^*(\mathcal{V}^{\mathrm{can}})$. In general this problem is far from being completely understood. However, by using the explicit imbeddings, defined by Blank [**2**], of discrete series representations in certain induced representations, one can define arithmetically interesting Eisenstein classes in coherent cohomology. These classes correspond to *cusp forms on cuspidal maximal parabolic subgroups.*

6.1. We briefly recall the construction of Blank. Let $P = MAN$ be a cuspidal maximal rational parabolic subgroup of G and assume that $K_M = K \cap M$ contains a maximal compact subgroup of $M_0 = M^{der}(\mathbb{R})^0$. For simplicity, we assume G^{ad} to be $\mathbb{R}$-simple, although this hypothesis is unnecessary. Thus $\dim A = 1$ and $M = Z_G(A)$ has the property that rank $M =$ rank K_M. We assume that $\mathfrak{h}_M = H \cap \mathrm{Lie}(M)$ is a Cartan subalgebra of K_M, hence of M.

Let $\lambda \in \mathcal{F}+\rho \subset \mathfrak{h}_{\mathbb{C}}^*$ (notation 3.3) be the Harish-Chandra parameter of the discrete series representation π_λ of G_0 (cf. 3.3.2) and let λ_M be the restriction of λ to $\mathfrak{h}_M$. Then λ_M is a Harish-Chandra parameter for M_0 [2, Prop. 4.1]. We would like to define the corresponding discrete series representation of $M(\mathbb{R})$. Since $M(\mathbb{R})$ is disconnected, this can be done in more than one way; Blank picks one out, which we denote π_{λ_M}.

Let $\mathfrak{a} = \mathrm{Lie}(\mathbf{A})$; then any $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ defines a character e^ν of A. Extend $\pi_{\lambda_M} \otimes e^\nu$ trivially to N, and let $I(P, \lambda_M, \nu)$ denote the representation induced from P to G of $\pi_{\lambda_M} \otimes e^\nu$, normalized so that $I(P, \lambda_M, \nu)$ is unitary when ν is imaginary. Order $\mathfrak{a}_{\mathbb{R}}^*$ so that the roots of A on N are positive. In [**2**], Blank gives an explicit analytic proof of the following unpublished theorem of Schmid:

6.2. THEOREM. *There exists a unique positive $\nu(\lambda) \in \mathfrak{a}_{\mathbb{R}}^*$ such that there exists an imbedding $S_\lambda : \pi_\lambda \to I(P, \lambda_M, \nu(\lambda))$ as representations of G_0.*

Although the uniqueness of $\nu(\lambda)$ is not mentioned by Blank, it is determined by the condition that $I(P, \lambda_M, \nu(\lambda))$ and π_λ have the same infintesimal character.

6.3. Since $\dim(A) = 1$ we may identify $\mathfrak{a}_{\mathbb{R}}^* \cong \mathbb{R}$, with the given ordering. If $\nu(\lambda)$ is sufficiently large (this is true for λ far from the walls), then $I(P, \lambda_M, \nu(\lambda))$ is in the range of absolute convergence for Eisenstein series. More precisely, let M_P be the Levi factor of P such that $M_P(\mathbb{R}) = MA$, and let $(\sigma, V_\sigma), \sigma(\nu) = \otimes_v \sigma_v(\nu)$, be a cuspidal automorphic representation of the algebraic group M_P, where v runs through the places of $\mathbb{Q}$, such that $\sigma_\infty(\nu) \cong \pi_{\lambda_M} \otimes e^\nu, \nu \in \mathbb{R}$. Extend σ trivially to a representation of $P(\mathbf{A})$, and let $I_P(\sigma, \nu)$ denote the (normalized) induced representation from $P(\mathbf{A})$ to $G(\mathbf{A})$ of σ. Thus

$$(6.3.1) \quad I_P(\sigma, \nu) = \{\phi \in C^\infty(G(\mathbf{A}), V_\sigma) | \phi(pg) = \sigma(\nu)(p)\delta_P(p)\phi(g), p \in P(\mathbf{A}), g \in G(\mathbf{A})\},$$

where δ_P is the square root of the modulus character for $P(\mathbf{A})$.

Let $I_P(\sigma, \nu)_0$ be the space of K_h-finite vectors in $I_P(\sigma, \nu)$. For $\phi \in I_P(\sigma, \nu)_0$ and $g \in G(\mathbf{A}), \phi(g)(\cdot)$ is a cusp form on M_P. If we let $f_\phi(g) = \phi(g)(1)$, then f_ϕ is a function on $N(\mathbf{A}) \cdot P(\mathbb{Q}) \backslash G(\mathbf{A})$. If ν is sufficiently large, the Eisenstein series

$$(6.3.2) \qquad E(\phi, \nu, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_\phi(\gamma g)$$

converges absolutely to an element of $\mathcal{A}(G) \subset C_{si}(G)$, thus defining a morphism

$$(6.3.3) \qquad E : I_P(\sigma, \nu)_0 \to C_{si}(G)$$

of $(\mathfrak{g}, K_h) \times G(\mathbf{A}^f)$-modules.

Let $I_P(\sigma, \pi_\lambda)$ be the $(\mathfrak{g}, K_h) \times G(\mathbf{A}^f)$-submodule of $I_P(\sigma, \nu)_0$ of elements whose archimedean components belong to $S_\lambda(\pi_\lambda) \subset I(P, \lambda_M, \nu(\lambda)) \cong I_P(\sigma, \nu)_\infty$. We may write $I_P(\sigma, \pi_\lambda) \cong \pi_{\lambda,0} \otimes I_P^f$, where $\pi_{\lambda,0}$ is the $(\mathfrak{g}, K_h)$-module associated to π_λ and I^f is a representation of $G(\mathbf{A}^f)$. Then the map

$$
\begin{aligned}
&I_P^f \to \mathrm{Hom}_{(\mathfrak{g},K_h)}(\pi_{\lambda,0}, C_{si}(G)),\\
&v^f \mapsto (v_\infty \mapsto E(v_\infty v^f))
\end{aligned}
\tag{6.3.4}
$$

is intertwining for the $G(\mathbf{A}^f)$-actions. Moreover, we have

6.3.5 LEMMA. *Let*

$$(\cdot) \mapsto (\cdot)_P : \mathcal{A}(G) \to C^\infty(N(\mathbf{A}) \cdot P(\mathbf{Q})\backslash G(\mathbf{A}))$$

be the map which takes an element of $\mathcal{A}(G)$ to its constant term along P. Then

$$E(\phi, \nu, g)_P = f_\phi, \forall \phi \in I_P(\sigma, \pi_\lambda).$$

PROOF (SKETCH): For general $\phi \in I_P(\sigma, \nu)_0$, $E(\phi, \nu, g)_P$ is the sum of f_ϕ and (possibly) a term $M(\nu) f_\phi$, where $M(\nu) = \otimes_v M_v(\nu)$ is the global intertwining operator from the theory of Eisenstein series. But, for ν positive, the image of the local intertwining operator $M_\infty(\nu)$ is the Langlands quotient of $I_P(\sigma, \nu)_\infty$; it follows that $M(\nu)$ vanishes on $I_P(\sigma, \pi_\lambda)$.

By Theorem 3.4, there exists a unique $\Lambda' \in \mathfrak{h}_{\mathbf{C}}^*$ and $q_\lambda \in \mathbf{Z}$ such that π_λ has $\bar{\partial}$-cohomology, in degree $n - q_\lambda$, with coefficients in $\sigma_{\Lambda'}$. Composing (6.3.4) with (2.4.5), we obtain

6.3.6. PROPOSITION. *There is a natural homomorphism of $G(\mathbf{A}^f)$-modules*

$$\mathrm{Eis} : I_P^f \to \tilde{H}^{n-q_\lambda}(E_{\Lambda'}^{\mathrm{can}}).$$

6.4. There exists a unique $w \in W^1(q_\lambda)$ (notation 4.1) such that $\mu = w^{-1}(\lambda) - \rho$ is R^+-dominant and integral. Let V_μ be the corresponding finite-dimensional representation of $\mathfrak{g}$; then $H^n(\mathfrak{g}, K_h, \pi_\lambda \otimes V_\mu)$ is canonically isomorphic, by 4.4.13, to $H^{n-q_\lambda}(\mathfrak{P}_h, K_h, \pi_\lambda \otimes W_{\Lambda'})$. Via Theorem 4.5, 6.3.6 defines a natural homomorphism

$$\mathrm{Eis}^\nabla : I_P^f \to H^n(M, V_\mu^\nabla)$$

of $G(\mathbf{A}^f)$-modules. This is not the map used by Harder and Schwermer. However, Lemma 6.3.5, combined with Satz 1.10 of [**42**], implies that Eis^∇ does provide a lifting of cohomology classes on the Borel-Serre boundary ∂Sh of Sh to classes in $H^n(Sh, V_\mu^\nabla)$. In particular, when the cusp form $f_\phi|_{M(\mathbf{A})}$ defines a non-trivial class in the corresponding component of ∂Sh, it follows from the preceding argument that $\mathrm{Eis}^\nabla(I_P^f)$, and hence $\mathrm{Eis}(I_P^f)$, is *non-trivial.*

For λ sufficiently regular, it can be shown that the $G(\mathbf{A}^f)$-action separates the $\mathrm{Eis}(I_{P'}^f)$ from $\mathrm{Eis}(I_{P'}^f)$, when P and P' are not associate, and from $\bar{H}^{n-q_\lambda}(E_{\Lambda'}) \subset \tilde{H}^{n-q_\lambda}(E_{\Lambda'}^{\mathrm{can}})$ (cf. [**18**]). In some cases (e.g., when it is known that $H^n(Sh, V_\mu^\nabla)$ is represented entirely by automorphic forms) information of this type suffices to show that $\mathrm{Eis}(I_P^f)$ is a $k(\Lambda')$-rational subspace of $\tilde{H}^{n-q_\lambda}(E_{\Lambda'}^{\mathrm{can}})$, and that the $k(\Lambda')$-rational elements are those which restrict to $k(\Lambda')$-rational classes in the coherent cohomology, with appropriate coefficients, of the boundary of some toroidal compactification ${}_KSh \hookrightarrow {}_KSh_\Sigma$.

For certain groups more explicit answers can be obtained. The following results represent work in progress on Eisenstein classes in coherent cohomology, in the case $G = GSp(2)$. Let P_0 be a minimal parabolic of G, and let P and Q be the two standard maximal parabolics of G, with abelian and non-abelian unipotent radicals, respectively.

6.5. *The parabolic Q.* Starting from an antiholomorphic elliptic modular cusp form f of weight $k+2$, one can define spaces of Eisenstein classes $\mathrm{Eis}(I_Q^f) \subset \tilde{H}^2(E_{\Lambda'}^{\mathrm{can}}), \Lambda' = (k, -\ell-2, c)$, where the central character c coincides with that of f, and where ℓ is any large integer such that $k + \ell \equiv c (\mathrm{mod}\, 2)$. These are non-holomorphic analogues of the Eisenstein series of Klingen [**27**]. Unpublished results of Schwermer, mentioned briefly in [**43**], seem to imply that the Eisenstein map (6.3.5) is *injective* in this case, provided Λ' is sufficiently regular. In any case, the image $\mathrm{Eis}(I_Q^f)$ is non-trivial and $\mathbf{Q}$-rational for highly regular Λ'. Given a level subgroup K and a toroidal compactification ${}_KSh \hookrightarrow {}_KSh_\Sigma$, one can show that the elements of $\mathrm{Eis}(I_Q^f)$ of level K restrict non-trivially to the genus one component of the boundary of ${}_KSh_\Sigma$, whose typical connected component is a toroidal compactification of the universal elliptic fibration over the modular curve of some level.

Moreover, if $\phi \in I_Q(\sigma, \pi_\lambda)_0$ belongs to the minimal K_h-type of π_λ and has the property that the function f_ϕ is rational in the sense of

(5.4.1), then $E(\phi)$ is also rational. One can apply this fact, and the results of 5.4, to express the special values of certain triple product L-functions (cf. Shahidi's article in this volume) in terms of periods.

6.6. *The parabolic P.* In this case $M(\mathbf{R})^{der} \cong SL(2,\mathbf{R})^{\pm}$, the group of 2×2 matrices with determinant ± 1. One is thus forced to lift holomorphic and anti-holomorphic forms together; in other words, one is lifting elements of the cohomology of a locally symmetric space attached to $M(\mathbf{R})$ with coefficients in a local system V_{μ}^{∇}. Since $M_0 = SL(2,\mathbb{R})$, we may take μ to be a positive integer; then these forms give rise to Eisenstein classes in $\tilde{H}^1(E_{\Lambda'}^{\mathrm{can}}), \Lambda' = (b,-a,c)$, where $a+b=\mu$, $a > b$, and c is determined by the central character. Again, Schwermer's unpublished results suggest that the Eisenstein map is injective for sufficiently regular Λ', and one can use Theorem 4.5 to show that $\mathrm{Eis}(I_P^f)$ is non-trivial and $\mathbf{Q}$-rational for highly regular Λ'. Identifying rational classes is more difficult in this case, however, since the elements of $\mathrm{Eis}(I_P^f)$ of level K are associated to the genus zero component of the boundary of ${}_KSh_{\Sigma}$, whose typical connected component is a union of rational surfaces. Heuristic arguments suggest that the rational elements of $\mathrm{Eis}(I_P^f)$ are the Eisenstein series attached to rational classes in $H^*(V_{\mu}^{\nabla})$. This argument is supported, in a sense, by the expression of the Whittaker coefficients of elements of $\mathrm{Eis}(I_P^f)$ in terms of special values of standard L-functions for $GL(2)$ (cf. [**44**]).

§7. Arithmetic applications

We describe three applications of the theory developed thus far. Together with the results mentioned in 6.5–6.6, those presented here are in some sense typical of the arithmetic applications of the representation of automorphic forms of $\bar{\partial}$-cohomology type as coherent cohomology classes. An application to the arithmetic of Maass waveforms is discussed in Blasius' article in this collection.

7.1. Period invariants of Hilbert modular forms, and special values of L-functions. Let F be a totally real number field of degree n, and let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be the set of real imbeddings of F. For any subset $I \subset \Sigma$, let $|I|$ denote its cardinality. Let $G = R_{F/\mathbf{Q}}GL(2)_F$. Let $\mathcal{M} = Sh(G,X)$ be the corresponding Shimura variety (or stack). In [**52**], Shimura conjectured that, to any cuspidal modular eigenform f on G, one can associate two collections of numerical invariants in $\mathbf{C}^{\times}$, defined up to multiplication by elements of $\bar{\mathbf{Q}}^{\times}$. These invariants are expected to satisfy certain hypotheses which imply, among other things, that they can be used to express the critical values of all Hecke- and Rankin-Selberg-type L-functions.

Let f be a holomorphic cuspidal Hilbert modular eigenform for F of weight $\underline{k} = (k_1, \ldots, k_n, c)$ on G (c is the central character, assumed algebraic). If $\alpha, \beta \in \mathbb{C}^\times$, we write $\alpha \sim \beta$ if $\alpha/\beta \in \bar{\mathbb{Q}}^\times$. More generally, if $L \subset \bar{\mathbb{Q}}$, we write $\alpha \sim_L \beta$ if $\alpha/\beta \in L^\times$. We paraphrase Shimura's hypotheses in the case of the first set of invariants, which Shimura denotes $Q(\chi, \alpha)$, and which we denote $Q(\pi, I)$, where $\pi = \pi_\infty \otimes \pi_f$ is the automorphic representation of $G(\mathbf{A})$ which corresponds to f and I runs through subsets of Σ:

(7.1.1)(Q1) $Q(\pi, \emptyset) \sim 1$.

(Q2) $Q(\pi, I) \cdot Q(\pi, J) \sim Q(\pi, I \cup J) \cdot Q(\pi, I \cap J)$.

(Q3) Let B be a quaternion algebra over F unramified at places in I and ramified at places in $\Sigma - I$. Suppose there is an automorphic representation π^B of $B^\times(\mathbf{A})$ which transfers to π under the Jacquet-Langlands correspondence. Let g and h be two arithmetic holomorphic automorphic forms in π^B, and let $<,>_B$ be the Petersson inner product on $B^\times(\mathbf{A})$, normalized as in [**52**]. If $< g, h >_B \neq 0$, then $< g, h >_B \sim Q(\pi, I)$.

(Q4) The $Q(\pi, I)$ are related to special values of certain Dirichlet series introduced by Shimura (see [**52**], p. 279 for details).

(Q5) Let E be a totally imaginary quadratic extension of F, Φ a CM type of E, and let ξ be an algebraic Hecke character of $E^\times(\mathbf{A})$ with the property that $\xi_\infty = \sum_{\sigma \in \Phi} \xi_\sigma \sigma$, with $\xi_\sigma \geq 0$. Let $\pi = \pi(E, \xi)$ be the base change of ξ to an automorphic representation of $G(\mathbf{A})$. Then $(2\pi i)^{|I|} Q(\pi, I) \sim p_E(\xi, 2\sum_{\sigma \in \eta} \sigma)$. Here $\eta = \{\sigma \in \Phi | \sigma_{|F} \in I\}$ and $p_E(\cdot, \cdot)$ is the period invariant introduced by Shimura in [**48**]; it is a product of certain periods of abelian varieties of CM type (E, Φ).

When B and π^B exist as in (Q3), one is forced by the hypotheses to propose $< g, g >_B$ as a candidate for $Q(\pi, I)$. One then has to verify that one obtains the same $Q(\pi, I)$ if π comes from two distinct quaternion algebras B_1 and B_2; this is proved under a rather mild hypothesis by Shimura as Theorem 5.6 of [**52**], and conjectured to be true in general.

The theory of arithmetic higher coherent cohomology suggests a natural set of candidates for these invariants. These candidates are defined for all f, and are determined up to multiplication by a nonzero element of the field generated by the Hecke eigenvalues of f.

Let f, $\underline{k}$, and $\pi = \pi_\infty \otimes \pi_f$ be as above; let $\mathcal{E}(\underline{k})$ be the automorphic vector bundle (possibly in the sense of stacks) of which f is a section, and let $\mathbb{Q}(\pi)$ be the field generated by the Hecke eigenvalues of f. Let $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$. As $U(\mathfrak{g}) \times G(\mathbf{A}^f)$-module, π splits up as the direct sum of 2^n representations π^I, parametrized by the subsets $I \subset \Sigma$ in such a way that, for any $v \in \Sigma$, the component π_v^I has a holomorphic (resp. anti-holomorphic) vector ω_v^I if and only if $v \notin I$ (resp. $v \in I$). We let $\omega^I = \bigotimes_{v|\infty} \omega_v^I$.

Start with $I = \emptyset$. We use the same letter f to designate the function on $G(\mathbf{A})^+ = G(\mathbb{R})^+ \times G(\mathbf{A}^f)$ and the holomorphic function on the hermitian symmetric space. In the former setting, f is an element of the subspace $\mathcal{V}(\emptyset) = \mathbb{C}\omega^\emptyset \otimes \pi_f \subset \pi^\emptyset$ of the space $\mathcal{A}_0(G)$ of cusp forms, and is assumed to be *normalized*, in the following sense: If $W_\psi(f)(g) = \bigotimes_v W_{\psi,v}(f)(g)$ is the Whittaker function attached to f and some additive character ψ, then $\bigotimes_{v|\infty} W_{\psi,v}(f)(1) = c_\infty$ has the property that, for all $v|\infty$, the local zeta integral

(7.1.2)
$$\int_{F_v^\times} W_{\psi,v}(f)\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s+\frac{1}{2}} d^\times a = (2\pi)^{-s-\alpha_v(\underline{k})}\Gamma(s+\alpha_v(\underline{k}))$$

where $\alpha_v(\underline{k})$ is an integer (or half integer) determined by $\underline{k}$ and various normalizations. We further assume $W_{\psi,v}(f)(1) = 1$ for v non-archimedean. Let $\mathcal{V}(\emptyset)_1$ denote the $\mathbb{Q}(\pi)$-rational form of $\mathcal{V}(\emptyset)$ generated by the $G(\mathbf{A}^f)$-translates of f. Let $\mathcal{V}(\emptyset)_2$ denote the $\mathbb{Q}(\pi)$-rational form of $\mathcal{V}(\emptyset)$ generated by elements of $\mathcal{V}(\emptyset)$ which define $\mathbb{Q}(\pi)$-rational sections of $\mathcal{E}(\underline{k})$. In this case $\mathcal{V}(\emptyset)_2 = \mathcal{V}(\emptyset)_1$, by the q-expansion principle.

For general $I \subset \Sigma$, the space $\mathcal{V}(I) = \mathbb{C}\omega^I \otimes \pi_f \subset \pi^I$ injects, by the theory of §2–3, into a cohomology space of the form $\bar{H}^{|I|}(\mathcal{M}, \mathcal{E}(\underline{k}^I))$, for some $\mathcal{E}(\underline{k}^I)$. We let $\mathcal{V}(I)_2$ denote the $\mathbb{Q}(\pi)$-rational form of $\mathcal{V}(I)$ generated by elements of $\mathcal{V}(I)$ which define $\mathbb{Q}(\pi)$-rational elements of $\bar{H}^{|I|}(\mathcal{M}, \mathcal{E}(\underline{k}^I))$. Again, we let $\mathcal{V}(I)_1$ denote the $\mathbb{Q}(\pi)$-rational form of $\mathcal{V}(I)$ generated by functions whose Whittaker functions have the value c_∞ at the identity. As representations of $G(\mathbf{A}^f)$, the spaces $\mathcal{V}(I)_i, i = 1, 2$ contains primitive vectors ("new forms") f_1^I, f_2^I; we write $f^I = f_1^I$, $f^{I,\mathrm{arith}} = f_2^I$. Let $\nu^I(\pi) \in \mathbb{C}^\times$ be the constant such that
$$\nu^I(\pi) \cdot f^{I,\mathrm{arith}} = f^I.$$

The function f^I is essentially what you obtain by conjugating the

I-variables in f, viewed as a holomorphic function. The $\nu^I(\pi)$ are well-defined up to multiplication by $\mathbb{Q}(\pi)^\times$.

7.1.3. CONJECTURE. *The quantities $Q(\pi, I) = \nu^I(\pi)$ satisfy Shimura's conjecture (7.1.1). Moreover, in (Q1–3), $\sim$ may be replaced by $\sim_{\mathbb{Q}(\pi,I)}$, where $\mathbb{Q}(\pi, I)$ is a certain finite extension of $\mathbb{Q}(\pi)$ depending only on I and $\underline{k}$.*

At least the $\nu^I(\pi)$ satisfy Shimura's Conjecture (Q1), that $\nu^\emptyset(\pi) = 1$. More interestingly, it is easy to prove (Q2) in the special case $I \cap J = \emptyset$, $I \cup J = \Sigma$. Indeed, for such I and J, $\mathcal{E}(\underline{k}^I) \otimes \mathcal{E}(\underline{k}^J) \cong \mathbb{K}$ (cf. 2.4) and Serre duality defines a cup product pairing

$$(\cdot,\cdot) : \bar{H}^{|I|}(\mathcal{M}, \mathcal{E}(\underline{k}^I)) \otimes \bar{H}^{|J|}(\mathcal{M}, \mathcal{E}(\underline{k}^J)) \to \mathbb{C}$$

which is (at least) rational over $\bar{\mathbb{Q}}$. In particular

$$(7.1.4) \quad (f^I, f^J) = \nu^I(\pi) \cdot \nu^J(\pi)(f^{I,\mathrm{arith}}, f^{J,\mathrm{arith}}) \sim \nu^I(\pi) \cdot \nu^J(\pi).$$

On the other hand, one can show directly that $(f^I, f^J) \sim (f^{I'}, f^{J'})$ if $I' \cap J' = \emptyset$, $I' \cup J' = \Sigma$; the point is that f^I is essentially obtained by conjugating f in the variables in I. In particular, we may take $I' = \emptyset$, $J' = \Sigma$.

By similar considerations, we can prove the following relation with special values of L-functions, which may be regarded as a partial analogue of Shimura's (Q4): We consider cusp forms f and f', of weights $\underline{k} = (k_1, \ldots, k_n, c), \underline{\ell} = (\ell_1, \ldots, \ell_n, c')$, respectively, and let I (resp. J) be the subset of $\sigma_i \in \Sigma$ for which $k_i > \ell_i$ (resp. $\ell_i > k_i$). Let π and π' be the corresponding automorphic representations. We assume (i) $I \cup J = \Sigma$, and (ii) $k_i - \ell_i \equiv k_j - \ell_j (\mathrm{mod}\, 2)$ for all i, j. Let s_0 be a *critical point* of the Rankin-Selberg tensor product L-function $L(s, \pi \times \pi')$, in the sense of Deligne's conjecture [**13**]. Then the methods of [**47**] show easily that

7.1.5. PROPOSITION. *There exists a positive integer $d(s_0)$ depending only on $\underline{k}$, $\underline{\ell}$ and s_0, such that*

$$L(s_0, \pi \times \pi') \sim (2\pi i)^{d(s_0)} \nu^I(\pi) \cdot \nu^J(\pi').$$

*One can actually demonstrate more precise results about the field of rationality of the quotient of the left hand side by the right hand side, along the lines of Theorem 4.2 of [**47**].*

If we admit (Q3), then Proposition 7.1.5 implies Theorem 5.3 of [**52**] as a special case. On the other hand, it is possible to reduce (Q3), at least when all the weights $k_i \geq 3$, to Proposition 7.1.5 and (Q5), the special case of binary theta functions. Now a generalization of Corollary 5.3.2, together with some results of Shimura in [**50**], implies (Q5) when $|I| = 1$ or $|I| = n-1$. Using this special case, (Q5) can be reduced to some assertions about special values of L-functions which are, in principle, known.

This leaves the general case of (Q2). We first remark that, if B, I, and π^B are as in (Q3), then, for any $J \subset I$, the theory of arithmetic automorphic forms on $B^\times(\mathbf{A})$ and the theory of L-functions of automorphic forms on $B^\times(\mathbf{A})$ permit us to define invariants $\nu^J(\pi^B)$, in analogy with the definition for B split. Again, $\nu^J(\pi^B)$ is well-defined up to multiplication by $\mathbb{Q}(\pi)^\times$.

7.1.6. Conjecture. *For all* $J \subset I, \nu^J(\pi^B) \sim_{\mathbb{Q}(\pi,J)} \nu^J(\pi)$.

Using the methods developed by Shimura in [**49**],[**50**], it is not difficult to reduce (Q2) and Conjecture 7.1.6 to the special case of 7.1.6 in which π is of the form $\pi(E,\xi)$ (notation as in (Q5)). In any case, the above remarks imply

7.1.7. Proposition. *Assume* $n = 2$. *Then the quantities* $\nu^I(\pi)$ *satisfy Shimura's conjectures (Q1), (Q2), (Q3), and (Q5).*

Proof of (Q2) in general will imply, among other things, the transcendental part of the Birch-Swinnerton-Dyer conjecture for the L-function of a factor of the Jacobian of a Shimura curve over F, lifted to a quadratic CM extension of F. This is far from what one wants, but may still be of interest. Similarly, the conjecture provides a (conjectural) analytic definition for the periods of an F-rational differential on an arbitrary elliptic curve over F, including (for the first time) the case of elliptic curves with good reduction everywhere. This refines the (conjectural) analytic definition of the Hodge structure on such curves, due to Oda [**36**] and Murty-Ramakrishnan (to appear).

7.2. Arithmeticity of certain non-holomorphic thetafunctions [23]. Let W be a vector space of dimension 4 over $\mathbb{Q}$, endowed with a non-degenerate symmetric bilinear form $(\cdot,\cdot)_W$ of signature $(2,2)$ over $\mathbb{R}$. Let H be the connected component of the group $GO(W)$ of similitudes of W. Then there is a (non-connected) hermitian symmetric space X_H such that (H, X_H) is the datum defining a Shimura variety. Shimura shows in [**50**] that the theta correspondence for the dual reductive pair [**25**] $SL(2) \times O(W) \subset Sp(W^2)$ defines a lifting from arithmetic holomorphic cusp forms on $\mathcal{M}_1$ to arithmetic holomorphic cusp forms on $Sh(H, X_H)$. When H is quasi-split, this was known [**35**]; but when H is anisotropic one needs to use Shimura's characterization of arithmetic automorphic forms in terms of their values at CM points.

One can also consider the theta correspondence for the dual reductive pair $Sp(2) \times O(W) \subset Sp(W^4)$. The theta-lifting for this pair takes cusp forms on H to automorphic forms on $G = GSp(2)$, and one verifies that cusp forms on H whose archimedean components are of sufficiently regular discrete series type go to cusp forms on G belonging to the non-holomorphic discrete series of G. Thus, the choice of an appropriate theta kernel defines a homomorphism from a coherent cohomology group of $Sh(H, X_H)$ to higher coherent cohomology of $\mathcal{M}_2$.

Consider first the case in which $(\cdot,\cdot)$ is the split form. Then $H \cong GL(2) \times GL(2)$ modulo centers; an automorphic form on H corresponds to a pair (f_1, f_2) of automorphic forms on $GL(2)$ whose central characters are inverse to one another. Such a pair defines a class in coherent cohomology if and only if f_1 and f_2 are each either holomorphic or anti-holomorphic. (We are identifying a holomorphic modular form with its lift to $GL(2)$.)

Thus, let f_i be holomorphic cusp forms of weight $k_i, i = 1, 2$, which are eigenfunctions of the Hecke operators T_p for almost all p. Suppose that f_1 and f_2 have algebraic Fourier coefficients and that their central characters coincide. Let $< \cdot, \cdot >$ be the classical Petersson inner product. Then $(f_1, C(f_2)/ < f_2, f_2 >)$ (notation 5.1) defines a cusp form ϕ on H.

The Weil representation of the dual reductive pair $(Sp(2), O(W))$ in may be realized on the Schwarz space $\mathcal{S}((W_{\mathbf{A}})^2)$ [**56**]. Thus $\Phi \in \mathcal{S}((W_{\mathbf{A}})^2)$ defines, in the usual way, a theta-kernel θ_Φ on $Sp(2,\mathbb{Q})\backslash\widetilde{Sp}(2,\mathbf{A}) \times O(W,\mathbb{Q})\backslash\tilde{O}(W,\mathbf{A})$, where $\tilde{}$ denotes a two-fold cover, which for our purposes may be disregarded. With ϕ as above, define $\theta_\Phi(\phi) \in \mathcal{A}(Sp(2))$ by the integral

$$\theta_\Phi(\phi)(g) = \int_{O(W,\mathbb{Q})\backslash O(W,\mathbf{A})} \theta_\Phi(g,h)\phi(h)dh. \tag{7.2.1}$$

If $k_1 \neq k_2$, then $\theta_\Phi(\phi)$ is necessarily a cusp form [**26**]. We say Φ is arithmetic if it is a tensor product $\Phi_\infty \otimes \Phi_f$, where Φ_f is a $\bar{\mathbb{Q}}$-valued function in $\mathcal{S}((W_{\mathbf{A}^f})^2)$ and $\Phi_\infty \in \mathcal{S}((W_{\mathbb{R}})^2)$ is the product of a standard exponential by a polynomial with coefficients in $\bar{\mathbb{Q}}$. The following theorem, based on the rationality criterion 5.4.2 and the method of seesaw dual reductive pairs [**29**], is typical of the results of [**23**] (joint work with S. Kudla):

7.2.2. THEOREM. *With ϕ as above, suppose $k_1 = a - b - 2, k_2 = a + b + 2$, for some $a, b \in \mathbb{Z}, a - 4 > b > 0$. Let E_Λ be the automorphic vector bundle on $\mathcal{M}_2$ with $\Lambda = (b, -a; c)$, where c is determined by the central character of $f_i, i = 1, 2$. Then there exists $\eta \in \mathbb{Z}$, depending only on (a, b, c), such that for any arithmetic Φ such that Φ_∞ has the right K_∞-type (see Remark 7.2.3), $(2\pi i)^\eta \theta_\Phi(\phi)$ defines a $\bar{\mathbb{Q}}$-rational element of $\bar{H}^1(E_\Lambda)$.*

7.2.3. Remark. We are assuming that Φ_∞ transforms according to a certain representation of the maximal compact subgroup of $Sp(2, \mathbb{R}) \times O(W, \mathbb{R})$; otherwise the cohomology class of $\theta_\Phi(\phi)$ is trivial. Non-trivial Φ_∞'s with the given K_∞-type do exist, and it follows from the results of Howe and Piatetski-Shapiro [**26**] that, possibly after twisting ϕ by a quadratic character, one can guarantee non-vanishing of the cohomology class of $\theta_\Phi(\phi)$.

The non-triviality in [**26**] is a consequence of the fact that the standard Whittaker functions of $\theta_\Phi(\phi)$ are linear combination with coefficients in $(2\pi i)^e\bar{\mathbb{Q}}$, for some fixed $e \in \mathbb{Z}$, of the Whittaker functions of ϕ. With our normalization above, this implies that, up to a fixed power of $(2\pi i)$, the values at the identity of the standard Whittaker functions of $\theta_\Phi(\phi)$ are algebraic multiples of $< f_2, f_2 >$. The cohomology classes on $\mathcal{M}_2$ defined by theta lifts from $O(W)$ are far from typical, but this suggests nevertheless that the Whittaker functions of Hecke eigenforms in $\bar{H}^1(E_\Lambda)$ contain interesting arithmetic information.

7.2.4. Remark. The analogue of Theorem 7.2.2 is apparently true for general W of signature (2,2); as of the date of this writing, this has been verified when W is globally isotropic, or when W is the space of a quaternion division algebra B over $\mathbb{Q}$ with the quadratic form given by the reduced norm. The latter case has the following consequence. Let F be a real quadratic field, and let f be a holomorphic Hecke eigenform on the Shimura variety attached to the algebraic group $B^\times F$. Let f' be its Jacquet-Langlands transfer to a holomorphic cusp form on $GL(2, F)$. Following the method of Oda [**36**], one can attach "motives" $M(f)$ and $M(f')$, of dimension 4 over the field of

Hecke eigenvalues, to f and f'. These motives have obvious Betti rational structures, and the constructions in §4 define de Rham rational structures as well (cf. 4.3.2). Under a suitable regularity hypothesis on the archimedean component of the automorphic representation attached to f, the analogue of Theorem 7.2.4 provides algebraic relations between the period matrices of $M(f)$ and $M(f')$, relative to these two rational structures. Details will appear in [**23**].

We note that similar results hold for theta-liftings to $\bar{H}^2$ of automorphic vector bundles over $\mathcal{M}_2$.

7.3. Limit multiplicities of discrete series. In this section (G, X) is arbitrary. Let $\lambda \in \mathcal{F} + \rho$ be the Harish-Chandra parameter of a discrete series representation of G_0. If K is a level subgroup, let $m(K, (\pi_\lambda)^*) = \dim \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}((\pi_\lambda)^*, \mathcal{A}_0(G, K))$, where $\mathcal{A}_0(G, K)$ is the space of cusp forms on $G(\mathbb{Q})\backslash G(\mathbf{A})/K$. Let $\Lambda = \lambda - \rho$, and let $\mathcal{E}_\Lambda$ (resp. E_Λ) be the corresponding homogeneous vector bundle on $\check{X}$ (resp. automorphic vector bundle on M). Let $\tilde{\chi}({}_KSh, E_\Lambda)$ (resp. $\chi(\check{X}, \mathcal{E}_\Lambda)$) denote the Euler characteristic $\chi(\overline{Sh}, E_\Lambda^{\mathrm{can}})$ of the vector bundle $[\mathcal{V}]^{\mathrm{can}}$ over some SNC toroidal compactification $\overline{Sh}$ of ${}_KSh$ (resp. the Euler characteristic of $\mathcal{E}_\Lambda$ over $\check{X}$). By Proposition 2.4, $\tilde{\chi}({}_KSh, E_\Lambda)$ is independent of the choice of $\overline{Sh}$.

It has recently been proved by Savin [**40**], using previous work of Rohlfs and Speh [**38**] that

$$\lim_{\mathrm{vol}(K)\to 0}[\mathrm{vol}({}_KSh)^{-1}(m(K, (\pi_\lambda)^*) - (-1)^{n+q_\lambda}\chi(\check{X}, \mathcal{E}_\Lambda)] = 0. \tag{7.3.1}$$

Savin's proof works for general discrete series representations; the pair (G, K_h) is not assumed to be of hermitian type.

Using Theorem 3.6.3 and Mumford's generalization of the Hirzebruch Proportionality Theorem [**34**], we may obtain a proof of a somewhat weaker version of (7.3.1), in the hermitian symmetric case, when λ is sufficiently regular. Although our result is considerably less general than Savin's, it provides some information on the error term, and may thus be of some interest. In this connection, it should be mentioned that the error terms have been extensively studied by Satake [**39**] as functions of Λ.

References

1. Ash, A., Mumford, D., Rapoport, M., Tai, Y-S., "Smooth Compactification of Locally Symmetric Varieties," Math. Sci. Press, Brookline, Mass, 1975.

2. Blank, B.E., *Knapp-Wallach Szegö Integrals and Generalized Principal Series Representations: The Parabolic Rank One Case*, J. Fun. Analysis **60** (1985), 127–145.
3. Blasius, D., Harris, M., Ramakrishnan, D., *Coherent cohomology, limits of discrete series, and Maass forms of Galois type.*
4. Borel, A., *Stable real cohomology of arithmetic groups*, Ann. Scient. Ec. Norm. Sup. **7** (1974), 235–272.
5. Borel, A., *Stable real cohomology of arithmetic groups II*, in "Manifolds and Lie Groups, Papers in honor of Y. Matsushima," Progress in Math., Birkhaüser, Boston:, 1981, pp. 21–55.
6. Borel, A., *Regularization theorems in Lie algebra cohomology. Applications*, Duke Math. J. **50** (1983), 605–624. Correction and complement (to appear).
7. Borel, A., Casselman, W., *L_2 cohomology of locally symmetric manifolds of finite volume*, Duke Math. J. **50** (1983), 625–647.
8. Borel, A., Wallach, N., "Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups," Ann. of Math. Studies 94, Princeton Univ. Press, Princeton, NJ, 1980.
9. Borel,A. et al., "Algebraic D-Modules.," Perspectives in Math., **2**, Academic Press, Orlando, FL, 1987.
10. Chai, C-L., Faltings, G., Forthcoming book on arithmetic compactification of Siegel moduli schemes.
11. Deligne, P., "Equations différentielles à points singuliers réguliers," Lect. Notes in Math. 163, 1970.
12. Deligne, P., *Variétés de Shimura: Interprétation modulaire et techniques de construction de modèles canoniques*, Proc. Symp. Pure Math. XXXIII, part 2 (1979), 247–290.
13. Deligne, P., *Valeurs de fonctions L et périodes d'intégrales.*, Proc. Symp. Pure Math. XXXIII, part 2 (1979), 313–346.
14. Faltings, G., "Arithmetic theory of Siegel modular forms, New York Number Theory Seminar," Lect. Notes in Math. 1240, 1987, pp. 101–108.
15. Faltings, G., "On the cohomology of locally symmetric hermitian spaces," Lect. Notes in Math. 1029, 1984, pp. 55–98.
16. Garrett, P.B., *Arithmetic and structure of automorphic forms on bounded symmetric domains*, Am. J. Math. **105** (1983), 1171–1194.
17. Harder, G., *Eisenstein cohomology of arithmetic groups: The case GL_2*, Invent. Math. **89** (1987), 37–118.
18. Harris, M., *Eisenstein series on Shimura varieties*, Ann. of Math. **119** (1984), 59–94.
19. Harris, M., *Arithmetic vector bundles and automorphic forms on Shimura varieties. I.*, Inventiones Math. **82** (1985), 151–189; II. Compositio Math. **60** (1986), 323–378.
20. Harris, M., *Formes automorphes "géométriques" non-holomorphes: Problèmes d'arithméticité*, Progress in Math. 63, 109–129, Sém de Théorie des Nombres, Paris (1986), Birkhaüser, Boston.
21. Harris, M., *Automorphic forms of $\bar{\partial}$-cohomology type as coherent cohomology classes*, J. Diff. Geom. (to appear).
22. Harris, M., *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. Lon. Math. Soc. (to appear).

23. Harris, M., Kudla, S., "Arithmetic automorphic forms for the non-holomorphic discrete series of $GSp(2)$" (to appear).
24. Harris, M., Phong, D. H., *Cohomologie de Dolbeault à croissance logarithmique à l'infini*, C. R. Acad. Sci. Paris **302** (1986), 307–310.
25. Howe, R., *θ-series and Invariant Theory*, Proc. Symp. Pure Math. XXXIII, part 1 (1979), 275–286.
26. Howe, R., Piatetski-Shapiro, I.I., *Some examples of automorphic forms on* $Sp(4)$, Duke Math. J. **50** (1983), 55–106.
27. Klingen, H., *Zum Darstellungssatz für Siegelsche Modulformen*, Math. Z. **102** (1967), 30–43; *Berichtigung*, Math. Z. **105** (1968), 399–400.
28. Knapp, A. W., "Representation Theory of Semisimple Groups.," Princeton University Press, Princeton, 1986.
29. Kudla, S., *Seesaw dual reductive pairs.*, in "Automorphic forms of Several Variables, Taniguchi symposium, Katata (1983)," Birkhaüser, Boston, 1984, pp. 244–268.
30. Matsushima, Y., Murakami, S., *On vector bundle valued harmonic forms and automorphic forms on symmetric spaces*, Ann. of Math. **78** (1963), 365–416.
31. Milne, J.S., *The action of an automorphism of* $\mathbb{C}$ *on a Shimura variety and its special points*, Progr. Math. J. **35** (1983), 234–265, Birkhaüser, Boston.
32. Milne, J.S., *Automorphic vector bundles on connected Shimura varieties*, Invent. Math. **92** (1987), 91–128.
33. Mirković, I., "Localization for singular infinitesimal characters," manuscript, 1988.
34. Mumford, D., *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.
35. Oda, T., *On modular forms associated with indefinite quadratic forms of signature* $(2, n-2)$, Math. Ann **231** (1977), 97–144.
36. Oda, T., *Periods of Hilbert modular surfaces*, Progress in Math. **19** (1982), Birkhaüser, Boston.
37. Repka, J., *Tensor products of unitary representations of* $SL_2(\mathbb{R})$, Amer. J. Math. **100** (1978), 747–774.
38. Rohlfs, J., Speh, B., *On limit multiplicities of representations with cohomology in the cuspidal spectrum*, Duke Math. J. **55** (1987), 199–211.
39. Satake, I., *On numerical invariants of arithmetic varieties of* $\mathbb{Q}$*-rank one*, Automorphic Forms of Several Variables, Progress in Math. **46** (1984), 353–369, Birkhaüser, Boston.
40. Savin, G., *Limit multiplicities of discrete series*, Invent. Math. (to appear).
41. Schmid, W., L_2 *cohomology and the discrete series*, Ann. of Math. **103** (1976), 375–394.
42. Schwermer, J., "Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen," Lecture Notes in Math. 988, 1983.
43. Schwermer, J., *On arithmetic quotients of the Siegel upper half space of degree two*, Compositio Math. **58** (1986), 233–258.
44. Shahidi, F., *On certain L-functions*, Amer. J. Math. **103** (1981), 297–355.
45. Shimura, G., *On canonical models of arithmetic quotients of bounded symmetric domains I,II*, Ann. of Math. **91** (1970), 144–222;, **92** (1970), 528–549.
46. Shimura, G., *On certain reciprocity laws for theta functions and modular forms*, Acta Math. **141** (1978), 35–71.

47. Shimura, G., *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J . **45** (1978), 637–679.
48. Shimura, G., *Automorphic forms and the periods of abelian varieties*, J. Math. Soc. Japan **31** (1979), 561–592.
49. Shimura, G., *The arithmetic of certain zeta functions and automorphic forms on orthogonal groups*, Ann. of Math. **111** (1980), 313–375.
50. Shimura, G., *On certain zeta functions attached to two Hilbert modular forms I, II*, Ann. of Math. **114** (1981), 569–607, 127–164.
51. Shimura, G., *The critical values of certain zeta functions associated with modular forms of half-integral weight*, J. Math. Soc. Japan **33** (1981), 649–672.
52. Shimura, G., *Algebraic relations between critical values of zeta functions and inner products*, Am. J. Math. **104** (1983), 253–285.
53. Vogan,D., Jr, *Representations of Real Reductive Lie Groups*, Progress in Math. **15** (1981), Birkhaüser, Boston.
54. Vogan, D., Jr., Zuckerman, G., *Unitary representations with non-zero cohomology*, Compositio Math. **53** (1984), 51–90.
55. Wallach, N., *On the constant term of a square integrable automorphic form, (Neptun, 1980)*, Operator Algebras and Group Representations, Vol II, Monographs Stud. Math. **18** (1984), 227–237.
56. Weil, A., *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.
57. Williams, F. L., *The n-cohomology of limits of discrete series*, J. Fun. Analysis (1988).
58. Zucker, S., *Locally homogeneous variations of Hodge structure*, L'Enseignement Math. **27** (1981), 243–276.

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p-adic L-functions for base change lifts of GL_2 to GL_3.

By Haruzo Hida

Contents

§0. Introduction

There is a growing number of examples of p-adic L-functions for modular forms on $GL(2)$ whose domain of (p-adic) continuation is given by the spectrum of the p-adic *nearly ordinary* Hecke algebras. At the conference, we discussed several examples of such L-functions including those of standard L-functions of $GL(2)$ due to Mazur and Kitagawa [**Ki**] and Rankin product L-functions [**H1**]. Here we would like to present another example of such p-adic interpolation. In fact, we treated in [**H3**,§4] and [**H5**], as a first example of such L-functions on the spectrum of the *ordinary* Hecke algebra (and hence of one variable), the symmetric square L-functions attached to the base change lift established by Jacquet and Gelbart [**G-J**] of modular forms on $GL(2)_{/\mathbf{Q}}$ to $GL(3)_{/\mathbf{Q}}$. However, the p-adic L-function given there is *a priori* a characteristic power series of the module of congruence of each irreducible component of the Hecke algebra and hence has a defect that it is defined only up to multiple of units in the Hecke algebra, although it has, as a merit, direct connection to the arithmetic object (see also [**M-T**]). In this paper, we present another (analytic) method for the p-adic interpolation of this type of L-functions which even yields p-adic interpolation on the spectrum of *nearly ordinary* Hecke algebra (hence of two variable) including a variable corresponding to cyclotomic twists (or p-adic derivatives) of modular forms. This result (for L-functions given by Shimura integrals [**Sh2**], see Theorem 5.1 in the text) actually goes back to 1983 and the author gave a

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series of lectures about this first at Université de Paris VII in May of 1985 and later at the Ecole Normale in January 1986. The delay of publication is due to the author's inability of proving the expected holomorphy for the *primitive* p-adic L-functions exactly corresponding to the lift to $GL(3)$. Such holomorphy result for the cyclotomic continuation was recently supplied by Schmidt [**Sch**]. Although there is some restriction for the result in [**Sch**], a small trick is sufficient to remove this restriction and to prove the expected holomorphy even for the two variable L-function outside the congruence primes. The nearly ordinary Hecke algebras do not play explicit role in the course of proof, because over the base field $\mathbf{Q}$ it coincides with the profinite completion of the tensor product over $\mathbf{Z}_p$ of the ordinary Hecke algebra and the Iwasawa algebra $\mathbf{Z}_p[[X]]$. This fact is true only for $\mathbf{Q}$, and we need the full strength of the theory of nearly ordinary Hecke algebras to carry out the same objectives in the case of totally real fields, which is yet to come. The two variable L-function is supposed to have an arithmetic tie with the projective limit of congruence modules of Hecke algebras over layers of the cyclotomic $\mathbf{Z}_p$-extension (as already discussed in part in [**H6**, Remark 5.6]). The author hopes to clarify this point in near future.

Our analytic method of obtaining two variable interpolation is the p-adic Rankin convolution method developed in [**H1**]. In this sense, this paper is a continuation of [**H1**]. We first summarize some basic notation and terminology given in [**H1**] and then state the exact result. Throughout the paper, we fix a prime $p \geq 5$ and two positive integers N and J prime to p. We take the algebraic closure $\overline{\mathbf{Q}}$ in $\mathbf{C}$ and also fix an algebraic closure $\overline{\mathbf{Q}}_p$ of the p-adic field $\mathbf{Q}_p$. We fix once and for all a field embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$ and hence one can consider any algebraic number as a complex number and a p-adic number simultaneously. The normalized p-adic absolute value of $x \in \overline{\mathbf{Q}}_p$ will be written as $|x|_p$. We take a finite extension K of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$ and let $\mathcal{O}$ denote its p-adic integer ring. Let Λ be the continuous group algebra over $\mathcal{O}$ of the topological group $\Gamma = 1 + p\mathbf{Z}_p$. We fix a topological generator u of Γ and identify Λ with the one variable power series ring $\mathcal{O}[[X]]$ via $u \mapsto 1 + X$. Let $\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O})$ be the (universal) ordinary Hecke algebra of (prime-to-p) level N defined in [**H1**, I,§4], [**H2**,§2] and [**H3**,§1], which is an algebra finite flat over Λ. We fix an irreducible component of $\mathrm{Spec}(\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}))$, which is thus the spectrum of an integral domain $\mathbf{I}'$ finite over Λ. Let

$\mathbf{I}$ be the integral closure of $\mathbf{I}'$ in its quotient field, which is known to be finite flat over Λ (e.g. [**H1**, II, Lemma 3.1]). We denote by $\lambda : \mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \to \mathbf{I}$ the Λ-algebra homomorphism which is the projection to the irreducible component. We may assume that $\mathrm{Spec}(\mathbf{I})$ is *defined* over $\mathcal{O}$, i.e., $\mathbf{I} \cap \overline{\mathbf{Q}}_p = \mathcal{O}$ replacing $\mathcal{O}$ by its finite extension if necessary. We denote by $\mathcal{X} = \mathcal{X}(\mathbf{I})$ the (p-adic) space of all $\overline{\mathbf{Q}}_p$-valued points of $\mathrm{Spec}(\mathbf{I})_{/\mathcal{O}}$, i.e. $\mathcal{X} = \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathbf{I}, \overline{\mathbf{Q}}_p)$. The subset $\mathcal{A} = \mathcal{A}(\mathbf{I})$ of arithmetic points in $\mathcal{X}$ is defined to be the set of all $\mathcal{O}$-algebra homomorphisms $\phi : \mathbf{I} \to \overline{\mathbf{Q}}_p$ which coincide, on a small neighborhood in Γ of the identity, with the group homomorphism $\phi_k : \Gamma \to \mathbf{Z}_p^\times$ given by $\phi_k(\gamma) = \gamma^k$ for some $k \geq 0$. For each $P \in \mathcal{A}$, the integer k as above will be denoted by $k(P)$ and is called the weight of P. The character $P\phi_{k(P)}^{-1}$ is then a finite order character of Γ. This character will be written as ε_P and its order is denoted by $p^{r(P)-1}$. Of course, $\mathcal{A}$ is Zariski dense in $\mathrm{Spec}(\mathbf{I})$ and any algebraic functions, i.e., functions on $\mathrm{Spec}(\mathbf{I})$ in the global section $\mathbf{I}$ of the structure sheaf on $\mathrm{Spec}(\mathbf{I})$, are determined by its value on $\mathcal{A}$. Combining each $P \in \mathcal{X}$ with λ, we get an $\mathcal{O}$-algebra homomorphism $\lambda_P = P \circ \lambda : \mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \to \overline{\mathbf{Q}}_p$. Then the formal q-expansion $f_P = \sum_{n=1}^{\infty} \lambda_P(T(n))q^n \in \overline{\mathbf{Q}}_p[[q]]$ has a meaning as a p-adic ordinary modular form of (prime-to-p) level N which satisfies $f_P|T(n) = \lambda_P(T(n))f_P$ for all $n > 0$. Especially when $P \in \mathcal{A}$ and $k(P) \geq 2$, then f_P is known to be classical; i.e., the q-expansion of f_P actually falls in $\overline{\mathbf{Q}}[[q]]$ and gives the q-expansion of a complex cusp form in $S_{k(P)}(\Gamma_0(Np^{r(P)}), \varepsilon_P\psi\omega^{-k(P)})$, where ψ is a Dirichlet character modulo Np independent of P (and only depending on λ) and ω is the Teichmüller character. Write the restriction of ψ to $(\mathbf{Z}/p\mathbf{Z})^\times$ (resp. $(\mathbf{Z}/N\mathbf{Z})^\times$) as ψ_1 (resp. ψ') and put $\psi_P = \varepsilon_P\psi_1\omega^{-k(P)}$, which is a finite order character of $\mathbf{Z}_p^\times$. Let f_P° be the primitive form associated with f_P for $P \in \mathcal{A}$ and $\pi(P)$ be the automorphic representation of $GL_2(\mathbf{A})$ spanned by the right shifts of f_P°. There is a unique base change lift $\hat{\pi}(P)$ of $\pi(P)$ to $GL_3(\mathbf{A})$ shown in [**G-J**]. The L-function $L(s, \hat{\pi}(P))$ is an Euler product of degree 3 whose Euler factor is given for almost all prime l by $D_l(l^{-k+1-s})^{-1}$ for

(0.1)

$$D_l(X) = (1-\psi'\psi_P(l)^{-1}\alpha^2X)(1-\psi'\psi_P(l)^{-1}\alpha\beta X)(1-\psi'\psi_P(l)^{-1}\beta^2X),$$

where α and β are two roots of $X^2 - \lambda_P(T(l))X + \psi'\psi_P(l)l^{k(P)-1} = 0$. The explicit Euler factors of $L(s, \hat{\pi}(P))$ are computed in [**G-J**, §1] and a summary of them can be found [**Sch**, §1]. The L-function we are concerned is defined by

$$L(s, f_P, \chi) = L(s - k(P) + 1, \hat{\pi}(P) \otimes \chi),$$

where in the left-hand side we consider χ as a Dirichlet character and in the right-hand side, we regard it as an idele character so that for almost all primes l the value of the idele character at the prime element l is given by the value of the Dirichlet character χ at l. This L-function is independent of the twist of f_P by Dirichlet characters (see §1, i6 in the text for the twisting by characters) and its Euler factor is given for almost all prime l by $D_l(\chi(l)l^{-s})^{-1}$. Thus we may assume that λ is minimal in the sense of [**H5**, §7] without losing much generality. Thus f_P° is minimal (i.e. having the minimal conductor among its twists) of conductor N or $Np^{r(P)}$ for all $P \in \mathcal{A}$. We write the conductor of ψ_P as $p^{r_0(P)}$. Then $r_0(P) = r(P)$ if ψ_P is non-trivial but $r_0(P) = 0$ if ψ_0 is trivial. As for the explicit Euler factors of $L(s, f_P, \chi)$ valid for all l, see again [**G-J**, §1] and [**Sch**, §1]. The L-function $L(s, f_P, \chi)$ is critical at integers $1 \leq n \leq k(P) - 1$ with $\chi(-1) = (-1)^{n+1}$ (there is another half of the range of critical values in the interval $[k(P), 2k(P) - 2]$ but they are essentially equal to the values in $[1, k(P) - 1]$ by the functional equation). By the work of Sturm [**St1**], [**St2**] (see also [**Sch**, Th.2.3]), we have the algebraicity property:

$$\frac{L(n, f_P, \chi)}{(2\pi \mathrm{i})^{n-2}\Omega(P)} \in \overline{\mathbf{Q}},$$

where $\Omega(P) = (2\mathrm{i})^{k(P)+1}\pi^2\langle f_P^\circ, f_P^\circ\rangle$ as in [H1, II, (4.13)].

Now let us define several Euler p-factors which appear as modification factors for the p-adic interpolation. For each Dirichlet character η, hereafter if not otherwise indicated, η stands for the primitive Dirichlet chracter associated with η and thus $\eta(n)$ stands for the value of the primitive Dirichlet character η. We also always write $C(\eta)$ (resp. $G(\eta)$, η' and η_1) for the conductor (the Gauss sum, the prime-to-p part and the p-part) of η. We fix a Dirichlet character ξ modulo Jp whose conductor is divisible by J. Then we define, writing $\eta = \psi\psi'_P\eta_Q^{-1}$ and $\eta_Q = \xi\varepsilon_Q\omega^{1-k(Q)}$ for $(Q, P) \in \mathcal{A}(\Lambda) \times \mathcal{A}(\mathbf{I})$,

$$E(Q, P) = E_1(Q, P)E_2(Q, P) \tag{0.2a}$$

with

$$E_1(Q, P) = \left(1 - \eta^{-1}\psi'\psi_P(p)\lambda_P\left(T(p)\right)^{-2}p^{k(Q)-1}\right)$$
$$\left(\psi'(p)^{-1}\lambda_P\left(T(p)\right)^{-2}\right)^{\delta(Q)},$$

and

$$E_2(Q,P)=\begin{cases}\left(1-\eta(p)p^{k(P)-k(Q)-1}\right)\\ \qquad\left(1-\eta\psi'\psi_P(p)\lambda_P\left(T(p)\right)^{-2}p^{2k(P)-k(Q)-2}\right), & \text{if } f_P\neq f_P^\circ,\\ 1, & \text{if } f_p=f_P^\circ.\end{cases}$$

and we define

(0.2b)

$$S(P)\;=\;\begin{cases}-1 \qquad \text{if } \psi_P=1 \text{ and } f_P=f_P^\circ \ \ (\text{then } k(P)=2)\\ \left(\frac{\psi'(p)\lambda_P(T(p))^{2\rho}}{p^{k(P)}}\right)^{r_0(\rho)}\left(1-\frac{\psi'\psi_P(p)p^{k(P)-1}}{\lambda_P(T(p))^2}\right)\\ \qquad\left(1-\frac{\psi'\psi_P(p)p^{k(P)-2}}{\lambda_P(T(p))^2}\right)\\ \qquad \text{if either } \psi_P \text{ is nontrivial or } f_P\neq f_P^\circ,\end{cases}$$

where ρ denotes complex conjugation and $p^{\delta(Q)}$ gives the conductor of the p-part of η_Q. Finally, let $C_0(\lambda;\mathbf{I})$ be the congruence module over $\mathbf{I}$ for the natural extension $\hat{\lambda}:\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O})\otimes_\Lambda\mathbf{I}\to\mathbf{I}$ defined in [**H5**, §6], which is a torsion module over $\mathbf{I}$ of finite type. Then our result is

THEOREM. *Let $\lambda:\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O})\to\mathbf{I}$ be the primitive and minimal Λ-algebra homomorphism as above. Let ξ be a Dirichlet character with $\xi(-1)=1$ modulo Jp whose conductor is divisible by J. Then, unless $\xi'\psi'^{-1}$ is imaginary quadratic and λ has complex multiplication under the imaginary quadratic field corresponding to $\xi'\psi'^{-1}$ in the sense of [**H3**, Proposition 2.3], there exists a unique element L in the quotient field of $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ such that*

(i) for any $0\neq H\in\mathbf{I}$ which annihilates $C_0(\lambda;\mathbf{I})$, $HL\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$,

(ii) For all pairs of points $(Q,P)\in\mathcal{A}(\Lambda)\times\mathcal{A}(\mathbf{I})$ satisfying $1\leq k(Q)<k(P)-1$,

$$L(Q,P)=c(Q,P)S(P)^{-1}E(Q,P)\frac{L(k(Q),f_P,\psi'\psi_P\eta_Q^{-1})}{(2\pi\mathrm{i})^{k(Q)-2}\Omega(P)},\tag{0.3}$$

where

$$c(Q,P)=\Gamma(k(Q))(C(\eta_Q))^{k(Q)-1}\\ \qquad G(\eta_Q)N^{-k(P)/2}W'(f_P)^{-1}G(\psi_P)^{-1}\psi'(p)^{\delta(Q)}.$$

Moreover the above evaluation formula holds for almost all Q with $k(Q) = k(P) - 1$.

This theorem will be proven in §6. The restriction L_p of L to $\mathcal{X}(\Lambda) \times P$ for $P \in \mathcal{A}(\mathbf{I})$ gives essentially the distribution μ in [**Sch**, Th.5.3] for $f = f_P^\circ$ and $\lambda = \psi'\xi'^{-1}$, which is in fact a measure (i.e., holomorphic everywhere) by the above theorem. Thus we can now remove the assumption made in [**Sch**] for μ to be a measure on the conductor of f_P° and the prime p. (Note here that the L-function L_P has a pole if $\xi'\psi'^{-1}$ is imaginary quadratic and λ has complex multiplication under the field corresponding to the character $\xi'\psi'^{-1}$). The evaluation formula (0.3) is expected to hold for *all* (Q, P) in $\mathcal{A}(\Lambda) \times \mathcal{A}(\mathbf{I})$ satisfying $1 \leq k(Q) \leq k(P) - 1$, but this remains to be an open question in general for (Q, P) with $k(Q) = k(P) - 1$.

Here is a summary of this paper. In §1, we collect for our later use some results from the theory of p-adic modular forms of integral weight. In §2, these results valid for integral weight (given in §1) will be generalized to the case of half integral weight. Then in §3, we will construct the p-adic Eisenstein measure of half integral weight, which is a key to carry out the p-adic Rankin convolution in our half integral case. In §4, we state and prove the interpolation theorem for any arithmetic measure having values in the space of p-adic modular forms of half integral weight. In §5, the result in §4 will be specialized to the theta measure of one variable. This yields two variable meromorphic interpolation of L-functions as in the theorem. Finally, in §6, we will show the holomorphy of L as in the theorem by adopting an idea of Schmidt in [**Sch**, §§4 and 5] in our two variable case. Besides this, we shall make in §6 some corrections to misstatements given in [**H3**] and [**H5**].

At the conference, the author presented a survey of results concerning p-adic interpolation of modular L-functions on the spectrum of Hecke algebras, including those of standard L-functions due to Mazur and Kitagawa [**Ki**] and Rankin product type L-functions [**H1**]. Since these results have already been or will be published, we have taken this opportunity to add a new example to our class of p-adic L-functions. In addition to this, we dicussed at the conference the generalization of our methods to totally real fields. This generalization will be discussed in our subsequent papers. One of the reasons for not including the results for totally real fields in the present account is that the full theory of nearly ordinary Hecke algebras and p-adic modular forms of half integral weight is not yet written down and we depend very much on it over totally real fields to develop the theory like the one presented here.

Notation. We follow the notation introduced in the introduction and [**H1**, II] (we quote the second part of [**H1**] as [**H1**, II]). However as already done, the (universal) ordinary p-adic Hecke algebra will be denoted by $\mathbf{h}^{\mathrm{ord}}(N,\mathcal{O})$ (it is denoted as $\mathbf{h}^0(N,\mathcal{O})$ in [**H1**, II]). As already mentioned, if not otherwise indicated, Dirichlet characters are always assumed to be primitive. We denote by χ_m for any integer m, the quadratic (or trivial) character corresponding to the extension $\mathbf{Q}(\sqrt{m})/\mathbf{Q}$. Thus, strictly speaking, according to our convention, we denote by χ_m the primitive character corresponding to this extension. For each character χ and an integer L, we denote by $L_L(s,\chi)$ the L-function obtained from the Dirichlet L-function $L(s,\chi)$ excluding Euler l-factors for primes l dividing L. We always denote by μ_N the group of N-th roots of unity in $\overline{\mathbf{Q}}$ and write $\omega : \mathbf{Z}_p^\times \to \mu_{p-1}$ for the Teichmüller character; hence, the projection of $z \in \mathbf{Z}_p^\times$ to $\Gamma = 1+p\mathbf{Z}_p$ is given by $\langle z\rangle = \omega(z)^{-1}z$.

§1. p-ADIC MODULAR FORMS OF INTEGRAL WEIGHT

We refer to our previous paper [**H1**, I, II, §1] for the notation and the definition for p-adic modular forms (of integral weight) and p-adic Hecke algebras. Especially, for each extension $K/\mathbf{Q}_p$ (inside $\overline{\mathbf{Q}}_p$) and for each subgroup Δ of $SL_2(\mathbf{Z})$ containing the principal congruence subgroup $\Gamma(N)$ of level N, the space $\mathcal{M}_k(\Delta;K)$ of modular forms of weight k over K (with respect to Δ) is by definition the subspace of $K[[q_N]]$ ($q_N = \exp(2\pi i z/N)$) spanned over K by usual modular forms on Δ of weight k with coefficients in $\overline{\mathbf{Q}} \cap K$. Suppose that N is prime to p and write $\Delta(p^r) = \Delta \cap \Gamma_1(p^r)$. We take the inductive limit inside $K[[q_N]]$:

$$\mathcal{M}_k(\Delta(p^\infty);K) = \cup_{r=1}^{\infty}\mathcal{M}_k(\Delta(p^r);K)$$

and define, inside $K[[q_N]]$,

$$\mathcal{M}(\Delta(p^r);K) = \sum_{k=0}^{\infty}\mathcal{M}_k(\Delta(p^r);K) \qquad (r = 0,1,\ldots,\infty).$$

Writing $\sum_{n=0}^{\infty} a(n/N, f)q_N^n$ for the q-expansion of each $f \in \mathcal{M}(\Delta(p^\infty); K)$, we define the uniform norm:

$$|f|_p = \sup_n \left| a\left(\frac{n}{N}, f\right) \right|_p . \tag{1.1}$$

This norm is well defined, and we denote by adding "-" the completion of these spaces inside $K[[q_N]]$. It is well known (e.g. [**H2**, §1], [**H4**, Cor.5.4]) that

(1.2) $\overline{\mathcal{M}_k}(\Delta(p^\infty); K)$ *for* $k \geq 2$ *and* $\overline{\mathcal{M}}(\Delta(p^r); K)$ *are independent of* k *and* r

as a subspace of $K[[q_N]]$ (e.g. [**H4**, Th.3.2, Cor.5.4] and [**G**, III.3]). We thus write $\overline{\mathcal{M}}(\Delta(p^\infty); K)$ for the spaces in (1.2) and put, for the p-adic integer ring $\mathcal{O}$ of K,

$$\overline{\mathcal{M}}(\Delta(p^\infty); \mathcal{O}) = \overline{\mathcal{M}}(\Delta(p^\infty); K) \cap \mathcal{O}[[q_N]],$$

$$\mathcal{M}_k(\Delta(p^\infty); \mathcal{O}) = \mathcal{M}_k(\Delta(p^r); K) \cap \mathcal{O}[[q_N]].$$

Similarly we define $\overline{\mathcal{S}}(\Delta(p^\infty); \mathcal{O})$ and $\mathcal{S}_k(\Delta(p^r); \mathcal{O})$ out of the spaces of cusp forms (see for details [**H1 II.1**]). When $\Delta = \Gamma_1(N)$, we write $\overline{\mathcal{M}}(N; \mathcal{O})$ and $\overline{\mathcal{S}}(N; \mathcal{O})$ for $\overline{\mathcal{M}}(\Delta(p^\infty); \mathcal{O})$ and $\overline{\mathcal{S}}(\Delta(p^\infty); \mathcal{O})$.

As is summarized in [**H1**, II.1], we have the following operators acting on $\overline{\mathcal{M}}(\Delta(p^\infty); \mathcal{O})$:

i1. The action of $\mathbf{Z}_p^\times$.

Let $\Delta_0(p^r) = \Delta \cap \Gamma_0(p^r)$ for $\Delta \supset \Gamma(N)$ with N prime to p. Then $\Delta_0(p^r)/\Delta(p^r)$ is isomorphic to $(\mathbf{Z}/p^r\mathbf{Z})^\times$ via

$$\Delta_0(p^r) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{p^r}.$$

Thus we can let $\varprojlim_r \Delta_0(p^r)/\Delta(p^r) \cong \mathbf{Z}_p^\times$ $(z \mapsto z_p \in \mathbf{Z}_p^\times)$ act on $f \in \mathcal{M}_k(\Delta(p^r); \mathcal{O})$ via

$$f|z = z_p^k f|_k \sigma_z, \tag{1.3a}$$

where $\sigma_z \in \Delta_0(p^r)$ is such that $\sigma_z \equiv \begin{pmatrix} z^{-1} & * \\ 0 & z \end{pmatrix} \pmod{p^r}$. This action extends to a continuous action of $\mathbf{Z}_p^\times$ on $\overline{\mathcal{M}}(\Delta(p^\infty); \mathcal{O})$ and is independent of k. When $\Delta = \Gamma_1(N)$ for N prime to p, we can extend this action to $z \in Z_N = \varprojlim_r (\mathbf{Z}/Np^r\mathbf{Z})^\times = \mathbf{Z}_p^\times \times (\mathbf{Z}/N\mathbf{Z})^\times$ by (1.3a), taking $\sigma \in \Gamma_0(Np^r)$ with $\sigma_z \equiv \begin{pmatrix} z^{-1} & * \\ 0 & z \end{pmatrix}$ (mod Np^r) and writing z_p for the projection of z to $\mathbf{Z}_p^\times$. We can consider $(\mathbf{Z}/Np\mathbf{Z})^\times$ as a subgroup of Z_n naturally. Then we define

$$\begin{aligned}\overline{\mathcal{M}}(N;\mathcal{O})[\psi] \\ &= \{f \in \overline{\mathcal{M}}(N;\mathcal{O}) \quad \big| \quad f|\zeta = \psi(\zeta) \quad \text{for} \quad \zeta \in (\mathbf{Z}/Np\mathbf{Z})^\times\},\end{aligned}$$

where $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ is a Dirichlet character. Then by [**H1**, II.1.1], we know that for each character $\varepsilon : (1+p\mathbf{Z}_p)/(1+p^r\mathbf{Z}_p) \to \mathcal{O}^\times$

$$\overline{\mathcal{M}}(N;\mathcal{O})[\psi] \supset \mathcal{M}_k(\Gamma_0(Np^r), \varepsilon\psi\omega^{-k}; \mathcal{O}),$$

where ω is the Teichmüller character of $\mathbf{Z}_p^\times$.

i2. The Hecke operator $T(n)$.

We have an operator $T(n)$ for each integer $n > 0$ acting on $\mathcal{M}_k(\Gamma_1(Np^r);\mathcal{O})$ and $\overline{\mathcal{M}}(N,\mathcal{O})$ which is given by

$$a(m, f|T(n)) = \sum_{\substack{0<q|m, \\ q|n \\ (q,Np)=1}} q^{-1} a(mn/q^2, f|q), \tag{1.3b}$$

where $f|q$ is the image of f under the action of $q \in Z_N$.

i3. The operator $[t]$ for $0 < t \in \mathbf{Q}^\times$.

Write $t = n/m$ with $(m,n) = 1$ and put

$$\Delta' = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}^{-1} \Delta \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} \cap SL_2(\mathbf{Z}).$$

Then $[t] : \overline{\mathcal{M}}(\Delta(p^\infty);\mathcal{O}) \to \overline{\mathcal{M}}(\Delta'(p^\infty);\mathcal{O})$ is given by

$$f|[t] = \sum_{n=1}^{\infty} a\left(\frac{n}{N}, f\right) q^{nt/N} \qquad \left(q^{nt/N} = q_N^{nt}\right).$$

i4. The action of $SL_2(\mathbf{Z}/N\mathbf{Z})$.

For $f \in \mathcal{M}_k(\Gamma(N)(p^r);\mathcal{O})$, we define $f|\bar\gamma \in \mathcal{M}_k(\Gamma(N)(p^r);\mathcal{O})$ for $\bar\gamma \in SL_2(\mathbf{Z}/N\mathbf{Z})$ by $f|_k\gamma$ where $\gamma \in \Gamma_1(p^r)$ lifting $\bar\gamma$, i.e., $\gamma \equiv \bar\gamma$ mod N. This action is defined independently of the choice of γ and extends to a continuous action of $SL_2(\mathbf{Z}/N\mathbf{Z})$ on $\bar{\mathcal{M}}(\Gamma(N)(p^\infty);\mathcal{O})$ (which is also independent of k).

i5. The twisted trace operator $T_{L/N}$..

Let L be a positive integer prime to p and N be a divisor of L. Then we have a continuous operator

$$T_{L/N} : \overline{\mathcal{M}}(L;\mathcal{O})[\psi] \to \overline{\mathcal{M}}(N;\mathcal{O})[\psi]$$

for each character $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$, which is characterized by the following commutative diagram:

$$\begin{array}{ccc}
\overline{\mathcal{M}}(L;\mathcal{O})[\psi] \supset & \mathcal{M}_k(\Gamma_0(Lp^r), \varepsilon\psi\omega^k;\mathcal{O}) & \\
{\scriptstyle T_{L/N}}\downarrow & \downarrow{\scriptstyle (L/N)\left[\Gamma_0(Lp^r)\begin{pmatrix}1 & 0\\ 0 & L/N\end{pmatrix}\Gamma_0(Np^r)\right]} & \\
\overline{\mathcal{M}}(N;\mathcal{O})[\psi] \supset & \mathcal{M}_k(\Gamma_0(Np^r), \varepsilon\psi\omega^{-k};\mathcal{O}) &
\end{array}$$

for each character ε of $1+p\mathbf{Z}_p$ of order p^{r-1} and each positive integer k.

i6. The twisting operator for each Dirichlet character.

Let $\chi : (\mathbf{Z}/Mp^r\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character for a positive integer M prime to p. Then we can define an operator

$$\chi : \overline{\mathcal{M}}(N;\mathcal{O}) \to \overline{\mathcal{S}}(NM^2;\mathcal{O})$$

by $a(n, f|\chi) = \chi(n)a(n, f)$ for all $n \geq 0$, where we agree that $\chi(n) = 0$ if n has a non-trivial common divisor with Mp. The cuspidality of $f|\chi$ follows from [**H1**, II.2.2] (see also the proof of [**H1**, II.2.3]).

i7. Differential operators.

We have a differential operator

$$d : \overline{\mathcal{M}}(N;\mathcal{O}) \to \overline{\mathcal{S}}(N;\mathcal{O})$$

given by $a(n, df) = na(n, f)$ for all n. The cuspidality of df follows from [**H1**, II.2.3].

i8. Ordinary projection e.

There is a continuous operator

$$e : \overline{\mathcal{M}}(N,\mathcal{O}) \to \overline{\mathcal{M}}(N;\mathcal{O})$$

such that (i) $\mathcal{M}_k(\Gamma_1(Np^r);\mathcal{O})$ is stable under e for all r and $k > 0$, (ii) $e = e^2$ and on $\mathcal{M}_k(\Gamma_1(Np^r);\mathcal{O})$, there exists an integer $m > 0$ such that

$$e = \lim_{n\to\infty} T(p)^{p^{mn}(p^m-1)} \text{ in } \mathrm{End}_{\mathcal{O}}(\mathcal{M}_k(\Gamma_1(Np^r);\mathcal{O})).$$

For each subspace X of $\overline{\mathcal{M}}(N;K)$ stable under e, we write X^{ord} for eX. Then $\overline{\mathcal{M}}^{\mathrm{ord}}(N;\mathcal{O})$ is the maximal subspace of $\overline{\mathcal{M}}(N;\mathcal{O})$ on which $T(p)$ is invertible. Then the following fact is known (see [**H1**, II.2]):

(1.4) $$\chi \circ d = d \circ \chi, \quad nd \circ T(n) = T(n) \circ d, \quad d \circ [t] = t[t] \circ d$$
$$\text{and} \quad z_p^2 d(f|z) = (df)|z \text{ for } z \in Z_N.$$

§2. p-ADIC MODULAR FORMS OF HALF INTEGRAL WEIGHT

In this section, we define the space of p-adic modular forms of half integral weight and generalize results in §1 in the case of half integral weight.

We first consider the theta series

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z) \qquad (e(z) = \exp(2\pi i z) = q)$$

associated with the quadratic form: $x \mapsto x^2$ on $\mathbf{Z}$. Define for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ an automorphic factor

(2.1a) $$j(\gamma, z) = \Theta(\gamma(z))/\Theta(z).$$

Then $j(\gamma, z)$ is a holomorphic function on $\mathcal{H}$ satisfying

(2.1b) $$j(\gamma, z)^2 = \chi_{-1}(d)(cz+d),$$

where χ_m for $m \in \mathbf{Q}^\times$ is the Dirichlet character corresponding to the extension $\mathbf{Q}\left(\sqrt{m}\right)/\mathbf{Q}$. We refer to [**Sh1**] for the details of these facts and for the transformation formula of Θ. Let Δ be a congruence subgroup of $\Gamma_0(4)$. Then for each odd positive integer k, the space $\mathcal{G}_{k/2}(\Delta;\ \mathbf{C})$ of modular forms of weight $k/2$ consists of holomorphic functions $f : \mathcal{H} \to \mathbf{C}$ satisfying

(2.2) $$f|_{k/2}\gamma(z) = f(\gamma(z))j(\gamma, z)^{-1}(cz+d)^{-(k-1)/2}$$
$$= f(z) \textit{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$$

(2.2b) $\quad f$ *is holomorphic at all cusps of* Δ

in the sense of [**Sh1**, §1]. Replacing (2.2b) by the cuspidal condition:

(2.2c) *f vanishes at every cusp of Δ*

we define the space of cusp forms $\mathcal{P}_{k/2}(\Delta;\mathbf{C})$. Under the Fourier expansion at ∞, which we write as q-expansion (by replacing $\exp(2\pi i n z/N)$ by $q_N^n = q^{n/N}$), we consider $\mathcal{G}_{k/2}(\Delta;\mathbf{C})$ as a subspace of the power series ring $\mathbf{C}[[q_N]]$ if $\Delta \supset \Gamma(N)$ (this means that $4|N$). We define, for a subring A of $\mathbf{C}$,

$$\begin{aligned}\mathcal{G}_{k/2}(\Delta;A) &= \mathcal{G}_{k/2}(\Delta;\mathbf{C}) \cap A[[q_N]],\\ \mathcal{P}_{k/2}(\Delta;A) &= \mathcal{P}_{k/2}(\Delta;\mathbf{C}) \cap A[[q_N]].\end{aligned} \tag{2.3}$$

We then define $\mathcal{G}_{k/2}(\Delta;\bar{\mathbf{Q}}_p)$ as the linear span of $\mathcal{G}_{k/2}(\Delta;\bar{\mathbf{Q}})$ in $\bar{\mathbf{Q}}_p[[q_N]]$; i.e., we have

$$\begin{aligned}\mathcal{G}_{k/2}(\Delta;\bar{\mathbf{Q}}_p) &= \mathcal{G}_{k/2}(\Delta;\bar{\mathbf{Q}}) \otimes_{\bar{\mathbf{Q}}} \bar{\mathbf{Q}}_p \text{ in } \bar{\mathbf{Q}}_p[[q_N]],\\ \mathcal{P}_{k/2}(\Delta;\bar{\mathbf{Q}}_p) &= \mathcal{P}_{k/2}(\Delta;\bar{\mathbf{Q}}) \otimes_{\bar{\mathbf{Q}}} \bar{\mathbf{Q}}_p \text{ in } \bar{\mathbf{Q}}_p[[q_N]].\end{aligned} \tag{2.4}$$

Then we define $\mathcal{G}_{k/2}(\Delta;A)$ and $\mathcal{P}_{k/2}(\Delta;A)$ for each subring A of $\bar{\mathbf{Q}}_p$ replacing $\mathbf{C}$ in (2.3) by $\bar{\mathbf{Q}}_p$. If A is a subfield in $\bar{\mathbf{Q}}_p$ or $\mathbf{C}$, we have:

$$\mathcal{G}_{k/2}(\Gamma_1(N);A) = \mathcal{G}_{k/2}(\Gamma_1(N);\mathbf{Q}) \otimes_{\mathbf{Q}} A, \tag{2.5}$$

$$\mathcal{G}_{k/2}(\Gamma_0(N),\xi;A) = \mathcal{G}_{k/2}(\Gamma_0(N),\xi;\mathbf{Q}(\xi)) \otimes_{\mathbf{Q}(\xi)} A \text{ if } A \supset \mathbf{Q}(\xi),$$

where

$$\mathcal{G}_{k/2}(\Gamma_0(N),\xi;A) = \left\{f \in \mathcal{G}_{k/2}(\Gamma_1(N);A) \,\middle|\, f|_{k/2}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \xi(d)f \right.$$

$$\left. \text{for } \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(N)\right\}$$

for each Dirichlet character ξ modulo N. This fact is shown by [**Sh4**, Lemma 4] or can be proven similarly to [**H4**, Cor.4.5] (using the result in [**Sh3**, Th.1.2]).

Let K be an extension of $\mathbf{Q}_p$ (inside $\bar{\mathbf{Q}}_p$) and $\mathcal{O}$ be its p-adic integer ring. We then put (for $\Gamma_0(4) \supset \Delta \supset \Gamma(N)$)

$$\mathcal{G}_{k/2}(\Delta(p^\infty);K) = \bigcup_{r=1}^{\infty} \mathcal{G}_{k/2}(\Delta(p^r);K) \text{ in } K[[q_N]], \tag{2.6}$$

$$\mathcal{G}_{k/2}(\Delta(p^\infty);\mathcal{O}) = \mathcal{G}_{k/2}(\Delta(p^\infty);K) \cap \mathcal{O}[[q_N]],$$

$$\mathcal{G}(\Delta(p^r);K) = \sum_{m=0}^{\infty} \mathcal{G}_{m+(1/2)}(\Delta(p^r);K),$$

$$\mathcal{G}(\Delta(p^r);\mathcal{O}) = \mathcal{G}(\Delta(p^r);K) \cap \mathcal{O}[[q_N]] \qquad (r = 1, 2, \ldots, \infty).$$

Actually, $\mathcal{G}(\Delta(p^r);K)$ is the direct sum: $\oplus_{m=0}^{\infty}\mathcal{G}_{m+(1/2)}(\Delta(p^r);K)$. Hereafter suppose that K is a finite extension of $\mathbf{Q}_p$ and hence is complete under the p-adic topology. By adding "-", we denote the completion of these spaces under the norm (1.1). Especially $\bar{\mathcal{G}}(\Delta(p^\infty);K)$ denotes the space of p-adic modular forms of half integral weight. Similarly we define the spaces $\mathcal{P}_{k/2}(\Delta(p^r);\mathcal{O})$ and $\bar{\mathcal{P}}(\Delta(p^\infty);\mathcal{O})$, etc., out of cusp forms.

THEOREM 2.1. *Let A be one of the rings K and $\mathcal{O}$, and suppose that $\Gamma_0(4) \supset \Delta \supset \Gamma(N)$ for an integer N prime to p. Then, as a subspace of $A[[q_N]]$, $\bar{\mathcal{G}}(\Delta(p^r);A)$ and $\bar{\mathcal{P}}(\Delta(p^r);A)$ are independent of $r \geq 1$ and coincide with $\bar{\mathcal{G}}(\Delta(p^\infty);A)$ and $\bar{\mathcal{P}}(\Delta(p^\infty);A)$ respectively.*

As in (1.2), one may conjecture that

$$\bar{\mathcal{G}}_{k/2}(\Delta(p^\infty);A)$$

and

$$\bar{\mathcal{P}}_{k/2}(\Delta(p^\infty);A)$$

coincide respectively with

$$\bar{\mathcal{G}}(\Delta(p^\infty);A)$$

and

$$\bar{\mathcal{P}}(\Delta(p^\infty);A)$$

if $k \geq 3$, but what we know from the above theorem is the inclusion of the former space in the latter.

PROOF: We prove the assertion only for $A = \mathcal{O}$, $r = 1$ and for modular forms. The other cases can be dealt with in a similar fashion. Note that

(i) Θ as a power series in $\mathbf{Z}[[q]]$ is a unit in the ring $\mathbf{Z}[[q]]$,
(ii) Θ has no zeros inside $\mathcal{H}$.

The assertion (ii) follows from the well known infinite product expansion of Θ (for example in [W, (30), p.31]) convergent on $\mathcal{H}$. We write $\mathcal{M}$ for $\overline{\mathcal{M}}(\Delta(p^\infty);\mathcal{O})$, which is an integral domain, and $\mathcal{A}$ for the quotient field of $\overline{\mathcal{M}}(\Delta(p^\infty);\mathcal{O})$. Let

$$\mathcal{G} = \overline{\mathcal{G}}(\Delta(p^\infty);\mathcal{O}) \text{ and } \mathcal{G}_0 = \mathcal{G}(\Delta(p);\mathcal{O}).$$

It is known by Serre and Katz (see [**H2**, §1]) that $\mathcal{M}$ is also the completion under the norm (1.1) of the ring $\mathcal{M}_0 = (\sum_{m=0}^{\infty}\mathcal{M}_m(\Delta(p);K))\cap \mathcal{O}[[q_N]]$ in $K[[q_n]]$. Then naturally $\mathcal{G}_0$ (resp. $\mathcal{G}$) is a module over $\mathcal{M}_0$ (resp. $\mathcal{M}$) under the multiplication in $\mathcal{O}[[q_N]]$. We consider $\Theta^{-2}\mathcal{M}_0$ inside $\mathcal{A}$. By (i), $\Theta^{-2}\mathcal{M}$ is embedded into $\mathcal{O}[[q_N]]$ and is the completion of $\Theta^{-2}\mathcal{M}_0$ under the norm (1.1). Let $Y_r = \Delta(p^r)\backslash\mathcal{H}$ and X_r be the smooth compactification of Y_r. Then $C_r = X_r - Y_r$ is the set of cusps of X_r, which is a finite set. A cusp $P \in C_r$ is called unramified if it is unramified over X_0. Let S_r (for $r > 0$) be the subset of unramified cusps in C_r. If we denote by $\bar{\Delta}$ the natural image of Δ in $SL_2(\mathbf{Z}/N\mathbf{Z})$ and put $U = \left\{\pm\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} \middle| u \in \mathbf{Z}/N\mathbf{Z}\right\}$, then $C_0 \cong \bar{\Delta}\backslash SL_2(\mathbf{Z}/N\mathbf{Z})/U$ and either $S_r \cong C_0 \times (\mathbf{Z}/p^r\mathbf{Z})^\times$ or $S_r \cong C_0 \times (\mathbf{Z}/p^r\mathbf{Z})^\times/\{\pm 1\}$ naturally according as $-1 \notin \Delta$ or $-1 \in \Delta$. Let $\mathcal{C}(S_0 \times \mathbf{Z}_p^\times;\mathcal{O})$ be the space of all continuous functions on $S_0 \times \mathbf{Z}_p^\times$ with values in $\mathcal{O}$. Since we have natural action of $SL_2(\mathbf{Z}/N\mathbf{Z})$ and $\mathbf{Z}_p^\times$ on $\overline{\mathcal{M}}(\Delta(p^\infty);\mathcal{O})$ (i1 and i4 in §1), we can consider an embedding

$$\iota : \Theta^{-2}\mathcal{M} \to \mathcal{C}(S_0 \times \mathbf{Z}_p^\times;\mathcal{O})((q_N))$$

such that the n-th coefficient of $\iota(f)$ is the function:

$$S_0 \times \mathbf{Z}_p^\times \ni (s,z) \mapsto a(n, f|sz) \in \mathcal{O},$$

where $\mathcal{C}(C_0 \times \mathbf{Z}_p^\times;\mathcal{O})((q_N))$ is the ring of Laurent series of indeterminate q_N with coefficients in $\mathcal{C}(S_0 \times \mathbf{Z}_p^\times;\mathcal{O})$ For any point $s \in S_0$, we put

$$v_s(f) = \min\{n | z \mapsto a(n, f|sz) \text{ is non-zero as a function on } \mathbf{Z}_p^\times\}$$

and define

$$\mathcal{X} = \{f \in \Theta^{-2}\mathcal{M} | v_s(f) \geq -v_s(\Theta^2)/2 \text{ for all } s \in S_0\},$$

$$\mathcal{X}_0 = \{f \in X \cap \Theta^{-2}\mathcal{M}_0 | f \text{ is holomorphic at all } t \in C_1 - S_1\}.$$

As already seen in [**H1**, II] in the proof of Theorem 2.1,

(2.7) *there exists a modular form* $G_n \in \mathcal{M}_r(\Gamma_0(p))$ *for each positive integer* n *such that (i)* $r = p^{n-1}(p-1)$, *(ii)* $G_n \equiv 1 \pmod{p}^{n-\epsilon}$ *for* $0 \leq \epsilon \in \mathbf{Z}$ *independent of* n *and (iii)* $a(0, G_n|\gamma) = 0$ *if* $\gamma \in SL_2(\mathbf{Z}) - \Gamma_0(p)$.

Namely for all $t \in C_1 - S_1$, the function $z \mapsto a(0, G_n|tz)$ on $\mathbf{Z}_p^\times$ is identically zero. Thus if $f \in \mathcal{X}$ and $f = \lim_{n\to\infty} f_n$ with $f_n \in \Theta^{-2}\mathcal{M}_0$, then for sufficiently large m (which may depend on f_n), $G_n^m f_n \in \mathcal{X}_0$ and $|G_n^m f_n - f_n|_p < p^{\varepsilon - n}$. Namely $f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} G_n^m f_n$, and $\mathcal{X}_0$ is dense in $\mathcal{X}$. Then, for any $g \in \mathcal{G}$, we have by (ii), $g\Theta^{-1} \in \mathcal{X}$. Thus $\Theta\mathcal{X} \supset \mathcal{G}$. On the other hand, by definition, $\mathcal{G}_0 \supset \Theta X_0$ and hence $\mathcal{G} = \Theta X = \overline{\mathcal{G}}(\Delta(p); \mathcal{O})$, which finishes the proof.

We simply write $\overline{\mathcal{G}}(N; \mathcal{O})$ for $\overline{\mathcal{G}}(\Gamma_1(N)(p); \mathcal{O})$. We then have the following operations defined on $\overline{\mathcal{G}}(N; \mathcal{O})$.

h1. The action of $\mathbf{Z}_p^\times$.

Suppose that $\Gamma_0(4) \supset \Delta \supset \Gamma(N)$ for N prime to p. It is known (see [**Sh3**, Th.1.2]) that the action: $f \mapsto f|_{k/2}\sigma$ for $\sigma \in \Gamma_0(4)$ preserves $\mathcal{G}_{k/2}(\Delta; \bar{\mathbf{Q}})$ if σ normalizes Δ. Thus this operator induces an action of σ on $\mathcal{G}_{k/2}(\Delta; \bar{\mathbf{Q}}_p)$ which we again write as $f \mapsto f|_{k/2}\sigma$. We now define two kinds of action of $z \in \mathbf{Z}_p^\times$ by

$$\text{(2.8a)} \qquad f\|z = z^{(k+1)/2} f|_{k/2}\sigma, \quad f|z = z^{(k-1)/2} f|_{k/2}\sigma$$

for $f \in \mathcal{G}_{k/2}(\Delta(p^r); \bar{\mathbf{Q}}_p)$, where $\sigma \in \Gamma_0(4)$ with $\sigma \equiv \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$ $\pmod{p^r}$ and $\sigma \equiv 1 \mod N$. When $\Delta = \Gamma_1(N)$ or $\Gamma_0(N)$, we can extend this action to $(z, \zeta) \in Z_N = \mathbf{Z}_p^\times \times (\mathbf{Z}/N\mathbf{Z})^\times$ by

$$\text{(2.8b)} \quad f\|(z,\zeta) = \chi(\zeta) z^{(k+1)/2} f|_{k/2}\sigma, \qquad f|(z,\zeta) = z^{(k-1)/2} f|_{k/2}\sigma = \chi(\zeta) z^{-1} f\|z,$$

where $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \{\pm 1\}$ is the Legendre symbol corresponding to $\mathbf{Q}\left(\sqrt{-1}\right)/\mathbf{Q}$ and $\sigma = \sigma_{(z,\zeta)} \in \Gamma_0(N)$ such that $\sigma \equiv \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$ $\pmod{p^r}$ and $\sigma \equiv \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \pmod{N}$. We then extend these actions to $\mathcal{G}(\Delta(p^r); \bar{\mathbf{Q}}_p)$ $(0 \leq r \leq \infty)$ diagonally; for example, for

$f = \sum_k f_k$ with $f_k \in \mathcal{G}_{m+(1/2)}(\Gamma_1(Np^r); \bar{\mathbf{Q}}_p)$, we have

$$f|z = \sum_k z^{(k-1)/2} f_k|_{k/2}\sigma,$$

since $\mathcal{G}(\Delta(p^r); \bar{\mathbf{Q}}_p)$ is actually the direct sum

$$\oplus_{m=0}^{\infty} \mathcal{G}_{m+(1/2)}(\Delta(p^r); \bar{\mathbf{Q}}_p).$$

THEOREM 2.2. *The action (2.8a) (resp. (2.8b)) preserves* $\mathcal{G}(\Delta(p^r); A)$ $(0 \leq r \leq \infty)$ *and* $\mathcal{G}_{k/2}(\Delta(p^r); A)$ *(resp.* $\mathcal{G}(\Gamma_1(Np^r); A)$ $(0 \leq r \leq \infty)$ *and* $\mathcal{G}_{k/2}(\Gamma_1(Np^r); A)$*) for* $A = K$ *and* $\mathcal{O}$ *and hence extends to its completion under the norm (1.1).*

PROOF: We only prove the result for the action: $f \mapsto f\|(z,\zeta)$ in (2.8b) and $\mathcal{G}(\Gamma_1(Np^r); \mathcal{O})$, since the other cases can be treated similarly. Because of the formula: $j(\sigma, z)^2 = \chi(\zeta)(cz+d)$ if $\sigma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for $f \in \mathcal{G}_{k/2}(\Gamma_1(Np^r); \bar{\mathbf{Q}})$, we see, with the notation of (2.8b),

$$\begin{aligned}
&(f\|(z,\zeta))\Theta \\
&\quad = \chi(\zeta) z^{(k+1)/2} f(\sigma(z)) j(\sigma, z)^{-1} \\
&\qquad\qquad (cz+d)^{-(k-1)/2} \Theta(\sigma(z)) j(\sigma, z)^{-1} \\
&\quad = (f\Theta)|(z,\zeta),
\end{aligned}$$

where at the left-hand side, (z,ζ) acts via the action in (i1) in §1. Thus generally for $f \in \mathcal{G}(\Gamma_1(Np^r); \mathcal{O})$, we have $f\|(z,\zeta))\Theta = (f\Theta)|(z,\zeta)$. Since the action of (z,ζ) on $\overline{\mathcal{M}}(N; \bar{\mathbf{Q}}_p)$ preserves $\overline{\mathcal{M}}(N; \mathcal{O})$, $(f\|(z,\zeta))\Theta \in \mathcal{O}[[q]]$. Since Θ is a unit in $\mathbf{Z}[[q]]$, $f\|(z,\zeta) \in \mathcal{O}[[q]]$, which proves the assertion.

h2. The module structure of $\overline{\mathcal{G}}(N; \mathcal{O})$ over $\overline{\mathcal{M}}(N; \mathcal{O})$..

Picking an integer κ and an odd k, we have a product

$$\mathcal{M}_{\kappa}(\Delta(p^r); \mathcal{O}) \times \mathcal{G}_{k/2}(\Delta(p^r); \mathcal{O}) \rightarrow \mathcal{G}_{\kappa+(k/2)}(\Delta(p^r); \mathcal{O})$$

induced by the multiplication in $\mathcal{O}[[q_N]]$. This induces a product

$$\mathcal{M}(\Delta(p^r); \mathcal{O}) \times \mathcal{G}(\Delta(p^r); \mathcal{O}) \rightarrow \mathcal{G}(\Delta(p^r); \mathcal{O})$$

which is uniformly continuous under the norm (1.1). Thus by continuity, we obtain the product:

$$\overline{\mathcal{M}}(\Delta(p^{\infty}); \mathcal{O}) \times \overline{\mathcal{G}}(\Delta(p^{\infty}); \mathcal{O}) \rightarrow \overline{\mathcal{G}}(\Delta(p^{\infty}); \mathcal{O}), \tag{2.9}$$

which of course coincides with the usual multiplication in $\mathcal{O}[[q_N]]$ and satisfies

(2.9a) $(fg)|z = (f|z)(g\|z)$ for all $z \in \mathbf{Z}_p^\times$ or $z \in Z_N$ if $\Delta = \Gamma_1(N)$.

The product (2.9) also induces a module structure on $\overline{\mathcal{P}}(N;\mathcal{O})$ over $\overline{\mathcal{M}}(N;\mathcal{O})$. Similarly, the multiplication in $\mathcal{O}[[q_N]]$ induces a product

$$\overline{\mathcal{G}}(N;\mathcal{O}) \times \overline{\mathcal{G}}(N;\mathcal{O}) \to \overline{\mathcal{M}}(N;\mathcal{O}),$$

which again satisfies

$$(fg)|z = (g|z)(f\|z) \qquad (f, g \in \overline{\mathcal{G}}(N;\mathcal{O})), \tag{2.9b}$$

where the action of $z \in Z_N$ on the right-hand side is the one given in (1.3a).

h3. The action of $[t]$ for $0 < t \in Z$.

As shown in [**Sh1**, Prop.1.3], we can define a linear operator

$$[t] : \overline{\mathcal{G}}(N;\mathcal{O}) \to \overline{\mathcal{G}}(Nt;\mathcal{O}) \text{ by } f|[t] = \sum_{n=1}^{\infty} a(n,f)q^{nt}$$

and especially $[t]$ induces

$$[t] : \mathcal{G}_{k/2}(\Gamma_0(Np^r), \xi; \mathcal{O}) \to \mathcal{G}_{k/2}(\Gamma_0(Ntp^r), \xi\chi_t; \mathcal{O}),$$

where χ_t is the Dirichlet character corresponding to the extension $\mathbf{Q}\left(\sqrt{t}\right)/\mathbf{Q}$.

h4. The involution τ.

We can define as in [**Sh1**, 1.4] the involution $\tau = \tau(Np^r)$ on $\mathcal{G}_{k/2}(\Gamma_1(Np^r);\mathbf{C})$ and $\mathcal{M}_\kappa(\Gamma_1(Np^r);\mathbf{C})$ as follows:

$$f|\tau = \begin{cases} f(-1/Np^rz)(Np^r)^{-k/4}(-\mathrm{i}z)^{k/2} & \text{for } f \in \mathcal{G}_{k/2} \\ f(-1/Np^rz)(Np^r)^{-\kappa/2}z^{-\kappa} & \text{for } f \in \mathcal{M}_\kappa. \end{cases} \tag{2.10a}$$

This action preserves the space of modular forms over $\bar{\mathbf{Q}}$ and thus induces an action on those over $\bar{\mathbf{Q}}_p$. We especially have

$$\tau^2 = \begin{cases} (-1)^\kappa \text{ on } \mathcal{M}_\kappa \\ 1 \text{ on } \mathcal{G}_{k/2}. \end{cases} \tag{2.10b}$$

h5. Twisting operator.

Let $\phi : (\mathbf{Z}/Mp^s\mathbf{Z}) \to \mathcal{O}$ be a function, where M is an integer prime to p. In the same manner as in [**Sh1**, 3.6] (see also [**H1**, I.8.1]), we can define an operator

$$\phi : \overline{\mathcal{G}}(N;\mathcal{O}) \to \overline{\mathcal{G}}(NM^2;\mathcal{O})$$

which is given by

$$a(n, f|\phi) = \phi(n)a(n, f) \text{ for all } n \geq 0. \tag{2.11}$$

This can be generalized to any continuous function ϕ on $\mathbf{Z}_p \times (\mathbf{Z}/M\mathbf{Z})$ with values in $\mathcal{O}$. In fact, by taking locally constant functions ϕ_n : $\mathbf{Z}_p \times \mathbf{Z}/M\mathbf{Z} \to \mathcal{O}$ uniformly converging to ϕ, we can define $f|\phi = \lim_{n\to\infty} f|\phi_n$ in $\overline{\mathcal{G}}(NM^2;\mathcal{O})$. Especially, taking as ϕ the identity map : $\mathbf{Z}_p \cong \mathbf{Z}_p$, we have the differential operator

(2.12)
$d : \overline{\mathcal{G}}(n;\mathcal{O}) \to \overline{\mathcal{G}}(N;\mathcal{O})$ *such that* $a(n, df) = na(n, f)$ *for all* $n \geq 0$.

The same proof as in [**H1**, II.2.4] gives

$$e(f(g|\chi)) = \chi(-1)e((f|\chi)g), \qquad e(fdg) = -e(gdf) \tag{(2.13)}$$

for $f, g \in \overline{\mathcal{G}}(N;\mathcal{O})$, where $\chi : (\mathbf{Z}/Mp^s\mathbf{Z})^\times \to \mathcal{O}^\times$ is a Dirichlet character. Here we agree to put $\chi(n) = 0$ if n is not prime to Mp^s.

§3. Eisenstein measure of half integral weight

For any compact topological space T, we write $\mathcal{C}(T;\mathcal{O})$ (resp. $\mathcal{LC}(T;\mathcal{O})$ for the normed space of all continuous (resp. locally constant) functions $\phi : T \to \mathcal{O}$ with the uniform norm:

$$|\phi|_p = \sup_t(|\phi(t)|_p).$$

For each positive integer L divisible by 4, we now define an $\mathcal{O}$-linear map

$$E : \mathcal{C}(Z_L;\mathcal{O}) \to \overline{\mathcal{G}}(L;\mathcal{O})$$

with the following properties:

$$E(\phi)|z = zE(\phi|z) \text{ for } z \in Z_L, \tag{E1.}$$

where $(\phi|z)(z') = \phi(zz')$;

(E2.) *For each character* $\xi : (\mathbf{Z}/Lp^r\mathbf{Z})^\times \to \mathcal{O}^\times$ *and for each odd integer* $k \geq 3$,

$$\int_{Z_L} \xi(z) z_p^{(k-3)/2} dE \in \mathcal{G}_{k/2}(\Gamma_0(Lp^{r+1}), \xi)$$

(Especially, this implies $E(\phi) = 0$ if $\phi(-z) = \phi(z)$);

(E3.) *For each* $\phi \in \mathcal{C}(Z_L; \mathcal{O})$, $a(n, E(\phi)) = 0$ *if* p *divides* n. *Especially we have*

$$a(0, E(\phi)) = 0 \textit{ and } E(\phi)|\iota_p = E(\phi)$$

where $\iota_p : \mathbf{Z}/p\mathbf{Z} \to \{0, 1\}$ is the trivial Dirichlet character modulo p; i.e. $\iota_p(n) = 0$ or 1 according as n is divisible by p or not.

Before constructing E explicitly, we recall the result in [**Sh2**, Prop.1] on the Fourier coefficients of Eisenstein series of half integral weight. We define, for each odd integer $k > 0$ and for a character $\xi : (\mathbf{Z}/Lp^r\mathbf{Z})^\times \to \bar{\mathbf{Q}}^\times$ with $\xi(-1) = (-1)^{(k-1)/2}$,

$$(3.1) \quad E^*_{k/2}(z, s; \xi) = L_{Lp}(2s + k - 1, \xi^2) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(Lp^r)} \xi\chi_{Lp^r}\chi^{(k-1)/2}(\gamma) j(\gamma, z)^{-k} |j(\gamma, z)|^{-2s},$$

$$E_{k/2}(z, m; \xi) = (2\pi)^{(m-k)/2}(Lp^r)^{(k-2m)/4} \Gamma\left(\frac{k-m}{2}\right)\left\{(2y)^{-m/2} E^*_{k/2}(z, -m; \xi)\right\}|_{k/2}\tau,$$

where $\tau = \tau(Lp^r)$ as in h4 in §2, $\xi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \xi(d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Lp^r)$, we have written $z = x + iy$ $(x, y \in \mathbf{R})$ and $\Gamma_\infty = \left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbf{Z}\right\}$. Then $E^*_{k/2}(z, s; \xi)$ is absolutely and locally uniformly convergent if the real part of s is sufficiently large and can be continued as a meromorphic function on the whole complex s-plane. By [**Sh2**, Prop.1] combined with the well known formula [H1, II, (6.4a,b,c)], we have

(3.2a) $$E_{k/2}(\xi) = E_{k/2}(z, 2-k; \xi) \in \mathcal{G}_{k/2}(\Gamma_0(Lp^r), \xi; \bar{\mathbf{Q}}),$$

(3.2b)
$$\begin{aligned} E_{k/2}(\xi) = L_{Lp}(2-k, \xi^2) &+ \sum_{n=1}^{\infty} q^n L_{Lp}\left(\frac{3-k}{2}, \xi\chi_n\right) \\ &\times \sum_{\substack{u^2t^2|n,\\ (ut,Lp)=1,\\ u>0,\\ t>0}} \mu(u)\xi(ut^2)\chi_n(u)t(ut^2)^{(k-3)/2}, \end{aligned}$$

where μ is the Möbius function and χ_n is the primitive character such that $\chi_n(m) = \left(\frac{n}{m}\right)$ if m is prime to n. Moreover we have
(3.2c)
$E_{k/2}(z, m; \xi) = \delta^i_{j/2} E_{j/2}(z, 2-j; \xi)$ *if* $m \in [(k-1)/2, k]$ *and* m *is odd,*

where $i = \frac{k-m}{2} - 1$, $j = 2m + 4 - k$ and δ^r_s is the Shimura's differential operator defined by

$$\delta_s = \frac{1}{2\pi i}\left(\frac{s}{2iy} + \frac{d}{dz}\right), \quad \delta^r_s = \delta_{s+2r-2} \cdot \cdots \cdot \delta_{s+2}\delta_s$$

for $s \in \mathbf{C}$ and $0 \le r \in \mathbf{Z}$. We need the following variant of the well known result of Kubota-Leopoldt and Mazur (cf. [**L**, Chapter 4]):

LEMMA 3.1. *Let* $\alpha : (\mathbf{Z}/Cp^r\mathbf{Z})^\times \to \mathcal{O}^\times$ *be a primitive character (of conductor* Cp^r *for* C *prime to* p*). For each integer* $b > 1$ *prime to* p*, there exists a unique* $\mathcal{O}$*-linear map* $\zeta^b_\alpha : \mathcal{C}(Z_L; \mathcal{O}) \to \mathcal{O}$ *such that for each finite order character* $\xi : Z_L \to \mathcal{O}$ *and for each integer* $n > 0$

$$\int_{Z_L} \xi(z) z_p^{n-1} d\zeta^b_\alpha = (1 - \xi_1\alpha_1(b)b^n) L_{Lp}(1-n, \xi\alpha),$$

where $\xi_1\alpha_1$ is the restriction of $\xi\alpha$ to $\mathbf{Z}_p^\times$.

A sketch of proof: As in [**L**, Chap.4], we have a distribution ζ on Z_{CL} such that

$$\int_{Z_L} \xi(z)z_p^{n-1}d\zeta = L_{CL}(1-n,\xi)$$

for all Dirichlet characters ξ on Z_{CL} and positive integer n. (In particular, this implies $\int_{Z_L} \xi(z)z_p^{n-1}d\zeta = 0$ if $\xi(-1) = (-1)^{n-1}$). Moreover, for each $x \in Z_{CL}$, if we define $\zeta^x : \mathcal{LC}(Z_{CL};\mathcal{O}) \to \mathcal{O}$ by

$$\int_{Z_L} \phi(z)d\zeta^x(z) = \int_{Z_L} (\phi(z) - x_p\phi(xz))d\zeta(z),$$

then ζ^x extends uniquely to a measure $\zeta^x : \mathcal{C}(Z_{CL};\mathcal{O}) \to \mathcal{O}$. Decomposing $Z_{CL} = \mathbf{Z}_p^\times \times (\mathbf{Z}/CL\mathbf{Z})^\times$ naturally, we put $x = (b,1)$ for $b \in \mathbf{Z}_p^\times$ and $1 \in (\mathbf{Z}/CL\mathbf{Z})^\times$. Then writing $\pi : \mathbf{Z}_{CL} \to \mathbf{Z}_L$ for the projection map, we define

$$\int_{Z_L} \phi(z)d\zeta_\alpha^b(z) = \int_{Z_{CL}} \alpha(z)\phi(\pi(z))d\zeta^x(z).$$

Especially, for ξ and n in the lemma,

$$\int_{Z_L} \xi(z)z_p^{n-1}d\zeta_\alpha^b = \int_{Z_{CL}} \alpha(z)\xi(z)z_p^{n-1}d\zeta^x = (1-\xi_1\alpha_1(b)b^n)L_{CLp}(1-n,\xi\alpha)$$

Here note that the conductor of $\xi\alpha$ is divisible by all prime factors of C outside L because α is primitive of conductor Cp^r. Hence we have

$$L_{CLp}(1-n,\xi\alpha) = L_{Lp}(1-n,\xi,\alpha),$$

which proves the desired assertion.

Hereafter, we fix $b > 1$ prime to Lp and define an $\mathcal{O}$-linear map $E : \mathcal{C}(Z_L;\mathcal{O}) \to \mathcal{O}[[q]]$ by

$$\text{(3.3a)} \quad \int_{Z_L} \phi dE = \sum_{\substack{n=1,\\(p,n)=1}}^{\infty} q^n \sum_{\substack{u^2t^2|n,\\(ut,Lp)=1,\\u>0,\\t>0}} \mu(u)\chi_n(u)t \int_{Z_L} (\phi|ut^2)d\zeta_{\chi_n}^b.$$

We now compute the value of E when $\phi(z) = \xi(z)z_p^{(k-3)/2}$ for an odd integer $k \geq 3$ and a character $\xi : (\mathbf{Z}/Lp^r\mathbf{Z})^\times \to \bar{\mathbf{Q}}^\times$ with $\xi(-1) = (-1)^{(k-1)/2}$: By definition, we have, for $\alpha = \chi_n$

$$\int_{Z_L} (\phi|ut^2)d\zeta_\alpha^b = \xi(ut^2)(ut^2)^{(k-3)/2}\int_{Z_L}\xi(z)z_p^{(k-3)/2}d\zeta_\alpha^b$$

$$= \xi(ut^2)(ut^2)^{(k-3)/2}(1-\xi_1(b)b^{(k-1)/2})L_{Lp}((3-k)2, \xi\chi_n),$$

because the restriction to $\mathbf{Z}_p^\times$ of ξ_n is trivial since $p \geq 5$ and n is prime to p. Since $\chi_n(-1) = 1$, $\int_{Z_L}\xi(z)z_p^{(k-3)/2}d\zeta_{\xi_n}^b = 0$ if $\xi(-1) = (-1)^{(k-1)/2}$. Thus we have

$$\text{(3.3b)} \quad \int_{Z_L}\xi(z)z_p^{(k-3)/2}dE = \begin{cases} (1-\xi_1(b)b^{(k-1)/2})E_{k/2}(\xi)|\iota_p & \text{if } \xi(-1) = (-1)^{(k-1)/2} \\ 0 & \text{if } \xi(-1) = (-1)^{(k+1)/2}. \end{cases}$$

By this fact, for any $\phi \in \mathcal{LC}(Z_L;\mathcal{O})$,

$$\int_{Z_L}\phi(z)z_p^n dE \in \mathcal{G}_{n+(3/2)}(\Gamma_1(Lp^\infty);\mathcal{O})$$

because any locally constant ϕ is a linear combination of characters $\xi \in \mathcal{LC}(Z_L;\bar{\mathbf{Q}})$. Since $\mathcal{LC}(Z_L;\mathcal{O})$ is dense in $\mathcal{C}(Z_L;\mathcal{O})$, E can be extended on $\mathcal{C}(Z_L;\mathcal{O})$ by continuity to an $\mathcal{O}$-linear map with values in $\overline{\mathcal{G}}(L;\mathcal{O})$). Then we can easily check the properties E1 to E3 for this E.

By using the differential operator in (2.12), we can extend E to a two variable measure $\mathcal{E} : \mathcal{C}(\mathbf{Z}_p^\times \times Z_L;\mathcal{O}) \to \overline{\mathcal{G}}(L;\mathcal{O})$ as follows:

$$\text{(3.4)} \quad \int_{\mathbf{Z}_p^\times \times Z_L} w^m\eta(w)z_p^n\xi(z)d\mathcal{E}(w,z) = d^m\left(\int_{Z_L}\xi(z)z_p^n dE(z)|\eta)\right),$$

where $\eta \in \mathcal{C}(\mathbf{Z}_p^\times;\mathcal{O})$, $0 \leq m, n \in \mathbf{Z}$ and d is the differential operator in (2.12).

§4. Arithmetic measures of half integral weight

In [**H1**, II.5], we have defined arithmetic measures with values in $\overline{\mathcal{M}}(J;\mathcal{O})$ for an integer J prime to p. Here we extend the definition to those with values in $\overline{\mathcal{G}}(J;\mathcal{O})$. Let T be a p-adic space; hence, T is a product of $\mathbf{Z}_p^r$ and a finite set T_0. An $\mathcal{O}$-linear map $\mu : \mathcal{C}(T;\mathcal{O}) \to \overline{\mathcal{G}}(J;\mathcal{O})$ is called arithmetic if μ satisfies the following three conditions:

A1. *There exists a positive (odd) integer k such that*

$$\int_T \phi d\mu \in \mathcal{G}_{k/2}(\Gamma_1(Jp^\infty);\bar{\mathbf{Q}})$$

for all $\phi \in \mathcal{LC}(T;\bar{\mathbf{Q}})$ (This half integer $k/2$ will be called the weight of μ),

A2. *There exists a finite order character $\varphi : Z_J \to \mathcal{O}^\times$ and a continuous action:$Z_J \times T \to T$ such that*

$$\mu(\phi)|z = z_p^{(k-1)/2}\varphi(z)\mu(\phi|z) \text{ for } \phi \in \mathcal{C}(T;\mathcal{O}),$$

where $\phi|z(t) = \phi(zt)$ for $t \in T$ and $z \in Z_J$,

A3. *There exists a continuous function $\nu : T \to \mathbf{Z}_p$ such that*

$$(\nu|z)(t) = z_p^2\nu(t) \text{ for } z \in Z_J \text{ and } d(\mu(\phi)) = \mu(\nu\cdot\phi)$$

for the differential operator d in (2.12).

We say that μ is *super-singular* if $\iota_p \circ \mu = \mu$ (i.e. $a(n,\mu(\phi)) = 0$ for all ϕ if p divides n). We say that μ is *cuspidal* if μ has values in $\overline{\mathcal{P}}(J;\mathcal{O})$.

Lemma 4.1. *Suppose μ to be arithmetic of weight $k/2$. Then μ is cuspidal if μ is super singular.*

Proof: By [**Sh1**, 1.5], we can define a Hecke operator $T(p) : \overline{\mathcal{G}}(J;\mathcal{O}) \to \overline{\mathcal{G}}(J;\mathcal{O})$ by $a(n, f|T(p)) = a(np, f)$ for all $n \geq 0$. Thus μ is super singular if and only if $\mu(\phi)|T(p) = 0$ for all $\phi \in \mathcal{C}(T;\mathcal{O})$. If $\phi \in \mathcal{LC}(T;\mathcal{O})$, then $\mu(\phi) \in \mathcal{G}_{k/2}(\Gamma_1(Jp^r);\mathcal{O})$ for some r. Since $\mu(\phi)|T(p)^m|\gamma = \mu(\phi)|\gamma|T(p)^m$ for sufficiently large m if $\gamma \in \Gamma_0(p^r)$ (see the proof of [**H1**, II.2.2]), we have

$$a(0, \mu(\phi)|\gamma) = 0 \text{ if } \gamma \in \Gamma_0(p^r).$$

We can find for each $n > 0$, a modular form $G'_r \in \mathcal{M}_r(\Gamma_0(p^r);\mathcal{O})$ such that $r = p^{n-1}(p-1)$, $G'_r \equiv 1 \pmod{p^n}$ and G'_r vanishes at all ramified cusps for $\Gamma_1(Jp^r)$. In fact, it is easy to check these properties for $G'_r = G_r|[p^{r-1}]$, where G_r is the modular form in (2.7). Then $G'_r\mu(\phi) \in \mathcal{P}_{r+(k/2)}(\Gamma_1(Jp^r);\mathcal{O})$ and $G'_r\mu(\phi) \equiv \mu(\phi) \pmod{p^n}$. Namely $\mu(\phi) = \lim_{n\to\infty} G'_r\mu(\phi) \in \overline{\mathcal{P}}(J;\mathcal{O})$ for all $\phi \in \mathcal{LC}(T;\mathcal{O})$. Then by the continuity of μ and the density of $\mathcal{LC}(T;\mathcal{O})$ in $\mathcal{C}(T;\mathcal{O})$, μ has values in $\overline{\mathcal{P}}(J;\mathcal{O})$.

We now define the Rankin product zeta function between a modular form of half integral weight and another of integral weight. Let

$$f = \sum_{n=1}^{\infty} a(n)q^n \in \mathcal{S}_k(\Gamma_0(Np^\beta),\psi)$$

and

$$g = \sum_{n=0}^{\infty} b(n)q^n \in \mathcal{G}_{l/2}(\Gamma_0(Jp^\beta),\xi).$$

Then we define according to [**Sh2**] and [**St1**, Addendum, p.782],

$$\mathcal{D}_{Lp}(s,f,g) = L_{Lp}(2s-2k-l+3,(\psi\xi)^2)\sum_{n=1}^{\infty} a(n)b(n)n^{-s/2},$$

where L is the least common multiple of N and J.

To state an interpolation theorem for $\mathcal{D}_{Lp}(m,f_p,\mu(\phi)|\tau)$, we recall some symbols introduced in [**H1**, II, §5]: Let f be a normalized common eigen form of weight k of all Hecke operators and f_0 be the primitive form associated to f. Let $0 < C \in \mathbf{Z}$ be the conductor of f. Then the root number $W(f) \in \mathbf{C}$ with $|W(f)| = 1$ is defined by

$$f_0|_k\tau(C) = W(f)f_0^\rho, \tag{4.1a}$$

where $\tau(C)$ is as in (2.10a) and ρ denotes complex conjugation. Then $W(f)$ is an algebraic number and can be decomposed as in [**H1**, II.5] into the product $W(f) = W_p(f)W'(f)$ of the p-part $W_p(f)$ and the prime-to-p part $W'(f)$. Similarly, for any Dirichlet character ξ, writing ξ_0 for the associated primitive character of conductor $C = C(\xi)$, we define the Gauss sum by

$$G(\xi) = \sum_{r=1}^{C-1} \xi(r)\exp(2\pi ir/C). \tag{4.1b}$$

Let $\mathbf{I}$ be the integral closure of Λ in a finite extension $\mathbf{K}$ of the quotient field $\mathbf{L}$ of Λ. Let $\lambda : \mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \to \mathbf{I}$ be a primitive Λ-algebra homomorphism and $f_P \in \mathcal{S}_k(\Gamma_0(Np^r), \psi'\psi_P)$ be the classical cusp form belonging to λ at $P \in \mathcal{A}(\mathbf{I})$ as defined in the introduction. Here $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ is the character of λ as in the introduction, ψ_1 (resp. ψ') is the restriction of ψ to $(\mathbf{Z}/p\mathbf{Z})^\times$ (resp. $(\mathbf{Z}/N\mathbf{Z})^\times$) and $\psi_P = \varepsilon_P\psi_1\omega^{-k}$. We recall here the definition of the congruence module $C_0(\lambda;\mathbf{I})$ of λ given in [**H5**, §§0 and 6]: Since λ is primitive, λ induces an algebra decomposition: $\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathbf{K} = \mathbf{K} \oplus \mathbf{B}$. Let $\mathbf{h}(\mathbf{K})$ (resp. $\mathbf{h}(\mathbf{B})$) be the projection of $\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathbf{I}$ to $\mathbf{K}$ (resp. $\mathbf{B}$). Then we define

$$C_0(\lambda;\mathbf{I}) = \mathbf{h}(\mathbf{K}) \oplus \mathbf{h}(\mathbf{B})/\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathbf{I},$$

which is a torsion $\mathbf{I}$-module of finite type. Let $\mu : \mathcal{C}(T;\mathcal{O}) \to \overline{\mathcal{G}}(J;\mathcal{O})$ be a supersingular arithmetic measure of weight $l/2$ and with character $\psi : Z_J \to \mathcal{O}^\times$. We denote by L the least common multiple of J and N.

Now we state

THEOREM 4.2. *Let $b \in \mathbf{Z}_p^\times$. Fix an element $0 \neq H \in \mathbf{I}$ which annihilates the congruence module $C_0(\lambda;\mathbf{I})$ of λ. Then there exists a unique generalized measure $\Phi \in \mathrm{Meas}(T;\mathcal{O})\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in the sense of [**H1**, II.3], which is characterized by the following interpolation property: For each pair (P,m) ($P \in \mathcal{A}(\mathbf{I})$ and $m \in \mathbf{Z}$) with $0 \le 2m < k(P)-(l+1)/2$ (and $H(P) \neq 0$), if $\phi \in \mathcal{LC}(T;\mathcal{O})$ satisfies $\phi(zt) = \xi(z)\phi(t)$ for a finite order character $\xi : \mathbf{Z}_J \to \mathcal{O}^\times$, then*

$$(1 - \psi_P\xi_1^{-1}\varphi_1^{-1}(b)b^{k(P)-(l+1)/2-2m})^{-1}S(P)H(P)^{-1}\int_T \phi(t)\nu(t)^m d\Phi_P$$
$$= c(l,P,m)W'(f_P)^{-1}G(\psi_P)^{-1}a(p,f_P)^{-\beta}p^{\beta(2m+(l/2))/2} \times\frac{\mathcal{D}_{Lp}(l+2m, f_P, \mu(\phi)|\tau(Jp^\beta))}{(2\pi\mathrm{i})^{2m-1+((l-1)/2)}\Omega(P)},$$

$$c(l,P,m) = J^{m+(l/4)}N^{-k(P)/2} \left(\Gamma(m+1)\Gamma(m+(l/2))/\sqrt{2\pi}\right)\left(\sqrt{-1}\right)^{(l-1)/2},$$

where Φ_P is the image of Φ under the natural map

$$\mathrm{id}\otimes P : \mathrm{Meas}(T;\mathcal{O})\hat{\otimes}_{\mathcal{O}}\mathbf{I} \to \mathrm{Meas}(T;\mathcal{O}')$$

for $\mathcal{O}' = P(\mathrm{I})$, $\Omega(P) = (2\mathrm{i})^{k(P)+1}\pi^2\langle f_P^\circ, f_P^\circ\rangle$, $S(P)$ *is the Euler factor given in (0.2b), and* β *is the positive integer such that* $\mu(\phi) \in \mathcal{G}_{1/2}(\Gamma_0(Jp^\beta), \xi\varphi)$.

Before proving the theorem, we shall give some examples of arithmetic measures and also several lemmas concerning the holomorphic projection of nearly holomorphic modular forms. As a first example, we can offer the Eisenstein measure

$$\mathcal{E} : \mathcal{C}(\mathbf{Z}_p^\times \times Z_L; \mathcal{O}) \to \overline{\mathcal{G}}(L; \mathcal{O})$$

defined in the previous section. Especially, by E1-3 and (3.4), we have

(4.2) $\mathcal{E}$ *is supersingular and arithmetic of weight 3/2 and with the trivial character. The function* $\nu : \mathbf{Z}_p^\times \times Z_L \to \mathbf{Z}_p$ *in A3 is given by* $\nu(w, z) = w$ *and the action of* Z_L *on* $\mathbf{Z}_p^\times \times Z_L$ *is given by*

$$z(w, z') = (z_p^2 w, zz')$$

for $z \in Z_L$.

Next we shall construct arithmetic measures by using theta series: Let V be a $\mathbf{Q}$-vector space of odd dimension l and $\nu : V \to \mathbf{Q}$ be a positive definite quadratic form. The corresponding $\mathbf{Q}$-bilinear form $S : V \times V \to \mathbf{Q}$ is given by

$$S(x, y) = \nu(x + y) - \nu(x) - \nu(y).$$

Choose a lattice L in V such that $\mathbf{Z} \supset \nu(L)$, and put

$$L^* = \{x \in V | \mathbf{Z} \supset S(x, L)\}$$

Then $L^* \supset L$. We write Δ for $[L^* : L]$ and put

$$\mathcal{W} = \{x \in L^* | \nu(x) \in \mathbf{Z}\}.$$

Let M be the smallest positive integer such that $\mathbf{Z} \supset M\nu(L^*)$. It is known that $4|M$. Then for any function $\phi : \mathcal{W}/p^r L \to \bar{\mathbf{Q}}$, it is classically known that

$$\theta(\phi) = \frac{1}{2} \sum_{w \in \mathcal{W}} \phi(w) q^{\nu(w)} \in \mathcal{G}_{l/2}(\Gamma_1(M); \bar{\mathbf{Q}}) \tag{4.3}$$

(see [**Sh1**, §2]). Thus if we define $W = \varprojlim_r \mathcal{W}/p^r L$, then W is a p-adic space and the measure $\theta : \mathcal{C}(W;\mathcal{O}) \to \mathcal{O}[[q]]$ given by the formula (4.3) has values in $\overline{\mathcal{G}}(M;\mathcal{O})$. The function $\nu : \mathcal{W} \to \mathbf{Z}$ extends to $\nu : W \to \mathbf{Z}_p$ by continuity. Note that the natural multiplication: $\mathbf{Z} \times \mathcal{W} \to \mathcal{W}$ gives by continuity the action of Z_M on W. We put

$$W^\times = \{x \in W | \nu(x) \in \mathbf{Z}_p^\times\}.$$

Then we can easily verify by [**Sh1**, Prop.2.1] that

(4.4a) *the restriction of θ to $W^\times$ is supersingular and arithmetic of weight $l/2$. The character $\varphi : (\mathbf{Z}/M\mathbf{Z})^\times \to \{\pm 1\}$ of θ is given by* $\varphi(d) = \left(\dfrac{(-1)^{(l-1)/2}2\Delta}{d}\right)$.

Moreover, if $\eta : V \to \bar{\mathbf{Q}}$ is a spherical function of order α in the sense of [**Sh1**, §2] (see also [**H1**, I.1]), then η induces by continuity a polynomial function $\eta : W \to \bar{\mathbf{Q}}_p$. Then we have

(4.4b) $\eta \cdot \theta : \mathcal{C}(W^\times;\mathcal{O}) \to \overline{\mathcal{G}}(M;\mathcal{O})$ *given by* $\eta \cdot \theta(\phi) = \theta(\eta\phi)$ *is supersingular and arithmetic of weight* $\alpha + (l/2)$ *and with character* ϕ *as in (4.4a).*

When $V = \mathbf{Q}$ (i.e. $l = 1$), $\nu(x) = x^2$ and $L = J\mathbf{Z}$ with an integer J prime to p, we see easily

(4.4c) $\qquad \varphi = \text{ id and } M = 4J^2 \text{ and } W^\times = Z_J.$

We record here the speci al case of Theorem 4.2 when $\mu = \theta$.

COROLLARY 4.3. *When $\mu = \theta$ as above in Theorem 4.1, then we have the following evaluation formula: if $0 \le 2m < k(P) - \alpha - (l+1)/2$ and if $\eta : W \to \mathcal{O}$ is a spherical function of order α, then*

$$\begin{aligned}(1 - \psi_P \xi_1^{-1}\varphi_1^{-1}(b) b^{k(P)-\alpha-((l+1)/2)-2m})^{-1} & \\ S(P)H(P)^{-1} \int_{W^\times} \eta\phi\nu^m d\Phi_P & \\ = c(l+2\alpha, P, m) W'(f_P)^{-1} G(\psi_P)^{-1} a(p, f_P)^{-\beta} p^{\beta((l/2)+\alpha+2m)/2} & \\ \times \frac{\mathcal{D}_{Lp}(l+2\alpha+2m, f_P, \theta(\phi\eta)|_{\alpha+(l/2)}\tau(Mp^\beta))}{(2\pi i)^{\alpha+2m+((l-1)/2)-1}\Omega(P)},\end{aligned}$$

where

$$c(j,P,m) = M^{m+(j/4)}N^{-k(P)/2}\left(\Gamma(m+1)\Gamma(m+(j/2))/\sqrt{2\pi}\right)\left(\sqrt{-1}\right)^{(j-1)/2}$$

Here, if $l=1$, then

(4.5)
$$\Gamma(m+1)\Gamma\left(m+\alpha+\frac{1}{2}\right)/\sqrt{\pi} = \begin{cases} 2^{-2m}(2m)! & \text{if } \alpha=0 \\ 2^{-2m-1}(2m+1)! & \text{if } \alpha=1. \end{cases}$$

Now we recall some result from [**H1**, II.6]. Let $\mathcal{N}^m(A)$ $(0<m\in\mathbf{Z})$ for any subring A of $\mathbf{C}$ be the space of functions on $\mathcal{H}$ with Fourier expansion of the following form:

$$f=\sum_{n=1}^{\infty} a(n,y)e\left(\frac{n(z+h)}{M}\right) \text{ for some } 0<M\in\mathbf{Z} \text{ and } 0\le h<1,$$

where $a(n,y)$ is a polynomial in $(4\pi y)^{-1}$ with coefficients in A of degree less than or equal to m; thus, f is nearly holomorphic (and A-integral) in the sense of Shimura. Let Δ be a congruence subgroup of $SL_2(\mathbf{Z})$. For an integer or a half integer $0<k\in\frac{1}{2}\mathbf{Z}$, we write $\mathcal{N}_k^m(\Delta;A)$ (resp. $\mathcal{N}_k^m(\Delta,\psi;A)$ for a finite order character $\psi:\Delta\to A^\times$) for a subspace of $\mathcal{N}_m(A)$ consisting of functions $F:\mathcal{H}\to\mathbf{C}$ such that

(i) $f(\gamma(z))\varphi(z)^{-2}\in\mathcal{N}_k^m(\Delta;\mathbf{C})$ *(resp.* $\mathcal{N}_k^m(\mathrm{Ker}(\psi);\mathbf{C})$*) for all* $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in SL_2(\mathbf{Z})$ *and for all holomorphic functions* $\varphi:\mathcal{H}\to\mathbf{C}$ *with* $\varphi^2(z)=cz+d$,

(ii) $f|_k\gamma=f$ *(resp.* $f|_k\gamma=\psi(\gamma)f$*) for all* $\gamma\in\Delta$.

When $2m<k\in\mathbf{Z}$, we define the holomorphic projection according to Shimura [**Sh5**, Lemma 7]

$$H:\mathcal{N}_k^m(\Delta;A)\to\mathcal{M}_k(\Delta;A)$$

as in [**H1**,II, §6]. Then we have

LEMMA 4.4. *(i) (Shimura). If $f \in \mathcal{S}_k(\Gamma_0(N), \psi; \mathbf{C})$ with $k \in \mathbf{Z}$ (for a character $\psi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$), then $\langle f, g\rangle_N = \langle f, H(g)\rangle_N$ for all $g \in \mathcal{N}_k^m(\Gamma_0(N), \psi; \mathbf{C})$ with $2m < k \in \mathbf{Z}$. Moreover, H sends $\mathcal{N}_k^m(\Gamma_0(N), \psi; \bar{\mathbf{Q}})$ to $\mathcal{M}_k(\Gamma_0(N), \psi; \bar{\mathbf{Q}})$ if $k > 2m$.*
(ii) If $h \in \mathcal{G}_{k/2}(\Gamma_1(N); \mathbf{C})$ and $g \in \mathcal{G}_{l/2}(\Gamma_1(N); \mathbf{C})$ then

$$H(g\delta^r_{k/2}h) = (-1)^r H(h\delta^r_{l/2}g).$$

(iii) If $g \in \mathcal{G}_{l/2}(\Gamma_1(Np^\beta); \bar{\mathbf{Q}})$ and $h \in \mathcal{G}_{k/2}(\Gamma_1(Np^\beta); \bar{\mathbf{Q}})$, then

$$e(H(g\delta^r_{k/2}h)) = e(gd^r h),$$

where d is the differential operator in (2.12).

We omit the proof since we can prove this in exactly the same way as the proof of [**H1**, II, Lemma 6.5]. Here we quote a lemma from [**St1**, Addendum, p.782]:

LEMMA 4.5. *Suppose $f = \sum_{n=1}^\infty a(n)q^n \in \mathcal{S}_k(\Gamma_0(Lp^\beta), \psi)$ and $g = \sum_{n=0}^\infty b(n)q^n \in \mathcal{G}_{l/2}(\Gamma_0(Lp^\beta), \xi)$ for $k > l/2$. Then, we have*

$$\begin{aligned}
&(4\pi)^{-s/2}\Gamma(s/2)\mathcal{D}_{Lp}(s, f, g)\\
&\quad= \langle f^\rho, gE^*_{k-(l/2)}(z, s+2-2k; \xi\psi\chi_{-Lp^\beta})y^{(s/2)+1-k}\rangle_{Lp^\beta}\\
&\quad= \left(-\sqrt{-1}\right)^k \langle f^\rho|_k\tau, g|_{l/2}\tau\\
&\qquad\quad \left(E^*_{k-(l/2)}(z, s+2-2k; \xi\psi\chi_{-Lp^\beta})y^{(s/2)+1-k}\right)|_{k-(l/2)}\tau\rangle_{Lp^\beta},
\end{aligned}$$

where $\tau = \tau(Lp^\beta)$ as in (2.10a).

The first equality in the lemma follows from [**St1**] and the second is a consequence of the following formulae:

$$\langle f|_k\tau, h|_k\tau\rangle = \langle f, h\rangle \text{ and } (g|_{l/2}\tau)(E|_{k-(l/2)}\tau) = \left(\sqrt{-1}\right)^k (gE)|_k\tau$$

for g as above and $E \in \mathcal{G}_{k-(l/2)}(\Gamma_1(Lp^\beta))$.

PROOF OF THEOREM 4.2: We shall apply the result in [**H1**, II.8] in the following situation: With the notation introduced there, we take $\overline{\mathcal{G}}(L; \mathcal{O})$ as U^* and $\overline{\mathcal{P}}(L; \mathcal{O})$ as V^*, where L is the least common multiple of N and J. We let Z_L act on $M^* = \mathcal{C}(T; \mathcal{O})$ by

$$\phi\|z(t) = z_p^{(l+1)/2}\phi''(z)\phi(z_J t), \tag{4.6a}$$

where $\phi''(m) = \phi(m)\left(\frac{-L/J}{m}\right)$ and $z_J \in Z_J$ is the natural projection of $z \in Z_L$ to Z_J. Then $\mu^L = [L/J] \circ \mu : \mathcal{C}(R;\mathcal{O}) \to \overline{\mathcal{P}}(L;\mathcal{O})$ is compatible with the action of Z_L; i.e., we have

$$\text{(4.6b)} \qquad \mu^L(\phi)\|z = \mu^L(\phi\|z) \textit{ for } z \in Z_L \textit{ and } \phi \in \mathcal{C}(T;\mathcal{O}).$$

We take $\mu^L : M^* = \mathcal{C}(T;\mathcal{O}) \to \overline{\mathcal{P}}(L;\mathcal{O}) = V^*$ as $\varphi : M^* \to V^*$ in [**H1**, II.8]. We let Z_L act on $\mathcal{C}(Z_L;\mathcal{O})$ by $\phi\|z(z') = \phi(zz')$. Then for $b \in \mathbf{Z}_p^\times$ as in the theorem, we take the Eisenstein measure

$$E : \mathcal{C}(Z_L;\mathcal{O}) \to \overline{\mathcal{G}}(L;\mathcal{O}) \text{ as in §3.}$$

We take this measure as E in [**H1**, II.8].

We now show that $\Phi = L^{-1}(E *_\lambda \mu^L)$ defined in [H1, II, (8.5)] satisfies the requirement of the theorem. By [**H1**, II, Th.8.5], we have

$$\text{(4.7)} \quad H(P)^{-1}L\int_T \phi\nu^m d\Phi_P$$
$$= \ell_P \circ T_{L/N} \circ e\Big(\int_T\int_{Z_L} \varepsilon\psi\omega^{-k}(z)z_p^{k-1}(\phi\nu^m)\|z^{-1}(t)dE(z)d\mu^L(t)\Big),$$

where $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}\times$ is the character of λ as in [**H1**, II, §4], $k = k(P)$, $\varepsilon = \varepsilon_P$ and $\ell_P : \mathcal{S}_k^{\mathrm{ord}}(\Gamma_0(Np^r), \varepsilon\psi\omega^{-k};\mathcal{O}) \to \mathcal{O}$ is the linear form defined in [H1, II, (7.6)]. Similarly to the computation done in [H1, II, (9.3)], we know from (4.6a), (4.7) is equal to

$$\text{(4.8)} \qquad \ell_P \circ T_{L/N} \circ e\left(\left(\int_T \phi\nu^m d\mu^L \cdot \int_{Z_L} \eta(z)z_p^{(2k-j-3)/2}dE(z)\right)\right),$$

where $j = L + 4m$ and $\eta = \varepsilon\psi\chi_{-L/J}(\varphi\xi\omega^k)^{-1}$. By h3 and h4 and A3, we have

$$d(\mu^L(\phi)) = (L/J)\mu^L(\phi\nu).$$

As seen in (2.13), we have

$$e(hdf) = -e(fdh) \text{ and } e(h(f|\iota_p)) = e((h|\iota_p)f),$$

where $\iota_p : \mathbf{Z}/p\mathbf{Z} \to \{1, 0\}$ such that $\iota_p(z) = 1$ if and only if $z \in (\mathbf{Z}/p\mathbf{Z})^\times$. Applying these formulae to (4.8) in order and keeping in mind the fact: $\mu(\phi)|\iota_p = \mu(\phi)$, we know that (4.8) is equal to

(4.9)
$$(-L/J)^{-m}\ell_P \circ T_{L/N} \circ e\left(\mu^L(\phi) \cdot d^m\left(\int_{Z_L} \eta(z)z_p^{(2k-j-3)/2}dE\right)\right)$$

Then, by applying (3.3b), Lemma 4.4 (iii) and (3.2c) in order, under the assumption that $0 \le 2m < (2k-l-1)/2$ we reduce (4.9) to

$$(4.10) \quad (-J/L)^m \Big(1 - \eta_1(b) b^{(2k-j-1)/2}\Big) \ell_P \circ T_{L/N} \circ e \left(H \left(\mu^L(\phi) \delta^m_{(2k-j)/2} E_{(2k-j)/2}(\eta) \right) \right).$$

We write g for $H\left(\mu^L(\phi)\delta^m_{(2k-j)/2}E_{(2k-j)/2}(\eta)\right)$ and assume the level of g is Lp^β for $\beta > 0$. We can compute ℓ_P by [H1, II, (7.6)] as

$$(4.11) \quad H(P)^{-1} L(1 - \eta_1(b) b(2k-j-1)/2)^{-1} \int_T \phi \nu^m d\Phi_P$$
$$= (-J/L)^m a(p, f_p)^{r-\beta} p^{(\beta-r)(k-1)} \langle h_P \,|[p^{\beta-r}], g|\, T_{L/N} \rangle_{Np^\beta} / \langle h_P, f_P \rangle_{Np^r},$$

where r is the exponent of p in the conductor of ψ_P and $h_P = f_P^\rho | \tau(Np^r)$. Moreover, according to the computation done between (9.4) and (9.5) in [**H1**, II], we have

$$\langle h_P | [p^{\beta-r}], g | T_{L/N} \rangle_{Np^\beta}$$
$$= (L/N)^{k/2} p^{k(r-\beta)/2} \langle f_P^\rho | \tau(Lp^\beta), \mu^L(\phi) \delta^m_{k-(j/2)} E_{k-(j/2)}(\eta) \rangle_{Lp^\beta}$$

and

$$\mu^L(\phi) = (L/J)^{-l/4} (\mu(\phi)|_{l/2} \tau(Jp^\beta)|_{l/2} \tau(Lp^\beta)).$$

Applying these formulae and (3.2c), we know from Lemma 4.5 that if $0 \le 2m < k - ((l+1)/2)$,

$$(4.12) \quad \langle f^\rho | t(Lp^\beta), \mu^L(\phi) \delta^m_{k-(j/2)} E_{k-(j/2)}(\eta) \rangle_{Lp^\beta}$$
$$= \left(\sqrt{-1}\right)^k 2^{-(j/2)-k} \pi^{-(j/2)-1} \Gamma(m+1)$$
$$\Gamma(m + (l/2)) (Lp^\beta)^{(j/4)-(k/2)+1} (L/J)^{-l/4}$$
$$\times \mathcal{D}_{Lp}(l + 2m, f, \mu(\phi)|_{l/2} \tau(Jp^\beta)).$$

On the other hand, by [H1, II, (5.5b), (9.5)], we have, for S(P) in (0.2b)

$$(4.13) \quad \begin{aligned} &\langle h_P, f_P \rangle_{Np^r} / \langle f_P^\circ, f_P^\circ \rangle_{Np^r} \\ &\quad = (-1)^k W'(f_P) a(p, (f_p)^\rho)^{-r} S(P) p^{kr/2} G(\psi_P) \\ &\quad\quad \text{if } f_P = f_P^\circ \text{ and } \psi_P \neq \mathrm{id}, \\ &\langle h_P, f_P \rangle_{Np^r} / \langle f_P^\circ, f_P^\circ \rangle_{Np^r} \\ &\quad = (-1)^k W'(f_P) p^{(2-k)/2} a(p, f_P) S(P) \\ &\quad\quad \text{if either } f_P = f_P^\circ \text{ and } \psi_P = \mathrm{id} \text{ or } f_P \neq f_P^\circ. \end{aligned}$$

We then apply (4.13) and (4.12) to (4.10) and obtain the desired formula in the theorem.

§5. A special case of a quadratic form of one variable

We fix a Λ-algebra homomorphism $\lambda : \mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \to \mathbf{I}$ with character $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathbf{C}^\times$ as in Theorem 4.2, to which we can attach the congruence module $C_0(\lambda;\mathbf{I})$. We suppose that $\mathbf{I}$ is defined over $\mathcal{O}$ (i.e. the integral closure of $\mathcal{O}$ in $\mathbf{I}$ coincides with $\mathcal{O}$). Then $\Lambda \otimes_{\mathcal{O}} \mathbf{I}$ is still a local ring having the unique maximal ideal $\mathbf{m}$. We consider the profinite completion $\Lambda \otimes_{\mathcal{O}} \mathbf{I}$ of $\Lambda \otimes_{\mathcal{O}} \mathbf{I}$, which is the completion of $\Lambda \otimes_{\mathcal{O}} \mathbf{I}$ under the $\mathbf{m}$-adic topology. Thus $\Lambda \hat{\otimes}_{\mathcal{O}} \Lambda$ is a subring of $\Lambda \hat{\otimes}_{\mathcal{O}} \mathbf{I}$. We fix a topological generator u of $1 + p\mathbf{Z}_p$ and identify Λ with $\mathcal{O}[[X]]$ via $u \mapsto 1 + X$. Thus we can identify $\Lambda \hat{\otimes}_{\mathcal{O}} \Lambda$ with $\mathcal{O}[[X,Y]]$. For each $z \in \mathbf{Z}_p^\times$, we define a power series $A_z(X) \in \mathbf{Z}_p[[X]]$ by

$$A_z(X) = (1+X)^{\log(\langle z \rangle)/\log(u)},$$

where $(1+X)^s = \sum_{n=0}^{\infty} \binom{s}{n} X^n$ for $s \in \mathbf{Z}_p$ and $\log(1+X) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n$ is the p-adic logarithm map. We now define (possibly imprimitive) L-function which can be given by the Shimura integral: We put

$$\mathcal{L}(s, f_p^\circ, \chi) = \prod_l D_l(\chi(l) l^{-s})^{-1}$$

where l runs all primes and $D_l(X)$ is the cubic polynomial defined in (0.1). We prove in this section the following weaker version of Theorem I:

Theorem 5.1. *Let the notation be as in Theorem 4.2. Let $\xi : (\mathbf{Z}/Jp\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character for a positive integer J prime to p. Suppose that $\xi(-1) = -1$ and that the conductor of ξ is divisible by J. Then there exists an element $\mathcal{L}$ in the quotient field of $\Lambda \hat{\otimes}_{\mathcal{O}} \mathbf{I}$ such that:*

(i) Define $D \in \Lambda \hat{\otimes}_{\mathcal{O}} \mathbf{I}$ as follows:

$D = (X - Y)(A_2(X)\langle 2 \rangle - A_2(Y))$ if $\psi_1 = \xi_1 \omega$, and $\psi \xi^{-1} \omega^{-2}(2) = \pm 1$ and N is odd,

$D = (A_2(X)\langle 2 \rangle - A_2(Y))$ if $\psi \xi^{-1} \omega^{-2}(2) = \pm 1$ and N is odd but $\psi_1 \neq \xi_1 \omega$,

$D = (X - Y)$ if $\psi_1 = \xi_1\omega$ but either $\psi\xi^{-1}\omega^{-2}(2) \neq \pm 1$ or N is even and $D = 1$, otherwise.

Then we have, for any element $H \in \mathbf{I}$ in the annihilator of $C_0(\lambda; \mathbf{I})$, $DH\mathcal{L} \in \Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$;

(ii) if $1 \leq n < k(P)$ ($n \in \mathbf{Z}$) for $P \in \mathcal{A}(\mathbf{I})$ and if either $n \neq k(P) - 1$ or $\psi^2(2) \neq \xi^2\omega^4(2)$, then

$$\mathcal{L}(P_{n,\varepsilon}, P) = c(P_{n,\varepsilon}, P)S(P)^{-1}E(n, \psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^{n-1}) \frac{\mathcal{L}(n, f_p^\circ, \psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^{n-1})}{(2\pi \mathrm{i})^{n-2}\Omega(P)},$$

where, writing $C(\varepsilon\xi\omega^{-n+1}) = Jp^\delta$ for J prime to p,

$$c(P_{n,\varepsilon}, P) = (n-1)!\left(C(\epsilon\xi\omega^{-n+1})\right)^{n-1} G(\varepsilon\xi\omega^{-n+1})N^{-k(P)/2}W'(f_P)^{-1}G(\psi_P)^{-1}\psi'(p)^\delta$$

and $E(n,\eta) = E_1(n,\eta)E_2(n,\eta)$ with

$$E_1(n,\eta) = (1 - \eta^{-1}\psi'\psi_P(p)a(p, f_P)^{-2}p^{n-1})(\psi'(p)^{-1}a(p, f_p)^{-2})^\delta,$$

$$E_2(n,\eta) = (1 - \eta(p)p^{k-n-1})(1 - \eta\psi'\psi_P(p)a(p, f_P)^{-2}p^{2k-n-2}) \text{ if } f_P \neq f_P^\circ,$$

$$\text{and } E_2(n,\eta) = 1 \text{ if } f_P = f_P^\circ.$$

The idea to prove this theorem is to specialize Corollary 4.3 to the case where $l = 1$. Before starting the proof, we review some formulae for theta series. Let η be a Dirichlet character of $(\mathbf{Z}/Jp^{\delta'}\mathbf{Z})^\times$ with values in $\mathcal{O}$ for a positive integer J prime to p. Write Cp^δ for the conductor of η for C prime to p. We consider the quadratic form $\nu : \mathbf{Q} \to \mathbf{Q}$ given by $\nu(x) = x^2$ and the lattice $L = J\mathbf{Z}$. Then by (4.4a), the character of the corresponding θ-measure is trivial, the level M of L is equal to $4J^2$, and $W^\times = Z_J$. Let $\eta_0 : \mathbf{Z} \to \mathcal{O}$ be the primitive Dirichlet character associated with η, and we put

$$\theta_J(\eta) = \int_{W^\times} \eta(w)w_p^\alpha d\theta \text{ for } \alpha = 0, 1 \text{ with } \eta(-1) = (-1)^\alpha$$

$$\theta_C(\eta) = \frac{1}{2}\sum_{n\in\mathbf{Z}} \eta_0(n)n^\alpha q^{n^2}.$$

Then, the following formula is easy:

$$\text{(5.1a)} \qquad \theta_J(\eta) = \sum_{0<d|(Jp/C)} \mu(d)\eta_0(d)d^\alpha\theta_C(\eta)|[d^2],$$

where μ is the Möbius function. We also know from [**Sh1**, Prop.2.2]

$$\text{(5.1b)} \quad \theta_C(\eta)|\tau(4C^2p^{2\delta}) = \left(-\sqrt{-1}\right)^\alpha C^{-1/2}p^{-\delta/2}G(\eta)\theta_C(\eta^{-1}).$$

Thus if $C = J$, we have

$$\text{(5.1c)} \qquad \theta_J(\eta)|\tau(4C^2p^{2\delta'}) =$$

$$\begin{cases} -\eta_0(p)(Jp)^{-1/2}\left(-\sqrt{-1}\right)^\alpha G(\eta)\{\theta_C(\eta^{-1}) \\ \qquad -p^{1+\alpha}\eta_0^{-1}(p)\theta_C(\eta^{-1})|[p^2]\}, & \text{if } \delta = 0 \text{ and } \delta' = 1 \\ \left(-\sqrt{-1}\right)^\alpha (Jp^\delta)^{-1/2}G(\eta)\theta_C(\eta^{-1}), & \text{if } \delta = \delta' > 0. \end{cases}$$

Now we start proving Theorem 5.1. Let $\xi : (\mathbf{Z}/Jp\mathbf{Z})^\times \to \mathcal{O}^\times$ be the Dirichlet character as in the theorem. Therefore, the conductor of ξ is equal either to J or Jp. Let $\Phi \in \mathcal{O}[[W^\times]]\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ be the generalized measure as in Corollary 4.3 for θ as above. We compute $\int_{W^\times} \varepsilon\xi\omega^{-n}(w)w^{\alpha+2m}d\Phi_P$, writing, for $0 \le n < k(P) - 1$, $n \equiv \alpha \pmod 2$ for $\alpha = 0$ or 1 and $\alpha + n = 2\alpha + 2m$ with $0 \le 2m < k(P) - \alpha - 1$; namely, we shall compute

(5.2)

$$\mathcal{D}_{Lp}\left(1+\alpha+n, f_P, \int_{W^\times} \epsilon(\langle w\rangle)\langle w\rangle^\alpha\xi(w)d\theta|_{\alpha+(1/2)}\tau(4J^2p^{2\delta'})\right),$$

where $\epsilon : (1 + p\mathbf{Z}_p) \to \mu_{p^\infty}$ is a character with $\mathrm{Ker}(\varepsilon) = 1 + p^{\delta'}\mathbf{Z}_p$ (thus $\delta' \ge 1$) and $\langle w\rangle$ is the projection in $1 + p\mathbf{Z}_p$ of $w \in W^\times$. Then by (5.1c) applied to $\eta = \varepsilon\xi\omega^{-n}$ (thus $C(\varepsilon\xi\omega^{-n})$ is written as Jp^δ) and by an easy computation, (5.2) is equal to, if $C(\varepsilon\xi\omega^{-n})$ is prime to p (i.e. $\delta = 0$),

(5.3a)

$$\left(-\sqrt{-1}\right)^\alpha (Jp)^{-1/2}p^{-n}a(p, f_P)^2G(\varepsilon\xi\omega^{-n})(1 - \varepsilon\xi\omega^{-n}(p)a(p, f_p)^{-2}p^n)$$

$$\times(1 - (\psi_P\psi'(2)\varepsilon^{-1}\xi^{-1}\omega^n(2))^2 2^{2k-4-2n})\mathcal{L}(n+1, f_P, \psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^n).$$

If $p|C(\varepsilon\xi\omega^{-n})$, then (5.2) can be rewritten as

(5.3b)
$$\left(-\sqrt{-1}\right)^{\alpha}(Jp^{\delta})^{-1/2}G(\varepsilon\xi\omega^{-n})(1-(\psi_P\psi'(2)\varepsilon^{-1}\xi^{-1}\omega^{n}(2))^2 2^{2k-4-2n}) \\ \times \mathcal{L}(n+1, f_P, \psi'\psi_p\epsilon^{-1}\xi^{-1}\omega^n).$$

This combined with Corollary 4.3 (see also (4.5)) yields

$$(5.4)\quad (1-\psi_1\xi_1^{-1}\omega^{-1}\varepsilon_P\varepsilon^{-1}(b)\langle b\rangle^{k-1-n})^{-1} \\ (1-(\psi\varepsilon_P(2)\varepsilon^{-1}\xi^{-1}\omega^{-2}(2))^2\langle 2\rangle^{2k-4-2n})^{-1} \\ \times S(P)H(P)^{-1}\int_{Z_J}\xi(w)\varepsilon(\langle w\rangle)\langle w\rangle^n d\Phi_P \\ = c(P_{n+1,\varepsilon},P)E(n+1,\psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^n) \\ \frac{\mathcal{L}(n+1,f_p^{\circ},\psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^n)}{(2\pi \mathrm{i})^{n-1}\Omega(P)}.$$

Now we define, for $\phi\in\mathcal{C}(\Gamma;\mathcal{O})$, an element $\Psi\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ so that

$$\int_{\Gamma}\phi(w)d\Psi_P=\frac{1}{2}\int_{Z_J}\xi(w)\phi(\langle w\rangle)\langle w\rangle^{-1}d\Phi_P \text{ for all } P\in\mathcal{X}(\mathbf{I}).$$

Then choosing $b\in\mathbf{Z}_p^{\times}$ so that b generate topologically $\mathbf{Z}_p^{\times}$ and $\langle b\rangle=u$, we put

$$H_1=1-\psi_1\xi_1^{-1}\omega^{-1}(b)(1+Y)(1+X)^{-1},$$
$$H_2=1-(\psi\xi^{-1}\omega^{-2}(2))^2\langle 2\rangle^{-2}A_2(Y)^2A_2(X)^{-2},$$
$$\text{and } \mathcal{L}=(H_1H_2H)^{-1}\Psi.$$

Then $\mathcal{L}$ satisfies the assertion of Theorem 5.1 by (5.4) because H_1 (resp. H_2) is non-unit if and only if $\psi_1=\xi_1\omega$ (resp. $\psi\xi^{-1}\omega^{-1}(2)=\pm1$) and H_2 is a unit multiple of $A_2(X)\langle 2\rangle-A_2(Y)$ if $\psi\xi^{-1}\omega^{-1}(2)=\pm1$.

Here we add another result for our later use:

PROPOSITION 5.2. *Let the notation be as in Theorem 5.1. Then $\mathcal{L}$ is finite at $X-Y$ unless $\psi'\xi'^{-1}$ is imaginary quadratic and λ has complex multiplication under the imaginary quadratic field corresponding to $\psi'\xi'^{-1}$.*

PROOF: We may assume that $\psi_1=\xi_1\omega$ (i.e. $\psi\xi^{-1}\omega^{-1}=\psi'\xi'^{-1}$), because otherwise the desired assertion is already contained in Theorem 5.1. We may also assume that λ is minimal in the sense of [**H5**, p.380]. Let L be the least common multiple of N and $4J^2$, where the conductor of ξ' is given by J. We consider the theta measure θ on Z_J of level $4J^2$ used in the proof of the above theorem. Now we replace, in the formula (4.8), ξ by $\varepsilon\xi\omega^{k-1}$, φ by the identity character, j by $1+2\alpha+4m$ and $2k-1$ by $2n+1$. Then what we need to show is the vanishing of

$$(*) \qquad \ell_p \circ T_{L/N} \circ e\left(\theta^L(\phi) \cdot d^m\left(\int_{Z_L} \psi'\xi'^{-1}\chi_{-L}(z)z_p^{-1}dE\right)\right).$$

Write $L = x^2M$ for a positive integer x and a square-free M. Since $\int_{Z_L} \eta(z)z_p^{-1}dE = 0$ if η is neither real quadratic nor the identity by the following well known formula:

$$\int_{Z_L} \eta(z)z_p^{-1}d\zeta_\alpha^b = \begin{cases} -\{\varphi(C(\alpha)Lp)/C(\alpha)Lp\}\log(\langle b\rangle) & \text{if } \alpha = \eta^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for the Euler function φ, we may assume that $\phi'\xi'^{-1} = \chi_{-M/t}$ for a divisor t of M. Hence, we know

$$(5.5) \qquad a\left(n, \int_{Z_L} z_p^{-1}dE\right) = 0 \textit{ unless } n/t \textit{ is a square.}$$

On the other hand, $a(n, \theta^L(\phi)) = 0$ unless $n = m^2L/4J^2$ for some integer m. Then we see that $a\left(n, \theta^L(\phi)\cdot d^m\left(\int_{Z_L} z_p^{-1}dE\right)\right) = 0$ unless n is of the form $Ma^2 + tb^2$ for integers a and b; namely,

$$(5.6) \quad a\left(n, \theta^(\phi)\cdot d^m\left(\int_{Z_L} z_p^{-1}dE\right)\right)$$
$$= 0 \textit{ unless } n/t \textit{ is a norm of an integer in } \mathbf{Q}\left(\sqrt{-M/t}\right).$$

Thus if λ does not have complex multiplication by $F = \mathbf{Q}\left(\sqrt{-M/t}\right)$, then there exists a prime l such that l remains prime in F and $\ell_P \circ T(l) = a\ell_P$ with $a \neq 0$ (cf. [**R**, Th.2.3]). On the other hand, (5.6) implies that $\theta^L(\phi)\cdot d^m\left(\int_{Z_L} z_p^{-1}dE\right)$ is annihilated by $T(l)$. Thus we know even in this case (*) vanishes.

§6. PRIMITIVE p-ADIC L-FUNCTIONS AND PROOF OF THEOREM I

Before starting the proof of Theorem I, we want to make correction to [**H3**, Lemma 10.3] and [**H5**, Proposition 7.3 and Corollary 7.12]. At the same time, we shall show the identity of two primitive L-functions: the one introduced in [**C-S**] and the one constructed by Gelbart-Jacquet [**G-J**] (see also [**H3**] and [**Sch**, §1]). Of course, these two L-functions must coincide because they have the same Euler factor at almost all primes and satisfy the functional equation of the same type. However, the point here is to show the identity directly without using the analytic continuation and the functional equation. For a minimal primitive form f of conductor C and for its automorphic representation π, the L-function for the base change lift $\hat{\pi}$ (of π to GL_3 introduced in §0) is given by

$$L(s,f) = L(s+k-1,\hat{\pi}) = \prod_{l\in\Sigma}(1+l^{k-1-s})^{-1}\prod_{l\in\Sigma'}(1-l^{k-1-s})^{-1}\mathcal{L}(s,f),$$

where Σ (resp. Σ') is the set of all primes $l \neq p$ at which the local factor π_l of π is supercuspidal and satisfies $\pi_l \cong \pi_l \otimes \eta$ for the non-trivial unramified quadratic character η of $\mathbf{Q}_l$ (resp. is principal but not spherical). For the above result, we refer [**G-J**, §1] and [**H3**, Lemma 10.3]. However, there is a misstatement in [**H3**, Lemma 10.3]. Namely, the Euler factor in (iii) of Lemma 10.3 there should read

$$D_l(s,f) = \left[\left(1-\psi_o^{-1}(l)\alpha_l^2 l^{-s}\right)\left(1-l^{k-1-s}\right)\left(1-\psi_0^{-1}(l)\beta_l^2 l^{-s}\right)\right]^{-1}$$

although it is written as

$$D_l(s,f) =$$
$$\left[\left(1-\psi_0^{-1}(l)\alpha_l^2 l^{-s}\right)\left(1-\psi_0^{-1}(l)\alpha_l\beta_l l^{-s}\right)\left(1-\psi_0^{-1}(l)\beta_l^2 l^{-s}\right)\right]^{-1}.$$

This is clear from the description of $D_l(s,f)$ below (10.1) in p.604 of [**H3**]. This mistake caused a trouble in [**H5**, Prop.7.3 and Cor.7.12], and all the result there is valid if we replace H_0 in [**H5**, Cor.7.12] by $\prod_{l\in\Sigma}(1+l^{-1})^{-1}\prod_{l\in\Sigma'}(1-l^{-1})^{-1}H$ and $D(s,f)$ in [**H5**, Prop.7.3] by $L(s,f)$ defined above. Now we shall prove

PROPOSITION 6.1. *Let f be a minimal form of conductor C and of character ψ. Let $C(\psi)$ be the conductor of ψ and write $C/C(\psi) = \prod_l l^{e(l)}$ for primes l. Then, we have*

$$\Sigma = \{l : \textit{ prime} \mid e(l) > 0 \textit{ and } e(l) \equiv 0 \pmod 2\},$$

$$\Sigma' = \{l : \text{ prime} \mid l \mid C \text{ but } e(l) = 0\}.$$

By this result, the question raised in [**H5**, 7.13] is solved affirmatively, and the result in [**H5**, Cor.7.12] is true for all arithmetic point P with $k(P) \geq 2$ after the above correction that H_0 there should be replaced by $\prod_{l\in\Sigma}(1+l^{-1})^{-1}\prod_{l\in\Sigma'}(1-l^{-1})^{-1}H$.

PROOF: The assertion about Σ' is obvious from [**H3**, Lemma 10.1]. Thus we shall prove the assertion concerning Σ. Let Σ_0 be the set defined by the right-hand side of the above formula about Σ. Let π be a supercuspidal representation of $GL_2(\mathbf{Q}_l)$ and K be the unique unramified quadratic extension of $\mathbf{Q}_l$. Then, by [**La**, Lemma 7.17], $\pi \otimes \eta \cong \pi$ for the unramified quadratic character η of $\mathbf{Q}_l^\times$ if and only if there exists a ramified quasi-character θ of $K^\times$ such that π corresponds to θ via the Langlands correspondence [**K3**, §2]. Then if the conductor of θ is equal to l^m, then the conductor of π is given by $l^{2m} = \mathcal{N}_{k/\mathbf{Q}_l}(l^m)$. Namely $\Sigma_0 \supset \Sigma$. On the other hand, there is a classification due to Kutzko (in our case of GL(2) [**K1**],[**K2**] and due to Carayol [**C**] for general $GL(n)$) of supercuspidal (minimal) representation of $GL_2(\mathbf{Q}_l)$ using induction from subgroups compact modulo the center. Namely let

$$K_m = \begin{cases} GL_2(\mathbf{Z}_l) & \text{if } m = 0 \\ 1 + l^m M_2(\mathbf{Z}_l) & \text{if } m > 0, \end{cases} \quad A_m = l^m M_2(\mathbf{Z}_l).$$

Then $K_{m-1}/K_m \cong A_{m-1}/A_m$ if $m \geq 2$ via $\gamma \mapsto \gamma - 1$, and the character group $(A_{m-1}/A_m)^*$ of the finite group A_{m-1}/A_m is given by A_{-m}/A_{1-m} via the natural pairing $(x,y) = \mathrm{Tr}(xy)$. Thus we can naturally identify $(K_{m-1}/K_m)^*$ $(m \geq 2)$ with A_{-m}/A_{1-m}. Thus each character of K_{m-1}/K_m corresponds to an element of A_{-m}. A character of K_{m-1}/K_m is called cuspidal if for the corresponding element u, the characteristic polynomial of $l^m u$ modulo l is irreducible over $\mathbf{F}_l$ (especially, $\mathbf{Q}_l(u)$ is the unique unramified extension K of $\mathbf{Q}_l$). A representation of $\mathbf{Q}_l^\times GL_2(\mathbf{Z}_l)$ is called very cuspidal of type $m \geq 2$ if it is trivial over K_m and its restriction to K_{m-1}/K_m is decomposed into the sum of cuspidal characters. Then every irreducible super cuspidal and minimal representation π of conductor l^{2m} of $GL_2(\mathbf{Q}_l)$ is induced from a very cuspidal representation of type m (see [**C**, Th.8.1]). On the other hand, by [**K2**] (see also [K3, §3, (3.10) Remark 3] and [**C**, 3.6]), there exists a character θ of K such that π as above corresponds to θ in the sense of Langlands. This shows in particular, $\Sigma \supset \Sigma_0$.

We now prove Theorem I. Let $\lambda : \mathbf{h}^{\mathrm{ord}}(N;\mathcal{O}) \to \mathbf{I}$ be a Λ-algebra homomorphism as in Theorem I. Let $\hat{\lambda} : \mathbf{h}^{\mathrm{ord}}(N;\mathcal{O})\otimes_\Lambda \mathbf{I} \to \mathbf{I}$ be the $\mathbf{I}$-algebra homomorphism naturally induced from λ. Thus $\hat{\lambda}$ is primitive and minimal in the sense of [**H5**, §7]. Let f_P be the cusp form belonging to $\hat{\lambda}$ at $P \in \mathcal{A}(\mathbf{I})$. Then the primitive form f_P° associated with f_P is a minimal form in the sense of [**H5**, 7.2]. Let $\pi(P) = \otimes\pi(P)_l$ be the automorphic representation associated with f_P°. Write $\alpha = \xi\omega\psi^{-1}$ for a given character ξ modulo Jp whose conductor is divisible by J (J is an integer prime to p). When α_l^2 is unramified but α_l itself ramifies at l, we denote by μ_l the quadratic character of $\mathbf{Q}_l^\times$ such that $\mu_l(l) = 1$ and $\mu_l|_{\mathbf{Z}_p^\times} = \alpha_l|_{Z_p^\times}$. Then define sets of prime factors of N as follows:

$$\Sigma = \{l|\pi(P)_l \text{ is super cuspidal}\}$$

$$\Sigma_0 = \{l \in \Sigma|\alpha_l \text{ is unramified, and } \pi(P)_l \cong \pi(P)_l \otimes \eta_l\},$$

$$\Sigma_1(P) = \{l \in \Sigma|\alpha_l^2 \text{ is unramified but } \alpha_l \text{ is ramified and } \\ \pi(P)_l \cong \pi(P)_l \otimes \mu_l \text{ but } \pi(P)_l \not\cong \pi(P) \otimes \mu_l\eta_l\},$$

$$\Sigma_2(P) = \{l \in \Sigma|\alpha_l^2 \text{ is unramified but } \alpha_l \text{ is ramified and } \\ \pi(P)_l \cong \pi(P)_l \otimes \mu_l\eta_l \text{ but } \pi(P)_l \not\cong \pi(P)_l \otimes \mu_l\}$$

$$\Sigma_3(P) = \{l \in \Sigma|\alpha_l^2 \text{ is unramified but } \alpha_l \text{ is ramified and } \\ \pi(P)_l \cong \pi(P)_l \otimes \mu_l \text{ but } \pi(P)_l \cong \pi(P)_l \otimes \mu_l\eta_l\},$$

$$\Xi = \{l|N|\pi(P)_l \text{ is principal}\} = \{l|N|(N/C(\psi_l), l) = 1\}.$$

where η_l is the unique unramified quadratic character of $\mathbf{Q}_l^\times$. Here we think of ξ as an idele character in the way that $\xi(\bar{\omega}_l)$ is equal to the value $\xi(l)$ of the Dirichlet character ξ for the prime element $\bar{\omega}_l$ of $\mathbf{Z}_l$ for each prime l outside Jp. The sets Σ, Σ_0 and Ξ are independent of P as already remarked. We can show the independence of $\Sigma_i(P)$ from P as follows. By [**La**, Lemma 7.17], the base change lift of super cuspidal $\pi(P)_l$ to the quadratic extension $F_l/\mathbf{Q}_l$ corresponding to a quadratic character α becomes principal if and only if $\pi(P)_l \otimes \alpha \cong \pi(P)_l$. Thus taking a real quadratic extension F_1 (resp. F_2) whose l-adic completion corresponds to ξ_l (resp. $\xi_l\eta_l$) for all $l \in \Sigma_1(P) \cup$

$\Sigma_2(P) \cup \Sigma_3(P)$. Then we can think of the global lift λ_i of λ to the p-adic ordinary Hecke algebra $\mathbf{h}^{\mathrm{ord}}(N;\mathcal{O})_{/F_i}$ for F_i as in [**H6**, Remark 5.6] (the author hopes to discuss the base change lift of λ in detail in a future occasion). Let

$$\Xi_i = \{l|N \mid \text{the lift of } \pi(P)_l \text{ to } F_i \text{ is principal}\}.$$

Then, we see Ξ_i is independent of P by the same reasoning in the beginning of this section. Then $\Sigma_3(P) = (\Xi_1 \cap \Xi_2) - \Xi$, $\Sigma_1(P) = \Xi_1 - \Xi_2$, $\Sigma_2(P) = \Xi_2 - \Xi_1$. In order to prove the independence of $\Sigma_i(P)$ for almost all P, we can do it in the similar manner as in the proof of [**H5**, Th.7.11] without introducing the lift. Hereafter we write Σ_i for $\Sigma_i(P)$. Note that $A_z(X)$ stands for $(1+X)^{\log(\langle z\rangle)/\log(u)}$. Now we put, as an element of $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$, $E = E_0E_1E_2E_\Xi$, $E' = E'_0E'_1E'_2E'_\Xi$ with

(6.1)

$$\begin{aligned}
E_0 &= \prod_{l\in\Sigma_0}(1+\psi\xi^{-1}\omega^{-2}(l)\langle l\rangle^{-1}A_l(Y)/A_l(X)),\\
E'_0 &= \prod_{l\in\Sigma_0}(1+\psi^{-1}\xi\omega(l)A_l(X)/A_l(Y)),\\
E_1 &= \prod_{l\in\Sigma_1\cup\Sigma_3\cup\Xi}(1-\psi\xi^{-1}\omega^{-2}(l)\langle l\rangle^{-1}A_l(Y)/A_l(X))\\
E'_1 &= \prod_{l\in\Sigma_1\cup\Sigma_3\cup\Xi}(1-\psi^{-1}\xi\omega(l)A_l(X)/A_l(Y)),\\
E_2 &= \prod_{l\in\Sigma_2\cup\Sigma_3}(1+\psi\xi^{-1}\omega^{-2}(l)\langle l\rangle^{-1}A_l(Y)/A_l(X)),\\
E'_2 &= \prod_{l\in\Sigma_2\cup\Sigma_3}(1+\psi^{-1}\xi\omega(l)A_l(X)/A_l(Y)),\\
E_\Xi &= \prod_{l\in\Xi}(1-\psi^2\xi^{-1}\omega^{-3}(l)\langle l\rangle^{-2}A_l(Y)^{-2}/\lambda(T(l))^2A_l(X)),\\
E'_\Xi &= \prod_{l\in\Xi}(1-\xi(l)\langle l\rangle^{-1}A_l(X)/\lambda(T(l))^2),
\end{aligned}$$

where $\alpha(l)$ is the value at l of the primitive Dirichlet character associated to α. Then, as computed in [**Sch**, §1], the special values of the primitive L-function $L(s,f_P,\xi)$ is given by

$$L(n,f_P,\psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^{n-1}) = E(P_{n,\varepsilon},P)^{-1}\mathcal{L}(n,f_P^\circ,\psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^{n-1})$$

if $E(Pn, \varepsilon, P) \neq 0$ and $1 \leq n < k(P)$. We now fix $P \in \mathcal{A}(\mathbf{I})$ and consider f_P and the minimal form f_P°. We write $E_P(Q) = E(Q, P)$ and $\mathcal{L}_P = \mathcal{L}(Q, P)$, which are elements of the quotient field $\mathbf{L}$ of Λ. Then we define an element L in the quotient field of $\Lambda \hat{\otimes}_{\mathcal{O}} \mathbf{I}$ by $E^{-1}\mathcal{L}$. Its specialization L_P at P is given by $E_P^{-1}\mathcal{L}_P$ (note that E_P is always non-zero for $P \in \mathcal{A}(\mathbf{I})$). Thus if $E_P(P_{n,\varepsilon})D(P_{n,\varepsilon}, P) \neq 0$ for D as in Theorem 5.1,

$$\begin{aligned} &L_P(P_{n,\varepsilon}) \\ &\quad = c(P_{n,\varepsilon}, P)E(n, \psi'\psi_P\varepsilon^{-1}\xi^{-1}\omega^{n-1})\frac{L(n, f_P^\circ, \psi'\psi_P\epsilon^{-1}\xi^{-1}\omega^{n-1})}{(2\pi \mathrm{i})^{n-2}\Omega(P)}, \end{aligned}$$

where, writing $C(\varepsilon\xi\omega^{-n+1}) = Jp^\delta$ for J prime to p,

$$\begin{aligned} c(P_{n,\varepsilon}, P) = (n-1)!(C(\varepsilon\xi\omega^{-n+1}))^{n-1} \\ G(\varepsilon\xi\omega^{-n+1})N^{-k(P)/2}W'(f_P)^{-1}G(\psi_P)^{-1}\psi'(p)^\delta \end{aligned}$$

and $E(n, \eta) = E_1(n, \eta)E_2(n, \eta)$ with

$$E_1(n, \eta) = (1 - \eta^{-1}\psi'\psi_P(p)a(p, f_P)^{-2}p^{n-1})(\psi'(p)^{-1}a(p, f_P)^{-2})^\delta,$$

$$\begin{aligned} &E_2(n, \eta) \\ &= (1 - \eta(p)p^{k-n-1})(1 - \eta\psi'\psi_P(p)a(p, f_P)^{-2}p^{2k-n-2}) \text{ if } f_P \neq f_P^\circ, \end{aligned}$$

and

$$E_2(n, \eta) = 1 \text{ if } f_P = f_P^\circ.$$

Now define a unit $U \in \Lambda \hat{\otimes}_{\mathcal{O}} \mathbf{I}$ by

$$U(X, Y) = \frac{\psi_1^{-1}\xi_1\varepsilon(C')G(\psi'^{-1}\xi')\langle CC'\rangle A_{C'}(X)}{\xi_1(C)G(\xi')A_C(X)A_{C'}(Y)},$$

where $C' = C(\psi'^{-1}\xi')$ and $C = C(\xi')$. Then, if ψ_P is trivial, we see easily that

$$U(P_{n,\varepsilon}, P) = \frac{\psi_1^{-1}\xi_1\varepsilon(C')G(\psi'^{-1}\xi')\langle C'\rangle^{n-k+1}}{\xi_1\varepsilon(C)G(\xi')\langle C\rangle^{n-1}}.$$

and thus UL_P satisfies, if ψ_P is trivial,

$$U(P_{n,\varepsilon}, P)L_P(P_{n,\varepsilon}) = (-1)^{k-1}2^{2k(P)-2}N^{-k(P)/2}W'(f_P)^{-1}S(P)^{-1}$$

$$\times\Gamma(n)C(\psi'\xi'^{-1})^{n-k+1}$$
$$E(n,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})I(n-k+1,f_P^\circ,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1}),$$

where $I(m,f_P^\circ,\eta)$ with $\eta(-1)=(-1)^{m+\alpha}$ ($\alpha=0$ or 1) is defined as in [**Sch**, Cor.2.6] as follows:

$$I(m,f_P^\circ,\eta)=\left(\frac{G(\eta^{-1})}{(2\pi\mathrm{i})^m}\right)^{1+\alpha}L(m+k-1,f_P^\circ,\eta)/\pi^{k-1}\langle f_P^\circ,f_P^\circ\rangle$$

for $2-k(P)\le m\le 0$ is $\alpha=0$ and $1\le m\le k-1$ if $\alpha=1$. Writing μ the measure given in [**Sch**, Th.5.3] for $\lambda=\psi'^{-1}\xi'$ and $f=f_P^\circ$, we have, if the level of f is prime to p,

$$U(P_{n,\varepsilon},P)L_P(P_{n,\varepsilon})=(-1)^{k-1}2^{2k-2}N^{-k(P)/2}W'(f_P)^{-1}S(P)^{-1}$$
$$\times\int_{\mathbf{Z}_p^\times}\varepsilon\xi\omega^{1-n}(x)x^{n-k+1}d\mu(x).$$

Here, perhaps, the following two remarks may help the reader to understand this formula: (i) Although the measure μ is given in [**Sch**, Th.5.3] only when p is not exceptional in the sense described in [**Sch**, p.627], the distribution μ as an element of the total quotient ring of the algebra of p-adic measures on $\mathbf{Z}_p^\times$ always exists and the above formula remains true except for finitely many ε's; (ii) There is a misprint in the formula (3.15) in [**Sch**] and $i^{1/2}$ in the formula should read $(-1)^{\rho_\chi}(-i)^{1/2}=(-1)^{m+k}(-i)^{1/2}$. This causes a little trouble there. In fact, one has to multiply $(-1)^{m+k-1}i$ to μ'_m defined there to have [**Sch**, Th.3.8] to be true. Since i does not depend on m and k, to fix it, we have actually divided μ'_m by $(-1)^{m+k-1}$ and written the divided measure μ'_m. Then we go on to construct the measure μ_m and μ as in [**Sch**, Th.5.3]. Then, replacing $Q_{m,\lambda}$ in p.608 of [**Sch**] by $(-1)^{m+k-1}Q_{m,\lambda}$, we have the evaluation formula [Sch, (5.10)]. Then every result in [**Sch**] stands well without additional change for this definition of μ. Now we put, for $f=f_P^\circ$ with trivial ψ_P,

$$\tilde I(m,f,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})$$
$$=\Gamma(k+m-1)C(\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})^{m+(m-1)\delta}I(m,f,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})).$$

Then the following inversion formula is deduced from the functional equation of [**G-J**] in [**Sch**, Prop.2.7]:

$$\tilde I(m,f,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})$$
$$=C(\hat\pi\otimes\psi'\xi'^{-1})M(\psi'\xi'^{-1})^{-m}\varepsilon^{-1}\xi_1^{-1}\omega^{n-1}(M(\psi'\xi'^{-1}))$$
$$\times 2\Gamma(1-m)\tilde I(1-m,f,\psi'^{-1}\varepsilon\xi\omega^{1-n}),$$

where $\hat{\pi}$ is the base change lift to $GL(3)$ given in [**G-J**] of the automorphic representation π attached to f_P°,

$$C(\hat{\pi}\otimes\psi'\xi'^{-1}) = \varepsilon(0,\hat{\pi}\otimes\psi'\xi'^{-1})\left(\frac{G(\psi'^{-1}\xi')}{\sqrt{\psi'^{-1}\xi'(-1)}C(\psi'^{-1}\xi')}\right)^3$$

for the ε-factor, $\varepsilon(s,\hat{\pi}\otimes\psi'\xi'^{-1})$ normalized as in [**Sch**, p.601] and

$$M(\psi'\xi'^{-1}) = C(\psi'\xi'^{-1})^{-3}C(\hat{\pi}).$$

Here we made a change of sign for

$$M(\psi'\xi'^{-1})$$

and

$$\tilde{I}(m,f,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})$$

and the one given in [**Sch**] are

$$-M(\psi',\xi'^{-1})$$

and

$$(-1)^m\tilde{I}(m,f,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})$$

with our notation here. This sign change does not alter the form of the inversion formula. This shows that, if ψ_P is trivial, then

$$\begin{aligned}
&U(P_{n,\varepsilon},P)L_P(P_{n,\varepsilon})\\
&\quad = 2^{2k(P)-2}N^{-k(P)/2}W'(f_P)^{-1}S(P)^{-1}E(n,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})\\
&\qquad \times \tilde{I}(n-k+1,f,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})\\
&\quad = 2^{2k(P)-1}N^{-k(P)/2}W'(f_P)^{-1}S(P)^{-1}E(n,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})\\
&\qquad \times C(\hat{\pi}\otimes\psi'\xi'^{-1})M(\psi'\xi'^{-1})^{-1+k-n}\varepsilon^{-1}\xi_1^{-1}\omega^{n-1}(M(\psi'\xi'^{-1}))\\
&\qquad \times \Gamma(k-n)\tilde{I}(k-n,f,\psi'^{-1}\varepsilon\xi\omega^{1-n}).
\end{aligned}$$

Namely, the numbers:

$$E(2k-1-n,\psi'^{-1}\varepsilon\xi\omega^{1-n})M(\psi'\xi'^{-1})^{-1+k-n}\varepsilon^{-1}\xi_1^{-1}\omega^{n-1}(M(\psi'\xi'^{-1}))$$

$$\times\Gamma(k-n)\tilde{I}(k-n,f,\psi'^{-1}\varepsilon\xi\omega^{1-n})$$

depends p-adic meromorphically on n and ε. Since p is prime to $2NC(\hat{\pi}\otimes\psi'\xi'^{-1})$, we can find a unit $V(X,Y)\in\mathbf{Z}_p[[X,Y]]$ such that

$$\begin{aligned}V(P_{n,\varepsilon},P) &= 2^{2k(P)-1}N^{-k(P)/2}C(\hat{\pi}\otimes\psi'\xi'^{-1})\\ &\quad M(\psi'\xi'^{-1})^{-1+k-n}\varepsilon^{-1}\xi_1^{-1}\omega^{n-1}(M(\psi'\xi'^{-1}))\\ &= 2^{-1}\xi_1^{-1}(M(\psi'\xi'^{-1}))C(\hat{\pi}\otimes\psi'\xi'^{-1})\\ &\quad M(\psi'\xi'^{-1})^{-2}\{4N^{-1/2}M(\psi'\xi'^{-1})\}^{k(P)}\\ &\quad\times\langle M(\psi'\xi'^{-1})\rangle^{1-n}\varepsilon^{-1}(M(\psi'\xi'^{-1}))\end{aligned}$$

whenever ψ_P is trivial and $f_P^\circ\neq f_P$ (this condition holds if ψ_P is trivial and $k(P)>2$), because $k(P)$ for P with $\psi_P=1$ stays in one residue class modulo $p-1$, i.e. the class of a when $\psi^1=\omega^a$. Thus for $L'=V^{-1}UL$,

$$\begin{aligned}&L'(P_{n,\varepsilon},P)\\ &\quad= W'(f_P)^{-1}S(P)^{-1}E(n,\psi'\varepsilon^{-1}\xi^{-1}\omega^{n-1})\\ &\qquad\qquad\Gamma(k-n)\tilde{I}(k-n,f,\psi'^{-1}\varepsilon\xi\omega^{1-n}).\end{aligned}$$

Then, defining L'_P in the quotient field of $\mathcal{O}[[X]]$ by $L'_P(Q)=L'(Q,P)$, it has been shown in [**Sch**, Th.4.1] that there exists a non-zero constant $C_{P,n}\in K$ such that

$$C_{P,n}E'_PL'_P\in\mathcal{O}[[X]]$$

if either $\psi'\xi^{-1}$ is not a quadratic character with $\psi'\xi^{-1}(-1)=-1$ or $\xi_1\omega\neq\psi_1$, and otherwise

$$(1-(1+Y)/(1+X))C_{P,n}E'_PL'_P\in\mathcal{O}[[X]].$$

We shall say that we are in Case I if either $\psi'\xi^{-1}$ is not a quadratic character with $\psi'\xi^{-1}(-1)=-1$ or $\xi_1\omega\neq\psi_1$ and in Case II otherwise. Then there exists $0\neq H'\in\mathbf{I}$ such that

$H'E'\ L'\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case I and $(1-(1+Y)/(1+X))H'E'_PL'\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$.

In fact, for example, in Case I, writing the ideal $(E'L')=\frac{N}{D_0D_1}$ for mutually prime divisors D_0, D_1 and N of $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ so that D_0 is an ideal in $\mathbf{I}$ and D_1 does not have such prime factors. As we will see in the proof of the next lemma, the prime divisor of D_1 is of the form $(1+X)-\alpha$ for $\alpha \in \mathbf{I}$, and hence the specialization $(D_1+P)/P$ in $\Lambda\hat{\otimes}_{\mathcal{O}}(\mathbf{I}/P)$ is prime to p. Then for almost all $P \in \mathcal{A}(\mathbf{I})$, the images $N(P)$ and $D_1(P)$ in $\Lambda\hat{\otimes}_{\mathcal{O}}(\mathbf{I}/P)$ are mutually prime divisors in $\Lambda\hat{\otimes}_{\mathcal{O}}(\mathbf{I}/P)$. In fact, if $\sqrt{D_1+N}$ contains $P\otimes\Lambda$ for all $P \in \mathcal{A}(\mathbf{I})$ with $k(P)>2$ (i.e. $D_1+N+P\otimes\Lambda/P\otimes\Lambda$ is of height 1 for all $P \in \mathcal{A}(\mathbf{I})$), then $\sqrt{D_1+N}$ contains $\mathbf{m}\otimes\Lambda$ for the maximal ideal $\mathbf{m}$ of $\mathbf{I}$. Thus $(D_1+N)/(P\otimes\Lambda)$ for any $P\in\mathcal{A}(\mathbf{I})$ has common factor with $p\Lambda$, because $\mathbf{m}\otimes\Lambda/P\otimes\Lambda=\mathbf{m}_P\otimes\Lambda$ for the maximal ideal $\mathbf{m}_P$ of $\mathbf{I}/P$. Namely $(D_1+N)/(P\otimes\Lambda)$ contains p^m for m large. On the other hand, $(D_1+N)/(P\otimes\Lambda)$ contains an element prime to p and hence any prime ideal containing $(D_1+N)/(P\otimes\Lambda)$ is of height 2, which is a contradiction. This shows that if D_1 is non-trivial, then $D_1(P)$ is prime to $N(P)$ for almost all $P\in\mathcal{A}(\mathbf{I})$ and $E'_PL'_P \notin \mathcal{O}[[X]]\otimes\mathbf{Q}_p$ a contradiction. Thus D_1 is trivial. Since $L'=V^{-1}UL$, we may assume that $H'=H$ for $H\in\mathbf{I}$ as in Theorem 5.1. Note that

$$E'^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \ni HL' = V^{-1}UHL \in E^{-1}D^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \text{ in Case I}$$

and

$$(X-Y)^{-1}E^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \ni HL' = V^{-1}UHL \in E^{-1}D^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \text{ in Case II}$$

for D as in Theorem 5.1. Thus we now know

$$E'^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \ni HL' = V^{-1}UHL \in E^{-1}D^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \text{ in Case I}$$

and

$$\begin{aligned}(X-Y)^{-1}E^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} &\ni HL' \\ &= V^{-1}UHL \in (X-Y)^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \text{ in Case II.}\end{aligned}$$

Thus if E and E' are mutually prime in $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ we have $HL\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case I and $HL\in(X-Y)^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case II.

LEMMA 6.2. *$(A_2(X)-\langle 2\rangle^{-1}A_2(y))E$ and $(X-Y)E'_0E'_1E'_2$ (resp. $(A_2(X)-\langle 2\rangle^{-1}A_2(Y))\ E_0E_1E_2$ and E') are mutually prime in $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$.*

PROOF: First we suppose that E_Ξ has a factor $\bar{\omega}$ for the prime element $\bar{\omega}$ in $\mathcal{O}$. Namely suppose that

$$(1-\psi^2\xi^{-1}\omega^{-3}(l)\langle l\rangle^{-2}A_l(Y)^2/\lambda(T(l))^2A_l(X))$$

has a factor $\bar{\omega}$ for $l \in \Xi$. Then, writing a for $\psi^2\xi^{-1}\omega^{-3}(l)\langle l\rangle^{-2}$,

$$A_l(X) \equiv aA_l(Y)^2/\lambda(T(l))^2 \pmod{\bar{\omega}}.$$

Note that $A_l(X) \in \mathcal{O}[[X]]$ and $A_l(Y)^2/\lambda(T(l))^2 \in \mathbf{I}$. Thus as a power series in $\mathbf{F}[[X]]$ for the residue field $\mathbf{F}$ of $\mathcal{O}$, $A_l(X) \pmod{\bar{\omega}}$ must be a constant, which is well known to be impossible (because the functions $s \mapsto \binom{s}{n}$ for $s \in \mathbf{Z}_p$ span the space of continuous functions on $\mathbf{Z}_p$). Thus $\bar{\omega}$ cannot divide E_Ξ. Similarly E and E' are not divisible by $\bar{\omega}$. For any $P \in \mathcal{A}(\mathbf{I})$ with ψ_P is trivial and $f_P^\circ \neq f_P$, the specialization of $DE_0E_1E_2$ and $E_0'E_1'E_2'$ at P are mutually prime away from $\bar{\omega}$ as already shown in the proof of [**Sch**, Th.5.1]. In fact, the prime factor Q of D, E_i, and E_i' is of the form $(1+X) - z(1+Y)$ for some $z \in \mathcal{D} = \{x \in \bar{\mathbf{Q}}_p | \, |x-1|_p < 1\}$, and $z \in \mu_{p^\infty}$ or $z \in u^{-1}\mu_{p^\infty}$ according as $Q|E_0'E_1'E_2'(X-Y)$ or $Q|E_0E_1E_2(A_2(X) - \langle 2\rangle^{-1}A_2(Y))$. On the other hand, the prime factor of E_Ξ and E_Ξ' is of the form $(1+X) - \alpha(Y)$ with $\alpha(Y) \in \mathbf{I}$ which is obviously prime to $(1+X) - z(1+Y)$ unless $\alpha(Y) = z(1+Y)$. Now suppose that $P = (1+X) - \alpha(Y)$ divides E_Ξ and show that P is prime to $E_0'E_1'E_2'(X-Y)$. Then, $z \in \mu_{p^\infty}$ and for a prime $q \in \Xi$, $z^s(1+Y)^s = \alpha(Y)^s = \zeta\langle q\rangle^{-2}A_q(Y)^2\lambda(T(q))^{-2}$ for $s = \log(\langle q\rangle)/\log(u)$ and a root of unity ζ. Namely $\lambda(T(q))^2 = z^{-s}\zeta\langle q\rangle^{-2}(1+Y)^s$. If one specilaize it at $P \in \mathcal{A}(\mathbf{I})$, then $\lambda(T(q))^2(P) = \zeta' q^{k(P)-2}$ with a root ζ' of unity. Note that the complex absolute value of $\lambda(T(q))(P)$ is given by $\left|\lambda(T(q))^2(P)\right| = q^{k(P)-1}$. Thus this is impossible and P does not divide $E_0'E_1'E_2'(X-Y)$. Next suppose that $P = (1+X) - \alpha(Y)$ divides E_Ξ' and show that P is prime to $E_0E_1E_2(A_2(X) - \langle 2\rangle^{-1}A_2(Y))$. Then, $z \in u^{-1}\mu_{p^\infty}$ and writing $z = \zeta u^{-1}$ for $\zeta \in \mu_{p^\infty}$,

$$\zeta^s u^{-s}(1+Y)^s = \alpha(Y)^s = \zeta'\langle q\rangle\lambda(T(q))^2$$

for $s = \log(\langle q\rangle)/\log(u)$ and roots of unity ζ and ζ'. Namely $\lambda(T(q))^2 = \zeta''\langle q\rangle^{-2}(1+Y)^s$ for a root of unity ζ''. If one specializes it at $P \in \mathcal{A}(\mathbf{I})$, then $\left|\lambda(T(q))^2(P)\right| = q^{k(P)-2}$. This again contradicts to the fact: $\left|\lambda(T(q))^2(P)\right| = q^{k(P)-1}$. Namely P does not divide $E_0E_1E_2(A_2(X) - \langle 2\rangle^{-1}A_2(Y))$. This finishes the proof of the lemma.

COROLLARY 6.3. *(i) E_Ξ' (resp. E_Ξ) has no prime factors of the form:*

$$(1+X) - z(1+Y) \text{ with } z \in u^{-1}\mu_{p^\infty} (\text{resp.} z \in \mu_{p^\infty}).$$

(ii) The zeros in $\mathcal{A}(\Lambda)$ of

$$E_\Xi(X,P)E_0(X,P)E_1(X,P)E_2(X,P)(A_2(X)-\langle 2\rangle^{-1}A_2(P))$$

for $P \in \mathcal{A}(\mathbf{I})$ with $k(P) \geq 2$ are not of the form ζu^n for integers $n < k(P)-1$,

(iii) If a factor $(1-\psi^2\xi^-1\omega^{-3}(l)\langle l\rangle^{-2}A_l(Y)^2/\lambda(T(l))^2A_l(X))$ of E_Ξ (resp. $(1-\xi(q)\langle q\rangle^{-1}A_q(X)/\ \lambda(T(q))^2)$ of E'_Xi) is not a unit, then $\psi^{-2}\xi\omega^3(l)\lambda(T(l))^2 \equiv 1 \pmod{\mathbf{m}}$ and $\xi(q)\lambda(T(q))^2 \equiv 1 \pmod{\mathbf{m}}$ for the maximal ideal $\mathbf{m}$ of $\mathbf{I}$.

The assertions (i) and (iii) follows from the proof of the lemma given above. As for the assertion (ii), it is obvious for the factors $E_0(X,P)E_1(X,P)E_2(X,P)(A_2(X)-\langle 2\rangle^{-1}A_2(P))$. If ζu^n is a zero of $\zeta\langle l\rangle^n = \zeta'\langle l\rangle^{2k-2}\ /a(l,f_p)^2$ for some $l \in \Xi$ with roots of unity ζ and ζ', which is only possible when $n = k(P)-1$ because of $|a(l,f_p)^2| = l^{k(p)-1}$.

By the lemma, we know that $HL \in C^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case I and $HL \in (X-Y)^{-1}C^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case II, where C is the greatest common divisor of E_Ξ and E'_Ξ. We now show that $HL \in \Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case I and $HL \in (X-Y)^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case II. By Cor.6.3, (i), it is sufficient to prove that HL have singularity possibly only at prime factors of C of the form $(1+X)-\zeta u^{-1}(1+Y)$ for $\zeta \in \mu_{p^\infty}$. By the above argument, we know that for $T = E_0E_1E_2$, $T^{-1}H\mathcal{L}$ is in $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case I and $T^{-1}\mathcal{L}$ is in $(X-Y)^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ in Case II. Since $\mathcal{L}$ is constructed out of λ and ξ, we write $\mathcal{L}_{\lambda,\xi}$ to indicate this dependence. We now number primes in Ξ as $\Xi = \{q_1, q_2, \ldots, q_r\}$ and let ψ_1 be the restriction of ψ to $\mathbf{Z}_{q_i}^\times$. Let 2^Ξ be the set of all subsets of Ξ. For $J \in 2^\Xi$, define $\psi_J = \prod_{i\in J}\psi_i$, $\lambda_J = \lambda\otimes\psi_J^{-1}$ and $\xi_J = \xi\psi_J^{-2}$. Here $\lambda_J = \lambda\otimes\psi_J^{-1}$ is a primitive algebra homomorphism which is the twist by ψ_J^{-1} of λ defined in [**H5**, 7.8]. This process does not affect the conductor, and the conductor of λ_J is equal to that of λ, and λ_J is still minimal (cf. [**H5**, Cor.7.10], [**H3**, Lemma 10.1] and the proof of [**Sch**, Prop.5.2]). Now we write L_J for $T^{-1}H\mathcal{L}_{\lambda_J,\xi_J}$, $f_P|J$ for the cusp form belonging to λ_J at P and $\Omega_j(P)$ for the period $\Omega(P)$ for λ_J, $c_J(P_{n,\varepsilon},P)$ for the constant $c(P_{n,\varepsilon},P)$ as in Theorem 5.1 for λ_J and ξ_J and L for L_ϕ. Then we have, writing $C(\varepsilon\xi\omega^{-n+1}) = Jp^\delta$,

$$\begin{aligned} D_J^{-1}L/D_j'^{-1}L_J(P_{n,\varepsilon},P) &= \frac{\psi'\psi_J^{-2}(p)^\delta a(p,f_P|J)^{2\delta}c_\phi(P_{n,\varepsilon},P)\Omega_J(P)}{\psi'(p)^\delta a(p,f_P)^{2\delta}c_J(P_{n,\varepsilon},P)\Omega_\phi(P)} \\ &= \frac{G(\xi')W'(f_P|J)\Omega_J(P)}{G(\xi'\psi_J^{-2})W'(f_P)\Omega_\phi(P)}, \end{aligned}$$

where

$$D_J = \prod_{i \in J}(1 - \psi^2\xi^{-1}\omega^{-3}(q_i)\langle q_i\rangle^{-2}A_{q_i}(Y)^2/\lambda(T(q_i))^2A_{q_i}(X)),$$

$$D'_J = \prod_{i \in J}(1 - \xi^{-1}\omega^{-1}(q_i)\lambda(T(q_i))^2/A_{q_i}(X)).$$

Here, to simplify the right-hand side of the above formula, we have used the facts:

$$a(p, f_P|J)\psi_J(p) = a(p, f_P)$$

and

$$G(\varepsilon\xi_J\omega^{-n+1}) = G(\xi'\psi_J^{-2})G(\varepsilon\xi_1\omega^{-n+1})\varepsilon\xi_1\omega^{-n+1}(J)\xi'\psi_J^{-2}(p)^\delta.$$

Therefore, the quotient $D_J^{-1}L/D_J'^{-1}L_J$ is independent of the variable X, and hence, it is a unit in $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \otimes_{\mathbf{I}} \mathbf{K}$. For any element A, B in $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$, we write $A \approx B$ if A/B is a unit in $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I} \otimes_{\mathbf{I}} \mathbf{K}$. By the same argument, we now know that $D_J^{-1}L_I \approx D_J'^{-1}L_{I\cup J}$ if $I \cap J = \emptyset$. If P is a prime factor of D_i, then P is of the form:

$$1 + X - u^{-2}(1+Y)^2/\{\zeta\lambda_i(T(q_i))\}^{2\log(u)/\log(\langle q_i\rangle)}$$

with a root of unity ζ, and if P is a prime factor of D'_i, then P is of the form:

$$1 + X - \{\zeta\lambda_i(T(q_i))\}^{2\log(u)/\log(\langle q_i\rangle)}.$$

Thus if P is a common prime factor of D_i and D'_i, then P is of the form $(1+X)-\zeta u^{-1}(1+Y)$ for $\zeta \in \mu_{p^\infty}$. Let $\mathbf{P}$ be the set of all prime divisor of $\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ of the form:

$$(1+X) - \zeta u^{-1}(1+Y) \text{ for } \zeta \in \mu_{p^\infty},$$

Then, by $D_i^{-1}L_J \approx D_i'^{-1}L_{\{i\}\cup J}$, $D_i^{-1}L_J$ $(i \notin J)$ has only singularity at $\mathbf{P} \cup \{X - Y\}$. Then we see $D_{\{i,j\}}^{-1}L \approx D_j^{-1}D_i^{-1}L \approx D_j'^{-1}D_i^{-1}L_j$ has only singularity at $\mathbf{P} \cup \{X - Y\}$, because this fact is true for $D_i^{-1}L$ and $D_i^{-1}L_j$ and because D_j and D'_j has only common factors in $\mathbf{P}$. Now supposing that $D_J^{-1}L_I$ has only singularity at $\mathbf{P}\cup\{X-Y\}$ for all subset J with $\#(J) < n$ and all subset I disjoint from J, we shall prove that $D_{J\cup\{i\}}^{-1}L_I$ has only singularity at $\mathbf{P}$ in Case I and at $\mathrm{P}\cup\{X-Y\}$ in Case II for $i\notin J\cup I$. We know that

$$D_{J\cup\{i\}}^{-1}L_I \approx D_i^{-1}D_J^{-1}L_I \approx D_i'^{-1}D_J^{-1}L_{I\cup\{i\}} \text{ for } i\notin J\cup I.$$

Since we already know that $D_J^{-1}L_I$ and $D_J^{-1}L_{I\cup\{i\}}$ has no singularity outside $\mathbf{P}$ in Case I and $\mathbf{P}\cup\{X-Y\}$, we have the desired assertion because D_i and D_i' has common zeros only in $\mathbf{P}$. Thus $HL = D_{\Xi}^{-1}L$ has only singularity at $\mathbf{P}$ and hence by Corollary 6.3, (i), $HL\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathrm{I}$ in Case I and $HL\in(X-Y)^{-1}\Lambda\hat{\otimes}_{\mathcal{O}}\mathrm{I}$ in Case II.

Now let us prove that $HL\in\Lambda\hat{\otimes}_{\mathcal{O}}\mathbf{I}$ even in Case II unless $\xi'\psi'^{-1}$ is imaginary quadratic and λ has complex multiplication by the field corresponding to $\xi'\psi'^{-1}$. By Lemma 6.2 and Corollary 6.3, (i), we know that E is prime to $X-Y$. Thus we only needs to prove that $\mathcal{L}$ is finite at $X-Y$ under the above condition, but this has already been seen in Proposition 5.2. This finishes the proof of Theorem I.

References

[C] H. Carayol, Représentations cuspidales du groupe linéaire, Ann. Scient. Éc. Norm. Sup., 4e-serie, 17 (1984), 191-225.

[C-S] J. Coates and C.-G. Schmidt, Iwasawa theory for the symmetric square of an elliptic curve, J. Reine Angew. Math. 375/376 (1987), 104-156

[G-J] S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), Ann. Scient. Éc. Norm. Sup., 4e-serie, 11 (1978), 471-542.

[G] F. Q. Gouvêa, *Arithmetic of p-adic modular forms*, Lecture notes in Math. 1304 (1988), Springer.

[H1] H. Hida, A p-adic measure attached to the zeta functions associated with two elliptic modular forms, I, Inventiones Math. 79 (1985), 159-195; II, Ann. l'institut Fourier, Tome 38 Fascicule 3 (1988), 1-83.

[H2] H. Hida, Iwasawa modules attached to congruences of cusp forms, Ann. Scient. Ec. Norm. Sup. 4e-serie 19 (1986), 231-273.

[H3] H. Hida, Galois representations into $GL_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms, Inventiones Math. 85 (1986), 545-613.

[H4] H. Hida, On p-adic Hecke algebras for GL_2over totally real fields, Ann. of Math. 128 (1988), 295-384.

[H5] H. Hida, Modules of congruence of Hecke algebras and L-functions associated with cusp forms, Amer. J. Math. 110 (1988), 323-382.

[H6] H. Hida, Hecke algebras for GL_1 and GL_2, Sém. Th. des Nombres, Paris 1984-85, Progress in Math. 63 (1986), 131-163.

[Ki] K. Kitagawa, On p-adic L-functions of families of elliptic cusp forms, in preparation.

[K1] P. C. Kutzko, The Langlands conjecture for GL_2 of a local field, Ann. of Math. 112 (1980), 381-412.

[K2] P. C. Kutzko, The exceptional representations of GL_2, Compositio Math. 51 (1984), 3-14.

[K3] P. C. Kutzko, On the supercuspidal representations of GL_n and other p-adic groups, Proc. ICM86 (Berkeley), 853-861.

[L] S. Lang, *Cyclotomic fields*, Grad. Texts in Math. No.59, Springer 1978.

[La] R. P. Langlands, *Base Change for GL(2)*, Ann. of Math. Studies 96 (1980), Princeton University Press.

[M-T] B. Mazur and J. Tilouine, Représentations galoisiennes, différentielles de Köhler et conjectures principales, preprint.

[R] K. A. Ribet, Galois representations attached to eigenforms with nebentypus, Lecture notes in Math. 601 (1977), 17-52.

[Sch] C.-G. Schmidt, p-adic measures attached to automorphic representations of GL(3), Invent. Math. 92 (1988), 597-631.

[Sh1] G. Shimura, On modular forms of half integral weight, Ann. of Math. 97 (1973), 440-481.

[Sh2] G. Shimura, On the holomorphy of certain Dirichlet series, Proc. London Math. Soc. 31 (1975), 79-98.

[Sh3] G. Shimura, On certain reciprocity laws for theta functions and modular forms, Acta. Math. 141 (1978), 35-71.

[Sh4] G. Shimura, The critical values of certain zeta functions associated with modular forms of half integral weight, J. Math. Soc. Japan, 33 (1981), 649-672.

[Sh5] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. Pure Appl. Math. 29 (1976), 783-804.

[St1] J. Sturm, Special values of zeta functions and Eisenstein series of half integral weight, Amer. J. Math. 102 (1980), 219-240; Addendum, ibid. 781-783.

[St2] J. Sturm, Evaluation of the symmetric square at the near center point, preprint.

[W] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Ergeb. der Math. 88, 1976, Springer.

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Exterior square L-functions

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1. INTRODUCTION

Let π be an automorphic irreducible representation of $G(F_{\mathbf{A}})$, where $G = GL(n)$ and F is a number field. The corresponding L-group ${}^LG^0$ is just the group $GL(n, \mathbb{C})$; let ρ be the natural representation of degree n of ${}^LG^0$. Then $L(s, \pi, \rho)$ is the standard L-function attached to π, also noted $L(s, \pi)$, and $L(s, \pi, \rho \otimes \rho)$ the "convolution" of this L-function with itself; it is also noted $L(s, \pi \times \pi)$. In turn, the representation

$$\rho \otimes \rho$$

decomposes into the direct sum

$$\rho \otimes \rho = \sigma \oplus \tau,$$

where σ is the representation of $GL(n, \mathbb{C})$ on the space of symmetric tensors and τ the representation on the space of antisymmetric tensors. The L-function $L(s, \pi, \rho \otimes \rho)$ decomposes accordingly into a product:

$$L(s, \pi, \rho \otimes \rho) = L(s, \pi, \sigma)L(s, \pi, \tau).$$

Now suppose π is self contragredient. Then $L(s, \pi, \rho \otimes \rho)$ has a simple pole at $s = 1$ ([**J-S I**]). Thus one of the two functions on the right has a pole at $s = 1$. They cannot both have a pole, otherwise the L-function on the left would have a double pole. Furthemore, the results of Shahidi (see [**G-S**]) show that the L-functions on the right do not vanish at $s = 1$; thus if one of the two functions has a pole at $s = 1$, the L-function on the left has also a pole and π is self contragredient ([**J-S I**]).

The self contragredient representations are precisely those which are invariant under the outer automorphism $g \mapsto {}^t g^{-1}$; thus they can be studied via the twisted trace formula. One hopes then to show they are the functorial image of certains automorphic representations of classical groups.

In this paper we discuss an integral representation for the function $L(s, \pi, \tau)$, that is, the exterior square L-function. In particular, if n is even, this L-function (or rather the partial version of it) has a pole

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at $s = 1$ if and only if certain period integrals—which are residues of the integral representation—are non-zero. This is the main result of this paper (Theorem 1 in Section 8).

In more detail, the representation π is attached to an irreducible representation r of a certain large group H into ${}^LG^0$; for the purpose of this heuristic introduction, we may as well take for H the Galois group of F. Then $L(s,\pi)$ is the L-function $L(s,r)$ and similarly:

$$L(s,\pi,\tau) = L(s,\tau \circ r).$$

Now suppose that $L(s,\pi,\tau)$ has a pole at $s = 1$. Thus $L(s,\tau \circ r)$ has a pole at $s = 1$. This means that the representation $\tau \circ r$ contains the trivial representation of H, or, what amounts to the same, the image of r is contained in a conjugate of the symplectic group. This then suggests that π is the functorial image of some automorphic representation π' of the group G' whose L-group is the symplectic group.

When $n = 4$ one can use the Weil representation to carry out this idea; the group G' is then the group $GSp(4)$ and the groups G and G' form a dual reductive pair. A forthcoming paper with Piatetski-Shapiro will contain the details. See also **[So]**. The result has been used in an essential way in **[B-C-R]** I and II.

A model for an alternate approach can be found in **[J]**. There we reprove a result of Waldspurger: roughly speaking, the forms on GL(2) whose associated L-function does not vanish at one half are those in the functorial image of the correspondance with the metaplectic group. We use a form of the trace formula that we call the relative trace formula. There is evidence that a similar method will work in the present context. The advantage of such a method is that it is not limited to the case $n = 4$.

An integral representation for $L(s,\pi,\tau)$ was discussed in **[Sh]** and **[G-J]** for $n = 2$ and more recently by Patterson and Piatetski-Shapiro for $n = 3$ (**[P-P.S.]**).

The paper is arranged as follows. Section 2 contains the combinatorics needed to compute the local integral in the unramified situation. Section 3 contains elementary material on certain spaces of meromorphic functions. It is used in Section 4 to establish convenient estimates on Whittaker functions. We study a global integral in Sections 5 and 6. In particular it is shown that it is a product of local integrals which are studied in Section 7. The main theorem is then proved in section

8. Finally, in Section 9 we briefly discuss the case where n is odd. We show that the L-function is then holomorphic at $s = 1$.

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2. Local combinatorics

2.1. If B is a square matrix then

$$\det(1 - xB)^{-1} = \sum_{p\geq 0} x^p \mathrm{Tr}(\mathrm{S}^p B),$$

where $\mathrm{S}^p B$ denotes the p-th symmetric power of B. Now we fix an integer n and apply the previous identity to $B = \bigwedge^2 A$, where A is an $n \times n$ matrix and $\bigwedge^2$ means the exterior square. We have:

$$\det(1 - x\textstyle\bigwedge^2 A)^{-1} = \displaystyle\sum_{p\geq 0} x^p a_p,$$

where

$$a_p = \mathrm{Tr}(\mathrm{S}^p \textstyle\bigwedge^2 A).$$

The purpose of this section is to compute the numbers a_p. In order to do that we need to decompose the representation $\mathrm{S}^p \bigwedge^2 r$, where r is the standard representation of $GL(n, \mathbb{C})$, into a sum of irreducible representations.

To that end, we introduce the following notations: if

$$a_1 \geq a_2 \geq a_3 \ldots \geq a_n$$

is an increasing n-tuple of integers, we denote by $r(a_1, a_2, a_3, \ldots, a_n)$ the irreducible representation r' of $GL(n, \mathbb{C})$ having a vector e' such that:

$$r'(p)e' = t_1^{a_1} t_2^{a_2} t_3^{a_3} \cdots t_n^{a_n} e',$$

if p is an upper triangular matrix with eigenvalues

$$t_1, t_2, t_3, \ldots, t_n.$$

Such a vector is said to be dominant of weight $(a_1, a_2, a_3, \ldots, a_n)$.

PROPOSITION 1. *With the above notations, if n is even and $n = 2m$, then the representation of $GL(n, \mathbb{C})$ on*

$$\mathrm{S}^p\bigwedge^2 r$$

is the sum with multiplicity one of the irreducible representations

$$r(a_1, a_1, a_2, a_2, \cdots a_m, a_m)\,,$$

where

$$a_1 + a_2 + \cdots + a_m = p \text{ and } a_1 \geq a_2 \geq \cdots a_m \geq 0\,.$$

If n is odd and $n = 2m + 1$, then the representation

$$\mathrm{S}^p\bigwedge^2 r$$

is the sum with multiplicity one of the irreducible representations

$$r(a_1, a_1, a_2, a_2, \ldots, a_m, a_m, 0)\,,$$

where

$$a_1 + a_2 + \cdots + a_m = p \text{ and } a_1 \geq a_2 \geq \cdots a_m \geq 0\,.$$

Let V be the space $\mathbb{C}^n$ and consider the space $\otimes^p V$. Then $GL(n, \mathbb{C})$ operates on this space by the representation $\otimes^p r$. The symmetric group $\mathfrak{S}_p$ operates on $\otimes^p V$ by permuting the factors of a pure tensor. The two representations commute with eachother. In particular we may identify $\bigwedge^p V$ with the space of tensors v such that

$$\sigma v = \chi_p(\sigma) v\,,$$

for all σ in $\mathfrak{S}_p$, where we denote by χ_p the signature character of $\mathfrak{S}_p$. Similarly, we may identify $\mathrm{S}^p V$ to the space of tensors v such that

$$\sigma v = v\,,$$

for all σ in $\mathfrak{S}_p$.

Now the irreducible representations of $GL(n, \mathbb{C})$ contained in $\otimes^p r$ are exactly the representations of the form:

$$\pi = r(a_1, a_2, \ldots, a_n)$$

with

$$a_1 \geq a_2 \geq \cdots a_n \geq 0$$

and

$$a_1 + a_2 + \cdots + a_n = p\,.$$

We now apply these notions to the case of an even integer $2p$. We regard $\mathfrak{S}_{2p}$ as the group of permutations of the set $\{1, -1, 2, -2, \ldots, p, -p\}$ and we imbed $\mathfrak{S}_p \times \mathfrak{S}_p$ into $\mathfrak{S}_{2p}$ by the rule:

$$(\sigma, \tau)(i) = \sigma(i), \;\; (\sigma, \tau)(-i) = -\tau(i)\,.$$

We denote by $\mathfrak{S}$ the diagonal in $\mathfrak{S}_p \times \mathfrak{S}_p$; it is isomorphic to $\mathfrak{S}_p$. We denote by (i, j) the element of $\mathfrak{S}_{2p}$ which permutes i and j and leaves all other elements invariant. Then we let H_0 be the subgroup generated by the permutations

$$(1, -1),\, (2, -2), \ldots, (p, -p)\,.$$

It is isomorphic to the product

$$\underbrace{\mathfrak{S}_2 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_2}_{p}\,.$$

Clearly $\mathfrak{S}$ normalises H_0 so that $H = \mathfrak{S} H_0$ is a subgroup. Furthemore, an element (σ, σ) may be regarded as the product of the two elements $(\sigma, 1)$ and $(1, \sigma)$, which have the same signature. It follows that the signature character χ_{2p} of $\mathfrak{S}_{2p}$ is trivial on the diagonal $\mathfrak{S}$.

We index the components of a pure tensor in $\otimes^{2p} V$ by $1, -1, 2, -2, \ldots, p, -p$ and we define a linear operator P on $\otimes^{2p} V$ by

$$\begin{aligned} &P(v_1 \otimes v_{-1} \otimes v_2 \otimes v_{-2} \cdots \otimes v_p \otimes v_{-p}) \\ &\quad = \tfrac{1}{\# H} \textstyle\sum_{g \in H} \; \chi_{2r}(g) v_{g(1)} \otimes v_{g(-1)} \\ &\quad\;\; \otimes v_{g(2)} \otimes v_{g(-2)} \cdots \otimes v_{g(p)} \otimes v_{g(-p)}. \end{aligned}$$

If we set

$$w_i = \frac{1}{2}(v_i \otimes v_{-i} - v_{-i} \otimes v_i)\,,$$

then this antisymmetric tensor is an element of $\bigwedge^2 V$ and the previous element can be written as:

$$\frac{1}{\# \mathfrak{S}} \sum_{h \in \mathfrak{S}} w_{h(1)} \otimes w_{h(2)} \cdots \otimes w_{h(p)}\,,$$

so that we may view $S^p\bigwedge^2(V)$ as the range of P, or, what amounts to the same, as the space of tensors v such that

$$hv = \chi_{2p}(h)v$$

for all $h \in H$. Thus we may view

$$\bigwedge^{2p} V$$

as a subspace of:

$$S^p\bigwedge^2 V\,.$$

We denote by $e_1, e_2, \dots e_n$ the canonical basis in V. Then, for every i with $i \le s = [\frac{n}{2}]$, the vector

$$\epsilon_i = e_1 \wedge e_2 \wedge \cdots \wedge e_{2i}$$

is a dominant vector in $S^i\bigwedge^2(V)$. The vector (symmetric product)

$$\epsilon_1^{m_1}\epsilon_2^{m_2}\cdots\epsilon_i^{m_i}$$

is thus also a dominant vector in $S^p\bigwedge^2 V$, provided

$$2m_1 + 4m_2 + \cdots + 2sm_s = 2p\,.$$

Its weight has the form:

$$(m_1, m_1, 0, 0, \dots) + (m_2, m_2, m_2, m_2, 0, 0, \dots) + \cdots\,.$$

It follows that the representation

$$r(a_1, a_1, a_2, a_2 \dots, a_m, a_m)\,,$$

if $n = 2m$ and the representation

$$r(a_1, a_1, a_2, a_2 \dots, a_m, a_m, 0)\,,$$

if $n = 2m + 1$, where

$$a_1 \ge a_2 \ge \cdots a_m \ge 0 \text{ and } a_1 + a_2 + \cdots a_m = p\,,$$

is contained in

$$S^p\bigwedge^2 V\,.$$

To complete the proof we use an argument communicated to us by S. Rallis. Let $\mathrm{Sym}\bigwedge^2 V$ the symmetric algebra of $\bigwedge^2 V$. Thus $GL(n,\mathbb{C})$ operates on this algebra and we have proved that the irreducible representations

$$r(a_1,a_1,a_2,a_2\dots,a_m,a_m)\,,$$

or

$$r(a_1,a_1,a_2,a_2\dots,a_m,a_m,0)\,,$$

where

$$a_1\geq a_2\geq\cdots a_m\geq 0$$

occur. We have to show no other representation occurs and the multiplicity is one. To that end we let T be the vector space of skew symmetric forms on V. We regard $GL(n,\mathbb{C})$ as acting on the right on T. We may view $\mathrm{Sym}\bigwedge^2 V$ as the algebra $\mathbb{C}[T]$ of polynomial functions on T. Let Ω be the Zariski open set formed by the skew bilinear forms of maximal rank. Then $GL(n,\mathbb{C})$ is transitive on Ω. Thus if u is any form of maximal rank and H its isotropy group in $G=GL(n,\mathbb{C})$, we may identify the algebraic varieties Ω and $H\backslash G$. In particular, the restriction from T to Ω allows us to identify $\mathbb{C}[T]$ with a subalgebra of $\mathbb{C}[\Omega]$ or $\mathbb{C}[H\backslash G]$.

In addition, if n is even, then Ω is the affine open set where the discriminant Δ (with respect to a basis of V) does not vanish. Thus we may identify $\mathbb{C}[H\backslash G]$ with the localization of $\mathbb{C}[T]$ at Δ. In this identification, the discriminant becomes the square of the determinant. It follows that all representations of the form

$$r(a_1,a_1,a_2,a_2\dots,a_m,a_m)\,,$$

where

$$a_1\geq a_2\geq\cdots a_m\,,$$

occur in the representation of G (by right shifts) on $\mathbb{C}[H\backslash G]$. If n is odd, then all representations of the form

$$r(a_1,a_1,a_2,a_2\dots,a_m,a_m,0)\,,$$

where

$$a_1\geq a_2\geq\cdots a_m\geq 0$$

occur in the representation of G on $\mathbb{C}[H\backslash G]$. It will suffice to show no other representation occur and the multiplicity is one. To that end we choose for u the form with the following matrix (with respect to the canonical basis). If $n = 2m$ the matrix has the form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

If $n = 2m + 1$ the matrix has the form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

Let B be the group of triangular matrices in G and A the group of diagonal matrices. Then HB is a Zariski open set in G and the intersection of H and A consists of all diagonal matrices of the form

$$\operatorname{diag}(t_1, t_1^{-1}, t_2, t_2^{-1}, \dots, t_m, t_m^{-1}),$$

if n is even and of the form

$$\operatorname{diag}(t_1, t_1^{-1}, t_2, t_2^{-1}, \dots, t_m, t_m^{-1}, t_{m+1}),$$

if m is odd. Now suppose that π is an irreducible represenation of g which occurs in

$$\mathbb{C}[H\backslash G].$$

Then evaluation on H provides us with a non zero linear form v on the space of π invariant under H. Let e be the dominant vector in the space of π. Then the matrix coefficient $(\pi(g)e, v)$ is determined by its values on HB thus by (e, v). This already shows that v is unique, that is, the multiplicity is one. Also $(\pi(a)e, v) = (e, v) \neq 0$ for $a \in H \cap A$. This forces the dominant weight to have the required form and we are done.

2.2. We now transcribe the previous results in terms of local integrals. Accordingly, we let F be a local, non-archimedean field, R its ring of integers, q the cardinality of its residual field, ω a uniformizer, and ψ an additive character of conductor R. Let G_n be the group $GL(n)$, regarded as an algebraic group, and K_n the compact group $GL(n,R)$. Let π be an irreducible representation of $GL(n,F)$ with a K_n-fixed vector. Then to π is associated a certain conjugacy class A in $GL(n,\mathbb{C})$, its Langlands class. By definition:

$$L(s,\pi,\bigwedge^2 r) = \det(1 - x\bigwedge^2 A)^{-1}\,,$$

where $x = q^{-s}$ and q is the cardinality of the residual field of F.

Let A_n be the group of diagonal matrices in G_n, N_n the group of upper triangular matrices with diagonal entries equal to one and $B_n = A_n N_n$ the group of upper triangular matrices. Let ψ be an additive character of F whose conductor is R. Define a character θ of N_n by the formula:

$$\theta(u) = \prod \psi(u_{j,j+1})\,.$$

Assume π is generic. Then the Whittaker model $\mathcal{W}(\pi,\psi)$ is the unique space of functions transforming on the left under θ, invariant under right shifts, the representation of $G_n(F)$ on $\mathcal{W}(\pi,\psi)$ being equivalent to π. It contains a unique vector W taking the value 1 on K_n. The value of W on the diagonal matrix with eigenvalues

$$\omega^{a_1}, \omega^{a_2}, \dots, \omega^{a_n}\,,$$

where

$$a_1 \geq a_2 \geq \dots \geq a_n$$

is

$$\delta_n^{1/2}(a)\mathrm{Tr}(r(a_1,a_2,...a_n)(A))\,,$$

where δ_n is the module of B_n. If the above inequality is not satisfied then the value of W on the diagonal matrix is 0 (see **[C-S]** for instance).

Say n is even with $n = 2m$. By Proposition 1, we have:

$$L(s,\pi,\bigwedge^2 r) \\ = \sum_{p\geq 0} x^m \sum_{a_1+a_2+\cdots+a_c=p} \mathrm{Tr}(r(a_1,a_1,a_2,a_2,\dots a_m,a_m)(A))\,,$$

where it is understood that the sum is for

$$a_1 \geq a_2 \geq \ldots \geq a_m \geq 0 .$$

This can also be written as

$$\sum_{a_1 \geq a_2 \geq \ldots \geq a_m \geq 0} x^{a_1} x^{a_2} \ldots x^{a_m} \mathrm{Tr}(r(a_1, a_1, a_2, a_2, \ldots, a_m, a_m)(A)) .$$

By the results we have just recalled, this can be written as an integral of W, namely:

$$L(s, \pi, \textstyle\bigwedge^2 r) =$$

$$\int_{|b_m| \leq 1} W \delta_n^{-1/2} \begin{pmatrix} b_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & b_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & b_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & b_2 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & b_m & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & b_m \end{pmatrix}$$

$$| b_1 b_2 \ldots b_m |^s \, d^\times b_1 d^\times b_2 \ldots d^\times b_m .$$

If n is odd and $n = 2m + 1$ the formula reads:

$$L(s, \pi, \textstyle\bigwedge^2 r) =$$

$$\int W \delta_n^{-1/2} \begin{pmatrix} b_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & \ldots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \ldots & b_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & b_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$| b_1 b_2 \ldots b_m |^s \, d^\times b_1 d^\times b_2 \ldots d^\times b_m .$$

3. Mellin transforms

In this section F is a local field and we review elementary facts about Mellin transforms (see also [I] Chapter I). A finite function, on an locally compact abelian group, is a continuous function whose translates span a finite dimensional vector space. For instance, the finite functions on $F^\times$ are the finite linear combinations of functions of the form

$$f(x) = \chi(x) \mid x \mid^u (\log \mid x \mid)^n ,$$

where χ is a character of module 1, u is real and $n \geq 0$ is an integer.

3.1. We first discuss the archimedean case. We set $\mid x \mid_F = \mid x \mid$ if $F = \mathbb{R}$ and $\mid x \mid_F = x\overline{x}$ if $F = \mathbb{C}$. Whenever convenient, we also write:

$$\alpha_F(x) = \mid x \mid_F .$$

We also set $\chi_0(x) = x \mid x \mid^{-1}$ if $F = \mathbb{R}$ and $\chi_0(x) = x(x\overline{x})^{-1/2}$ if $F = \mathbb{C}$. Then any character of $F^\times$ has the form:

$$\chi(x) = \mid x \mid_F^s \chi_0^m(x) ,$$

where s is complex and m is an integer. In the real case we may take m to be 0 or 1, or to be an integer modulo 2. We often write $\chi = (s, m)$ and $\Re(\chi) = \Re(s)$. The set of characters is thus a complex manifold of dimension one, with two connected components in the real case, and infinitely many in the complex case.

The Mellin transform of a function f on $F^\times$ is defined by the integral

$$\hat{f}(\chi) = \int f(a)\chi(a) d^\times a .$$

If n is an integer, we also define:

$$\hat{f}(\chi; n) = \int f(a)\chi(a)(\log \mid a \mid_F)^n d^\times a .$$

If $\chi = (s, m)$ we also write $\hat{f}(s, m)$ for $\hat{f}(\chi)$ and $\hat{f}(s, m; n)$ for $\hat{f}(\chi; n)$.

3.2. For convenience we recall the formal properties of the Mellin transform. In the real case, we have:

$$\text{if } f(a) = ag(a) \text{ then } \hat{f}(s, m; n) = \hat{g}(s + 1, m + 1; n) .$$

In the complex case, we have the following formulas:

$$\text{if } f(a) = ag(a) \text{ then } \hat{f}(s,m;n) = \hat{g}(s+\frac{1}{2}, m+1; n);$$

$$\text{if } f(a) = \overline{a}g(a) \text{ then } \hat{f}(s,m;n) = \hat{g}(s+\frac{1}{2}, m-1; n).$$

We also introduce the Euler operator:

$$Df(a) = \frac{d}{dt} f(ae^t)\Big|_{t=0}.$$

In the real case we have

$$Df(a) = a\frac{df}{da},$$

and:

$$\widehat{Df}(s,m;n) = -s\hat{f}(s,m;n) - n\hat{f}(s,m;n-1).$$

In the complex case we have

$$Df(a) = a\frac{\partial f}{\partial a} + \overline{a}\frac{\partial f}{\partial \overline{a}},$$

and

$$\widehat{Df}(s,m;n) = -2s\hat{f}(s,m;n) - 2n\hat{f}(s,m;n-1).$$

In the real case the Lie algebra of $F^\times$ is generated by the differential operator D. In the complex case, the Lie algebra of $F^\times$ (regarded as a real Lie group) is generated by the operator D and by another operator R, defined by:

$$Rf(a) = \frac{1}{i}\frac{d}{dt} f(ae^{it})\Big|_{t=0}.$$

We have:

$$Rf = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial \overline{a}},$$

and

$$\widehat{Rf}\,(s,m;n) = -m\hat{f}(s,m;n).$$

3.3. We now review the analytic properties of the Mellin transform. Let ϕ be a Schwartz function on F. Suppose F is real. Then $\hat{\phi}(s,m;n)$ has poles of order (at most) n at the points $-2k-m$, where $k \geq 0$ is an integer. Furthermore, the coefficient of

$$\frac{1}{(s+2k+m)^n}$$

in the Laurent expansion at $-2k-m$ is proportional to

$$\frac{d^{2k+m}\phi}{da^{2k+m}}(0)\,.$$

Now suppose F is complex. Then $\hat{\phi}(s,m;n)$ has poles of order n at the points $-\frac{|m|}{2}-k$, where $k \geq 0$ is an integer. Furthermore, the coefficient of

$$\frac{1}{(s+\frac{|m|}{2}+k)^n}$$

in the Laurent expansion at $-\frac{|m|}{2}-k$ is proportional to

$$\frac{\partial^{p+q}\phi}{\partial a^p \partial \overline{a}^q}(0)\,,$$

where

$$p = k + m^- \,, q = k + m^+\,.$$

In particular, for a, b given, there are only finitely many m for which the functions $\hat{f}(s,m;n)$ has a pole in the strip $a \leq \Re s \leq b$.

The function $\hat{f}(\chi;n)$ decays rapidly in any vertical strip. In a precise way, let $P(\chi)$ be a function on the set of characters of the following form:

$$P(\chi) = P_m(s) \text{ if } \chi = (s,m)\,,$$

where P_m is a polynomial of degree $\leq d$ and d is independent of m. Choose a norm on the vector space of polynomials of degree $\leq p$. Suppose furthermore that the norm of P_m is $0(m^N)$ and that $P(\chi)\hat{f}(\chi,n)$ has no pole for $a \leq \Re\chi \leq b$. Then there is a constant C such that

$$\mid P(\chi)\hat{f}(\chi;n) \mid \leq C \text{ for } a \leq \Re\chi \leq b\,.$$

Indeed our assertion is clear for $a > 0$ and $P = 1$, for the integral defining f is then absolutely convergent. The general case follows from the formal properties of the Mellin transform.

3.4. We will need an extension of the previous remarks. Recall the standard L-function $L(\chi)$. Up to exponential factors it is given by:

$$L(\chi) = L(s,m) = \Gamma(\frac{s+1+m}{2}),$$

if F is real, $\chi = (s,m)$ with $m = 0,1$. If F is complex and $\chi = (s,m)$, up to exponential factors, it is given by

$$L(\chi) = L(s,m) = \Gamma(s + \frac{|m|}{2}).$$

For any finite-dimensional semi-simple representation σ of $F^\times$ we can define a function $L(\chi,\sigma)$ on the set of characters by the rule:

$$L(\chi, 0) = 1,$$

and

$$L(\chi,\pi) = L(\chi\zeta)L(\chi,\tau) \text{ if } \sigma = \zeta \oplus \tau,$$

where ζ is one-dimensional. We will define a certain space $\mathcal{M}(\sigma)$ of meromorphic functions on the set of characters of $F^\times$. These functions have the form

$$M(\chi) = L(\chi,\sigma)h(\chi)$$

where h is an entire function on the set of characters. They are to satisfy a certain growth condition which we now explain. Suppose $a < b$ are given. Let $P(\chi)$ be a function of the form

$$P(\chi) = P_m(s),$$

where P_m is a polynomial of degree $\leq d$. Suppose that the norm of P_m in the space of polynomials of degree $\leq d$ is $O(m^N)$. Finally suppose that $P(\chi)L(\chi,\sigma)$ has no pole in the strip $a \leq \Re\chi \leq b$. Then the product $P(\chi)M(\chi)$ is bounded in $a \leq \Re\chi \leq b$. Note that the function $L(\chi,\sigma)$ belongs to $\mathcal{M}(\sigma)$.

Now suppose that X is a finite set of finite functions on $F^\times$. Then there is a representation σ of $F^\times$ with the following property: consider a function of the form:

$$f(a) = \sum_{\xi \in X} \phi_\xi(a)\xi(a)$$

where each ϕ_ξ is a Schwartz function on F; then the Mellin transform of f belongs to $\mathcal{M}(\sigma)$.

3.5. We are going to prove the converse. To that end we introduce a space $\mathcal{S}(F^\times)$ of functions on $F^\times$. We set:

$$\| a \| = | a |^{-1/2} + | a |^{1/2},$$

where $| a |$ means the usual absolute value. Then the elements of our space are the indefinitely differentiable functions f on $F^\times$ such that for any p, q, N the product

$$\| a \|^N \frac{\partial^{p+q} f}{\partial a^p \partial \overline{a}^q}(a)$$

is uniformly bounded. We can view $\mathcal{S}(F^\times)$ as a subspace of the Schwartz space $\mathcal{S}(F)$; it is then the subspace of Schwartz functions which are divisible by any power of a or $\overline{a}$.

PROPOSITION 1. *If σ is zero-dimensional then $\mathcal{M}(\sigma)$ is the space of the Mellin transforms of the functions in $\mathcal{S}(F^\times)$. If σ has positive dimension, then there is a finite set X of finite functions on $F^\times$ such that any M in $\mathcal{M}(\sigma)$ is the Mellin transform of a function f of the form:*

$$f(a) = \sum_{\xi \in X} \phi_\xi(a) \xi(a),$$

where each ϕ_ξ is a Schwartz function on F.

Consider first the case of a zero-dimensional representation. A function M in the space is then entire and the Mellin transform of a bounded continuous function f on $F^\times$. The product of f by any power of a, positive or not, has the same property. Also, by the formal properties of the Mellin transform, f is differentiable on $F^\times$ and Rf and Df have the same properties as f. It follows that f is in $\mathcal{S}(F^\times)$. Note that given any character χ we may write f in the form $f = \phi\chi$ where ϕ is a Schwartz function.

Now we prove the proposition by induction on the degree d of σ. We have proved our assertion for $d = 0$. We assume it is true for a representation of degree less than d and prove it for a representation σ of degree d. We write $\sigma = \chi_0 \oplus \tau$ where χ_0 is one-dimensional. At a pole of $L(\chi\chi_0)$ a function M in the space has a pole of order at most d. We claim there is a sequence of smooth functions of compact support on F, say ϕ_j with $0 \le j \le d$, such that the Mellin transform $\hat{f}(\chi)$ of the function

$$f(a) = \sum_j \phi_j(a) \chi_0(a) (\log | a |_F)^j$$

has the same polar part as M, at all poles of $L(\chi\chi_0)$. Indeed, by the analytic properties of the Mellin transform, the required properties of $\hat{f}$ are equivalent to a triangular sytem of linear equations for the derivatives of the functions ϕ_j at zero. However, by the Borel lemma, these derivatives are arbitrary. The existence of the functions follows. Now the difference $M - \hat{f}(\chi)$ is then in $L(\tau)$ and our assertion follows from the induction hypothesis (and the remark that the functions in $\mathcal{S}(F^\times)$ can be written in the above form for any X).

3.6. We will need a complement to the proposition. We will write $\Re\sigma > a$ if all the one-dimensional components χ of σ satisfy $\Re\chi > a$.

PROPOSITION 2. *Suppose that the Mellin transform of a function f is in $\mathcal{M}(\sigma)$. Suppose furthemore that $\hat{f}(\chi)$ is holomorphic for $\Re s \geq 0$. Then $\hat{f}(\chi)$ is in some space $\mathcal{M}(\tau)$ with $\Re\tau > 0$. Furthemore, we can write f in the form*

$$f(a) = \sum_{j,n} \phi_{j,n}(a)\chi_j(a)\log(\mid a \mid)_F^n\,,$$

where the sum is finite and $\Re\chi_j > 0$.

It is easily verified that there is τ with $\Re\tau > 0$ such that

$$\hat{f}(\chi) = L(\chi,\tau)h(\chi)\,,$$

where h is entire. Let us check that $\hat{f}$ belongs to $\mathcal{M}(\tau)$. We only need to verify the required growth condition is satisfied. Accordingly, let us consider $a, b, P(\chi)$ satisfying the conditons of 1.4. for the representation τ. In particular $P(\chi)L(\chi,\tau)$ has no pole in the strip $a \leq \Re\chi \leq b$. Let $Q(\chi)$ be a polynomial function of χ, equal to one in almost all connected components, such that the product $P(\chi)Q(\chi)L(\chi,\sigma)$ is holomorphic in the same strip. Then the product

$$\hat{f}(\chi)P(\chi)Q(\chi)$$

is uniformly bounded in the same strip. At the cost of slightly enlarging the strip, we may assume that $L(\chi,\tau)$ and $L(\chi,\sigma)$ have no pole on the boundary of the strip. Choose $C > 0$ so large that $Q(\chi)$ has no zero for $\Im\chi \geq\mid C \mid$. Then $Q(\chi)$ is bounded below in the following three regions

$$\Im\chi \geq\mid C \mid,$$

$$\Re\chi = a\,,$$
$$\Re\chi = b\,.$$

Thus the product

$$\hat{f}(\chi)P(\chi)$$

is uniformly bounded above in the same regions. By the maximum principle, it is then uniformly bounded in the whole strip.

The remaining assertion of the proof is a consequence of the proof of the previous proposition.

3.7. We now discuss a multivariate generalization. Accordingly, let r be an integer and $\sigma_1, \sigma_2, \ldots, \sigma_r$ a sequence of representations of $F^\times$. We define a space of meromorphic functions M in the variables $\chi_1, \chi_2, \ldots, \chi_r$. They have the form:

$$\prod_j L(\chi_j, \sigma_j) h(\chi_1, \chi_2, \ldots, \chi_r)\,,$$

where h is entire. They satisfy a condition of decay in a multi-strip which we now describe. For $1 \leq j \leq r$, let $a_j < b_j$ and let P_j be a function on the set of characters which satisfies the condtions of section 1.4 with respect to a_j, b_j, σ_j. Then the product

$$MP_1P_2\cdots P_r$$

is bounded in the multistrip:

$$a_j \leq \Re\chi_j \leq b_j\,.$$

We denote this space by $\mathcal{M}(\sigma_1, \sigma_2, \cdots \sigma_r)$.

We will consider the multivariate Mellin transform of a function f on $(F^\times)^r$. It will be noted

$$\hat{f}(\chi_1, \chi_2, \ldots, \chi_r)\,.$$

Suppose that X is a finite set of finite functions on $(F^\times)^r$. Then we can choose finite-dimensional representations σ_j, $1 \leq j \leq r$, of $F^\times$ with the following property. Consider a function of the form:

$$f = \sum_{\xi \in X} \phi_\xi \xi\,,$$

where the functions ϕ_ξ are Schwartz functions on F^r. Then the Mellin transform of f is in $\mathcal{M}(\sigma_1, \sigma_2, \cdots \sigma_r)$.

3.8. We wish to prove the converse:

PROPOSITION 3. *Given representations $\sigma_1, \sigma_2, \cdots \sigma_r$ of $F^\times$ there is a finite set X of finite functions on $(F^\times)^r$ such that any element of $\mathcal{M}(\sigma_1, \sigma_2, \cdots \sigma_r)$ is the Mellin transform of a function f of the form:*

$$f = \sum_{\xi \in X} \phi_\xi \xi \,,$$

where the functions ϕ_ξ are Schwartz functions on F^r.

In order to prove the proposition conveniently we introduce certain spaces of functions. If G is an algebraic linear group, we define a norm on G to be any function of the form $\| g \| + \| g \|^{-1}$ where $\| g \|$ is the norm in a faithful finite dimensional representation. We will denote by $\| g \|$ such a function. Following Casselman ([**C**]) we will denote by $\mathcal{S}(G)$ the space of indefinitely differentiable functions f on $G(F)$ such that, for any left-invariant differential operator D on $G(F)$ and any N, the product

$$\| g \|^N \, (Df)(g)$$

is bounded. We remark that in the definition we may replace left-invariant operators by right-invariant differential operators. We will apply these notions to the abelian groups $(F^\times)^j \times F^{r-j}$. If $j = 0$ the space we just defined is simply the Schwartz space $\mathcal{S}(F^r)$. In general the space $\mathcal{S}((F^\times)^j \times F^{r-j})$ may be viewed as a subspace of $\mathcal{S}(F^r)$.

The proposition will be a consequence of the following more precise proposition:

PROPOSITION 4. *Suppose $\sigma_1, \sigma_2, \cdots \sigma_r$ is a sequence of representations of $F^\times$; suppose further that the first j representations are zero-dimensional and the others are not. Then there is a finite set X of finite functions on F^{r-j} such that any element of $\mathcal{M}(\sigma_1, \sigma_2, \cdots \sigma_r)$ is the Mellin transform of a function f of the form:*

$$f = \sum_{\xi \in X} \phi_\xi \xi \,,$$

where the functions ξ are regarded as functions on $(F^\times)^r$ depending only on the last $r - j$ variables, and the ϕ_ξ are in

$$\mathcal{S}((F^\times)^j \times F^{r-j}) \,.$$

The proof is by induction on r. We have proved the assertion of the proposition for $r = 1$. We assume $r > 1$ and the assertion true for $r - 1$ and prove it for r. Next, we remark that the assertion of the proposition is elementary for $j = r$. We therefore assume $j \leq r$ and the assertion true for j and we prove it for $j - 1$. Let Y be a finite set associated with $\mathcal{M}(0, 0, \ldots, 0, \sigma_{j+1}, \cdots \sigma_r)$. A function f on $(F^\times)^r$ will be written as a function of two variables $f(a_j, b)$ where b is in $(F^\times)^{r-1}$. Similarly, a function M of the set or r-tuples of characters will be written as $M(\chi_j, \rho)$, where ρ is an $(r-1)$-tuple of characters. Finally, the functions η in Y will be viewed as functions $\eta(b)$ depending only the last $r - j$ coordinates.

We shall first prove there is a finite set Z of finite functions on $F^\times$ such that for any M in $\mathcal{M}(\sigma_1, \sigma_2, \cdots \sigma_r)$, there exist functions $\phi_{\zeta,\eta}$ in $\mathcal{S}(F^\times j - 1 \times F^{r-j+1})$ with $\zeta \in Z$ and $\eta \in Y$, such that the difference between M and the Mellin transform of

$$\sum_{\zeta,\eta} \phi_{\zeta,\eta}(a_j, b)\zeta(a_j)\eta(b)$$

has no singular hyperplane of the form $\chi_j = (z, p)$. To that end we consider, as before, a one-dimensional representation χ_0 contained in σ_j. Then the poles of $L(\chi_j, \sigma_j)$ which are poles of $L(\chi_j \chi_0)$ have order $\leq d$. We claim there is a function f of the form

$$f(a_j, b) = \sum_{0 \leq j \leq d, \eta \in Y} \phi_{j,\eta}(a_j, b)\chi_0(a_j)(\log \mid a_j \mid_F)^j \eta(b)\,,$$

with $\phi_{j,\eta}$ as above, such that the difference between M and the Mellin transform of f has no singular hyperplane of the form $\chi_j = (z, p)$, where (z, p) is a pole of $L(\chi_j \chi_0)$.

Let us show the existence of such an f, say in the complex case. After a translation, we may assume $\chi_0 = 1$. Consider a pole (z, t) of $L(\chi_j)$. Thus $z = -k- \mid t \mid /2$ where $k \geq 0$ is an integer. For $\chi_j = (s, t)$ we claim we can write:

$$M(\chi_j, \rho) = \sum_{1 \leq j \leq d} \frac{h_j(\rho)}{(s - z)^j} + g(\chi_j, \rho)\,;$$

here g is a meromorphic function of the form

$$\prod_{k \geq j} L(\chi_k, \sigma_k) h\,,$$

where h is entire but g does not have the hyperplane $\chi_j = (z, p)$ as a singular hyperplane; the h_j are in

$$\mathcal{M}(\underbrace{0, \ldots, 0}_{j-1}, \sigma_{j+1}, \ldots, \sigma_r).$$

The Weirstrass division theorem shows the existence of g and the h_j, except for the growth condition that the h_j have to satisfy. However the h_j may be obtained as residues, thus by integration over a compact path, and that can be used to prove the required estimates. Of course, outside the connected component defined by the integer t the formula is to be interpreted as saying that $g = M$. By the induction hypothesis for $r-1$, each h_j is the Mellin transform of a function of the form:

$$\sum_{\eta \in Y} \psi_{j,\eta}(b)\eta(b),$$

where each function $\psi_{j,\eta}$ is in the space

$$\mathcal{S}((F^\times)^{j-1}) \times F^{r-j}).$$

The condition required of the $\phi_{j,\eta}$ is simply that the derivatives

$$\frac{\partial^{p+q}\phi_{j,\eta}}{\partial^p a \partial^q \overline{a}}(0, b),$$

where

$$p = k + t^- \ , \ q = k + t^+ ,$$

satisfy a triangular system of linear equations, with right hand side expressed in terms of the $\psi_{j,\eta}$. The relative Borel lemma shows that the system can be solved.

Repeating the argument for each character contained in σ_j, we can prove there is a set X of finite functions on $(F^\times)^{r-j+1}$ and a function f of the form prescribed by the proposition such that the difference $M - \hat{f}$ has no singular hyperplane of the form $\chi_j = (z, p)$. The difference $M - \hat{f}$ is then in some space

$$\mathcal{M}(\underbrace{0, \ldots, 0}_{j}, \sigma'_{j+1}, \ldots, \sigma'_r),$$

where the representations σ'_j are determined by the finite sets Y which appear in the proof. Indeed it is clear these differences have the form

$$\prod_{k>j} L(\chi_k, \sigma'_k)h\,,$$

where h is entire. To prove they satisfy the required growth condition we use the maximum principle (in one variable) just as in the proof of Proposition 2. Finally we may apply the induction hypothesis for j to the differences; thus they have the form:

$$\sum_{\zeta} \phi_\zeta(a)\zeta(a)$$

where the ϕ_ζ are in $\mathcal{S}((F^\times)^j \times F^{r-j})$ and the $\zeta(a)$ are finite functions depending only on the $r-j$ last variables. To conclude we simply remark that every function ϕ_ζ is divisible by an arbitrary character depending only on a_j.

3.9. Just as in the case $r = 1$, we can prove that if the Mellin tranform of f is in one of the spaces $\mathcal{M}$ and this Mellin transform is holomorphic in the multi-half-plane defined by $\Re\chi_j \geq 0$ for all j, then the Mellin transform of f is in a space $\mathcal{M}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ with $\Re\sigma_j > 0$ for each j. Furthermore, we can choose the finite functions in the expression for f to be products of logarithmic terms with characters whose real parts are positive. In particular, f is then square-integrable on the group $(F^\times)^r$.

We will prove a (partial) converse:

PROPOSITION 5. *Suppose that the Mellin transform of f is in one of the spaces $\mathcal{M}$. Suppose further that f (resp. the product of f and any polynomial in $a_j, \overline{a}_j$ if F is archimedean) is square-integrable on the group $(F^\times)^r$. Then $\hat{f}$ is holomorphic in the multi-half-plane defined by the inequalities:*

$$\Re\chi_j \geq 0 \text{ for all } j\,.$$

For simplicity, let us prove the proposition in the complex case. We set

$$g = f\overline{f}\,.$$

Then g has an expression of the form:

$$g = \sum_{\xi \in Y} \phi_\xi \xi \,,$$

where Y is a suitable set of finite functions and the ϕ_ξ are Schwartz functions. In particular, $\hat{g}$ belongs to another space $\mathcal{M}$. We choose an integer $x \geq 0$, larger than the opposite of the real part of the characters appearing in Y. Then $\hat{g}$ is holomorphic in the product of the half-planes $\Re\chi_j > x$. We claim that the integral defining the Mellin transform of g converges absolutely in the product of the following half-planes:

$$\Re\chi_1 \geq 0\,,\ \Re\chi_j > x \text{ for } j \neq 1\,,$$

uniformly in any multistrip of finite width. To see that we decompose the integral into the sum of two integrals, one over the set $\mid a_j \mid > 1$ and the other over the set $\mid a_j \mid < 1$. The first integral is finite because of the choice of x and the decomposition of g. In the second integral, the factor $\chi_j(a_j)$ is bounded by one. The second integral is thus finite, because the product of f by any polynomial is square integrable. Since $\hat{g}$ is in some space $\mathcal{M}$, this implies that $\hat{g}$ is actually holomorphic in a product of half-planes of the form

$$\Re\chi_1 > -c\,,\ \Re\chi_j > x \text{ for } j \neq 1\,,$$

with $c > 0$. It follows from Hartogs'lemma that $\hat{g}$ is actually holomorphic in a product of half-planes of the form:

$$\Re\chi_j > -c \text{ for all } j\,,$$

with $c > 0$. We will show that $\hat{f}(\chi)$ is holomorphic in the product of the half-planes defined by

$$\Re\chi_j > -\frac{c}{2}\,,$$

which will prove the proposition.

To that end, we fix a character η of module one, we set

$$u = (u_j)\,,$$

and we consider the function

$$f_u(a) = f(a) \prod_j \mid a_j \mid_F^{u_j} \eta(a)\,.$$

Suppose x is larger than the opposite of the real part of the characters appearing in the decomposition of f. Then for $\Re u_j > x$ and μ of module one, $\widehat{f_u}(\mu)$ is defined and is a square-integrable function on the set of characters of module one. Let us denote it by h_u. Let $\mathcal{H}$ be the Hilbert space of the square integrable functions on the dual group of $(F^\times)^r$. Then $u \mapsto h_u$ is an holomorphic function of u, in the product of the half planes $\Re u_j > x$, with values in $\mathcal{H}$. It will suffice to show it extends to an holomorphic function, with values in the same space, in the product of the half planes $\Re u_j > -\frac{c}{2}$ For then, by a lemma of Warner and Osborn ([**O.W.**], Theorem p.113) the Mellin transform $\hat{f}(\alpha^{u_j}\eta_j)$ will be holomorphic (as a scalar function), in the same domain.

At this point let us assume $r = 2$. The general case is only notationally more difficult. Let a and b be complex numbers with $\Re a > x$ and $\Re b > x$. Let P and Q be the largest discs of center a and b contained in the half-plane $\Re s > -\frac{c}{2}$. Consider the scalar product $B(u_1, u_2 : v_1, v_2) = (h_u, h_v)$. A priori, it is defined for

$$\Re u_j > x \ , \ \Re v_j > x \, .$$

However, it can expressed in terms of the Mellin tranform of g and has thus an analytic continuation to the product of the half-planes

$$\Re u_j > -\frac{c}{2} \ , \ \Re v_j > -\frac{c}{2} \, .$$

In the polydisc $P \times P \times Q \times Q$ we have therefore a convergent power series:

$$\sum_{n,m} \frac{(u_1 - a)^i \overline{(v_2 - a)}^{\,j} (u_1 - b)^k \overline{(v_2 - a)}^{\,l}}{i!j!k!l!}$$
$$\frac{\partial^{i+j+k+l}}{\partial u_1^i \partial \overline{v_1}^{\,j} \partial u_2^k \partial \overline{v_2}^{\,l}} B \Big|_{u_1 = a\ ,\ v_1 = b\ , u_2 = a\ ,\ v_2 = b} \, .$$

A fortiori, the series

$$\sum_{n,m} \frac{|(u_1 - a)^n (u_2 - b)^m|^2}{n!^2 m!^2} \ \| \ \frac{\partial^{n+m}}{\partial u_1^n \partial u_2^m} \ h_u|_{u_1 = a\ ,\ u_2 = b} \ \|^2$$

converges absolutely in $P \times Q$. Then the series

$$\sum_{n,m} \frac{u_1^n u_2^m}{n!m!} \frac{\partial^{n+m}}{\partial u_1^n \partial u_2^m} \ h_u|_{u_1 = a, u_2 = b}$$

converges absolutely, in the space $\mathcal{H}$, for (u_1, u_2) in $P \times Q$. This does imply that h_u has an holomorphic extension to the product of the half planes $\Re u_1 > -\frac{c}{2}$ and $\Re u_2 > -\frac{c}{2}$. So we are done.

3.10. Let us say a representation τ of $F^\times$ is generic if it has form:

$$\tau = \bigoplus n_j \chi_j \,,$$

where the functions $L(\chi\chi_j)$ have no common pole. If τ is generic, let X be the set of finite functions of the form $\chi_j(a)\log(\mid a \mid_F)^k$ with $0 \leq k \leq n_j$. Let also $\mathcal{K}_X$ be the space of functions of the form:

$$\sum_{\xi \in X} \phi_\xi \xi \,,$$

where the ϕ_ξ are Schwartz functions. Then the Mellin transform defines a bijection of $\mathcal{K}_X$ onto $\mathcal{M}(\tau)$.

Let $\tau_1, \tau_2, \ldots, \tau_r$ be a sequence of generic representations and X_j the corresponding sets of finite functions. Let X be the functions on $(F^\times)^r$ which are products of functions in the sets X_j. Finally, let $\mathcal{K}_X$ be the set of functions of the form:

$$\sum_{\xi \in X} \phi_\xi \xi \,,$$

where the ϕ_ξ are Schwartz functions. Then the Mellin transform defines a bijection of $\mathcal{K}_X$ onto the space

$$\mathcal{M}(\tau_1, \tau_2, \ldots, \tau_r) \,.$$

The space $\mathcal{K}_X$ is a quotient of a direct sum of finitely many copies of the Schwartz-space

$$\mathcal{S}(F^r) \,.$$

As such, it has a natural topology. The space $\mathcal{M}(\tau_1, \tau_2, \ldots, \tau_r)$ has also a natural toplogy, imposed by the condition of growth at infinity. The closed graph theorem shows the bijection is an homeomorphism. We also remark that, in the decomposition:

$$f = \sum_{\xi \in X} \phi_\xi \xi \,,$$

the functions ϕ_ξ may be chosen to depend continuously on f. More precisely, the space $\mathcal{K}_X$ is a closed direct factor of a direct sum of copies of the Schwartz-space $\mathcal{S}(F^r)$.

We remark that if $L(\chi\chi_1)$ and $L(\chi\chi_2)$ have a common pole, then one of the two functions is equal to the product of the other by an entire function. It follows that, given σ, there is a generic representation τ such that $L(\chi,\sigma) = L(\chi,\tau)h(\chi)$ where h is entire. Then $\mathcal{M}(\sigma)$ is contained in $\mathcal{M}(\tau)$. Similarly, given $\sigma_1, \sigma_2, \dots, \sigma_r$ there are generic representations $\tau_1, \tau_2, \dots, \tau_r$ such that $\mathcal{M}(\sigma_1, \sigma_2, \dots, \sigma_r)$ is contained in $\mathcal{M}(\tau_1, \tau_2, \dots, \tau_r)$.

3.11. We need a complement to Proposition 5. If σ is a representation of $F^\times$, let us write $\Re\sigma > a$ if all the one-dimensional components χ of σ satisfy $\Re\chi > a$. Similarly, if ξ is a finite function on $F^\times$ let us write $\Re\xi > a$ if all the character components of ξ satisfy the same inequality.

PROPOSITION 6. *Suppose that $\sigma_1, \sigma_2, \dots, \sigma_r$ is a sequence of generic representations of $F^\times$. Then there are finite sets of finite functions X_j, $1 \le j \le r$, with $\Re X_j > 0$ and the following property. Let X be the finite functions which are products of the functions in the X_j. Let f be a function whose Mellin transform $\hat{f}$ is in $\sigma_1, \sigma_2, \dots, \sigma_r$; suppose $\hat{f}$ is holomorphic in the product of the half-planes $\Re\chi_j > 0$. Then f can be written in the form:*

$$f = \sum_{\xi \in X} \phi_\xi \xi ,$$

where the functions ϕ_ξ are Schwartz functions.

As in the one-dimensional case, we first prove that the Mellin transform of f is in a space:

$$\mathcal{M}(\tau_1, \tau_2, \dots, \tau_r)$$

with $\tau_j > 0$. Then we use Proposition 4 (or rather its proof). We remark that we may choose the ϕ_ξ to depend continuously on f.

3.12. The previous definitions extend to the case of a non-archimedean field F. Every character χ has the form:

$$\chi(a) = | a |^u \chi_0(a) ,$$

where u is real and χ_0 has module one. We write $u = \Re\chi$. We define the function $L(\chi)$ by:

$$L(\chi) = 1\,,$$

if χ is ramified and

$$L(\chi) = (1 - q^{-s})^{-1}\,,$$

if $\chi(a) = \alpha(a)^s$. Here α denotes the absolute value and q the cardinality of the residual field. We can then define the functions $L(\chi, \sigma)$ as before. If $\sigma_1, \sigma_2, \dots, \sigma_r$ is a sequence of finite-dimensional representations of $F^\times$ we can define the space:

$$\mathcal{M}(\sigma_1, \sigma_2, \dots, \sigma_r)\,.$$

It consists of all functions m of the form:

$$\prod_j L(\chi_j, \sigma_j) h(\chi_1, \chi_2, \dots, \chi_r)\,;$$

here h vanishes on all but finitely many connected components and, for each $\chi_1, \chi_2, \dots, \chi_r$,

$$h(\chi_1\alpha^{s_1}, \chi_2\alpha^{s_2}, \dots, \chi_r\alpha^{s_r})$$

is a polynomial in the variables

$$q^{-s_1}, q^{-s_1}, \dots, q^{-s_r}$$

and their reciprocals. Finally, the space $\mathcal{S}((F^\times)^j \times F^{r-j})$ is simply the space of locally constant functions of compact support on $(F^\times)^j \times F^{r-j}$.

All the previous propositions extend to the non-archimedean case. We leave the proof to the reader.

4. Estimates for the Whittaker functions

In this section we let F be a local field and ψ a non-trivial additive character of F. We derive the estimates that we need for the Whittaker functions. We use in an essential way results and ideas of Casselman and Wallach ([**C**], [**W**] I, [**W**] II). In a precise way, we let π be an irreducible unitary generic representation of $G_r(F) = GL(r, F)$. We denote by K_r the standard maximal compact subgroup of $G_r(F)$, by V the space of smooth vectors for π and by V_0 the space of K_r-finite vectors in V. We let θ be the character of $N_r(F)$ defined by :

$$\theta(n) = \prod \psi(n_{i,i+1}) .$$

There is a linear form $\lambda \neq 0$ on V such that for any $v \in V$:

$$\lambda(\pi(n)v) = \theta(n)\lambda(v) \text{ for } n \in N_r(F) .$$

We denote by $\mathcal{W}(\pi, \psi)$ the space of functions of the form:

$$W(g) = \lambda(\pi(g)v) ,$$

with $v \in V$ and by $\mathcal{W}_0(\pi, \psi)$ the subspace of those W for which v is in V_0.

We also denote by A_r the group of diagonal matrices, by B_r the group of upper triangular matrices, by N_r the group of upper triangular matrices with unit diagonal and by Z_r the center of G_r. Let ω be the central character of π. In what follows, we will consider functions on $G_r(F), B_r(F)$ and $A_r(F)$ which transform under the character ω.

The matrix

$$a = \operatorname{diag}(a_1 a_2 a_3 \cdots a_{r-1}, a_2 a_3 \cdots a_{r-1}, \ldots, a_{r-2}a_{r-1}, a_{r-1}, 1) ,$$

will also be denoted

$$m(a_1, a_2, \ldots, a_{r-1}) .$$

Note that that $\alpha_i(a) = a_i$, where α_i denote the simple roots of A_r with respect to B_r.

4.1. PROPOSITION 1. *Fix an irreducible unitary representation π. There is a finite set X of finite functions on F^{r-1} with the following property: for any W in $\mathcal{W}_0(\pi, \psi)$, there are Schwartz-Bruhat functions in $r-1$ variables, ϕ_ξ , $\xi \in X$, such that:*

$$W(a) = \sum_{\xi \in X} \phi_\xi(a_1, a_2, \ldots, a_{r-1}) \xi(a_1, a_2, \ldots, a_{r-1}) ,$$

for

$$a = m(a_1, a_2, \ldots, a_{r-1}) .$$

We will prove this result in the complex case. The proof in the real case is similar. The non-archimedean case has already be treated in **[J-P-S]** (Proposition 2.2).

Fix an index $1 \leq j \leq r$ and let P be the parabolic subgroup of type $(j, r-j)$. Let U be its unipotent radical, $\mathfrak{u}$ the Lie algebra of U (as a real Lie-group), M the standard Levi-factor of P, $\mathfrak{m}$ the Lie algebra of M. Then the group A_j of matrices of the form:

$$\mathrm{diag}(\underbrace{a_j, a_j, a_j, \ldots, a_j}_{j}, 1, \ldots, 1)$$

is contained in the center of M. The lie algebra of A_j is generated by the following elements:

$$H_j = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{j}, 0 \ldots, 0)$$

and

$$K_j = \frac{1}{i}\mathrm{diag}(\underbrace{i, i, \ldots, i}_{j}, 0 \ldots, 0),$$

where $i = \sqrt{-1}$. The Lie algebra $\mathfrak{m}$ operates by an admissible representation on the quotient $V_0/\mathfrak{u}V_0$ (**[W]** II 4.1). In particular, there is a finite set X of complex numbers and an integer n such that any vector v in the quotient can be written as a finite sum

$$v = \sum_{x \in X} v_x,$$

where

$$(H_j - x)^n v_x = 0.$$

Similarly, there is a finite set M_j of integers such that any v can be written as a sum of eigenvectors of K_j with eigenvalues $m \in M_j$.

Let W be in $\mathcal{W}_0(\pi, \psi)$ and ϕ the function on $(F^\times)^{r-1}$ determined by

$$\phi(a_1, a_2, \ldots, a_{r-1}) = W(a),$$

where, as above,

$$a = m(a_1, a_2, \ldots, a_{r-1}).$$

We have

$$H_j W(a) = D_j \phi(a_1, a_2, \ldots, a_{r-1}),$$

where D_j is the Euler operator:

$$D_j = a_j \frac{\partial}{\partial a_j} + \overline{a}_j \frac{\partial}{\partial \overline{a}_j} .$$

Similarly:

$$K_j W(a) = \frac{\partial}{\partial a_j} \phi(a_1, a_2, \ldots, a_{r-1}) - \frac{\partial}{\partial \overline{a}_j} \phi(a_1, a_2, \ldots, a_{r-1}) .$$

Finally, suppose that α is a root whose root space is contained in $\mathfrak{u}$. Then for any X in that root space, we have $XW(a) = 0$ unless α is the simple root α_j. On the other hand, there is a basis X, Y of the root space for α_j such that:

$$XW(a) = a_j W(a) \text{ and } YW(a) = \overline{a}_j W(a) .$$

Let $\mathcal{K}$ be the space spanned by the functions ϕ corresponding to the functions W. Thus $\mathcal{K}$ is a space of C^∞-functions on $(F^\times)^{r-1}$. The previous observations have the folloing consequences for the space $\mathcal{K}$.

It is stable under the action of the operators D_j and R_j, and under multiplication by a_j and $\overline{a}_j$. Furthemore, for each j we can write each element ϕ of $\mathcal{K}$ as a finite sum

$$\phi = \sum_{x \in X_j} \phi_x ,$$

where ϕ_x has the following property: there are two elements θ_1 and θ_2 of $\mathcal{K}$ such that

$$(D_j - x)^n \phi_x = a_j \theta_1 + \overline{a}_j \theta_2 .$$

Every ϕ can be written as finite sum of eigenvectors for R_j. Suppose that ϕ is an eigenvector with an eigenvalue not in the finite set M_j. Then there are two elements θ_1 and θ_2 of $\mathcal{K}$ such that

$$\phi = a_j \theta_1 + \overline{a}_j \theta_2 .$$

As before, we denote by $\| g \|$ the norm of g in a faithful representation of $PGL(r, \mathbf{C})$. Then, ([**J.P.S.**] lemma 8.3.1), there is $N > 0$ and for any $W \in \mathcal{W}(\pi, \psi)$ a constant C such that

$$| W(g) | \leq C \| g \|^N .$$

It follows that any ϕ in $\mathcal{K}$ is bounded by a multiple of

$$\prod \left[1 + a_j \overline{a}_j + (a_j \overline{a}_j)^{-1}\right]^N .$$

4.2. Let us check that the previous properties of $\mathcal{K}$ imply that the Mellin transforms of the functions in $\mathcal{K}$ belong to one of the spaces $\mathcal{M}$ defined in the previous section. This will establish the proposition.

Indeed, applying the previous majorization to the product of a function ϕ in the space by a suitable power of the $a_j\overline{a_j}$, we see that each ϕ is majorized, for each $M > 0$ by a constant multiple of a product:

$$\prod \left[m_1(a_j)(a_j\overline{a_j})^{-N} + m_2(a_j)(a_j\overline{a_j})^{-M}\right] ,$$

where m_1, m_2 is a partition of unity on $F^\times$ with $m_1 = 1$ near 0. Consider the multivariate Mellin transform of a function ϕ in $\mathcal{K}$. The previous majorization shows that the Mellin transform is defined by a convergent integral in a product of half-planes of the form $\Re\chi_j > A$. Furthermore the Mellin transform is bounded in any product of vertical strips of the form

$$A < a_j \leq \Re\chi_j \leq b_j \, .$$

Finally, the Mellin transform vanishes in all but a finite number of connected components.

Fix an index j. We are going to fix the characters χ_k for $k \neq j$ and study the Mellin transform as a function of $\chi_j = (s, m)$. We will suppress the other variables from the notation and write $\hat{\phi}(s, m)$ for this Mellin tansform. We see that any ϕ in the space can be written as a finite sum

$$\phi = \sum_{x \in X_j} \phi_x \, ,$$

where the Mellin transform of each ϕ_x satisfies a difference equation:

$$(2s + x)^n \hat{\phi}_x(s, m) = \hat{\theta}_1(s + \frac{1}{2}, m + 1) + \hat{\theta}_2(s + \frac{1}{2}, m - 1) \, ,$$

where the θ_i are in $\mathcal{K}$.

On the other hand, we may assume the finite set M_j contains 0 and furthermore that $-M_j = M_j$. Now suppose that ϕ is an eigenvector of R_j with an eigenvalue not in M_j. Then we have :

$$\hat{\phi}(s, q) = 0 \text{ unless } q = -m \, ,$$

and

$$\hat{\phi}(s, -m) = \hat{\theta}_1(s + \frac{1}{2}, 1 - m) + \hat{\theta}_1(s + \frac{1}{2}, -1 - m) \, ,$$

where θ_1 is an eigenvector with eigenvalue $m+1$ and θ_2 an eigenvector with eigenvalue $m-1$. An easy inductive argument shows that the Mellin transform of ϕ has the form

$$\hat{\phi}(s,-m) = \prod_{x \in X_j} \Gamma(2s+x+p)^n H(s)\,,$$

where n is the distance from m to the set M_j and H is entire. It is then easy to show that there is a finite-dimensional representation σ_j of $F^\times$ such that, for each $\phi \in \mathcal{K}$,

$$\hat{\phi}(s,m) = L(\chi_j,\sigma)H(s,m)\,,$$

where $H(s,m)$ is entire. In fact H depends on all the variables χ_k and, as such, it is holomorphic in the region defined by the inequalities:

$$\Re\chi_k > A \ \text{ for } \ k \neq j\,.$$

Applying this result for all j and Hartogs' theorem we conclude that for each $\phi \in \mathcal{K}$ the Mellin transform $\hat{\phi}$ is the product of

$$\prod_j L(\chi_j,\sigma_j)$$

and an entire function.

It remains to check that the Mellin transforms satisfy the growth condition which defines the space $\mathcal{M}(\sigma_1,\sigma_2,\ldots,\sigma_{r-1})$. Here the Mellin transforms vanish on all but a finite number of connected components of the space of $(r-1)$-tuples of characters. Thus, we need only check that for $\chi_j = (s_j, m_j)$ with m_j fixed, the following condition is satisfied: for each j, let $P_j(s_j)$ be a polynomial such that the product

$$P_j(s_j)L(\chi_j,\sigma_j)$$

is bounded in the strip $a_j \leq \Re s_j \leq b_j$; then for each $\phi \in \mathcal{K}$ the product

$$\hat{\phi}\prod_j P_j(s_j)$$

is bounded in the product of the strips. This is certainly so for $P_j = 1$ and $A < \Re a_j$. The general case follows from the recurrence relations satisfied by $\hat{\phi}$.

4.3. We now assume F is real or complex. The previous results extends to the larger space $\mathcal{W}(\pi, \psi)$. We sketch a proof; the method is due to Casselman.

At the cost of enlarging the space $\mathcal{M}(\sigma_1, \sigma_2, \dots, \sigma_{r-1})$, we may assume that the representations σ_j are generic (see 3.10). We let X_j be the finite set of finite functions attached to σ_j and X the set of finite functions on $(F^\times)^r$ which are products of functions in the various X_j. We also let V_X be the space of linear combinations of functions in X. It is invariant under translations.

Recall we have fixed a character ω of the center of $G_r(F)$. If $\mathcal{V}$ is a space of functions on $(F^\times)^{r-1}$ invariant under translations, we define a representation of $B_r(F)$ on $\mathcal{V}$ as follows: the center operates by ω; a diagonal matrix b of the form

$$b = m(b_1, b_2, \dots, b_{r-1})$$

operates by translations:

$$bf(a_1, a_2, \dots, a_{r-1}) = f(a_1 b_1, a_2 b_2, \dots, a_{r-1} b_{r-1}) .$$

A matrix n in $N_r(F)$ operates by multiplication:

$$nf(a_1, a_2, \dots, a_{r-1}) = f(a_1, a_2, \dots, a_{r-1}) \prod \psi(a_j n_{j,j+1}) .$$

We may view $\mathcal{V}$ as a space of functions on $A_r(F)$ transforming under ω. The group $A_r(F)$ operates then by translations on $\mathcal{V}$ and the previous representation is formally a representation induced to $B_r(F)$ by the character θ.

For instance, we can take for $\mathcal{V}$ the space of Schwartz functions, with ω trivial. It is then easy to see that the corresponding representation is continuous and differentiable. Furthemore, it satisfies the condition of slow growth introduced by Casselman ([**C**]): let as before $\| g \|$ be a norm on $PGL(r, F)$, that is, the norm of the matrix g in a faithful representation of $PGL(r, F)$. Then, for any continuous semi-norm β on $\mathcal{V}$, there is an integer N and a constant C such that:

$$\beta(gv) \leq C \| g \|^N \beta(v) ,$$

for any $v \in \mathcal{V}$.

We may also view V_X as a space of functions on $A_r(F)$, transforming under the character ω. We have then a finite dimensional representation of $B_r(F)$ on the space V_X, the diagonal matrices operating by translations and $N_r(F)$ operating trivially. Clearly, this representation satisfies the condition of Casselman. The same is true for its tensor product with the previous representation.

Now let $\mathcal{K}$ be the space of functions of the form:

$$\sum_{\xi \in X} \phi_\xi \xi ,$$

where the ϕ_ξ are Schwartz functions. We may view $\mathcal{K}$ as a quotient of the tensor product $\mathcal{V} \otimes V_X$, with the quotient topology. In particular, the representation of $B_r(F)$ on $\mathcal{K}$ satisfies the condition of Casselman.

Next we consider the space $\mathcal{U}$ of all smooth functions f on G such that, for any $k \in K_r$ and any X in the Lie algebra of K_r, the function:

$$b \mapsto (f * X)(bk)$$

belongs to $\mathcal{K}$. Because $\mathcal{K}$ is a closed direct factor of the tensor product $\mathcal{V} \otimes V_X$, it is easy to see that any element f of $\mathcal{U}$ has the form:

$$f = \sum_\xi f_\xi ,$$

where

$$f_\xi(nak) = \theta(n)\phi_\xi(a,k)\xi(a) ,$$

and the functions ϕ_x are in the space $\mathcal{S}(F^{r-1} \times K_r)$ (viewed as a space a functions on $A_r \times K_r$); in addition the functions ϕ_ξ verify:

$$\phi_\xi(ah,k)\xi(h) = \phi_\xi(a,hk)$$

for $h \in A_r \cap K_r$. We can view $\mathcal{U}$ as a space of smooth functions from K_r to $\mathcal{K}$. As such, it has a natural topology. Furthemore, the space $\mathcal{U}$ is invariant under right translations and the representation of $G_r(F)$ on $\mathcal{U}$ is differentiable and satisfy the slow growth condition of Casselman.

By construction the space $\mathcal{W}_0(\pi,\psi)$ is contained in $\mathcal{U}$. Let $\mathcal{U}_1$ be its closure. By the usual argument, each element W of $\mathcal{W}_0(\pi,\psi)$ is an analytic vector in $\mathcal{U}$. Thus its right translates also belong to $\mathcal{U}_1$ and it follows that $\mathcal{U}_1$ is invariant under right translations. Thus it is a smooth representation of slow growth, infinitesimally equivalent to π. By a fundamental result of Casselman and Wallach ([**C**]) $\mathcal{U}_1$ is homeomorphic to the space of smooth vectors for π. It follows that $\mathcal{U}_1 = \mathcal{W}(\pi, \psi)$. We have thus proved the following proposition:

PROPOSITION 2. *For each j there is a finite set C_j of characters, and, for each χ in the set C_j, an integer n_χ with the following property: let X_j be the set of finite functions of the form $\chi(a)(\log \mid a \mid_F)^n$ with $\chi \in C_j$ and $n \leq n_j$ and let X be the finite functions on $(F^\times)^{r-1}$ which are products of functions in the X_j. Then for any W in $\mathcal{W}(\pi, \psi)$ there are functions ϕ_ξ in $\mathcal{S}((F^{r-1}) \times K_r)$ such that:*

$$W(g) = \sum_\xi \phi_\xi(a_1, a_2, \ldots, a_{r-1}, k)\xi(a_1, a_2, \ldots, a_{r-1})$$

for $g = ak$ and

$$a = m(a_1, a_2, \ldots, a_{r-1}).$$

4.4. Let us go back to the case where F is an arbitrary local field. Let δ_r be the module of the group $B_r(F)$. We may view the function δ_{r-1} as a function on $A_r(F)$ invariant under the center. Recall ([**J.S.** I] 3.8) that for any W we have:

$$\int_{N_{r-1}(F)\backslash G_{r-1}(F)} \left| W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right]\right|^2 dg < +\infty .$$

Suppose W is K_r-finite and consider the function:

$$(a_1, a_2, \ldots, a_{r-1}) \mapsto \delta_{r-1}^{-1/2}(a)W(a),$$

where

$$a = m(a_1, a_2, \ldots, a_{r-1}).$$

Then this function is square-integrable on the group $(F^\times)^{r-1}$. Moreover, if F is archimedean, replacing W by appropriate transforms of W under the Lie algebra of $N_r(F)$, we see its product by any polynomial in $a_j, \overline{a}_j$ is also square-integrable. Then by Proposition 5 of the previous section, the Mellin transform of this function is holomorphic in the product of the halfplanes

$$\Re\chi_j \geq 0.$$

It follows that the same is true for an arbitrary W. We now appeal to Proposition 6 of the previous section (or rather to a version of this proposition with parameters). After a change of notation, we arrive at the following proposition:

PROPOSITION 3. *For each j there is a finite set C_j of characters with positive real parts, and, for each χ in the set C_j, an integer n_χ with the following property: let X_j be the set of finite functions of the form $\chi(a)(\log \mid a \mid_F)^n$ with $\chi \in C_j$ and $n \leq n_j$ and let X be the finite functions on $(F^\times)^{r-1}$ which are products of functions in the X_j. Then for any W in $\mathcal{W}(\pi, \psi)$ there are functions ϕ_ξ in $\mathcal{S}((F^{r-1}) \times K_r)$ such that:*

$$W(g) = \delta_{r-1}^{1/2}(a) \sum_{xi} \phi_\xi(a_1, a_2, \ldots, a_{r-1}, k)\xi(a_1, a_2, \ldots, a_{r-1})$$

for $g = ak$ and

$$a = m(a_1, a_2, \ldots, a_{r-1}) .$$

5. GLOBAL MAJORIZATIONS

The results of Section 2 suggest the existence of a global integral representation for the exterior square L-function. We discuss it in this section and the next.

We let F be a number field, ψ a non-trivial additive character of $F_{\mathbf{A}}/F$. We consider an automorphic unitary cuspidal representation π of $GL(r, F_{\mathbf{A}})$. We demote by ω_π its central character and we let ϕ be a form in the space of π. Finally, we let χ be an idele-class character of F, of module one.

5.1. From now on we assume $r = 2n$ is an even integer. The case of an odd integer is treated in section 9. We let $P_{n-1,n}$ be the parabolic subgroup of type $(n-1, 1)$ in $G_n = GL(n)$, A_n the group of diagonal matrices, B_n the group of upper triangular matrices, N_n the group of upper triangular matrices with unit diagonal and Z_n the center of G_n. We consider the group V_0 of matrices of the form:

$$v = \begin{pmatrix} 1_n & X \\ 0 & 1_n \end{pmatrix},$$

where X is in M_n, the ring of $n \times n$ matrices. Then

$$\theta(v) = \psi(\mathrm{Tr}X)$$

is a character of $V_0(F_{\mathbf{A}})$ trivial on $V_0(F)$, fixed by the conjugation by elements of the form:

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \; g \in G_n(F_{\mathbf{A}}) .$$

We let Φ be a Schwartz-Bruhat function in n variables. We define an Eisenstein series on G_n as follows. We first set:

$$f(g,s) = \int_{F_{\mathbf{A}}^\times} \Phi(\epsilon t g) \mid t \mid^{ns} \chi^n \omega_\pi(t)\, d^\times t\, \chi(\det g) \mid \det g \mid^s ,$$

where

$$\epsilon = (\underbrace{0,0,\ldots,0}_{n-1},1) .$$

Then we set:

$$E(g,s) = \sum_\gamma f(\gamma g, s) ,$$

the sum over $P_{n-1,n}(F)\backslash G_n(F)$.

The integral we want to consider is then:

$$I = I(s,\chi,\phi,\Phi) ,$$

where

$$I = \int\int \phi\left[v \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right] \theta(v)\, dv E(g,s)\, dg ;$$

the integral in v is over $V_0(F)\backslash V_0(F_{\mathbf{A}})$ and the integral in g over

$$G_n(F)\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}}) .$$

In this section, we establish the various majorizations which will be needed to show that I is an Eulerian integral.

5.2. We first prove the integral I converges for all s. To that end we estimate

$$\mid \phi \mid \left[\begin{pmatrix} 1_n & X \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right] ,$$

where X is in a compact set, g in a Siegel set of $G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}})$. We may as well assume

$$g = am ,$$

where m is in a compact set and a is a diagonal matrix of the form:

$$a = \operatorname{diag}(a_1, a_2, \ldots, a_{n-1}, a_n);$$

here the a_i are ideles whose finite components are one and whose infinite components are all equal to some postive real number; furthermore:

$$t_i = | a_i/a_{i+1} | \geq c \text{ and } a_n = 1,$$

where c is a constant. We write

$$X = Y + Z,$$

where the last column of Y is 0 and all columns of Z are 0 except the last one, the entries of which we denote by $z_1, z_2, \ldots, z_n$. Then the previous expression can also be written:

$$|\phi| \left[\begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_n & a^{-1}Za \\ 0 & 1_n \end{pmatrix} m \right],$$

where m is another matrix in some compact set. The non-zero entries of $a^{-1}Za$ are the quantities

$$a_1^{-1} z_1, a_2^{-1} z_2, \ldots, a_{n-1}^{-1} z_{n-1}, z_n.$$

They remain in a compact set. Thus the above expression has the form:

$$|\phi| \left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} m \right],$$

where m is again in a compact set and h is a matrix in G_{r-1} with determinant equal to $(\det a)^2$. For all $N > 0$ this is majorized by a constant multiple of

$$\inf(| \det h |^{-N}, | \det h |^{N})$$

([**J-S**] II p.799) hence by a constant multiple of:

$$t_1^{-2N} t_2^{-4N} \cdots t_{n-1}^{-2(n-1)N}.$$

Since the Eisenstein series is slowly increasing ([**J-S**] I Lemma (4.2)) this estimate implies the convergence of the integral I.

As a consequence, the singularities of the integral are those of the Eisenstein series. We will study the behavior at $s = 1$. We recall the properties of the Eisenstein series ([**J-S**] I §4). The Eisenstein series is holomorphic at $s = 1$ if $\chi^n \omega_\pi \neq 1$. If on the contrary $\chi^n \omega_\pi = 1$, then the Eisenstein series has a pole of order (at most) 1 with a residue proportional to

$$\Phi(0)\chi(\det g).$$

This implies at once the following result:

PROPOSITION 1. *The integral I is holomorphic at $s = 1$ if $\chi^n \omega_\pi \neq 1$. If on the contrary $\chi^n \omega_\pi = 1$, then the integral I has a pole of order (at most) 1 with residue proportional to*

$$\int\int \phi\left[v\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]\theta(v)\,dv\chi(\det g)\,dg;$$

the integral in v is over $V_0(F)\backslash V_0(F_{\mathbf{A}})$ and the integral in g over

$$G_n(F)\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}}).$$

We remark that our estimates also implies the convergence of the integral in the proposition. We also remark that similar results are true in the function field case.

5.3. In order to show that I is an Eulerian integral which represents the exterior square L-function we introduce auxiliary subgroups and integrals. For $0 \leq a \leq n$ we denote by $P_{a,n}$ the parabolic subgroup of type

$$(a, \underbrace{1, 1, \ldots, 1}_{n-a})$$

in G_n and by $U_{a,n}$ its unipotent radical. We also denote by $\mathfrak{p}_{a,n}$ and $\mathfrak{u}_{a,n}$ the Lie algebras of these two groups. We remark that $\mathfrak{p}_{a,n}$ is actually an associative algebra (for the ordinary matrix multiplication) and $\mathfrak{u}_{a,n}$ an ideal in this algebra. In particular, $P_{1,n} = P_{0,n} = B_n$ and $U_{1,n} = U_{0,n} = N_n$. Also $P_{n,n} = G_n$ and $U_{n,n} = 1$. The notations $\overline{P}_{a,n}$, $\overline{U}_{a,n}$, $\overline{\mathfrak{p}}_{a,n}$, $\overline{\mathfrak{u}}_{a,n}$ denote the opposed subgroups and subalgebras, that is, the images under transposition. If $0 \leq k \leq n$ we will often identify G_k with the subgroup of G_n formed of the matrices of the type:

$$\begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix}, g \in G_k.$$

Similarly, we can view $P_{a,k}$ and $U_{a,k}$ as subgroups of G_n and $\mathfrak{p}_{a,k}$ and $\mathfrak{u}_{a,k}$ as subalgebras of M_n.

For $0 \leq j \leq n-1$ we will denote by V_j the subgroup of G_n of matrices of the form

$$v = \begin{pmatrix} n_1 & y \\ t & n_2 \end{pmatrix},$$

where

$$n_1, n_2 \in U_{n-j,n}\,,\ y \in \mathfrak{p}_{n-j,n}\,,\ t \in \mathfrak{u}_{n-j,n}\,.$$

We will denote by $\mathrm{Ts}(t)$ the sum of the entries of the matrix t which are just above the diagonal. Then

$$\theta(v) = \psi(\mathrm{Tr}(s) + \mathrm{Ts}(t))$$

is a character of $V_j(F_{\mathbf{A}})$ trivial on $V_j(F)$.

The group V_0 is the group introduced previously. The group V_{n-1} is a maximal unipotent subgroup. In fact let $\sigma \in \mathfrak{S}_{2n}$ be the permutation which changes the sequence

$$(1, 2, 3, \ldots, n, n+1, n+2, n+3, \ldots, 2n)$$

into the sequence

$$(1, 3, 5, \ldots, 2n-1, 2, 4, 6, \ldots, 2n)\,.$$

We also denote by σ the corresponding permutation matrix. Then:

$$N_{2n} = \sigma V_{n-1} \sigma^{-1}\,.$$

Furthemore σ transforms the character θ of V_{n-1} into the character θ of N_r defined by:

$$\theta(u) = \prod \psi(u_{j,j+1})\,.$$

We consider the following integrals:

$$I_j = \int\int\int \phi\left[v \begin{pmatrix} 1_n & X \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]$$
$$\theta(v) dv \psi(\mathrm{Tr} X) dx f(g, s) dg\,.$$

Here v is integrated over

$$V_j(F) \backslash V_j(F_{\mathbf{A}})\,,$$

and X is integrated over

$$\mathfrak{p}_{n-j,n}(F_{\mathbf{A}})\backslash M_n(F_{\mathbf{A}})\,.$$

Finally, g is integrated over the quotient:

$$G_{n-j-1}(F)U_{n-j-1,n}(F)U_{n-j,n}(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}})\,.$$

The integral exists only as an iterated integral; in a precise way, we shall see that:

$$\int\int\left|\int\phi\left[v\begin{pmatrix}1_n & X\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}g & 0\\ 0 & g\end{pmatrix}\right]\theta(v)dv\right|$$

$$dX\mid f(g,s)\mid dg<+\infty\,.$$

5.4. In order to establish the convergence of the integrals I_j we first establish a partial result.

PROPOSITION 2. *Given ϕ and a compact set M of $G_n(F_{\mathbf{A}})$ there is a constant $C>0$ such that:*

$$\int\left|\int\phi\left[v\begin{pmatrix}1_n & Z\\ 0 & 1_n\end{pmatrix}g\right]\theta(v)dv\right|dz<C\,,$$

for $g\in M$. Here v is integrated over:

$$V_j(F)\backslash V_j(F_{\mathbf{A}})\,,$$

and Z over the quotient

$$\mathfrak{p}_{n-j,n}(F_{\mathbf{A}})\backslash M_n(F_{\mathbf{A}})\,,$$

or, what amounts to the same, over:

$$\overline{\mathfrak{u}}_{n-j,n}(F_{\mathbf{A}})\,.$$

We let V' be the subgroup of V_j of matrices of the following form:

$$v=\begin{pmatrix}u_1 & y\\ t & u_2\end{pmatrix},$$

where

$$u_1,u_2\in U_{n-j,n}\;,\;t\in\mathfrak{u}_{n-j,n}\;,\;y\in\mathfrak{p}_{1,n}\;.$$

We also let V'' be the subgroup of matrices of the form:

$$v = \begin{pmatrix} 1_n & y \\ 0 & 1_n \end{pmatrix},$$

where

$$y \in \overline{\mathfrak{u}}_{1,n-j}\,.$$

Then V_j is the semi-direct product of V' and V'', with V' normal. Furthemore V'' normalizes the subgroup of matrices of the form:

$$\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix}, Z \in \overline{\mathfrak{u}}_{n-j,n}(F_{\mathbf{A}})\,.$$

Thus it suffices to majorize the integral

$$\int \left| \int \phi \left[v \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} g \right] \theta(v) dv \right| dz\,,$$

where v is integrated over:

$$V'(F)\backslash V'(F_{\mathbf{A}})\,.$$

Set $V = \sigma V' \sigma^{-1}$ and

$$u_Z = \sigma \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \sigma^{-1}\,.$$

We remark that V is contained in N_r and the conjugate under σ of the restriction of the character θ to V' is the restriction to V of the character θ of N_r introduced previously. Set

$$W(g) = \int \phi(vg)\theta(v)dv\,,$$

the integral over

$$V(F)\backslash V(F_{\mathbf{A}})\,.$$

It will then suffice to majorize

$$\int |W(u_Z g)|\, dZ$$

for g in a compact set. Now we may assume that ϕ is a convolution:

$$\phi(g) = \int \phi_0(gh) f(h) dh\,,$$

where ϕ_0 is also in the space of π and f is a smooth function of compact support on $G_r(F_{\mathbf{A}})$. Indeed, in general, ϕ is a finite sum of such convolutions ([**D-M**]). For the corresponding function W we have:

$$W(g) = \int W_0(gh)f(h)dh\,.$$

We introduce the Iwasawa decomposition of u_Z:

$$u_Z = t_Z n_Z k_Z\,,$$

with t_Z diagonal. We remark that V contains the unipotent radical U of the parabolic subgroup of type

$$(n-2j+1, \underbrace{1, 1, \ldots, 1}_{2j-1})\,.$$

We have:

$$W(u_Z g) = \int W_0(t_Z n_Z h)f(g^{-1}k_Z^{-1}h)dh\,.$$

We can break up the integral in h into an integral for $u \in U(F_{\mathbf{A}})$ followed by an integral for

$$h \in U(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})\,.$$

Using the fact that t_Z and u_Z normalize U, we find:

$$W(u_Z g) = \int W_0(t_Z n_Z h)f(g^{-1}k_Z^{-1}uh)du\overline{\theta}(t_Z u t_Z^{-1})dh\,.$$

Since the cusp form ϕ_0 is bounded uniformly, the same is true of W_0. Thus the previous expression is bounded by a constant multiple of the absolute value of

$$\int f(g^{-1}k_Z^{-1}uh)du\overline{\theta}(t_Z u t_Z^{-1})dh\,.$$

Regard

$$f(g^{-1}k_Z^{-1}uh)$$

as a Schwartz-Bruhat function of the entries of u above the diagonal. As g and k_Z remain in compact sets, this Schwartz-Bruhat function remains in a bounded set. In the previous integral, we can integrate first with respect to the entries of u corresponding to root spaces for non-simple roots of A_n; the resulting function is a Schwartz-Bruhat function of the remaining variables,the ones corresponding to the simple root spaces. The integral of this function can be interpreted as a Fourier transform. It follows that we have a bound:

$$|W(u_Z g)| \leq \Phi(\frac{t_{2n-2j+1}}{t_{2n-2j+2}}, \frac{t_{2n-2j+2}}{t_{2n-2j+3}}, \ldots, \frac{t_{2n-1}}{t_{2n}}),$$

where $\Phi \geq 0$ is a fixed Schwartz-Bruhat function and the t_k are the diagonal entries of the diagonal matrix t_Z. Thus, we need only prove that the following integral is finite:

$$\int \Phi(\frac{t_{2n-2j+1}}{t_{2n-2j+2}}, \frac{t_{2n-2j+2}}{t_{2n-2j+3}}, \ldots, \frac{t_{2n-1}}{t_{2n}})dZ .$$

We may assume that Φ is a product of local functions. Then the integral is itself the product of local analogous integrals. We will prove in the next subsection that each one of the local integrals converges, and, furthermore, that almost all of them are equal to one. This will imply the above integral is finite and complete the proof of Proposition 2. We will need also the convergence of a slightly different integral. Let W_j be the subgroup of V_j formed of the matrices of the type:

$$v = \begin{pmatrix} u_1 & y \\ t & u_2 \end{pmatrix},$$

where

$$u_1, u_2 \in U_{n-j,n} \ , \ t \in \mathfrak{u}_{n-j+1,n} \ , \ y \in \mathfrak{p}_{n-j,n} \ .$$

PROPOSITION 3. *Given ϕ and a compact set M of $G_n(F_{\mathbb{A}})$ there is a constant $C > 0$ such that:*

$$\int \left| \int \phi \left[v \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} g \right] \theta(v) dv \right| dz < C ,$$

for $g \in M$. Here v is integrated over:

$$W_j(F) \backslash W_j(F_{\mathbb{A}}),$$

and Z over the quotient

$$\mathfrak{p}_{n-j,n}(F_{\mathbb{A}}) \backslash M_n(F_{\mathbb{A}}),$$

or, what amounts to the same, over:

$$\overline{\mathfrak{u}}_{n-j,n}(F_{\mathbb{A}}) .$$

The proof is the same: indeed the group W_j is the semi-direct product of V'' and $V' \cap W_j$ and the conjugate of the later group under σ contains U.

5.5. For this subsection we go back to a local situtation. We let F be a local field. We again set

$$u_Z = \sigma \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \sigma^{-1},$$

where at first Z is in $\overline{\mathfrak{u}}_{1,n}$, that is, is a lower triangular matrix. We consider the Iwasawa decomposition of u_Z:

$$u_Z = n_Z t_Z k_Z ,$$

and denote by t_j the entries of the matrix t_Z.

PROPOSITION 4. *With the previous notations, we have:*

$$\begin{aligned} | \, t_k \, | &\geq 1 \quad \text{for} \quad k \text{ odd} \\ | \, t_k \, | &\leq 1 \quad \text{for} \quad k \text{ even} \end{aligned}$$

Furthemore:

$$t_1 = 1 \text{ and } t_{2n} = 1 .$$

It will be helpful to vizualize the shape of the matrix u_Z. We illustrate the case $n = 4$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & * & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let e_k with $1 \leq k \leq 2n$ be the canonical basis of the space of row vectors. Then we have for k odd:

$$e_k u_Z = e_k + \sum_{h<k, h \in 2\mathbf{Z}} x_k^h e_h .$$

On the other hand, for k even:

$$e_k u_Z = e_k \,.$$

Similarly:

$$e_k n_Z^{-1} = e_k + \sum_{h>k} y_k^h e_h \,.$$

It follows that for k odd:

$$e_k n_Z^{-1} u_Z = e_k + \sum_{h \neq k} z_h e_h \,.$$

In other words, the odd diagonal entries of the product

$$n_Z^{-1} u_Z$$

are one. Thus the odd diagonal entries of

$$t_Z^{-1} n_Z^{-1} u_Z = k_Z$$

are the numbers t_k^{-1} with k odd. Since the entries of k_Z must be less than one in absolute value, we already get our assertion for k odd.

For simplicity, let us finish the proof in the case where F is real or non-archimedean. The complex case diffrs only in notation. To continue we use the formula:

$$| \, t_k t_{k+1} \cdots t_{2n} \, | = \|(e_{2n} u_Z) \wedge (e_{2n-1} u_Z) \wedge \cdots \wedge (e_k u_Z)\| \,.$$

Here we have for k even

$$e_k u_Z = e_k \,.$$

Now it is clear that for any vector v in any exterior power:

$$\|v \wedge e_k\| \leq \|v\| \,.$$

Our assertion for k even follows at once.

Before going back to our integral, we prove one more proposition:

PROPOSITION 5. *Set*

$$m(Z) = \sqrt{1 + \|Z\|}$$

if F is archimedean, and

$$m(Z) = \sup(1, \|Z\|)$$

if F is not archimedean. Fix j. Then there is a constant $\alpha > 0$ such that for $Z \in \overline{\mathfrak{u}}_{n-j,n}$

$$\prod_{2n-2j+1 \leq k, k \text{ odd}} | t_k | \geq m(Z)^\alpha .$$

Again we disregard the complex case. Set:

$$s_k = | t_k t_{k+1} \cdots t_{2n} | ;$$

this is the norm of a tuple whose entries are the number 1 and certain minors of the matrix u_Z. Now each entry of u_Z appears as a minor in at least one s_k with $k \geq 2n - 2j + 1$. It follows that

$$\prod_{k \geq 2n-2j+1} s_k \geq m(Z) .$$

On the other hand, by the previous proposition, we have:

$$\prod_{2n-2j+1 \leq k, k \text{ odd}} | t_k | \geq \left(\prod_{k \geq 2n-2j+1} s_k \right)^\alpha ,$$

for some $\alpha > 0$. The proposition follows.

We now go back to our integral:

$$\int \Phi(\frac{t_{2n-2j+1}}{t_{2n-2j+2}}, \frac{t_{2n-2j+2}}{t_{2n-2j+3}}, \ldots, \frac{t_{2n-1}}{t_{2n}}) dZ .$$

where Z is integrated over $\overline{\mathfrak{u}}_{n-j,n}(F)$. Suppose first that F is non archimedean and that Φ is the appropriate characteristic function of the integers. Then, if the integrand is non-zero, we must have:

$$1 \leq | t_{2n-2j+1} | \leq | t_{2n-2j+2} | \leq | t_{2n-2j+3} | \leq \cdots \leq | t_{2n-1} | \leq 1 .$$

Thus the t_k with $2n - 2j + 1 \leq k \leq 2n$ are actually units if the integrand is non-zero. By the previous proposition the entries of Z are then integral. It follows that the integral is one. A similar argument shows that the integral converges for F non-archimedean.

Suppose that F is real. Then we have by the previous propositions:

$$\prod_{2n-2j+1\leq k\leq 2n-1}(1+|t_k/t_{k+1}|)\geq \prod_{2n-2j+1\leq k\leq 2n-1,k\,\mathrm{odd}} |\, t_k \,|\geq m(Z)^{\alpha}\,.$$

Thus for any $N>0$, the integrand is bounded by a constant multiple of $m(Z)^{-N}$. It follows that the integral converges. This concludes the proof of Proposition 2.

5.6. We now prove the convergence of the integral I_{n-1}:

PROPOSITION 6. *For* $s>1+2(n-1)$ *we have:*

$$\int\int\left|\int\phi\left[v\begin{pmatrix}1_n & X\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}g & 0\\ 0 & g\end{pmatrix}\right]\theta(v)dv\right|$$

$$dX\mid f(g,s)\mid dg<+\infty\,.$$

We will replace V_{n-1} by N_r, its conjugate under σ. Let us set:

$$W(g)=\int\phi(ug)\theta(u)du\,,$$

where the integral is over

$$N_r(F)\backslash N_r(F_{\mathbb{A}})\,.$$

We have to establish the finiteness of the following integral:

$$\int\int\int\left|W\left[\sigma\begin{pmatrix}1_n & Z\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}k\right]\right|\delta_n^{-1}(a)\mid \det a\mid^s dadZdk\,,$$

where δ_n denotes the module of the Borel subgroup $P_{0,n}$ and

$$a=\mathrm{diag}(a_1,a_2,\ldots,a_{n-1},1)\,.$$

Here k is integrated over the standard maximal compact subgroup; each a_i is integrated over $F_{\mathbb{A}}^{\times}$ and Z is integrated over $\overline{\mathfrak{u}}_{1,n}(F_{\mathbb{A}})$. Simple formal manipulations bring this integral into the form:

$$\int|W(bu_Zk)|\,\delta_n(a)^{-2}\mid\det a\mid^s dadZdk\,,$$

where

$$b = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, 1, 1)\,,$$

and, as before:

$$u_Z = \sigma \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \sigma^{-1}\,.$$

We again introduce the Iwasawa decomposition of u_Z:

$$u_Z = n_Z t_Z k_Z\,.$$

Then the previous integral can be written:

$$\int |W(bt_Z k)|\,\delta_n(a)^{-2} \mid \det a \mid^s dadZdk\,.$$

Just as before, we can write W as a convolution product:

$$W(g) = \int W_0(gh)f(h)dh\,,$$

where f is a smooth function of compact support. Then we can write:

$$|W(bt_Z k)| = \left| \int W_0(bt_Z h) f(k^{-1}h)dh \right|\,.$$

The integral in h can be broken into an integral over $u \in N_r(F_{\mathbf{A}})$ followed by an integral over the quotient:

$$h \in N_r(F_{\mathbf{A}}) \backslash G_r(F_{\mathbf{A}})\,.$$

We get in this way for the previous expression:

$$\left| \int W_0(bt_Z h) \left[\int \phi(k^{-1}uh)\bar{\theta}(bt_Z u t_Z^{-1} b^{-1})du \right] dh \right|\,.$$

Since W_0 is bounded, this is bounded by:

$$\Phi\left(\frac{b_1 t_1}{b_2 t_2}, \frac{b_2 t_2}{b_3 t_3}, \dots, \frac{b_{2n-1} t_{2n-1}}{b_{2n} t_{2n}} \right)\,,$$

where $\Phi \geq 0$ is a fixed Schwartz-Bruhat function. Of course the b_j are the diagonal entries of b. Finally, we see that we have to show the following integral is finite:

$$\int \Phi\left(\frac{b_1 t_1}{b_2 t_2}, \frac{b_2 t_2}{b_3 t_3}, \dots, \frac{b_{2n-1} t_{2n-1}}{b_{2n} t_{2n}} \right) \delta_n(a)^{-2} \mid \det a \mid^s dadZ\,.$$

As before, we may assume Φ is a product of local functions. The integral is then a product of local integrals. In the next subsection, we show that each local integral is finite and, furthermore, that almost all local integral are ones.

5.7. In this subsection we go back to a local situation. Thus F is now a local field. We consider the following integral:

$$\int \Phi\left(\frac{b_1 t_1}{b_2 t_2}, \frac{b_2 t_2}{b_3 t_3}, \ldots, \frac{b_{2n-1} t_{2n-1}}{b_{2n} t_{2n}}\right) \delta_n(a)^{-2} \mid \det a \mid^s da dZ\,,$$

Here Φ is a Schwartz-Bruhat function, the t_j are the diagonal entries of t_Z, the diagonal component of the Iwasawa decomposition of u_Z. Finally:

$$b = \operatorname{diag}(a_1, a_1, a_2, a_2, \ldots, a_{n-1}, a_{n-1}, 1, 1)\,.$$

After suitable translations in the variables a_i and under the additional assumption that the local function Φ is a product of functions of each coordinate, we find the local integral is a product of two integrals:

$$\int \Phi_1\left(\frac{t_1}{t_2}, \frac{t_3}{t_4}, \ldots, \frac{t_{2n-1}}{t_{2n}}\right) \mu_s(t_Z) dZ\,,$$

and

$$\int \Phi_2\left(\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, a_{n-1}\right) \delta_n(a)^{-2} \mid \det a \mid^s da dZ\,.$$

Here Φ_1 and Φ_2 are Schwartz-Bruhat functions. In addition, μ_s is a certain character, depending only on the absolute values of the diagonal entries of t_Z.

Let us consider the first integral. Suppose first that F is non-archimedean and Φ_1 the characteristic function of the integers. Just as in section 4, if the integrand is non-zero, Proposition 3 shows the entries t_i are unit. By Proposition 4, this implies that the entries of Z are integers. The integral is then one. If F is non-archimedean, a similar argument shows the integrand is compactly supported. Now assume F is real. Then by Propositions 3 and 4 we have:

$$\prod_{k\,\mathrm{odd}} (1+ \mid \frac{t_k}{t_{k+1}} \mid) \geq \prod_{k\,\mathrm{odd}} \mid t_k \mid \geq M(Z)^{\alpha}\,.$$

On the other hand, we can express $\mu(t_Z)$ in terms of the quantities:

$$\prod_{j\geq k} t_j$$

which are of polynomial growth in Z. It follows that the first integral converges. A similar argument applies to the complex case.

As for the second integral, a simple change of variables puts it in the form:

$$\int \Phi_2(a_1, a_2, \ldots, a_{n-1}) \prod_j \mid a_j \mid^{js-2j(n-j)} da\,.$$

This multiple Tate integral converges for $s > 2(n-1)$. Their product, over all places, converges absolutely for $s > 1 + 2(n-1)$. Proposition 5 is thus proved.

6. Global computations

In this section, our goal will be to prove the equality of the integrals I_j defined in the previous section (for $\Re s$ sufficiently large). Since the original integral I is clearly equal to I_0 this will show that I is actually equal to I_{n-1}: this will give the integral representation of the exterior square L-function we were looking for.

6.1. In this subsection and the next we prove preliminaries results. Recall the group V_j. Its elements are the matrices of the form:

$$v = \begin{pmatrix} u_1 & y \\ t & u_2 \end{pmatrix},$$

where

$$u_1, u_2 \in U_{n-j,n}\,,\ y \in \mathfrak{p}_{n-j,n}\,,\ t \in \mathfrak{u}_{n-j,n}\,.$$

Recall also the subgroup W_j of V_j. It is the subgroup of matrices v for which

$$t \in \mathfrak{u}_{n-j+1,n}\,.$$

It will be convenient to denote by V_j^* the quotient

$$V_j(F)\backslash V_j(F_{\mathbb{A}})\,,$$

and to use a similar notation for other nilpotent groups.

PROPOSITION 1. *We have:*

$$\sum_{\mu} \int_{V_j^*} \phi\left[v \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} g\right] \theta(v) dv$$

$$= \int_{W_j^*} \phi(vg)\theta(v)dv \,,$$

the sum for

$$\mu \in G_{n-j-1}(F)U_{n-j-1,n}(F)\backslash G_{n-j}(F) \,.$$

The expression on the left converges in the sense that:

$$\sum_{\mu} \left| \int_{V_j^*} \phi\left[v \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} g\right] \theta(v) dv \right| < +\infty \,.$$

We first remark that W_j is a normal subgroup of V_j. Furthermore the element

$$\xi = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

normalizes V_j. Indeed, we have:

$$\xi^{-1} v \xi = \begin{pmatrix} \mu^{-1} u_1 \mu & \mu^{-1} y \mu \\ \mu^{-1} t \mu & \mu^{-1} u_2 \mu \end{pmatrix} ,$$

and μ, being in $P_{n-j,n}$, normalizes the unipotent radical $U_{n-j,n}$ as well as the Lie algebras $\mathfrak{p}_{n-j,n}$ and $\mathfrak{u}_{n-j,n}$. If in addition v is in W_j, then t is in $\mathfrak{u}_{n-j+1,n}$ and

$$\mu^{-1} t \mu = t \,.$$

Thus ξ normalizes W_j. Furthermore ξ fixes the restriction of the character θ to the subgroup W_j. The left hand side of the equality can thus be written:

$$\sum_{\mu} \int_{V_j^*} \phi(\xi^{-1} v \xi g)\theta(v) dv \,,$$

or, after changing variables:

$$\sum_{\mu} \int_{V_j^*} \phi(vg)\theta(\xi v \xi^{-1}) dv \,.$$

Now let U be the subgroup of V_j of matrices of the form:

$$u = \begin{pmatrix} 1_n & 0 \\ t & 1_n \end{pmatrix},$$

where $t \in \mathfrak{u}_{n-j,n-j+1}$. In other words, t has the form:

$$t = \begin{pmatrix} 0_{n-j} & a & 0 \\ 0 & 0_1 & 0 \\ 0 & 0 & 0_{j-1} \end{pmatrix},$$

where 0_k is the 0 matrix in the ring of $k \times k$ matrices and a is a column of size $n-j$. Then V_j is the semi-direct product of U and W_j. Thus the previous expression can be written as:

$$\sum_\mu \int_{U^*} \left[\int_{W_j^*} \phi(wug)\theta(w)dw \right] \theta(\xi u \xi^{-1})du\,.$$

Now we have:

$$\theta(u) = \psi(\epsilon a)\,,$$

where

$$\epsilon = (\underbrace{0, 0, \dots, 0}_{n-j-1}, 1)\,.$$

Similarly:

$$\theta(\xi u \xi^{-1}) = \psi(\epsilon \mu a)\,.$$

Thus the previous expression can be written as the sum of the Fourier coefficients, except the constant one, of the function

$$\mathcal{F}(a) = \int_{W_j^*} \phi(wug)\theta(w)dw\,.$$

This already establishes the second assertion of the proposition. The first assertion will be proved if we show that the function $\mathcal{F}$ has a zero constant Fourier coefficient. In other words, let θ' be the character of V_j equal to one on W_j and to one on U; we have to show that:

$$\int_{V_j^*} \phi(vg)\theta'(v)dw = 0\,.$$

Now it is easily checked that V_j contains the group U', conjugate under σ^{-1} of the unipotent radical of the parabolic subgroup of type

$(2j, 2n-2j)$, and that θ' is trivial on U'. Since V_j is unipotent, it is contained in a maximal unipotent subgroup. Thus in fact the unipotent radical of a parabolic subgroup contained in V_j is normal in V_j. Hence U' is normal in V_j. It follows that the above integral factors trough the integral of ϕ on U'^*, which is zero because ϕ is cuspidal. This completes the proof of the proposition.

6.2. PROPOSITION 2. *Let $j \geq 1$. Then:*

$$\int \left\{ \int_{W_j^*} \phi \left[w \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} g \right] \theta(w) dw \right\} dZ =$$

$$\int \left\{ \left\{ \int_{V_{j-1}^*} \phi \left[v \begin{pmatrix} 1_n & Z' \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} g \right] \theta(v) dv \right\} dZ' \right\} du\,.$$

Here Z is integrated over

$$\overline{\mathfrak{u}}_{n-j,n}(F_{\mathbb{A}})\,,$$

Z' is integrated over

$$\overline{\mathfrak{u}}_{n-j+1,n}(F_{\mathbb{A}})\,,$$

u over

$$U_{n-j,n-j+1}(F) \backslash U_{n-j,n-j+1}(F_{\mathbb{A}})\,.$$

We first remark that by Propositions 2 and 3 of Section 5, both integrals exist, as iterated integrals. Since u is in G_{n-j+1} it normalizes $P_{n-j+1,n}$, its unipotent radical U_{n-j+1}, the Lie algebras of these two groups and $\overline{\mathfrak{u}}_{n-j+1,n}$. It follows that an element:

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$$

normalizes the group of matrices of the form:

$$\begin{pmatrix} 1_n & Z' \\ 0 & 1_n \end{pmatrix}.$$

and the group V_{j-1}. It also fixes the character θ of V_{j-1}. This gives a meaning to the second integral.

We remark that we can write

$$Z = Z' + Y$$

with $Y \in \overline{\mathfrak{u}}_{n-j,n-j+1}(F_{\mathbb{A}})$ and break up the integral in Z into an integral in Z' and an integral in Y. Thus it suffices to prove the following equality:

PROPOSITION 3. *With the previous notations:*

$$\int \left\{ \int_{W_j^*} \phi \left[w \begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} g \right] \theta(w) dw \right\} dY =$$

$$\int \left\{ \int_{V_{j-1}^*} \phi \left[v \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} g \right] \theta(v) dv \right\} du \,.$$

We again remark that by (the proof of) Proposition 3 in Section 5, the integral on the left exists, as an interated integral. In order to establish this proposition, we break the integration in Y into a summation in

$$\eta \in \overline{\mathfrak{u}}_{n-j,n-j+1}(F)$$

followed by an integration in

$$Y \in \overline{\mathfrak{u}}^*_{n-j,n-j+1} \,.$$

The integral on the left hand side takes then the form:

$$\int_{\overline{\mathfrak{u}}^*_{n-j,n-j+1}} J \left[\begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} g \right] dY \,,$$

where we have set:

$$J(g) = \sum_{\eta} \int_{W_j^*} \phi(w\xi g) \theta(w) dw \,,$$

and

$$\xi = \begin{pmatrix} 1_n & \eta \\ 0 & 1_n \end{pmatrix} .$$

Next we remark that W_j is the semi-direct product of two subgroups W and U, with W normal: the group W is the group of matrices of the form:

$$v = \begin{pmatrix} u_1 & y \\ t & u_2 \end{pmatrix} ,$$

with

$$u_1, u_2 \in U_{n-j+1,n}\,,\ t \in \mathfrak{u}_{n-j+1,n}\,,\ y \in \mathfrak{p}_{n-j,n}\,.$$

The group U is the group of matrices of the form:

$$v = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

with

$$u_i \in U_{n-j,n-j+1}\,.$$

Thus we can write:

$$J(g) = \sum_\eta \int_{U^*} \int_{W^*} \phi\left[w \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \xi g\right] \theta(w) dw du\,.$$

Now write:

$$u_i = \begin{pmatrix} 1_{n-j} & v_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{j-1} \end{pmatrix}.$$

Similarly write:

$$\eta = \begin{pmatrix} 0_{n-j} & 0 & 0 \\ \lambda & 0_1 & 0 \\ 0 & 0 & 0_{j-1} \end{pmatrix}.$$

Then:

$$u_1 \eta u_2^{-1} = \begin{pmatrix} v_{1\lambda} & -v_1 \lambda v_2 & 0 \\ \lambda & -\lambda v_2 & 0 \\ 0 & 0 & 0_{j-1} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \xi =$$

$$\begin{pmatrix} 1_n & z \\ 0 & 1_n \end{pmatrix} \xi \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where $z \in \mathfrak{p}_{n-j,n}$ and

$$\mathrm{Tr} z = \lambda(v_1 - v_2)\,.$$

Using this identity in the expression for J and changing variables, we find:

$$J(g) = \sum_\lambda \int_{U^*} \int_{W^*} \phi\left[w\xi \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} g\right] \theta(w) dw \psi(-\lambda(v_1 - v_2)) du\,.$$

Now ξ normalizes W and fixes the restriction of θ to W. Thus we may finally write:

$$J(g) = \sum_{\lambda} \int_{U^*} \int_{W^*} \phi \left[w \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} g \right] \theta(w) dw \psi(\lambda(v_1 - v_2)) du .$$

By Fourier analysis, this reduces to:

$$J(g) = \int \int_{W^*} \phi \left[w \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} g \right] \theta(w) dw du ,$$

the integral in u being over

$$U^*_{n-j,n-j+1} .$$

The original integral we had to transform was:

$$\int J \left[\begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} g \right] dY .$$

It is thus equal to:

$$\int \int \int \phi \left[w \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} g \right] dw du dY .$$

We now have the following commutation relation:

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} =$$

$$\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} ,$$

where $z \in \mathfrak{p}_{n-j,n-j+1}$ and $\mathrm{Tr} Z = 0$. The first matrix on the right hand side is thus in W and θ trivial on it. Finally V_{j-1} is the semi-direct product of the group W and the group of matrices of the form:

$$\begin{pmatrix} 1_n & Y \\ 0 & 1_n \end{pmatrix} ,$$

where $Y \in \overline{\mathfrak{u}}_{n-j,n-j+1}$. Thus we finally find our integral is actually equal to:

$$\int \left\{ \int_{V^*_{j-1}} \phi \left[v \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} g \right] \theta(v) dv \right\} du ,$$

as required. This concludes the proof of Proposition 3 and 2.

6.3. We now state the main result of this section. Recall the integrals I_j defined in 5.2.

PROPOSITION 4. *For $j \geq 1$ we have $I_j = I_{j-1}$.*
We first compute formally and justify our steps later, in particular the convergence of the integrals I_j for $j < n-1$. We have (we write 1 for 1_n):

$$I_j = \int\int f(g,s)\left\{\int \phi\left[v\begin{pmatrix} 1 & Z \\ 0 & 1\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g\end{pmatrix}\right]\theta(v)dv\right\} dg\psi(\mathrm{Tr}Z)dZ\,,$$

where

$$v \in V_j^*\,,$$

and

$$Z \in \mathfrak{p}_{n-j,n}(F_{\mathbf{A}})\backslash M_n(F_{\mathbf{A}})\,,$$

and

$$g \in G_{n-j-1}(F)U_{n-j-1,n}(F)U_{n-j,n}(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}})\,.$$

We can view the Z integration as an integration over

$$\overline{\mathfrak{u}}_{n-j,n}(F_{\mathbf{A}})\,.$$

On the other hand the group:

$$G_{n-j-1}(F)U_{n-j-1,n}(F)U_{n-j,n}(F_{\mathbf{A}})$$

is the semi-direct product of

$$U_{n-j,n}(F_{\mathbf{A}})$$

and

$$G_{n-j-1}(F)U_{n-j-1,n-j}(F)\,.$$

With an abuse of notation, this allows us to write:

$$I_j = \int f(g,s)\sum_{\mu}\left\{\int\left\{\int\phi\left[v\begin{pmatrix} 1 & Z \\ 0 & 1\end{pmatrix}\begin{pmatrix} \mu g & 0 \\ 0 & \mu g\end{pmatrix}\right]\theta(v)dv\right\}dZ\right\}dg\,,$$

where

$$v \in V_j^*\,,$$

$$Z \in \overline{\mathrm{u}}_{n-j,n}(F_{\mathbf{A}})\,,$$

$$\mu \in G_{n-j-1}(F)U_{n-j-1,n-j}(F)\backslash G_{n-j}(F)\,,$$

and

$$g \in G_{n-j}(F)U_{n-j,n}(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}})\,.$$

Using the fact that the matrix

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

normalizes the group of matrices of the form:

$$\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix},$$

we can interchange the integration in Z and the summation in μ to arrive at the following expression:

$$\int f(g,s)\left\{\int \sum_{\mu} \left\{\int \phi \left[v\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]\theta(v)dv\right\}dZ\right\}dg\,.$$

By Proposition 1, this is equal to:

$$\int f(g,s)\left\{\int \left\{\int \phi\left[v\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]\theta(v)dv\right\}dZ\right\}dg\,.$$

where v is now integrated over W_j^*. Now we apply Proposition 2. This is also equal to:

$$\int f(g,s)\left\{\int \left\{\int_{V_{j-1}^*} \phi \left[v\begin{pmatrix} 1 & Z' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} ug & 0 \\ 0 & ug \end{pmatrix}g\right]\theta(v)dv\right\}du dZ'\right\}dg\,.$$

Here Z' is integrated over

$$\overline{\mathfrak{u}}_{n-j+1,n}(F_{\mathbf{A}}),$$

u over

$$U_{n-j,n-j+1}(F)\backslash U_{n-j,n-j+1}(F_{\mathbf{A}}).$$

and g over

$$g \in G_{n-j}(F)U_{n-j,n}(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}}).$$

We can now combine the integration in u and the integration in g to obtain an integration in g over:

$$g \in G_{n-j}(F)U_{n-j,n}(F)U_{n-j+1,n}(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}}).$$

The resulting integration is I_{j-1} and we are done. However, we have to justify the formal manipulations.

6.4. We prove by descending induction on j with $n-1 \geq j \geq 0$ that the following integral is finite:

$$\int |\, f(g,s)\, | \left| \phi\left[v \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right] \theta(v) dv dZ \right| dg\,.$$

In Proposition 5 of section 5, we have proved this integral is finite for $j = n-1$. We may therefore assume $j \leq n-1$ and the above integral is finite for j. We have to show the analogous integral with j replaced by $j-1$ is finite. Let us write $G_n(F_{\mathbf{A}})$ as the union of a sequence of compact sets Ω_k. Let m_k be the product of $|\, f(g,s)\, |$ and the characteristic function of the set

$$G_{n-j}(F)U_{n-j,n}(F_{\mathbf{A}})\Omega_k Z_n(F_{\mathbf{A}}).$$

Then the sequence of formal manipulations of the previous subsection can be used to prove the following inequality:

$$\int m_k(g) \left| \phi\left[v \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right] \theta(v) dv dZ \right| dg \geq$$

$$\int m_k(g) \left| \phi\left[v \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right] \theta(v) dv dZ \right| dg\,.$$

Here the integral on the left is for:

$$v \in V_j^*,$$

$$Z \in \overline{\mathfrak{p}}_{n-j,n}(F_{\mathbb{A}}),$$

$$g \in G_{n-j-1}(F)U_{n-j-1}(F)U_{n-j,n}(F_{\mathbb{A}})\backslash G_n(F_{\mathbb{A}})/Z_n(F_{\mathbb{A}}).$$

The integral on the right is for:

$$v \in V_{j-1}^*,$$

$$Z \in \overline{\mathfrak{p}}_{n-j+1,n}(F_{\mathbb{A}}),$$

$$g \in G_{n-j}(F)U_{n-j}(F)U_{n-j+1,n}(F_{\mathbb{A}})\backslash G_n(F_{\mathbb{A}})/Z_n(F_{\mathbb{A}}).$$

We should remark that by Proposition 2 of section 5, the expression on the right is meaningful and is finite. Now the integral on the left is majorized by the integral obtained by replacing m_k by the absolute value of $f(g,s)$, which is finite by the induction hypothesis. Letting k tends to infinity, we obtain our conclusion for $j-1$.

6.5. Let us now consider the integral

$$I = I(s, \chi, \phi, \Phi)$$

defined in 5.1. Taking s sufficiently large and replacing the Eisenstein series by its expression as a series, we obtain that $I = I_0$. By the previous propostion we have therefore $I = I_{n-1}$. Now in the integral I_{n-1} we will replace the group V_{n-1} by its conjugate under σ, namely the group N_{2n}. Set, as before:

$$W(g) = \int_{N_{2n}(F)\backslash N_{2n}(F_{\mathbb{A}})} \phi(ug)\theta(u)du.$$

We recall that θ is the character of $N_{2n}(F_{\mathbb{A}})$ defined by:

$$\theta(u) = \prod_j \psi(u_{j,j+1}).$$

Introduce the integral:

$$J = J(s, \chi, \phi, \Phi),$$

where

$$J = \int W\left[\sigma\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]\psi(\mathrm{Tr}Z)dZf(g,s)dg\,.$$

Here Z is integrated over the quotient:

$$\mathfrak{u}_{0,n}(F_{\mathbf{A}})\backslash M_n(F_{\mathbf{A}})\,,$$

and g over the quotient

$$N_n(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})/Z_n(F_{\mathbf{A}})\,.$$

Replacing f by its definition (given in 1.1) we find:

$$J = \int W\left[\sigma\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]$$
$$\psi(\mathrm{Tr}Z)dZ\Phi(\epsilon g)\chi(\det g) \mid \det g \mid^s dg\,,$$

where g is now integrated over

$$N_n(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})\,.$$

Proposition 5 of section 5 shows this integral converges absolutely for $\Re s$ sufficiently large. We have proved:

PROPOSITION 5. *For $\Re s$ sufficiently large:*

$$I(s,\chi,\phi,\Phi) = J(s,\chi,\phi,\Phi)\,.$$

7. LOCAL COMPUTATIONS

In this section we go back to a local situation. We let F be a local field, ψ a non-trivial additive character, $r = 2n$ an even integer, π a unitary irreducible generic representation of $G_r(F)$, χ a character of module 1. We denote by $\mathcal{W}(\pi,\psi)$ the Whittaker model of π. Let W be in $\mathcal{W}(\pi,\psi)$ and Φ a Schwartz-Bruhat function in n variables. We consider the integral:

$$J = J(s,\chi,W,\Phi)$$

defined by

$$J = \int W\left[\sigma\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right]$$
$$\psi(-\mathrm{Tr}Z)dZ\Phi(\epsilon g)\chi(\det g) \mid \det g \mid^s dg\,,$$

where Z is integrated over the quotient:

$$\mathfrak{p}_{0,n}(F)\backslash M_n(F)\,,$$

and g over the quotient

$$N_n(F)\backslash G_n(F)\,,$$

and we have set:

$$\epsilon = (\underbrace{0,0,\ldots,0}_{n-1},1)\,.$$

We recall M_n is the space of $n \times n$ matrices, $\mathfrak{p}_{0,n}$ the space of upper triangular matrices and N_n the group of upper triangular matrices with unit diagonal. In what follows, we often write G_n for $G_n(F)$ and use a similar notation for other groups and vector spaces.

7.1. We will need to know this integral converges absolutely for $\Re s \geq 0$:

PROPOSITION 1. *Given π there is $\eta > 0$ such that the integral J converges absolutely for $\Re s > 1 - \eta$.*

The proof is similar to the proof of Proposition 5 in section 5. We have to see that for a suitable $\eta > 0$ and $s > 1 - \eta$ the following integral is finite:

$$\int \left| W\left[\sigma \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} k\right]\right| dZ \mid \det a \mid^s \delta_n(a)^{-1} da dk\,,$$

where, as before, δ_n denotes the module of the Borel subgroup $P_{0,n}$ and

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_{n-1}, 1)\,.$$

The variables a_i are integrated over the multiplicative group $F^\times$ and Z over $\overline{\mathfrak{u}}_{0,n}$, the space of lower triangular matrices. As before we introduce the element

$$u_Z = \sigma \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \sigma^{-1}\,.$$

Then the integral can be written:

$$\int |W(bu_Z k)|\, \delta_n(a)^{-2} da dZ dk\,,$$

where we have set:

$$b = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, 1, 1)\,.$$

Next we introduce the Iwasawa decomposition of u_Z:

$$u_z = n_Z t_Z k_Z\,,$$

and we denote by t_j the diagonal entries of the diagonal component t_Z. Then the previous integral can be written:

$$\int |W(bt_Z k)|\, \delta_n(a)^{-2} \mid \det a \mid^s da dZ dk\,.$$

Now by Proposition 3 of section 4, there is a finite set X of finite functions in $n-1$ variables such that:

$$|W(bt_Z k)|$$

is bounded by a finite sum of expressions of the form:

$$\delta_{n-1}^{1/2}(bt_Z)\chi\Phi\left(\frac{b_1 t_1}{b_2 t_2}, \frac{b_2 t_2}{b_3 t_3}, \dots, \frac{b_{2n-1} t_{2n-1}}{b_{2n} t_{2n}}\right)\,,$$

where χ is the absolute value of some element of X and $\Phi \geq 0$ is a fixed Schwartz-Bruhat function. Of course the b_j are the diagonal entries of b. Thus it suffices to prove that the integral obtained by replacing W by this estimate is finite. Next we remark that:

$$\delta_{n-1}^{1/2}(b)\delta_n^{-2}(a) = \mid a_1, a_2, \dots, a_{n-1} \mid^{-1}\,.$$

Finally, we see it suffices to show the following integral is finite:

$$\int \chi\Phi\left(\frac{b_1 t_1}{b_2 t_2}, \frac{b_2 t_2}{b_3 t_3}, \dots, \frac{b_{2n-1} t_{2n-1}}{b_{2n} t_{2n}}\right) \mid a_1 a_2 \cdots a_{n-1} \mid^{s-1} \delta_{n-1}^{1/2}(t_Z) da dZ\,.$$

Now recall the definition of b in terms of the a_j:

$$b = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, 1, 1)\,.$$

After suitable translations in the variables a_i and under the additional assumption that the function Φ is a product of functions of each coordinate, we find the above integral is a sum of products of two integrals of the form:

$$\int \Phi_1\left(\frac{t_1}{t_2}, \frac{t_3}{t_4}, \dots, \frac{t_{2n-1}}{t_{2n}}\right) \mu(t_Z) dZ\,,$$

and

$$\int \chi\Phi_2\left(\frac{a_1}{a_2}, \frac{a_2}{a_3}, \dots, a_{n-1}\right) \mid a_1 a_2 \cdots a_{n-1} \mid^{s-1} da\,.$$

Here Φ_1 and Φ_2 are Schwartz-Bruhat functions and χ belongs to X. In addition, μ is the absolute value of a certain finite function (depending on s).

As in 5.6, the first integral converges. As for the second integral, a simple change of variables shows it is a sum of integrals of the form:

$$\int \chi\Phi_2(a_1, a_2, \dots, a_{n-1}) \prod_j \mid a_j \mid^{j(s-1)} da\,.$$

Now by Proposition 4 of section 3, we can choose X so that any χ in it is the product of a polynomial in the logarithms of the absolute values of the variables, times a character of the form:

$$\chi_1(a_1)\chi_2(a_2)\cdots\chi_{n-1}(a_{n-1})\,,$$

with $\Re\chi_j > 0$, for each j. It follows that this multiple Tate integral converges in some strip $s > 1 - \eta$, with $\eta > 0$. This completes the proof of the proposition.

7.2. In the unramified situation, the integral J is equal to the local L-factor for the exterior square:

PROPOSITION 2. *Assume F is local, non archimedean. Assume ψ has for conductor the ring of integers and the Haar measures are normalized in the usual way. Suppose χ is unramified and the representation π contains the unit representation of the maximal compact K_r. Assume Φ is the characteristic function of the integers, W the Whittaker function which is invariant under K_r and takes the value one on K_r. Then:*

$$J = L(s, \pi, (\textstyle\bigwedge^2\rho) \otimes \chi)\,.$$

Taking in account the invariance of W under K_r, we get that J is equal to the following expression, where we write 1 for the matrix 1_n:

$$\int W\left[\sigma\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right] \delta_n^{-1}(a)\chi(\det a) \mid \det a \mid^s \Phi(\epsilon a_n) da \psi(-\mathrm{Tr} Z) dZ\,,$$

with

$$a = \mathrm{diag}(a_1, a_2, \dots, a_{n-1}, a_n)\,.$$

After a formal manipulation this becomes

$$J = \int W(bu_Z)\delta_{2n}^{-1/2}(b)\chi(\det a) \mid \det a \mid^s \Phi(\epsilon a_n)dadZ\,,$$

where we have set

$$b = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, a_n)\,,$$

and used the relation

$$\delta_n^2(a) = \delta_{2n}^{1/2}(b)\,.$$

Let us once more introduce the Iwasawa decomposition of u_Z:

$$u_z = n_Z t_Z k_Z\,.$$

We have then

$$J = \int W(bt_Z)\theta(bn_Zb^{-1})\delta_{2n}^{-1}(b)\chi(\det a) \mid \det a \mid^s \Phi(\epsilon a_n)dadZ\,.$$

Now if $W(bt_z) \neq 0$ then the diagonal entries of b and t_Z must satisfy

$$\mid b_j t_j \mid \leq \mid b_{j+1} t_{j+1} \mid\,.$$

This gives in fact

$$\mid t_j \mid \leq \mid t_{j+1} \mid$$

for j odd. However, by Proposition 4 of section 5, we have

$$\mid t_j \mid \geq 1 \text{ for } j \text{ odd}\,,$$

$$\mid t_j \mid \leq 1 \text{ for } j \text{ even}\,.$$

It follows that $W(bt_z) \neq 0$ implies that the t_j are units and then, by Proposition 5 of section 5, that the entries of Z are integers. We may take then $t_Z = n_Z = 1$ and we obtain finally:

$$J = \int W(b)\delta_{2n}^{-1/2}(b)\chi(\det a) \mid \det a \mid^s \Phi(\epsilon a_n)da\,.$$

By the results of Section 3 this integral is equal to the required L-factor.

7.3. We will need one more local result:

PROPOSITION 3. *There are Φ and W such that:*

$$J(1,\chi,\Phi,W)\neq 0\,.$$

Assume on the contrary that $J=0$ for all choices of W and Φ. Then we get:

$$\int W\left[\sigma\begin{pmatrix}1_n & Z\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}g & 0\\ 0 & g\end{pmatrix}\right]$$
$$\psi(-\mathrm{Tr}Z)dZ\chi(\det g)\mid \det g\mid \Phi(\epsilon_n g)dg=0\,,$$

for all Φ, where we have set:

$$\epsilon_n=(\underbrace{0,0,\dots,0}_{n-1},1)\,.$$

As before Z is integrated over

$$\mathfrak{p}_{0,n}\backslash M_n\,,$$

and g over

$$N_n\backslash G_n\,.$$

We can choose for Φ a function whose support is contained in the orbit of ϵ_n under G_n. Then $g\mapsto\Phi(\epsilon_n g)$ is arbitary among the functions invariant on the left under the subgroup

$$G_{n-1}U_{n-1,n}\,,$$

of compact support modulo that subgroup. It follows that

$$\int W\left[\sigma\begin{pmatrix}1_n & Z\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}g & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & g & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right]\psi(-\mathrm{Tr}Z)dZ\chi(\det g)dg=0\,,$$

where Z is integrated as before and g is now integrated over

$$N_{n-1}\backslash G_{n-1}\,.$$

We will show by descending induction on k with $0\leq k\leq n-1$ that the following integral:

$$I_k=$$

$$\int W\left[\sigma\begin{pmatrix} 1_{k+1} & 0 & Z & 0 \\ 0 & 1_{n-k-1} & 0 & 0 \\ 0 & 0 & 1_{k+1} & 0 \\ 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix} \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\chi(\det g)\mid \det g\mid^{2(k+1-n)}\, dg$$

is zero for all W. Here Z is integrated over

$$\mathfrak{p}_{0,k+1}\backslash M_{k+1}\,,$$

and g over

$$N_k\backslash G_k\,.$$

We have just seen that $I_{n-1}=0$. We may therefore assume $k\le n-1$ and $I_k=0$ for all W. We have to show that $I_{k-1}=0$ for all W.

To that end, in the definition of I_k, we replace the matrix on the left by the following matrix:

$$\begin{pmatrix} 1_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & 1_{n-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix}.$$

The integration is now for

$$Z\in\mathfrak{p}_{0,k}\backslash M_k$$

and

$$Y\in F^k\,;$$

in other words, Y is a row of size k. After simple matrix multiplications and the change of Y to Yg^{-1}, we find for the integral I_k the expression:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} 1_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & 1_{n-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\chi(\det g) \mid \det g \mid^{2(k-n)+1} dg\,.$$

This expression is thus 0 for all W. We apply this fact to the function W_1 defined by:

$$W_1(g) = \int W\left[g\begin{pmatrix} 1_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{n-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k & u & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix}\right]\Phi(u)du\,,$$

where u is a column of size k and Φ a smooth function of compact support. After simple matrix multiplications, we see that the integral can be written in the form:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & Yu & 0 \\ 0 & 0 & 1_{n-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k & gu & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix}\right.$$

$$\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}$$

$$\left.\begin{pmatrix} 1_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & 1_{n-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZdY\Phi(u)du\chi(\det g) \mid \det g \mid^{2(k-n)+1} dg\,.$$

The conjugate under σ of the last matrix on the left is in N_{2n}. Taking into account the fact that W transforms under the character θ of N_{2n}, we find for the previous integral the alternate expression:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} 1_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & 1_{n-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n-k-1} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\hat{\Phi}(Y)dY\chi(\det g)\mid \det g\mid^{2(k-n)+1} dg\,,$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ:

$$\hat{\Phi}(Y) = \int \Phi(u)\psi(Yu)du\,.$$

This expression is thus zero for all W and for any function $\hat{\Phi}$ which is the Fourier transform of a smooth function of compact support. By (the proof of) Proposition 1, this integral converges if we replace $\hat{\Phi}$ by one. It follows that we can replace $\hat{\Phi}$ by any Schwartz-Bruhat function and still obtain a zero integral. In particular, we may replace it by a smooth function of compact support. It follows that the following integral is zero:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\chi(\det g)\mid \det g\mid^{2(k-n)+1} dg\,.$$

Next, we apply this relation to the function W_1 defined by:

$$W_1(g) = \int W\left[g\begin{pmatrix} 1_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{n-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & u & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{n-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]\Phi(u)du\,,$$

where u is a column of size k and Φ a smooth function of compact support. After simple matrix multiplications, we obtain the following expression for the previous integral:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & Zgu & 0 & 0 & 0 \\ 0 & 1_{n-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & gu & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{n-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\Phi(u)du\chi(\det g)\mid \det g\mid^{2(k-n)+1} dg\,.$$

Again the conjugate of the last matrix on the left under σ is in N_{2n}. Using the invariance property of W, we obtain the following alternate expression for the previous integral:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\hat{\Phi}(\epsilon_k g)\chi(\det g)\mid \det g\mid^{2(k-n)+1} dg\,.$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ: for a row t of size k,

$$\hat{\Phi}(t) = \int \Phi(u)\psi(tu)du\,.$$

This expression is thus zero for all Φ and all W. We can again replace in this integral $\hat{\Phi}$ by any Schwartz-Bruhat function and still obtain a zero integral. This implies then that the following integral is zero:

$$\int W\left[\sigma\begin{pmatrix} 1_k & 0 & Z & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & 1_{n-k+1} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1_{n-k+1} \end{pmatrix}\right]$$

$$\psi(-\mathrm{Tr}Z)dZ\chi(\det g)\mid \det^{2(k-n)}\mid dg\,;$$

here Z is integrated over

$$\mathfrak{p}_{0,k}\backslash M_k$$

and g over

$$N_{k-1}\backslash G_{k-1}\,.$$

However, this integral is precisely the integral I_{k-1} and we are done.

For $k=0$ the relation $I_0=0$ reads $W(e)=0$ for all W, a contradiction. This proves Proposition 3.

8. THE GLOBAL PERIOD INTEGRAL

In this section, we go back to a global situation. We let F be a number field, ψ a non-trivial character of $F_{\mathbb{A}}/F$. We consider the exterior square L-function attached to an automorphic cuspidal (unitary) representation π of $G_r(F_{\mathbb{A}})$ with central character ω_π. We let S be a finite set of places containing all places at infinity and all places where the representation π ramifies. We also assume that for $v\notin S$ the character ψ_v has the ring of integers for conductor. We consider an idele-class character χ of module 1, unramified outside S, and we set

$$L^S(s,\pi,(\bigwedge^2\rho)\otimes\chi)=\prod_{v\notin S}L(s,\pi_v,(\bigwedge^2\rho)\otimes\chi_v)\,.$$

We will write simply $L^S(s)$ for this partial L-function. Our main result is the following theorem:

THEOREM 1. *The function L^S extends as a meromorphic function to a half plane $\Re s>1-\eta$ with $\eta>0$. It has a pole at $s=1$ if and only if $\chi^n\omega_\pi=1$ and there is a K_r-finite vector ϕ in the space of π such that the following integral is not zero:*

$$\int\phi\left[\begin{pmatrix}1_n & X\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}g & 0\\ 0 & g\end{pmatrix}\right]\psi(\mathrm{Tr}X)dX\chi(\det g)dg\,.$$

Here g is integrated over

$$G_n(F)\backslash G_n(F_{\mathbb{A}})/Z(F_{\mathbb{A}})\,,$$

and X over

$$M_n(F)\backslash M_n(F_{\mathbb{A}})\,.$$

8.1. Let ϕ be a smooth vector in the space of π. Assume that the corresponding Whittaker function W is a product of local ones (notations are as in 6.5). Let also Φ be a Schwartz-Bruhat function in n variables which is a product of local functions. Assume that Φ_v is the characteristic function of the lattice of integral points for all $v \notin S$. According to the results of Sections 5 and 6, the global integral $I(s, \chi, \phi, \Phi)$ can be written, for $\Re s$ sufficiently large, as the product:

$$I(s, \chi, \phi, \Phi) = L^S(s) \prod_{v \in S} J(s, \chi_v, W_v, \Phi_v) .$$

By Proposition 1 in 7.1, we can choose $\eta > 0$ such that the local integrals J (for $v \in S$) converge and are holomorphic in the half-plane $\Re s > 1 - \eta$. Furthemore the local integrals are not identically zero for at least one choice of the local data (for instance by Proposition 3 in 7.3). It follows that $L^S(s)$ extends to a meromorphic function to that half-plane. Of course from the results of Shahidi ([S]) we know it actually extends to the whole complex plane.

If $\chi^n \omega_\pi \neq 1$ the global integral is holomorphic at $s = 1$ for all choices of Φ and ϕ (Proposition 1 in 5.1). Choose ϕ and Φ as before, but, in addition, choose the local data W_v and Φ_v for $v \in S$ in such a way that the local integrals (for $v \in S$) do not vanish at $s = 1$ (Proposition 3 in 7.3). Then the previous relation shows that L^S must be holomorphic at $s = 1$.

Now assume that $\chi^n \omega_\pi = 1$. The period integral of the theorem converges for any smooth ϕ and depends continuously on ϕ (for the topology of the smooth vectors in the unitary representation π). It follows that the period integral vanishes for all smooth ϕ if and only if it vanishes for all K_r-finite ϕ.

If the period integral in the theorem is zero for all ϕ, then the global integral I is actually holomorphic at $s = 1$ for all choices of the data. Again this implies that L^S is holomorphic at $s = 1$.

Finally, suppose that the integral of the theorem is non-zero for at least one choice of ϕ. By linearity and continuity, it must be non-zero for a function ϕ whose associated function W is a product. Then W_v is K_v invariant for all $v \notin T$ where $T \supseteq S$. Let Φ be a Schwartz-Bruhat function which is a product of local functions; assume $\Phi_v(0) \neq 0$ for all v and Φ_v is the characteristic function of the lattice of integral points for all $v \notin T$. The global integral $I(s, \chi, \phi, \Phi)$ has a pole at

$s = 1$. Since

$$I(s,\chi,\phi,\Phi) = L^T(s) \prod_{v\in T} J(s,\chi_v,W_v,\Phi_v)\,,$$

and the local integrals are holomorphic at $s = 1$ the function $L^T(s)$ has a pole at $s = 1$.

Suppose the order of this pole is higher than one. Then we can choose a ϕ whose associate function W is a product and a function Φ which is a product satisfying the following conditions: for $v \notin T$ the function W_v is K_r invariant and the function Φ_v the characteristic function of the lattice of integral points; for $v \in T$ the local integral $J(s,\chi_v,W_v,\Phi_v)$ is not zero at $s = 1$. Then, by the previous relation, the global integral has a pole of order higher than 1, a contradiction. Thus the function L^T has a pole of order 1 at $s = 1$. Now we have:

$$L^S(s) = L^T(s) \prod_{v\in T-S} L(s,\pi_v,(\textstyle\bigwedge^2\rho)\otimes\chi_v)\,.$$

In this formula the local L-factors are non-zero and holomorphic at $s = 1$, because of the convergence of the local integral. It follows that L^S must have a simple pole at $s = 1$. This concludes the proof of the theorem.

9. The odd case

In this section we briefly discuss the exterior square L-function for the group G_r, where $r = 2n+1$ is an odd integer. We simply indicate what changes must be made in the proofs we have given in the case of an even integer r.

9.1. Let us discuss the global situation first. Thus we let π be a cuspidal unitary representation of $G_r(F_{\mathbf{A}})$ and χ an idele-class character of module one. For ϕ in the space of π, we define an integral

$$I = I(s,\chi,\phi)$$

as follows:

$$I = \int \phi\left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$\psi(\mathrm{Tr}Z)dZdY\chi(\det g) \mid \det g \mid^{s-1} dg\,,$$

where the matrix Z is integrated over

$$M_n(F)\backslash M_n(F_{\mathbf{A}})\,,$$

the column Y is integrated over

$$F^n\backslash F^n_{\mathbf{A}}\,,$$

and g over

$$G_n(F)\backslash G_n(F_{\mathbf{A}})\,.$$

PROPOSITION 1. *The following integral is finite for all real s:*

$$\int\left|\int\phi\left[\begin{pmatrix}1_n & Z & Y\\ 0 & 1_n & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & 1\end{pmatrix}\right]\psi(\mathrm{Tr}Z)dZdY\right| \mid \det g \mid^{s-1} dg\,.$$

We will first show that for $s<1$ the following integral is finite:

$$\int\left|\phi\left[\begin{pmatrix}1_n & Z & Y\\ 0 & 1_n & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & 1\end{pmatrix}\right]\right| dZdY \mid \det g \mid^{s-1} dg\,.$$

To that end, we replace the integral in g by an integral over a Siegel domain. Thus, it will suffice to show that, given a compact set M, there is a constant C such that:

$$\int\left|\phi\left[\begin{pmatrix}1_n & Z & Y\\ 0 & 1_n & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}az & 0 & 0\\ 0 & az & 0\\ 0 & 0 & 1\end{pmatrix}m\right]\right|$$

$$dZdY \mid \det a \mid^{s-1}\mid z \mid^{2n(s-1)} \delta_n(a)^{-1}dadz < C\,,$$

for $m \in M$. Here

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_{n-1}, a_n)\,,$$

z is a scalar; the a_i and z are ideles whose finite components are 1 and whose infinite components are all equal to the same positive number; in addition:

$$\mid a_i/a_{i+1} \mid \geq c \text{ and } a_n = 1\,,$$

where $c > 0$ is some constant. Let us write

$$Z = U + V,$$

where V is upper triangular and U lower triangular with 0 diagonal entries. Then $a^{-1}Va$ remains in a fixed compact set. At the cost of enlarging the compact set M, we see it suffices to prove our assertion for the integral obtained by replacing the integration in Z by an integration in U. Now let τ be the following permutation matrix:

$$\tau = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1_{2n-2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\tau \begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \tau^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix},$$

where $u \in G_{2n-1}$ and $\det u = 1$. On the other hand,

$$\tau \begin{pmatrix} az & 0 & 0 \\ 0 & az & 0 \\ 0 & 0 & 1 \end{pmatrix} \tau^{-1} = z \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & z^{-1} \end{pmatrix},$$

where b is a diagonal matrix with $\det b = \det a^2$. Altogether, using the invariance properties of ϕ, we see that the integral takes the form:

$$\int \left| \phi\left[\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \tau m \right] \right| dU dY \mid \det a \mid^{s-1} \delta_n(a)^{-1} \mid z \mid^{2n(s-1)} da dz,$$

where h is in G_{n-1} and

$$\det h = z^{-1} \det a^2.$$

We appeal again to **[J.S]** II p.799: the integrand is majorized, for all $N > 0$, by a constant multiple of

$$\mid \det a \mid^{s-1} \mid z \mid^{2n(s-1)} \delta_n^{-1}(a) \inf \left(\mid \det a \mid^{2N} \mid z \mid^{-N}, \mid \det a \mid^{-2N} \mid z \mid^{N} \right).$$

If we integrate this estimate over z we find a constant multiple of:

$$\mid \det a \mid^{(4n+1)(s-1)} \delta_n^{-1}(a).$$

Thus it will suffice to show the following integral is finite:

$$\int \mid \det a \mid^{(4n+1)(s-1)} \delta_n^{-1}(a)da\,.$$

However if we express the determinant in terms of the simple roots α_j we have:

$$\det a = \prod \alpha_j^{r_j}(a)\,,$$

where $r_j > 0$. Thus the factor

$$\mid \det a \mid^{(4n+1)(s-1)}$$

is bounded over the range of integration and this implies our assertion.

To prove the proposition for arbitrary s we will establish an identity which gives the functional equation for the integral I; in the functional equation s goes to $1-s$ and ϕ to the function $\tilde{\phi}$ defined by:

$$\tilde{\phi}(g) = \phi({}^t g^{-1})\,.$$

The identity we have in mind is the following one:

$$\int \left\{ \phi \left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & X & 1 \end{pmatrix} g \right] \psi(\mathrm{Tr} Z) dZ dY \right\} dX$$

$$= \int \phi \left[\begin{pmatrix} 1_n & Z & 0 \\ 0 & 1_n & 0 \\ 0 & X & 1 \end{pmatrix} g \right] \psi(\mathrm{Tr} Z) dZ dX dY\,.$$

Here Z and Y are integrated as before; the row X is integrated over $F_{\mathbb{A}}^n$ in the left hand side, and over the quotient $F^n \backslash F_{\mathbb{A}}^n$ in the right hand side.

To prove our identity, we start with the left hand side. We decompose the integration in X into a summation in $\xi \in F^n$ followed by an integration in $X \in F^n \backslash F_{\mathbb{A}}^n$. Because of the matrix identity:

$$\begin{pmatrix} 1_n & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & -\xi & 1 \end{pmatrix} \begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & \xi & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1_n & Z + Y\xi & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find, after a change of variables, that the integral on the left hand side of the equality is equal to:

$$\int \sum_{\xi} \int \phi \left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & X & 1 \end{pmatrix} \right]$$

$$\psi(\mathrm{Tr} Z) dZ \psi(-\xi Y) dY dX .$$

By Fourier analysis this is equal to the right hand side of our identity.

We will use a variant of this identity. Let Φ be a Schwartz-Bruhat function in n variables; we view Φ as a function on the space of rows vectors. Its Fourier transform is the function on the space of columns vectors defined by:

$$\hat{\Phi}(Y) = \int \Phi(-X) \psi(XY) dY .$$

Define functions ϕ_1 and ϕ_2 by:

$$\phi_1(g) = \int \phi \left[g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & X & 1 \end{pmatrix} \right] \Phi(X) dX$$

and

$$\phi_2(g) = \int \phi \left[g \begin{pmatrix} 1 & 0 & Y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \hat{\Phi}(Y) dY .$$

Because we can view ϕ as a smooth vector in a unitary representation, ϕ_1 and ϕ_2 are also smooth vectors in the space of π. We have then:

$$\int \phi_1 \left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\psi(\mathrm{Tr} Z) dZ dY =$$

$$\int \phi \left[\begin{pmatrix} 1_n & Z & 0 \\ 0 & 1_n & 0 \\ 0 & X & 1 \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\psi(\mathrm{Tr} Z) dZ dX \mid \det g \mid .$$

Here Z is integrated as before, the column Y and the row X over $F^n \backslash F^n_{\mathbb{A}}$.

Let us introduce the permutation matrix $\rho \in G_n$ defined by:

$$\rho = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and the permutation matrix $\tau \in G_r$ defined by:

$$\tau = \begin{pmatrix} 0_n & \rho & 0 \\ \rho & 0_n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The previous identity may also be written:

$$\int \phi_1 \left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\psi(\mathrm{Tr} Z) dZ dY =$$

$$\int \tilde{\phi}_2 \left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho^t g^{-1} \rho & 0 & 0 \\ 0 & \rho^t g^{-1} \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\psi(\text{-}\mathrm{Tr} Z) dZ dY \mid \det g \mid .$$

This identity implies the functional equation:

$$I(s, \chi, \phi_1) = I(1 - s, \chi^{-1}, \phi'),$$

where ϕ' is a suitable translate of $\tilde{\phi}_2$. It follows from the previous discussion applied to ϕ' that the integral $I(s, \chi, \phi_1)$ converges for $s > 0$ (in the sense indicated in the proposition). Now by the lemma of **[D-M]** ϕ can be written as a sum of functions of the form ϕ_1 (with Φ smooth of compact support). The proposition follows.

9.2. Next we compute the global integral I in terms of an integral which is a product of local ones. As before, we consider the character θ of N_r and then the Whittaker function W attached to ϕ. We let σ be the permutation which changes the sequence

$$(1, 2, 3, \ldots, n, n+1, n+2, n+3, \ldots, 2n, 1)$$

into the sequence

$$(1,3,5,\ldots,2n-1,2,4,6,\ldots,2n,1)\,.$$

We also denote by σ the corresponding permutation matrix. Then the integral we have in mind is the following one:

$$J = J(s,\chi,W)\,,$$

where

$$J = \int W\left[\sigma\begin{pmatrix} 1_n & Z & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$
$$\psi(\mathrm{Tr}Z)dZ\chi(detg)\mid \det g\mid^{s-1} dg\,.$$

Here Z is integrated over

$$\mathfrak{u}_{0,n}(F_{\mathbf{A}})\backslash M_n(F_{\mathbf{A}})\,,$$

and g over

$$N_n(F_{\mathbf{A}})\backslash G_n(F_{\mathbf{A}})\,.$$

PROPOSITION 2. *The integral J converges absolutely for $\Re s$ large enough and is then equal to the integral I.*

We only indicate the starting point. The function $\mathcal{F}$ (of a column vector) defined by:

$$\mathcal{F}(U) = \int \phi\left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & U \\ 0 & 0 & 1 \end{pmatrix}\right]\psi(\mathrm{Tr}Z)dZdY$$

has a zero constant Fourier coefficient, because of the cuspidality of ϕ. Thus its Fourier series reads:

$$\mathcal{F}(0) = \sum_{\gamma\in P_{n-1,n}(F)\backslash G_n(F)}\int \mathcal{F}(U)\psi(\epsilon_n\gamma U)dU$$

$$= \sum_{\gamma}\int \phi\left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & U \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\psi(\mathrm{Tr}Z)dZdY\psi(\epsilon_n U)dU\,,$$

where, as before,

$$\epsilon_n = (\underbrace{0,0,\ldots,0}_{n-1},1)\,.$$

Using this identity inside the expression for I we find:

$$I = \int \phi\left[\begin{pmatrix} 1_n & Z & Y \\ 0 & 1_n & U \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$\psi(\mathrm{Tr}Z)dZdY\psi(\epsilon_n U)dU\chi(\det g) \mid \det g \mid^{s-1} dg\,.$$

Here g is integrated over the quotient

$$P_{n-1,n}(F)\backslash G_n(F_{\mathbf{A}})\,.$$

9.3. We now discuss the local situation. Accordingly, we let F be a local field and π a unitary irreducible generic representation of $G_r(F)$. For each W in the Whittaker model $\mathcal{W}(\pi,\psi)$ of π we define a local integral $J = J(s,\chi,W)$ by:

$$J = \int W\left[\sigma\begin{pmatrix} 1_n & Z & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$\psi(\text{-}\mathrm{Tr}Z)dZ\chi(det g) \mid \det g \mid^{s-1} dg\,.$$

Here Z is integrated over

$$\mathfrak{p}_{0,n}(F)\backslash M_n(F)$$

and g over

$$N_n(F)\backslash G_n(F)\,.$$

PROPOSITION 3. *There is $\eta > 0$ such that the integral J converges absolutely for $\Re s > 1-\eta$.*

The proof is similar to the proof of proposition 1 in section 1.1: we are reduced to proving the convergence of the integral

$$\int \mid W(bt_Z k) \mid \delta_n(a)^{-2} \mid \det a \mid^{s-1} dadZdk\,,$$

where we have set:

$$b = \mathrm{diag}(a_1, a_1, a_2, a_2, \ldots, a_n, a_n, 1)$$

$$u_Z = \sigma\begin{pmatrix} 1_n & Z & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & 1 \end{pmatrix}\sigma^{-1}\,,$$

and t_Z denotes the diagonal part of the Iwaswa decomposition of Z. At this point we note the relation:

$$\delta_{2n}^{1/2}(b) = \delta_n^2(a)\,.$$

Thus the above integral can be written:

$$\int \mid W(bt_Z k) \mid \delta_{2n}(b)^{-1/2} \mid \det a \mid^{s-1} da dZ dk\,,$$

and the rest of the proof is unchanged.

9.4. We now examine the unramified situation. Accordingly, we let F be a local non-archimedean field. We assume the conductor of ψ is the ring of integers and the character χ is unramified. We assume π contains the unit representation of the maximal compact subgroup K_r and we let W be the Whittaker function which is invariant under K_r and takes the value 1 on K_r.

PROPOSITION 4. *Under the above assumptions*

$$J(s,\chi,W) = L(s,\pi,(\bigwedge^2\rho)\otimes\chi)\,.$$

With the same notations as in the previous subsection, we use the relation

$$\delta_{2n+1}^{1/2}(b) = \delta_n^2(a) \mid \det a \mid\ .$$

We have then:

$$J = \int \mid W(bt_Z) \mid \delta_{2n+1}(b)^{-1/2}\chi(\det a) \mid \det a \mid^s da dZ\,,$$

and the rest of the proof is similar to the proof of Proposition 2 in section 7.2.

9.5. Now we go back to the general local situation: We have the following analogue of Proposition 3 in section 7.3:

PROPOSITION 5. *There is a* W *such that:*

$$J(s,\chi,W) \neq 0\,.$$

We assume that on the contrary $J(s,\chi,W)=0$ for all choices of W. We apply this relation to the function defined by:

$$W_1(g)=\int W\left[g\begin{pmatrix}1_n & 0 & 0\\ 0 & 1_n & U\\ 0 & 0 & 1\end{pmatrix}\right]\Phi(U)dU\,,$$

where Φ is a smooth function of compact support on the space of column vectors. Using matrix multiplications we find the following integral vanishes:

$$\int W\left[\sigma\begin{pmatrix}1_n & 0 & ZgU\\ 0 & 1_n & gU\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1_n & Z & 0\\ 0 & 1_n & 0\\ 0 & 0 & 1\end{pmatrix}\right.$$
$$\left.\begin{pmatrix}g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & 1\end{pmatrix}\right]\psi(\mathrm{Tr}Z)dZ\chi(detg)\Phi(U)dU\,.$$

Taking in account the invariance property of W this gives:

$$\int W\left[\sigma\begin{pmatrix}1_n & Z & 0\\ 0 & 1_n & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & 1\end{pmatrix}\right]$$
$$\psi(\mathrm{Tr}Z)dZ\chi(\det g)\hat{\Phi}(\epsilon_n g)dg=0\,,$$

where $\hat{\Phi}$ is the Fourier transform of Φ. Just as before this implies that:

$$\int W\left[\sigma\begin{pmatrix}1_n & Z & 0\\ 0 & 1_n & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & 1\end{pmatrix}\right]$$
$$\psi(\mathrm{Tr}Z)dZ\chi(\det g)\mid \det g\mid^{-1}dg=0\,,$$

where g is now integrated over $N_{n-1}(F)\backslash G_{n-1}(F)$. The rest of the proof is not changed.

9.6. Going back to a global situation we can now prove the following result. We let π be a unitary cuspidal representation of $G_r(F_{\mathbf{A}})$. We let S be a finite set of places containing all the places at infinity and all places where the representation π ramifies. We set

$$L^S(s)=\prod_{v\notin S}L(s,\pi_v,(\textstyle\bigwedge^2\rho)\otimes\chi_v)\,.$$

THEOREM 2. *The function $L^S(s)$ is holomorphic at $s=1$.*

REFERENCES

[B-c-r I] I D. Blasius, L. Clozel And D. Ramakrishnan, *Algébricité de l'action des opérateurs de Hecke sur certaines formes de Maass*, C. R. Acad. Sci. Paris 305, Sér. I, pp. 705–708 (1987).

[B-c-r Ii] D. Blasius, L. Clozel And D. Ramakrishnan, *Opérateurs de Hecke et formes de Maass: application de la formule des traces*, C. R. Acad. Sci. Paris 306, Sér. I, pp. 59–62 (1988).

[B-r] D. Blasius and Ramakrishnan, *Maass forms and Galois representations*, preprint.

[B-h-r] D. Blasius, M. Harris and D. Ramakrishnan, *Coherent cohomology, limits of discrete series and Maass forms*, in preparation.

[C] W. Casselman, *Canonical extensions of Harish Chandra modules*, preprint, 1988.

[C-s] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups*, Comp. Math., Vol. 41, 1980, pp. 207-231.

[D-m] J. Dixmier and P. Malliavin, *Factorizations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math., Ii. Sér. 102 (1978) pp. 471-542.

[G-j] S. Gelbart and H. Jacquet, *A relation between automorphic forms on* $Gl(2)$ *and* $Gl(3)$, Ann. Sci. Ecole Normale Sup., 4ᵉ série, Vol. 11 (1978), pp. 471-552.

[G-s] S. Gelbart and F. Shahidi, "Analytic Properties of Automorphic L-functions", Perspectives in Mathematics, Vol. 6, Academic Press, New York, 1988.

[I] J. I. Igusa, "Lectures on Forms of Higher Degree", Tata Institute of Fundamental Research, Bombay, 1978.

[J] H. Jacquet, *On the nonvanishing of some L-functions*, Proc. Indian Acad. Sci. (Math. Sci.), Vol. 97, Nos 1–3, (1987), pp. 117–155.

[J-s] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations*, I and Ii, American Journal of Math., Vol. 103, No. 3, pp. 499–558 and vol. 103, No. 4, pp. 777-815.

[J-p-s] H. Jacquet, I.i. Piatetski-shapiro and J. Shalika, *Automorphic forms on Gl(3)*, I and Ii, Annals of Math, 109 (1979), pp. 169-212 and pp. 213-258.

[O-w] S. Osborne and Warner, "The theory of Eisenstein Systems", Academic Press, New York, London, 1981.

[P-p] Patterson and Piatetski-shapiro, *The symmetric-square L-function attached to a cuspidal automorphic representation of* Gl_3, preprint.

[S] F. Shahidi, *On the Ramananujan conjecture and finiteness of poles for certain L-functions*, Annals of Math., 127 (1988), pp. 547–584.

[Sh] G. Shimura, *On the holomorphy of certain Dirichlet Series*, Proc. London Math. Soc. 3, (1975), pp. 171-193.

[So] D. Soudry, *A uniqueness Theorem for representations of* $GsO(6)$ *and the strong multiplicity one theorem for generic representations of* $Gsp(4)$, Israel Journal of Math., Vol. 58, No. 3, 1987, pp. 257-287.

[W I] N. Wallach, *Asymptotic expansion of generalized matrix entries of real reductive groups*, in "Lie groups representations" I (Proceedings, University of Maryland, 1982-83), Lecture Notes in Mathematics 1024, Springer-verlag, New York, 1983.

[**W Ii**] N. Wallach, "Real reductive groups I", Academic Press, New York, 1988.

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Problems Arising from the Tate and Beilinson Conjectures in the context of Shimura Varieties

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The object of this paper is to briefly highlight some of the problems raised in the setting of Shimura varities X by the general conjectures of Tate, Beilinson, Deligne, Bloch, et al, relating the poles and zeros of L-functions of X to the existence of algebraic cycles and "motivic cohomology" classes on X. Certain simple examples are mentioned on the way, and there is no pretension whatsoever of being exhaustive in the choice of the questions raised or the works cited.

A lot of beautiful arithmetic, and geometry, is encoded in the (mysterious) nature of special values of L-functions. Shimura varieties (and associated objects), owing to their rich structural relationship to reductive groups and to the moduli of abelian varieties and Hodge structures, and their deep (conjectural) relationship ([**La**]) to automorphic forms, provide an interesting testing ground. We refer to [**De2**] and [**Mil1**] for the basic facts used on Shimura varieties, to [**Sem**] and [**Ko**] for some recent results on their zeta functions, and to [**Ra1**] and [**RSS**] for two very different, detailed introductions to the conjectures for general varieties. There is very little overlap between the material treated here and in [**Ra1**]; in fact, this article is intended to complement the other, while begin self-contained. We also refer to the recent preprint [**BlKa**] for a Tamagawa number conjecture for motifs with $\mathbb{Q}$-coefficients, which gives an interpretation of the rational numbers involved.

1. PRELIMINARIES ON MOTIFS AND L-FUNCTIONS

Depending on the type of correspondences one uses, there are (at least) three good candidates for the category of (pure) motifs, and important invention of A. Grothendieck.

For any field k, let $\underline{V}(k)$ denote the category of smooth projective varieties X over k, with morphisms being maps of algebraic varieties. For every integer $m \geq 0$, denote by $C^m(X)$ the group of k-rational algebraic cycles of codimension m on X. By definition, every

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Automorphic Forms, Shimura
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Z in $C^m(X)$ is a finite (formal) $\mathbb{Z}$-linear combination, invariant under $\mathrm{Gal}(\bar{k}/k)$, of closed irreducible subvarieties of codimension m. By an algebraic correspondence from X to Y one means an algebraic cycle on $X \times Y$ of codimension $d = \dim(Y)$. Say that two codimension m cycles Z, Z' on X are rationally (or linearly) equivalent if there exist (a finite set of) closed irreducible subvarieties Y_i of X of codimension $m-1$, and functions of f_i on Y_i, such that $Z - Z'$ is the cycle defined by the sum of $\mathrm{div}(f_i)$. The m-th Chow group $CH^m(X)$ of X is the quotient of $C^m(X)$ by rational equivalence over k, so that $CH^1(X) \cong \mathrm{Pic}(X)$, which equals: $\mathrm{Pic}(X)^{\mathrm{Gal}(\bar{k}/k)}$. For arbitrary m, one has Galois descent only after tensoring with $\mathbb{Q}$. Define the group of *motivic correspondences* (from X to Y in $\underline{V}(k)$) to be:

$$(1.1) \qquad \mathrm{Hom}_r(X,Y) = CH^d(X \times Y) \otimes \mathbb{Q}, \quad \text{with } d = \dim(Y)$$

There is a product structure, defined by intersection of cycles (see [**Fu**]), on the Chow ring $CH^*(-)$. Given a correspondence Z from X to Y, we can then associate to any cycle U on X a cycle (modulo rational equivalence) $Z(U)$ on Y by taking it to be $p_{2*}(p_1^*(U).Z)$, where p_1 (resp. p_2) signifies the projection of $X \times Y$ onto X (resp. Y).

The category of correspondences $\underline{C}_r(k)$ (relative to rational equivalence) is obtained from $\underline{V}(k)$ by replacing maps by motivic correspondences. The category of effective motifs $\underline{M}_{eff,r}(k)$ is the result of formally splitting (in $\underline{C}_r(k)$) idempotent correspondences. When $X = \mathbb{P}^1$, if we fix a k-rational point Q, there are two obvious idempotents, namely $\mathbb{P}^1 \times Q$ and $Q \times \mathbb{P}^1$, and the effective motif $/k$ representing $\mathbb{P}^1$ decomposes into a direct sum of the trivial (effective) motif $\mathbb{Q} = \underline{\mathbb{Q}}_k$, which represents a point ($/k$) and another object (=*Lefschetz motif*), denoted by $\underline{\mathbb{Q}}(-1)$. Set: $\underline{\mathbb{Q}}(-n)$=$\underline{\mathbb{Q}}(-1)^{\otimes n}$, for $n > 0$, which is a sub-object of $(\mathbb{P}^1)^n$. By definition, the *Tate motif* is the formal inverse, denoted by $\underline{\mathbb{Q}}(1)$, of $\underline{\mathbb{Q}}(-1)$. The category of motifs $\underline{M}_r(k)$ is what one obtains by adjoining to $\underline{M}_{eff,r}(k)$ the Tate motif (and its tensor powers). For any motif M and integer n, its n-th "Tate twist" $M(n)$ is defined to be $M \otimes \underline{\mathbb{Q}}(n)$.

For fields k with parameters, the Chow groups of an X in $\underline{V}(k)$ can be very large ([**Mum**],[**Bl1**],[**Sch**]). However, one hopes that if k is a *global field*, then $CH^m(X) \otimes \mathbb{Q}$ is finite-dimensional, for $m \geq 0$.

We will take k from now on to be a finitely generated field. For every X in $\underline{V}(k)$, denote by $h_r(X)$ the object in $\underline{M}_r(k)$ representing X.

Let H^* be a good cohomology theory, satisfying Poincaré duality, etc. Principal examples are the étale cohomology with $\mathbb{Q}_\ell$ coefficients, singular (Betti) cohomology with $\mathbb{Q}$ coefficients, algebraic de Rham cohomology and cristalline cohomology. Then H^* defines in a natural way a contravariant functor on $\underline{M}_r(k)$, with motivic correspondences being associated to homological correspondences defined by the cycle classes in $H^{2*}(-)(*)$. We note in particular the cycle class maps:

$$(1.3)\quad cl_B : CH^m(X) \underset{(\text{when } k\subset\mathbb{C})}{\longrightarrow} Hg^m(X(\mathbb{C})) = H_B^{2m}(X(\mathbb{C}),\ \mathbb{Q}(m)) \cap H^{m,m}(X(\mathbb{C}))$$

and

$$cl_\ell : CH^m(X) \underset{(\text{when } \ell\neq \text{char}(k))}{\longrightarrow} T_\ell^m(X) = H_{et}^{2m}(X\otimes \bar{k}, \mathbb{Q}_\ell)^{\text{Gal}(\bar{k}/k)}$$

The classes in $Hg^m(X(\mathbb{C}))$ and $T_\ell^m(X)$ are respectively called the Hodge cycles and the (k-rational) Tate cycles of codimension m. In the former setting, $\mathbb{Q}(m)$ denotes $(2\pi i)^m\mathbb{Q}$, the rational Tate Hodge structure of rank 1 and pure bidegree $(-m,-m)$, while in the latter, $\mathbb{Q}_\ell(m)$ denotes the one-dimensional ℓ-adic Galois representation $(\varprojlim_n \mu_{\ell^n}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

CONJECTURE 1.4.
(H) (Hodge) Every Hodge cycle on $X(\mathbb{C})$ belongs to $cl_B(CH^*(X\otimes k'))\otimes\mathbb{Q}$, for some finite extension of k' of k; and
(T1) (Tate) Every (k-rational) Tate cycle on X belongs to $cl_1(CH^*(X))\otimes\mathbb{Q}_\ell$.

CONJECTURE 1.5. *Ker$(c\ell_\ell)\otimes\mathbb{Q}$ is independent of $\ell(\neq$ char$(k))$.*

If $k \subset \mathbb{C}$, then there is a comparison isomorphism between $H_B^*(X(\mathbb{C}),\mathbb{Q}(m))\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ with $H_{et}^*(X\otimes\mathbb{C},\mathbb{Q}_\ell(m))$, which by the proper base change theorem is isomorphic to $H_{et}^*(X\otimes\bar{k},\mathbb{Q}_\ell(m))$; and $c\ell_\ell$ factors (minus the Galois action) through $c\ell_B$. Hence the conjecture 1.5 holds in characteristic zero. Set:

(1.6)
(*i*) $CH^m(X)^0 = \text{Ker}(c\ell_\ell)$, $\quad C_h^m(X) = (CH^m(X)/CH^m(X)^0)\otimes\mathbb{Q}$;
and

(*ii*) For X, Y in $\underline{V}(k)$, $\quad \mathrm{Hom}_h(X,Y) = C_h^{\dim(Y)}(X \times Y)$

We call the classes in $CH^m(X)^0$ the (k-rational) *homologically trivial cycles* (of codimension m), and those in $\mathrm{Hom}_h(X,Y)$ *correspondences modulo homological equivalence.* Clearly, in characteristic $p > 0$, for 1.6 to make sense, one needs the conjecture 1.5.

We get a different category of motifs $\underline{M}_h(k)$, say, by repeating the construction of $\underline{M}_r(k)$ with $\mathrm{Hom}_r(X,Y)$ replaced by $\mathrm{Hom}_h(X,Y)$. In his original definition (see [**Gro**], [**K**] and [**Ma**]), Grothendieck used *correspondences modulo numerical equivalence.* Recall that two cycles (of the same codimension) are numerically equivalent if they have the same intersection number with all the cycles of complementary dimension, meeting them properly. Standard conjectures predict that the numerical and homological equivalences coincide (upto torsion), and so the objects of $\underline{M}_h(k)$ are morally Grothendieck motifs.

For X in $\underline{V}(k)$, let $h_h(X)$ be the corresponding object in $\underline{M}_h(k)$, and let Δ be the diagonal correspondence on $X \times X$. Then the Künneth components of $cl_B(\Delta)$ (resp. $c\ell_\ell(\Delta)$) are all easily seen to be Hodge (resp. Tate) classes. An important special case of Conjecture 1.4 asserts that the Künneth components of the diagonal are algebraic. If this holds, then they define idempotents $e_j, 0 \leq j \leq 2d, \quad d = \dim(X)$, such that $H(e_j(h_h(X))) = H^j(X)$, for every j, with H denoting H_B, $H_{\acute{e}t}$ or H_{dR}. This will lead to a decomposition:

$$h_h(X) = \oplus_{0 \leq j \leq 2d}\, h^j(X), \tag{1.7}$$

where each $h^j(X)$ is *pure of weight* j. There should be and analogous decomposition of $h_r(X)$ as well. For $\dim(X) \leq 2$, 1.7 is known to hold (see [**Ma**] and [**K**] for h_h and [**Mur**] for h_r).

Suppose k is of characteristic zero. Then, for every embedding σ of k in $\mathbb{C}$, we have a comparison isomorphism:

$$I : H_B^*(X(\mathbb{C}), \mathbb{C}) \to H_{dR}^*(X/k) \otimes_{k,\sigma} \mathbb{C}, \tag{1.8}$$

which does not preserve the rational structures. Call, following Deligne ([**Del**]), a class Z in $H_{dR}^{2m}(X/k)(m) \times H_{et}^{2m}(X \otimes \bar{k}, \mathbb{A}_f(m))$, $\mathbb{A}_f = \Pi'_\ell \mathbb{Q}_\ell$, *absolutely Hodge* (of codimension m) if, for every imbedding σ of k in $\mathbb{C}$, its image in $H_B^{2m}(X(\mathbb{C}), \mathbb{Q}(m)) \otimes (k \times \mathbb{A}_f)$ is a Hodge class. Set:

$$\mathrm{Hom}_{aH}(X,Y) = \{\text{absolutely Hodge classes on } X \times Y \text{ of codim.} = \dim .Y\} \tag{1.9}$$

CONJECTURE 1.10 [**Del**]. *Every Hodge class is absolutely Hodge.*

It is known to hold for abelian varieties ([**DMOS**]) in every codimension, while the Hodge and Tate conjectures are open for $m > 1$. The Hodge conjecture is known for divisors on any X by Lefschetz ([**GH**]), while the Tate conjecture for divisors on any abelian variety is known by the work of Faltings ([**Fa**]).

Define a category $\underline{C}_{aH}(k)$ by replacing the varieties X in $\underline{V}(k)$ by the set of their "realizations" in the Betti, de Rham, ℓ-adic (and cristalline) cohomology, with comparison isomorphisms, and by replacing maps (of varieties) by absolutely Hodge correspondences. Then, after formally inverting idempotent morphisms and inserting Tate twists, we get a third candidate $\underline{M}_{aH}(k)$ for the category of (sums of pure) motifs over k. (One can get another variant by using absolutely Hodge, Hodge-Tate correspondences.) It can be checked that the Künneth components of the diagonal are absolutely Hodge, and so one gets a decomposition of an M in $\underline{M}_{aH}(k)$ as a sum of its components of pure weight.

In all three settings, one conjectures, following Grothendieck, that the category of motifs is semi-simple and ($\mathbf{Q}$-linear) Tannakian, with a fiber functor over $\bar{k}$. Hence it should be the category of semi-simple representations of a pro-reductive group $\mathfrak{g}_M/k$, called the motivic Galois group $/k$.

The disadvantage of working with absolutely Hodge correspondences, or with algebraic correspondences modulo homological equivalence, is that they do not act on the Chow groups, or on the *motivic cohomology* groups, defined for any X in $\underline{V}(k)$ (and $m, n \geq 0$) to be:

$$H^n_{\mathcal{M}}(X, \mathbf{Q}(m)) = Gr^m_\gamma K_{2m-n}(X) \otimes \mathbf{Q}, \tag{1.10}$$

where $K_*(X)$ denotes the algebraic K-theory of X, and Gr^m_γ the m-th graded piece relative to the γ-filtration [**So**]. A theorem of Grothendieck gives an isomorphism:

$$H^{2m}_{\mathcal{M}}(X, \mathbf{Q}(m)) = CH^m(X) \otimes \mathbf{Q} \tag{1.11}$$

One can evidently define $H^n_{\mathcal{M}}(M, \mathbf{Q}(m))$ for any M in $\underline{M}_r(k)$, and $H^*(-, \mathbf{Q}(^{**}))$ should presumably be the universal Tate-twisted cohomology theory on $\underline{M}_r(k)$.

Let k be a *number field.* Then, for a motif M (in either of the formalisms, there is, for every $j \geq 0$, an associated L-function $L^{(j)}(M, s)$, which is an Euler product over the finite places of k, with the unramified factors given by the characteristic polynomials of the geometric Frobenii F_v on $H^j_{et}(M \otimes \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ [**De1**]. If M is pure of weight w, we write $L(M,s)$ for $L^{(w)}(M,s)$, which converges absolutely in $\{Re(s) > 1 + (w/2)\}$ by Deligne ([**De3**]). (We assume here that bad Euler factors do not have a pole in this region.) The *Hasse-Weil-Serre hypothesis* is that $L(M,s)$ extends to a meromorphic function in the whole s-plane with no pole outside the "edge of absolute convergence": $s = 1 + w/2$, and that there is a functional equation (with $L^*(M,s) = L(M,s)L_\infty(M,s)$):

(1.12)

$$L^*(M^\vee, s) = \varepsilon(M,s)L^*(M, 1+w-s), \text{ with } \varepsilon(M,s) \neq 0, \infty, \forall s.$$

CONJECTURE 1.12. *Let M be pure of weight w. Then ([**T**])*

(T2)(Tate)

$$-\, ord_{s=m+1} L(M,s) = \dim_{\mathbb{Q}_\ell} Hom_{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}(1, H(M \otimes \bar{\mathbb{Q}}, \mathbb{Q}_\ell)(m)), \quad \textit{if } w = 2m$$

and

(T3)

$$ord_{s=1+w/2} L(M,s) = 0, \quad \textit{if } w = 2m+1$$

In conjunction with (T1), (T2) says that the order of pole of $L(M,s)$ at $s = m$ is the dimension of $C^m_h(M)$. Since $\underline{\mathbb{Q}}(r)$ is, for any r, of even weight, no Tate twist of an odd weight motif can contain the trivial motif, and this is the moral basis for (T3). We get, using [**JS1**], [**JPSS2**]:

PROPOSITION 1.13. *Let M be a (semi-simple) motif $/k$ of rank n and weight w such that $L(M,s) = L(\pi, s - w/2)$ for an isobaric ([**La**]) automorphic form π of $GL(n, \mathbb{A}_k)$. Then (T2), (T3) hold for M.*

For a number field T, the category of motifs $/k$ with coefficients in T is constructed as above, with $\mathrm{Hom}_r(X,Y)$ replaced by $\mathrm{Hom}_r(X,Y) \otimes T$ and with $\underline{\mathbb{Q}}(-1)$ replaced by $\underline{T}(-1)$, the non-trivial direct factor of $T\mathbb{P}^1$. The associated L-functions are $(T \otimes \mathbb{C})$-valued. One often works in this enlarged setup.

Langlands conjectures ([**La**]) that the category of isobaric automorphic forms on $GL(\mathbb{A}_k)$ should be Tannakian, equivalent to the category of completely reducible representations of a pro-reductive group $\mathfrak{g}_A/k$, and that there is a surjective morphism $/\bar{\mathbb{Q}} : \mathfrak{g}_A \to \mathfrak{g}_M$.

(For important insight into the structure of non-tempered automorphic representations, see [**Ar1**].) Thus motifs should be attached to automorphic forms of *arithmetic type* (see [**Cl1**], [**B3**]), but the automorphic theory is not sensitive to coefficients and so there is no satisfactory bijection over k.

Denoting by $H^*_{\mathcal{M}}(M_{\mathbf{Z}}, \mathbb{Q}(**))$ a suitable integral structure of $H^*_{\mathcal{M}}(M, \mathbb{Q}(**))$ ([**Be1**]), Beilinson makes the following

CONJECTURE 1.14. *For $0 \leq j \leq 2m-1$, and M in $\underline{M}_r(k)$,*

$$ord_{s=j+1-m}\, L^{(j)}(M,s) = \dim H^{j+1}_{\mathcal{M}}(M_{\mathbf{Z}}, \mathbb{Q}(m), \quad \textit{if } j < 2m-2$$
$$ord_{s=m-1}\, L^{2m-2}(M,s) = \dim H^{2m-1}_{\mathcal{M}}(M_{\mathbf{Z}}, \mathbb{Q}(m)) + \dim C^{m-1}_h(M)$$

and

$$ord_{s=m}\, L^{(2m-1)}(M,s) = \dim H^{2m}_{\mathcal{M}}(M_{\mathbf{Z}}, \mathbb{Q}(m))^0$$

Furthermore, the leading coefficient of $L^{(j)}(M,s)$ at the respective point is expected to be a rational multiple of the *volume of a regulator* (in the first two cases) or a *height pairing* (in the last case) involving the corresponding group on the right (cf. [**Be1**],[**Bl2**],[**Ra1**],[**RSS**]). All of this extends to motifs with coefficients, and is consistent with Deligne's conjectures at critical points. When M is, for example, defined by spec k or by an algebraic Hecke character, there are positive results (cf. [**Bo**],[**Be1**],[**B1**],[**B2**],[**HaS**], [**Den1**],[**Den2**]).

The groups $H^n_{\mathcal{M}}(M, \mathbb{Q}(m))$ are expected to be $\mathrm{Ext}^n(M, \underline{\mathbb{Q}}(m))$, with the extensions taking place in a convenient category of *mixed motifs*, i.e., motifs which admit an increasing (weight) filtration with successive quotients being pure. So the philosophy (of Beilinson, Deligne et al.) is that, even if one wants to restrict oneself to pure motifs M, the special values of the associated L-functions are controlled by suitable groups of extensions of M by Tate motifs. It is this author's hope that (more general than isobaric) automorphic forms of *arithmetic type* should be associated to mixed motifs, having in general a complicated weight filtration, with splittings governed by the vanishing of certain associated L-functions. This should be done in a way consistent with the programs of Arthur ([**Ar1**],[**Ar2**]) and Harder ([**Ha1**],[**Ha2**]). We wish for a formalism over local fields F as well; for instance the Steinberg representation of $GL(2,F)$ should (presumably) correspond to a generator of $\mathrm{Ext}^1(\underline{\mathbb{Q}}, \underline{\mathbb{Q}}(1))$. We hope to try and come back to this question elsewhere.

See [**De2**], [**Ja1**] and [**Be2**] for different approaches to the (conjectural) category of mixed motifs, and [**BeMS**], [**BeGSV**], [**Bl2**], [**Li**] and [**Mil2**] for various candidates for motivic cohomology.

2. Questions concerning Shimura varieties

Let G be a connected reductive (algebraic) group $/\mathbb{Q}$, admitting a non-trivial $\mathbb{R}$-morphism $h : \mathbb{C}^* \to G_{\mathbb{R}}$ satisfying the axioms of [**De2**], and let K_∞ be the centralizer of h in $G(\mathbb{R})$. Then $\mathcal{X} = G(\mathbb{R})/K_\infty$ is of hermitian symmetric type, and for K : a (neat) open compact subgroup of $G_f = G(\mathbb{A}_f)$, $\mathbb{A}_f = (\varprojlim_n \mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Q}$, there is an associated Shimura variety S_K defined over a canonical number field E (see [**Mil1**]) such that $S_K(\mathbb{C})$ identifies with: $G(\mathbb{Q})\backslash\mathcal{X} \times G_f/K = \cup_{1\leq i\leq h(K)}\Gamma_i\backslash\mathcal{X}^+$ where each Γ_i is a congruence subgroup of $G^{ad}(\mathbb{Q})$. If we write $h_{\mathbb{C}} = (\mu, \bar{\mu})$, then E is the field determined by the stabilizer of the conjugacy class of μ in Aut $\mathbb{C}$. Let S_K^* be the (often singular) Baily-Borel-Satake compactivifcation $/E$ of S_K. Denote by $\tilde{S}_K(\mathbb{C})$ a smooth toroidal compactification of $S_K^*(C)$ ([**AMRT**]). When S_K is a curve, this is canonical, and when it is a (non-rational) surface, there is a minimal one such as the Hirzebruch compactification in the Hilbert modular case. In general, there is no way of choosing one in a natural way among a family of such compactifications. However, the rational polyhedral cone decompositions $\sum$ defining $\tilde{S}_K(C)$ can be chosen compatibly with the action of Aut$(\mathbb{C}/E)$ on $S_K^*(\mathbb{C})$ to obtain: For a suitable (infinite family of) $\sum$'s there is a smooth resolution $\tilde{S}_K = S_K \cup D$ of S_K^* over E, where D is a divisor with normal crossings ([**H2**]).

Question 1. *Is there a motivic splitting* $(*)$ $h(\tilde{S}_K) = [IS_K^*] \oplus [S_K^\infty]$, *for* $h = h_r$, h_h *or* h_{aH}, *such that, for any good cohomology theory* H *(such as* H_B *or* H_{et}*)* : $H([IS_K^*]) = IH(S_K^*))$ *and* $H(^*)$ *is given by the decomposition theorem of [**BeBD**]?*

Here IH denotes the intersection cohomology of Goresky MacPherson and Deligne. If such a splitting exists with $h = h_r$, then the motivic cohomology decomposes accordingly. Note that the action of the Hecke algebra $\mathcal{H}(G_f, K)$ on S_K (by correspondences) extends to S_K^*, but not (in general) to $\tilde{S}_K$, for any choice of the cone decomposition $\sum$. (One of the exceptions is the case of a quasi-split unitary group G with $G(\mathbb{R}) = U(2,1)$, $\tilde{S}_K$ is a Picard modular surface on which the Hecke algebra acts because there is a unique decomposition of $\mathbb{R}_+^*$— [**Sem**]). However, in general, we can choose a projective family of

$\sum$'s such that $\mathcal{H}(G_f, K)$ acts on the inverse limit $\hat{S}_K$ of the $S_K(\sum)$'s. Taking the limit over K, we get a G_f action on the double projective limit $\hat{S}$ ([**Ra2**]), and we get a *weak* splitting (even in $\underline{M}_r$). It will be very interesting to determine, already at the Betti or étale cohomological level, what possible intertwining (as G_f-modules) there can be between (the limits of) $H([IS_K^*])$ and $H([S_K^\infty])$. For example, let G denote the restriction of scalars to $\mathbb{Q}$ of $GL(2)$ over a real quadratic field F. Then S_K is a finite set of points ("cusps"). Moreover, $\tilde{S}_K = S_K \cup D$, with D being a finite union of cycles of rational curves, and $\mathrm{Aut}\,\mathbb{C}$ acts on the cusps via its abelian quotient. Consequently, $[S_K^\infty]^{(2)}$, which is H^2 of $\tilde{S}_K$ with supports in S_K^∞, decomposes into a sum of $H(\omega(-1))$'s, with each ω a Dirichlet character of $\mathbb{Q}$. On the other hand, $IH^2(S_K^*) = Im(H^2(S_K) \to H^2(S_K))$ is spanned by cusp forms of weight two and by one-dimensional automorphic forms corresponding to cyclotomic motifs ([**HLR**], [**Ra3**]).

QUESTION 2. *Assume a positive answer to Question 1. Then does the decomposition (1.7) hold for $[IS_K^*]$? More reasonably, can it be proven that the primitive part $PH^d(S_K)$ of the intersection cohomology (in the middle degree $d = \dim(S_K)$) corresponds to a split submotif of $[IS_K^*]$?*

Note that the decomposition (1.7) is shown in [**KaM**] for any smooth projective variety over $\mathbb{F}_q$ as a consequence of Deligne's proof of the Weil conjectures. It is natural to wonder if the Hecke correspondences can be used globally (in the context of Shimura varieties) to separate out the different weight pieces. (Can one use the relationship between Hecke and Frobenius?) By the proof of Zucker's conjecture ([**Lo**], [**SS**]), the (Betti) intersection cohomology (with $\mathbb{C}$-coefficients) can be replaced by the L^2-cohomology of $S_K(\mathbb{C})$, which can be described in terms of the (Lie $G_{\mathbb{C}}, K_\infty$)-cohomology with coefficients in discrete automorphic forms ([**BoW**]). (One wishes that the whole cohomology of $S_K(\mathbb{C})$ is automorphic; it will be interesting to try and prove that, if this wish were granted, then there can be no ghost classes.) The restriction to the primitive part comes in because of the compounding problem that an automorphic representation $\pi = \pi_\infty \otimes \pi_f$ (of $G(\mathbb{A}) = G(\mathbb{R}) \times G_f$) can contribute in more than one degree (by cupping with the hyperplane section.) Of particular importance are those π's with π_∞ in the discrete series, which do not contribute in any degree but d.

When S_K parametrizes abelian varieties of PEL type, one can consider the question of motivic splitting also for the fiber products of the universal family over S_K. When S_K is a modular curve, the forms of higher weight enter, and for a proof that they split off motifs, see [**Ja**] (in $\underline{M}_{aH}$) and [**Sc**].

QUESTION 3. *Let π_f be an irreducible admissible G_f-module, admitting a K-fixed vector, and let $S(\pi_f)$ denote the "motif" $Hom_{G_f}(\pi_f, [IS^*]), [IS^*] = \lim_K [IS_K^*]$, cut out by the Hecke correspondences. Can it be proven, when $S(\pi_f)$ is pure of even weight, that it contains a Tate motif as a direct summand whenever $L(S(\pi_f), s)$ has a pole at an integral point? When $S(\pi_f)$ is of odd weight w, can it be shown that $L(S(\pi_f), s)$ is entire and invertible at the critical edge $s = 1 + w/2$?*

An example, in the *even* weight case, is supplied by an infinite dimensional π_f contributing to IH^2 of a Hilbert modular surface S_K^* (for K such that the space of π_f admits a K-fixed vector). There is a unique π_∞, namely the discrete series of weight $(2,2)$ of $GL(2, F \otimes \mathbb{R})$, such that $\pi = \pi_\infty \otimes \pi_f$ occurs in the space of cusp forms, and $S(\pi_f)$ has rank $4/\mathbb{Q}$, of Hodge type $(2,0)$, $(1,1)$, $(0,2)$. One knows that $L(S(\pi_f), s)$ is an Asai L-function $L(\pi, r, s)$ ([**HLR**]), which has a pole at the edge when π_f is a base change of a cuspidal π'_f (of $GL(2, \mathbb{A}_f)$). (For general Shimura varieties, the automorphic L-functions which should intervene in the explication of the Zeta function of $[IS_K^*]$ are associated to the representation r of ${}^LG$ determined by the coweight μ—see [**La**].) Using the involution which flips the two factors of $\mathcal{X}$, it can be proved (in the base change situation) that $S(\pi_f)$ splits as a motif (in $\underline{M}_r$) into a sum of $\mathbb{Q}(-1)$ and the symmetric square of the motif $S(\pi'_f)$ (occurring in the degree 1 piece of a modular curve.) ([**Ra2**]). For general G, it will be interesting to devise a procedure for π_f's which come by lifting from a smaller group.

It is useful to observe that $S(\pi_f)$ is in general *not* the conjectural motif, say $M(\pi_f)$, attached to π_f (of arithmetic type.) In the Hilbert modular case (of $GL(2)$) over a totally real number field F, the étale, resp. Grothendieck, realization of $M(\pi_f)$ is constructed in [**Tay**], resp. [**BRo1**]. In the weight 2 case, $M(\pi_f)$ has weight one, and it is still an (important) open problem to associate an abelian variety $A(\pi_f)/F$ such that $M(\pi_f) = h^1(A(\pi_f))$. When F is of odd degree $/\mathbb{Q}$, there is a construction of $A(\pi_f)$ as a factor of a Shimura curve ([**Ca**]).

In the *odd* weight case, suppose $\pi = \pi_\infty \otimes \pi_f$ is a globally generic, cuspidal representation of some $G(\mathbb{A})$ contributing to the middle primitive cohomology of $\tilde{S}_K(\mathbb{C})$ in some degree. Suppose further that $S(\pi_f)$ is a multiple of some M with $L(M,s)$ the standard L-function of π. Then (by [**Shah**]) $L(S(\pi_f),s)$ has no zero at the critical edge. An example to think of is the unitary group $/\mathbb{Q}$ of signature $(2m-1,1)$ at infinity defined by a division algebra with an involution of the second kind over an imaginary quadratic field ([**Ko**]), with π_∞ being in the (generic) discrete series (contributing to H^{2m-1}). (In this case $S_K(\mathbb{C})$ is a finite union of compact arithmetic quotients of the unit ball in $\mathbb{C}^n$.) In general, if π lifts to a cusp form on some $GL(n)/\mathbb{Q}$, then we can appeal to Proposition 1.13.

QUESTION 4. *Given an arbitrary Shimura variety of dimension d, can one construct non-trivial, primitive algebraic cohomology classes in $IH^{2m}(S_K)$, for every m not ruled out by the vanishing theorems, by means of the Hecke translates of Shimura subvarieties? Can these classes be chosen to not come from $G(\mathbb{R})$-invariant forms on $\mathcal{X}$? If $2m \neq d$, then do all such classes arise this way? In the middle dimension (when $d = 2m$), are all the Tate classes over E^{ab} exhausted this way? Is there any additional contribution from the cycle classes of non-congruence quotients of sub-hermitian domains of $\mathcal{X}$?*

We note first that the relevant cohomology vanishes below the real rank of G ([**BoW**]). Better, if G is $\mathbb{R}$-simple, then there is a precise integer $r(G) \geq rk_{\mathbb{R}}(G)$ below which one has vanishing (see [**Kum**], [**VZ**].) If $G(\mathbb{R}) = SU(p,q)$ or $SO(2,n)$, there exists in every even dimension outside the vanishing range a non-trivial class, not associated to an invariant differential form, represented by a sub-Shimura variety attached to a sub-group H, with $H(\mathbb{R})$ being of the form $SU(k,q)$ or $SO(2,k)$. The construction of such "geodesic" cycles is addressed from the Weil representation point of view in [**Ku1**] and [**KuM**]. A different construction of Hodge classes on (compact) unitary Shimura varieties comes from (suitable) algebraic Hecke characters, and is given by combining [**Cl1**] and [**Cl2**].

It is in general fruitful to study cycles coming from a subgroup defined by the fixed points of an involution. Suppose we are in the Hilbert modular surface case. Then one knows [**HLR**] that the Hirzebruch-Zagier cycles, i.e., the Hecke translates of embeddings of modular curves ([**HZ**]), exhaust all the Tate classes over $\mathbb{Q}^{\mathrm{ab}}$, but not over $\bar{\mathbb{Q}}$. Even though the Tate conjectures ((T1) and (T2)) are known in the remaining (CM) cases ([**MuRa1**], [**Kl**]), the problem of explicit construction of these exotic divisors is very much open. This problem, if solved, will have important consequences, because then one can intersect these exotic curves with the diagonal (modular) curve X and get interesting algebraic points on X. When G is a quasi-split group with $G(\mathbb{R}) = U(2,1)$, the Tate conjectures ((T1) for all number fields k, and (T2) for k/E abelian) for H^2 has been proved in [**BRo2**]. It is also shown that there are no exotic classes in this case. If $\pi = \pi_\infty \otimes \pi_f$ is a globally generic, cuspidal representation of $G(\mathbb{A})$ contributing to H^2, with π_∞ in the discrete series, then it seems reasonable to expect to prove, as in the Hilbert case, that $S(\pi_f)$ has a class represented by a curve coming from $U(1,1)$. The complications here arise from L-indistinguishability ([**Ro**]). Finally, when $G = GSp(4)/\mathbb{Q}$, it has been shown in [**Wei**] that all of H^2 of the threefold $\tilde{S}_K$ is algebraic, represented by cycles coming from translates of embedded modular surfaces.

QUESTION 5. *Let G, G' be inner forms such that $G(\mathbb{R}) \cong G'(\mathbb{R})$, with Shimura varieties S, S' respectively. In case $\pi_f, \pi_f{}'$ are cuspidals of G_f, G'_f such that their L-packets correspond (by an instance of the principle of functoriality), what is the relationship between $S(\pi_f)$ and $S'(\pi_f{}')$? When are they multiples of isomorphic motifs (in $\underline{M}_r$, $\underline{M}_h$, or $\underline{M}_{aH}$)? What about when G, G' are inner forms, but $G(\mathbb{R}) \not\cong G'(\mathbb{R})$?*

When $\pi_f, \pi_f{}'$ correspond as above, we get an identity of the relevant L-functions. The Galois representations on the étale realizations of $S(\pi_f)$ and $S'(\pi_f{}')$ should be semisimple, so that (by the Tate conjectures) there should be an algebraic correspondence between $\tilde{S}_K$ and $\tilde{S}'_{K'}$ which induces an isomorphism of the appropriate pieces of the ℓ-adic cohomology of $\tilde{S}_K \otimes \bar{\mathbb{Q}}$ and $\tilde{S}_{K'} \otimes \bar{\mathbb{Q}}$. It is already a (forbiddingly) difficult task to exhibit a Hodge correspondence, which, when the L-values coincide, is tantamount to proving certain period relations, which are predicted by the conjectures of Shimura ([**Sh1**]) and Deligne ([**De**]). (See [**Sh2**], [**Sh3**], [**H**] for a sample of positive results on preiod relations). An important case is when G, G' are defined by the multiplicative groups of quaternion algebras B, B' over a totally real number field F. When B is ramified at r infinite places with $0 < r < [F : \mathbb{Q}] - 1$, there is also this problem of not knowing the truth of Langlands's conjecture for the computation of the points of S_K over $\mathbb{F}_q$. For the surfaces defined by a totally indefinite B over a real quadratic F, the Tate conjectures ((T1) for arbitrary k and (T2) for abelian number fields k) have been proved in [**MuRa2**] by transferring, using period relations, the problem to Hilbert modular surfaces (defined by a split B'/F), where one knows the algebraicity of Tate classes ([**HLR**]+[**MuRa1**] or [**Kl**]). If $B = B_0 \otimes_Q F$, for some $B_0/\mathbb{Q}$, this was proved earlier by [**Lai**] using the relative trace formula developed in [**JaLai**]. (The Shimura curve associated to B_0 defines an analog of the Hirzebruch-Zagier cycle in this situation, with no similar fact known when B does not come from B_0.) When B is a quaternion algebra over a totally real F, which is split at exactly one infinite place, and $B' = M(2,F)$, S is a Shimura curve $/F$ and S' is a Hilbert modular variety $/\mathbb{Q}$ of dimension $d = [F : \mathbb{Q}]$. When (weight 2) π_f and $\pi_f{}'$ correspond, one expects there to be an isomorphism: $S(\pi_f) = \otimes_\sigma S(\pi_f)^\sigma$, $\sigma \in \mathrm{Hom}(F,\mathbb{C})$. (The CM aspect of this is discussed in [**MuRa3**].) In the case of a quasi-split G with $G(\mathbb{R}) = U(2,1)$, the relative trace formula method should help nail down a basis of the Néron-Severi group $C_h^1(S_K \otimes \mathbb{Q})$, by comparing the period integrals of generic and non-generic forms over (algebraic 1-cycles coming from) different forms of $U(1,1)$.

For general G, G' (with $G(\mathbb{R}) = G'(\mathbb{R})$), when there is an algebraic correspondence T between $\tilde{S}_K$ and $\tilde{S}'_{K'}$, for suitable $k \subset G_f$ and $K' \subset G'_f$, it induces maps (for every $n > 0$) between the motivic cohomology of n-fold self products of $\tilde{S}_K$ and $\tilde{S}_{K'}$. For example, let $G = B^*$, with B an indefinite quaternion division algebra $/\mathbb{Q}$, and let $G' = GL(2)/\mathbb{Q}$. Then $S(\pi_f)$ (resp. $S'(\pi_f)$) is attached to a factor of the Jacobian of a Shimura (resp. modular) curve $/\mathbb{Q}$. In this case, by [**Ri**] or by [**Fa**], there is an isogeny over $\mathbb{Q}$ from $S(\pi_f)$ to $S'(\pi_f{}')$, which gives rise to an isomorphism, for every $n > 0$ and (j,m): (see [**Ra5**])

$$H^j_{\mathcal{M}}(S(\pi_f)^{\otimes n}, \mathbb{Q}(m)) = H^j_{\mathcal{M}}(S'(\pi_f{}')^{\otimes n}, \mathbb{Q}(m))$$

This allows us, for instance, to deduce part of Beilinson's conjectures (the inequality $\leq$ in 1.14 + volume of the regulator) for S_K (at all the *non-positive* integers) and for $S_K \times S_K$ (at $s = 1$) from the corresponding theorems for modular curves ([**Be1**], [**Be3**], [**ScSc**]). The situation for Shimura curves over totally real $F \neq \mathbb{Q}$ is completely open.

QUESTION 6. *For any Shimura variety $\tilde{S}_K$, let $A^m(\tilde{S}_K)$ denote, for every $m > 0$, the algebraic part of the m-th intermediate Jacobian $J^m(\tilde{S}_K(\mathbb{C}))$. Can one determine the types of abelian varieties $/E$ which occur as factors of $A^m(\tilde{S}_K)$? More concretely, is there a simple non-CM abelian variety factor (of some $A^m(\tilde{S}_K)$) which is not a factor* *(possibly over a finite extension) of the Jacobian of a modular or Shimura curve? If S_K is attached to a unitary group G with $G(\mathbb{R}) = U(p,q), p < q$, then is every simple factor of $A^m(\tilde{S}_K)$, for $2m-1 < pq$, isogenous to a factor of $A^m(S'_{K'})$, where $S'_{K'}$ is a sub-Shimura variety attached to and endoscopic group G' with $G'(\mathbb{R}) = U(p,q-1) \times U(1)$?*

Recall that, for any smooth projective variety $X\ /E$ of dimension d, the m-th (Griffiths) intermediate Jacobian is:

$$J^m(X(\mathbb{C})) \\ = (\oplus_{p \geq d-m+1} H^{p,2d-2m+1-p}(X(\mathbb{C})))\check{}\ / H_{2d-2m+1}(X(\mathbb{C}), \mathbb{Z}(m)),$$

which is a complex torus. There is an (Abel-Jacobi) homomorphism

$$\mathrm{ab} : CH^m(X)^0 \to J^m(X(\mathbb{C})),$$

given by representing a homologically trivial cycle Z of codimension m by the linear form defined by integrating over a $2d-2m+1$-chain C with boundary $(2\pi i)^m \mathbb{Z}$. (If C' is another such chain, then $C - C'$ represents a class in $H_{2d-2m+1}(X(\mathbb{C}), \mathbb{Z}(m)))$. Let $\mathrm{Alg}^m(X)$ denote the subgroup of $CH^m(X)^0$ consisting of cycles which are *algebraically trivial*. Then it is known ([**Mur1**]) that $\mathrm{ab}(\mathrm{Alg}^m(X_{\mathbb{C}}))$ is an abelian variety, which we denote by $A^m(X)_{\mathbb{C}}$, admitting a model $A^m(X)$ over E. For divisors (and zero cycles) algebraic and homological equivalences coincide, and thus $A^1(X)$ (resp. $A^d(X)$) identifies with the Picard (resp. Albanese) variety of X.

For Shimura varieties $\tilde{S}_K$ attached to unitary groups G with no $U(1,1)$ factors at infinity, it is proven in [**MuRa4**] (using [**Ro**] and ideas from [**Od**]) that the Albanese variety (and hence its dual, the Picard variety) of $\tilde{S}_K$ is of (potential) CM type. More precisely, it is shown there that every factor (over an explicitly determinable finite extension) of the Jacobian of a Shimura curve ([**MuRa3**]). As remarked by Blasius and Rogawski, we can then conclude, using the classification of G's with real rank 1 (and with a little work at infinity), that the tensor category $/\bar{\mathbb{Q}}$ of Abelian varieties generated by the Albaneses of all the Shimura varieties coincides with that generated by Jacobians of modular and Shimura curves and by CM abelian varieties. It is this author's fond hope that the conjectural abelian varieties attached to Hilbert modular cusp forms of weight two occur as factors of suitable $A^m(\tilde{S}_K)$'s.

Let $\mathcal{E}$ be a CM elliptic curve $/E$ occurring as a factor $S(\pi_f)$ (up to isogeny) of the Albanese of a Picard modular surface $\tilde{S}_K$. (There exist such $\mathcal{E}$'s—see [**MuRa4**] & [**Ro**].) A natural question to ask is whether the special (CM) points on $S_K(\bar{\mathbf{Q}})$, not lying on a modular curve $\subset S_K \otimes \bar{\mathbf{Q}}$, can be used to construct some new rational zero cycles on $\mathcal{E}$? Let $\tilde{\mathcal{S}}_K$ be a proper scheme over 0_E, regular outside a finite set of primes, such that $\tilde{\mathcal{S}}_K \otimes \mathbf{Q} = \tilde{S}_K$. If Z_0 is a zero cycle of degree zero on S_K, then its scheme-theoretic closure Z, say, in $\tilde{\mathcal{S}}_K$ is an arithmetic 1-cycle, representing a class in $CH^2(\tilde{\mathcal{S}}_K)^0 \otimes \mathbf{Q}$. If Z' is another such, then we get a linking number $< Z, Z' > \in \mathbf{R}$, where $<,>$ is a height pairing ([**Be1**], [**Be4**], [**Bl4**]), defined as a sum of local terms $<,>_v$, as v runs over all the places of E. It is known that this pairing factors through the Néron-Tate height pairing (between the Albanese and Picard). Conjecturally, the determinant of $<,>$ relative to a π_f-basis of Z's modulo rational, or Albanese (see below), equivalence should be a rational multiple of (the inverse of a Deligne period times) the leading coefficient of $L(S(\pi_f), s)$ at the critical center. To compute $< Z, Z' >$ explicitly, in a manner analogous to the (groundbreaking) treatment of Heegner points in [**GrZ**], however, one needs a better hold on the bad fibers of $\tilde{\mathcal{S}}_K$. A model with many of the desired properties has been constructed by M. Larsen ([**Lar**]).

QUESTION 7. *For F/E finite, let $B^m(S_K/F)$ denote the subgroup of $CH^m([IS_K^*] \otimes_E F)$ generated by the F-rational classes coming from the Hecke translates of Shimura subvarieties of $\tilde{S}_K \otimes \bar{\mathbf{Q}}$. Then is $B^m(S_K/F) \otimes \mathbf{Q}$ finite dimensional $/\mathbf{Q}$? Is the torsion subgroup of the integral part of $B^m(S_K/F)$ finite, and, if so, what is its structure?*

The first part is a (hopefully more accesible) special case of what one expects from Beilinson's conjecture 1.14 (and the expectation that $CH^m(\tilde{S}_K) \otimes \mathbf{Q} = CH^m(\tilde{\mathcal{S}}_K) \otimes \mathbf{Q}$ for a proper model $\tilde{\mathcal{S}}_K$ over 0_E as above). A weaker thing one can ask for is the finite dimensionality over $\mathbf{Q}$ of $\mathrm{ab}(B^m(S_K)^0) \otimes \mathbf{Q}$. Recall that the algebraically trivial cycles $/E$ get mapped by ab into the E-rational points of the abelian variety $A^m(\tilde{S}_K)$. Hence, by the Mordell-Weil theorem, the question "reduces" to one about the rank of the Abel-Jacobi image of (part of the "Griffiths group" of) homologically trivial cycles modulo algebraic equivalence coming from Shimura subvarieties.

The expectation underlying the second part is parallel to a general conjecture of H. Bass which asserts the finite generation of K-groups of any scheme of finite type $/\mathbf{Z}$. Recall from (1.10) that $CH^m(-)$ is up to torsion the weight m piece, for the γ-filtration, of $K_0(-)$.)

Of special importance is the case of *zero cycles* ($m = d = \dim S_K$), and the contribution of the *special CM points* (defined by embeddings of anisotropic tori in G). Neither part of the question is clear in this setup, except for special geometric types of $\tilde{S}_K$ where one know *a priori* that $CH^d(\tilde{S}_K \otimes \mathbf{C})^0$ is isomorphic to $\mathrm{Alb}(X_{\mathbf{C}})$ (see [**Bl5**], [**CoS**]), for example), and even then not much is known about the structure of the torsion group. (The best result known in the non-rational situation is the famous theorem of B. Mazur [**Maz**] on the modular curve $X_0(N)$.) Can one at least prove a weak Mordell Weil type theorem for $B^d(S_K/F)$? As a small step in the positive direction, it can be shown ([**Ra6**]) that for certain Hilbert modular surfaces $S_K/\mathbf{Q}$, the $\mathbf{Q}$-subspace of $B^m(S_K/\mathbf{Q})$ generated by special points corresponding to compositums of pairs of imaginary quadratic fields is zero. (The first Betti number of such S_K is zero, and hence the Albanese is trivial.) Of use in the proof is the fact that every such *very* special point is at the intersection of Hirzebruch-Zagier cycles. The Neron-Severi groups of Hilbert-Blumenthal surfaces are (often) torsion-free, and it will be interesting to determine the torsion subgroup of $B^2(S_K)$ in terms of modular information. (See [**Ras**] for a survey of results on the torsion in Chow groups.)

It is useful to note the following consequence of the Bloch-Beilinson conjecture for zero cycles of degree zero. By (1.1.4), we must have, for every F/E finite: $\dim CH^d(\tilde{S}_K \otimes F)^0 \otimes \mathbf{Q} = \mathrm{ord}_{s=d} L^{(d)}(\tilde{S}_K \otimes F, s) = \mathrm{ord}_{s=1} L^{(1)}(\tilde{S}_K \otimes F, s)$, which is expected, by the Birch and Swinnerton-Dyer conjecture [**T**], to be the rank of the group of F-rational points of the Picard variety of $\tilde{S}_K$. By duality, it should also be the rank of the F-rational points of the Albanese. Using the surjectivity of the Albanese map and the fact ([**Roi**]) that the torsion subgroups of $CH^d(\tilde{S}_K \otimes \mathbf{Q})$ and $\mathrm{Alb}(\tilde{S}_K \otimes \bar{\mathbf{Q}})_0$ are isomorphic, one is led to the expectation that $CH^d(\tilde{S}_K \otimes F) \otimes \mathbf{Q}$ is isomorphic to $\mathrm{Alb}(\tilde{S}_K \otimes F) \otimes \mathbf{Q}$, for every number field $F \supset E$.

QUESTION 8. *Is there a general procedure to construct, for F/E: finite, classes in $H^m_{\mathcal{M}}(\tilde{S}_K \otimes F, \mathbf{Q}(m))$ (resp. $H^{2m-1}_{\mathcal{M}}(S_K \otimes F, \mathbf{Q}(m))$), by making use of functions with divisorial support in $B^1(S_K \otimes F)$ (resp. cycles in $B^{m-1}(S_K \otimes F)$ and functions thereon?) Do they generate a finite dimensional space $/\mathbf{Q}$? When do they come from an integral model of $\tilde{S}_K$?*

If $f_1, f_2, \ldots f_m$ are functions in $F(\tilde{S}_K)^*$, they define a symbol

$\{f_1, f_2, \dots f_m\}$ in $K_m^M(F(\tilde{S}_K))$, which is by definition the quotient of the m-fold tensor product of $F(S_K)^*$ (with itself) modulo the group generated by the relations: $x_1 \otimes \cdots \otimes x_m = 0$, whenever $x_j = 1 - x_i$, for some $i \neq j$. By **[Su]**, $H^m_{\mathcal{M}}(F(\tilde{S}_K), \mathbf{Q}(m))$ equals: $K_m^M(F(\tilde{S}_K)) \otimes \mathbf{Q}$. If D is the union of divisors of the f_i, then a multiple of $\{f_1, f_2, \dots, f_m\}$ defines a class $s(f_1, \dots, f_m)$, say, in $H^m_{\mathcal{M}}(U, \mathbf{Q}(m))$, where U is the complement of D in $S_K \otimes F$. It globalizes to a class in $H^m_{\mathcal{M}}(\tilde{S}_K \otimes F, \mathbf{Q}(m))$ if $\partial(s(f_1, \dots, f_m)) = 0$, where ∂ is the boundary map: $H^m_{\mathcal{M}}(U, \mathbf{Q}(m)) \to H^{m-1}_{\mathcal{M}}(D, \mathbf{Q}(m-1))$. For instance, let $m = 2$ and $\dim S_K = 1$. In this case, ∂ is $T \otimes \mathbf{Q}$, where T is the familiar tame symbol taking values in $\mathrm{div}^0(S_K \otimes F) \otimes \bar{\mathbf{Q}}^*$. A very useful construction of Bloch is the following: Suppose every divisor of degree zero supported on D is torsion of order r in the Jacobian. Then, after modifying by an "elementary" symbol of the form $\{h, a\}$ with $h \in F(\tilde{S}_K)^*$ and $a \in \bar{\mathbf{Q}}^*$, $\{f_1, f_2\}^r$ globalizes to $K_2(\tilde{S}_K \otimes F)$. This construction is used to great effect in Beilinson's work on modular curves (**[Be1]**, **[ScSc]**), where he takes D to lie in the cusps, satisfying Bloch's hypothesis by the Manin-Drinfeld theorem. The question is what one can say when D lies in the CM points. (For Shimura curves over totally real $F \neq \mathbf{Q}$, there is no recourse but to try and make use of the special points. Even for modular curves, since the cusps are all defined over $\mathbf{Q}^{\mathrm{ab}}$, there is no known construction for non-abelian fields.) Let π be a holomorphic cuspidal representation of $GL(2, \mathbb{A}_{\mathbf{Q}})$ of conductor N with $\mathbf{Q}$-coefficients. Then $S(\pi_f)$ is a rank 2 motif occurring in the modular curve $X_0(N)/\mathbf{Q}$. Suppose $L(\pi_f, s)$ vanishes to odd order > 1 at the critical center $s = 1$. Then, by **[GrZ]**, there is a special zero cycle Z of degree zero, whose class in the π_f-component of $\mathrm{Jac}(X_0(N)) \otimes \mathbf{Q}$ is trivial. (If $\mathcal{E}$ is the elliptic curve factor $/\mathbf{Q}$ of $X_0(N))$ determined by π_f upto $\mathbf{Q}$-isogeny, then Z determines a rational point P on $\mathcal{E}(\mathbf{Q})$ which is torsion.) It will be interesting to try to understand the symbols made up of functions with support in such special cycles. Recent work of R. Ross (**[Rs]**) has exhibited the following instructive example where one can use special points of *infinite* order: Let $\mathcal{E}$ be the elliptic curve $/\mathbf{Q}$ defined by: $y^2 = x^3 + x$, which has conductor 64 and CM by $\mathbb{Z}[i]$. Set: $\underline{o} =$ point at infinity, $P_2 = (0,0)$, $P_u = (u, -u)$ for $u = (1 + i\sqrt{3})/2$, and $P_{\bar{u}} = (\bar{u}, -\bar{u})$. If $f(x,y) = x - y$ and $g(x,y) = x$, then $\mathrm{div}(f) = P_2 + P_u + P_{\bar{u}} - 3\underline{o}$, and $\mathrm{div}(g) = 2P_2 - 2\underline{o}$. It turns out that $\{f, g\}^6$ is in $\mathrm{Ker}(T)$. On the other hand, P_u is not a torsion point, because any torsion in E over $\mathbf{Q}(i\sqrt{3})$ divides 4 and

$4P_u \neq 0$. (Note that $2P_u = (-3/4,^*)$.) Let $p : X_0(N) \to \mathcal{E}$ denote the modular parametrization $/\mathbb{Q}$. Then $p^{-1}(P_u)$ is a CM point. Furthermore, if r denotes the regulator map: $K_2(\mathcal{E}) \to H^1(\mathcal{E}(\mathbb{C}), \mathbb{R}(1))^+ \cong \mathbb{R}$, the image under r of (the class $< f, g >$ defined by $\{f, g\}^6$) is nontrivial in $H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(2)) = K_2(\mathcal{E}) \otimes \mathbb{Q}$. It may also be useful to remark here that, since $\mathcal{E}$ has CM and thus has no split multiplicative reduction, $< f, g >$ comes from any regular model ([**BlG**]) of $\mathcal{E}$ over $\mathbb{Z}$. We hope that $r(< f, g >)$ can be (at least numerically) checked to be a rational multiple of $L'(\mathcal{E}, 0)$, as it should be; if not the conjectures will have to be revised.

Incidentally, for any smooth projective curve X over $\mathbb{Q}$, we conjecture that the regulator map r is injective on $H^2_{\mathcal{M}}(X, \mathbb{Q}(2))$, not only on the integral subspace. For underlying theoretical reasons (which are consistent with [**BlG**]) see [**Ra1**], sec.4.7, and for evidence in the case of modular curves see [**ScSc**].

It is known ([**Q**], [**Ra1**]) that, for every $m > 0$, the classes in $H^{2m-2}_{\mathcal{M}}(\tilde{S}_K, \mathbb{Q}(m))$ are represented by $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$-invariant formal linear combinations: $\sum (Z_i, f_i)$, $i \in$ finite set, with each Z_i: irreducible subvariety of codimension $m-1$, and f_i a function invertible at the generic point of Z_i, *such that*: $(^*)$ $\mathrm{div}(f_i)$ is zero as a codimension m cycle on $\tilde{S}_K$. A simple way to satisfy $(*)$ is to take constants for f_i, and this induces a natural map $g : H^{2m-2}_{\mathcal{M}}(\tilde{S}_K, \mathbb{Q}(m-1)) \otimes E^* \to H^{2m-1}_{\mathcal{M}}(\tilde{S}_K, \mathbb{Q}(m))$. It will be useful to construct classes in $\mathrm{coker}(g)$ which come from Shimura subvarieties. This setup can be enlarged to include products of S_K's and also the univeral families of abelian varieties $\mathcal{A}$ over S_K (for G's of symplectic type). Suppose, for example, X is the 2-fold product of a modular curve S_K with itself. Then, with P and Q being two distinct rational cusps on S_K, Beilinson considers the sum: $b(P, Q) = (S_K \times P, f) + (\Delta, f^{-1}) + (Q \times S_K, f)$, wehre Δ is the diagonal curve, and f is the function on S_K with $\mathrm{div}(f) = k((P) - (Q))$, for some $k > 0$. Then $(^*)$ is satisfied, and we get a class in $H^3_{\mathcal{M}}(\tilde{S}_K, \mathbb{Q}(2))$ not in the image of g. Let r denote the regulator map, which takes values in $H^{1,1}(X(\mathbb{C}), \mathbb{R}(1))^+$. If we denote by $[,]$ the self-dual pairing on $H^{1,1}(X(\mathbb{C}))$, then for w, w' in $H^{1,0}(\tilde{S}_K(\mathbb{C}))$, noting: $w \otimes \bar{w}'|_{\tilde{S}_K \times P} = w \otimes \bar{w}'|_{Q \times \tilde{S}_K} = 0$ we get:

$$[r(b(P, Q), w \wedge w'] = (1/2\pi i) \int_{S_K(\mathbb{C})} \log |f| \quad w \wedge \bar{w}'.$$

Writing w (resp. w') as the differential attached to a cusp form π

(resp. π') on $GL(2)/\mathbb{Q}$ of weight 2, Beilinson interprets the regulator integral as an explicit multiple of the Rankin L-function of (π, π') at $s = 2$. When $S(\pi_f) \neq S(\pi_f{}') \otimes \chi$, one then deduces the conjectured relationship to the derivative at $s = 1$, and the non-vanishing of the integral shows the non-triviality of $b(P, Q)$ for some (P, Q). It should be noted that Beilinson's proof of integrality of $b(P, Q)$ is not quite correct as given in [**Be1**], due to the problem found in [**ScSc**]. However, the argument can be modified to give the same final result by closely analyzing hte boundary map. There are analogous results for Hilbert modular surfaces ($m = 2, s = 1$) ([**Ra2**]; see [**Ra3**], [**Ra4**] for a sketch) and for a product of two elliptic modular surfaces ($m = 3, s = 3$) ([**Ra6**]).

QUESTION 9. *For m : odd > 0, can one give criteria for cycles in $B^m(S_K)$ to be homologically trivial? For π_f contributing to $H^{2m-1}([IS_K^*])$, can one find examples where $L^{(2m-1)}(S(\pi_f), s)$ vanishes at the critical center $s = m$? Can they sometimes be matched with cycles in $B^m(S_K)^0$ having infinite order in the π_f-component of $J^m(S_K(\mathbb{C}))$? Can the height pairing be evaluated on any (one) of these special cycles for $m > 1$? Can one try to understand systematically the analytic expressions for the derivatives of the relevant L-functions?*

Clearly, when $2m$ is in the vanishing range, one gets homologically trivial cycles for free. More interesting examples arise when one knows enough about the intersection numbers with the algebraic classes in H^{2d-2m}. Such a situation arises, for example, for divisors on Hilbert modular, resp. Picard modular, surfaces in [**HZ**], resp. [**Ku1**], where the intersection numbers arise as coefficients of certain modular forms. It will be interesting to understand systematically the intersection numbers of algebraic 1-cycles Z lying on a Siegel modular threefold $\tilde{S}_K$ (attached to $GSp(4)\mathbb{Q}$) relative to the modular surfaces with non-trivial classes in H^2. Some examples of Z are the Shimura curves $/\mathbb{Q}$ (which parametrize abelian surfaces with multiplication by an indefinite quaternion algebra $/\mathbb{Q}$) and the modular curves via the embedding of $GL(2) \times GL(1)$ in $GSp(4)$. Here are two examples of cuspidal $\prod$'s contributing to H^3, both making use of the characterization ([**JPSS1**], [**JS2**]) of the forms on $GL(4)$ which descend to $GSp(4)$:

(i) Let π be a holomorphic cuspidal representation of $GL(2)$ over a real quadratic field F of weight $(2, 4)$ at infinity. Then there exists

a cuspidal $\prod$ of $GSp(4)/\mathbf{Q}$ such that (the degree $4/\mathbf{Q}$ L-function—[**PS**]) $L(\prod, s)$ coincides upto a finite number of factors with $L(\pi, s)$, and such that $\prod_\infty$ corresponds to a generic discrete series contributing to H^3.
(ii) Let $\mathcal{E}$ be an elliptic curve $/\mathbf{Q}$ with CM corresponding to a Hecke character χ of weight 1 of an imaginary quadratic field K. Then there exists a cuspidal $\prod$ of $GSp(4)/\mathbf{Q}$ such that $L(\prod, s) = L(\mathrm{Sym}^3(H^1(\mathcal{E})), s) = L(\chi^3, s)L(\chi^2\chi^\rho)$, ρ : non-trivial automorphism of K, with $\prod_\infty$ the same type as in (i).

In either case, root number calculations give examples where the L-function vanishes at the critical center. The generic forms give the ((2,1),(1,2))-part of H^3, while the L-equivalent holomorphic ones, when they exist as they are expected to for stable forms ([**Ar1**]), give the $((3,0),(0,3))$-part. When there is no holomorphic contribution for a specific π_f, then the intermediate Jacobian becomes algebraic, isogenous in case (ii) to $\mathcal{E}(-1)$. It will be exciting to get *modular* examples (in codimension > 1) of homologically trivial cycles of infinite order modulo algebraic equivalence as in [**Bl2**], [**Harr**]. One wonders if the see-saw pairs formalism of [**Ku2**] could be used to understand the Abel-Jacobi periods. On the analytic side, one has an understanding of the L-functions on $GSp(4)\times GL(2)$ when the form on $GSp(4)$ is generic or of special Bessel type ([**PSSo**], [**GePS**]), but one wishes for an integral representation when $\prod$ is holomorphic. (It is not clear if every holomorphic form has a special Bessel model *globally*.) See [**GeSh**] for a survey of automorphic L-functions and their integral representations. Finally, for elliptic cusp forms of higher (even) weight $2k$, the derivative formula of [**GrZ**] provides good support for conjecture (1.14) at the critical center ($s = k$), and a theory of heights in local systems is developed in [**Br**] to interpret this formula. It will be striking to gain a further understanding in terms of the height of (algebraic) Heegner cycles in the fiber product of the universal elliptic curve over the modular curve.

We have completely ignored here the spectacular successes achieved in the context of the Birch and Swinnerton-Dyer conjecture for different class of modular elliptic curves $/\mathbf{Q}$. For results in this direction, see, for example, [**CW**], [**G**], [**GrZ**], [**GrKZ**], [**Ru**], [**Ko**], (and also [**BFH**] and [**MuMu**]).

References

[**Ar1**] J. Arthur, *On some problems suggested by the trace formula*, in "Lie

Group Representations II," *SLN* **1041**, 1983, pp. 1–49.
[**Ar2**] J. Arthur, *Unipotent automorphic representations: global motivation*, in this volume.
[**AMRT**] A. Ash, D. Mumford, M. Rapoport and Y. Tai, "Smooth compactification of locally symmetric varieties," Math. Sci. Press, 1975.
[**Be1**] A. Beilinson, *Higher regulators and values of L-functions*, (English translation), Journal of Soviet Math **30, no.2** (1985), 2036–2070.
[**Be2**] ________, *Notes on absolute Hodge cohomology*, Contemp. Math **55, part I** (1986), 35–68.
[**Be3**] ________, *Higher regulators for modular curves*, Contemp. Math **55, part I** (1986), 1–34.
[**Be4**] ________, *Height pairing for algebraic cycles*, in "K-Theory and arithmetic," *SLN* **1289** (1987), 1–26.
[**BeBD**] A. Beilinson, J. N. Bernstein and P. Deligne, *Faisceaux pervers*, Asterisque **100** (1982).
[**BeMS**] A. Beilinson, R. MacPherson and V. V. Schechtman, *Notes on motivic cohomology*, Duke Math Journal **54, no. 2** (1987), 679–701.
[**BeGSV**] A. Beilinson, A. B. Goncharev, V. V. Schechtman, A. N. Varchenko, *Aomoto dilogarithms, mixed Hodge structures, and motivic cohomology of pairs of triangles on the plane*, preprint.
[**B1**] D. Blasius, *On the critical values of Hecke L-series*, Annals of Math **124** (1986), 23–63.
[**B2**] ________, *Period relations and the critical values of L-functions*, MSRI preprint (1987).
[**B3**] ________, *Automorphic forms and Galois representations: Some examples*, in this volume.
[**BRo1**] D. Blasius and J. Rogawski, *Galois representations for Hilbert modular forms*, Bulletin of the AMS (to appear).
[**BRo2**] ________, *Tate classes and arithmetic quotients of the unit ball*, to appear in [**Sem**] below.
[**Bl1**] S. Bloch, "Lectures on algebraic cycles," Duke Mathematics series IV, 1980.
[**Bl2**] ________, *Algebraic cycles and values of L-functions*, Crelle's Journal **350** (1984), 94–108.
[**Bl3**] ________, *Algebraic cycles and higher K-theory*, Advances in Math **61, no.3** (1986), 267–304.
[**Bl4**] ________, *Height pairings for algebraic cycles*, Journal of Pure and Applied Algebra **34** (1984), 119–145.
[**Bl5**] ________, *On the Chow groups of certain rational surfaces*, Annales Scient. ENS **14** (1981), 41–59.
[**BlG**] S. Bloch and D. Grayson, *K_2 and elliptic curves: Computer calculations*, Contemp. Math **55 part I** (1986), 79–88.
[**Bo**] A. Borel, *Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers*, Ann. Sc. Norm. Pisa **4, no.4** (1977), 613–636.
[**BoW**] A. Borel and N. Wallach, "Continuous cohomology discrete subgroups, and representations of reductive groups," *Annals of Math. Studies* **94** (1980).
[**Br**] J.-L. Brylinski, *Heights for local systems on curves*, preprint (1987).

[BFH] D. Bump, S. Friedberg and J. Hoffstein, *Nonvanishing theorems for L-functions of modular forms and their derivatives*, preprint (1989).

[Ca] H. Carayol, *Sur les représentations L-adiques associées aux formes modulaires de Hilbert*, Annales Scient. ENS **4^e sér., t. 19** (1986), 409-468.

[Cl1] L. Clozel, *Motifs et formes automorphes: Applications de fonctorialité*, in these proceedings, .

[Cl2] ———, *On the cuspidal cohomology of arithmetic subgroups of $SL(2n)$ and the first Betti number of arithmetic three manifolds*, Duke Math. Journal **55, no.2** (1987), 475–486.

[CW] J. Coates and A. Wiles, *On the conjectures of Birch and Swinnerton-Dyer*, Inventiones Math **39, no.3** (1977), 223–251.

[CoS] J.-L. Colliot-Thélène and J.-J. Sansuc, *On the Chow groups of certain rational surfaces: a sequel to a paper of Bloch*, Duke Math. Journal **48** (1981), 421–477.

[De1] P. Deligne, *Périodes d'integrales et les valeurs de fonctions L*, Proc. Symp. Pure Math **33, part 2** (1979), 313–346.

[De2] ———, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Proc. Symp. Pure Math **33, part 2** (1979), 247–289.

[De3] ———, *La conjecture de Weil*, Publications Math. IHES **43** (1974), 273–307.

[De4] ———, *Le groupe fondamental de la droite projectif moins trois points*, preprint (1989).

[DMOS] P. Deligne, J. S. Milne, A. Ogus and Shih, "Hodge cycles, motives and Shimura varieties," Springer LN 900, 1982.

[Den1] C. Deninger, *Higher regulators and Hecke L-series of imaginary quadratic fields I*, Invent. Math **96** (1989), 1–69.

[Den2] ———, *Higher regulators and Hecke L-series of imaginary quadratic fields II*, preprint.

[Fa] G. Faltings, *Endlichkeitssatze für abelsche varietäten über Zahlkörpern*, Inventiones Math **73** (1983), 349–366.

[Fu] W. Fulton, "Intersection theory," Ergebnisse ser., Springer, 1984.

[GeSh] S. Gelbart and F. Shahidi, *Analytic properties of automorphic L-functions*, Perspectives in Math, Acad. Press.

[GePSR] S. Gelbart, I. Piatetski-Shapiro and S. Rallis, "Explicit constructions of automorphic L-functions," *SLN* **1254** (1987).

[G] R. Greenberg, *On the Birch and Swinnerton-Dyer conjecture*, Inventiones Math **72** (1983), 241–265.

[GH] P. Griffiths and J. Harris, "Principles of Algebraic Geometry," John Wiley & Sons, NY, 1978.

[GrZ] B. Gross, *Heegner points and derivatives of L-series*, Inventiones Math **84** (1986), 225–320.

[GrKZ] B. Gross, W. Kohnen and D. Zagier, *Heegner points and derivatives of L-series II*, Math. Annalen **278** (1987), 497–562.

[Gro] A. Grothendieck, *Standard conjectures on algebraic cycles*, International Colloquium (Bombay) on Algebraic Geometry, Oxford University Press (1969), 193–199.

[Ha1] G. Harder, *Arithmetische Eigenschaften von Eisenstein-klassen, die modulare Konstruktion von gemischten Motiven und von Erweiterungen endlicher Galoismoduln*, preprint.

[Ha2] ———, *Eisenstein-Kohomologie arithmetischer Gruppen: Allgemeine Aspekte*, preprint.

[HLR] G. Harder, R. P. Langlands and M. Rapoport, *Algebraische Zykeln auf Hilbert-Blumenthal Flachen*, Crelle's Journal **366** (1986), 53–120.

[HaS] G. Harder and N. Schappacher, *Special values of Hecke L-functions and abelian integrals*, SLN **1111** (1985), 17–49.

[Har] B. Harris, *Homological versus algebraic equivalence in a Jacobian*, Proc. Natl. Acad. Sci. USA **80** (1983), 1157–1158.

[H1] M. Harris, *Period invariants of Hilbert modular forms*, preprint.

[HZ] F. Hirzenbruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Inventiones Math **36** (1976), 57–113.

[JLai] H. Jacquet and F. Lai, *A relative trace formula*, Compositio Math **54, no.2** (1985), 2243–310.

[JPSS1] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *A converse theorem for $GSp(4)$*, in preparation.

[JPSS2] ———, *On Euler products and the classification of automorphic forms II*, American Journal of Math **103, no.4** (1981), 777–815.

[JS1] H. Jacquet and J. Shalika, *A non-vanishing theorem for GL_n*, Inventiones Math **38** (1976), 1–16.

[JS2] ———, *Exterior square L-functions*, in these proceedings .

[Ja] U. Jannsen, *Mixed motives and the conjectures of Hodge and Tate*, preprint; see also the article in [**RSS**] below.

[KaM] N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Inventiones Math **23** (1974), 73–77.

[K] S. Kleiman, *Motives*, 5th Nordic summer school in Math., Oslo, Norway.

[Kl] C. Klingenberg, *Die Tate-Vermutungen für Hilbert-Blumenthal Flachen*, Inventiones Math **89** (1987), 319–345.

[Kol] V. A. Kolyvagin, *Finiteness of $E(\mathbb{Q})$ and $III(E/\mathbb{Q})$ for a subclass of Weil curves*, Izv. Akad. Nauk SSSR ser. Matem **52** (1988), in Russian.

[Ko] R. Kottwitz, article in this volume.

[Ku1] S. Kudla, *Intersection numbers for quotients of the complex 2-ball and Hilbert modular forms*, Inventiones Math **47** (1978), 189–208.

[Ku2] ———, *Period integrals for $SU(n,1)$*, Compositio Mathematica **50** (1983), 3–63.

[KuM] S. Kudla and J. Millson, *Geodesic cycles and the Weil representation I; Quotients of hyperbolic space and Siegel modular forms*, Compositio Math **45** (1982), 207–271.

[Kum] S. Kumaresan, *On the canonical K-types in the irreducible unitary $\mathfrak{g}$-modules with non-zero relative cohomology*, Inventiones Math **59** (1980), 1–11.

[Lai] F. Lai, *Algebraic cycles on compact Shimura surfaces*, Math. Zeitschrift **189** (1985), 593–602.

[La1] R. P. Langlands, *Automorphic representations, Shimura varieties and Motive. Ein marchen*, Proc. Symp. Pure Math **33 part 2** (1979), 205–246.

[**Lar**] M. Larsen, *Arithmetic compactification of unitary Shimura surfaces*, to appear in [**Sem**] below.

[**Li**] S. Lichtenbaum, *The construction of weight two arithmetic cohomology*, Inventiones Math **88** (1987), 183–215.

[**Lo**] E. Looijenga, *L^2-cohomology of locally symmetric varieties*, Compositio Mathematica (1988).

[**Ma**] J. I. Manin, *Correspondences, motifs and monoidal transformations*, Mathematics of the USSR, Sbornik **6, no.4**, 439–470.

[**Maz**] B. Mazur, *Rational isogenies of prime degree*, Inventiones Math **44 no.2** (1978), 129–162.

[**MR**] J. Millson and M. S. Raghunathan, *Geometric construction of cohomology for arithmetic groups*, Proc. Indian Acad. Sci. (Math. Sci.) **90 no.2** (1981), 103–123.

[**Mil1**] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, this volume.

[**Mil2**] ________, *Motivic cohomology and values of zeta-functions*, Compositio Math. **68** (1988), 58–102.

[**Mum**] D. Mumford, *Rational equivalence of O-cycles on surfaces*, J. Math. Kyoto University **9** (1968), 195–204.

[**Mur1**] J. P. Murre, *Applications of Algebraic K-theory to the theory of algebraic cycles*, Springer Lec. Notes **1124** (1985),.

[**Mur2**] ________,, preprint.

[**MuMu**] V. K. Murty and M. R. Murty, *Mean values of derivatives of modular L-series*, preprint 1989.

[**MuRa1**] V. K. Murty and D. Ramakrishnan, *Period relations and the Tate conjecture for Hilbert modular surfaces*, Inventiones Math **89** (1987), 319–345.

[**MuRa2**] ________, *Tate conjecture for quaternionic Shimura surfaces*, in preparation.

[**MuRa3**] ________, *Shimura curves, algebraic cycles, and complex multiplication*, in preparation.

[**MuRa4**] ________, *On the Albanese of unitary Shimura varieties*, to appear in [Sem] below.

[**Od**] T. Oda, *A note on the Albanese variety of an arithmetic quotient of the complex hyperball*, Fac. Sci. Univ. Tokyo Sect.1A Math **28, no.3** (1982), 481–486.

[**PS**] I. Piatetski-Shapiro, *L-functions for $GSp(4)$*, preprint.

[**PSSo**] I. Piatetski-Shapiro and D. Soudry, *L and ϵ-functions for* $GSp(4) \times GL(2)$, Proc. Natl. Acad. Sci. USA **81** (1984), 3924–3927.

[**Ra1**] D. Ramakrishnan, *Regulators, algebraic cycles, and values of L-functions*, Contemp. Math **83** (1989), 183–310.

[**Ra2**] ________, *Periods of integral arising from K_1 of Hilbert-Blumenthal surfaces*, preprint (85); being revised.

[**Ra3**] ________, *Arithmetic of Hilbert-Blumenthal surfaces*, CMS Proceedings **I** (1987), 285–370.

[**Ra4**] ________, *Valeurs de fonctions L des surfaces d'Hilbert-Blumenthal en $s = 1$*, CR Acad. Sci. Paris **t.301, Ser. I, no.18** (1985), 809–812.

[Ra5] ________________, *Higher regulators for quaternionic Shimura curves, and values of L-functions*, Contemporary Math **55, part I** (1986), 377–387.

[Ra6] ________________, *Two results concerning Beilinson's conjectures*, in preparation.

[RSS] M. Rapoport, P. Schneider and N. Schappacher, *Beilinson's conjectures on special values of L-functions*, in "Perspectives in Math **4**," Academic Press (1988).

[Ras] W. Raskind, *Algebraic K-theory, Etale cohomology and Torsion algebraic cycles*, Contemp. Math **83** (1989), 311–341.

[Ri] K. Ribet, *Sur les varietés abeliennes à multiplications reelles*, CR Acad. Sc. Paris **ser. A, t.291** (1980), 121–123.

[Ro] J. Rogawski, "Automorphic representations of unitary groups in three variables," to appear (1989), Annals of Math. Studies.

[Roi] Roitman, *The torsion of the group of O-cycles modulo rational equivalence*, Ann. Math **111, no.3** (1980), 553–569.

[Rs] R. Ross, Ph. D. thesis, Rutgers University (1989/90).

[Ru] K. Rubin, *Tate Shafarevich groups and L-functions of elliptic curves with complex multiplication*, Inventiones Math **89** (1987), 527–559.

[SS] L. Saper and M. Stern, *L^2 cohomology of arithmetic varieties*, preprint.

[Sch] C. Schoen, *Zero cycles modulo rational equivalence for some varieties over fields of transcendence degree one*, Proc. Symp. Pure Math **46 part 2** (1987), 463–473.

[ScSc] N. Schappacher and A. J. Scholl, *Beilinson's theorem on modular curves*, article in [RSS] abovel.

[Sc] A. J. Scholl, *Motives for modular forms*, preprint.

[Sem], "Seminar on the Zeta functions of Picard modular surfaces," ed. by J. Arthur, R. P. Langlands and D. Ramakrishnan, CRM Proceedings, Montreal (to appear).

[Shah] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Annals of Math **127** (1988), 547–584.

[Sh1] G. Shimura, *Algebraic relations between critical values of zeta functions and inner products*, American Journal of Math **104** (1983), 253–285.

[Sh2] ___________, *The periods of certain automorphic forms of arithmetic type*, J Fac. Sci. Univ. Tokyo IA Math **28, no.3** (1981), 605–632.

[Sh3] ___________, *On certain zeta functions attached to two Hilbert modular forms I & II*, Annals of Math **114** (1981), 127–164 & 569–607.

[So] C. Soulé, *Operations en K-theorie algebrique*, Canadian Journal of Math **XXXVII** (1985), 488–550.

[Su] A. A. Suslin, *Homology of GL_n, characteristic classes and Milnor K-theory*, SLN **1046** (1984), 357–375.

[T] J. Tate, *On the conjectures of Birch and Swinnerton-Dyer, and a geometric analog*, Seminaire Borbaki, 18[e] année Exp. **306** (Février 1966),.

[Tay] R. Taylor, *Galois representations attached to Hilbert modular forms*, article in this volume.

[VZ] D. Vogan and G. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math **53** (1984), 51–90.

[**Wei**] Weissauer, Crelle's Journal.

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On the Bad Reduction of Shimura Varieties

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INTRODUCTION

In this survey article we are concerned with the reduction behaviour and the local zeta function of Shimura varieties at primes of bad reduction. More specifically we are interested in the structure of a Shimura variety $S(G,X)_C$, where $C \subset G(\mathbf{A}_f)$ is a (sufficiently small, i.e. neat) compact open subgroup and $C_p \subset G(\mathbb{Q}_p)$ a parahoric subgroup, and its reduction behaviour at primes dividing p. Naturally, the rational prime number p is fixed throughout.
The case "$\Gamma_0(p)$" for $G = GL_2$ has a long history which probably starts with Kronecker and involves the names of Eichler, Shimura and (more to the point) Igusa, but it was Deligne who determined completely the structure at p of this simplest Shimura variety. The next significant step was taken in 1975 by Cherednik who proved that for G the multiplicative group of a quaternion algebra over $\mathbb{Q}$ which is unramified at the infinite prime and ramified in p the corresponding Shimura variety possesses a p-adic uniformization. Drinfeld, in 1976, gave a direct and conceptual proof of Cherednik's theorem which opened the way to various generalizations. In the *first section* we review some of the structure theorems obtained so far. This section which also presents results of Langlands, Zink and myself is by no means exhaustive and in various places only gives glimpses of the full truth. Its main purpose, besides recording some of the progress made in this direction since the time of the Corvallis meeting, is to convince the reader that the global geometric structure of the reduction is in general so complicated that it cannot be effectively used to calculate the local factor of the zeta function.
We turn to the problem of the determination of the local zeta function and its semi-simple variant in the *second section.* We explain the relation of the two which relies on Deligne's conjecture on the purity of the monodromy filtration and make some remarks on its connection with the Ramanujan conjecture.
The rest of the article is concerned with a class of specific examples, namely the "fake" unitary groups defined by a division algebra with

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involution of the second kind which stays a division algebra after localization at p. In the *third section* we explain the moduli problem connected with this Shimura variety and reduce the problem of determining the **local** structure of its reduction to a problem on formal groups. We also introduce here the concept of a **local model** of the Shimura variety which is supposed to describe the singularities arising in the reduction. In an *appendix* to this section we focus our attention on a special class of these examples which we call the Drinfeld case since this class possesses a p-adic uniformization by Drinfeld's upper half space $\widehat{\Omega}$. This establishes a connection between the conjecture of Drinfeld giving a geometric construction based on the cohomology of $\widehat{\Omega}$ of the "Langlands correspondence" between representations of the general linear groups, central division algebras and Galois representations on the one hand, and the determination of the local zeta function of the Shimura variety on the other hand. (This conjecture is considered in more detail also in Carayol's contribution to this conference). I also state a conjecture on the vanishing of holomorphic cohomology up to the middle dimension for certain Shimura varieties arising from unitary groups. This conjecture can be proved in some cases.

The *fourth section* breaks up the points in the reduction into "isogeny classes". There is a conjectural description of those, due to Langlands and myself. However, a conjecture in local harmonic analysis which plays here the role of the "fundamental lemma" on spherical functions in the case of good reduction implies that only the **basic** isogeny classes yield a non-zero contribution to the Lefschetz fixed point formula. In the *fifth section* we explain how one can use an important observation of Kottwitz on the nature of L-indistinguishability in the groups involved to deduce the local zeta function from the various conjectures and assumptions made along the way. In particular we recover the result of Zink and myself on the local factor of the zeta function of a quaternionic Shimura surface. At the end of this last section I have defined some explicit functions in the Iwahori algebra of GL_d. I conjecture that their non-elliptic twisted orbital integrals vanish. If this conjecture could be proved then the program outlined in these notes very likely could be turned into a solid theorem. STOP PRESS: J.-L. Waldspurger has just proved this conjecture (see note at the end of the last section).

Proofs in these notes are very sketchy and sometimes entirely omitted but I hope that the reader can follow the line of development and nourish a sympathy for my excitement. In dealing with the problems posed by bad reduction one is indeed confronted with fascinating questions in algebraic geometry (and here of two sorts, the ones on abelian varieties, the others in ℓ-adic cohomology), in global harmonic analysis (Selberg trace formula), in local harmonic analysis (calculation of orbital and twisted orbital integrals), in continuous cohomology and probably some more — and it is wonderful how they all blend.

In writing these notes I have incorporated many ideas of others. Langlands' ideas on the zeta function of a Shimura variety in general and on the Selberg trace formula have been decisive. No less important were the influence of Drinfeld and T. Zink on the analysis of the varieties obtained by reduction modulo p, and Kottwitz's ideas on the combinatorial problems arising in the theory of Shimura varieties. All these influences should be obvious. It may be less obvious how much I learned from conversations with others, and I particularly wish to record my gratitude to R.P. Langlands, T. Zink and R.E. Kottwitz. I also thank L. Clozel for his help with some problems in local harmonic analysis.

These notes have their origin in a series of lectures which I gave in March 1983 at the Ecole Normale Superieure des Jeunes Filles (Paris). I wish to thank J. Coates and M.-F. Vigneras for inviting me to give these lectures, as well as L. Clozel and J. Milne for organizing this conference which gave me an occasion to think once again about this material.

§1 The geometry of the reduction in some examples

To put things into perspective we start with Deligne's result which concerns

$$G = GL_2, \quad C_p = \left\{ \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \operatorname{mod} p \right\} \subset GL_2(\mathbb{Z}_p) \subset G(\mathbb{Q}_p) \quad ,$$

(and where the conjugacy class X of homomorphisms $h : S \to G_{\mathbb{R}}$ is the usual one). Then the Shimura variety $S(G, X)_C$ for $C = C^p \cdot C_p$ is the parameter space of alternatively

elliptic curves together with a (cyclic) subgroup of order p

or

a (cyclic) isogeny of degree p between two elliptic curves,

together with a level structure prime to p (depending on C^p which, as always, is assumed sufficiently small). This moduli problem possesses a solution $\mathcal{M}_C$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$. A first version of Deligne's theorem is the following [**9**], V, 1.16.

1.1 THEOREM. *The scheme $\mathcal{M}_C$ is a regular 2-dimensional scheme with special fibre a reduced divisor with normal crossings. The singularities in the special fibre occur precisely in the points corresponding to the supersingular isogeny class.*

A refined version of this theorem describes the global structure of the special fibre. Let

$$C'_p = GL_2(\mathbb{Z}_p) \subset G(\mathbb{Q}_p) \quad \text{and} \quad C' = C^p \cdot C'_p \quad .$$

To C' there corresponds the moduli problem which parametrizes elliptic curves with a level structure depending on C^p (no additional structure at p). A refinement of the statement above is ([**9**], V. 1.18).

1.2 THEOREM. *The scheme $\mathcal{M}_C \otimes \mathbb{F}_p$ is obtained by glueing two copies of $\mathcal{M}_{C'} \otimes \mathbb{F}_p$ along the supersingular points, where the supersingular point x of the second copy is identified with the point $x^{(p)}$ (image of x under Frobenius) of the first copy.*

We note that this particular Shimura variety is not compact but that there is a complete and explicit description of a compactification of $\mathcal{M}_C$ (loc. cit.).

We next turn to Cherednik's result. Let D be a quaternion algebra over $\mathbb{Q}$ which is unramified at the infinite prime and ramified at p. Let G be the multiplicative group of D considered as an algebraic group over $\mathbb{Q}$ equipped with a conjugacy class X as in Deligne's example. Let

$$C_p = \text{unique maximal compact subgroup of } G(\mathbb{Q}_p),$$

and $C = C^p \cdot C_p$ as before. Then the Shimura variety $S(G,X)_C$ is again the parameter space of a moduli problem, namely **roughly speaking**

> abelian varieties of dimension 4 together with an action of a certain fixed order in D which is maximal in p such that the trace of an element of this order operating on the Lie algebra equals its reduced trace,

together with a level structure prime to p. As such it has a model $\mathcal{M}_C$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$. To state Cherednik's result we need Drinfeld's upper half space $\widehat{\Omega}^2$ for $\mathbb{Q}_p$ ([**12**]). For further remarks on this (formal) $\mathbb{Z}_p$-scheme compare the appendix to §3. Here we only mention that $PGL_2(\mathbb{Q}_p)$ acts on $\widehat{\Omega}^2$. We denote by $\mathcal{K}$ the completion of the maximal unramified extension of $\mathbb{Q}_p$ and by $\mathcal{O}$ its ring of integers.

1.3 THEOREM. *There exists an inner form G_- of G with isomorphisms*

$$G_{-ad}(\mathbb{Q}_p) \simeq PGL_2(\mathbb{Q}_p)$$
$$G_-(\mathbf{A}_f^p) \simeq G(\mathbf{A}_f^p)$$

and such that $G_{-ad}(\mathbb{R})$ is compact, such that if C^p is the image of C^p under the above isomorphism, there is an isomorphism of $\mathbb{Z}_p$-schemes

$$\mathcal{M}_C \otimes \mathbb{Z}_p \simeq G_-(\mathbb{Q})\backslash[G_-(\mathbf{A}_f^p)/C_-^p \times (\widehat{\Omega}^2 \widehat{\otimes} \mathcal{O})] \ .$$

Here the action of $G_-(\mathbb{Q})$ is diagonal; the action on the second factor is through its p-component via

$$\begin{array}{ccccc} G_-(\mathbb{Q}_p) & \longrightarrow & PGL_2(\mathbb{Q}_p) \times \mathbb{Z} & \longrightarrow & Aut(\widehat{\Omega}^2 \widehat{\otimes} \mathcal{O}) \\ g & \longmapsto & (g_{ad}, \operatorname{ord} \circ \det) & & \\ & & (g, v) & \longmapsto & g \otimes \sigma^{-v} \end{array} \quad .$$

[Here σ denotes the Frobenius substitution].

Here the right side of the isomorphism may be identified with a disjoint sum of schemes of the form $\Gamma\backslash\widehat{\Omega}^2 \widehat{\otimes} \mathcal{O}$ where Γ is a discrete subgroup of $GL_2(\mathbb{Q}_p)$ with compact quotient, a formal scheme which may be algebraicized. The local structure of $\widehat{\Omega}^2$ implies the consequence.

1.4 COROLLARY. *$\mathcal{M}_C$ is a regular scheme with special fibre a reduced divisor with normal crossings. Furthermore, $\mathcal{M}_C \otimes \mathbf{F}_{p^2}$ is the* *union of two closed smooth subschemes $\mathcal{M}_{C,i}$, $i \in \mathbb{Z}/2$, whose intersection, which is transversal, is the set of double points and which are permuted under the action of* $\mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$.

We mention that Cherednik [**5**] has also given a generalization to the case where G comes from a quaternion algebra over a totally real field in which p is inert which is unramified at precisely one of the infinite primes and ramified at p. [Experience has shown that the cases where p is allowed to split in the totally real field (but is still required to be unramified) behave essentially like products of copies of the varieties for inert p.]
We next come to the result of Langlands [**24**] and T.Zink [**50**]. They consider the following generalization of Cherednik's situation. Let D be a quaternion algebra over a totally real field F of degree n in which p stays prime. Assume that D is unramified at all infinite primes of F and ramified in p. Let $S(G,X)_C$ be the associated Shimura variety (again $C_p \subset G(\mathbb{Q}_p)$ is the unique maximal compact subgroup). Again there is a moduli problem entirely analogous to the one considered in Cherednik's case which is solved by this Shimura variety and which defines a model $\mathcal{M}_C$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$. Its structure is dictated by the representation of the maximal order $\mathcal{O}_{D_p}$ on the Lie algebra of the abelian varieties parametrized by $\mathcal{M}_C \otimes \mathbf{F}_p$. The global structure of $\mathcal{M}_C$ for higher n is combinatorially so difficult that we content ourselves with stating the precise result in the case $n = 2$ only. In the statement there appear subsets $S \subset \mathbb{Z}/2n\mathbb{Z}$ such that for all i at least one element of $\{i, i+n\}$ lies in S. Such S are called **admissible**.

1.5 THEOREM. *Let $n = 2$. (i) $\mathcal{M}_C \otimes \mathbf{F}_{p^4}$ is the union of closed subschemes $\mathcal{M}_{C,S}$, for S ranging over the admissible subsets of $\mathbb{Z}/4\mathbb{Z}$, with*

$$\mathcal{M}_{C,S} \subset \mathcal{M}_{C,S'} \iff S \supset S'$$
$$\mathcal{M}_{C,S} \cap \mathcal{M}_{C,S'} = \mathcal{M}_{C,S\cup S'}$$

The scheme $\mathcal{M}_{C,S}$ has dimension $4 - |S|$. The subschemes $\mathcal{M}_{C,S}$ intersect transversely. If $x \in \mathcal{M}_C \otimes \mathcal{O}_{\mathbb{Q}_{p^4}}$ is a closed point and S_x is the maximal admissible subset S with $x \in \mathcal{M}_{C,S}$, then $\mathcal{M}_C \otimes \mathcal{O}_{\mathbb{Q}_{p^4}}$ is locally in x for the étale topology isomorphic to a product of $|S_x| - 2$ ordinary double points and a smooth scheme of dimension $4 - |S_x|$. The Frobenius element in $\mathrm{Gal}(\mathbf{F}_{p^4}/\mathbf{F}_p)$ *takes $\mathcal{M}_{C,S}$ to $\mathcal{M}_{C,S+1}$.*
(ii) There are morphisms $\pi_i : \mathcal{M}_{C,\{i,i+1\}} \to \mathcal{M}_{C,\{i,i+1,i+2\}}$ with smooth generic fibre and whose reduced geometric fibres are non-

singular rational curves. The restriction of π_i to $\mathcal{M}_{C,\{i,i+1,i+2\}}$ is an isomorphism and the restriction of π_i to $\mathcal{M}_{C,\{i-1,i,i+1\}}$ is a purely inseparable morphism of degree p.
(iii) There are universal homeomorphisms

$$\mathcal{M}_{C,\{i,i+1,i+2\}} \longrightarrow \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\mathbb{Q}_{p^4}}} \mathbf{F}_{p^4}$$

where $\widetilde{\mathcal{M}}$ is a twisted form of a Shimura variety with **good reduction** *associated to a quaternion algebra over F which splits at p and is ramified at precisely one of the two infinite primes (depending on i) and has the same ramification behaviour as D elsewhere.*

Here we have denoted by $\mathbb{Q}_{p^4}$ the unramified extension of degree 4 of $\mathbb{Q}_p$ and by $\mathcal{O}_{\mathbb{Q}_{p^4}}$ its ring of integers. Roughly speaking, the curves $\mathcal{M}_{C,\{i,i+1,i+2\}}$ are good reductions of Shimura varieties closely related to $\mathcal{M}_C$. Something similar is true for the zero-dimensional subscheme $\mathcal{M}_{C,\mathbb{Z}/4\mathbb{Z}}$. In the case of arbitrary n the statement (i) of the above theorem which describes the local structure of $\mathcal{M}_C$ continues to hold with the obvious modifications. As to the global structure, Zink [**50**] introduces the concept of a **saturated** admissible subset $S \subset \mathbb{Z}/2n\mathbb{Z}$ (for $n = 2$, any S with $|S| \geq 3$ is saturated) and proves that the corresponding subschemes $\mathcal{M}_{C,S}$ are homeomorphic to twisted forms of the good reduction of Shimura varieties associated to quaternion algebras split at p, with a certain prescribed ramification behaviour at the infinite primes (depending on S) and with the same ramification behaviour as D elsewhere. (For $S = \mathbb{Z}/2n\mathbb{Z}$ this statement needs qualification.) He furthermore proves that if S is an arbitrary admissible subset and $S' = S \cup \{i_1, \ldots, i_m\}$ is a minimal saturated subset which contains S then there is a sequence of morphisms

$$\mathcal{M}_{C,S} \longrightarrow \mathcal{M}_{C,S\cup\{i_1\}} \longrightarrow \ldots \longrightarrow \mathcal{M}_{C,S\cup\{i_1,\ldots,i_m\}}$$

which are $\mathbb{P}^1$-fiberings similar to the morphisms π above (again for $S' = \mathbb{Z}/2n\mathbb{Z}$ this statement needs qualification). Roughly speaking, all $\mathcal{M}_{C,S}$ are $(\mathbb{P}^1)^m$-fiberings over good reductions of Shimura varieties. The enumeration of the saturated subsets of $\mathbb{Z}/2n\mathbb{Z}$ is for $n > 2$ a combinatorially complicated business.
As the final example we consider a fake unitary group in 3 variables over $\mathbb{Q}$. (I obtained these results almost 10 years ago, cf. [**33**]). Let $E \subset \mathbb{C}$ be an imaginary-quadratic field in which p splits into

two primes $\wp$ and $\overline{\wp}$. Let D be a central division algebra of degree 9 over E and let τ be an involution of the second kind. We make the assumption that the signature of τ (relative to the complex embedding of E) is (1,2) and that D stays a division algebra at p with invariants

$$\mathrm{inv}_{\wp} D = \frac{1}{3} \quad , \quad \mathrm{inv}_{\overline{\wp}} D = \frac{2}{3} \quad .$$

We let G be the associated unitary group over $\mathbb{Q}$,

$$G(\mathbb{Q}) = \{d \in D^{\times} \mid d \cdot d^{\tau} \in \mathbb{Q}\}$$

and X the canonical conjugacy class of homomorphisms $h : S \to G_{\mathbb{R}}$. The associated Shimura variety $S(G, X)_C$ where C_p is the unique maximal compact subgroup of $G(\mathbb{Q}_p)$ (note that $G_{ad}(\mathbb{Q}_p)$ is anisotropic), is defined over E. Again it is the moduli space of abelian varieties with additional structures (for details compare §3) and as such has a model $\mathcal{M}_C$ over $\operatorname{Spec} \mathcal{O}_{E_{(p)}}$. Interestingly enough, the structure of $\mathcal{M}_C$ at the primes $\wp$ and $\overline{\wp}$ (note that they are well-distinguished by the conditions above) are quite distinct.

1.6 THEOREM. (structure in $\wp$): *There exists an inner form G_- of G with isomorphisms*

$$G_{-ad}(\mathbb{Q}_p) \simeq PGL_3(\mathbb{Q}_p)$$
$$G_-(\mathbf{A}_f^p) \simeq G(\mathbf{A}_f^p)$$

and with $G_{-ad}(\mathbb{R})$ compact such that if C_-^p is the image of C^p under the above isomorphism, there is an isomorphism of schemes over $\operatorname{Spec} \mathcal{O}_{E_\wp}$

$$\mathcal{M}_C \otimes \mathcal{O}_{E_\wp} \simeq G_-(\mathbb{Q}) \backslash \left[G_-(\mathbf{A}_f^p)/C_-^p \times (\widehat{\Omega}^3 \widehat{\otimes} \mathcal{O}) \right] \quad .$$

Here $\widehat{\Omega}^3$ is Drinfeld's upper half space for $\mathbb{Q}_p$ of dimension one higher than in 1.3. The explanation for 1.3. applies here as well; also the corollary 1.4. has an obvious analogue (here $\mathcal{M}_C \otimes \mathbb{F}_{p^3}$ is a union of three closed subsets permuted by $\mathrm{Gal}(\mathbb{F}_{p^3}/\mathbb{F}_p)$).

1.7 THEOREM. (structure in $\overline{\wp}$): *(i) $\mathcal{M}_C \otimes \mathbb{F}_{p^3}$[1] is the union of closed subschemes $\mathcal{M}_{C,S}$ for S ranging over the non-empty subsets of $\mathbb{Z}/3\mathbb{Z}$*

[1]Here the homomorphism $\mathcal{O}_E \to \mathbb{F}_{p^3}$ is supposed to factor through $\mathcal{O}_{E_{\overline{\wp}}}$.

with the inclusion and intersection properties as in 1.5. The closed subschemes $\mathcal{M}_{C,\{i\}}$ ($i \in \mathbb{Z}/3$) are divisors in $\mathcal{M}_C \otimes \mathcal{O}_{\mathbb{Q}_{p^3}}$ [1)] *crossing transversely. The Frobenius carries $\mathcal{M}_{C,S}$ into $\mathcal{M}_{C,S+1}$.*
(ii) There are morphisms $\pi_i : \mathcal{M}_{C,\{i\}} \to \mathcal{M}_{C,\{i,i+1\}}$ which are $\mathbb{P}^1$-fiberings as in 1.5 (ii) and whose restriction to $\mathcal{M}_{C,\{i,i+1\}}$ is an isomorphism and whose restriction to $\mathcal{M}_{C,\{i-1,i\}}$ is purely inseparable of degree p.

It should be pointed out that whereas the zero-dimensional scheme $\mathcal{M}_{C,\mathbb{Z}/3}$ can be identified with the good reduction of a Shimura variety, a similar identification (or any other) of the curves $\mathcal{M}_{C,\{i,i+1\}}$ is not known. Also, a global structure theorem for such unitary groups in 3 variables over a totally real field F (instead of $\mathbb{Q}$) is not known. However, T. Zink does have a generalization to the case where the signature of τ at one infinite prime of F is $(1,2)$ and is $(0,3)$ at the remaining infinite primes.

This concludes our review of some of the cases where the global structure of the reduction has been investigated. For a more general result on p-adic uniformization compare the appendix to §3. Finally, I wish to mention that these geometric descriptions have spectacular applications (spectacular even in the public domain) to the construction of Goppa codes ([**46**], [**51**]).

§2 The local factor of the Hasse-Weil zeta function

We now turn to the problem of the determination of the local factor of the zeta function. In this present section we make some general remarks on the methods used and Shimura varieties will not appear explicitly.

Let K be a number field and X a smooth projective variety over K (not necessarily connected). The Hasse-Weil zeta function is at first defined as a product over almost all places of K (all except for a finite set S, containing the infinite places)

$$Z_{(S)}(s, X/K) = \prod_{\wp \notin S} Z_\wp(s, X/K) \quad . \tag{2.1}$$

If $\wp$ is a non-archimedian place where X has good reduction $X(\wp)$ the local factor at $\wp$ is the zeta function of $X(\wp)$.

$$Z_\wp(T, X(\wp)) = \exp\Big(\sum_{j=1}^{\infty} \frac{N_j}{j} T^j\Big) \quad ,$$

(2.2)
$$N_j = \operatorname{card} X(\wp)(\kappa_{\wp^j}) \quad .$$

Here $\kappa_{\wp^j}$ denotes the extension of degree j of the residue field at $\wp$. We obtain a function of the complex variable s by substituting $T = N\wp^{-s}$.

The product (2.1) is convergent for all s with $\operatorname{Re} s > \dim X + 1$ and is in this domain a holomorphic Dirichlet series with integer coefficients. The Hasse-Weil conjecture states that, with a suitable definition of the local factors at the missing places, the Hasse-Weil zeta function possesses an analytic continuation as a meromorphic function to the whole complex plane and admits a functional equation of the usual sort relating $Z(s, X/K)$ with $Z(\dim X + 1 - s, X/K)$. Tanyama suggested dividing this problem into two. The first is to prove that $Z(s, X/K)$ is a product of **automorphic L-functions**. The second is to prove for these L-functions analytic continuation and functional equation. We shall be concerned here only with the first problem in the case of Shimura varieties, and even only with the local factor at a **non-archimedean bad** prime. The local factor at an **arbitrary** non-archimedean prime $\wp$ is defined [38] through the ℓ-adic representation. It mimics Artin's definition [2] of his non-abelian L-series. Fix an algebraic closure $\overline{K}_\wp$ of the local field of K at $\wp$ and let $K_\wp^{un}$ be the maximal unramified subfield. The Galois group is an extension by the inertia subgroup.

$$(2.3) \qquad 1 \longrightarrow I \longrightarrow \operatorname{Gal}(\overline{K}_\wp/K_\wp) \longrightarrow \operatorname{Gal}(K_\wp^{un}/K_\wp) \longrightarrow 1 \quad .$$

The Galois group acts by transport of structure on the étale cohomology groups $H^i(X \times_K \overline{K}_\wp, \mathbb{Q}_\ell)$. We form
(2.4)
$$Z_\wp(T, X/K) = \prod_{i=0}^{2\dim X} \det(1 - T \cdot \sigma^* \mid H^i(X \times_K \overline{K}_\wp, \mathbb{Q}_\ell)^I)^{(-1)^{i+1}} \quad .$$

Here $\sigma \in \operatorname{Gal}(\overline{K}_\wp/K_\wp)$ denotes an arbitrary lifting of the inverse of the Frobenius substitution in $\operatorname{Gal}(K_\wp^{un}/K_\wp)$ and the upper index I signifies the invariants under the inertia subgroup. Again, to obtain a function of the complex variable s we make the substitution $T = N\wp^{-s}$. If $\wp$ is a good prime, I acts trivially on all cohomology groups, we have $H^i(X \times_K \overline{K}_\wp, \mathbb{Q}_\ell) = H^i(X(\wp) \otimes_{\kappa_\wp} \overline{\kappa}_\wp, \mathbb{Q}_\ell)$, and (2.4) is simply Grothendieck's cohomological expression of the zeta function (2.2).

In the general case, even though it is not known whether (2.4) is independent of the prime number ℓ used to form ℓ-adic cohomology, it is expected that this is the correct definition of the local factors. Recall that, if σ is an endomorphism of a finite-dimensional vector space over a field of characteristic zero, we have

$$\log\det(1 - T\cdot\sigma \mid V) = -\sum_{j=1}^{\infty} \frac{\operatorname{Tr}\sigma^j}{j}\cdot T^j \quad . \tag{2.5}$$

Thus the determination of the local factor at $\wp$ is equivalent with the determination of the alternating trace of σ^{*j} on the I-invariants in the cohomology for all $j = 1, 2, \ldots$ This problem is approached through the method of vanishing cycles. I shall recall briefly the essence of this method [39], as it motivates much of what follows.

We consider a diagram as follows in which all squares are cartesian.

$$\begin{tikzcd}
\widetilde{X}^* \arrow[r] \arrow[d] \arrow[rr, bend left, "\widetilde{j}"] & X^* \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] & X_s \arrow[l, hook', "\iota"'] \arrow[d] \\
\widetilde{D}^* \arrow[r] & D^* \arrow[r, hook] & D & s \arrow[l]
\end{tikzcd}$$

In the classical case D is the unit disc, s is the origin and D^* is its complement in D, and $\widetilde{D}^*$ is the universal covering of D^* which may be identified with the upper half plane via the map $z \mapsto \exp(2\pi i z)$. The arrows are morphisms of analytic spaces and one assumes that the morphism f is proper and that its restriction to X^* is smooth. Then X_s is a deformation retract of X and the fibering of $\widetilde{X}^*$ over the contractible topological space $\widetilde{D}^*$ is topologically trivial. We consider the Leray spectral sequence for the morphism $\widetilde{j}$, $H^p(X, R^q\widetilde{j}_*\mathbb{Q}) \Rightarrow H^{p+q}(\widetilde{X}^*, \mathbb{Q})$. Using the facts mentioned above we may rewrite this spectral sequence as follows

$$H^p(X_s, \iota^* R^q\widetilde{j}_*\mathbb{Q}) \Longrightarrow H^{p+q}(\widetilde{X}^*, \mathbb{Q}) \quad .$$

The sheaves on the special fibres are the **vanishing cycle sheaves** for the constant sheaf $\mathbb{Q}$,

$$R^q\Psi = \iota^* R^q\widetilde{j}_*\mathbb{Q} \quad .$$

Their stalks at a point $x \in X_s$ are calculated as follows:

$$R^q\Psi_x = H^q(X_{(x)} \cap \widetilde{X}^*_t, \mathbb{Q}) \quad .$$

Here $X_{(x)}$ is a small open neighbourhood of x in X and $\widetilde{X}_t^*$ is the fibre of $\widetilde{X}^*$ over a point $t \in \widetilde{D}^*$ which is mapped to a point close to the origin in D^*.

In the abstract case, D is the spectrum of a henselian discrete valuation ring, s is its special point and D^* its general point η. By $\widetilde{D}^*$ we denote the spectrum of a geometric point $\overline{\eta}$ over η. The morphism f is now a morphism of schemes about which we make the same assumption as in the classical case. Replacing the coefficient field $\mathbb{Q}$ by $\mathbb{Q}_\ell$ and the topological arguments used in the classical case by theorems in ℓ-adic cohomology we obtain the spectral sequence of vanishing cycles in étale cohomology.

$$H^p(X_{\overline{s}}, R^q\Psi) \Longrightarrow H^{p+q}(X_{\overline{\eta}}, \mathbb{Q}_\ell) \quad . \tag{2.6}$$

Here $\overline{s}$ denotes the geometric point over s determined by $\overline{\eta}$. This spectral sequence is equivariant with respect to the action of $\mathrm{Gal}(\overline{\eta}/\eta)$. For our purposes it is enough, instead of going up all the way to the (spectrum of the) algebraic closure $\overline{\eta}/\eta$ to pass instead to the maximal tame extension η_t. Therefore if the residue field s is finite there is an exact sequence

$$1 \longrightarrow \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \longrightarrow \mathrm{Gal}(\eta_t/\eta) \longrightarrow \hat{\mathbb{Z}} \longrightarrow 1 \quad .$$

If $P \subset I$ denotes the kernel of the map to $\mathrm{Gal}(\eta_t/\eta)$, then, with obvious notation (**tame** vanishing cycles)

$$R^q\Psi_t = (R^q\Psi)^P$$

(taking P-invariants is an exact functor). The calculation of the sheaves of tame vanishing cycles has been effected in a few cases only. Here is one of them.

2.7 THEOREM. *Suppose that the special fibre is a* **reduced** *divisor with normal crossings. Assume also for simplicity that $X_{\overline{s}}$ is globally the union of smooth irreducible divisors. Let $x \in X_s$ and let S_x be the set of irreducible components of $X_{\overline{s}}$ passing through x. Then*

$$R^1\Psi_{t_x} = \mathrm{Ker}\left(\bigoplus_{S_x} \mathbb{Q}_\ell(-1) \xrightarrow{\Sigma} \mathbb{Q}_\ell(-1)\right)$$

$$R^q\Psi_{t_x} = \Lambda^q R^1\Psi_{t_x} \quad . \quad \textit{(exterior power)}$$

In particular the inertia group operates trivially on the sheaves $R^q\Psi_t$.

This statement is a consequence of Grothendieck's purity conjecture for the inclusion of each of the irreducible components of the special fibre in X. In the equicharacteristic case, and when S is excellent, this is a classical theorem [40] in étale cohomology, and then in fact an arbitrary divisor with normal crossings can be treated (then the inertia group acts through a finite group on $R^q\Psi$). In the unequal characteristic case multiplicities divisible by p create an obstacle to the proof in [33] in the general case. However, under certain finiteness hypotheses which are satisfied in all reasonable cases, R. Thomason [44] has proved the purity conjecture in general so that the theorem above may be formulated also in the case of arbitrary multiplicities.

The existence of the spectral sequence of vanishing cycles leads naturally to the following concepts. We first recall some general facts about ℓ-adic representations ([43], [7]). Let F be a non-archimedian local field, and $W_F \subset \mathrm{Gal}(\overline{F}/F)$ its Weil group. There are three kinds of "representations" in a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space that one may consider.

(i) an ℓ-adic representation $\varrho_\ell : W_F \to GL(V)$ (i.e. continuous for the ℓ-adic topology).

(ii) a pair $\varrho' = (\varrho, N)$, where $\varrho : W_F \to GL(V)$ is continuous when V is given the discrete topology, and where N is a nilpotent endomorphism of V such that

$$\varrho(w) \cdot N \cdot \varrho(w)^{-1} = \|w\| \cdot N \quad , \quad w \in W_F \ .$$

Such a pair ϱ' is called σ-semisimple if $\varrho(\sigma)$ is a semi-simple automorphism of V for one and hence all $\sigma \in W_F \setminus I$. There is a functor $\varrho' \mapsto \varrho'^{ss}$ (σ-semi-simplification).

(iii) a homomorphism $\widetilde{\varrho} : W_F \times SL_2(\overline{\mathbb{Q}}_\ell) \to GL(V)$ which is semi-simple and whose restriction to the SL_2-factor is algebraic.

Then there is a bijection between isomorphism classes of **objects of type** (i) **and** (ii), given by

$$\varrho_\ell(\sigma^n \cdot \tau) = \varrho(\sigma^n \cdot \tau) \cdot \exp(t_\ell(\tau) \cdot N) \quad , \quad \tau \in I \ .$$

where $t_\ell : I \to \mathbb{Z}_\ell$ is a fixed non-zero homomorphism, and σ a fixed geometric Frobenius (cf. [7], 4.1.9).

σ-semi-simple objects of type (ii) **and objects of type** (iii), given by integrating N into a representation of SL_2 (Jacobson-Morosov).

The nilpotent endomorphism N defines the **associated Schmid filtration** (increasing), characterized by the following two properties:

(i) $NW_k \subset W_{k-2}$

(ii) N^k induces an isomorphism

$$gr_k^W V \longrightarrow gr_{-k}^W V \quad .$$

In terms of the integrated SL_2-representation and the eigenvalues of the diagonal matrices,

$$W_k = \Sigma \text{ eigenspaces of eigenvalue } \leq k \quad .$$

In case N comes from an ℓ-adic representation, N is called the **monodromy operator** (or rather its logarithm) and the filtration $W.$ the **monodromy filtration**. In this case the Weil group W_F acts on the associated graded, $gr_{\cdot}^W V$ and hence we may speak of weights. More precisely if $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ is a homomorphism, and if (V, ϱ_ℓ) is an ℓ-adic representation of W_F it is called **ι-pure of ι-weight** $s_\iota \in \mathbb{R}$ if

$$|\iota(\alpha_\sigma)| = q^{s_\iota/2}$$

for every eigenvalue α_σ of $\varrho(\sigma)$. [This definition is independent of the choice of the lifting of the geometric Frobenius σ]. The representation is called **pure of weight** s if it is ι-pure of ι-weight s for every ι. The following conjecture is central to the subject.

2.8. CONJECTURE. (Deligne) *Suppose that the ℓ-adic representation ϱ_ℓ comes from $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, where the notations are as in the beginning of this section. Then the associated monodromy filtration $W.$ is pure of weight i, i.e. the Galoismodule $gr_j^W H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ is pure of weight $i + j$.*

If X is smooth over D, then the monodromy filtration is trivial and the conjecture is true by Deligne's solution of the Weil conjectures. The following cases are solved.

2.9. THEOREM. *a) [8]. In the equal characteristic case the conjecture is true.*
b) [33]. Assume that the relative dimension of X over D is at most 2 and that the special fibre is a reduced divisor with normal crossings. Then the conjecture is true.

c) Assume that the special fibre is a reduced divisor with normal crossings which can be deformed into a smooth projective variety in char s. *Then the conjecture is true.*

The proof of b) is an imitation of the proof by Steenbrink [41] of an analogue in Hodge theory. In fact, if ℓ is prime to the multiplicities occurring in the special fibre, and making use of the remarks after 2.7., the assumption made here that the multiplicities all be one may be dropped.

Returning again to an arbitrary ℓ-adic representation V of W_F, we call an increasing filtration $\mathcal{W}$ on V **admissible**, if it is stable under the action of W_F and such that I operates through a finite quotient group on the associated graded $gr^{\mathcal{W}}_{\cdot}(V)$. We define the **semi-simple L-function**

$$L^{ss}(T,V) = \prod_k \det(1-\sigma\cdot T; gr_k(V)^I)^{-1} \quad . \tag{2.10}$$

Determining the semi-simple L-function is equivalent to determining the **semi-simple traces** of all powers of the Frobenius, i.e.

$$Tr^{ss}(\sigma^j;V) = \sum_k Tr(\sigma^j; gr_k(V)^I) \quad . \tag{2.11}$$

It is easy to see that the semi-simple zeta function is independent of the choice of $\mathcal{W}$. Similarly we define in the situation of the beginning of this section the **semi-simple zeta function** $Z^{ss}(s, X/K)$ as the alternating product of the semi-simple L-functions associated to the various $H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$, as well as $Tr^{ss}(\sigma^{*j}; H^*(X_{\overline{\eta}}, \mathbb{Q}_\ell))$. The semi-simple zeta function is not the correct local factor for the functional equation but lends itself more easily to calculation. Indeed, let us assume that the inertia group I operates through a finite quotient group on the sheaves of vanishing cycles. Then the filtration on $H^*(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ induced by the spectral sequence of vanishing cycles is admissible. We obtain, using the fact that invariants under a finite group in a vector space over a field of characteristic zero is an exact functor,

$$\begin{aligned}
&\sum_{i=0}^{\infty}(-1)^i Tr^{ss}(\sigma^{*n} \mid H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)\\
&\quad = \sum_{q=0}^{\infty}(-1)^q \cdot \sum_{p=0}^{\infty}(-1)^p Tr(\sigma^{*n} \mid H^p(X_{\overline{s}}, R^q\Psi)^I)\\
&\quad = \sum_{q=0}^{\infty}(-1)^q \cdot \sum_{p=0}^{\infty}(-1)^p Tr(\sigma^{*n} \mid H^p(X_{\overline{s}}, R^q\Psi^I)) \quad .
\end{aligned}$$

We may apply to every summand indexed by q on the right the Lefschetz fixed point formula on the special fibre, so that at least in principle these summands are accessible to explicit computation.

Note that by 2.7. and the remarks following it the assumptions are satisfied in a large number of cases. In the case of a **reduced** divisor with normal crossings the upper index I may be dropped on the right since then I operates trivially on the sheaves of vanishing cycles.

How can one recover the **true** local factor from the semi-simple zeta function? It is for this problem that one would like to apply Deligne's conjecture on the purity of the monodromy filtration. Indeed, assume that the monodromy filtration on $V = H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ is pure of weight i, and let $\mathcal{W}$ be any admissible filtration on V. Here is how one can recover the trace $Tr(\sigma^{*j}; V^I)$ from $Tr^{ss}(\sigma^{*j}; V)$. Let

$$EV_0 = EV = \{\alpha; \alpha \text{ generalized eigenvalue of } \sigma^{*j} \text{ on } (gr^{\mathcal{W}} V)^I \} \quad \text{(counted with multiplicity)}.$$

Define inductively for $k = 0, 1, \dots$

$$\begin{aligned} EV_{k\,\min} &= \{\alpha \in EV_k;\ \text{weight } (\alpha) \text{ minimal among those for } \alpha \in EV_k \} \\ EV_{k+1} &= EV_k \setminus \{\alpha \in EV_k; \alpha = q^r \cdot \beta \text{ for some } \beta \in EV_{k\,\min} \text{ and } r \in \mathbb{Z} \} \end{aligned}$$

Then, using the avatar (iii) of the ℓ-adic representation V and in particular the explicit description of the monodromy filtration in terms of SL_2-weights mentioned earlier, we obtain using the fact that $V^I = (\operatorname{Ker} N)^I$,

$$Tr(\sigma^{*j}; V^I) = \sum_{k=0}^{\infty} \sum_{\alpha \in EV_{k\,\min}} \alpha \ .$$

We omit the proof of the following lemma.

2.12. LEMMA. *Let ϱ and ϱ' be two ℓ-adic representations with $L^{ss}(s, \varrho) = L^{ss}(s, \varrho')$. We assume that the sets of ι-weights of ϱ and ϱ' (with multiplicities) are identical. Then under either of the following hypotheses we may conclude that $L(s, \varrho) = L(s, \varrho')$.*

(i) ϱ and ϱ' have ι-pure monodromy filtrations.

(ii) We have

$$L^{ss}(s,\varrho) = L^{ss}(s,\varrho') = L(s, \tau \otimes (\alpha^{s_1} \oplus \ldots \oplus \alpha^{s_r}))$$

with τ irreducible and where the s_i are **real** *numbers.*

Here α denotes the cyclotomic character. Note that (i) is a variant of a result of Deligne, [7], 8.9.. We thus see that the purity of the monodromy filtration helps us to recover the zeta function from the semi-simple zeta function. The semi-simple version of an L-function may also be introduced on the automorphic forms side. This is done as follows.

Let F be a non-archimedian local field. Let $\{n_1, \ldots, n_r\}$ be a partition of n and $\pi_1, \ldots, \pi_r$ essentially square-integrable representations of $GL(n_i, F)$ (= square-integrable modulo center after twisting with a quasi-character). If ω_i denotes the central character of π_i we write

$$|\omega_i(z)| = |z|^{s_i} \quad , z \in F^x$$

for some real number s_i. Changing the order of the partition we suppose that $s_1 \geq \ldots \geq s_r$. The partition defines a standard parabolic subgroup P of $GL(n,F)$ and the π_i define an essentially square-integrable representation $\sigma = \otimes \pi_i$ of its Levi component. The induced representation I_σ of $GL(n,F)$ may not be irreducible but has a unique irreducible quotient which we denote by $\pi_1 \boxplus \ldots \boxplus \pi_r$. Every irreducible admissible representation of $GL(n,F)$ is of this form, and this in an essentially unique way. The collection of real numbers $(2s_1, \ldots, 2s_r)$ is called the weight of $\pi_1 \boxplus \ldots \boxplus \pi_r$. There is a standard L-function $L(s,\pi)$ that comes with π [14]. The above construction suggests introducing the **semi-simple** L-function of which $L(s,\pi)$ is a factor. We put

$$L^{ss}(s, \pi_1 \boxplus \ldots \boxplus \pi_r) = L^{ss}(s,\pi_1) \cdot \ldots \cdot L^{ss}(s,\pi_r) \quad .$$

We still have to define the semi-simple L-function of an essentially square-integrable representation. Such a representation π is the quotient of an induced representation I_σ where $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_j$ is a representation of the standard parabolic corresponding to the partition $\{m, \ldots, m\}$ of n of the following sort:

$$\sigma_{i+1} = \sigma_i \otimes |\ | \quad , \quad \sigma_1 \text{ supercuspidal} \ ; \ i = 1, \ldots, j-1 \quad .$$

We then put

$$L^{ss}(s,\pi) = \prod_{i=1}^{j} L(s,\sigma_i) \quad .$$

Conjecturally at least, the irreducible admissible representations of $GL(n,F)$ are classified by the (equivalence classes of) representations $\widetilde{\varrho} : W_F \times SL_2(\overline{\mathbb{Q}}_\ell) \to GL(V)$ of the type encountered earlier of degree n. Here it is convenient to regard via an isomorphism ι the representation of $GL(n,F)$ as taking place in a $\overline{\mathbb{Q}}_\ell$-vector space. Irreducible $\widetilde{\varrho}$ correspond to essentially square-integrable π, and the monodromy filtration of $\widetilde{\varrho}$ is ι-pure of weight equal to the weight of π. In the general case, when $\widetilde{\varrho}$ decomposes as a direct sum of irreducible representations, the weight of π is the collection of ι-weights of the Schmid filtrations of the various constituents of $\widetilde{\varrho}$.

We finally indicate the connection between the Ramanujan conjecture and Deligne's conjecture on the purity of the monodromy filtration. Let now F be a number field. An automorphic representation $\pi = \otimes \pi_v$ of $GL(n, \mathbf{A}_F)$ is called **isobaric** [25] if the weight of π_v is independent of the place v. The Ramanujan conjecture states that a cuspidal automorphic representation is isobaric with single weight $2s$ if $|\omega_\pi(z)| = |z|^s$. If this were true then all other isobaric automorphic representations would arise by an induction procedure entirely analogous to the local case. The corresponding uniqueness result in the global case is proved by Jacquet and Shalika.

To return to algebraic geometry, let X be a smooth projective variety over a number field K. What we are trying to suggest is to establish an expression of the **semi-simple** zeta function of X as a product of **semi-simple** automorphic L-functions.

$$Z^{ss}(s, X/K) = \prod_{\pi} L^{ss}(s,\pi) \quad . \tag{2.13}$$

Here the left hand side should be amenable to explicit calculation through the Lefschetz fixed point formula, whereas the right hand side should be accessible to the Selberg trace formula. Once this is accomplished we still have to pass from the semi-simple zeta function resp. L-function to the true Euler products. On the right hand side this corresponds to the passage from an automorphic form to an isobaric automorphic form, a process which is only poorly understood [25]. The corresponding problem on the left hand side is also nontrivial, even if we assume the purity of the monodromy filtration, but this is the impact of lemma 2.12 above. Summarizing, we see that **the Ramanujan conjecture** (in the formulation above) **is the "automorphic version" of the purity of the monodromy conjecture.** To conclude this section I refer to Clozel's contribution to these proceedings for further information on the automorphic side, and also to Kottwitz's article [22] where a conjectural formula of the type (2.13) is given in the case of a Shimura variety. More precisely, Kottwitz formulates a conjecture for the partial Euler products ranging over all **good** primes. It seems reasonable to extend the conjecture to **all** non-archimedian primes by simply adding the suffix "ss" to both sides of the identity.

§3 Presentation of the examples

Fix a totally real extension F of degree n over $\mathbb{Q}$ in which p stays prime. Let K be a purely imaginary quadratic extension of F in which p splits into two primes, $p = \wp \cdot \overline{\wp}$. Let D be a central division algebra of degree d^2 over K and $*$ a positive involution on D which is then necessarily of the 2^{nd} kind. We demand that $D \otimes \mathbb{Q}_p$ be a product of division algebras. We fix a free D-module of rank 1, V, and a non-degenerate alternating F-bilinear form

$$\psi : V \times V \longrightarrow F$$

satisfying

$$\psi(dx, y) = \psi(x, d^* y) \quad , \quad d \in D \quad .$$

As customary we let $\overline{\mathbb{Q}}$ stand for the field of algebraic numbers in $\mathbb{C}$. We fix a Langlands diagram $\varphi : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$,

$$\begin{tikzcd}
\mathbb{C} \arrow[dr, dash] & & \overline{\mathbb{Q}}_p \arrow[dd, dash] \\
& \overline{\mathbb{Q}} \arrow[ur, dash] \arrow[dd, dash] & \\
& & \mathbb{Q}_p \\
& \mathbb{Q} \arrow[ur, dash] &
\end{tikzcd}$$

We fix embeddings $\sigma_i : K \to \overline{\mathbb{Q}}, i = 1, \ldots, n$ such that $\sigma_1, \overline{\sigma}_1, \ldots, \sigma_n, \overline{\sigma}_n$ forms a complete set of embeddings of K and such that all n p-adic embeddings $\varphi \circ \sigma_i$ determine one and the same place $\wp$ of K.

Let $\sigma = \sigma_i \in \{\sigma_1, \dots, \sigma_n\}$. There is an isomorphism

$$D \otimes_{K,\sigma} \mathbb{C} \simeq M_d(\mathbb{C})$$

such that the involution $*$ becomes the standard involution on $M_d(\mathbb{C})$: $X \mapsto {}^t\overline{X}$. We may choose such a generator of V that the bilinear form ψ is written as

$$\psi(x, y) = \mathrm{Tr}^{\circ}_{D/F}(a^{-1} \cdot x^* \cdot y)$$

with

$$a = \begin{pmatrix} i \cdot I_r & 0 \\ 0 & -i \cdot I_{d-r} \end{pmatrix} \quad .$$

The integer $r = r_\sigma$ is independent of these normalizations. We shall suppose that the integers $r_{\sigma_i}, i = 1, \dots, n$, are all identical and satisfy

$$1 \leq r_{\sigma_i} \leq d - 1 \quad .$$

We denote by r their common value. Let $S = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be Deligne's pet-group and define for $\sigma \in \{\sigma_1, \dots, \sigma_n\}$ a homomorphism

$$h_\sigma : S \longrightarrow Gl(V \otimes_{F,\sigma} \mathbb{R})$$

by sending $\mathbb{R}^* \subset \mathbb{C}^*$ into the center in the obvious way and $i \in \mathbb{C}^*$ into the matrix $a = a_\sigma$ above. It may be verified that the bilinear form $\psi(x, h_\sigma(i)y)$ is symmetric and positive definite and that h_σ defines on $V \otimes_{F,\sigma} \mathbb{R}$ a Hodge structure of type $(-1, 0) + (0, -1)$. We have

$$\mathrm{Tr}(d \mid V \otimes_{F,\sigma} \mathbb{C}/V^{0,-1}) = r \cdot \sigma(\mathrm{Tr}^{\circ}_{D/K}(d)) + (d - r) \cdot \overline{\sigma}(Tr^{\circ}_{D/K}(d)) \quad .$$

We introduce the algebraic group over F

$$G' = \{g \in Gl_D(V) \mid \psi(gx, gy) = \mu(g) \cdot \psi(x, y) \quad , \quad \mu(g) \in F^*\} \quad .$$

Using restriction of scalars we obtain an algebraic group G over $\mathbb{Q}$. It is easy to see that $G_{\mathbb{Q}_p}$ is anisotropic modulo center. The collection of homomorphisms h_{σ_i} above may be interpreted as one single homomorphism

$$h_0 : S \longrightarrow G_{\mathbb{R}} \quad ,$$

whose $G(\mathbb{R})$-conjugacy class X is independent of all choices. The pair (G, X) satisfies the axioms defining a Shimura variety. There is a canonical model of $S(G,X)$ (in the sense of the Shimura conjecture) over the following subfield $E = E(G,X)$ of $\mathbb{C}$:

$$E = \mathbb{Q}(\sum_i r_{\sigma_i} \cdot \sigma_i(k) + (d - r_{\sigma_i}) \cdot \overline{\sigma}_i(k) \mid k \in K)$$
$$= \begin{cases} \mathbb{Q}(\sum_i \sigma_i(k) \mid k \in K) & \text{if } 2r \neq d \\ \mathbb{Q} & \text{if } 2r = d \end{cases} .$$

Thus E is either the field $\mathbb{Q}$ or an imaginary quadratic field in which p splits into two primes. The Langlands diagram distinguishes one of these at most two primes above p. If we denote it by $\wp_1$, then in all cases $E_{\wp_1} = \mathbb{Q}_p$. Let $C \subset G(\mathbf{A}_f)$ be an open compact subgroup.

3.1 THEOREM. *$S_C(G,X)$ is the set of complex points of the coarse moduli scheme of the following moduli problem.*
M1: The points with values in a $\mathbb{C}$-scheme T consist of isomorphism classes of quadruples $(A, \iota, \overline{\lambda}, \overline{\eta})$.

a) A is an abelian scheme over T up to isogeny and ι is an injection

$$\iota : D \longrightarrow \mathrm{End}(A)^\circ$$

such that

$$\mathrm{Tr}(\iota(d) \mid \mathrm{Lie}\, A) = \mathrm{Tr}(d \mid V_{\mathbb{C}}/V^{0,-1}(h_0)) \quad , \quad d \in D \ .$$

b) $\overline{\lambda}$ is a F-homogeneous polarization of A such that the Rosati-involution of $\overline{\lambda}$ induces on D via ι the given involution $*$.

c) $\overline{\eta}$ is an equivalence class modulo C of $D \otimes \mathbf{A}_f$-linear symplectic similitudes

$$(\prod_l T_l(A)) \otimes \mathbb{Q} \xrightarrow{\sim} V \otimes \mathbf{A}_f \ .$$

Implicit in the statement of this theorem is the assertion that this moduli problem does indeed possess a coarse moduli scheme. By the very definition of E the trace on the right side of the identity above lies in E. The moduli problem may be formulated and solved over $\mathrm{Spec}\, E$.

We let O_F resp. O_K stand for the rings of integers. We fix an order O_D which contains O_K and such that $O_D \otimes \mathbb{Z}_p$ is the product of the maximal orders $O_{D_\wp}$ and $O_{D_{\overline{\wp}}}$ in the central division algebras $D_\wp$ and $D_{\overline{\wp}}$ over F_p. Let $V_{\mathbb{Z}}$ be a lattice in V which is preserved under the action of O_D and such that $\psi \mid V_{\mathbb{Z}} \times V_{\mathbb{Z}}$ takes values in O_F and indeed yields a **perfect** bilinear form

$$\psi \otimes \mathbb{Z}_p : (V_{\mathbb{Z}} \otimes \mathbb{Z}_p) \times (V_{\mathbb{Z}} \otimes \mathbb{Z}_p) \longrightarrow O_F \otimes \mathbb{Z}_p \quad .$$

Let C_p be the maximal compact subgroup of $G(\mathbb{Q}_p)$,

$$C_p = \{g \in G(\mathbb{Q}_p) \mid g \cdot V_{\mathbb{Z}_p} \subset V_{\mathbb{Z}_p}\} \quad .$$

We shall consider only subgroups $C = C^p \cdot C_p$ of $G(\mathbf{A}_f)$ where C_p is the group fixed above and where C^p takes $V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$ into itself. We may formulate another moduli problem, this time over $\mathrm{Spec}(O_{E_{(p)}})$. Here $O_{E_{(p)}}$ denotes the ring extension of O_E where all elements prime to p are made invertible.

M2: The points with values in a scheme T over $\mathrm{Spec}\, O_{E_{(p)}}$ *are the isomorphism classes of quadruples $(B, \iota, \overline{\lambda}, \overline{\eta}^p)$.*

a) B is an abelian scheme over T and ι is an injection

$$\iota : O_D \longrightarrow \mathrm{End}\, B$$

such that for all geometric points $\mathrm{Spec}\, \overline{k} \to T$ *the representation of $O_D \otimes \overline{k}$ on* $\mathrm{Lie}\, B \otimes \overline{k}$ *satisfies the following condition. Let $F' \subset D$ be a field extension of K of degree d which is unramified at p and let $\mathcal{O}'$ be an order of F' containing O_K and contained in O_D and maximal at p. Then in* $\mathrm{Lie}\, B \otimes \overline{k}$ *any character of $\mathcal{O}'$ inducing σ_i on O_K $(i = 1, \ldots, n)$ occurs precisely r times and any character inducing $\overline{\sigma}_i$ on O_K occurs precisely $d - r$ times.*

b) $\overline{\lambda}$ is an F-homogeneous polarization containing in its class a polarization of degree prime to p and such that the Rosati-involution of $\overline{\lambda}$ induces through ι on D the given involution $*$.

c) $\overline{\eta}^p$ is an equivalence class modulo C^p of O_D-linear symplectic similitudes

$$\eta^p : \prod_{l \neq p} T_l(B) \simeq V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^p \quad .$$

3.2 THEOREM. *There is a coarse moduli scheme $\mathcal{M}_C$ for the moduli problem M2. It is a projective scheme over* $\mathrm{Spec}\, O_{E_{(p)}}$ *whose set of complex points is the Shimura variety $S_C(G, X)$. If C^p is sufficiently small, then $\mathcal{M}_C$ is a fine moduli scheme.*

I should point out that I have been unable to give an explicit congruence condition on C^p to turn $\mathcal{M}_C$ into a fine moduli scheme. This can be done for the fibre of $\mathcal{M}_C$ in characteristic zero.

We shall take $\mathcal{M}_C$ for our **model** of $S_C(G, X)$. The analysis of the reduction behaviour of this model will proceed along the lines of Drinfeld [12] and Zink [50]. We sketch the main points. We shall use Cartier theory. Let us briefly summarize some of the results of this theory (compare [29], [36]; but most importantly the book by T. Zink [49]). For an arbitrary $\mathbb{Z}_p$-algebra R we let $\operatorname{Cart} R$ be the Cartier-ring

$$\operatorname{Cart} R = \{ \sum_{r,s \geq 0} V^r [x_{r,s}] F^s \mid x_{r,s} \in R \text{ ; for fixed } r, x_{r,s} = 0 \text{ for almost all } s\} \quad .$$

The $\mathbb{Z}_p$-algebra structure on $\operatorname{Cart} R$ is given by functoriality and the following relations:

$$\begin{aligned} 1 &= [1] \\ F \cdot V &= p \\ F[x] &= [x^p] \cdot F \\ [x] V &= V \cdot [x^p] \\ [x] \cdot [y] &= [x \cdot y] \\ [x] + [y] &= [x + y] + \sum_{r \geq 1} V^r \cdot [z_r] \cdot F^r \end{aligned}$$

for certain elements $z_r \in R$.

The "diagonal elements" $\sum V^r [z_r] F^r$ form the subring of Witt vectors $W(R)$.

A (left) $\operatorname{Cart} R$-module M is called **reduced**, if V operates injectively on M, if M/VM is a projective R-module of finite rank and if M is V-complete, i.e. $M = \varprojlim_i M/V^i M$. There is an equivalence of categories of the category of commutative, smooth formal groups over $\operatorname{Spec} R$ and the category of reduced Cartier modules. Under this equivalence the Lie-algebra of the formal group may be identified with M/VM. The equivalence comes about as follows. Let $\hat{W}$ be the formal Witt group "scheme". Then if X is a formal group,

$$M_X = \operatorname{Hom}(\hat{W}, X)$$

is a left module under the ring $\operatorname{End} \hat{W}$. One may identify this ring with $\operatorname{Cart} R$ and then M_X becomes a reduced Cartier module. Conversely, to a reduced Cartier module M one associates the formal group

$$X = \hat{W} \otimes_{\operatorname{Cart} R} M \quad .$$

To apply this to the study of $\mathcal{M}_C$ we change notations slightly. We shall now let F stand for an unramified extension of degree n over $\mathbb{Q}_p$. Let D be a central division algebra of degree d^2 over F and with invariant s/d. We describe this algebra explicitly. Let F' be an unramified extension of degree d of F contained in D. We let $s' \bmod d$ be such that $s \cdot s' \equiv 1 \bmod d$. We can write

$$O_D = O_{F'}[\Pi] : \Pi^d = p \quad , \quad \Pi \cdot a = a^{\tau^{s'}} \cdot \Pi \quad .$$

Here $\tau \in \mathrm{Gal}(F'/F)$ denotes the Frobenius substitution. We have denoted by O_F, $O_{F'}$, O_D the rings of integers.

3.3 DEFINITION. *Fix an integer r with $0 \leq r \leq d$. A formal O_D-module of type r (r-formal O_D-module) over a $\mathbb{Z}_p$-scheme T is a formal group X (always smooth and commutative) of dimension $r \cdot n \cdot d$, together with an embedding $\iota : O_D \to \mathrm{End}\, X$ such that in the action of $O_{F'}$ on $\mathrm{Lie}\, X$, at each geometric point of T, every one of the $n \cdot d$ characters appears exactly r times.*

This definition generalizes a concept introduced by T.Zink [50], which in turn generalizes the original definition of Drinfeld [12]. We shall comment on this in the appendix to this section. The Cartier modules of $r-f.$ O_D-modules are described by the following theorem.

3.4 THEOREM. *Let R be an $O_{F'}$-algebra. The category of $r-f.$ O_D-modules over $\mathrm{Spec}\, R$ is equivalent to the category of $\mathbb{Z}/n \cdot d$-graded reduced Cartier modules $M = \bigoplus_{i \in \mathbb{Z}/nd} M_i$, equipped with an endomorphism Π of degree $n \cdot s'$ with $\Pi^d = p$ such that*

i) $\deg V = +1 \quad , \quad \deg F = -1 \quad , \quad \deg[x] = 0$ *for* $x \in R$.

ii) $M_i / V M_{i-1}$ *is a projective R-module of rank r for all $i \in \mathbb{Z}/nd$.*

This is proved as follows (comp. [50]). Let M be the Cartier module of an $r-f.$ O_D-module. Since R is an $O_{F'}$-algebra, there is a canonical homomorphism $O_{F'} \to \mathrm{Cart}\, R$. We now put

$$M_i = \{m \in M \mid \iota(a) \cdot m = a^{\sigma^{-i}} \cdot m \text{ for all } a \in O_{F'}\} \quad .$$

Here $\sigma \in \mathrm{Gal}(F'/\mathbb{Q}_p)$ denotes the Frobenius substitution. The element Π induces by functoriality an endomorphism of M. One checks easily that the conditions i) and ii) are satisfied. The converse is proved similarly.

Suppose that in the above theorem R is a perfect field L of characteristic p. Over a perfect field Cartier theory becomes Dieudonné theory. The next statement describes the $r-f.\ O_D$-modules of finite height $h(=\dim M/pM)$.

3.5 PROPOSITION. *Let X be an $r-f.\ O_D$-module of height h over L. There is an integer ν such that $h = \nu \cdot n \cdot d^2$ and*

i) All M_i are free $W(L)$-modules of rank $\nu \cdot d$.

ii) $p \cdot M_{i+ns'} \subset \Pi M_i \subset M_{i+ns'}$ and $[M_{i+ns'} : \Pi M_i] = \nu$ for all i.

One shows that ii) is satisfied with an integer ν independent of i. Since $\Pi^d = p$, this concludes the proof (comp. [50]).

We are interested in the case $\nu = 1$. To visualize the possible configurations of the M_i we use the Bruhat-Tits building. Let $\mathcal{K} = \mathcal{K}(L)$ be the fraction field of $W(L)$. We fix an embedding $\psi_0 : M_0 \to \mathcal{K}^d$. We propagate this into σ^i-linear embeddings ψ_i for $i = 0, \ldots, n-1$ via the following commutative diagram

$$\begin{array}{ccccccc}
M_0 & \xrightarrow{V} & M_1 & \xrightarrow{V} & \cdots & \xrightarrow{V} & M_{n-1} \\
{\scriptstyle\psi_0}\downarrow & & {\scriptstyle\psi_1}\downarrow & & & & {\scriptstyle\psi_{n-1}}\downarrow \\
\mathcal{K}^d & = & \mathcal{K}^d & = & \cdots & = & \mathcal{K}^d
\end{array}$$

We define σ^i-linear embeddings $\psi_{i+kn} : M_{i+kn} \to \mathcal{K}^d$ through the following commutative diagram:

$$\begin{array}{ccccccc}
\longrightarrow & M_i & \xrightarrow{\Pi} & M_{i+s'n} & \xrightarrow{\Pi} & \cdots \\
 & \downarrow{\scriptstyle\psi_i} & & \downarrow{\scriptstyle\psi_{i+s'n}} & & \\
\longrightarrow & \mathcal{K}^d & = & \mathcal{K}^d & = & \cdots
\end{array}$$

We note that $V : M_{n-1} \to M_n$ induces a σ^{-n}-linear endomorphism $U : \mathcal{K}^d \to \mathcal{K}^d$. We put $A_k^i = \operatorname{Im} \psi_{i+ks'n}$, for $i = 0, \ldots, n-1$ and any $k \in \mathbb{Z}$, and let a_k^i be the class of the lattice A_k^i in the Bruhat-Tits-building of $PGL_d(\mathcal{K})$. It only depends on the class of k modulo d. We fix a number s with $1 \leq s \leq d-1$ such that s/d is the invariant of D.

3.6. PROPOSITION. *Let X be an $r-f.\ O_D$-module of height $n \cdot d^2$ over L. We fix an embedding $\psi_0 : M_0 \to \mathcal{K}^d$. Then U and $\{a_k^i\}$ satisfy the following conditions:*

i) $\operatorname{ord}\det U = n \cdot r - s$

ii) For every $i = 0, \ldots, n-1$, $\Delta^i = \{a_j^i\}_j$ is a simplex of maximal dimension.
iii) a_k^i is a neighbour of a_k^{i+1} and the type is given by

$$[a_k^{i+1} : a_k^i] = r \bmod d \quad .$$

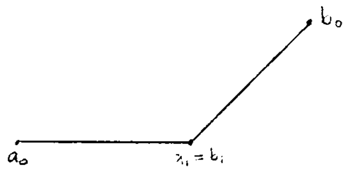
Similarly Ua_k^{n-1} and a_{k+s}^0 are neighbours and

$$[a_{k+s}^0 : Ua_k^{n-1}] = r \bmod d \quad .$$

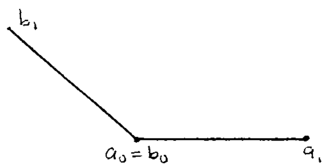
Conversely, let U and $\Delta^i (i = 0, \ldots, n-1)$ be given and number the vertices a_k^i of each simplex Δ^i in such a way that $[a_{k+1}^i : a_k^i] = 1 \bmod d$. Suppose that the conditions i)-iii) are satisfied. Then there is a $r - f$. O_D-module X of height $n \cdot d^2$ over L and an embedding $\psi_0 : M_0 \to \mathcal{K}^d$, giving rise to U, $\{a_k^i\}$. Furthermore, X is unique up to isomorphism and ψ_0 is unique up to a scalar. If U is given, then two gadgets $\{a_k^i\}$ and $\{b_k^i\}$ determine isomorphic $r - f$. O_D-modules if and only if there is a matrix $A \in GL_d(\mathcal{K})$ such that $UA = A\,U$ and $Aa_k^i = b_k^i$.

This is almost obvious (comp. [50]). To visualize the possibilities for the positions of the simplices let us consider the case $n = 1$. The first case is $d = 2$. There are 3 possibilities. We omit the upper index i. We denote Ua_j by b_j.

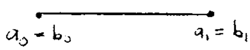
(I)



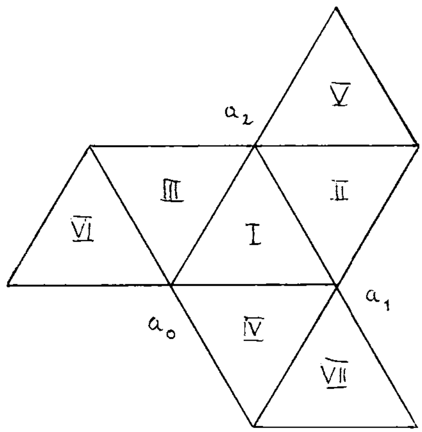
(II)



(III)



For $d = 3$ there are already 7 possibilities. We put them all into one "simultaneous" diagram.



Here we have put the simplex $\{a_0, a_1, a_2\}$ in the center and have numbered from I to VII the possible positions of the simplex $\{b_0, b_1, b_2\}$. For instance, in possibility II we have if $s = 1$, that $a_1 = b_1$, $a_2 = b_2$, $b_0 =$ north-east vertex. The cases where $r = 1$ are more or less understood; indeed, one can enumerate all possibilities in this case (see end of §5). Drinfeld [12] has studied the case where $n = r = s = 1$. All of his results are based on the observation that in this case all $r - f.$ O_D-modules are isogenous (O_D-linear isogeny, see appendix to this section). In fact, this is the deeper reason why **in this case** one can parametrize the $r - f.$ O_D-modules by the p-adic upper half-space. I have found that, essentially, in no other case is there a similar phenomenon to be observed. To determine the isogeny classes one uses the following addendum to the proposition above.

3.7 PROPOSITION. *In the notation of 3.6, the isogeny class of X is uniquely determined by $(M_0 \otimes \mathcal{K}, V^{s'n}\Pi^{-1})$, or equivalently by the σ^{-n}-linear operator U on $\mathcal{K}^d$ up to a change of basis.*

We now return to our moduli problem. Recall that we fixed a Langlands diagram so that one of the at most two prime ideals of E over p, namely $\wp_1$, is distinguished. Also, by virtue of our choice of the half system $\sigma_1, \ldots, \sigma_n$, a prime ideal $\wp$ of K over p is distinguished.

3.8 THEOREM. *Let L be a perfect field of characteristic p which is an $O_{E_{\wp_1}}$-algebra. Let B be an abelian variety over L and $\iota : O_D \to$*

End B *an injection which satisfies the trace-condition formulated in M2. Then B has p-rank zero. Let X be the formal group of B and let $X = X_\wp \times X_{\overline{\wp}}$ be the decomposition corresponding to the action of*

$$O_D \otimes \mathbb{Z}_p = O_{D_\wp} \times O_{D_{\overline{\wp}}} \quad .$$

Then $X_\wp$ is an $r-f$. $O_{D_\wp}$-module and $X_{\overline{\wp}}$ is a $(d-r)-f$. $O_{D_{\overline{\wp}}}$-module.

The last assertion follows immediately from the definitions. To prove the first assertion we have to show that equality holds in the following relation between height and dimension:

$$ht(X) \leq 2 \cdot \dim B = 2 \cdot n \cdot d^2 \quad .$$

This follows since, by 3.5, $ht\ X_\wp = \nu \cdot nd^2$ and $ht\ X_{\overline{\wp}} = \overline{\nu} \cdot nd^2$ with $\nu, \overline{\nu} > 0$.

3.9 COROLLARY. *In the notation of the previous theorem, $ht\ X_\wp = ht\ X_{\overline{\wp}} = nd^2$.*

We next determine the local structure of $\mathcal{M}_C \otimes_{O_E} O_{E_{\wp_1}}$. Let T be an $O_{E_{\wp_1}}$-scheme on which p is locally nilpotent. Let (B, ι) be an abelian scheme over T with O_D-action as above. The **Barsotti-Tate-group** X of B decomposes as $X = X_\wp \times X_{\overline{\wp}}$, this decomposition being induced by the splitting

$$O_D \otimes \mathbb{Z}_p = O_{D_\wp} \times O_{D_{\overline{\wp}}} \quad .$$

3.10 THEOREM. *Let T and $\widetilde{T}$ be $O_{E_{\wp_1}}$-schemes on which p is locally nilpotent and let $T \subset \widetilde{T}$ be a nilembedding. Let (B, ι) be an abelian scheme over T with an action of O_D as above. Let λ be a polarization which is principal at p and whose associated Rosati involution induces on D the given involution $*$.*

To lift the Barsotti-Tate-group $X_\wp$ to $\widetilde{T}$ (with its $O_{D_\wp}$-action) is equivalent to lifting the triple (B, ι, λ) to $\widetilde{T}$.

Since Cartier theory is not capable of dealing effectively with duality this is proved using crystalline theory [30]. We may clearly assume that $T \subset \widetilde{T}$ admits divided powers. Let $M = M_{T,\widetilde{T}}$ be the value of the crystal associated to B. This is an $O_D \otimes O_{\widetilde{T}}$-module which decomposes as usual

$$M = M_\wp \oplus M_{\overline{\wp}} \quad .$$

The polarization λ defines an alternating bilinear form $\Phi : M \times M \to O_{\widetilde{T}}$ which satisfies

$$\Phi(dm, m') = \Phi(m, d^*m') \quad , \quad d \in O_D \quad .$$

This implies that $M_\wp$ and $M_{\overline{\wp}}$ are isotropic w.r.t. Φ. Hence Φ determines and is determined by a linear map

$$\varphi : M_\wp \longrightarrow M^*_{\overline{\wp}}$$

which is equivariant with respect to the actions of $O_{D_\wp}$ on $M_\wp$ and of $O_{D_{\overline{\wp}}}$ on the dual module $M^*_{\overline{\wp}}$:

$$\varphi(d \cdot m) = d^* \cdot \varphi(m) \quad .$$

The hypothesis that λ be principal at p signifies that φ is an isomorphism. By the theorem of Serre and Tate, the liftings of the abelian variety B correspond precisely to liftings of the p-divisible group which in turn, by the theorem of Grothendieck and Messing, correspond precisely to liftings of the Hodge filtration. However, a lifting of the Hodge filtration corresponding to $(\widetilde{B}, \widetilde{\lambda})$ decomposes as $F = F_\wp \oplus F_{\overline{\wp}}$ and the second summand is determined by the first, via $F_{\overline{\wp}} = \mathrm{Ker}(M_{\overline{\wp}} \to \varphi(F_{\overline{\wp}})^*)$. Therefore, the liftings of (B, λ) correspond precisely to liftings of $F_\wp$, i.e. to liftings of $X_\wp$.

We thus have reduced the problem of determining the local structure of $\mathcal{M}_C$ to a question in the deformation theory of $r - f.\ O_D$-modules, which in turn may be translated into a problem in Cartier theory. This problem is far from being trivial; in fact, only the case $r = 1$ can be explicitly solved.

3.11 THEOREM. *We suppose $r = 1$. Let $x \in \mathcal{M}_C \otimes O_{E_{\wp_1}}$ be a point with values in an algebraically closed field L of characteristic p. Let X be the corresponding $r - f.\ O_D$-module and $M = \oplus_{i \in \mathbb{Z}/nd} M_i, \Pi$ be its Cartier module. Let*

$$S = \{i \in \mathbb{Z}/nd \mid \Pi : M_i \longrightarrow M_{i+s'n} \text{ factors through } VM_{i+s'n-1}\} \quad .$$

We partition S into n subsets S_k,

$$S_k = \{i \in S \mid i \equiv k \bmod n\} \quad .$$

Then every one of the S_k is non-empty. The formal completion of the local ring at x is isomorphic to the following $W(L)$-algebra:

$$\hat{O}_x \simeq W(L)[|X_i|]_{i\in\mathbb{Z}/nd}/(\prod_{i\in S_k} X_i - p)_{k\in\mathbb{Z}/n} \quad .$$

For the first statement one merely has to remark that $\Pi^d = p$ induces the zero map on the Lie algebra of X but factors as a composition of maps between L-vector spaces of dimension 1,

$$M_i/VM_{i-1} \longrightarrow M_{i+s'n}/VM_{i+s'n-1} \longrightarrow \ldots \longrightarrow M_i/VM_{i-1} \quad .$$

The second assertion is proved using the structure theorem of Cartier theory (compare [50]). **The theorem implies that the special fibre locally is the product of n singularities with reduced normal crossings.**

We have stated the result only for **geometric** points of the special fibre, but it is in fact easy to analyze the general case. We then obtain the following result.

3.12 THEOREM. *In the situation of 3.11, let $x \in \mathcal{M}_C \otimes O_{E_{\wp_1}}$ be a point with values in $\mathbb{F}_q$, with $q = p^j$. Then the set $S \subset \mathbb{Z}/nd$ is stable under the translation $T : i \mapsto i + j$. Define a twisted action $\widetilde{F}$ of the Frobenius (over $\mathbb{F}_q$) on G_m^S and on $G_m^{\mathbb{Z}/n}$ by composing the standard action with the shift operator $(x_i) \mapsto (x_{i+j})$.*

The alternating trace of the geometric Frobenius over $\mathbb{F}_q$ on the sheaves of vanishing cycles in x is equal to

$$\mathrm{Tr}_{x,q} = \frac{\mathrm{Tr}(\widetilde{F} \mid H^*(G_m^S, \mathbb{Q}_l))}{\mathrm{Tr}(\widetilde{F} \mid H^*(G_m^{\mathbb{Z}/n}, \mathbb{Q}_l))} \quad .$$

If j is divisible by $n \cdot d$, this expression equals

$$\mathrm{Tr}_{x,q} = \prod_{k\in\mathbb{Z}/n} \frac{1 - q^{|S_k|}}{1 - q} \quad .$$

The proof of this theorem uses the calculation of the sheaves of vanishing cycles in the case of a divisor with normal crossings (2.7) and an appropriate Künneth formula.

When $r > 1$ the local structure of $\mathcal{M}_C$ is not explicitly known. In fact, it seems hopeless to calculate the vanishing cycles through an explicit description by equations of the special fibre. We may however reformulate the problem.

Let F be the unramified extension of degree n of $\mathbb{Q}_p$. The algebraic group $G_{\mathbb{Q}_p}$ arises by restriction of scalars from the algebraic group G'_F. Let $\overline{\Delta}'$ be the unique simplex in the Bruhat-Tits building of $G'_{ad}(\mathcal{K}(\overline{\mathbb{F}}_p))$ invariant under $\mathrm{Gal}(\mathbb{Q}_p^{un}/F)$. Then $\overline{\Delta}'$ is a simplex of maximal dimension in the Bruhat-Tits building of PGL_d which we may represent by a chain of inclusions of O_F-lattices in F^d,

$$\ldots \subset A_0 \subset A_1 \subset \ldots A_{d-1} \subset \frac{1}{p} A_0 \subset \ldots \subset F^d \ .$$

Define as follows a functor on (Sch/O_F):
To an O_F-scheme T associate the set of isomorphism classes of commutative periodic diagrams of the following type

$$\begin{array}{ccccccccccc}
\ldots & \subset & A_i & \subset & A_{i+1} & \subset & \ldots & \subset & \frac{1}{p}A_i & \subset & \ldots \\
 & & {\scriptstyle\alpha_i}\downarrow & & {\scriptstyle\alpha_{i+1}}\downarrow & & & & {\scriptstyle p\cdot\alpha_i}\downarrow & & \\
\ldots & \longrightarrow & \mathcal{E}_i & \longrightarrow & \mathcal{E}_{i+1} & \longrightarrow & \ldots & \longrightarrow & \mathcal{E}_i & \longrightarrow & \ldots \ ,
\end{array}$$

such that the $\mathcal{E}_i$ are vector bundles of rank r over T and such that α_i are O_F-linear maps such that $\alpha_i \otimes O_T$ is surjective for every i.

It is easy to see that this functor is representable by a projective scheme $M(\overline{\Delta}', X')$ over $\mathrm{Spec}\, O_F$, which is in fact a closed subscheme of the relative cartesian product over $\mathrm{Spec}\, O_F$ of the Grassmannians of r-dimensional quotient spaces,

$$M(\overline{\Delta}', X') \subset \underset{i \in \mathbb{Z}/d}{\times} Gr_r(A_i) \ .$$

The generic fibre of $M(\overline{\Delta}', X')$ is a twisted form of the Grassmannian $Gr_r(F^d)$. Roughly speaking the twisted action of the Frobenius in $\mathrm{Gal}(\mathbb{Q}_p^{un}/F)$ on $Gr_r(F^d)(\mathbb{Q}_p^{un})$ differs from the standard action by translation by a matrix in GL_d whose dth power is central and which carries A_i into A_{i+s}. Via restriction of scalars from O_F to $\mathbb{Z}_p$ we obtain the $\mathbb{Z}_p$-scheme

$$M(\overline{\Delta}, X) = R_{O_F/\mathbb{Z}_p} M(\overline{\Delta}', X') \ .$$

This $\mathbb{Z}_p$-scheme may also be constructed directly, starting with the unique polysimplex $\overline{\Delta}$ in the Bruhat-Tits building of $G_{ad}(\mathcal{K}(\overline{\mathbb{F}}_p))$ stable under $\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$, by a construction entirely analogous to that of $M(\overline{\Delta}', X')$.

3.13 DEFINITION. *The $\mathbb{Z}_p$-scheme $M(\overline{\Delta}, X)$ is called the* **local model** *of the Shimura variety $S_C(G, X)$.*

The idea behind this definition is that the local structure of the local model contains all the information on the singularities appearing in the special fibre of $\mathcal{M}_C$. Therefore it would be very important to better understand the local model. If $r = 1$, this is quite easy.

3.14 PROPOSITION. *Assume $r = 1$. Then $M(\overline{\Delta}', X')$ is the join of $\mathbb{P}(A_i)$, $i \in \mathbb{Z}/d$ (i.e. the closure of the common generic fibre in $\underset{i}{\times} \mathbb{P}(A_i)$). The special fibre of this O_F-scheme is a reduced divisor with normal crossings. Therefore the local model $M(\overline{\Delta}, X)$ is locally a product of n singularities with reduced normal crossings.*

This proposition is related to matters which will be touched upon in the appendix to this section. It is connected with the works of Mumford [32], Mustafin [31], Kurihara [23] and Drinfeld [12].

For general r virtually nothing is known, not even whether $M(\overline{\Delta}', X')$ is flat over O_F (which would be the case if one could also identify $M(\overline{\Delta}', X')$ as a join).

Let $x \in \mathcal{M}_C \otimes O_{E_{\mathfrak{p}_1}}$ be a point with values in $\mathbb{F}_q$ with $q = p^j$. Assume first that $n = 1$ and that $d \mid j$. Let $M = \oplus M_i$ be its Cartier module. There is such an embedding $\psi_0 : M_0 \to \mathcal{K}(\mathbb{F}_q)^d$ that the lattices $A_k^0 \subset \mathcal{K}(\mathbb{F}_q)^d$ obtained by the procedure appearing before 3.6 form the simplex $\overline{\Delta}'$ in the Bruhat-Tits building of $G'_{ad}(\mathcal{K}(\overline{\mathbb{F}}_p))$. Then the residue class maps

$$\alpha_k : A_k^0 \longrightarrow A_k^0 / U A_{k-s}^0$$

form a point x_0 of the functor represented by $M(\overline{\Delta}', X')$. We therefore obtain a point

$$x_0 \in M(\overline{\Delta}, X)(\mathbb{F}_q) \quad .$$

This point is only well-determined up to the action of $G_{ad}(\mathbb{Z}_p)$. This procedure extends to the case where d doesn't divide j by first extending scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$ and then taking invariants. The procedure also extends to arbitrary n. For this one has to observe in particular that an $\mathbb{F}_q$-valued point of a scheme obtained by restriction of scalars from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ is the same as a collection of n $\mathbb{F}_q$-valued points of the original scheme. This is of course also implicit in the formulation of 3.6.. We call x_0 **an associated point in the local model.**

For the next statement we shall formulate a hypothesis. The first part of this hypothesis is quite speculative. The second part is one of the standard tenets of ℓ-adic cohomology, applied to this case.

3.15 HYPOTHESIS. *Let $x \in \mathcal{M}_C \otimes O_{E_{\mathfrak{p}_1}}(\mathbf{F}_q)$ resp. $x_0 \in M(\overline{\Delta}, X)(\mathbb{F}_q)$. Then the inertia group acts through a finite factor group on the sheaves of vanishing cycles at x resp. x_0 and the alternating trace of the Frobenius over $\mathbf{F}_q$ on the inertia invariants in the sheaves of vanishing cycles in x resp. x_0 only depends on the formal completion of $\mathcal{M}_C$ in x resp. of $M(\overline{\Delta}, X)$ in x_0.*

We note that by 3.11 resp. 3.14 together with 2.7 and the Künneth formula, this hypothesis is satisfied when $r = 1$. We can now make precise the motivating remarks made earlier.

3.16 PROPOSITION. *Let $x \in \mathcal{M}_C \otimes O_{E_{\mathfrak{p}_1}}(\mathbf{F}_q)$ and let $x_0 \in M(\overline{\Delta}, X)(\mathbb{F}_q)$ be an associated point. Assume hypothesis 3.15. Then the alternating traces of the Frobenius over $\mathbb{F}_q$ on the inertia invariants in the sheaves of vanishing cycles in x resp. x_0 of the two $\mathbb{Z}_p$-schemes $\mathcal{M}_C \otimes O_{E_{\mathfrak{p}_1}}$ and $M(\overline{\Delta}, X)$ coincide,*

$$\mathrm{Tr}_{x,q} = \mathrm{Tr}_{x_0,q} \quad .$$

Even though outside the case $r = 1$ explicit formulae for $\mathrm{Tr}_{x_0,q}$ are lacking we can calculate their sum.

3.17 PROPOSITION. *Assume hypothesis 3.15. Let $q = p^j$. Then*

$$\mathrm{Tr}(\sigma^{*j}; H^*(M(\overline{\Delta}, X)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)) = \sum_{x \in M(\overline{\Delta}, X)(\mathbb{F}_q)} \mathrm{Tr}_{x,q} \quad .$$

There is an abuse of notation on the left side of this identity since the Frobenius σ doesn't lie in the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. However, the generic fibre $M(\overline{\Delta}, X)_{\overline{\mathbb{Q}}_p}$ is an unramified form of the Grassmannian and has therefore good reduction. Therefore the cohomology of $M(\overline{\Delta}, X)_{\overline{\mathbb{Q}}_p}$ is an unramified Galois module. The assertion of the proposition is an immediate consequence of the existence of the spectral sequence of vanishing cycles.

The alternating sum on the left may be explicitly calculated. We first note that the representations of the Galois group over F on the Grassmannian and its twisted form are equivalent:

3.18 LEMMA. *Let j be divisible by n. Then*

$$\mathrm{Tr}(\sigma^{*j} \mid H^*(M(\overline{\Delta}, X)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_l)) = \mathrm{Tr}(\sigma^{*j} \mid H^*(Gr_r(F^d)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_l)) \quad .$$

Indeed, the standard action of the Frobenius and the exotic one differ by an element of the Weyl group of GL_d. A routine homotopy argument [11] using the fact that GL_d is a connected algebraic group immediately yields the assertion.

The representation of the **full** Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is obtained from $H^*(Gr_r(F^d)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_l)$ by forming the n-fold tensor product and letting σ^* act as follows

$$\sigma^*(x_1 \otimes \ldots \otimes x_n) = (\sigma^{*n} x_n, x_1, \ldots, x_{n-1}) \quad .$$

The identity appearing in 3.17 will turn out in section 5 to yield the calculation of one twisted orbital integral.

APPENDIX TO §3:
THE DRINFELD CASE

Of the formal groups encountered in the previous section Drinfeld [12] singled out the case where $n = r = 1$ and where $s/d = 1/d$. We shall call these formal groups **special formal** (s.f.) O_D-**modules**. Drinfeld's discovery was that s.f. O_D-modules may be classified through "p-adic uniformization". The purpose of this appendix is to explain briefly what is meant by this and deduce some consequences from this fact. The following lemma is critical.

A.3.1 LEMMA. *All s.f. O_D-modules over an algebraically closed field of characteristic p are isogenous to one another (O_D-linearly isogenous).*

We apply the criterion 3.7. However, U has $\mathrm{ord}\det U = 0$, as follows from 3.6 and U fixes a vertex in the Bruhat-Tits-building. Indeed, this follows from the fact (proved in the same fashion as the nonemptyness of S_i in 3.11) that there is an index i with $\Pi M_i = V M_i$. All such σ^{-1}-linear operators are equivalent.

In fact, for the previous argument we didn't need that the ground field is algebraically closed; all that was needed was that it is an $O_{F'}$-algebra where F' denotes the unramified extension of degree d of $\mathbb{Q}_p$. It is easy to see that there is a s.f. O_D-module Φ over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$. We now define a functor $\mathcal{N}$ on the category of $\mathbb{Z}_p$-schemes on which p is locally nilpotent. If S is such a scheme, we denote by $\overline{S}$ its reduction modulo p. Then $\mathcal{N}(S)$ consists of the isomorphism classes of triples (ψ, X, ρ) where:

1) ψ is a homomorphism $\psi : \overline{\mathbb{F}}_p \to O_{\overline{S}}$.
2) X is a s.f. O_D-module over S.
3) $\rho : \psi_*(\Phi) \to X \otimes_S \overline{S}$ is an O_D-quasi-isogeny of height zero.

We write $\mathcal{O} = W(\overline{F}_p)$. Drinfeld's theorem is

A.3.2 THEOREM. *The functor $\mathcal{N}$ is representable by the formal scheme $\hat{\Omega}^d \hat{\otimes} \mathcal{O}$.*

Here $\hat{\Omega} = \hat{\Omega}^d$ is Drinfeld's upper half space over $\mathbb{Z}_p$. It is a formal scheme, only locally of finite type but with all irreducible components of its special fibre proper over $\mathbb{F}_p$, which has the remarkable property that for every finite extension K of $\mathbb{Q}_p$

$$\hat{\Omega}(O_K) = \mathbb{P}^{d-1}(K) \setminus \bigcup_{H/\mathbb{Q}_p} H(K) \quad .$$

The union on the right is over all hyperplanes of $\mathbb{P}^{d-1}$ defined over $\mathbb{Q}_p$. Clearly the point set on the right cannot be the set of K-valued points of a scheme. There are several ways to construct $\hat{\Omega}$.

One way [31] is to start with projective $(d-1)$-space over $\mathbb{Z}_p$ and to blow up all rational points in its special fibre, then to blow up the inverse images of all rational lines in its special fibre and so on. One checks that the special fibre of the scheme thus obtained is a union of blown-up copies of $\mathbb{P}^{d-1}$. To each of these irreducible components one again applies the previous procedure. **Continuing indefinitely** we obtain a scheme Ω locally of finite type over $\mathbb{Z}_p$ which is regular, with general fibre $\mathbb{P}^{d-1}_{\mathbb{Q}_p}$ and with special fibre a reduced divisor with normal crossings. To make sense of this construction one shows that Ω is the union of **open subschemes** each of which is contained in one of the $\mathbb{Z}_p$-schemes after a finite number of operations above. The completion of Ω along its special fibre is $\hat{\Omega}$.

Another way of describing this construction uses the Bruhat-Tits-building of $PGL_d(\mathbb{Q}_p)$. To a vertex represented by a lattice $M \subset \mathbb{Q}_p^d$ there is a well-defined projective space $\mathbb{P}(M)$ over $\mathbb{Z}_p$ whose generic fibre is **equal** to $\mathbb{P}^{d-1}_{\mathbb{Q}_p}$. For every finite convex subcomplex Δ in the Bruhat-Tits-building we may form the join (compare 3.14)

$$\bigvee_{M \in \Delta} \mathbb{P}(M) \quad .$$

Letting Δ grow indefinitely we obtain Ω. To see the relation to the previous construction note that after the first stage in that construction we have obtained $\bigvee_{\sigma} \mathbb{P}(M)$, where σ is a simplex of maximal dimension.

The next construction, due to Deligne, is the most useful one in connection with proving A.3.2. Let σ be a simplex in the Bruhat-Tits-building of $PGL_d(\mathbb{Q}_p)$ which we may represent as an infinite **cyclic** chain of lattices in $\mathbb{Q}_p^d$:

$$\ldots \subset M_i \subset M_{i+1} \subset \ldots \subset \frac{1}{p} M_i \subset \ldots \quad .$$

Define as follows a functor F_σ on the category of $\mathbb{Z}_p$-schemes on which p is locally nilpotent. F_σ associates to S the set of isomorphism classes of commutative "periodic" diagrams:

$$\begin{array}{ccccccccccc} \subset & M_i & \subset & M_{i+1} & \subset & \ldots & \subset & \frac{1}{p} M_i & \subset \\ & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i+1}} & & & & \downarrow{\scriptstyle p\alpha_i} & \\ \longrightarrow & \mathcal{L}_i & \xrightarrow{c_i} & \mathcal{L}_{i+1} & \xrightarrow{c_{i+1}} & \ldots & \longrightarrow & \mathcal{L}_i & \longrightarrow \end{array}$$

where the $\mathcal{L}_i$ are invertible O_S-modules and the α_i are $\mathbb{Z}_p$-linear maps with $\alpha_i \otimes O_S$ surjective, such that

$(*)$ for all points s, denoting by $\kappa(s)$ the residue field of s,

$$\operatorname{Ker}(\alpha_i : M_i/pM_i \longrightarrow \mathcal{L}_i \otimes \kappa(s)) \subset M_{i-1} \quad .$$

Note that if c_i is invertible then it follows from $(*)$ that

$$\begin{aligned} &\operatorname{Ker}(\alpha_{i+1} : M_{i+1}/pM_{i+1} \longrightarrow \mathcal{L}_{i+1} \otimes \kappa(s)) \\ &\qquad \subset \operatorname{Im}(\operatorname{Ker} \alpha_i : M_i/pM_i \longrightarrow \mathcal{L}_i \otimes \kappa(s)). \end{aligned}$$

Thus if τ is a face of σ obtained by dropping the lattices M_{i_α}, then $F_\tau(S)$ is the part of $F_\sigma(S)$ with c_{i_α} invertible. We thus have for any two faces τ', τ'' of σ

$$F_{\tau'}(S) \cap F_{\tau''}(S) = F_{\tau' \cap \tau''}(S) \quad .$$

Putting $F_\emptyset = \emptyset$, this remains true if $\tau' \cap \tau'' = \emptyset$ since not all c_i can be invertible, their product being equal to p.

For every simplex σ the functor F_σ is representable by a formal scheme $\hat{\Omega}_\sigma$. For instance, if σ consists of a single vertex given by a lattice M, then

$$\hat{\Omega}_{\{M\}} \simeq \mathbb{P}(M) \setminus \bigcup_{H/\mathbb{F}_p} H \quad \text{(completion of)} \ .$$

The union on the right is over all hyperplanes **in the special fibre** defined over $\mathbf{F}_p$. Using the intersection property above we may define

$$\hat{\Omega} = \bigcup_\sigma \hat{\Omega}_\sigma \ .$$

One checks that $\hat{\Omega}$ is a **separated** formal scheme, and is indeed the same formal scheme that was defined before. Drinfeld uses the following functor description of $\hat{\Omega}$. The points of $\hat{\Omega}$ with values in a scheme S on which p is locally nilpotent consists in the isomorphism classes of **commutative periodic**[2] **diagrams**:

$$\begin{array}{ccccccccc}
\eta_i & \longrightarrow & \eta_{i+1} & \longrightarrow & \cdots & \longrightarrow & \eta_{i+d-1} & \longrightarrow & \eta_{i+d} \\
\downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i+1}} & & & & \downarrow{\scriptstyle \alpha_{i+d-1}} & & \downarrow{\scriptstyle \alpha_{i+d}} \\
\mathcal{L}_i & \xrightarrow{c_i} & \mathcal{L}_{i+1} & \xrightarrow{c_{i+1}} & \cdots & \longrightarrow & \mathcal{L}_{i+d-1} & \longrightarrow & \mathcal{L}_{i+d}
\end{array}$$

(note that here the index set ranges over **all** of $\mathbb{Z}$), where the η_i are locally constant flat $\mathbb{Z}_p$-sheaves for the Zariski topology on S and the $\mathcal{L}_i$ are invertible O_S-modules and where the α_i are $\mathbb{Z}_p$-linear homomorphisms with $\alpha_i \otimes O_S$ surjective, **together with a $\mathbb{Z}_p$-linear injection** $r : \eta_0 \to \mathbb{Q}_p^d$ such that the following conditions hold.

a) Let $S_i \subset S$ be the set of zeros of c_i. Then $\eta_i \mid S_i$ is a constant sheaf with fibre isomorphic to $\mathbb{Z}_p^d$ and, denoting by $r(\eta_i)$ the image of η_i under the rational extension of r, we have

$$[\mathbb{Z}_p^d : r(\eta_i)] = -i$$

(index of lattices in $\mathbb{Q}_p^d$).

b) for all points $s \in S$,

$$\eta_i/\eta_{i-1} \longrightarrow (\mathcal{L}_i/\mathcal{L}_{i-1}) \otimes \kappa(s)$$

[2]that is, $\eta_{i+d} = \eta_i$ and $\mathcal{L}_{i+d} = \mathcal{L}_i$, and the d-fold composition of horizontal arrows is equal to p.

is injective.

To see that this is the correct description of $\hat{\Omega}$ take a point of this functor over a scheme S such that all η_i are constant sheaves. Via r they define lattices M_i in $\mathbb{Q}_p^d$. The condition b) implies that $M_{i-1} = M_i$ if c_{i-1} is invertible. We obtain a simplex σ, not necessarily of maximal dimension, in the Bruhat-Tits building. Identifying η_i with M_i, the condition b) tells us that we are given a point of $\hat{\Omega}_\sigma$.

Let L be an algebraically closed field of characteristic p. We show how to pass from an object $(\psi, X, \rho) \in \mathcal{N}(L)$ to a point of $\hat{\Omega}\hat{\otimes}\mathcal{O}(L)$. The "second coordinate" of the point is given by ψ.

Let $\oplus M_i, \Pi$ be the Cartier module of X. Trivializing once and for all the Cartier module of Φ, the quasi-isogeny ρ defines an embedding $\psi_o : M_0 \to \mathcal{K}(L)^d$. We thus have defined the σ^{-1}-linear operator U and the lattices $A_k \subset \mathcal{K}(L)^d$ (we drop the second index $i = 0$). It is easy to see that $k \in \mathbb{Z}/d$ lies in the set S of critical indices (compare 3.11) if and only if $UA_k = A_k$ (k is a representant of k modulo d). Hence the vertices $\{a_k\}$, $k \in S$, define a simplex σ in the Bruhat-Tits building of $PGL_d(\mathbb{Q}_p)$ (invariants under U). The homomorphisms

$$\alpha_k : A_k \longrightarrow \mathcal{L}_k := A_k/A_{k-1} \quad , \quad k \in S$$

define (by restriction to the invariants under U in A_k) what is needed to have a point of $\hat{\Omega}_\sigma$ with values in L. It is straightforward to see that this construction in fact defines a **bijection**

$$\mathcal{N}(L) \longrightarrow \hat{\Omega}\hat{\otimes}\mathcal{O}(L) \ .$$

To see, however, that the functor $\mathcal{N}$ is representable and that there is an isomorphism as claimed in A.3.2 is still a long way off. Indeed, Drinfeld's proof of these assertions which takes him two pages is a technical masterpiece. The isomorphism that he constructs is equivariant w.r.t. the action $GL_d(\mathbb{Q}_p)$ which acts in a natural way on the two functors in question (compare [12]). Here $Gl_d(\mathbb{Q}_p)$ acts on $\hat{\Omega}\hat{\otimes}\mathcal{O}$ through its obvious action on the first factor and through

$$a \longmapsto \sigma^{-\nu}(a) \quad , \quad \nu = \operatorname{ord} \det g$$

on the second factor.

We now return to the global case, i.e. to the Shimura varieties considered in §3. We suppose that $n = r = 1$ and that the invariant of D at the distinguished prime $\wp$ of K is $1/d$.

A.3.3 THEOREM. *There is an inner twisting G_- of G over $\mathbb{Q}$ together with isomorphisms*

$$G_-(\mathbf{A}_f^p) \simeq G(\mathbf{A}_f^p)$$
$$G_{-ad}(\mathbb{Q}_p) \simeq PGL_d(\mathbb{Q}_p)$$

such that $G_{-ad}(\mathbb{R})$ is compact. We use the first isomorphism to define $C_-^p \subset G_-(\mathbf{A}_f^p)$ as the inverse image of C^p. There is an isomorphism (compatible with changes of C^p)

$$\mathcal{M}_C \otimes O_{E_{\wp_1}} \simeq G_-(\mathbb{Q})\backslash \left[G_-(\mathbf{A}_f^p)/C_-^p \times \hat{\Omega}_{\mathcal{O}}\right] \quad .$$

Here we have identified $E_{\wp_1}$ with $\mathbb{Q}_p$. The action of $G_-(\mathbb{Q})$ on $\hat{\Omega}\hat{\otimes}\mathcal{O}$ is through $G_-(\mathbb{Q}_p)$: On the first factor $G_-(\mathbb{Q}_p)$ acts through $G_{-ad}(\mathbb{Q}_p) = PGL_d(\mathbb{Q}_p)$. On the second factor $g \in G_-(\mathbb{Q}_p)$ acts through $a \mapsto \sigma^{-\nu}(a)$ with $\nu = \mathrm{ord}_\wp \kappa(g)$ where $\kappa : G/G_{\mathrm{der}} \xrightarrow{\sim} K^\times$ is the canonical isomorphism. The isomorphism above may be interpreted as an isomorphism between the formal scheme on the right and the completion along its special fibre of the scheme on the left.

This theorem is a consequence of Drinfeld's theorem A.3.2 and 3.10. The case $d = 2$ is (essentially) the discovery of Cherednik [5] (compare section 1), and the proof above mimics Drinfeld's proof in that case.

Note that the scheme on the right may be written as a finite disjoint sum of unramified forms of quotients which may be algebraicized

$$\Gamma\backslash\hat{\Omega} \quad .$$

Here $\Gamma \subset G_-(\mathbb{Q}_p)$ is a discrete cocompact group. For such quotients Mustafin [31] has proved, using the vanishing theorem for the cohomology of discrete subgroups of p-adic groups, due to Casselman and Garland, that for the cohomology of the structure sheaf

$$H^i((\Gamma\backslash\hat{\Omega})_{\mathbb{Q}_p}, \mathcal{O}) = 0 \quad \text{for} \quad 0 < i < d-1 \quad .$$

We thus obtain the following consequence of A.3.3.

A.3.4 COROLLARY. *In the situation of A.3.3,*

$$H^{i,0}(S_C(G,X)(\mathbb{C})) = 0 \quad \text{for} \quad 0 < i < d-1 \quad .$$

On the left there appear the spaces of harmonic forms of type $(i,0)$ on the compact Kähler variety $S_C(G,X)(\mathbb{C})$. We thus have obtained a purely transcendental vanishing theorem by means of an investigation of the p-adic structure of the Shimura variety. Since cohomology should not depend in an essential way on such subtle information as the invariant of D we are led to the following conjecture.

A.3.5 CONJECTURE. *Assume $r = 1$, but let n be arbitrary. Then*

$$H^{i,0}(S_C(G,X)(\mathbb{C})) = 0 \quad \text{for } 0 < i < n \cdot (d-1) = \dim S_C(G,X) \ .$$

Here, as a single exception, C stands for an **arbitrary** open compact subgroup in $G(\mathbf{A}_f)$ (not necessarily maximal compact in p). Except for $i = 1$ this conjecture says nothing about the vanishing of $H^i(S_C(\mathbb{C}), \mathbb{Q})$. This conjecture has been proved by Langlands and Rogawski in the case that $d = 3$ and $i = 1$. I don't know what to expect when $r > 1$. To understand what this conjecture says one should, as a first step, try to decide what distinguishes a unitary representation of $G(\mathbb{R})$ having continuous cohomology of type $(i,0)$ from all others. The unitary representations with cohomology are all known [48], [4], and it may be profitable to contemplate their list.

In conclusion I mention that Drinfeld formulates and proves his theorem for an arbitrary local field of characteristic 0, not just $\mathbb{Q}_p$. He advances the very deep conjecture that the cohomology of certain sheaves on $\hat{\Omega}$ should yield all discrete series representations of GL_d. Using theorem A.3.3 and the naive conjecture [26] on the zeta function of a Shimura variety one could even write down a precise version of the resulting "Langlands correspondence" between representations of $GL_d(\mathbb{Q}_p), D^\times$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. More about this may be found in Carayol's contribution to these proceedings [**6**].

Finally I should mention that the formal scheme $\hat{\Omega}$ was first constructed (in the case $d = 2$) by Mumford [32] and that it was he who taught us that formal schemes are better than rigid-analytic spaces (compare also [35]). Yet another construction of $\hat{\Omega}$, using toroidal concepts, is due to Kurihara [23].

§4 DESCRIPTION OF ISOGENY CLASSES AND THEIR CONTRIBUTION TO THE SEMI-SIMPLE ZETA FUNCTION

We form the logarithm of the semi-simple zeta function at the prime

$\wp_1$ of E.

$$\log Z^{ss}_{\wp_1}(s, S_C) = \sum_{j=1}^{\infty} N_j \cdot \frac{p^{-js}}{j} \tag{4.1}$$

According to §2 and, assuming hypothesis 3.15, as we shall always do, we may use the Lefschetz fixed point formula to write

$$N_j = \sum_{x \in \mathcal{M}_C \otimes O_{E_{\wp_1}}(\mathbb{F}_{p^j})} Tr_{x,p^j} \tag{4.2}$$

We shall group the points appearing in the index set into "isogeny classes over $\overline{\mathbb{F}}_p$" and describe the set of points in a given isogeny class with the action of the Frobenius on them, together with the sheaves of vanishing cycles. In fact, there is a conjectural description of these data in purely group theoretical terms [28]. Except in special cases [34] this conjecture is completely open. It turns out, however, that it is not really necessary to know the truth of this conjecture to proceed further. Indeed, modulo a conjecture stated in the next section, it turns out that almost all isogeny classes contribute zero to the sum in (4.2) and that the contribution of the few remaining isogeny classes can be written in a way which may be compared with the Selberg trace formula. This will be done in section 5. In the present section we shall first give a description of the set of points in an isogeny class and then express its contribution to (4.2).

We shall call two points of $\mathcal{M}_C$ over $\overline{\mathbf{F}}_p$, represented by quadruples $(B, \iota, \overline{\lambda}, \overline{\eta}^p)$ and $(B', \iota', \overline{\lambda}', \overline{\eta}^{p'})$ **isogenous** if there is an isogeny between B and B' which respects the O_D-actions and the polarization classes. There is no condition on the level structures. We fix an isogeny class $\mathcal{J}$ and introduce the group of self-isogenies

$$I_{\mathcal{J}}(\mathbb{Q}) = I(\mathbb{Q}) = \mathrm{End}(B_0, \iota_0, \lambda_0)^{\times}_{\mathbb{Q}}$$

where $(B_0, \iota_0, \overline{\lambda}_0, \overline{\eta}^p)$ is a point in $\mathcal{J}$. This group only depends on $\mathcal{J}$ and not on the choice of this base point. It is the group of rational points of an algebraic group over $\mathbb{Q}$.

4.3. THEOREM. *There is a bijection between the set of points in the isogeny class $\mathcal{J}$ of $\mathcal{M}_C(\overline{\mathbb{F}}_p)$ and the set*

$$I(\mathbb{Q}) \backslash \left[G(\mathbf{A}^p_f)/C^p \times X_p \right] .$$

The action of the Frobenius in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ *on* $\mathcal{J}$ *corresponds to the operation* $M \mapsto V^{-1}M$ *on the* X_p*-component of this coset space.*

This statement requires some explanations. The existence of the relative Frobenius

$$(B, \iota, \overline{\lambda}, \overline{\eta}^p) \longrightarrow (B, \iota, \overline{\lambda}, \overline{\eta}^p)^{(p)}$$

shows that the Frobenius indeed operates on $\mathcal{J}$. Fix a level structure $\eta_0^p \in \overline{\eta}_0^p$. Any element $g \in G(\mathbf{A}_f^p)$ defines a quasi-isogeny of degree prime to p, $\alpha : (B_0, \iota_0, \overline{\lambda}_0) \to (B, \iota, \overline{\lambda})$, such that α induces an identification of $\prod_{\ell \neq p} T_\ell(B)$ with $g(V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^p)$, via η_0^p. The resulting point $(B, \iota, \overline{\lambda}, \overline{\eta}^p)$ in $\mathcal{J}$ is independent of the choice of η_0^p and only depends on the class of g in $G(\mathbf{A}_f^p)/C^p$. The set X_p is the set of quasi-isogenies with source $(B_0, \iota_0, \overline{\lambda}_0)$ of degree a power of p, or what amounts to the same by Dieudonné theory, the set of Dieudonné submodules M of the rational Dieudonné module $M_0 \otimes \mathcal{K}(\overline{\mathbb{F}}_p)$ stable under the operation of O_D which induces a representation of O_D on M/VM satisfying the familiar trace condition appearing before 3.2. and such that the dual module with respect to any $\lambda_0 \in \overline{\lambda}_0$ satisfies

$$M^* = c \cdot M \quad , \quad c \in F \otimes W(\overline{\mathbf{F}}_p) \quad .$$

Any self-isogeny in $I(\mathbb{Q})$ induces the identity of $\mathcal{J}$ which explains why we divide out by its action. The assertion of the theorem is that conversely if two isogenies α and α' given by elements g resp. g' and M resp. M' define the same element of $\mathcal{J}$ they differ by an element of $I(\mathbb{Q})$. This is not difficult to show (e.g. [28]). The assertion that the Frobenius induces the indicated operation on the set of Dieudonné modules is a consequence of Dieudonné theory.

We now wish to give an expression for the contribution of the isogeny class $\mathcal{J}$ to (4.2). We note that the procedure of the previous section associates to a point $x \in X_p$ a point $x_0 \in M(\overline{\Delta}, X)(\overline{\mathbb{F}}_p)$. In fact, if an element of $\mathcal{J}$ with X_p-component x yields a point of $\mathcal{M}_C$ over $\mathbb{F}_{p^j}$, then the associated point on the local model of the Shimura variety is also defined over $\mathbf{F}_{p^j}$ and we have equality of the alternating traces of the Frobenius on the inertia invariants on the sheaves of vanishing cycles,

$$Tr_{x,p^j} = Tr_{x_0,p^j} \quad .$$

One further piece of notation will be needed. We let

$$J = J(\mathbb{Q}_p) = \mathrm{End}(M_0 \otimes \mathcal{K}(\overline{\mathbf{F}}_p), \iota_0, \overline{\lambda}_0)^{\times} \quad , \tag{4.4.}$$

with the obvious inclusion $I(\mathbb{Q}) \to J(\mathbb{Q}_p)$. It should be pointed out that J is the group of rational points of an algebraic group defined over $\mathbb{Q}_p$ and that it coincides, up to an algebraic torus, with the group appearing in the statement of 3.6., the centralizer of the σ^{-n}-linear operator U in $GL_d(\mathcal{K}(\overline{\mathbf{F}}_p))$. Clearly, $J(\mathbb{Q}_p)$ operates on X_p.

After these preliminary remarks we are now ready to imitate the procedure that Kottwitz [18] used in the case of good reduction.

Suppose that

$$(g, x) \in G(\mathbf{A}_f^p) \times X_p$$

represents a point of $\mathcal{J}$ lying in $\mathcal{M}_C \otimes O_{E_{\wp_1}}(\mathbf{F}_{p^j})$. There is thus an $h \in I(\mathbb{Q})$ and a $k \in C^p$ such that

$$g = h \cdot g \cdot k \quad , \quad (V^{-1})^j x = hx$$

For small enough C^p [26, p. 1171] the conjugacy class $\{h\}$ of h in $Z(\mathbb{Q}) \cap C \backslash I(\mathbb{Q})$ is well-determined by the point in $\mathcal{M}_C \otimes O_{E_{\wp_1}}$. Here $Z(\mathbb{Q})$ is the center of $G(\mathbb{Q})$ which is contained in $I(\mathbb{Q})$. We gather the contributions of all fixed points yielding a given $\{h\}$. Let $ch = ch^{(j)}$ be the characteristic function of the set

$$\{(g, x) \in J(\mathbb{Q}_p) \times X_p \mid g \cdot x = (V^{-1})^j x\} \quad .$$

Define for $h \in J = J(\mathbb{Q}_p)$:

$$\varphi^{(j)}(h) = \sum_{x \in X_p \bmod J} \frac{1}{\mathrm{vol}\ J_x} \cdot Tr_{x,p^j} \cdot \int_{J_h \backslash J} ch^{(j)}(h, gx) \frac{dg}{dg_h} \quad .$$

Here J_x is the stabilizer of x in J. The expression depends on the Haar measure dg_h on the centralizer J_h, but not on the Haar measure on J.

4.5 LEMMA. *Let $f^p = \frac{1}{\mathrm{vol}\ C^p} \cdot \mathrm{char}\ C^p$ and $O_g(f^p)$ be the orbital integral over $g \in G(\mathbf{A}_f^p)$. Let $Z_C = Z(\mathbf{A}_f) \cap C$. The contribution of h to the contribution of $\mathcal{J}$ to (4.2) is equal to*

$$\frac{\mathrm{vol}(Z_C \cdot I_h(\mathbb{Q}) \backslash I_h(\mathbf{A}_f))}{\mathrm{vol}\ Z_C} \cdot O_h(f^p) \cdot \varphi^{(j)}(h) \quad .$$

For this we have to take C^p sufficiently small. This lemma is stated in a slightly different form in [26, p.1172].

To appeal to the result stated there, we have to convince ourselves that if the expression above is non-zero, then

$$(i) \qquad I_h(\mathbf{A}_f^p) = G_h(\mathbf{A}_f^p)$$

$$(ii) \qquad I_h(\mathbb{Q}_p) = J(\mathbb{Q}_p) \ .$$

However, the assertion (i) follows from Tate's theorem [42] characterizing $I(\mathbf{A}_f^p)$ inside $G(\mathbf{A}_f^p)$ as the centralizer of the Frobenius endomorphism and the fact that the centralizer of h in $G(\mathbf{A}_f^p)$ and the common centralizer of h and the Frobenius endomorphism in $G(\mathbf{A}_f^p)$ coincide, as follows from the second defining equation for h. The assertion (ii) is proved in a similar way using the analogue of Tate's theorem for Dieudonné modules.

In the case of good reduction Kottwitz [18] has shown how to express $\varphi^{(j)}(h)$ as a twisted orbital integral. His method carries over to the case under consideration.

4.6 LEMMA.

$$\varphi^{(j)} = \sum_{x \in X_p \bmod J_x} \frac{1}{\mathrm{vol}_{dg_h}(J_h \cap J_x)} \cdot Tr_{x,p^j} \cdot ch^{(j)}(h,x) \ .$$

Indeed,

$$\frac{1}{\mathrm{vol}\ J_x} \cdot \int_{J_h \backslash J} ch(h, gx) \frac{dg}{dg_h} = \sum_{y \in J_x \backslash Jx} ch(h,y) \cdot \frac{1}{\mathrm{vol}_{dg_h}(J_h \cap J_x)} \ .$$

Multiplying both sides with Tr_{x,p^j} and summing over $X \bmod J$ we obtain the result.

The next lemma is critical. Let $\mathcal{K} = \mathcal{K}(\overline{\mathbb{F}}_p)$ and denote by $C_p(\mathcal{K}) \subset G(\mathcal{K})$ the corresponding Iwahori subgroup.

4.7. LEMMA. *Let $g \in C_p(\mathcal{K})$. There is an $h \in C_p(\mathcal{K})$ with*

$$g = h \cdot \sigma^j(h^{-1}) \ .$$

This follows from Lang's theorem since $C_p(\mathcal{K})$ may be interpreted as the set of **integral** points of a smooth group scheme **with connected fibres** over $\mathrm{Spec}\, W(\overline{\mathbb{F}}_p)$.

We denote by $\mathbb{Q}_{p^j}$ the unramified subfield of $\overline{\mathbb{Q}}_p$ of degree j over $\mathbb{Q}_p$. We shall define a function in the Hecke-algebra with respect to the Iwahori subgroup,

$$\phi \in \mathcal{H}(G(\mathbb{Q}_{p^j}), C_p(\mathbb{Q}_{p^j})) \ .$$

Its support is to be contained in the set of elements

$$\{g \mid |\chi(g)|_p = p^{-<\chi,\mu>} \text{ for all characters } \chi \text{ of } G\} \ . \tag{4.8.}$$

Here $\mu = \mu_{h_0}$ denotes the cocharacter of $G_{\mathbb{C}}$ defined by the first component of h_0. Note that if G/G_{der} is identified with (the algebraic group defined by) $K_\wp^\times \times K_{\overline{\wp}}^\times$, this set consists of those elements in $G(\mathbb{Q}_{p^j})$ whose valuation at $\wp$ equals 1 and at $\overline{\wp}$ equals 0. If g lies in the set (4.8) let $\overline{x} = g \cdot \overline{x}_0$ where $\overline{x}_0 = \overline{\Delta}$ is similarly to section 3 the unique polysimplex in the Bruhat-Tits building of $G_{ad}(\mathbb{Q}_{p^j})$ stabilized by the Galois group $\mathrm{Gal}(\mathbb{Q}_{p^j}/\mathbb{Q}_p)$. If then $\overline{x}$ and $\overline{x}_0$ are in such relative position that $\overline{x}$ defines a point $\overline{x}(g)$ of the local model $M(\overline{\Delta}, X)$ (then necessarily with values in $\mathbf{F}_{p^j}$), we put

$$\phi(g) = Tr_{\overline{x}(g),p^j} \ . \tag{4.9.}$$

Otherwise we put $\phi(g) = 0$.

We illustrate this in the examples appearing after 3.6. In particular let $n = 1$. Let us first consider the case $d = 2$. In the notations introduced in loc. cit., let $\overline{x}_0 = \overline{\Delta} = (a_0, a_1)$ and $\overline{x} = g \cdot \overline{x}_0 = (b_0, b_1)$. Then only if $\overline{x}$ is in position (I), (II), or (III) can we have $\phi(g) \neq 0$. If however g is in the set (4.8) and $\overline{x}$ is in position (I), (II), or (III) then we have the following table of values for ϕ.

j	I	II	III
even	1	1	$1 - p^j$
odd	/	/	$1 + p^j$

Here we have incorporated the fact that for odd j the positions (I) and (II) don't occur. Similar remarks apply to the case $d = 3$. Here the value of ϕ is equal to $(1 - p^j)^{m-1}$ where m is the number of vertices common to $\overline{x}$ and $\overline{x}_0$ (supposed to be $\neq 0$; otherwise the value is zero) provided that j is divisible by 3. If j is prime to 3, only position (III) can occur and then the value is equal to $(1 + p^j + p^{2j})$.

To give a group theoretical expression for $\varphi^{(j)}(h)$ we fix an isomorphism of $M_0 \otimes \mathcal{K}(\overline{\mathbb{F}}_p)$ with $V \otimes \mathcal{K}(\overline{\mathbb{F}}_p)$ which respects the actions of D and the symplectic forms and such that the Dieudonné modules of elements of $\mathcal{J}$ lie in the orbit under $G(\mathcal{K})$ of the lattice $V_{\mathbb{Z}} \otimes W(\overline{\mathbb{F}}_p)$. This allows us to identify X_p with a subset of $G(\mathcal{K})/C_p(\mathcal{K})$. We write $x = g \cdot x_0$ where x_0 is the base "point". The image $\overline{x}_0$ of x_0 in the Bruhat-Tits building of G_{ad} may be interpreted as a polysimplex $\overline{\Delta}$ as above. Using the fixed isomorphism we write the σ-linear operator V^{-1} in the form

$$V^{-1} = b \times \sigma \in G(\mathcal{K}) \times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \quad .$$

4.10. LEMMA. *Suppose that $h = N_j b = b \cdot \sigma(b) \dots \sigma^{j-1}(b)$, where $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^j}/\mathbb{Q}_p)$ denotes the Frobenius substitution. Then $b \in G(\mathbb{Q}_{p^j})$ and*

$$J_h = J \cap G(\mathbb{Q}_{p^j}) \quad .$$

Furthermore

$$\varphi^{(j)}(h) = \frac{1}{\mathrm{vol}\; C_p(\mathbb{Q}_{p^j})} \int_{J_h \backslash G(\mathbb{Q}_{p^j})} \phi(g^{-1} b \cdot \sigma(g)) \frac{dg}{dg_h}$$

PROOF: It is best to do these calculations in the semi-direct product $G(\mathcal{K}) \times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$. Then $(b \cdot \sigma)^j = N_j b \cdot \sigma^j$. By hypothesis

$$h^{-1} \cdot (b \cdot \sigma)^j = \sigma^j \quad .$$

But $b \cdot \sigma$ commutes with h, hence also with σ^j. So $b \in G(\mathbb{Q}_{p^j})$. Similarly, an element of $J(\mathbb{Q}_p)$ commutes with h precisely if it commutes with σ^j, that is, if it lies in $G(\mathbb{Q}_{p^j})$. A point $x \in X_p$ can yield a non-zero contribution to the sum in 4.6 only if

$$h \cdot x = (b \cdot \sigma)^j \cdot x \quad ,$$

i.e. by the first part of the proof, only if

$$\sigma^j \cdot x = x \quad .$$

Because of 4.7. this means $x \in G(\mathbb{Q}_{p^j})/C_p(\mathbb{Q}_{p^j})$. If now $x = g \cdot x_0$ then $x \in X_p$ precisely if g lies in the set (4.8.) and if

$$g \cdot \overline{x}_0 \quad \text{and} \quad b \cdot \sigma \cdot g \cdot \overline{x}_0$$

are in the correct relative position (cf. previous section) and the coefficient Tr_{x,p^j} then equals $\phi(g^{-1} \cdot b \cdot \sigma(g))$. Since

$$\operatorname{vol}(J_h \backslash J_h \cdot g \cdot C_p(\mathbb{Q}_{p^j})) = \operatorname{vol} C_p(\mathbb{Q}_{p^j}) \cdot \frac{1}{\operatorname{vol}(J_h \cap gC_p(\mathbb{Q}_{p^j})g^{-1})} \quad ,$$

we see that

$$\begin{aligned}
\varphi^{(j)}(h) &= \sum_{x \in X_p \bmod J} \frac{1}{\operatorname{vol}(J_h \cap J_x)} \cdot Tr_{x,p^j} \cdot ch^{(j)}(h,x) \\
&= \frac{1}{\operatorname{vol} C_p(\mathbb{Q}_{p^j})} \cdot \\
&\quad \cdot \sum_{g \in J_h \backslash G(\mathbb{Q}_{p^j})/C_p(\mathbb{Q}_{p^j})} \phi(g^{-1} b\sigma(g)) \cdot \operatorname{vol}(J_h \backslash J_h \cdot g \cdot C_p(\mathbb{Q}_{p^j})) \\
&= \frac{1}{\operatorname{vol} C_p(\mathbb{Q}_{p^j})} \cdot \int_{J_h \backslash G(\mathbb{Q}_{p^j})} \phi(g^{-1} b\sigma(g)) \frac{dg}{dg_h} \quad . \qquad \text{Q.E.D.}
\end{aligned}$$

We now obtain the following theorem.

4.11 THEOREM. *If $h^{-1} \cdot (V^{-1})^j$ has no fixed point in X_p, then $\varphi^{(j)}(h) = 0$. Otherwise, there is an element $c \in G(\mathcal{K})$ such that if we put*

$$\delta = cb\sigma(c)^{-1}$$

we have $\delta \in G(\mathbb{Q}_{p^j})$ and $N_j \delta = chc^{-1}$. Let

$$G_\delta^\sigma(\mathbb{Q}_p) = \{g \in G(\mathbb{Q}_{p^j}) \mid g \cdot \delta = \delta \cdot \sigma(g)\} \quad .$$

Then $G_\delta^\sigma(\mathbb{Q}_p) = cJc^{-1}$ and

$$\varphi^{(j)}(h) = \frac{1}{\operatorname{vol} C_p(\mathbb{Q}_{p^j})} \cdot \int_{G_\delta^\sigma(\mathbb{Q}_p) \backslash G(\mathbb{Q}_{p^j})} \phi(g^{-1} b\sigma(g)) \frac{dg}{dg_h^c} \quad .$$

PROOF: Suppose that $d^{-1}x_0$ is the fixed point,

$$d(h^{-1} \cdot (b \cdot \sigma)^j) d^{-1} \cdot x_0 = x_0 \quad ,$$

i.e. since σ^j fixes x_0,

$$d \cdot (h^{-1} \cdot N_j b) \cdot \sigma^j(d^{-1}) \in C_p(\mathbb{Q}_{p^j}) \quad .$$

By Lemma 4.7 we find a $k \in C_p(\mathcal{K})$ with

$$k \cdot \sigma^j(k^{-1}) = d \cdot (h^{-1} \cdot N_j b) \cdot \sigma^j(d^{-1}) \quad .$$

Setting $c = k^{-1} \cdot d$ we obtain

$$c^{-1} \cdot \sigma^j(c) = h^{-1} \cdot N_j b \quad .$$

In the semi-direct product $G(\mathcal{K}) \times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ this says

$$c \cdot h^{-1} \cdot (b \cdot \sigma)^j \cdot c^{-1} = \sigma^j \quad .$$

Put $\delta = cb\sigma(c)^{-1}$. It follows that

$$N_j\delta = N_j(cb\sigma(c)^{-1}) = c(N_j b)\sigma^j(c^{-1}) = chc^{-1} \quad .$$

Then lemma 4.10, applied to δ instead of b shows that $\delta \in G(\mathbb{Q}_{p^j})$. The final formula follows from 4.10.
We note that the σ-conjugacy class of $\delta \in G(\mathbb{Q}_{p^j})$ is uniquely determined by h. Indeed, if

$$cb\sigma c^{-1} = \delta\sigma \quad \text{and} \quad ucb\sigma c^{-1}u^{-1} = \delta'\sigma \quad ,$$

and

$$N_j\delta = chc^{-1} \quad \text{and} \quad N_j\delta' = uchc^{-1}u^{-1} \quad ,$$

then $\sigma^j(u) = u$, i.e., $u \in G(\mathbb{Q}_{p^j})$. Abbreviating the twisted orbital integral occurring in the expression in 4.11 into $TO_\delta(\phi)$, we may therefore write the contribution of an element $h \in I(\mathbb{Q})$, provided it does not vanish, as

$$(4.12.) \qquad \frac{\mathrm{vol}(Z_C \cdot I_h(\mathbb{Q})\backslash I_h(\mathbf{A}_f))}{\mathrm{vol}\, Z_C \cdot \mathrm{vol}\, C_p(\mathbb{Q}_{p^j})} \cdot O_h(f^p) \cdot TO_\delta(\phi) \quad .$$

To conclude this section we single out certain isogeny classes by looking at the isogeny class of the corresponding Dieudonné module or, what comes to the same, the isogeny class of the corresponding $r-f.\ O_{D_\wp}$-module $X_\wp$, i.e., equivalently, the σ^{-n}-linear operator U on $\mathcal{K}^d$, up to a change of basis (cf. 3.7.). Namely, we call the isogeny class of $X_\wp$ **isoclinic** or **basic** ([19]) if all slopes of the Newton polygon of U are identical (and then equal to $(n \cdot r - s)/d$). We call an isogeny class $\mathcal{J}$ isoclinic or basic if its "p-component" is so.

4.13. COROLLARY. *Let $\mathcal{J}$ be an isogeny class and assume that there exists an $h \in I(\mathbb{Q})$ with $\varphi^{(j)}(h) \neq 0$. Let $\delta \in G(\mathbb{Q}_{p^j})$ be the twisted conjugacy class given by the previous theorem. Consider its norm $N_j\delta$ as a conjugacy class of $G(\mathcal{K})$ stable under the Galois group. If there exists an element of $G(\mathbb{Q}_p)$ in this conjugacy class then $\mathcal{J}$ is isoclinic.*

PROOF: We have to show that the element of $B(G)$ (= set of σ-conjugacy classes in $G(\mathcal{K})$) given by δ is basic. Let

$$N_j\delta = c\varepsilon c^{-1} \quad , \quad c \in G(\mathcal{K})$$

with $\varepsilon \in G(\mathbb{Q}_p)$. Then ([28], p. 183) the element $b = c^{-1}\delta\sigma(c)$ centralizes ε and defines an element of $B(G_\varepsilon)$ which is basic (loc. cit., 5.15). Therefore [19], 5.3., this element lies in the image of the map $B(T) \to B(G_\varepsilon)$, where $T \subset G_\varepsilon$ is an elliptic torus over $\mathbb{Q}_p$. Since $G(\mathbb{Q}_p)$ is anisotropic modulo center, the torus T is elliptic in G as well. Therefore, by [19], 5.3. again, b defines an element of $B(G)$ which is basic.

§5 COMPARISON WITH THE SELBERG TRACE FORMULA

To formulate the result we are aiming for, we introduce the local L-group of G "at $\wp_1$" (where we identify as before $E_{\wp_1}$ with $\mathbb{Q}_p$),

$${}^LG^0 = \{(x_i, y_i) \in \prod_1^n GL_d(\mathbb{C}) \times GL_d(\mathbb{C}) \quad ; \quad x_i = \mu_i y_i \quad , \quad \mu_i \in \mathbb{C}^\times\} \quad .$$

The product is over all embeddings $\sigma_i : F_p \to \overline{\mathbb{Q}}_p$. Then

$${}^LG = {}^LG^0 \times \mathrm{Gal}(F_p/\mathbb{Q}_p) \quad .$$

The action of $\mathrm{Gal}(F_p/\mathbb{Q}_p)$ is by permutation of the factors. We denote by r the natural representation of dimension $n \cdot r(d-r)$ which on ${}^LG^0$ is the tensor product of the n representations

$$(x_i, y_i) \longmapsto \Lambda^r x_i$$

and which is extended to $\mathrm{Gal}(F_p/\mathbb{Q}_p)$ in the obvious way. The degree of r is equal to the dimension of the Shimura variety $S(G, X)$. In the statements below there appears the local factor at p of a (semi-simple version of a) Langlands L-function. Such a local factor makes sense in our situation, as we shall now explain. Any representation of $G(\mathbb{Q}_p)$ with a C_p-invariant vector is of the form

$$\pi_p = \pi_p^0 \otimes \chi \quad ,$$

where π_p^0 is the trivial representation and χ is an unramified character of $K_p^\times = (K \otimes \mathbb{Q}_p)^\times$. To π_p^0 there is associated the homomorphism

$$\varphi(\pi_p^0) : W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

whose restriction to the second factor is the n-fold tensor power (over the index set appearing in the definition of ${}^LG^0$) of the representation

$$\varrho \times \varrho : SL_2 \longrightarrow GL_d \times GL_d$$

where ϱ is the irreducible representation of degree d of SL_2, and whose restriction to the first factor maps an element $w \in W_{\mathbb{Q}_p}$ projecting to $\tau \in \mathrm{Gal}(F_p/\mathbb{Q}_p)$ to the permutation matrix corresponding to τ. The homomorphism $\varphi(\pi_p)$ for $\pi_p = \pi_p^0 \otimes \chi$ is obtained from $\varphi(\pi_p^0)$ by twisting. We may therefore introduce the L-functions

$$L_p(s, \pi, r) = L(s, r \circ \varphi(\pi_p))$$

and

$$L_p^{ss}(s, \pi, r) = L^{ss}(s, r \circ \varphi(\pi_p)) \quad ,$$

as well as their shifts.

Let now π_f be an irreducible admissible representation of $G(\mathbf{A}_f)$ which we assume realized in a vector space over $\overline{\mathbb{Q}}$ and which occurs in the action of the Hecke algebra $\mathcal{H}(G(\mathbf{A}_f)//C)$ (with coefficients in $\overline{\mathbb{Q}}$) on the cohomology $H^*(S_C(G,X)(\mathbb{C}), \overline{\mathbb{Q}})$. Let $m(\pi_f) = \dim \pi_f^C$. Kottwitz [21] has defined a "stable version" of a multiplicity at infinity (which also incorporates the multiplicity with which an automorphic representation π with finite component π_f occurs in the discrete spectrum). This definition uses his important observation that for the group with which we are dealing, even though there are locally phenomena connected with L-indistinguishability this is cancelled globally in a certain sense. We shall use his notation $a(\pi_f)$ for this integer. Kottwitz's result for the places of good reduction suggests the following formula (compare section 2).

$$(5.1) \quad Z_{\wp_1}^{ss}(s, S_C(G,X)E) = \prod_{\pi_f} L_p^{ss}(s - \frac{1}{2}\dim S_C, \pi, r)^{a(\pi_f)\cdot m(\pi_f)}.$$

As explained in section 2 we develop the logarithms of both sides in power series of p^{-s}. The coefficients in front of $\frac{1}{j \cdot p^{js}}$ of the left hand side is calculated through the Lefschetz fixed point formula. The coefficient in front of $\frac{1}{j \cdot p^{js}}$ of the right hand side is calculated through the Selberg trace formula for a certain function $f = f_\infty \cdot f^p \cdot f_{p^j}$. We shall not explain the correct choice of f_∞. The function f^p equals $f^p = \frac{1}{\mathrm{vol}\, C^p} \cdot \mathrm{char}\, C^p$. It is the choice of $f_{p^j} \in \mathcal{H}(G(\mathbb{Q}_p)//C_p)$ which is of interest to us. In contrast to the case of good reduction where the analogue of f_{p^j} is not given directly but rather is described through its Satake transform, here the function can be defined explicitly.

5.2. LEMMA. *Let*

$$\widetilde{\sigma} = \sigma \times \begin{pmatrix} p^{-\frac{1}{2}} & 0 \\ 0 & p^{\frac{1}{2}} \end{pmatrix} \in W_{\mathbb{Q}_p} \times SL_2(\mathbb{C})$$

be the image of the geometric Frobenius under the natural homomorphism ([25])

$$W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \quad .$$

Let $\check{X}$ be the compact dual of X, a projective algebraic variety defined over $\mathbb{Q}$. Then

$$Tr(\sigma^{*j}; H^*(\check{X}_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)) = p^{j \cdot \frac{1}{2} \dim X} \cdot Tr^{ss}(r \circ \varphi(\pi_p^0)(\widetilde{\sigma}^j)).$$

PROOF: To simplify notations assume that $F = \mathbb{Q}$. In the general case the proof uses restriction of scalars to reduce to a case which is virtually the same as this special case. When $F = \mathbb{Q}$, the compact dual is the Grassmanian $Gr_r(\mathbb{Q}^d)$. Its cohomology is all algebraic and has a basis consisting of Schubert cycles (e.g. [13], p. 196), which are defined over $\mathbb{Q}$. Each such cycle in codimension d contributes a summand $\mathbb{Q}_\ell(-d)$ in degree $2d$ to cohomology. Therefore the left hand side is equal to

$$q^{-\frac{1}{2} \cdot r(r-1)} \cdot Tr \wedge^r \begin{pmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{d-1} \end{pmatrix} \quad .$$

Here $\wedge^r$ denotes the r-th exterior power of the matrix, and we abbreviated p^j into q. The right hand side equals

$$q^{\frac{1}{2}\cdot r(d-r)} Tr \wedge^r \begin{pmatrix} q^{-\frac{1}{2}(d-1)} & & & \\ & q^{-\frac{1}{2}(d-1)+1} & & \\ & & \ddots & \\ & & & q^{\frac{1}{2}(d-1)} \end{pmatrix},$$

hence we have equality.

5.3. Remark. There is an obvious generalization of the assertion of this lemma to any Shimura variety, where $\varphi(\pi_p^0)(\widetilde{\sigma}^j)$ is replaced by $n \times \sigma^j \in {}^LG$, where $n \in {}^LG^0$ is the principal unipotent element. I have not understood the significance of this assertion, but it certainly must be related to J.Arthur's and R.Kottwitz's contributions to these proceedings.

We can now define the function f_{p^j}. We use the exact sequence

$$1 \longrightarrow C_p \longrightarrow G(\mathbb{Q}_p) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \tag{5.4.}$$

The copies of $\mathbb{Z}$ correspond to the primes $\wp$ and $\overline{\wp}$ in K. Let $\Pi = \Pi_\wp$ be a uniformizing element at $\wp$, i.e. an element of $G(\mathbb{Q}_p)$ which maps to $1 \in \mathbb{Z}$ in the copy corresponding to $\wp$ and to 0 in the copy corresponding to $\overline{\wp}$. We put

$$f_{p^j} = c^{(j)} \cdot \operatorname{char} C_p \Pi^j C_p \tag{5.5.}$$

where $c^{(j)}$ is the number appearing in the statement of lemma 5.2. This lemma immediately implies the following corollary.

5.6. Corollary. *For any irreducible admissible representation π_p of $G(\mathbb{Q}_p)$ we have*

$$Tr\, \pi_p(f_{p^j}) = p^{\frac{1}{2}j\cdot \dim S(G,X)} \cdot Tr^{ss}(r \circ \varphi(\pi_p)(\widetilde{\sigma}^j))$$

if $\pi_p^{C_p} \neq 0$. If $\pi_p^{C_p} = 0$ the left hand side is equal to zero.

Here we have normalized the Haar measure on $G(\mathbb{Q}_p)$ by giving C_p the volume 1.

The comparison between the Lefschetz fixed point formula and the Selberg trace formula needed for proving the identity (5.1) dictates a relation between the function f_{p^j} on $G(\mathbb{Q}_p)$ and the function ϕ on $G(\mathbb{Q}_{p^j})$ introduced in the previous section (4.9.). More precisely, we wish to **show that the identity (5.1.) follows from the following conjecture** (as well as — in the case $r > 1$ — the conjectures and assumptions made in section 3 (cf. 3.15.)).

5.7. CONJECTURE. *Suppose that $\delta \in G(\mathbb{Q}_{p^j})$ is a semi-simple element such that the conjugacy class of $N_j\delta$ is regular and contains no element of $G(\mathbb{Q}_p)$. Then the twisted orbital integral over δ (cf. 4.11) vanishes,*

$$TO_\delta(\phi) = 0 \ .$$

We shall reformulate this conjecture in terms of local harmonic analysis. For this purpose we change notations. Let then F denote a local field of characteristic zero and residue characteristic p. Let G be a reductive algebraic group over F and let G_* be its quasi-split inner form. We make the assumption that there is a normal subgroup of G_* with identical adjoint group which over an unramified extension is a product of copies of GL_d. Then

(i) Two elements in $G_*(F)$ conjugate in $G_*(\overline{F})$ are conjugate. Similarly for $G(F)$.

(ii) Any conjugacy class in $G_*(\overline{F})$ stable under $\mathrm{Gal}(\overline{F}/F)$ contains an element of $G_*(F)$.

Here (ii) is a general fact ([20], 4.4.), holding for any quasi-split group with simply-connected derived group, and (i) is easily reduced to the case where $G_* = GL_d$. [Note that the map $G_*(F) \to G_{*ad}(F)$ is surjective.] We might summarize the above statements by saying that for G there are no phenomena of L-indistinguishability. In particular, there is a **well-defined injective** map

$$\{\gamma\} \longrightarrow \{\gamma\}_* \tag{5.8}$$

of the set of conjugacy classes of $G(F)$ into the set of conjugacy classes of $G_*(F)$.

Let $\mathcal{H}_F = \mathcal{H}(G(F)//C(F))$ be the Hecke algebra with respect to an Iwahori subgroup $C = C(F)$ of $G(F)$. Let $[\mathcal{H}_F, \mathcal{H}_F]$ be the linear span of the commutators and put

$$\tau(\mathcal{H}_F) = \mathcal{H}_F/[\mathcal{H}_F, \mathcal{H}_F] \ .$$

After choosing a Haar measure on $G(F)$, every element γ in $G(F)$ defines after a choice of Haar measure on the centralizer $G_\gamma(F)$ a linear functional O_γ on $\tau(\mathcal{H}_F)$ (orbital integral) which only depends on the conjugacy class $\{\gamma\}$. All this applies equally well to G_* instead of G and defines $\mathcal{H}_{*F}$, $\tau(\mathcal{H}_{*F})$ etc.. We shall always normalize the measures such that the Iwahori subgroup has measure 1. Therefore the measures on $G(F)$ and $G_*(F)$ are **not** related in the sense of Jaquet-Langlands. We shall now state three theorems in local harmonic analysis. We shall comment on the proofs after stating the theorems.

5.9 THEOREM. *There is one and only one linear map*

$$\tau(\mathcal{H}_F) \longrightarrow \tau(\mathcal{H}_{*F})$$

such that an element represented by $f \in \mathcal{H}_F$ is mapped to an element represented by $f_* \in \mathcal{H}_{*F}$, satisfying for any regular semi-simple element $\gamma_* \in G_*(F)$ the following condition:

$$O_{\gamma_*}(f_*) = \begin{cases} O_\gamma(f) & \text{if } \{\gamma_*\} = \{\gamma\}_* \\ 0 & \text{if } \{\gamma_*\} \text{ is not in} \\ & \text{the image of the map (5.8)} \end{cases}$$

[Here the measures on $G_\gamma(F)$ and $G_{*\gamma_*}(F)$ are chosen in a compatible way]. *This map is injective. Its image consists precisely of those elements which are represented by functions f_* for which $O_{\gamma_*}(f_*) = 0$ for any regular semi-simple $\{\gamma_*\}$ not coming from a conjugacy class in $G(F)$.*

We also need a twisted version of the preceding considerations. Let E/F be an unramified extension of degree j. We form the semi-direct products

$$G(E) \times W_{E/F} \quad \text{resp.} \quad G_*(E) \times W_{E/F} \quad ,$$

where the action of the Weil group factors through the Galois group. Again there is a natural injection of the set of conjugacy classes in $G(E) \times W_{E/F}$ into the conjugacy classes in $G_*(E) \times W_{E/F}$, compatible with the projections onto the Weil group. **It depends on the choice of an inner twisting.** We consider the functions in the Hecke algebra $\mathcal{H}(G(E)//C(E))$ as having support in $G(E) \times \sigma$, where σ is a fixed representant of the Frobenius. To avoid confusion we shall denote this algebra $\mathcal{H}_{E/F}$. We also introduce

$$\tau(\mathcal{H}_{E/F}) = \mathcal{H}_{E/F}/[\mathcal{H}_{E/F}, \mathcal{H}_{E/F}] \quad ,$$

as well as linear forms TO_δ (twisted orbital integral), namely the orbital integral over the conjugacy class under $G(E)$ of $\delta \times \sigma$, for any $\delta \in G(E)$. Similar considerations apply to G_* instead of G and since there is a natural injection

$$\{\delta\sigma\} \longrightarrow \{\delta\sigma\}_*$$

of the sets of σ-conjugacy classes, we may formulate the following twisted analogue of 5.9.

"5.10 THEOREM". *There is one and only one linear map*

$$\tau(\mathcal{H}_{E/F}) \longrightarrow \tau(\mathcal{H}_{*E/F})$$

such that an element represented by $\phi \in \mathcal{H}_{E/F}$ is mapped to an element represented by $\phi_* \in \mathcal{H}_{*E/F}$ having the property that for any regular semi-simple $\delta_* \in G_*(E)$ we have

$$TO_{\delta_*}(\phi_*) = \begin{cases} TO_\gamma(\phi) & \text{if } \{\delta_*\sigma\} = \{\gamma\sigma\}_* \\ 0 & \text{if } \{\delta_*\sigma\} \text{ is not of the} \\ & \text{form } \{\gamma\sigma\}_* \ . \end{cases}$$

This map is injective. The image of the map is the set represented by functions ϕ_* for which $TO_{\delta_*}(\phi_*) = 0$ for any regular semi-simple $\{\delta_*\sigma\}$ not coming from a twisted conjugacy class of $G(E)$.

The third theorem relates the twisted and untwisted Hecke algebras (base change). Let $\delta_* \times \sigma \in G_*(E) \times W_{E/F}$ and form its power $(\delta_* \times \sigma)^j$ (recall that $j = [E : F]$) and take its component in $G_*(E)$. The conjugacy class $N_j\delta_* = N_{E/F}\delta_*$ of this element is independent of the choice of σ and is $\mathrm{Gal}(E/F)$-invariant, hence contains a rational element. We obtain by property (i) above a conjugacy class $\{N_j\delta_*\}$ of $G_*(F)$. We have obtained a map

$$\{\delta_*\sigma\} \longrightarrow \{N_j\delta_*\} \ .$$

5.11 THEOREM. *There is one and only one linear map*

$$\tau(\mathcal{H}_{*E/F}) \longrightarrow \tau(\mathcal{H}_{*F})$$

such that an element represented by ϕ_* is mapped to an element represented by f_* having the property that for any regular semi-simple $\gamma_* \in G_*(F)$ we have

$$O_{\gamma_*}(f_*) = \begin{cases} TO_{\delta_*}(\phi_*) & \text{if } \{\gamma_*\} = \{N_j\delta_*\} \\ 0 & \text{if } \{\gamma_*\} \text{ is not of the} \\ & \text{form } \{N_j\delta_*\} \ . \end{cases}$$

This map is injective. Its image is the set represented by functions f_* for which $O_{\gamma_*}(f_*) = 0$ for any regular semi-simple $\{\gamma_*\}$ not of the form $\{N_j\delta_*\}$.

We now shall comment on the proofs of these theorems, making use of some facts from harmonic analysis, at least insofar as they are available. To simplify the exposition we shall assume that $G_* = GL_d$. We start with the proof of theorem 5.9. We shall denote by $\mathcal{H}(G(F))$ the Hecke algebra of $G(F)$ of all compactly supported smooth functions. Similarly for $\mathcal{H}(G_*(F))$. To show that the element of $\tau(\mathcal{H}_{*F})$ represented by f_* is well-determined we use the obvious map (with self-evident notations)

$$\tau(\mathcal{H}_{*F}) \longrightarrow \tau(\mathcal{H}(G_*(F)) \quad .$$

By a theorem of Bernstein ([15], p. 178) this map is injective. Hence it suffices to show that the image of f_* in $\tau(\mathcal{H}(G_*(F)))$ is determined by its regular semi-simple orbital integrals. This is the content of a theorem of Harish-Chandra, comp. [16]. The same argument, this time applying the above facts to G instead of G_*, shows the injectivity of the map $\tau(\mathcal{H}_F) \to \tau(\mathcal{H}_{*F})$ we want to construct. For the existence of this map we shall use the fact that a "Jacquet-Langlands" correspondence exists (cf. [3]). Thus to every tempered representation π of $G(F)$ there corresponds a tempered representation π_* of $G_*(F)$ which satisfy character identities on matching functions $f \to f_*$, $f \in \mathcal{H}(G(F))$, $f_* \in \mathcal{H}(G_*(F))$. Now let $\widetilde{f}_* \in \mathcal{H}(G_*(F))$ be a function which matches $f \in \mathcal{H}_F$. Therefore for any tempered representation π_* of $G_*(F)$ we have

$$Tr\, \pi_*(\widetilde{f}_*) = \begin{cases} Tr\, \pi(f) & \text{if } \pi \text{ corresponds to } \pi_* \\ 0 & \text{if there is no such } \pi \end{cases}$$

5.12. Lemma. *Let π correspond to π_*. Then*

$$\pi^C \neq 0 \quad \textit{if and only if} \quad \pi_*^{C_*} \neq 0 \quad .$$

Proof: We first assume that π and π_* are discrete series representations. By the theory of Bernstein-Zelevinski [37], π_* is a generalized Steinberg representation $St(\omega, r)$ associated to a supercuspidal automorphic representation of $GL_{\frac{d}{r}}(F)$ (corresponding to the interval $[\omega, \omega \otimes |\,|^{r-1}]$). However $\pi_*^{C_*} \neq 0$ iff $\omega^{C_*} \neq 0$ and this is the case iff $d = r$ and ω is an unramified character of $GL_1(F)$. This last implication follows because ω is supercuspidal and using Casselman's theorem asserting that a representation with a fixed vector under the Iwahori subgroup is a subquotient of an induced representation $Ind_B^G \lambda$ where B is the Borel subgroup and λ an unramified character of the diagonal matrices. The conclusion is that $\pi_*^{C_*} \neq 0$ iff $\pi_* = St(\chi, d)$ is an ordinary Steinberg representation associated to an unramified character of $F^\times$. Something entirely analogous holds for $G(F)$ (cf. [3], p. 83).

We identify $G(F)$ with $GL_m(D)$, where D is a central division algebra of dimension $(d/m)^2$ over F. Then any discrete series representation π is a generalized Steinberg representation (in the sense of loc. cit.), $\pi = St(\omega, r)$, where ω is a supercuspidal representation of $GL_{\frac{m}{r}}(D)$. Again $\pi^C \neq 0$ iff $r = m$ and $\omega = \chi$ is an unramified character of $D^\times$. However, by loc. cit., the correspondence $\pi \to \pi_*$ induces a bijection between the discrete series representations. Furthermore, under this bijection, the ordinary Steinberg representations associated to a character of $F^\times$ correspond to one another, which finishes the proof in the case of discrete series representations. We reduce the general case to this special case. Any tempered representation π of $GL_m(D)$ is a subquotient of an induced representation associated to a partition $\underline{m} = (m_1, \ldots, m_r)$ of m and discrete series representations σ_i of $GL_{m_i}(D)$, $i = 1, \ldots, r$. However, by [3], p. 86, such an induced representation $Ind(\sigma_1, \ldots, \sigma_r)$ is irreducible, so that $\pi = Ind(\sigma_1, \ldots, \sigma_r)$. Furthermore, $\pi^C \neq 0$ iff $\sigma_i^C \neq 0$ for $i = 1, \ldots, r$. The conclusion follows now because under the correspondence

$$Ind(\sigma_1, \ldots, \sigma_r)_* = Ind(\sigma_{1*}, \ldots, \sigma_{r*})$$

(induced representation associated to the partition $(m_1 \cdot \frac{d}{m}, \ldots, m_r \cdot \frac{d}{m})$ of d).

We continue with the construction of the function $f_* \in \mathcal{H}_{*F}$. The lemma implies that

$$Tr \pi_*(\widetilde{f}_*) \neq 0 \Longrightarrow \pi_*^{C_*} \neq 0 \quad .$$

Therefore, the function

$$f_* = \text{char } C_*(F) \cdot \widetilde{f}_* \cdot \text{char } C_*(F) \in \mathcal{H}_{*F}$$

has the same trace as $\widetilde{f}_*$ on all tempered representations. Therefore, by Kazhdan's density theorem [16], Thm. 0, the orbital integrals of f_* and $\widetilde{f}_*$ coincide, hence by Harish-Chandra's theorem the images of f_* and $\widetilde{f}_*$ in $\tau(\mathcal{H}(G_*(F))$ are the same. Therefore the map appearing in 5.9 exists. To show that its image is characterized by the condition on the orbital integrals we use the fact ([3], p. 83) that if f_* satisfies this condition, then there exists a matching function $\widetilde{f} \in \mathcal{H}(G(F))$. The previous lemma implies that

$$Tr\ \pi(\breve{f}) \neq 0 \Longrightarrow \pi^C \neq 0 \quad .$$

Therefore $f = \text{char } C(F) \cdot \tilde{f} \cdot \text{char } C(F)$ has the same traces as $\tilde{f}$ on all tempered representations, and therefore by the same arguments as above, defines a preimage of f_*. This concludes the proof of theorem 5.9.

We next turn to the proof of theorem 5.11. The proof of the fact that the image of f_* in $\tau(\mathcal{H}_{*F})$ is well-determined is identical with the proof of the corresponding assertion in 5.9. For the injectivity of the searched-for map we need the twisted analogues of the theorems of Bernstein and of Harish-Chandra. A glance at the proof of Bernstein's theorem, [15], p. 178, which is based on the results of [3], p. 30, shows no impediment to paraphrasing it in the twisted case. Similarly, the proof in [47], §2, of Harish-Chandra's theorem carries over to the twisted case (comp. [1], I, §3).

For the existence of the map we use the theory of base change ([1]). Hence to every tempered representation π of $GL_d(F)$ there corresponds a σ-stable tempered representation Π of $GL_d(E)$ which satisfy (twisted) character identities on matching functions $\phi \to f$. Here $\phi \in \mathcal{H}(GL_d(E/F))$, the twisted version of the Hecke algebra of all compactly supported smooth functions on $GL_d(E)$ and $f \in \mathcal{H}(GL_d(F))$. We need a twisted version of lemma 5.12.

5.13 LEMMA. *Let Π be the base change of π. Then*

$$\pi^{C(F)} \neq 0 \quad \textit{if and only if} \quad \Pi^{C(E)} \neq 0 \quad .$$

PROOF: (L.Clozel): We easily reduce the proof to the case where j is prime. We first assume that π is a discrete series representation. This is equivalent to Π being a σ-discrete representation, i.e. ([1], I-2.8.) either Π is a σ-stable discrete series representation of $GL_d(E)$ or Π is an induced representation of the form

$$\Pi = Ind(\Pi_1, {}^{\sigma}\Pi_1, \ldots, {}^{\sigma^{j-1}}\Pi_1)$$

where Π_1 is a discrete series representation of $GL_{\frac{d}{j}}(E)$ which is not σ-stable. As we have seen in the proof of lemma 5.12, in the first case $\Pi^{C(E)} \neq 0$ iff Π is an ordinary Steinberg representation $St(\chi, d)$ associated to an unramified character of $E^\times$. We show that in the second case $\Pi^{C(E)} = 0$. This then concludes the proof of the lemma in the case of discrete series since, E being an unramified extension of F, the unramified characters of $E^\times$ all come via the norm map from unramified characters of $F^\times$ and since the base change of $St(\chi, d)$ is $St(\chi \circ N_{E/F}, d)$. Assume then that $Ind(\Pi_1, {}^\sigma\Pi_1, \ldots, {}^{\sigma^{j-1}}\Pi_1)$ has a fixed vector under $C(E)$. Then $\Pi_1^{C(E)} \neq 0$, hence $\Pi_1 = St(\chi, \frac{d}{j})$ for an unramified character of $E^\times$. However, then Π_1 would be σ-stable, a contradiction. The case of a general tempered representation is reduced as before to the case of discrete series representations.

Using this lemma the existence of the map in 5.11 is proved as the corresponding statement in 5.9. Similarly, making use of lemma 5.13, the characterization of the image of this map follows from the analogous characterization of the image of the matching correspondence, using the twisted versions of the theorems of Bernstein and Harish-Chandra and the twisted version of the Kazhdan density theorem, for which we refer to [1], I-2.7.

We now turn to the proof of theorem 5.10. The proofs of the facts that the map is well-defined and injective, if it exists, are identical with the corresponding proofs in 5.11. For the existence proof we would need a twisted version of the Jacquet-Langlands correspondence, as well as twisted analogues of its properties we used above. Here, as L.Clozel explained to me, there arises a problem: To even define such a correspondence, one would need to extend in a canonical way a σ-stable representation from $G(E)$ to the semi-direct product $G(E) \times \mathrm{Gal}(E/F)$. In the case of GL_d, this extension is produced by Arthur and Clozel [1] using a Whittaker model and this method fails. Therefore we have put this "theorem" in quotation marks. Its proof requires, as it seems, new ideas. In the sequel, however, we shall do as if this theorem were an acquired fact and proceed.

We add, however, one simple remark which shows the existence of the required map in the case where $G(E)$ is isomorphic to $G_*(E)$, i.e. when E splits G. In this case the action of $\mathrm{Gal}(E/F)$ on $G(E) = G_*(E)$ can be written in the form

$$\sigma(g) = c \cdot \sigma_*(g) \cdot c^{-1}$$

where σ_* is the Galois action on $G_*(E)$ and **where c normalizes the Iwahori subgroup**. It is easy to see that the map on twisted conjugacy classes is given as

$$\{\delta\sigma\} \longmapsto \{\delta \cdot c\sigma_*\} \quad .$$

The map which sends an element of $\mathcal{H}_{E/F}$ represented by ϕ to the element of $\mathcal{H}_{*E/F}$ represented by ϕ_* with

$$\phi_*(g) = \phi(g \cdot c^{-1})$$

establishes an isomorphism between $\tau(\mathcal{H}_{E/F})$ and $\tau(\mathcal{H}_{*E/F})$. This follows from the identity

$$TO_\delta(\phi) = TO^*_{\delta\cdot c}(\phi_*) \quad .$$

Putting these theorems together we obtain a diagram of linear maps

$$\begin{array}{ccc} \tau(\mathcal{H}_{E/F}) & \longrightarrow & \tau(\mathcal{H}_{*E/F}) \\ & & \downarrow \\ \tau(\mathcal{H}_F) & \longrightarrow & \tau(\mathcal{H}_{*F}) \end{array} \tag{5.12}$$

We now return to our original situation where $F = \mathbb{Q}_p$, $E = \mathbb{Q}_{p^j}$ and G is the group defining the Shimura variety. Recall the function ϕ on $G(\mathbb{Q}_{p^j})$ which we consider as an element of $\mathcal{H}_{\mathbb{Q}_{p^j}/\mathbb{Q}_p}$.

5.13 PROPOSITION. *Conjecture 5.7 on ϕ is equivalent to the statement that the element of $\tau(\mathcal{H}_{\mathbb{Q}_{p^j}/\mathbb{Q}_p})$ represented by ϕ is mapped under the composition of maps $\tau(\mathcal{H}_{\mathbb{Q}_{p^j}/\mathbb{Q}_p}) \to \tau(\mathcal{H}_{*\mathbb{Q}_{p^j}/\mathbb{Q}_p}) \to \tau(\mathcal{H}_{*\mathbb{Q}_p})$ to an element lying in the image of $\tau(\mathcal{H}_{\mathbb{Q}_p})$.*

PROOF: For any regular semi-simple $\{\gamma_*\}$ of the form $\{N_j\delta\}$,

$$O_{\gamma_*}(f_*) = TO_\delta(\phi) \quad ,$$

where f_* represents the image of ϕ in $\tau(\mathcal{H}_{*\mathbb{Q}_p})$. If γ_* does not come from $G(\mathbb{Q}_p)$, i.e. is not elliptic, then conjecture 5.7. implies that the right hand side vanishes. If $\{\gamma_*\}$ is not of the form $\{N_j\delta\}$ then $O_{\gamma_*}(f_*)$ vanishes anyhow. Hence by 5.9., f_* lies in $\tau(\mathcal{H}_{\mathbb{Q}_p})$. The converse is proved similarly.

5.14 THEOREM. *Assume conjecture 5.7. resp. its reformulation in 5.13. Then the image of ϕ in $\tau(\mathcal{H}_{\mathbb{Q}_p})$ is represented by the function $f_{p^j} \in \mathcal{H}(G(\mathbb{Q}_p)//C_p)$ (comp. 5.5. for the definition of f_{p^j}).*

PROOF: The exact sequence (5.4.) induces an isomorphism of algebras

$$\mathcal{H}(G(\mathbb{Q}_p)//C_p) \simeq \mathbb{C}[X^{\pm 1}, Y^{\pm 1}] \quad ,$$

where X resp. Y corresponds to the first resp. second summand $\mathbb{Z}$ in (5.4.). By its very definition

$$f_{p^j} = c^{(j)} \cdot X^j \quad .$$

From the condition 4.8. on the support of ϕ and the definition of the linear map from $\tau(\mathcal{H}_{\mathbb{Q}_{p^j}/\mathbb{Q}_p})$ to $\tau(\mathcal{H}_{*\mathbb{Q}_p})$ it is easy to see that if the image f_* of ϕ comes from a function $f \in \mathcal{H}(G(\mathbb{Q}_p)//C_p)$ then necessarily

$$f = c \cdot X^j \quad .$$

We have to prove that $c = c^{(j)}$. We do this by evaluating f on simple representations. Let $1_{\mathbb{Q}_p}$ resp. $1_{*\mathbb{Q}_p}$ be the trivial representation of $G(\mathbb{Q}_p)$ resp. $G_*(\mathbb{Q}_p)$. We have

$$c^{(j)} = Tr\ 1_{\mathbb{Q}_p}(f_{p^j}) \quad , \quad c = Tr\ 1_{\mathbb{Q}_p}(f) \quad .$$

The Weyl integration formula for $G_*(F)$ shows that

$$Tr\ 1_{*\mathbb{Q}_p}(f_*) = Tr\ 1_{\mathbb{Q}_p}(f) \quad .$$

Let ϕ_* be a representant of the image of ϕ in $\tau(\mathcal{H}_{*\mathbb{Q}_{p^j}/\mathbb{Q}_p})$. Let $\widetilde{1}_{\mathbb{Q}_{p^j}}$ resp. $\widetilde{1}_{*\mathbb{Q}_{p^j}}$ be the trivial representation of $G(\mathbb{Q}_{p^j}) \times W_{\mathbb{Q}_{p^j}/\mathbb{Q}_p}$ resp. $G_*(\mathbb{Q}_{p^j}) \times W_{\mathbb{Q}_{p^j}/\mathbb{Q}_p}$. Then a comparison of the Weyl integration formula for $G_*(\mathbb{Q}_p)$ and the twisted Weyl integration formula for $G_*(\mathbb{Q}_{p^j}) \times W_{\mathbb{Q}_{p^j}/\mathbb{Q}_p}$ [27, p. 100] shows that

$$Tr\ \widetilde{1}_{*\mathbb{Q}_{p^j}}(\phi_*) = Tr\ 1_{*\mathbb{Q}_p}(f_*) \quad .$$

Similarly we have

$$Tr\ \widetilde{1}_{\mathbb{Q}_{p^j}}(\phi) = Tr\ \widetilde{1}_{*\mathbb{Q}_{p^j}}(\phi_*) \quad .$$

We thus have obtained

$$c = Tr\ \widetilde{1}_{\mathbb{Q}_{p^j}}(\phi) = \int_{G(\mathbb{Q}_{p^j})} \phi(g)dg \quad .$$

With our normalization of measures we have

$$\int_{G(\mathbb{Q}_{p^j})} \phi(g)dg = \sum_{x \in M(\overline{\Delta},X)(\mathbb{F}_{p^j})} Tr_{x,p^j} \quad .$$

The right hand side was calculated in 3.17 and 3.18. Comparing the result with 5.2 the assertion follows.

Recall that the group G arose by restriction of scalars from an algebraic group G' over F_p, the unramified extension of degree n of $\mathbb{Q}_p$. Similarly, the local model $M(\overline{\Delta}, X)$ arose by restriction of scalars from a scheme $M(\overline{\Delta}', X')$ over $\operatorname{Spec} \mathcal{O}_{F_p}$. We abbreviate F_p to F. If E' is an unramified extension of degree j' of F, we may define a function on $G'(E')$,

$$\phi' \in \mathcal{H}'_{E'/F} \quad ,$$

completely analogously to the definition of ϕ. In particular, if $g \in G'(E')$ defines a point $\overline{x}(g)$ of $M(\overline{\Delta}, X')$ and has the correct absolute value (cf. 4.8), then

$$\phi'(g) = Tr_{\overline{x}(g), q^{j'}}$$

where the expression on the right is the alternating trace of the relative Frobenius in $\operatorname{Gal}(\mathbb{F}_{q^{j'}}/\mathbb{F}_q)$ over the residue field $\mathbb{F}_q$ of F on the inertia invariants of the sheaves of vanishing cycles at the point $\overline{x}(g) \in M(\overline{\Delta}', X')(\mathbb{F}_{q^{j'}})$.

With these definitions we may eliminate the restriction of scalars from conjecture 5.7 resp. 5.13 as follows. Write $j = k \cdot j'$, where $k = gcd(n, j)$. Then

$$\mathbb{Q}_{p^j} \otimes F \simeq E' \oplus \ldots \oplus E' \quad .$$

(k summands, $[E' : F] = j'$). Accordingly we have

$$G(\mathbb{Q}_{p^j}) = G'(E') \times \ldots \times G'(E') \quad .$$

Let $f_1, \ldots, f_k$ be smooth functions on $G'(E')$ and define the function f on $G(\mathbb{Q}_{p^j})$ by

$$f(g_1, \ldots, g_k) = f_1(g_1) \ldots f_k(g_k) \quad .$$

Let $\delta = (\delta_1, \ldots, \delta_k) \in G(\mathbb{Q}_{p^j})$ and $\overline{\delta} = \delta_1 \ldots \delta_k \in G'(E')$. Then [27]

$$TO_\delta(f) = TO_{\overline{\delta}}(f_1 * \ldots * f_k) \tag{5.15}$$

Here the right hand side is the twisted orbital integral of a convolution of functions on $G'(E')$ **relative to the extension** E'/F. Returning now to our functions $\phi \in \mathcal{H}_{\mathbb{Q}_{p^j}/\mathbb{Q}_p}$ resp. $\phi' \in \mathcal{H}_{E'/F}$, we have the following simple lemma.

5.16 LEMMA. *We assume that the Künneth formula holds for the sheaves of vanishing cycles with respect to the product decomposition*

$$M(\overline{\Delta}, X) \otimes W(\mathbb{F}_{p^j}) = \left[M(\overline{\Delta}', X') \otimes W(\mathbf{F}_{q^{j'}})\right]^k \quad .$$

We also assume that the inertia subgroup acts through a finite group on the sheaves of vanishing cycles. Then

$$\phi(g_1, \ldots, g_k) = \phi'(g_1) \cdot \ldots \cdot \phi'(g_k) \quad .$$

In particular, this identity holds in the case $r = 1$.

We may now reduce the conjecture 5.7 to the following statement.

5.17. PROPOSITION. *We assume that the hypotheses of the previous lemma are satisfied. Then conjecture 5.7 follows from the following statement. Let*

$$\phi'^{(k)} = \phi' * \ldots * \phi'$$

be the k-fold iterated convolution of ϕ'. Let $\delta \in G'(E')$ be a regular semi-simple element with regular non-elliptic norm $N_{j'}\delta$ (a conjugacy class in $G'_*(F)$). Then the twisted orbital integral vanishes,

$$TO_\delta(\phi'^{(k)}) = 0 \quad .$$

PROOF: Using the previous lemma we may apply 5.15. It remains to show that if $\delta = (\delta_1, \ldots, \delta_k)$ is a regular semi-simple element of $G(\mathbb{Q}_{p^j})$ with regular non-elliptic norm $N_j\delta \in G_*(\mathbb{Q}_p)$, then $\overline{\delta} = \delta_1 \ldots \delta_k$ has non-elliptic norm $N_{j'}\overline{\delta} \in G'_*(F)$. This is clear since under the embedding of $G_*(\mathbb{Q}_p)$ in $G_*(\mathbb{Q}_{p^j}) = G'_*(E')^k$, the element $N_j\delta$ goes into the diagonal element with entries $N_{j'}\overline{\delta}$.

I note one trivial case in which this assertion can be verified. If $gcd(j', d) = 1$, then $G'(E')$ is anisotropic. Therefore there are no regular semi-simple elements with non-elliptic norm, and hence the condition is empty.

The following theorem solves the problem in the case $d = 2$. We refer to [34] for the proof. (In fact, the group considered in loc. cit. differs from the group considered here by a central subgroup, but the same proof applies.)

5.18. THEOREM. *Let $d = 2$. Then the condition of 5.17. is satisfied for every $j = 1, 2, \ldots$. For the local semi-simple zeta function there is the formula*

$$Z_p^{ss}(s, S_C(G,X)/\mathbb{Q}) = \prod_{\pi_f} L_p^{ss}(s - \frac{1}{2}\dim S_C, \pi, r)^{a(\pi_f)\cdot m(\pi_f)}$$

The product on the right extends over all irreducible admissible representations of $G(\mathbf{A}_f)$ occurring in $H^*(S_C, \overline{\mathbb{Q}})$. Here $m(\pi_f) = \dim \pi_f^C$ and

$$a(\pi_f) = \begin{cases} 1 & \textit{if} \quad \dim(\pi) = 1 \\ (-1)^n & \textit{if} \quad \dim(\pi) = \infty \end{cases} .$$

The last statement is proved by showing that the sums expressing the coefficient in front of $\frac{1}{j\cdot p^{js}}$ of the logarithms of the left and right side of (5.1) range over the same index set and are termwise equal. In contrast to the cases where $d > 2$ there are no phenomena of L-indistinguishability even globally. As to the proof in loc. cit. of the condition in proposition 5.17., it is done on the dual side of representations. In other words, it is proved that the trace on a representation π'_* of $G'_*(F)$ of the image of $\phi'^{(k)}$ in $\tau(\mathcal{H}'_{*F})$ is zero when π'_* doesn't come from a representation of $G'(F)$. Another possible approach is to express the twisted orbital integral in combinatorial terms through the Bruhat-Tits building of $G'(E')$ (cf. [**17**]). It is conceivable that in pushing through this method one is led to a deeper understanding of the geometry of the local model of the Shimura variety.

We conclude this report with a corollary to the considerations in §2. For the proof we again refer to [34].

5.19 COROLLARY. *Let $d = 2$. Assume that the monodromy filtration of $H^*(S_C \otimes \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$ is pure. Then there is the following expression for the local zeta function.*

$$Z_p(s, S_C(G,X)/\mathbb{Q}) = \prod_{\pi_f} L(s - \frac{1}{2}\dim S_C, \pi_p^*, r)^{a(\pi_f)\cdot m(\pi_f)} ,$$

where π_p^* is the irreducible admissible representation of the quasi-split form $G_*(\mathbb{Q}_p)$ corresponding to the irreducible admissible representation π_f of $G(\mathbf{A}_f)$ as follows.

$$\pi_p^* = \begin{cases} (\pi_p)_* & \textit{if } \dim(\pi_f) = \infty \\ \chi_p & \textit{if } \dim(\pi_f) = 1 \text{ , } \textit{corresponding} \\ & \textit{to a Hecke character } \chi \textit{ of } K. \end{cases}$$

This in particular applies to the cases when $\dim S_C \leq 2$.

As any generalization of these theorems to the cases $d > 2$ depends on a positive solution of conjecture 5.7. I want to formulate explicitly what this conjecture says in the crucial case where d is arbitrary, but $r = 1$, the invariant of $D = D_\wp$ equals $1/d$, i.e. $s = 1$, and finally where j is divisible by d. Furthermore I shall neglect part of the center and do as if $G = D^\times$ (locally at p).

The question can be formulated purely in terms of a certain function in the Iwahori algebra of GL_d. Let F be a p-adic field and let E/F be an unramified extension of degree j divisible by d. Let

$$c = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \varpi & & & & 0 \end{bmatrix} \in GL_d(E) \quad ,$$

where ϖ is a uniformizing parameter. We shall define below a function ϕ_* in the Iwahori algebra of $GL_d(E)$ with respect to the standard Iwahori subgroup K of $GL_d(E)$,

$$K \equiv \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \operatorname{mod} \varpi \quad .$$

We also consider the translate $T_{c^{-1}}(\phi_*)$,

$$T_{c^{-1}}(\phi_*)(g) = \phi_*(g \cdot c) \quad .$$

The conjecture is that for any $\delta \in GL_d(E)$ with regular non-elliptic norm $N\delta$ (with respect to the standard action of the Galois group on $GL_d(E)$) the standard twisted orbital integral of the iterated convolution product $(T_{c^{-1}}(\phi_*))^{(n-1)} * \phi_*$ vanishes for any $n \geq 1$:

$$TO_\delta((T_{c^{-1}}(\phi_*))^{(n-1)} * \phi_*) = 0 \quad .$$

It remains to define the function ϕ_*. Its support will be contained in the set

$$\operatorname{ord}\det g = 0 \quad .$$

Let $\Delta = \{a_0, a_1, \dots, a_{d-1}\}$ be the standard simplex in the Bruhat-Tits building of $PGL_d(E)$, where a_i is the vertex represented by the lattice

$$\underbrace{\mathcal{O} \oplus \dots \oplus \mathcal{O}}_{i+1} \oplus \varpi \cdot \mathcal{O} \oplus \dots \oplus \varpi \cdot \mathcal{O} \subset E \quad .$$

For $g \in GL_d(E)$ with $\operatorname{ord}\det g = 0$ let

$$b_i = g \cdot a_i \quad .$$

Then $\phi_*(g)$ is zero unless the following condition is satisfied:

For any $i \in \mathbb{Z}/d$ a_i and b_{i-1} form an edge in the building and b_{i-1} corresponds to a sublattice of a_i of index 1.

If this condition is satisfied let

$$S(g) = \{i \in \mathbb{Z}/d \ ; \quad a_i = b_i\} \quad .$$

Then

$$\phi_*(g) = \frac{(1-q)^{|S(g)|}}{1-q} \quad .$$

Here q denotes the cardinality of the residue field of E.

To see that this is the correct translation of conjecture 5.7. in this special case one uses the fact that in this situation a twisted Jacquet-Langlands correspondence exists (compare the remarks on the proof of 5.10. above) and that under this correspondence the image of the n-fold iterate $\phi^{(n)}$ of our function ϕ on $G(E)$ equals

$$(\phi^{(n)})_* = T_{c^{-1}}(\phi_*)^{(n-1)} * \phi_*$$

(compare [34], §5). This function ϕ_* can in principle be written down. For $d = 2$ it is

$$\operatorname{char} K \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} K + (1-q)\operatorname{char} K + \operatorname{char} K \begin{bmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{bmatrix} K$$

For general d it is the sum of $2^d - 1$ scalar multiples of characteristic functions of double cosets by the Iwahori subgroup. It is possible to give a combinatorial expression of this function in terms of the affine Weyl group.

Note added after completion of this article: Using this combinatorial description (letter to Waldspurger of January 1989), J.-L. Waldspurger was able to prove the above conjecture (letter to the author of 25 february). His proof is valid regardless of whether d divides j.

Errata to [34]

p. 676: In the statement of 1.3. (iii) the condition for $i = n$ has to read

$$p \cdot A_1^0 \underset{\neq}{\subset} UA_0^{n-1} \underset{\neq}{\subset} A_1^0 \quad \text{and}$$

$$A_0^0 \underset{\neq}{\subset} UA_1^{n-1} \underset{\neq}{\subset} \frac{1}{p} \cdot A_0^0 \quad .$$

p. 680: In the statement of 2.5. the formula for B_j^0 has to read

$$B_0^0 = p \cdot UA_1^{n-1} \quad , \quad B_1^0 = UA_0^{n-1} \quad .$$

p. 693_8: The isomorphism $\widetilde{\alpha}$ sends $z \in L'^{\times} \subset W(L'/F)$ into $z \cdot 1 \times z \in G'_*(L') \times W(L'/F)$.

p. 694: In formula (5.c) the second summand has to read

$$(1-q) \cdot \text{char } K \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix} K \quad .$$

p. 695^{14}: In the formula for $T_{c_*}\widetilde{\phi}$ an expression like $\overline{\omega} \cdot W$ has to be read as

$$T_{\omega \cdot 1}(W) \quad .$$

References

1 Arthur, J., Clozel, L., Base Change for GL_n; to appear.

2 Artin, E, Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren, Abh. Hamb. **8**, 292-306 (1930).

3 Bernstein, J., Deligne, P., Kazhdan, D., Vignéras, M.-F., Représentations des groupes réductifs sur un corps local, Hermann, Paris, 1984.

4 Borel, A., Wallach, N, Continuous cohomology, discrete subgroups, and representations of reductive groups. Ann. of Math. Studies, Princeton, 1980.

5 Cherednik, I.V., Uniformization of algebraic curves by discrete subgroups of $\mathrm{PGL}_2(k_w)$ with compact quotients, Math. USSR Sbornik, **29** (1976), 55-78.

6 Carayol, H., Non-abelian Lubin-Tate theory. These proceedings..

7 Deligne, P, Les constantes des équations fonctionelles des fonctions L. Springer Lecture Notes, **349** (1973).

8 Deligne, P, La conjecture de Weil II. Pub. Math. IHES, **52** (1981).

9 Deligne, P., Rapoport, M, Modules des courbes elliptiques, in: Modular functions of one variable, II. Springer Lecture Notes **349** (1973).

10 Deligne, P., Travaux de Shimura, Sem. Bourbaki 389, Springer Lecture Notes **244** (1971).

11 Deligne, P., Lusztig, G., Representations of reductive groups over finite fields. Ann. Math. **103**, 103-161 (1976).

12 Drinfeld, V.G., Coverings of p-adic symmetric domains, Funct. Anal. Appl. **10**, 107-115, (1976).

13 Griffiths, P., Harris, J., Principles of algebraic geometry. J. Wiley, 1978.

14 Jacquet, H., Principal L-functions of the linear group. Proc. Symp. Pure Math. **33** (2), 63-86 (1979).

15 Kazhdan, D., Representations of groups over close local fields. J. d'analyse math. **47**, 175-179 (1986).

16 Kazhdan, D., Cuspidal geometry of p-adic groups. J. d'analyse math. **47**, 1-36 (1986).

17 Kottwitz, R.E., Orbital integrals on GL_3. Am. J. Math. **102**, 327-384 (1980).

18 Kottwitz, R.E., Shimura varieties and twisted orbital integrals. Math. Ann. **269**, 287-300 (1984).

19 Kottwitz, R.E., Isocrystals with additional structures, Compos. Math. **56**, 201-220 (1985).

20 Kottwitz, R.E., Rational conjugacy classes in reductive groups. Duke Math. J. **49**, 785-806 (1982).

21 Kottwitz, R.E., On the λ-adic representations associated to some simple Shimura varieties, to appear.

22 Kottwitz, R.E., Shimura varieties and λ-adic representations. These proceedings..

23 Kurihara, A., Construction of p-adic unit balls and the Hirzebruch proportionality. Amer. J. Math. **102**, 565-648 (1980).

24 Langlands, R.P., Sur la mauvaise réduction d'une variété de Shimura, Asterisque **65**, 125-154 (1979).

25 Langlands, R.P., Automorphic representations, Shimura varieties and motives. Ein Märchen. Proc. Symp. Pure Math. **33** (2), 205-246 (1979).

26 Langlands, R.P., On the zeta functions of some simple Shimura varieties. Can. J. Math. **31**, 1121-1216 (1979).

27 Langlands, R.P., Base change for GL(2). Ann. of Math. Studies, Princeton, 1979.

28 Langlands, R.P., Rapoport, M., Shimuravarietäten und Gerben. J. Reine Angew. Math. **378**, 113-220 (1987).

29 Lazard, M., Commutative formal groups. Springer Lecture Notes **443** (1975).

30 Messing, W., The crystals associated to Barsotti-Tate groups. Springer Lecture notes **264** (1972).

31 Mustafin, G., Nonarchimedian uniformization, Math. USSR-Sbornik, **34**, 187-214 (1978).

32 Mumford, D., An analytic construction of degenerating curves over complete local rings. Compos. Math. **24**, 129-174 (1972).

33 Rapoport, M., Zink, Th., Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration and verschwindende Zyklen in ungleicher Charakteristik, Invent. math. **68**, 21-101 (1980).

34 Rapoport, M., On the local zeta function of quaternionic Shimura varieties with bad reduction. Math. Ann. **279**, 673-697 (1988).

35 Raynaud, M., Géométrie analytique rigide d'après Tate, Kiehl,... Bull. Soc. math. France, Memoire **39-40**, 319-327 (1974).

36 Raynaud, M., p-torsion du schéma de Picard, Astérisque **64** 87-148 (1979).
37 Rodier, F., Répresentations de GL(n,k), où k est un corps p-adique, Sem. Bourbaki 587; Astérisque **92-93**, 201-218 (1982).
38 Serre, J.-P., Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures), Sem. Delange-Pisot-Poitou, 1970.
39 SGA 7 I, Springer Lecture Notes **288**.
40 SGA 4 III, Springer Lecture Notes **305**.
41 Steenbrink, J., Limits of Hodge structures, Invent. math. **31**, 229-257 (1976).
42 Tate, J., Endomorphisms of abelian varieties over finite fields. Invent. math. **2**, 134-144 (1966).
43 Tate, J., Number theoretic background. Proc. Symp. Pure Math. **33** (2), 3-26 (1979).
44 Thomason, R., Absolute cohomological purity. Bull. Soc. Math. France **112**, 397-406 (1984).
45 Tits, J., Reductive groups over local fields. Proc. Symp. Pure Math. **33** (1), 29-69 (1979).
46 Tsfasman, M.A., Vladut, S.G., Zink, T., Modular curves, Shimura curves and Goppa codes better than Varshamov-Gilbert bound; Math. Nachr. **109**, 21-28 (1982).
47 Vignéras, M.F., Caracterisation des intégrales orbitales sur un groupe réductif p-adique, J. Fac. Sci., Univ. Tokyo, **28** (3), 945-961 (1982).
48 Vogan, D., Zuckerman, G., Unitary representations with non-zero cohomology, Compos. Math. **53**, 51-90 (1984).
49 Zink, Th., Cartiertheorie kommutativer formaler Gruppen, Teubner-Texte zur Mathematik **68**, Leipzig 1984.
50 Zink, Th., Über die schlechte Reduktion einer Shimuravarietät. Compos. Math. **45**, 15-107 (1981).
51 Zink, Th., Examples of curves over $\mathbf{F}_{p^3}$ and coding theory, Proceedings of FCT, Cottbus 1983.

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Representations of Galois Groups Associated to Hilbert Modular Forms

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0.1. To holomorphic Hilbert modular newforms over a totally real field F one expects to be able to attach compatible systems of two dimensional λ-adic representations of $\mathrm{Gal}(\overline{F}/F)$. Techniques from algebraic geometry and analysis have been used by Eichler, Shimura, Deligne, Langlands, Ohta, Rogawski, Tunnel and Carayol to construct these representations in many but not all cases. More precisely let f be a holomorphic Hilbert modular newform (all whose weights are ≥ 2) then we may associate to f an automorphic representation $\pi = \otimes \pi_v$ of $GL_2(\mathbf{A}_F)$ such that π_v is discrete series for $v|\infty$. One can attach to f a number field E (generated by the coefficients of the Dirichlet series attached to f), and for each prime λ of E one would like to construct a continuous representation:

$$\rho_\lambda : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(\mathcal{O}_{E,\lambda})$$

which is unramified outside $\mathfrak{n}(\mathrm{N}\lambda)$, where $\mathfrak{n}$ is the level of f (or conductor of π); and such that if $\mathfrak{p}$ is a prime of F not dividing $\mathrm{N}\lambda$ then $\rho_\lambda|_{D_\mathfrak{p}}$, the decomposition group at $\mathfrak{p}$, is associated to $\pi_\mathfrak{p}$ via the local Langlands correspondence (or some twist thereof). For details see [**C**]. In particular this means that $\det \rho_\lambda$ corresponds to the central character of π, and that for $\mathfrak{p} \not| \mathfrak{n}(\mathrm{N}\lambda)$ then $\mathrm{tr}\,\rho_\lambda(\mathrm{Frob}_\mathfrak{p})$ is equal to the eigenvalue of the Hecke operator $T_\mathfrak{p}$ on f.

For $F = \mathbf{Q}$ these representations were found in the cohomology of elliptic modular curves. For $F \neq \mathbf{Q}$ Brylinski and Labesse found certain $2^{[F:\mathbf{Q}]}$ dimensional representations in the cohomology of the corresponding Hilbert-Blumental varieties. These are tensor products of Galois twists of the two dimensional representation one would like to construct. In the case $[F : \mathbf{Q}] = 2$ the tensor product can be factorised, but one apparently loses control of the trace of Frobenius on the factors—that is one can not ensure coherence between the trace of Frobenius at different primes. More precisely if $p = \mathfrak{p}_1\mathfrak{p}_2$ and $q = \mathfrak{q}_1\mathfrak{q}_2$ split in F then one can not rule out by this method that:

$$\begin{array}{ll} \mathrm{tr}\,\rho_\lambda \mathrm{Frob}_{\mathfrak{p}_1} = a_{\mathfrak{p}_1} & \mathrm{tr}\,\rho_\lambda \mathrm{Frob}_{\mathfrak{q}_1} = a_{\mathfrak{q}_2} \\ \mathrm{tr}\,\rho_\lambda \mathrm{Frob}_{\mathfrak{p}_2} = a_{\mathfrak{p}_2} & \mathrm{tr}\,\rho_\lambda \mathrm{Frob}_{\mathfrak{q}_2} = a_{\mathfrak{q}_1} \end{array}$$

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where $a_{\mathfrak{p}}$ is the expected trace of $\rho_\lambda \mathrm{Frob}_{\mathfrak{p}}$.

If $F/\mathbf{Q}$ is of odd degree or if some π_v is special or supercuspidal for some finite place v then by the correspondence of Jacquet-Langlands one can associate to π an automorphic representation of a quaternion algebra B/F which is ramified at all but one of the infinite places of F. The desired representations can then be found in the cohomology of Shimura curves associated to B. This theory was completed by Carayol [**C**], where he controlled the restriction of the representations ρ_λ to the decomposition groups of bad primes not dividing $\mathbf{N}\lambda$. However in the case $[F:\mathbf{Q}]$ even and π_v principal series at all finite places v the techniques of algebraic geometry and analysis do not appear to construct the desired two dimensional representations. The purpose of this talk is to describe the construction of these missing representations by using the theory of congruences between modular forms. Details of the argument can be found in [**T**].

This method was first employed by Wiles [**W**], who constructed λ-adic representations in the case that π is ordinary at λ, i.e. that the Hecke operators $T_{\mathfrak{p}}$ have eigenvalues which are λ-adic units for all primes $\mathfrak{p}$ of F dividing $\mathbf{N}\lambda$. One expects that such a π should be ordinary at infinitely many λ (in most cases in a set of λ of Dirichlet density one), but at present one can not show in general that it will be ordinary at any prime λ.

The idea of this method is that if f is the Hilbert modular newform corresponding to π, then one should be able to find forms, or linear combinations of forms, which are special at one place and arbitrarily congruent to f modulo powers of λ.

Throughout we shall restrict to the case $[F:\mathbf{Q}]$ even, as the problem is solved for the case $[F:\mathbf{Q}]$ odd.

After this article was written, Blasius and Rogawski [**BR**] found a completely different construction of the desired λ-adic representations. Their construction relies on the fact that SL_2 is an endoscopic group of $SU(2,1)$. Their method has the advantage that it shows that the representations obtained are of Hodge-Tate type. However they are unable to describe the restriction of the representation to the decomposition group at bad primes.

0.2. We first set up some notation. We shall consider Hilbert modular forms from a fairly classical point of view.

Let $F/\mathbf{Q}$ be totally real of even degree d. Let I denote the set of embeddings $F \hookrightarrow \mathbf{R}$. Let $\mathbf{A}_F = F_f \times F_\infty$ denote its adele ring. If $k \in \mathbf{Z}^I_{\geq 2}$ with each component of the same parity and if $U \subset GL_2(F_f)$ is an open compact subgroup, we define the space of cusp forms $S_k(U)$ to be those functions:

$$\phi : GL_2(F)\backslash GL_2(\mathbf{A}_F) \longrightarrow \mathbf{C}$$

which satisfy the following conditions:

1. $\phi(gu) = \phi(g)$ for $u \in U$
2. if $g \in GL_2(F_f)$ then the function $\mathcal{Z}^I \to \mathbf{C}$ defined by:

$$\gamma z_0 \mapsto j(\gamma, z_0)^k \det(\gamma)^{t-k-v} f(g\gamma)$$

for $\gamma \in GL_2(F_\infty)$ is well defined and holomorphic. Here:

- $\mathcal{Z}$ denotes the upper half complex plane
- $z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathcal{Z}^I$
- $t = (1, \ldots, 1) \in \mathbf{Z}^I$ and $v \in \mathbf{Z}^I$ is chosen such that each $v_\tau \geq 0$, some $v_\tau = 0$ and $k+2v = (\mu+2)t$ for some $\mu \in \mathbf{Z}_{\geq 0}$
- $j : GL_2(F_\infty) \times \mathcal{Z}^I \to \mathbf{C}^I$ by:

$$\begin{pmatrix} * & * \\ c_\tau & d_\tau \end{pmatrix} \times z_\tau \longmapsto (c_\tau z_\tau + d_\tau)$$

- if $z \in \mathbf{C}^I$ and $n \in \mathbf{Z}^I$ then $z^n = \prod_I z_\tau^{n_\tau}$

3. ϕ is cuspidal, i.e.

$$\int_{F\backslash \mathbf{A}_F} \phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) da = 0$$

for all $g \in GL_2(\mathbf{A}_F)$.

If $g \in GL_2(F_f)$ and if U and U' are open compact subgroups we define maps:

$$\begin{array}{rcl} x : S_k(U) & \longrightarrow & S_k(x^{-1}Ux) \\ \phi & \longmapsto & \phi(\_x^{-1}) \end{array}$$

and:

$$\begin{array}{rcl} [UxU'] : S_k(U) & \longrightarrow & S_k(U') \\ \phi & \longmapsto & \sum_i \phi(\_x_i^{-1}) \end{array}$$

where $UxU' = \coprod_i Ux_i$.

Now we fix an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ and let $U(\mathfrak{n})$ denote the subgroup of $\prod GL_2(\mathcal{O}_{F,v})$ consisting of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \in \mathfrak{n}$ and $(a-1) \in \mathfrak{n}$. Further if $\mathfrak{p}$ is a prime of $\mathcal{O}_F$ and $\mathfrak{p} \not| \mathfrak{n}$ then let $U(\mathfrak{n}, \mathfrak{p})$ denote the subgroup of $U(\mathfrak{n})$ consisting of those elements which also satisfy $c \in \mathfrak{p}$. We shall write $S_k(\mathfrak{n})$ (resp. $S_k(\mathfrak{n}, \mathfrak{p})$) for $S_k(U(\mathfrak{n}))$ (resp. $S_k(U(\mathfrak{n}, \mathfrak{p}))$). Let $T_\mathfrak{q}$ denote the Hecke operator $[U\begin{pmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{q} \end{pmatrix}U]$, where $\varpi_\mathfrak{q} \in F_f$ is 1 away from $\mathfrak{q}$ and a uniformiser at $\mathfrak{q}$. Also let $S_\mathfrak{a}$ denote $[U\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}U]$ where $\alpha = \prod_\mathfrak{q} \varpi_\mathfrak{q}^{v_\mathfrak{q}(\mathfrak{a})}$. These definitions are independent of the choice of $\varpi_\mathfrak{q}$ in every case where we shall use them. Let $\mathbf{T}(\mathfrak{n})$ denote the abstract Hecke ring generated over $\mathbf{Z}$ by the operators $T_\mathfrak{q}$ and $S_\mathfrak{a}$ for $\mathfrak{a}$ prime to $\mathfrak{n}$. Let $\mathbf{T}(\mathfrak{n}, \mathfrak{p})$ denote the subring generated by the $T_\mathfrak{q}$ for $\mathfrak{q} \neq \mathfrak{p}$ and the $S_\mathfrak{a}$ for $\mathfrak{a}$ prime to $\mathfrak{np}$.

The main result whose proof we shall outline is:

THEOREM 1. *Let $[F : \mathbf{Q}]$ be even, $k \in \mathbf{Z}^I_{\geq 2}$ with all the k_τ having the same parity, $f \in S_k(\mathfrak{n})$ an eigenform of $\mathbf{T}(\mathfrak{n})$ and $\mathcal{O}_f$ the integers of a number field such that there exists a morphism $\theta_f : \mathbf{T}(\mathfrak{n}) \to \mathcal{O}_f$ satisfying $f|T = \theta_f(T)f$. Then if λ is a prime of $\mathcal{O}_f$ there exists a continuous representation:*

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(\mathcal{O}_{f,\lambda})$$

which is unramified outside $\mathfrak{n}(\mathbf{N}\lambda)$ and such that:

1. $\det \rho = \chi$, *where χ is the continuous character unramified outside $\mathfrak{n}(\mathbf{N}\lambda)$, which is defined by:*

$$\chi(\mathrm{Frob}_\mathfrak{q}) = \theta_f(S_\mathfrak{q})(\mathbf{N}\mathfrak{q})$$

   for each prime $\mathfrak{q}$ of F not dividing $\mathfrak{n}(\mathbf{N}\mathfrak{q})$;

2. *if $\mathfrak{q} \not| \mathfrak{n}(\mathbf{N}\lambda)$ is a prime of F then:*

$$\mathrm{tr}\, \rho(\mathrm{Frob}_\mathfrak{q}) = \theta_f(T_\mathfrak{q})$$

3. *if $\mathfrak{q}$ is a prime of F dividing $\mathfrak{n}$ but not $\mathbf{N}\lambda$ then either $\theta_f(T_\mathfrak{q}) = 0$ or for every σ in the decomposition group at $\mathfrak{q}$ lying above* $\mathrm{Frob}_\mathfrak{q}$ *we have:*

$$\mathrm{tr}\, \rho(\sigma) = \theta_f(T_\mathfrak{q}) + \chi(\sigma)\theta_f(T_\mathfrak{q})^{-1}$$

This result is enough to remove the restriction in theorem A of [**C**] that in the case $[F:\mathbf{Q}]$ even π_v is essentially square integrable for some finite place v (see the second paragraph of section 0.3 of [**C**]). For if $\pi = \otimes \pi_v$ is a cuspidal automorphic representation of $GL_2(\mathbf{A}_F)$ with π_v discrete series for $v|\infty$ and π_v principal series for v finite, then we may associate to π a newform $f \in S_k(\mathfrak{n})$ for some k and $\mathfrak{n}$ as in the theorem, and hence a λ-adic Galois representation ρ as in the theorem. For $\mathfrak{q}$ a prime of F not dividing $\mathbf{N}\lambda$ the restriction of ρ to the decomposition group $D_\mathfrak{q}$ at $\mathfrak{q}$ is determined unless $\pi_\mathfrak{q} = \pi(\chi_1, \chi_2)$ with χ_1 and χ_2 both ramified. However in this case we may twist π by some finite character $\psi = \prod \psi_v$ such that $\chi_1 \psi_\mathfrak{q}$ is unramified. Then if we apply the above theorem to $\pi \otimes \psi$ we may deduce the restriction of $\rho \otimes \chi$ and hence ρ to $D_\mathfrak{q}$.

In this talk we shall not discuss the proof of part three of the theorem, it just requires slightly more care (see [**T**] for details).

The proof of theorem one is based on the following result about congruences. Recall that we have an embedding (for $\mathfrak{p} \not| \mathfrak{n}$ a prime):

$$S_k(\mathfrak{n})^2 \hookrightarrow S_k(\mathfrak{n}, \mathfrak{p})$$

given by:

$$(f_1, f_2) \longmapsto f_1 + f_2 | \eta_\mathfrak{p}$$

where $\eta_\mathfrak{p} = \begin{pmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}$. This is compatible with the actions of $\mathbf{T}(\mathfrak{n}, \mathfrak{p})$, and we have a decomposition over $\mathbf{T}(\mathfrak{n}, \mathfrak{p})$:

$$S_k(\mathfrak{n}, \mathfrak{p}) = S_k(\mathfrak{n})^2 \oplus S_k(\mathfrak{n}, \mathfrak{p})^{\text{new}}$$

We shall write $\mathbf{T}_k(\mathfrak{n}, \mathfrak{p})^{\text{old}}$ and $\mathbf{T}_k(\mathfrak{n}, \mathfrak{p})^{\text{new}}$ for the image of $\mathbf{T}(\mathfrak{n}, \mathfrak{p})$ in $End(S_k(\mathfrak{n})^2)$ and $End(S_k(\mathfrak{n}, \mathfrak{p})^{\text{new}})$ respectively. (Note that here "new" and "old" refer only to the place $\mathfrak{p}$.) Then we have:

THEOREM 2. *Let $[F:\mathbf{Q}]$ be even, $k \in \mathbf{Z}^I_{\geq 2}$ with all the k_τ having the same parity, $f \in S_k(\mathfrak{n})$ an eigenform of $\mathbf{T}(\mathfrak{n})$ and $\mathcal{O}_f$ the integers of a number field such that there exists a morphism $\theta_f : \mathbf{T}(\mathfrak{n}) \to \mathcal{O}_f$ satisfying $f|T = \theta_f(T) f$. Then if λ is a prime of $\mathcal{O}_f$ there exists a constant C such that for any prime $\mathfrak{p}$ of F with $\mathfrak{p} \not| \mathfrak{n}(\mathbf{N}\lambda)$ there is a homomorphism:*

$$\mathbf{T}_k(\mathfrak{n}, \mathfrak{p})^{\text{new}} \otimes_\mathbf{Z} \mathcal{O}_f \to \mathcal{O}_f / \lambda^{r_\mathfrak{p}}$$

given by:

$$T \longmapsto \theta_f(T)$$

and where:

$$r_{\mathfrak{p}} \geq v_\lambda(\theta_f(T_{\mathfrak{p}}^2 - S_{\mathfrak{p}}(1+\mathbf{N}\mathfrak{p})^2)) - v_\lambda(1+\mathbf{N}\mathfrak{p}) - C$$

Perhaps it would be useful to say a few words in explanation of the theorem. Roughly speaking the point is that the morphism $\theta_f : \mathbf{T}_k(\mathfrak{n},\mathfrak{p}) \to \mathcal{O}_f$ which a priori factors through $\mathbf{T}(\mathfrak{n},\mathfrak{p})^{\mathrm{old}}$, also factors through $\mathbf{T}(\mathfrak{n},\mathfrak{p})^{\mathrm{new}}$ when considered modulo a certain ideal, which may be essentially calculated as $\theta_f(T_{\mathfrak{p}}^2 - S_{\mathfrak{p}}(1+\mathbf{N}\mathfrak{p})^2)$. We can not quite prove this, infact it is probably not quite true, but we can show the result modulo an ideal that differs from this one by an easily controlled error term. Alternatively one could understand the theorem as asserting the existence of a form $g \in S_k(\mathfrak{n},\mathfrak{p})^{\mathrm{new}}$ with $f \equiv g$ modulo an ideal which is essentially $\theta_f(T_{\mathfrak{p}}^2 - S_{\mathfrak{p}}(1+\mathbf{N}\mathfrak{p})^2)$. For a discussion of these two ways of looking at congruences between modular forms the reader might like to consult [**R**] (this only considers the case $F = \mathbf{Q}$, but conceptually there is no difference).

It may be of interest to note that the expression $\theta_f(T_{\mathfrak{p}}^2 - S_{\mathfrak{p}}(1 + \mathbf{N}\mathfrak{p})^2)$ may be rewritten as:

$$-\theta_f(S_{\mathfrak{p}})(\alpha/\beta - \mathbf{N}\mathfrak{p})(\beta/\alpha - \mathbf{N}\mathfrak{p})$$

where α and β are the roots of the Hecke polynomial:

$$X^2 - \theta_f(T_{\mathfrak{p}})X + \theta_f(S_{\mathfrak{p}})\mathbf{N}\mathfrak{p}$$

As $\lambda \not| \mathbf{N}\mathfrak{p}$ the first term is a λ-adic unit and so λ divides this expression if and only if α/β or β/α is congruent to $\mathbf{N}\mathfrak{p}$ modulo λ, i.e. the parameters of the principal series representation of $GL_2(F_{\mathfrak{p}})$ corresponding to f look modulo λ as if they correspond to a special representation. Recall that if $g \in S_k(\mathfrak{n},\mathfrak{p})^{\mathrm{new}}$ is an eigenform of the Hecke algebra then it corresponds to an automorphic representation which is special at $\mathfrak{p}$, so in some sense this theorem is saying that if an automorphic representation (with a suitable infinity type) looks modulo λ as if it is special at $\mathfrak{p}$ then there really is an automorphic representation (of the same infinity type) which is special at $\mathfrak{p}$ and is congruent to the original one modulo λ.

Congruences of this sort were first studied by Ribet [**R**] in the case $F = \mathbf{Q}$ and $k = 2$. Similar results have been obtained in the case

$F = \mathbf{Q}$ and $k > 2$ by Fred Diamond and in special cases by Livne and Jordan. The latter (independently) employ a method that seems to be similar to ours.

0.3. We shall first outline how to derive theorem one from theorem two. The idea is that eigenforms in $S_k(\mathfrak{n}, \mathfrak{p})^{\text{new}}$ correspond to automorphic representations which are special at $\mathfrak{p}$ and thus to which we can associate the desired Galois representations. The principal difficulty is that theorem two does not imply that there is an eigenform $g \in S_k(\mathfrak{n}, \mathfrak{p})^{\text{new}}$ with $g \equiv f \bmod \lambda^{r_\mathfrak{p}}$. Thus we can not construct from theorem two a representation $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathcal{O}_f/\lambda^{r_\mathfrak{p}})$ with the properties described in theorem one. Another way of saying this is that we have a representation $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{T}_k(\mathfrak{n}, \mathfrak{p})^{\text{new}} \otimes \mathbf{Q}_l)$ but not one to $GL_2(\mathbf{T}_k(\mathfrak{n}, \mathfrak{p})^{\text{new}} \otimes \mathbf{Z}_l)$.

We get round this problem using Wiles' notion of a pseudo-representation, which we recall (see [W]). Let R be a ring and G a group with a distinguished element, c, of order two. By a pseudo-representation r of G into R we mean a collection of maps:

$$\begin{array}{lcl} A : G & \longrightarrow & R \\ D : G & \longrightarrow & R \\ T : G & \longrightarrow & R \\ X : G \times G & \longrightarrow & R \end{array}$$

satisfying the following polynomial conditions:

$$\begin{array}{l} 2A_{\sigma\tau} = A_\sigma A_\tau + X_{\sigma,\tau} \\ 2D_{\sigma\tau} = D_\sigma D_\tau + X_{\tau,\sigma} \\ A_\sigma = T_\sigma + T_{c\sigma} \\ D_\sigma = T_\sigma - T_{c\sigma} \\ T_1 = T_c = 2 \\ X_{c,\sigma} = X_{\sigma,c} = 0 \\ X_{\sigma,\tau} X_{\rho,\eta} = X_{\sigma,\eta} X_{\rho,\tau} \\ 4X_{\sigma\tau,\rho\tau} = A_\sigma A_\tau X_{\tau,\rho} + A_\eta D_\tau X_{\sigma,\rho} + A_\sigma D_\rho X_{\tau,\eta} + D_\tau D_\rho X_{\sigma,\tau} \end{array}$$

We define $\mathrm{tr}\,(r)$ to be T and $\mathrm{Det}\,(r)$ to be $\sigma \mapsto A_\sigma D_\sigma - X_{\sigma,\sigma}$. Note that a pseudo-representation is determined by its trace. The principal properties of pseudo-representations are the following:

1. If $\rho : G \to GL_2(R)$ is a representation with $\rho(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then there is a unique pseudo-representation r_ρ of G valued in R

(in fact in the subring generated by the traces of ρ) with $\operatorname{tr}\rho = \operatorname{tr} r_\rho$ (and $\operatorname{Det} r_\rho = 4\det\rho$). Explicitly if $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ then:

$$\begin{aligned} A_\sigma &= 2a_\sigma \\ D_\sigma &= 2d_\sigma \\ T_\sigma &= a_\sigma + d_\sigma \\ X_{\sigma,\tau} &= 4b_\sigma c_\tau \end{aligned}$$

2. If r is a pseudo-representation of G valued in a field R then there is a semi-simple representation $\rho_r : G \to GL_2(R)$ with $\operatorname{tr}\rho_r = \operatorname{tr} r$.

The proofs are elementary (see [**W**]). If ρ (resp. r) is continuous so is r_ρ (resp. ρ_r).

Now it follows from the results described in [**C**] that there is a continuous representation ρ of $\operatorname{Gal}(\overline{F}/F)$ into $GL_2(\mathbf{T}_k(\mathfrak{n},\mathfrak{p})^{\mathrm{new}} \otimes \mathbf{Q}_l)$, unramified outside $\mathfrak{n}\mathfrak{p}l$ and satisfying $\operatorname{tr}\rho(\operatorname{Frob}\mathfrak{q}) = T_\mathfrak{q}$ for $\mathfrak{q} \not| \mathfrak{n}\mathfrak{p}l$ and $\rho(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where c denotes complex conjugation. Thus we get a pseudo-representation of $\operatorname{Gal}(\overline{F}/F)$ into $\mathbf{T}_k(\mathfrak{n},\mathfrak{p})^{\mathrm{new}} \otimes \mathbf{Z}_l$ and so into $\mathcal{O}_f/\lambda^{r_\mathfrak{p}}$.

Suppose now that we can find $\mathfrak{p}$ with $r_\mathfrak{p}$ arbitrarily large. Then as a continuous pseudo-representation of $\operatorname{Gal}(\overline{F}/F)$, which is unramified at all but finitely many primes, is determined by its trace at all but finitely many Frobenius elements (by the Cebotarev Density Theorem) we could piece together these pseudo-representations to get one valued in $\mathcal{O}_{f,\lambda}$ with $\operatorname{tr} r(\operatorname{Frob}_\mathfrak{q}) = \theta_f(T_\mathfrak{q})$ for $\mathfrak{q} \not| \mathfrak{n}\mathrm{N}\lambda$. Then by the second property of pseudo-representations we would get the desired representation into GL_2 of the field of fractions of $\mathcal{O}_{f,\lambda}$, which we wanted to associate to f. This representation the stabilises a lattice and the proof would be complete.

Finally to see that as $\mathfrak{p}$ varies $r_\mathfrak{p}$ becomes arbitrarily large we use an argument of Wiles based which uses the representation constructed by Brylinski and Labesse. Let $\alpha_\mathfrak{p}, \beta_\mathfrak{p}$ denote the roots of $X^2 - \theta_f(T_\mathfrak{p})X + \theta_f(S_\mathfrak{p})\mathbf{N}\mathfrak{p} = 0$, then as we have remarked:

$$\theta_f(T_\mathfrak{p}^2 - S_\mathfrak{p}(1+\mathbf{N}\mathfrak{p})^2) = -\theta_f(S_\mathfrak{p})(\alpha_\mathfrak{p}/\beta_\mathfrak{p} - \mathbf{N}\mathfrak{p})(\beta_\mathfrak{p}/\alpha_\mathfrak{p} - \mathbf{N}\mathfrak{p})$$

Thus it will do to show that for any r we can find $\mathfrak{p}$ with $\mathbf{N}\mathfrak{p} \equiv \alpha_\mathfrak{p}/\beta_\mathfrak{p} \equiv 1 \bmod \lambda^r$. But for almost all $\mathfrak{p}$ which split in the normal closure of F $\alpha_{\mathfrak{p}}/\beta_{\mathfrak{p}}$ is the ratio of two roots of the characteristic polynomial of $\sigma(\mathrm{Frob}_{\mathfrak{p}})$, where σ is the λ-adic representation constructed by Brylinski and Labesse. Thus almost all primes $\mathfrak{p}$ which lie above a prime of $\mathbf{Q}$ which splits completely in the composite of the $(\mathrm{N}\lambda)^r$-cyclotomic field, the normal closure of F and the fixed field of σ mod λ^r will have the desired property.

0.4. We now explain the proof of theorem two. It is based on a method, implicit in Ribet's article [**R**], for finding congruences between the actions of commutative algebras on spaces which are orthogonal complements. In fact we have the following lemma whose proof is elementary:

LEMMA 1. *Let $V_1 \subset V_2$ be vector spaces and $L_1 \subset L_2$ $\mathbf{Z}$-lattices in V_1 and V_2 respectively. Let $\mathbf{T}$ be a commutative algebra which acts on V_2 and which preserves L_1 and L_2. Let $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$ be non-degenerate pairings on V_1 and V_2 respectively. If $T \in \mathbf{T}$ let T^* denote its adjoint with respect to $\langle\, ,\rangle_2$. Also let i^* denote the adjoint of the embedding $i : V_1 \hookrightarrow V_2$ (i.e. $\langle v_1, i^*v_2\rangle_1 = \langle v_1, v_2\rangle_2$). Assume that:*

1. $V_2 = V_1 \oplus V_1^{\perp_2}$
2. *If $T \in \mathbf{T}$ then T^* preserves V_1 and $T^*|_{V_1}$ is the adjoint of T with respect to $\langle\, ,\rangle_1$*
3. $L_2 \cap V_1 = L_1$
4. $\langle L_i, L_i\rangle_i \subset \mathbf{Z}$
5. *$\langle\, ,\rangle_2$ induces $L_2 \cong Hom(L_2, \mathbf{Z})$*

Then if $\mathbf{T}^{\mathrm{old}}$ and $\mathbf{T}^{\mathrm{new}}$ denote the images of $\mathbf{T}$ in $End(V_1)$ and $End(V_1^{\perp_2})$, respectively, we have that:

$$L_1/(i^*iL_1 \cap L_1)$$

is a module for $\mathbf{T}^{\mathrm{old}}$ and $\mathbf{T}^{\mathrm{new}}$ (a "congruence module").

COROLLARY 1. *With the same notation, if $v \in L_1$ is primitive (i.e. $\mathbf{Q}v \cap L_1 = \mathbf{Z}v$) and is an eigenvector of $\mathbf{T}$, say $Tv = \theta(T)v$; and if $i^*iv \in \alpha L_1$ then there is a map:*

$$\begin{array}{rcl} \mathbf{T}^{\mathrm{new}} & \to & \mathbf{Z}/\alpha\mathbf{Z} \\ T & \longmapsto & \theta(T) \end{array}$$

It should be noted that if $\langle\, , \rangle_1 = \langle\, , \rangle_2|_{V_1\times V_1}$ then the congruence module is trivial and we detect no congruences. At the other extreme if $\langle\, , \rangle_1$ induces $L_1 \cong Hom(L_1, \mathbf{Z})$ then we get the "best possible" congruence module. We can easily prove slight generalisations of this result where $\mathbf{Z}$ is replaced by the integers of a number field and properties 3),4),5) are replaced by the corresponding properties up to a fixed (non-zero) error C (e.g. 3) would become $C(L_2 \cap V_1) \subset L_1$), in which case we also introduce an error in the conclusion which depends only on C.

It may be useful to give an example of this lemma in a very simple situation. Take $\mathbf{T} = L_2$ to be the subring of $\mathbf{Z}^2$ consisting of pairs of integers which are congruent modulo n, for some integer n. Take L_1 to be the ideal consisting of pairs $(na, 0)$ for $a \in \mathbf{Z}$. Define:

$$\begin{aligned} \langle (a_1, a_2), (b_1, b_2)\rangle_2 &= (a_1b_1 - a_2b_2)/n \\ \langle (a_1, 0), (b_1, 0)\rangle_1 &= a_1b_1/n^2 \end{aligned}$$

Then $i^*(b_1, b_2) = (nb_1, 0)$ and $V_1^{\perp_2}$ consists of elements of the form $(0, a)$.It is easy to see that all the assumptions of the lemma are satisfied. $L_1/i^*L_1 \cong \mathbf{Z}/n\mathbf{Z}$ and indeed, as the lemma predicts, there is a map $\mathbf{T} \to \mathbf{Z}/n\mathbf{Z}$ ($(a_1, a_2) \mapsto a_1 \bmod n$) which factors via both $\mathbf{T}^{\text{old}} \cong \mathbf{Z}$ ($(a_1, a_2) \mapsto a_1$) and $\mathbf{T}^{\text{new}} \cong \mathbf{Z}$ ($(a_1, a_2) \mapsto a_2$).

We wish to apply this lemma to the case $V_1 = S_k(\mathfrak{n})^2$ and $V_2 = S_k(\mathfrak{n}, \mathfrak{p})$. It seems that with any reasonable choice of pairings (i.e. which behave well with respect to the action of $GL_2(F_f)$) 1) and 2) will be true and i^*i will be represented by the matrix (acting from the right):

$$\begin{pmatrix} 1 + \mathbf{N}\mathfrak{p} & (\mathbf{N}\mathfrak{p})^{\mu} S_{\mathfrak{p}}^{-1} T_{\mathfrak{p}} \\ T_{\mathfrak{p}} & (\mathbf{N}\mathfrak{p})^{\mu}(1 + \mathbf{N}\mathfrak{p}) \end{pmatrix}$$

for some integer μ (in fact $\mu = \min_I(k_\tau - 2)$). If we can find Hecke invariant lattices such that 3),4),5) hold, atleast up to an error independent of $\mathfrak{p}$, then applying the corollary to some multiple of:

$$((1 + \mathbf{N}\mathfrak{p})S_{\mathfrak{p}} f, -T_{\mathfrak{p}} f) \in S_k(\mathfrak{n})^2$$

will prove the theorem. Taking the lattice to be the usual space of integral cusp forms (defined via the Fourier expansion) and taking the product to be the Peterssen inner product, will fail to satisfy 4) and 5). Ribet used the integral structure coming from the Betti cohomology of modular curves. The only difficulty in this case was 3), which followed from a result of Ihara. But in any more general situation 3) seems to be very difficult. The approach we take for $[F : \mathbf{Q}]$ even is to interpret $S_k(\mathfrak{n})$ and $S_k(\mathfrak{n}, \mathfrak{p})$ as spaces of automorphic forms on totally definite quaternion algebras. This gives a natural integral structure and pairings. 1),2),4),5) are easy and 3) becomes combinatorics.

To explain this let D be the quaternion algebra over F ramified at exactly the infinite places. Pick a finite totally imaginary extension K/F which is Galois over $\mathbf{Q}$, a maximal order $\mathcal{O}_D$ in D and compatible isomorphisms:

$$\begin{aligned} \mathcal{O}_D \otimes \mathcal{O}_K &\cong M_2(\mathcal{O}_K) \\ \mathcal{O}_D \otimes \mathcal{O}_{F,v} &\cong M_2(\mathcal{O}_{F,v}) \end{aligned}$$

for each finite place v of F. In particular we can identify the finite adeles of $D^\times$ with $GL_2(F_f)$.

Let $k \in \mathbf{Z}^I$ be as above and $\mathcal{O}_K \subset R \subset \mathbf{C}$ we define an R-module $L_k(R)$ to be $\bigotimes_I S^{k_\tau - 2}(R^2)$, where S^i denotes the i^{th} symmetric power (i.e. the maximal symmetric quotient of the i^{th} tensor power). Let μ denote $\min_I(k_\tau - 2)$. Then we may define:

1. A pairing $\langle , \rangle : L_k(R)^2 \to R$, which induces $L_k(R) \cong Hom(L_k(R), R)$
2. If $R \supset K$ an action of $D^\times$ on $L_k(R)$, such that if $\alpha \in D^\times$ then $\langle x\alpha, y\alpha \rangle = (\mathbf{N}\nu\alpha)^\mu \langle x, y \rangle$ (ν denotes the reduced norm); and such that if $R \supset R' \supset \mathcal{O}_K$ and $\alpha \in \mathcal{O}_D$ then $L_k(R')\alpha \subset L_k(R')$
3. An action of $GL_2(F_f)$ on the $\mathcal{O}_K$ lattices in $L_k(K)$, such that on $D^\times$ this agrees with the definition coming from 2)

(see [**T**] for details). Then for U an open compact subgroup of $GL_2(F_f)$ we define a space of automorphic forms $S_k^D(U)$ to be the set of maps:

$$\phi : GL_2(F_f)/U \longrightarrow L_k(\mathbf{C})$$

such that $\phi(\alpha x) = \phi(x)\alpha^{-1}$ for $\alpha \in D^\times$. Alternatively if $X(U)$ denotes the finite set $D^\times \backslash GL_2(F_f)/U$ then:

$$S_k^D(U) = \bigoplus_{[g] \in X(U)} L_k(\mathbf{C})^{D^\times \cap gUg^{-1}}$$

where $D^\times \cap gUg^{-1}$ acts on $L_k(\mathbf{C})$ via a finite group. If $k_\tau = 2$ for all τ then $S_k^D(U)$ contains a "trivial" subspace of functions which factor through $\det : GL_2(F_f) \to F_f^\times$. We denote this space $I_k(U)$. In all other cases we set $I_k(U) = (0)$. Then we set $\tilde{S}_k^D(U) = S_k^D(U)/I_k(U)$. If $x \in GL_2(F_f)$ we get a map:

$$\begin{array}{ccc} S_k^D(U) & \longrightarrow & S_k^D(x^{-1}Ux) \\ \phi & \longmapsto & \phi(\_x^{-1}) \end{array}$$

and similarly we may define Hecke operators. These maps take the spaces $I_k(U)$ to each other. It is an amazing theorem of Jacquet and Langlands and of Shimitzu that there is a system of isomorphisms:

$$i_u : \tilde{S}_k^D(U) \xrightarrow{\sim} S_k(U)$$

which are compatible with the action of $GL_2(F_f)$ and of the Hecke operators. We use this to define a new pairing and integral structure on $S_k(U)$.

Firstly if $\mathcal{O}_K \subset R \subset K$ define $S_k^D(U;R)$ to be those elements of $S_k^D(U)$ which satisfy:

$$\phi(g) \in L_k(R)g^{-1}$$

for all $g \in GL_2(F_f)$. Equivalently:

$$S_k^D(U;R) = \bigoplus_{[g]\in X(U)} (L_k(R)g^{-1})^{D^\times \cap gUg^{-1}}$$

We see at once that $S_k^D(U;R)$ is an R-lattice in $S_k^D(U)$. This integral structure is preserved by $[UxU']$ if $UxU' \subset \prod_v M_2(\mathcal{O}_{F,v})$.

Secondly we define a pairing on $S_k^D(U)$ by setting:

$$\langle \phi_1, \phi_2 \rangle_U = \sum_{[g]\in X(U)} \langle \phi_1(g), \phi_2(g) \rangle |\mathbf{N}\nu g|$$

where $|.|$ is the usual morphism $\mathbf{Q}_f^\times \to \mathbf{Q}_{>0}^\times$. This is well defined and it is not difficult to check that:

1. $\langle \phi_1 | [UxU'], \phi_2 \rangle_{U'} = |\mathbf{N}\nu x|^\mu \langle \phi_1, \phi_2 | [U'x^{-1}U] \rangle_U$ and (hence) if $i : S_k^D(\mathfrak{n})^2 \to S_k^D(\mathfrak{n}, \mathfrak{p})$ (with the obvious notation) by $(\phi_1, \phi_2) \mapsto \phi_1 + \phi_2 | \eta_\mathfrak{p}$ then:
   - $(iS_k^D(\mathfrak{n})^2)^\perp \cong S_k(\mathfrak{n}, \mathfrak{p})^{\mathrm{new}}$
   - $i^* \circ i = \begin{pmatrix} 1 + \mathbf{N}\mathfrak{p} & (\mathbf{N}\mathfrak{p})^\mu S_\mathfrak{p}^{-1} T_\mathfrak{p} \\ T_\mathfrak{p} & (\mathbf{N}\mathfrak{p})^\mu (1 + \mathbf{N}\mathfrak{p}) \end{pmatrix}$

2. There is a constant $C \neq 0$ such that for any $U \subset \prod_v GL_2(\mathcal{O}_{F,v})$ we have:
$$C\langle S_k^D(U;R), S_k^D(U;R)\rangle_U \subset R$$
and:
$$\langle \phi, S_k^D(U;R)\rangle_U \subset R \Rightarrow C\phi \in S_k^D(U;R)$$

Thus to complete the proof of theorem two we need only the following lemma (modulo some care in the case $k_\tau = 2$ for all $\tau \in I$, i.e. when $I_k(U) \neq (0)$):

LEMMA 2. *There is a constant $C \neq 0$ such that if $\mathfrak{p}$ is any prime of F with $\mathfrak{p} \not| \mathfrak{n}$ and we set $R = \mathcal{O}_K[(\mathbf{N}\mathfrak{p})^{-1}]$, then:*
$$C^{-1}i(S_k^D(\mathfrak{n};R)^2) \supset S_k^D(\mathfrak{n},\mathfrak{p};R) \cap i(S_k^D(\mathfrak{n})^2) \supset i(S_k^D(\mathfrak{n};R)^2)$$

First suppose that for some $\tau \in I$, $k_\tau \neq 2$. Then if $i(\phi_1, \phi_2) \in S_k(\mathfrak{n},\mathfrak{p};R)$ one sees that:
$$\phi_1(gv) \equiv \phi_1(g) \bmod L_k(R)g^{-1}$$
for all v in the subgroup $V_\mathfrak{p}$ of $GL_2(F_f)$ generated by $U(\mathfrak{n})$ and $\eta_\mathfrak{p} U(\mathfrak{n})\eta_\mathfrak{p}^{-1}$; and hence:
$$\phi_1(g) \equiv \phi_1(g)\alpha \bmod L_k(R)g^{-1}$$
for all $\alpha \in D^\times \cap gV_\mathfrak{p} g^{-1}$. Let $g_1, \ldots, g_r$ represent the points of $X(\mathfrak{n})$. Then we need only show that there are constants $C_i \neq 0$, which are independent of $\mathfrak{p}$, such that if $x \in L_k(K)$ and:
$$x \equiv x\alpha \bmod L_k(R)g_i^{-1}$$
for all $\alpha \in D^\times \cap g_i V_\mathfrak{p} g_i^{-1}$ then $C_i x \in L_k(R)g_i^{-1}$. This is not difficult to show because $V_\mathfrak{p} \supset SL_2(F_\mathfrak{p}).g_i U(\mathfrak{n})g_i^{-1}$ and so by the strong approximation theorem $D^\times \cap g_i V_\mathfrak{p} g_i^{-1}$ is big (in a way which is independent of $\mathfrak{p}$).

For $k_\tau = 2$ for all τ the result is even easier. We are in the following general situation. We have finite sets X $(= X(\mathfrak{n},\mathfrak{p}))$, Y_1 and Y_2 (both $= X(\mathfrak{n})$) together with maps $\pi_i : X \to Y_i$. We have functions $\phi_i : Y_i \to K$ such that $\pi_1^*\phi_1 + \pi_2^*\phi_2$ is valued in R. We claim that there are functions $\phi_i' : Y_i \to R$ such that $\pi_1^*\phi_1' + \pi_2^*\phi_2' = \pi_1^*\phi_1 + \pi_2^*\phi_2$. To see this partition X into classes C_j which are minimal subject to the condition that if $x \in C_j$ and $\pi_i x = \pi_i y$ for either $i = 1$ or 2 then $y \in C_j$. We may treat each C_j separately, so without loss of generality $X = C_1$. Adding a constant function to ϕ_1 and subtracting it from ϕ_2 we may further suppose $\phi_1(y) = 0$ for some $y \in Y_1$. With these extra conditions it is easy to check that ϕ_1 and ϕ_2 are valued in R. (Use the twin observations that $\phi_1(\pi_1 x) \in R \Leftrightarrow \phi_2(\pi_2 x) \in R$; and that for each $i = 1, 2$ we have that $\phi_i(\pi_i x) \in R \& \pi_i(x) = \pi_i(x') \Rightarrow \phi_i(\pi_i x') \in R$.)

References

[**BR**] D.Blasius and J.Rogawski, *Galois Representations for Hilbert Modular Forms*, preprint

[**C**] H.Carayol, *Sur les Représentations p-adiques Associées aux Formes Modulaires de Hilbert*, Ann. Sci. Ecole Norm. Sup. (4^e série) 19 (1986), 409-468

[**R**] K.Ribet, *Congruence Relations between Modular Forms*, Proc. I.C.M. (1983), 503-514

[**T**] R.Taylor, *On Galois Representations Associated to Hilbert Modular Forms*, to appear in Invent. Math.

[**W**] A.Wiles, *On Ordinary λ-adic Representations Associated to Modular Forms*, Invent. Math. 94 (1988), 529-573

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The Lefschetz Trace Formula for an Open Algebraic Surface

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INTRODUCTION

Our purpose is to give evidence for a conjecture of Deligne on the Lefschetz trace formula. An application of this conjecture is explained in the work of Flicker and Kazhdan [6].

Let X be a smooth algebraic variety over an algebraically closed field k of characteristic $p > 0$, which is connected of dimension d. We assume that X is defined over a finite field $\mathbf{F}_q$ and we denote by $\mathrm{Fr}_X : X \to X$ the corresponding Frobenius morphism.

A correspondence of X is a diagram of schemes over k:

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X.$$

We assume that b_1 is proper and b_2 is finite. Moreover for simplicity assume that B is irreducible of dimension d.

If we replace b_1 by $\mathrm{Fr}_X^n b_1$ in the diagram above, we get a new correspondence, which we denote by B_n. If q^n is greater than the degree of b_2, the correspondence B_n has only finitely many fixed points u, i. e. closed points $u \in B_n$, such that $\mathrm{Fr}_X^n b_1(u) = b_2(u) = x$. We will denote the fixed points of B_n by $\mathrm{Fix}\, B_n$.

Consider a commutative coefficient ring Λ which is either finite of characteristic prime to p or $\mathbf{Q}_\ell$ or $\bar{\mathbf{Q}}_\ell$ for $\ell \neq p$. Let L be a locally free sheaf of Λ-modules on X, which is defined over $\mathbf{F}_q$.

Assume we are given a morphism of Λ-sheaves $\kappa : b_1^* L \to b_2^* L$. Since b_1 is proper, we have a map $b_1^* : H_c^{\cdot}(X, L) \to H_c^{\cdot}(B, b_1^* L)$. Assume for the moment that Λ is $\mathbf{Q}_\ell$ or $\bar{\mathbf{Q}}_\ell$. We fix once for all an isomorphism $\mathbf{Z}_\ell(1) \simeq \mathbf{Z}_\ell$, i.e. we forget the Tate twist. We denote by $\hat{L} = \mathrm{Hom}(L, \Lambda)$ the dual sheaf. For any locally free Λ- sheaf F on B the cup product gives a pairing:

$$H_c^i(B, F) \times H^{2d-i}(B, \hat{F}) \to H_c^{2d}(B, \Lambda) \cong \Lambda.$$

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If we replace B by the smooth variety X this pairing is nondegenerate. Using these pairings one readily verifies that the natural map b_2^* : $H^{\cdot}(X,\hat{L}) \to H^{\cdot}(B, b_2^*\hat{L})$ gives us a map $b_{2*} : H_c^{\cdot}(B, b_2^*L) \to H_c^{\cdot}(X, L)$. Hence we get a map $\bar{\kappa} = b_{2*}H^{\cdot}(\kappa)b_1^* : H_c^{\cdot}(X, L) \to H_c^{\cdot}(X, L)$.

Since our sheaf L is defined over $\mathbf{F}_q$, we can multiply the map κ with a power of the Frobenius $F_L : \mathrm{Fr}_X^* L \to L$:

$$\kappa_n : (\mathrm{Fr}_X^n b_1)^* L \xrightarrow{b_1^* F_L^n} b_1^* L \xrightarrow{\kappa} b_2^* L.$$

We get an endomorphism $\bar{\kappa}_n$ of $H_c^{\cdot}(X, L)$.

Let u be a fixed point of B, and put $x = b_2(u)$. Then κ defines an endomorphism of L_x:

$$\kappa_u : L_x = (b_1^* L)_u \to (b_2^* L)_u = L_x.$$

Of course we may replace here B by B_n and κ by κ_n.

We denote by $\deg_u B/X$ the multiplicity of u in the cycle $b_2^*(x)$.

Deligne's Conjecture. *Let L be a locally constant $\bar{\mathbf{Q}}_\ell$-sheaf on X and $\kappa : b_1^* L \to b_2^* L$ a homomorphism. There is a number n_0 such that for $n > n_0$ the following formula holds:*

$$\mathrm{Tr}(\bar{\kappa}_n | H_c^{\cdot}(X, L)) = \sum_{u \in \mathrm{Fix}\, B_n} \deg_u B/X \cdot \mathrm{Tr}(\kappa_{n,u} | L_{b_2(u)}).$$

Here the left hand side is by definition $\sum (-1)^i \mathrm{Tr}(\bar{\kappa}_n | H_c^i(X, L))$. If the coefficient ring Λ is finite, one has to replace $H_c^{\cdot}(X, L)$ by the perfect complex $R\Gamma_c(X, L)$. We will show in section 3 that Deligne's conjecture for a finite coefficient ring Λ or a $\bar{\mathbf{Q}}_\ell$-sheaf with finite monodromy, if it is true for the sheaf $\mathbf{Q}_\ell$ and for the identity correspondence $\kappa : b_1^* \bar{\mathbf{Q}}_\ell = b_2^* \bar{\mathbf{Q}}_\ell$. We note that the case of a $\bar{\mathbf{Q}}_\ell$-sheaf with finite monodromy would be enough for the application of Kazhdan and Flicker.

One can conjecture that n_0 depends only on B and not on the sheaf L. There are two cases where this is true:

Theorem (Grothendieck, Seminaire Bourbaki N°279). *Assume that $B = X$ and that b_1 and b_2 are the identity. Then Deligne's conjecture is true for $n_0 = 0$.*

Indeed this is the trace formula for the Frobenius morphism. It was remarked by Deligne-Lusztig [4] that the theorem is also true with the weaker condition that b_2 is the identity and b_1 is an automorphism of finite order. This is because $\mathrm{Fr}_X b_1$ is then also a Frobenius morphism for a different $\mathbf{F}_q$-structure on X.

THEOREM (ILLUSIE, SGA 5, III B, 1.2). *If X is a curve Deligne's conjecture is true for $n_0 \geq \log_q \deg B/X$.*

Another proof of this theorem follows from Alibert [1]. We give also a proof of Deligne's conjecture for curves, but our n_0 depends on the sheaf L (see example (4.2) below).

Brylinski and Labesse [2] have shown the conjecture of Deligne for the Hecke correspondences of Hilbert-Blumenthal varieties. They show that locally at infinity the Hecke correspondences behave essentially like products of correspondences of curves. This allows use of the theorem of Illusie above.

Our main result is:

THEOREM. *Let X be a surface. Assume that Λ is finite or that L is a $\bar{\mathbb{Q}}_\ell$-sheaf with finite monodromy. Then Deligne's conjecture is true.*

The proof of this theorem is as follows. By the previous result it is enough to consider the case where $L = \bar{\mathbb{Q}}_\ell$ and κ is the identity $b_1^*\bar{\mathbb{Q}}_\ell = b_2^*\bar{\mathbb{Q}}_\ell$. Moreover we can replace B by its normalization without changing the map $\bar{\kappa}$ on the cohomology. We choose a smooth compactification $X \to Y$, which is defined over $\mathbb{F}_q$, such that $Y - X = A$ is a divisor with normal crossings. Let D be the normalization of $Y \times Y$ in B. More precisely, let $\bar{B}$ be the closure of the image of the map $B \to X \times X \to Y \times Y$. Then $B \to \bar{B}$ is a dominant quasi finite map. D is by definition the normalization of $\bar{B}$ in the function field of B. We get a correspondence $Y \xleftarrow{d_1} D \xrightarrow{d_2} Y$. We assume that this correspondence has only isolated sufficiently high power of the Frobenius. Then we prove in section 4 a trace formula of the following type:

$$\operatorname{Tr}(\bar{\kappa}|H_c(X, \bar{\mathbb{Q}}_\ell)) = \sum_{u \in \operatorname{Fix} B} (B \cdot \Delta_X)_u + \sum_{\substack{u \in \operatorname{Fix} D \\ u \in \operatorname{Fix} B}} LT_u(\kappa)$$

Here $\Delta_X \subset X \times X$ is the diagonal. The numbers LT_u are defined in terms of intersection multiplicities on $Y \times Y$, on the desingularization of D, and on $A_i \times A_i$, where A_i are the components of A passing through u.

The trace formula of Verdier SGA 5 III 4.7 is exactly of the type above, but except for curves almost nothing is known about the local terms LT_u. It is possible to show that our local terms agree with those defined by Verdier.

In section 5 we prove a lemma from commutative algebra, which implies the vanishing of $LT_u(\kappa_n)$ for big n.

Section contains generalities on cohomological correspondences. We recommend the reader to start with section 2.

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1. Correspondences

Let S be the spectrum of an algebraically closed field k. We call a correspondence a diagram of schemes of finite type over S of the form:

$$Y \xleftarrow{b_1} B \xrightarrow{b_2} X \tag{1.1}$$

We will consider sheaves over a finite coefficient ring Λ of order prime to the characteristic of S. Sometimes we also use l-adic sheaves. In this case Λ will be $\mathbb{Q}_\ell$ or $\bar{\mathbb{Q}}_\ell$. We will work in the subcategory $D^b_{tfc}(X, \Lambda)$ of objects of the derived category, that may be represented by a bounded complex of flat Λ-modules with constructible cohomology. For the definition of the derived category in the case of l-adic sheaves we refer the reader to U. Jannsen [7] or P. Deligne [2]. In fact, we need D^b_{tfc} only for finite coefficients.

If $f : X \to Y$ is a morphism of finite type, we denote by f_*, $f_!$, f^*, $f^!$, the usual functors between the derived categories. If we use the direct image in the category of sheaves, we write $R^0 f_*$, etc..

Consider objects $F \in D^b_{tfc}(Y, \Lambda)$ and $G \in D^b_{tfc}(X, \Lambda)$. A Verdier-correspondence is a morphism:

$$b_1^* F \to b_2^! G$$

1.2 Lemma. *Let $b_2 : B \to X$ be a morphism of S-schemes of finite type. We assume that X is smooth and equidimensional. We put $c = \dim X - \dim B$. Let $j : U \to B$ be a smooth open subset, such that U is equidimensional and its complement has smaller dimension than B. Then we have $R^n b_2^! \Lambda = 0$ for $n < 2c$, and $R^{2c} b_2^! \Lambda = R^0 j_* \Lambda(-c)$.*

Proof: The vanishing $R^n b_2^! \Lambda = 0$ follows from SGA $4\frac{1}{2}$, Cycle 2.2.8. For the last assertion we consider first the case where B is smooth and equidimensional. We put $\dim B = N$ and $\dim X = n$. Denote by $\pi : B \to S$ and $\rho : X \to S$ the structural morphisms. By Poincaré duality we have canonical isomorphisms

$$\pi^! \Lambda = \Lambda[2N](N), \qquad \rho^! \Lambda = \Lambda[2n](n)$$

Hence we get:

$$b_2^! \Lambda = b_2^! \rho^! \Lambda[-2n](-n) = \pi^! \Lambda[-2n](-n) = \Lambda[2(N-n)](N-n).$$

This settles the case where B is smooth and equidimensional.

In the general case let T be the complement of U in B. Consider the diagram:

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "u_2"'] & B \arrow[d, "b_2"] & T \arrow[l, "t"'] \arrow[dl, "c_2"] \\ & X & \end{tikzcd}$$

We have the triangle: $t_* t^! b_2^! \Lambda \to b_2^! \Lambda \to j_* j^* b_2^! \Lambda$.

The previous case shows: $j_* j^* b_2^! \Lambda = j_* u_2^! \Lambda = j_* \Lambda[-2c](-c)$.

Therefore we have the following triangle:

$$t_* c_2^! \Lambda \to b_2^! \Lambda \to j_* \Lambda[-2c](-c).$$

By the vanishing result we have $R^k c_2^! \Lambda = 0$ for $k < 2(c+1)$. The exact cohomology sequence of the last triangle gives the lemma.

As an application we classify all possible Verdier-correspondences in the case where X is smooth and equidimensional, $\dim X = \dim B$, and $F = G = \Lambda$. We deduce from the lemma:

$$\operatorname{Hom}(b_1^* \Lambda, b_2^! \Lambda) = \operatorname{Hom}(\Lambda_U, \Lambda_U).$$

Hence, giving a Verdier-correspondence for the constant sheaf Λ is the same as giving for each irreducible component W of B, such that $\dim W = \dim B$ an element $n_W \in \Lambda$.

Let be $d = \dim X = \dim B$. We denote by $\mathcal{Z}_d(B, \Lambda)$ the cycles of dimension d on B with coefficients in Λ. We may formulate our remarks as follows.

(1.3) LEMMA. *Assume that X is smooth and equidimensional and that X and B have the same dimension d. Then we have a natural isomorphism:*

$$\mathrm{Hom}(b_1^*\Lambda, b_2^!\Lambda) = \mathcal{Z}_d(B,\Lambda)$$

We note that both groups are zero, if $\dim B < \dim X = d$. The correspondence associated to a cycle $z \in \mathcal{Z}_d(B,\Lambda)$ will be denote by γ_z.

Consider the general situation (1.2) with F and G as above. The canonical morphism $b_2^*G^L \otimes_\Lambda b_2^!\Lambda \to b_2^!G$ induces a morphism

$$\mathrm{Hom}(b_1^*F, b_2^*G) \otimes_\Lambda \mathrm{Hom}(b_1^*\Lambda, b_2^!\Lambda) \to \mathrm{Hom}(b_1^*F, b_2^!G). \tag{1.4}$$

Under the assumptions of (1.3) a correspondence in the image of this map is called a Gysin-correspondence.

We will review a few operations with correspondences, which we constantly need. For a more detailed discussion the reader is referred to Illusie SGA 5 III. Consider a diagram:

$$\begin{tikzcd} W \arrow[d, "w"'] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "u"'] & V \arrow[d, "v"'] \\ Y & B \arrow[l, "b_1"'] \arrow[r, "b_2"] & X \end{tikzcd} \tag{1.5}$$

Assume that the map $C \to B \times_X V$ is proper. Then if a correspondence $\kappa : c_1^*M \to c_2^!N$ is given, the direct image $u_*\kappa : b_1^*w_*M \to b_2^!v_*N$ is defined. This is done in two steps. First assume that $W = Y$, $V = X$, and that u is proper. Hence $c_1 = b_1u$, and $c_2 = b_2u$. Therefore we find a map

$$b_1^*M \to u_*c_1^*M \to u_*c_2^!N = u_!c_2^!N \to b_2^!N.$$

Hence taking first the direct image under the map $C \to B \times_X V$ reduces the problem defining direct images to the case where the second square in (1.5) is cartesian. In this case we have by the proper base change an isomorphism

$$u_*c_2^! = b_2^!v_*.$$

Therefore we can define the desired morphism:

$$b_1^*w_*M \to u_*c_1^*M \to u_*c_2^!N = b_2^!v_*N.$$

If for example the maps u and v in the diagram (1.5) are proper the direct image of a Verdier-correspondence is defined, because then $C \to B \times_X V$ is also proper.

Similarly one defines the direct image with compact support $u_! \kappa : b_1^* w_! M \to b_2^! v_! N$, if $C \to B \times_Y W$ is proper.

(1.6) Example: Assume that $Y = B = X = S$. Then the direct image is defined, iff c_2 is proper. In this case, we get a map $R\Gamma(W, M) \to R\Gamma(V, N)$.

Assume, moreover, that V, C, and W are proper, that C is irreducible, and that V is smooth and equidimensional of dimension $\dim C$. Let $\kappa = \gamma_C : c_1^* \mathbf{Q}_\ell \to c_2^! \mathbf{Q}_\ell$ be the correspondence defined by (1.3). The induced map $\bar{\gamma}_C : H^\cdot(W, \mathbf{Q}_\ell) \to H^\cdot(V, \mathbf{Q}_\ell)$ has the following interpretation.

Let $c_{2*} : H^\cdot(C, \mathbf{Q}_\ell) \to H^\cdot(V, \mathbf{Q}_\ell)$ be the Gysin homomorphism, which is defined by Poincaré duality:

$$\int_C \eta \cup c_2^* \epsilon = \int_V c_{2*} \eta \cup \epsilon$$

The pairing on the left hand side of this equation may be degenerate, since C is not assumed to be smooth. Nevertheless c_{2*} is uniquely defined by this equation. Then we have $\bar{\gamma}_C = c_{2*} c_1^*$. This equation may be rewritten in the following way. Let $c : C \to W \times V$ be the product of c_1 and c_2. Denote by $p : W \times V \to W$ and $q : W \times V \to V$ the projections. Let cls $C = c_* 1$ be the image of 1 under the Gysin homomorphism. We have:

$$\bar{\gamma}_C(\rho) = q_*(\text{cls } C \cup p^* \rho) \tag{1.7}$$

(1.8) Example. We assume that $Y = B = X = S$. Then the direct image with compact support $u_! \kappa$ is defined, iff c_1 is proper. In this case we get a map $R\Gamma_c(W, M) \to R\Gamma_c(V, N)$.

(1.9) Example. Another important example of the direct image of a correspondence is as follows. Consider any proper map $u : C \to B$. We get a commutative diagram:

$$\begin{tikzcd}
Y \arrow[d, equal] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "u"'] & X \arrow[d, equal] \\
Y & B \arrow[l, "b_1"'] \arrow[r, "b_2"] & X
\end{tikzcd}$$

Assume that X and Y are smooth and equidimensional of dimension d. Assume that $\dim B = \dim C = d$. Consider a cycle $z \in \mathcal{Z}_d(C, \Lambda)$. Then the direct image of the crrespondence γ_z is defined, and we have:

$$u_* \gamma_z = \gamma_{u_* z}$$

Here $u_* z$ denotes the direct image in the sense of cycles. Sometimes one can define an inverse image of a correspondence. Let V be a closed subscheme of X and W a closed subscheme of Y. Assume that $b_2^{-1}(V) \subset b_1^{-1}(W)$. Let $C = b_2^{-1}(V)$ be the closed subscheme of B with the reduced structure. We get a diagram (1.5), where the second square is cartesian up to nilpotent elements. The the inverse image of a correspondence $\kappa : b_1^* F \to b_2^! G$ is defined:

(1.10) $$u^* \kappa : c_1^* w^* F = u^* b_1^* F \to u^* b_2^! G \to c_2^! v^* G.$$

2. Deligne's conjecture

(2.1) Let X be a smooth, equidimensional scheme of finite type over S. We consider an irreducible correspondence $X \xleftarrow{b_1} B \xrightarrow{b_2} X$. We assume that $\dim X = \dim B = d$. Let $b = (b_1, b_2) : B \to X \times X$ be the natural map. We denote by $\Delta_X \subset X \times X$ the diagonal. The scheme of fixed points of the correspondence is $\operatorname{Fix} B = B \times_{(X \times X)} \Delta$. We say that the correspondence B has only isolated fixed points, if $\operatorname{Fix} B$ is of dimension zero. In this case the intersection multiplicity in a point $z \in \operatorname{Fix} B$ is defined:

$$(B \cdot \Delta_X)_z = \text{length } (O_B{}^L \otimes_{O_{X \times X}} O_{\Delta_X})_z.$$

(2.2) Theorem. *Assume that X and B are proper and that B has only isolated fixed points. Let $\gamma_B : b_1^* \mathbf{Q}_\ell \to b_2^! \mathbf{Q}_\ell$ be the correspondence defined by (1.3). Then the Lefschetz fixed point formula holds:*

$$\operatorname{Tr}(\bar{\gamma}_B | H^{\cdot}(X, \mathbf{Q}_\ell)) = \sum_{z \in \operatorname{Fix} B} (B \cdot \Delta_X)_z.$$

Proof: If B is a closed immersion, this follows from SGA $4\frac{1}{2}$ Cycle 3.3 and 2.3.8 and from (1.7) above. The general case follows from (1.9).

We assume now that X is obtained by base change from a finite field $\mathbf{F}_q$:

$$X = \mathbf{X} \times_{\operatorname{Spec} \mathbf{F}_q} S$$

We denote by $\mathrm{Fr}_X : X \to X$ the Frobenius morphism relative to $\mathbf{X}$. We can multiply our correspondence B by a power of the Frobenius:

$$X \xleftarrow{\mathrm{Fr}_X^n b_1} B \xrightarrow{b_2} X$$

We denote this correspondence by B_n.

(2.3) LEMMA. *Assume that b_2 is finite. If $q^n > \deg b_2$, the correspondence B_n has only isolated fixed points.*

PROOF: Let $C \subset B$ be an irreducible curve of fixed points of b_2. On the other hand the degree of C over its image under $\mathrm{Fr}_X^n b_1$ is greater or equal than q^n. Hence the maps b_2 and $\mathrm{Fr}_X^n b_1$ can not agree on C.

We assume that b_2 is finite. Let $z \in B$ be a closed point. We put $b_2(z) = x$. We define $\deg_z B/X$ by the equation of cycles:

$$b_2^*(x) = \sum_z (\deg_z B/X) \cdot z.$$

We also introduce the function $e(z) = \text{length } O_{B,z}/b_2^*(\mathbf{m}_{X,x})O_{B,z}$. It agrees with $\deg_z B/X$ if B is Cohen-Macaulay. We note that the function $e(z)$ is bounded on B.

(2.4) LEMMA. *Let z be an isolated fixed point of B_n for $q^n > e(z)$. Then we have:*

$$(B_n \cdot \Delta_X)_z = \deg_z B/X.$$

PROOF: Choose local parameters $u_1, \ldots, u_d$ on the scheme $\mathbf{X}/\mathbb{F}_q$ in the point that lies below $x = b_2(z)$. We put $\xi_i = b_1^*(u_i)$ and $\eta_i = b_2^*(u_i)$. The intersection multiplicity $(B_n \cdot \Delta_X)_z$ is by definition the Euler characteristic of the Koszul complex $K.(\xi_1^{q^n} - \eta_1, \ldots, \xi_d^{q^n} - \eta_d, O_{B,z})$. Let $(\boldsymbol{\eta})$ be the ideal generated by the η_i. By assumption we have $\xi_i^{q^n} \in \mathbf{m}_{B,z} \cdot (\boldsymbol{\eta})$. Hence the vector of the elements $\xi_i^{q^n} - \eta_i$ and the vector of the η_i differ by a matrix, whose determinant is a unit. It follows that the Koszul complex above is isomorphic to the Koszul complex $K.(\eta_1, \ldots, \eta_d, O_{B,z})$. The lemma follows.

(2.5) Let L be a smooth Λ-sheaf on X, which is obtained by base change from a Λ-sheaf on $\mathbf{X}/\mathbb{F}_q$. Let $\varphi : b_1^* L \to b_2^* L$ be a homomorphism. We denote by φ_n the map

$$b_1^*(\mathrm{Fr}_X^n)^* L \xrightarrow{b_1^*(F_L^n)} b_1^* L \to b_2^* L,$$

where F_L denotes the Frobenius morphism of L. Let $\kappa_n : b_1^*(\mathrm{Fr}_X^n)^* L \to b_2^! L$ be the image of $\varphi_n \otimes \gamma_{B_n}$ under (1.4). Let $z \in \operatorname{Fix} B_n$ be a fixed point. We put $\mathrm{Fr}_X^n b_1(z) = b_2(z) = x$. Then φ_n induces a map $\varphi_{n,z} : L_x = (b_1^*(\mathrm{Fr}_X^n)^* L) \to b_2^* L_z = L_x$.

(2.6) CONJECTURE OF DELIGNE. *In the situation (2.1) assume that b_1 is proper and b_2 is finite. Let L be a smooth $\bar{\mathbf{Q}}_\ell$-sheaf on X, and let $\varphi : b_1^* L \to b_2^! L$ be a morphism. Then there exists a number n_0 such that for $n > n_0$:*

$$\mathrm{Tr}(\bar{\kappa}_n | H_c^{\cdot}(X, L)) = \sum_{z \in \operatorname{Fix} b_n} \deg_z b/X \cdot \mathrm{Tr}(\varphi_{n,z} | L_{b_2(z)}) \tag{2.6.1}$$

By (2.3) the sum is finite for large n. If X is proper, (2.6.1) is true for the sheaf $L = \bar{\mathbf{Q}}_\ell$ and the correspondence γ_B by (2.2) and (2.4). In section 6 we will prove:

(2.7) THEOREM. *Deligne's conjecture is true, if L is a $\bar{\mathbf{Q}}_\ell$- sheaf with finite monodromy and X is a surface.*

Assume that Λ is finite and a $\mathbf{Z}_\ell$-algebra. Let L be a locally free Λ-module. By SGA $4\frac{1}{2}$ Finitude we know that $R\Gamma_c(X, L)$ is a perfect complex of Λ-modules, i.e. represented by a finite complex of free Λ-modules.

(2.8) WEAK CONJECTURE. *With the assumptions of (2.5) let Λ be finite and a $\mathbf{Z}_\ell$-algebra. Then there exists a number n_0 such that for $n > n_0$:*

$$\mathrm{Tr}(\bar{\kappa}_n | R\Gamma_c(X, L)) = \sum_{z \in \operatorname{Fix} B_n} \deg_z B/X \cdot \mathrm{Tr}(\varphi_{n,z} | L_{b_2(z)}).$$

We will show in the next section that Deligne's conjecture for the sheaf $\mathbf{Q}_\ell$ implies the weak conjecture (2.8) and (2.7).

3. THE METHOD OF NEILSEN-WECKEN

Let Λ be finite and a $\mathbf{Z}_\ell$-algebra. Assume we are given a correspondence (2.1), such that X is defined over a finite field $\mathbf{F}_q$. Let L be a locally free Λ-module of finite rank on X, which is defined over $\mathbf{F}_q$. The aim of this section is to show that (2.7) for the sheaf $\bar{\mathbf{Q}}_\ell$ and the correspondence γ_B implies the following:

(3.1) THEOREM. *The weak conjecture (2.8) holds for a smooth surface X.*

We formulate a more general proposition, which implies (3.1). We assume that our given correspondence B has only isolated fixed points and that b_1 is proper. Let L be as above. We choose a connected etale Galois covering $\pi : Y \to X$ with Galois group G, such that $\pi^* L$ becomes trivial. Let C be the fibre product:

$$\begin{array}{ccc} C & \xrightarrow{(c_1,c_2)} & Y \times Y \\ \downarrow & & \downarrow \\ B & \xrightarrow{(b_1,b_2)} & X \times X \end{array} \tag{3.2}$$

We assume that for any irreducible component D of C the following formula holds:

$$\mathrm{Tr}(\bar{\gamma}_D | H_c^{\cdot}(Y, \mathbf{Q}_\ell)) = \sum_{u \in \mathrm{Fix}\, D} (D \cdot \Delta_Y)_u.$$

(3.3) PROPOSITION. *With the assumptions made above we have for the correspondence $\kappa = \varphi \otimes \gamma_B$:*

$$\mathrm{Tr}(\bar{\kappa} | R\Gamma_c(X, L)) = \sum_{u \in \mathrm{Fix}\, B} (B \cdot \Delta_X)_u \cdot \mathrm{Tr}(\varphi | L_{b_2(u)})$$

This proposition tells us that Deligne's conjecture for the sheaf L follows from the conjecture for $\bar{\mathbf{Q}}_\ell$. The same is true for an l-adic sheaf $L = \varprojlim L_s$ if we may trivialize all L_s by the same finite covering, i.e., if L has finite monodromy.

We need some preparations for the proof. We denote by $\pi^*\varphi : c_1^* \pi^* L \to c_2^* \pi^* L$ the induced morphism. Let $\pi^*\kappa = \pi^*\varphi \otimes \gamma_C$ be the induced Verdier-correspondence (1.4), where we consider C as a cycle with multiplicities one. We get a map on the cohomology (1.8):

$$\pi^*\bar{\kappa} : R\Gamma_c(Y, \pi^* L) \to R\Gamma_c(Y, \pi^* L).$$

(3.4) PROPOSITION. *We have a commutative diagram:*

$$\begin{array}{ccc} R\Gamma_c(Y, \pi^* L) & \xrightarrow{\pi^*\bar{\kappa}} & R\Gamma_c(Y, \pi^* L) \\ \downarrow \mathrm{tr} & & \uparrow \mathrm{res} \\ R\Gamma_c(X, L) & \xrightarrow{\bar{\kappa}} & R\Gamma_c(X, L) \end{array}$$

PROOF: For $i = 1, 2$, define C_i as a fibre product:

$$\begin{tikzcd} C_i \arrow[r, "\gamma_i"] \arrow[d, "v_i"] & Y \arrow[d, "\pi"] \\ B \arrow[r, "b_i"] & X \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "u_1"] \arrow[d, "u_2"] & C_1 \arrow[d, "v_1"] \\ C_2 \arrow[r, "v_2"] & B \end{tikzcd}$$

The last diagram is also cartesian. Since the trace morphism commutes with base change, we get a commutative diagram:

$$\begin{tikzcd} R\Gamma_c(Y, \pi^* L) \arrow[r] \arrow[d, "\mathrm{tr}"] & R\Gamma_c(C_1, \gamma_1^* \pi^* L) \arrow[r] \arrow[d, "\mathrm{tr}"] & R\Gamma_c(C, c_1^* \pi^* L) \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{tr}"] & R\Gamma_c(C, c_2^! \pi^* L) \arrow[d, "\mathrm{tr}"] \\ R\Gamma_c(X, L) \arrow[r] & R\Gamma_c(B, b_1^* L) \arrow[r] \arrow[d, "\mathrm{tr}"] & R\Gamma_c(C_2, v_2^* b_1^* L) \arrow[r, "\mathrm{tr}"] & R\Gamma_c(C_2, v_2^* b_2^! L) \arrow[d, "\mathrm{tr}"] \\ & R\Gamma_c(B, b_2^! L) \arrow[r, "\mathrm{tr}"] \arrow[ur] & R\Gamma_c(X, L) \arrow[r] & R\Gamma_c(Y, \pi^* L) \end{tikzcd}$$

The result follows.

We recall that $R\Gamma_c(Y, \pi^* L)$ is a perfect complex of $\Lambda[G]$-modules. We denote the group cohomology respectively the group homology simply by $R\Gamma_c(Y, \pi^* L)^G$ respectively by $R\Gamma_c(Y, \pi^* L)_G$. The trace morphism in the diagram (3.4) induces an isomorphism $R\Gamma_c(Y, \pi^* L)_G \to R\Gamma_c(X, L)$ and the restriction induces an isomorphism $R\Gamma_c(X, L) \to R\Gamma_c(Y, \pi^* L)^G$. The composite of these two morphisms is the norm map.

For a projective $\Lambda[G]$ module (or a perfect complex) P and an endomorphism α of P we have the noncommutative trace (SGA $4\frac{1}{2}$ Rapport 5.5), which will be denoted by $\mathrm{Tr}_\Lambda^G(\alpha|P)$. More precisely this is the noncommutative trace evaluated in the conjugacy class 1. The following is elementary:

(3.5) LEMMA. *Let P be a projective $\Lambda[G]$-module. Let $\alpha : P_G \to P^G$ be a morphism of Λ-modules. The composite $\tilde{\alpha} : P \to P_G \to P^G \to P$ is clearly a G-module homomorphism. Denote by $\nu : P_G \to P^G$ the isomorphism induced by the norm map. Then we have:*

$$\mathrm{Tr}_\Lambda^G(\tilde{\alpha}|P) = \mathrm{Tr}_\Lambda(\nu^{-1}\alpha|P_G)$$

(3.6) COROLLARY.

$$\mathrm{Tr}_\Lambda^G(\pi^* \bar{\kappa}|R\Gamma_c(Y, \pi^* L)) = \mathrm{Tr}(\bar{\kappa}|R\Gamma_c(X, L)).$$

Let G act from the right on Y. We fix once for all an G-equivariant isomorphism $\pi^*L = Y \times E = E_Y$, where E is a $\Lambda[G]$-module. We get a morphism:

$$C \times E = c_1^* E_Y \xrightarrow{\pi^*\varphi} c_2^* E_Y = C \times E \tag{3.7}$$

For a morphism $\alpha \in \operatorname{End}_\Lambda E$ we denote by C_α the union of connected components of C, where (3.7) induces the morphism α.

Take a $g = (g_1, g_2) \in G \times G$. Since the map $\pi^*\phi$ is $G \times G$- equivariant, we have a commutative diagram:

$$\begin{array}{ccc} g^* c_1^* E_Y & \longrightarrow & g^* c_2^* E_Y \\ \downarrow & & \downarrow \\ c_1^* E_Y & \longrightarrow & c_2^* E_Y \end{array}$$

Hence we obtain: $C_\alpha g = C_{g_2^{-1}\alpha g_1}$.

Consider the Verdier-correspondence $\pi^*\kappa = \varphi \otimes \gamma_C : c_1^* E_Y \to c_2^! E_Y$. We denote its restriction to C_α by κ_α. We use also the notation $\gamma_\alpha = \gamma_{C_\alpha}$, where here again C_α is considered as a cycle with multiplicities one. Then we have the equation $\kappa_\alpha = \alpha \otimes \gamma_\alpha$.

From the equivariance we get $g^*\kappa_\alpha = \kappa_{g_2 \alpha g_1^{-1}}$. For the induced maps on the cohomology we have: $g_2 \bar{\kappa}_\alpha g_1^{-1} = \bar{\kappa}_{g_2\alpha g_1^{-1}}$ on $R\Gamma_c(Y, E)$ and $g_2 \bar{\gamma}_\alpha g_1^{-1} = \bar{\gamma}_{g_2 \alpha g_1^{-1}}$ on $R\Gamma_c(Y, \mathbf{Z}_\ell)$.

Let G operate on $\operatorname{End}_\Lambda E$ via $\alpha \to g\alpha g^{-1}$. If ϕ is an orbit for this operation, we define

$$C_\phi = \bigcup_{\alpha \in \phi} C_\alpha$$

Let κ_ϕ be the restriction of $\pi^*\kappa$ to C_ϕ, and $\gamma_\phi = \gamma_{C_\phi}$. These are G-equivariant correspondences, which induce G-morphisms on the cohomology.

(3.8) Lemma. *Let ϕ be a G-orbit and $\alpha \in \phi$. Then we have:*

$$\operatorname{Tr}_\Lambda^G(\bar{\kappa}_\phi | R\Gamma_c(Y, E)) = \operatorname{Tr}_\Lambda^G(\bar{\gamma}_\phi | R\Gamma_c(Y, \mathbf{Z}_\ell)) \cdot \operatorname{Tr}_\Lambda(\alpha | E).$$

Proof: For the problem to define the first factor on the right hand side see SGA $4\frac{1}{2}$Rapport 4.12. In the proof we may replace $\mathbf{Z}_\ell$ by $A = \mathbf{Z}/l^n$. By the universal coefficient theorem we have $R\Gamma_c(Y, E) =$

$R\Gamma_c(Y, A) \otimes_\Lambda E$. Let $Z_\alpha \subset G$ be the centralizer of α, From the considerations above we have:

$$\bar{\kappa}_\phi = \sum_{g \in G/Z_\alpha} g\bar{\gamma}_\alpha g^{-1} \otimes g\alpha g^{-1}.$$

Our lemma follows now from the following elementary lemma on G-modules, which we give without proof (compare loc.cit. 5.8):

(3.9) LEMMA. *let Λ be an A-algebra. Let E be a $\Lambda[G]$-module, which is projective as a Λ-module. Let H be a projective $A[G]$-module. Let $Z \subset G$ be a subgroup. Assume we are given Z-module homomorphisms $\alpha : E \to E$ and $\gamma : H \to H$. Then we have the formula:*

$$\begin{aligned}\operatorname{Tr}^G_\Lambda \Big(\sum_{g \in G/Z} g\gamma g^{-1} \otimes g\alpha g^{-1} | H \otimes_A E\Big) \\ = \operatorname{Tr}^G_A \Big(\sum_{g \in G/Z} g\gamma g^{-1} | H\Big) \cdot \operatorname{Tr}_\Lambda(\alpha|E).\end{aligned}$$

We return to the proof of (3.3). From (3.6) and (3.8) we deduce:

$$\begin{aligned}(3.10)\quad \operatorname{Tr}_\Lambda(\bar{\kappa}|R\Gamma_c(X, L)) &= \operatorname{Tr}^G_\Lambda(\pi^*\bar{\kappa}|R\Gamma_c(Y, E)) \\ &= \sum_\phi \operatorname{Tr}^G_\Lambda(\bar{\kappa}_\phi|R\Gamma_c(Y, E)) \\ &= \sum_\phi \operatorname{Tr}^G_\Lambda(\bar{\gamma}_\phi|R\Gamma_c(Y, \mathbf{Z}_\ell)) \cdot \operatorname{Tr}_\Lambda(\alpha|E) \\ &= \sum_\phi \left(\frac{1}{|G|} \sum_{v \in \operatorname{Fix} C_\phi} (C_\phi \cdot \Delta_Y)_v \right) \operatorname{Tr}_\Lambda(\alpha|E)\end{aligned}$$

Here we use our assumption for constant coefficients. We put $\pi(v) = u$. Since the map π is etale, we have $(C_\phi \cdot \Delta_Y)_v = (B \cdot \Delta_X)_u$. Since G acts diagonally on C_ϕ we have exactly $|G|$ fixed point v of C_ϕ lying over u. By definition we have $\operatorname{Tr}_\Lambda(\alpha|E) = \operatorname{Tr}_\Lambda(\varphi_u|L_{b_2(u)})$. We denote by $\operatorname{Fix}_\phi B$ the set of all fixed points of B, which lift to a fixed point of C_ϕ. We can now continue the equations (3.10):

$$\begin{aligned}&= \sum_\phi \sum_{u \in \operatorname{Fix}_\phi B} (B \cdot \Delta_X)_u \cdot \operatorname{Tr}_\Lambda(\varphi_u|L_{b_2(u)}) \\ &= \sum_{u \in \operatorname{Fix}_\phi B} (B \cdot \Delta_X)_u \cdot \operatorname{Tr}_\Lambda(\phi_u|L_{b_2(u)}).\end{aligned}$$

This proves proposition (3.3).

4. The Lefschetz formula for constant coefficients

Let X be a smooth, equidimensional surface of finite type over S. We consider a correspondence $X \xleftarrow{b_1} B \xrightarrow{b_2} X$, such that B is an irreducible normal surface, b_1 is proper, and b_2 is finite. Our aim is a formula for the trace of the Verdier correspondence γ_B (1.3) acting on $R\Gamma_c(X, \Lambda)$. We consider a smooth compactification $X \to Y$, such that the divisor at infinity A is a union of smooth irreducible curves A_i which intersect transversally. We denote by $\bar{D}$ the normalization of $Y \times Y$ in the function field of B (see: introduction). We get a commutative diagram:

$$\begin{tikzcd}
X \arrow[d] & B \arrow[l, "b_1"'] \arrow[r, "b_2"] \arrow[d] & X \arrow[d] \\
Y & \bar{D} \arrow[l, "\bar{d}_1"'] \arrow[r, "\bar{d}_2"] & Y
\end{tikzcd}$$

The vertical arrows are open immersions. The assumption b_1 and b_2 proper implies that both squares are cartesian. Hence we get $\bar{d}_1^{-1}(A) = \bar{d}_2^{-1}(A) = \bar{D} - B$. It will be necessary to replace $\bar{D}$ by a modification D, i.e., a proper birational morphism $\pi : D \to \bar{D}$, which is an isomorphism over B. We remark that by (1.9) the Verdier correspondences $\gamma_{\bar{D}}$ and γ_D induce the same endomorphism of $R\Gamma(Y, \Lambda)$. We put $C = D - B$.

$$\begin{tikzcd}
X \arrow[d, "j"'] & B \arrow[l, "b_1"'] \arrow[r, "b_2"] \arrow[d, "w"'] & X \arrow[d, "j"'] \\
Y & D \arrow[l, "d_1"'] \arrow[r, "d_2"] & Y \\
A \arrow[u, "a"] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[u, "u"] & A \arrow[u, "a"]
\end{tikzcd}$$

The inverse image $\kappa = u^*\gamma_D : c_1^*\Lambda \to c_2^!\Lambda$ is defined. Therefore we get a morphism of triangles:

$$\begin{tikzcd}
d_1^* j_! j^* \Lambda \arrow[r] \arrow[d, "j_!\gamma_B"'] & d_1^*\Lambda \arrow[r] \arrow[d, "\gamma_D"'] & d_1^* a_* a^* \Lambda \arrow[d, "a_*\kappa"'] \\
d_2^! j_! j^* \Lambda \arrow[r] & d_2^!\Lambda \arrow[r] & d_2^! a_* a^* \Lambda
\end{tikzcd}$$

On the cohomology this gives a morphism of triangles:

$$
\begin{tikzcd}
R\Gamma_c(X,\Lambda) \arrow[r] \arrow[d, "\bar\gamma_B"'] & R\Gamma_c(Y,\Lambda) \arrow[r] \arrow[d, "\bar\gamma_D"'] & R\Gamma_c(A,\Lambda) \arrow[d, "\bar\kappa"'] \\
R\Gamma_c(X,\Lambda) \arrow[r] & R\Gamma(Y,\Lambda) \arrow[r] & R\Gamma(A,\Lambda)
\end{tikzcd}
\tag{4.1}
$$

In the middle row we can replace $\bar\gamma_D$ by $\bar\gamma_{\bar D}$. If we assume that $\bar D$ has only isolated fixed points, we have a trace formula for $\gamma_{\bar D}$, because $\bar D$ is proper. Therefore we want to compute the trace of $\bar\kappa$. This we do by construction a Mayer-Vietoris sequence for κ.

(4.2) Example. So far we did not use that X is a surface. To show how our proof of (2.7) works, let us first consider the case of a curve X. Then A is a finite set and $\bar\kappa$ is just a map:

$$\bar\kappa : H^0(A,\Lambda) = \oplus_{b\in A}\Lambda \to H^0(A,\Lambda) = \oplus_{a\in A}\Lambda$$

We denote a vector of this direct sum by (λ_a).

LEMMA. *The map $\bar\kappa$ is given as follows:*

$$\lambda_a = \sum_{b\backslash u/a} \deg_u D/Y \cdot \mu_b.$$

Here u runs over all points such that $d_1(u) = b$ and $d_2(u) = a$.

PROOF: The morphism $\kappa : \Lambda_C \to c_2^!\Lambda_A$ is by SGA $4\frac{1}{2}$Cycle 2.3.4 adjoint to the weighted trace morphism $c_2 : \Lambda_C \to \Lambda_A$ belonging to $O_D{}^L\otimes_{d_2,O_Y} O_A$ (see also the proof of (4.3) below). But the length of this complex in u is $\deg_u D/Y$. The result follows.

From (4.1) we obtain the trace formula:

$$\mathrm{Tr}(\bar\gamma_{B_n}|R\Gamma_c(X,\Lambda)) = \sum_{u\in \mathrm{Fix}\, D_n} (D_n\cdot\Delta_Y)_u - \sum_{u\in\mathrm{Fix}\, C_n} \deg_u D/Y.$$

But if $q^n > \deg_u D/Y$ for a fixed point u of C_n, we have by (2.4):

$$(D_n\cdot\Delta_Y)_u - \deg_u D/Y = 0.$$

Since $\deg B/Y \geq \deg_u D/Y$ for any point u, we find that Deligne's conjecture (2.6) holds for the sheaf Λ, if $n_0 \geq \log_q \deg B/X$.

We will show how one can use (3.3) to obtain Deligne's conjecture for curves and l-adic coefficients. This has nothing to do with any result in this paper and may be skipped by the reader. Assume that our coefficient ring Λ is finite and that F is a locally constant sheaf on X, which is defined on $\mathbf{X}$ and tamely ramified in A. Take a tamely ramified Galois covering $\mathbf{X}' \to \mathbf{X}$ over $\mathbf{F}_q$ trivializing L. We get a morphism of the proper smooth models $\mathbf{Y}' \xrightarrow{\tau} \mathbf{Y}$. In the following diagram let D' be the normalization of the fibre product:

$$\begin{array}{ccc} D' & \xrightarrow{(d_1', d_2')} & Y' \times Y' \\ \downarrow & & \downarrow \\ D & \xrightarrow{(d_1, d_2)} & Y \times Y \end{array}$$

LEMMA. *Let u' be a fixed point of D'_n and let $u \in D$ be its image. Then we have $\deg_{u'} D'/Y' \leq \deg_u D/Y$.*

PROOF: The points $y_1' = d_1'(u')$ and $y_2' = d_2(u') = \mathrm{Fr}^n_{Y'}(u')$ lie over the same point $\mathbf{y}' \in \mathbf{Y}'$ and have therefore the same ramification index with respect to Y'/Y. Let K and K' be the rings of rational functions on Y and Y' respectively. Denote the image under τ of y_1' and y_2' by y_1 and y_2 respectively. Then K'/K is by assumption a tamely ramified Galois covering of the same ramification index m with respect to the valuations y_1 or y_2 of K.

Let $\eta : \mathrm{Spec} E \to D$ be the general point. We consider E as a valuated field with respect to u. We have $D'_q = \mathrm{Spec}(E \otimes_{K \otimes_k K} K' \otimes_k K')$. Consider the tamely ramified Galois extension $E_m = E(s^{1/m})/E$. Let $\eta_m = \mathrm{Spec} E_m$. We have:

$$\begin{aligned} D'_{\eta_m} &= \mathrm{Spec}(E_m \otimes_{K \otimes_k K} K' \otimes_k K') \\ &\qquad\qquad\qquad = \mathrm{Spec}\left((E_m \otimes_{d_1, K} K') \otimes_{E_m} (E_m \otimes_{d_2, K} K')\right). \end{aligned}$$

But it follows from the lemma of Abhyankar SGA 1 XIII, 5.2 that the extensions $E_m \otimes_{d_i, K} K'/E_m$ are unramified.

Consider now the irreducible component $\mathrm{Spec} C \subset D'_\eta$, whose closure contains u'. Let be $\mathrm{Spec} C_m \subset D'_{\eta_m}$ a component over $\mathrm{Spec} C$.

We get a diagram:

$$\begin{array}{ccccc} C_m & \longleftarrow & C & \xleftarrow{d_2'^*} & K' \\ {\scriptstyle\text{unramified}}\uparrow & & \uparrow & & \uparrow{\scriptstyle\text{Galois of ramification index } m} \\ E_m & \longleftarrow & E & \xleftarrow{d_2^*} & K \end{array}$$

Since E_m/E is also a Galois extention of ramification index m, we see that the ramification indices of C/K' are all smaller that m.

It follows from the lemma that Deligne's conjecture holds for the constant sheaf Λ and any irreducible component of D' with $n_0 \geq \log_q \deg B/X$. From the Proposition (3.3) we deduce:

THEOREM. *If F is a tamely ramified Λ-sheaf on $\mathbf{X}$, the weak conjecture (2.8) holds for $n_0 \geq \log_q \deg B/X$.*

Assume now we are given a locally free $\mathbf{Z}_\ell$-sheaf L, which we write as a projective limit $L = \varprojlim L_r$, where L_r is a locally free $\mathbf{Z}/l^r$- sheaf. We choose a finite etale covering $X' \leftarrow B' \rightarrow X'$ of our correspondence, which kills all wild ramification. If we go to a bigger tamely ramified covering X'', which trivializes L_r, we see that (2.6) holds for L_r and X, if $n^0 \geq \log_q \deg B'/X'$. Hence we have proven:

THEOREM. *Conjecture (2.6) holds for a curve.*

As we remarked in the introduction by a more precise result of Illusie, the conjecture holds even for any $n_0 \geq \log_q \deg B/X$.

We have now finished our example and return to the case where X is a smooth surface. We consider the commutative diagram:

$$\begin{array}{ccccc} A & \xleftarrow{e_1(i)} & C_i & \xrightarrow{e_2(i)} & A_i \\ {\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle v(i)}\downarrow & & {\scriptstyle a_i}\downarrow \\ A & \xleftarrow{c_1} & C & \xrightarrow{c_2} & A \end{array}$$

Here C_i is $c_2^{-1}(A_i)$.

We denote the inverse image of κ under $v(i)$ by $\kappa_i : e_1(i)^*\Lambda \rightarrow e_2(i)^!\Lambda$. By (1.3) the correspondence κ_i corresponds to a cycle from $\mathcal{Z}_1(C_i, \Lambda))$.

(4.3) LEMMA. *We view $\alpha(i) = d_2^* A_i$ as a 1-cycle on C_i. Then κ_i is the Gysin-correspondence $\gamma_{\alpha(i)}$.*

PROOF: Let W be an irreducible component of C_i. Consider a smooth open subset V of W, that meets no other component of C_i. Let be U an open subset of D, that intersects C_i in V. We get a cartesian diagram:

$$\begin{array}{ccc} V & \xrightarrow{\epsilon} & A_i \\ {\scriptstyle\nu}\downarrow & & \downarrow \\ U & \xrightarrow{\delta} & Y \end{array}$$

The correspondence γ_δ induces a morphism $\Lambda \to \delta^!\Lambda$, which is by definition the adjoint to the usual trace morphism $\delta_!\Lambda \to \Lambda$ (SGA $4\frac{1}{2}$Cycle 2.3.3). The morphism $\Lambda \to \epsilon^!\Lambda$ induced by the correspondence κ_i is obtained by base change from the morphism $\Lambda \to \delta^!\Lambda$ above. On the other hand we obtain by base change from the trace morphism $\delta_!\Lambda \to \Lambda$ the weighted trace morphism belonging to $O_V{}^L \otimes_{O_Y} O_{A_i}$ (SGA $4\frac{1}{2}$Cycle 2.3.4). But this is just $n_w \operatorname{Tr}_\epsilon : \Lambda \to \epsilon^!\Lambda$, where n_W is the multiplicity of W in $d_2^* A_i$, and where $\operatorname{Tr}_\epsilon$ is the usual trace morphism. Hence we get our result.

We denote by Λ_W the direct image of the constant sheaf by the morphism $W \to C$, and sometimes also by the morphism $W \to C_i$. The morphism $\gamma_W : \Lambda \to e_2(i)^!\Lambda$ factors $\Lambda \to \Lambda_W \to e_2(i)^!\Lambda$. Hence the direct image of this correspondence by $v(i)$ is $\Lambda \to \Lambda_W \to v(i)_* e_2(i)^!\Lambda = c_2^! a_{i*}\Lambda$. We denote by $\rho_{i,W}$ the morphism $\Lambda_W \to c_2^! a_{i*}\Lambda$. Define $n_{i,W}$ by the equation of cycles:

$$d_2^* A_i = \sum n_{i,W} \cdot W$$

The lemma (4.3) may be reformulated:

(4.5) COROLLARY. *There is a commutative diagram:*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \oplus \Lambda_W \\ \downarrow & & \downarrow {\scriptstyle \oplus n_{i,W}\rho_{i,W}} \\ c_2^!\Lambda & \longrightarrow & \oplus_i c_2^! a_{i*}\Lambda \end{array}$$

Here the upper direct sum runs over all components W of C and the middle direct sum over all i and W.

We denote by S the set of double points of the divisor A. Let $\iota_S : S \to A$ be the immersion. We denote by $Z \subset C$ the set of points, which lie on more than one component W. For a point $z \in Z$ let $\iota_z : z \to C$ be the immersion. We know that there is a morphism ξ making the following diagram into a morphism of triangles.

(4.6)

$$\begin{array}{ccccccc}
\Lambda & \longrightarrow & \oplus\Lambda_W & \longrightarrow & \oplus_{z\in Z}\iota_{z*}(\Lambda^{\mathrm{Comp}_z C}/\Lambda) & \longrightarrow & \Lambda[1] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \xi} & & \downarrow \\
c_2^!\Lambda & \longrightarrow & \oplus_i c_2^! a_{i*}\Lambda & \longrightarrow & c_2^!\iota_{S*}\Lambda & \longrightarrow & c_2^!\Lambda[1]
\end{array}$$

Here $\mathrm{Comp}_z C$ denotes the set components of C passing through z. The lower triangle depends on an ordering of the indices i, which we fix once for all. If A_i and A_j, with $i < j$ are the components passing through a point $x \in S$, the arrow $\oplus a_{i*}\Lambda \to \iota_{x*}\Lambda$ maps (λ_k) to $\lambda_i - \lambda_j$.

We know by (1.2) that the complex $c_2^! a_{i*}\Lambda$ has no cohomology in negative degrees. Hence $\mathrm{Hom}(\Lambda[1], \oplus c_2^! a_{i*}\Lambda) = 0$. Therefore the morphism ξ is unique.

We show an even stronger assertion. There is a unique ξ making the last right hand square in (4.6) commutative. Since we already know the existence we may assume that such a ξ is given. Let $\xi_z : \iota_{z*}(\Lambda^{\mathrm{Comp}_z C}/\Lambda) \to c_2^!\iota_{S*}\Lambda$ be the components of ξ. By adjunction of the functors ι_{z*} and $\iota_z^!$ we get a commutative diagram:

$$\begin{array}{ccc}
\Lambda^{\mathrm{Comp}_z C}/\Lambda & \longrightarrow & \iota_z^!\Lambda[1] \\
{\scriptstyle \bar{\xi}_z}\downarrow & & \downarrow \\
\iota_z^! c_2^! \iota_{S*}\Lambda & \longrightarrow & \iota_z^! c_2^!\Lambda[1]
\end{array} \tag{4.7}$$

Since $\bar{\xi}_z$ determines ξ_z by adjunction, it is enough to show that there is a unique morphism $\bar{\xi}_z$ making the diagram (4.7) commutative.

If the point z is not mapped to S, we have $\iota_z c_2^! \iota_{S*}\Lambda = 0$. Hence the uniqueness is trivial in this case. Assume that $c_2(z) = x \in S$. Since $\iota_x^! = \iota_z c_2^!$, the lower horizontal arrow (4.7) identifies with the connecting morphism δ provided by the Mayer-Vietoris sequence of the two components A_k and A_j of A passing through x:

$$\iota_x^!\Lambda \to \iota_x^! a_{k*}\Lambda \oplus \iota_x^! a_{j*}\Lambda \to \iota_{x*}\iota_x^!\Lambda \xrightarrow{\delta} \iota_x^!\Lambda[1].$$

We see that δ induces an isomorphism $\Lambda \to H_x^1(A, \Lambda)$. Applying H^0 to our diagram (4.7), we obtain that $\bar{\xi}_z$ must fit into the following commutative diagram:

$$\begin{tikzcd} \Lambda^{\mathrm{Comp}_z C}/\Lambda \arrow[r] \arrow[d, "\bar{\xi}_z"'] & H^1_z(C,\Lambda) \arrow[d] \\ \Lambda \arrow[r, "\simeq"] & H^1_x(A,\Lambda) \end{tikzcd} \tag{4.8}$$

The uniqueness of $\bar{\xi}_z$ is now obvious.

To obtain an explicit formula for ξ_z, we need to know the arrow:

$$H^1_z(C,\Lambda) \to H^1_x(A,\Lambda) \tag{4.9}$$

Denote by $Y_{(x)}$ and $D_{(z)}$ the henselizations in the points x and z respectively.

(4.10) LEMMA. *Assume that D is smooth in the points of C. Then the arrow (4.9) is dual to the canonical restriction map:*

$$H^2(Y_{(x)} - A, \Lambda) \to H^2(D_{(z)} - C, \Lambda).$$

PROOF: We have to recall the definition of (4.9). Consider the diagram:

$$\begin{tikzcd} z \arrow[r, "\iota_z"] \arrow[d, equal] & C \arrow[r, "u"] \arrow[d, "c_2"] & D \arrow[d, "d_2"] \\ x \arrow[r, "\iota_x"] & A \arrow[r, "t"] & Y \end{tikzcd}$$

The map (4.9) is obtained as follows. Consider the Gysin morphism $\Lambda \to d_2^!\Lambda$. Applying the functor $\iota_z^! u^*$, we get:

$$\iota_z^! u^*\Lambda \to \iota_z^! u^* d_2^!\Lambda \to \iota_z^! c_2^! t^*\Lambda \tag{4.10.1}$$

The map induced on H^1 is (4.9). Denote the dualizing complexes by K_Y, K_D, etc.. Ignoring the Tate twist we have canonical isomorphism $K_Y = \Lambda[4]$ and $K_D = \Lambda[4]$. The Verdier dual $d_2^* K_Y \to K_D$ of the Gysin morphism becomes under this identification the identity $\Lambda[4] = \Lambda[4]$. Hence the Verdier dual of (4.10.1) is:

$$\iota_z^* c_2^* t^!\Lambda[4] \to \iota_z^* u^!\Lambda[4] \to \iota_z^* u^!\Lambda[4].$$

Applying H^{-1} the desired result follows.

We need this kind of duality in a slightly more general situation.

(4.11) LEMMA. *Let $\kappa : Z \to D_{(z)}$ be a closed subscheme. Then the group $H^i_z(Z, \Lambda)$ is canonically dual to $H^{4-i}_Z(D_{(z)}, \Lambda) \cong H^{3-i}(D_{(z)} - Z, \Lambda)$.*

PROOF: Denote by D_Z the dualizing functor on a scheme Z. We have:

$$D_Z \iota_z \Lambda_Z = \iota_z^* D_Z \Lambda_Z = \iota_z^* \kappa^! \Lambda[4].$$

The lemma follows.

We consider the morphism (4.9). The local cohomology sequence gives an isomorphism:

$$\Lambda^{\mathrm{Comp}_z C}/\Lambda \to H^1_z(C, \Lambda)$$

Hence to any component P of $C_{(z)}$ corresponds an element $e_P \in H^1_z(C, \Lambda)$. Trivially the sum of the e_P is zero. We can get the elements e_P also from a Mayer-Vietoris sequence. Denote by P' the union of components of $C_{(z)}$ different from P. Let $j : C_{(z)} - z \to C_{(z)}$ be the open immersion. We have a diagram on $C_{(z)}$:

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d, equal] & \Lambda_P \oplus \Lambda_{P'} \arrow[r, "\delta_{P,P'}"] \arrow[d] & \Lambda_Z \arrow[d] \\
\Lambda \arrow[r] & j_* \Lambda \arrow[r] & \Lambda_Z^{\mathrm{Comp}_z C_{(z)}}/\Lambda_z
\end{tikzcd}$$

Here $\delta_{P,P'}(\lambda_P \oplus \lambda_{P'}) = \lambda_P - \lambda_{P'}$, in the obvious notation. Applying H^1_z to the upper row of this diagram, we get a connection morphism:

$$\partial_{P,P'} : \Lambda \to H^1_z(C, \Lambda).$$

It follows easily, that $e_P = \partial_{P,P'}(1)$.

Assume that the components of A passing through x are A_1 and A_2. We get elements $e_1 = e_{A_1}$ and $e_2 = e_{A_2}$ in the group $H^1_x(A, \Lambda)$. We denote also by e_1 the isomorphism $\Lambda \to H^1_x(A, \Lambda)$, which maps 1 to e_1.

(4.12) PROPOSITION. *Assume D is smooth in the points of C. Consider the map (4.9): $\alpha : H^1_z(C, \Lambda) \xrightarrow{e_1} H^1_x(A, \Lambda) \to \Lambda$. Then we have:*

$$\alpha(e_P) = \sum_{\substack{W/A_2 \\ W \neq P}} n_{1,P} n_{2,W} \cdot (P \cdot W)_z - \sum_{\substack{V/A_1 \\ V \neq P}} n_{1,V} n_{2,P} \cdot (V \cdot P)_z.$$

The symbol W/A_2 means, that W runs over all components of $C_{(z)}$, which lie over A_2. The number $n_{2,W}$ is the multiplicity of W in the divisor $\tilde{d}_2^*(A_2)$, where $\tilde{d}_2$ is the map $D_{(z)} \to Y_{(x)}$. We have a similiar meaning for V/A_1.

PROOF: For the proof we use the map (4.10), which is more straightfoward. The elements e_P correspond by duality to maps

$$\epsilon_P : H^2(D_{(z)} - C, \Lambda) \to \Lambda$$

. Dualizing the Mayer-Vietoris sequence used in the definition of e_P, we see that ϵ_P is obtained from the following sequence:

$$\begin{array}{l} H^2(D_{(z)} - P) \\ \otimes H^2(D_{(z)} - P') \xrightarrow{\rho_{P,P'}} H^2(D_{(z)} - C) \xrightarrow{\partial_{P,P'}} H^3(D_{(z)} - z) \xrightarrow{\text{cls } z} \Lambda. \end{array}$$

From now on we will often suppress the coefficients Λ. Before we continue, we need a lemma, which we leave to the reader.

(4.13) LEMMA. *Let T be a strictly henselian scheme with closed point z. Let P and P' be closed subschemes, such that $P \cap P' = z$. Then we have a commutative diagram for all $i, j \geq 1$:*

$$\begin{array}{ccc} H^i_P(T) \otimes H^j_{P'}(T) \xrightarrow{\cup} H^{i+j}_z(T) & \xleftarrow{\sim} & H^{i+j-1}(T - z) \\ \uparrow & & \uparrow \partial_{P,P'} \\ H^{i-1}(T - P) \otimes H^{j-1}(T - P') & \xrightarrow{\cup} & H^{i+j-2}(T - (P \cup P')) \end{array}$$

Assume T is the henselization of a local ring in a smooth point. We put $\dim T = d$. Let P and P' be irreducible subschemes of codimension p and p' respectively. We have by SGA4$\frac{1}{2}$Cycle 2.3.2 cohomology classes cls$P \in H^{2p}_P(T)$ and cls$P' \in H^{2p'}_{P'}(T)$. If $p + p' = d$ the cup product of these classes in $H^{2d}_z(T) = \Lambda$ is the intersection product $(P \cdot P')_z$ (loc. cit. 2.3.8). We will work with the preimage cls$\tilde{P} \in H^{2p-1}(T - P)$. Then we have the formula:

$$\partial_{P,P'}(\text{cls}\tilde{P} \cup \text{cls}\tilde{P}') = (P \cdot P')_z.$$

Note that the cup product and $\partial_{P,P'}$ depend on the order.

Let us return to the proof of (4.12). We have the map $\epsilon_1 : H^2(Y_{(x)} - A) \to \Lambda$, which is dual to e_1. Since A_1 and A_2 meet transversally in x, we get from (4.13) that $\epsilon_1(\mathrm{cls}\tilde{A}_1 \cup \mathrm{cls}\tilde{A}_2) = 1$.

Now consider the commutative diagram:

$$\begin{array}{ccc} H^2(Y_{(x)} - A) & \xrightarrow{\tau} & H^2(D_{(z)} - C) \\ \cup\uparrow & & \cup\uparrow \\ H^1(Y_{(x)} - A_1) \otimes H^1(Y_{(x)} - A_2) & \longrightarrow & H^1(D_{(z)} - C_1) \otimes H^1(D_{(z)} - C_2) \end{array}$$

Here C_i was the preimage of A_i (4.1).

All we need to know is the image of $\epsilon_1^{-1}(1)$ under τ. But this is the same as the image of the element $\mathrm{cls}\tilde{A}_1 \otimes \mathrm{cls}\tilde{A}_2$. The latter element is mapped by the lower horizontal map to the element:

$$\theta = \sum_{V/A_1, W/A_2} n_{1,V} \cdot n_{2,W} \mathrm{cls}\tilde{V} \otimes \mathrm{cls}\tilde{W}.$$

Therefore our problem is to compute $\epsilon_P(\mathrm{cls}\tilde{V} \cup \mathrm{cls}\tilde{W})$. We note that $\mathrm{cls}\tilde{V} \cup \mathrm{cls}\tilde{W}$ lies in the image of the restriction map:

$$\mathbf{r} : H^2(D_{(z)} - (V \cup W)) \to H^2(D_{(z)} - C).$$

But if $V = W = P$, or if $V \cup W \subset P'$ the image of $\mathbf{r}$ lies in the image of $\rho_{P,P'}$. Hence we get $\epsilon_P(\mathrm{cls}\tilde{V} \cup \mathrm{cls}\tilde{W}) = 0$ in these cases. Consider the remaining cases:

$$(A) \quad V = P, \quad W \subset P' \qquad\qquad (B) \quad V \subset P', \quad W = P.$$

In case (A) we have a commutative diagram:

$$\begin{array}{ccccc} H^2(D_{(z)} - P) \oplus H^2(D_{(z)} - P') & \to & H^2(D_{(z)} - C) & \xrightarrow{\partial_{P,P'}} & H^3(D_{(z)} - z) = \Lambda \\ \uparrow & & \uparrow & & \| \\ H^2(D_{(z)} - V) \oplus H^2(D_{(z)} - W) & \to & H^2(D_{(z)} - (V \cup W)) & \xrightarrow{\partial_{V,W}} & H^3(D_{(z)} - z) = \Lambda \end{array}$$

Hence we get: $\epsilon_P(\mathrm{cls}\tilde{V} \cup \mathrm{cls}\tilde{W}) = \partial_{P,P'}(\mathrm{cls}\tilde{V} \cup \mathrm{cls}\tilde{W}) = \partial_{V,W}(\mathrm{cls}\tilde{V} \cup \mathrm{cls}\tilde{W}) = (V \cdot W)_z$.

In case (B) we have to interchange V and W. Since $\partial_{W,V} = -\partial_{V,W}$, we get a minus sign. Therefore the final result for our element θ is:

$$\epsilon_P(\cup\theta) = \sum_{\substack{W/A_2 \\ W \neq P}} n_{1,P} \cdot n_{2,W} \cdot (P \cdot W)_z - \sum_{\substack{V/A_1 \\ V \neq P}} n_{1,V} \cdot n_{2,P} \cdot (V \cdot P)_z.$$

This proves proposition (4.12).

We can now write down the map $\bar{\xi}_z$ from (4.8). We have identified the upper horizontal map in (4.8) with the map $\Lambda^{\mathrm{Comp}_z C}/\Lambda \to \Lambda^{\mathrm{Comp}_z C_{(z)}}/\Lambda$, which is induced by the canonical map $\mathrm{Comp}_z C_{(z)} \to \mathrm{Comp}_z C$. We define for a component P of C passing through z a number $\mu_{P,z}$:
If z is not mapped to a double point of A, we put $\mu_{P,z} = 0$. If z is mapped to a double point x of A, and if A_i and A_j are the components of A passing through x with $i < j$, we define:

$$(4.14)\quad \mu_{P,z} = \sum_{\substack{W/A_j \\ W \neq P}} n_{i,P} \cdot n_{j,W} \cdot (P \cdot W)_z - \sum_{\substack{V/A_i \\ V \neq P}} n_{i,V} \cdot n_{j,P} \cdot (V \cdot P)_z$$

Here as above and below W/A_j means $c_2(W) \subset A_j$. Note that $\mu_{P,z} = 0$, if P is the only component passing through z.

Assume that P over A_i is finite i.e. c_2 is finite on P. Then $n_{j,P} = 0$, and the condition $W \neq P$ in the first sum is automatically satisfied. Hence we get:

$$(4.14.1)\qquad \mu_{P,z} = n_{i,P} \cdot (P \cdot d_2^* A_j)_z = n_{i,P} \cdot \deg_z P/A_i$$

Similiary if P is finite over A_j, we get:

$$(4.14.2)\qquad \mu_{P,z} = -n_{j,P} \cdot \deg_z P/A_j$$

Let $e_P \in \Lambda^{\mathrm{Comp}_z C}/\Lambda$ be the generator corresponding to the component P. Then it follows from (4.12), that $\bar{\xi}_z(e_P) = \mu_{P,z}$.

Let x be a double point of A, and let z be a point of the fibre $E = c_2^{-1}(x)$. We denote by $k_z : \iota_{z*}\Lambda = \Lambda_z \to K_E = c_2^!\Lambda_x$ the map, which is adjoint to the identity. If z is not mapped to a double point of A, let be k_z the map $\Lambda_z \to 0$. The result of our efforts is:

(4.15) PROPOSITION. *The unique map making (4.6) into a morphism of triangles is:*

$$\sum_{P,z} \mu_{P,z} k_z : \oplus_{z \in Z} \Lambda_z^{\mathrm{Comp}_z\, C} / \Lambda_z \to c_2^! \Lambda_S.$$

Our next aim is to build a morphism of triangles

$$\begin{array}{ccccccc} c_1^*\Lambda & \longrightarrow & \oplus c_1^* a_{j*}\Lambda & \longrightarrow & c_1^* \iota_{S*}\Lambda & \longrightarrow & c_1^*\Lambda[1] \\ {\scriptstyle \kappa}\downarrow & & {\scriptstyle \kappa_{i,j}}\downarrow & & {\scriptstyle \beta_{x,y}}\downarrow & & {\scriptstyle \kappa[1]}\downarrow \\ c_2^!\Lambda & \longrightarrow & \oplus c_2^! a_{i*}\Lambda & \longrightarrow & c_2^! \iota_{S*}\Lambda & \longrightarrow & c_2^!\Lambda[1] \end{array} \tag{4.16}$$

Contrary to (4.6) this morphism will not be uniquely determined by our map κ (4.1). To define the second vertical map, we choose for any component W of C an index j, such that $c_1(W) \subset A_j$. We denote our choice by $A_j \leftarrow W$. We define a map $\kappa_{i,j} : c_1^* a_{i*}\Lambda \to c_2^! a_{i*}\Lambda$ by the formula (4.5):

$$\kappa_{i,j} = \sum_{A_j \leftarrow W / A_i} n_{i,W} \rho_{i,W} \mathrm{res}_{W,j}.$$

Here $\mathrm{res}_{W,j}$ is the natural map $c_1^* a_{j*}\Lambda \to \Lambda_W$. It follows from (4.5) that the matrix $\kappa_{i,j}$ makes the first square in (4.16) commutative.

Consider a point $z \in C$. We put $c_1(z) = y$ and $c_2(z) = x$. We will describe this situation by $y \leftarrow z \to x$. Assume that x and y are double points. Let be A_i and A_j, with $i < j$ the components of A passing through y. We denote by $\mathrm{MinComp}_z\, C$ those components W of C passing through z, such that $A_i \leftarrow W$ is our choice. We define a map β_z as the composition of maps:

$$\beta_z : c_1^*\Lambda_y \xrightarrow{\oplus \mathrm{res}_z} \Lambda_Z^{\mathrm{Comp}_z\, C} \xrightarrow{\text{projection}} \Lambda_z^{\mathrm{MinComp}_z\, C} \xrightarrow{\oplus_P \mu_{P,z} k_z} c_2^! \Lambda_x.$$

We put $\beta_{x,y} = \sum_{y \leftarrow z \to x} \beta_z$.

PROPOSITION. *The maps $\kappa_{i,j}$ and $\beta_{x,y}$ make (4.16) into a morphism of triangles.*

PROOF: Because of (4.15) it is enough to construct a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \oplus c_1^* a_{j*}\Lambda & \longrightarrow & c_1^* \iota_{S*}\Lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \oplus \Lambda_W & \longrightarrow & \oplus_{z \in Z} \Lambda_z^{\mathrm{Comp}_z\, C} / \Lambda_z & \longrightarrow & 0 \end{array}$$

We choose for the middle vertical arrow $\oplus_{A_j \leftarrow W} \mathrm{res}_{W,j}$. The rest of the proof is straightforward.

The map $\beta_z : c_1^* \Lambda_y \to c_2^! \Lambda_x$ induces a map on the cohomology (1.6):

$$\begin{aligned} \bar{\beta}_z &: \Lambda \\ &= H^0(A, \Lambda_y) \to H^0(C, c_1^* \Lambda_y) \to H^0(C, c_2^! \Lambda_x) \to H^0(A, \Lambda_x) = \Lambda. \end{aligned}$$

(4.18) LEMMA. *The map $\bar{\beta}_z$ is the multiplication by*

$$\sum_{P \in \mathrm{MinComp}_z C} \mu_{P,z}$$

PROOF: The proof is obvious from the fact, that $k_z : \Lambda_z \to c_2^! \Lambda_x$ induces on the cohomology the identity $\Lambda = H^\circ(C, \Lambda_z) \to H^\circ(C, c_2^! \Lambda) \to H^\circ(A, \Lambda_x) = \Lambda$.

We want know, what the correspondence $\kappa_{i,i}$ does on the cohomology. Consider the scheme $C_{i,i} = \cup_{A_i \leftarrow W/A_i} W$. Clearly we have a correspondence $A_1 \leftarrow C_{i,i} \to A_i$. The cycle $\Omega = \sum_{A_i \leftarrow W/A_i} n_{i,W} \cdot W$ defines a Verdier correspondence γ_Ω. By definition $\kappa_{i,i}$ is the direct image of this correspondence. Hence $\kappa_{i,i}$ and γ_Ω induce the same map on the cohomology. We call a component W exceptional, if the map $W \to A_i \times A_i$ is constant. By (1.9) for such a component γ_W induces zero on the cohomology. If the non exceptional components W have only isolated fixed points, we get from the Lefschetz formula

$$\text{(4.20.1)} \quad \mathrm{Tr}(\bar{\kappa}_{i,i} | R\Gamma(A_i, \Lambda)) = \sum_{\substack{A_i \leftarrow W/A_i \\ W \text{ not except.}}} \sum_{z \in \mathrm{Fix}\, W} n_{i,W} (W \cdot \Delta_{A_i})_z$$

From the diagram (4.16) and from (4.18), we obtain

$$\text{(4.20.2)} \quad \mathrm{Tr}(\bar{\kappa} | R\Gamma(A, \Lambda)) = \mathrm{Tr}(\bar{\kappa}_{i,i}) - \sum_{z \in \mathrm{Fix}\, C} \sum_{P \in \mathrm{MinComp}_z C} \mu_{P,z}.$$

Note that $\mu_{P,z}$ is zero, if z does not lie over a double point of A.

Let us return to the situation (4.1). To get the formulas (4.20), we need to take a modification $\pi : D \to \bar{D}$, which is smooth outside B. We will now rewrite the formulas purely in terms of $\bar{D}$.

Let $\bar{C} = \bar{D} - B$. A component W of C, which is not exceptional is the proper transform under π of a component $\bar{W}$ of $\bar{D}$. Let $u \in \bar{W}$ be a fixed point. Any point $z \in W$ that maps to u is a fixed point of W. We have the formula:

$$\sum_{z \to u} (W \cdot \Delta_{A_i})_z = (\bar{W} \cdot \Delta_{A_i})_u.$$

Clearly we have $n_{i,W} = n_{i,\bar{W}}$, since these numbers depend only on the general point. Hence the (4.20.1) may be rewritten:

$$\mathrm{Tr}(\bar{\kappa}_{i,i} | R\Gamma(A_i, \Lambda)) = \sum_{A_i \leftarrow \bar{W}/A_i} \sum_{u \in \mathrm{Fix}\, \bar{W}} n_{i,\bar{W}} (\bar{W} \cdot \Delta_{A_i})_u. \tag{4.21}$$

We will now define the numbers $\mu_{P,z}$ in a slightly more general situation. Let be $\varphi : F \to Y$ a proper and generically finite map of normal surfaces. Assume that Y is smooth and that we are given smooth curves A_1 and A_2 on Y, which meet transversally in a point x. Let be $C = \varphi^{-1}(A_1 \cup A_2)$. We denote by $n_{i,W}$ the multiplicity of a component W of C in $\varphi^*(A_i)$. Choose a resolution of singularities $\pi : \tilde{F} \to F$ and put $\tilde{\varphi} = \varphi \cdot \pi : \tilde{F} \to Y$.

We define as under (4.14) for a component $\tilde{P}$ of C and a point $z \in \tilde{P}$:

$$\mu_{\tilde{P},z} = \sum_{\substack{\tilde{W}/A_2 \\ \tilde{W} \neq \tilde{P}}} n_{1,\tilde{P}} n_{2,\tilde{W}} (\tilde{P} \cdot \tilde{W})_z - \sum_{\substack{\tilde{V}/A_1 \\ \tilde{V} \neq \tilde{P}}} n_{1,\tilde{V}} n_{2,\tilde{P}} (\tilde{V} \cdot \tilde{P})_z.$$

The right hand side is zero, if $\tilde{P}$ is the only component through z or if z is not mapped to the point x.

(4.22) LEMMA. *Assume that F is smooth. Let be P a component of C, and let $u \in P$ be a point. We denote by P' the proper transform of P under π . Then we have the formula:*

$$\sum_{z \to u} \mu_{P',z} = \mu_{P,u}. \tag{4.22.1}$$

Moreover let be $\tilde{Q}$ a component of $\tilde{C}$, which is mapped to a point u of C. Then we have:

$$\sum_{z \to u} \mu_{\tilde{Q},z} = 0.$$

PROOF: We may assume that u is mapped to x. We write $\pi^*P = P'+E$. From the equation $\tilde{\varphi}^*(A_i) = \sum_{\substack{W/A_i \\ W\neq P}} n_{i,W}\pi^*(W)+n_{i,P}P'+n_{i,P}E$ we get:

$$\sum_{z\to u}\mu_{P',z} = \sum_{z\to u}\Big(\big(\sum_{\substack{W/A_i \\ W\neq P}} n_{1,P}n_{2,W}(\pi^*W\cdot P')_z + n_{1,P}n_{2,P}(E\cdot P')_z\big)- \big(\sum_{\substack{V/A_1 \\ V\neq P}} n_{1,V}n_{2,P}(\pi^*V\cdot P')_z + n_{1,P}n_{2,P}(E\cdot P')_z\big)\Big).$$

The formula (4.22.1) follows, because $\sum_{z\to u}(\pi^*V\cdot P')_z = (V\cdot P)_u$. The remaining assertion of the lemma is similiar.

(4.23) If F is not smooth we take (4.22.1) as a definition of $\mu_{P,u}$. By the last lemma it is clearly independent of the resolution $\pi : \tilde{F}\to F$. As above one verifies the following properties:

(4.23.1) If P/A_1 is finite, $\mu_{P,u} = n_{1,P}\deg_u P/A_1$.

2) If P/A_2 is finite, $\mu_{P,u} = -n_{2,P}\deg_u P/A_2$.

3) For any point u, $\sum_P \mu_{P,u} = 0$.

4) For any component P, such that $\varphi(P) = x$, $\sum_{u\in P}\mu_{P,u} = 0$.

Applying our remark to the resolution $D\to\bar{D}$ the formula (4.20.2) reads:

$$\mathrm{Tr}(\bar{\kappa}|R\Gamma(A,\Lambda)) = \mathrm{Tr}(\bar{\kappa}_{i,i}) - \sum_{u\in\mathrm{Fix}\,\bar{C}}\ \sum_{\bar{P}\in\mathrm{MinComp}_u}\mu_{\bar{P},u}.$$

From now on let be $D=\bar{D}$, because we don't need the resolution anymore. Assume that D has only isolated fixed points. We get a trace formula for the correspondence γ_B by (4.1) and (4.21):

$$\mathrm{Tr}(\bar{\gamma}_B|R\Gamma_c(X,\Lambda)) = \sum_{u\in\mathrm{Fix}\,B}(B\cdot\Delta_x)_u + \sum_{u\in\mathrm{Fix}\,C} LT_u \tag{4.24}$$

$$LT_u = (D\cdot\Delta_Y)_u - \sum_{A_i}\ \sum_{A_i\leftarrow W/A_i} n_{i,W}(W\cdot\Delta_{A_i})_u + \sum_{P\in\mathrm{MinComp}_u}\mu_{P,u}.$$

We call the last expression the local term in u.

We want to write the local term in such a way, that the independence of our choice $A_i\leftarrow W$ becomes apparent. If W is finite over A_i with respect to c_1, there is no choice. We will denote this case by $A_i\backslash^f W$.

(4.25) THEOREM. *Let* $X \xleftarrow{b_1} B \xrightarrow{b_2} X$ *be a normal correspondence of a smooth, irreducible surface* X, *such that* b_2 *is finite and* b_1 *is proper. Let* $X \to Y$ *be a smooth compactification of* X, *such that* $A = Y - X$ *is the union of smooth, irreducible curves* $A_i, i = 1, \ldots, N$, *which intersect transversally. Let* D *be the normalization of* $Y \times Y$ *in* B. *Then we have for the complement* $C = D - B$, *that* C *is mapped to* A *under both projections* $Y \xleftarrow{d_1} D \xrightarrow{d_2} Y$. *We assume that* D *has only isolated fixed points.*

Assume that $u \in C$ *is a fixed point, which is not mapped to a double point of* A. *Let be* A_i *the component passing through* $x = d_2(u)$. *We define:*

$$LT_u = (D \cdot \Delta_Y)_u - \sum_{A_i \backslash W / A_i} n_{i,W}(W \cdot \Delta_{A_i})_u \tag{4.25.1}$$

Here the first intersection product is taken on $Y \times Y$ *and the second on* $A_i \times A_i$.

Assume u *is mapped to a double point* x, *and that* A_i *and* A_j *for* $i < j$ *are the components passing through* x. *Then we define* Ω_u *to be the set of all components* P *of* C *passing through* u, *such that either* P *is finite over* A_i *with respect to* d_1, *or* $d_1(P) = x$ *and* $d_2(P) \subset A_j$, *i.e.* $A_i \backslash^f P$ *or* $x \backslash P / A_j$. *In this case we define:*

$$LT_u = (D \cdot \Delta_Y)_u - \sum_{A_i} \sum_{A_i \backslash^f W / A_i} n_{i,W}(W \cdot \Delta_{A_i})_u + \sum_{P \in \Omega_u} \mu_{P,u}. \tag{4.25.2}$$

The $\mu_{p,u}$ *are defined by (4.23).*

Then the formula (4.24) holds with the local terms (4.25.1) and (4.25.2).

Remark. It is easy to see from (4.23) that the last sum is independent of the order of the A_i.

PROOF: We deduce this from (4.24). In the case, where u is not mapped to a double point, the local terms defined by (4.25.1) and (4.24) agree. Consider the case, where $x = d_1(u) = d_2(u)$ is a double point of A. Assume A_1 and A_2 are the components passing through x. We know that the theorem holds, if we replace the last sum in (4.25.2) by

$$-\sum_{i=1,2} \sum_{\substack{x \backslash P / A_i \\ A_i \leftarrow P}} n_{i,P}(P \cdot \Delta_{A_i})_u + \sum_{P \in \operatorname{MinComp}_u C} \mu_{P,u}.$$

But if P appears in the first double sum, it is finite over A_i with respect to the second projection. Hence we have by (4.23): $n_{i,P}(P \cdot \Delta_{A_i})_u = (-1)^{i+1}\mu_{P,u}$. The theorem follows easily.

(4.26) Remark. Assume that X is obtained by base change from a finite field. Applying (2.3) to the correspondences B and $A \leftarrow W \rightarrow A$ for W/A finite, we see that after multipliying D by a power of the Frobenius greater than $\deg b_2$, it becomes a correspondence with isolated fixed points.

5. A LEMMA FROM COMMUTATIVE ALGEBRA

Let A be a regular local ring of dimension 2 with maximal ideal $\mathfrak{m}$. Let $\xi_1, \xi_2, \eta_1, \eta_2$ be elements of A, which generate an $\mathfrak{m}$-primary ideal. Assume that there exists a number s, such that $(\xi_1\xi_2)^s \in A\eta_1\eta_2$.

We write down the Weil divisors of these elements:

$$\operatorname{div}\eta_i = \sum n_{i,V}V, \quad \operatorname{div}\xi_i = \sum m_{i,V}V, \qquad \text{where } i = 1,2 \in \mathbb{Z}/2.$$

We denote by $\mathfrak{p}_V$ the prime ideal of V and by A_V the ring $A/\mathfrak{p}_V$. We write:

$$\begin{array}{rlrl} V/i, & \text{if } \eta_i \in \mathfrak{p}_V, & i\backslash V, & \text{if } \xi_i \in \mathfrak{p}_V \\ V/^f i, & \text{if } \eta_i \in \mathfrak{p}_V \text{ and } \eta_{i+1} \notin \mathfrak{p}_V, & i\backslash^f V, & \text{if } \xi_i \in \mathfrak{p}_V \text{ and } \xi_{i+1} \notin \mathfrak{p}_V \\ V/\cdot, & \text{if } V/1 \text{ and } V/2, & \cdot\backslash V, & \text{if } 1\backslash V \text{ and } 2\backslash V \end{array}$$

We introduce the following constants:

$$\begin{aligned} k_0 &= \max_i \max_{V/^f i} \operatorname{length} A_V/\eta_{i+1} \\ k_1 &= \min\{s \mid (\xi_1\xi_2)^s \in A\eta_1\eta_2\} \\ k_2 &= \min\{N \mid Nm_{i,V} \geq n_{i,V} \text{ if } m_{i,V} > 0\} \end{aligned}$$

Let be $k = k_0 + \max(k_1, k_2) \leq k_0 + \max_V(n_{1,V} + n_{2,V})$. We see that k is bounded by a constant depending only on η_1 and η_2.

Let be α_1 and α_2 elements of the fraction field of A, such that the support of $\operatorname{div}\alpha_i$ is contained in the support of $\operatorname{div}\xi_i$. Fix a natural number c, such that $\xi_i^c\alpha_i \in A$.

We write in this section for length simply $l(\dots)$.

(5.1) MAIN LEMMA. *For a natural number $N > k + c$ we have the formula:*

$$
\begin{aligned}
&l\left(A/\left(\alpha_1\xi_1^N - \eta_1, \alpha_2\xi_2^N - \eta_2\right)\right) \\
&\quad = \sum_{1\backslash V/\cdot} n_{1,V} l(A_V/\alpha_2\xi_2^N) + \sum_{2\backslash W/\cdot} n_{2,W} l(A_W/\alpha_1\xi_1^N) \\
&\qquad + \sum_{\substack{V/^{f}1 \\ 2\backslash V/\cdot}} \sum_{\substack{W/^{f}2 \\ 1\backslash W/\cdot}} n_{1,V} n_{2,W} (V \cdot W) - \sum_{1\backslash V/\cdot} \sum_{2\backslash W/\cdot} n_{1,V} n_{2,W} (V \cdot W).
\end{aligned}
$$

Hence for big N the left hand side is a linear function in N. We call the second row the fixed term. The reader should now start with the next section.

For the proof we need a few preparations. We denote by $\mathfrak{c}(N)$ the ideal generated by $\alpha_1\xi_1^N - \eta_1$ and $\alpha_2\xi_2^N - \eta_2$.

(5.2) For $N > k_1 + c$ we have $\eta_1\eta_2 \in \mathfrak{c}(N)$. Indeed, modulo $\mathfrak{c}(N)$ we have the equation: $\eta_1\eta_2 = \alpha_1\alpha_2(\xi_1\xi_2)^N = \alpha_1\alpha_2(\xi_1\xi_2)^{N-k_1} a\eta_1\eta_2$, for some $a \in A$. Hence the result.

(5.3) For $N > \max(k_0, k_1) + c$ the ideal $\mathfrak{c}(N)$ is $\mathfrak{m}$-primary.

Assume there is a prime ideal $\mathfrak{p}$ of height one, which contains $\mathfrak{c}(N)$. By (5.2) we may further assume, that for example $\eta_1 \in \mathfrak{p}$. Since the ξ_i and η_i generate a $\mathfrak{m}$-primary ideal, we get $\eta_2 \notin \mathfrak{p}$. By definition of k_0, we obtain $\xi_2^{N-c} = a\eta_2 \pmod{\mathfrak{p}}$ with $a \in \mathfrak{m}$, and hence $\alpha_2\xi_2^N = \alpha_2\xi_2^c a\eta_2 \pmod{\mathfrak{p}}$. This is a contradiction.

For an element $t \in A$ we introduce the function χ_t on the category of A-modules: $\chi_t(M) = l(\operatorname{Coker} t|M) - l(\operatorname{Ker} t|M)$. $\chi_t(M)$ is defined if both length are finite. It is additive in short exact sequences, if it is defined for two of the modules. For two elements s and t and an A-module M we have $\chi_{st}(M) = \chi_s(M) + \chi_t(M)$, if two terms of this equation are defined. For a module M of finite length we have $\chi_t(M) = 0$.

Proof of (5.1): Let be $d_i = \text{g.c.d. } (\eta_i, \xi_i^N)$. Since $N > k_2$ we have $\operatorname{div} d_i = \sum_{i\backslash V/i} n_{i,V} V$. Let $w_i(N) = \alpha_i\xi_i^N - \eta_i$ and $w_i'(N) = \frac{\alpha_i}{d_i} w_i(N) = \frac{\alpha_i}{d_i}\xi_i^N - \eta_i'$. We remark, that $d_i = \text{g.c.d. } (w_i(N), \eta_i)$, since $N > k_2 + c$. We have the exact sequence:

$$0 \to A/w_1'(N) \xrightarrow{d_1} A/w_1(N) \to A/d_1 \to 0.$$

Therefore we get:

$$\begin{aligned} l(A/\mathfrak{c}(N)) &= \chi_{w_2(N)}(A/w_1'(N)) + \chi_{w_2(N)}(A/d_1) \\ &= \chi_{w_2(N)}(A/w_1'(N)) + \sum_{1\backslash V/1} n_{1,V} l(A_V/w_2(N)) \end{aligned}$$

The same argument gives:

$$\begin{aligned} \chi_{w_2(N)}(A/w_1'(N)) &= \chi_{w_1'(N)}(A/w_2(N)) \\ &= \chi_{w_1'(N)}(A/w_2'(N)) + \chi_{w_1'(N)}(A/d_2) \\ &= \chi_{w_1'(N)}(A/w_2'(N)) + \chi_{w_1(N)}(A/d_2) - \chi_{d_1}(A/d_2). \end{aligned}$$

But by our choice of N we have $\frac{\alpha_i}{d_i}\xi_i^N \in \mathfrak{m}\cdot\eta_{i+1}'$. Indeed, by our choice of k_1 we have $(\xi_1\xi_2)^{k_1} \in A\eta_1\eta_2 \subset A\eta_2'$. Since η_2' and ξ_2 are relatively prime, we get $\xi_1^{k_1} \in A\eta_2'$. The assertion follows.

By the lemma of Nakayama we find:

$$\chi_{w_1'(N)}(A/w_2'(N)) = l(A/(w_1'(N), w_2'(N)) = l(A/(\eta_1', \eta_2')).$$

Hence so far we have proven the formula:

$$\begin{aligned} &l(A/\mathfrak{c}(N)) \\ &\qquad = l(A/\eta_1', \eta_2')) + \sum_{1\backslash V/1} n_{1,V} l(A_V/w_2(N)) \\ &\qquad\qquad + \sum_{2\backslash W/2} n_{2,W} l(A_W/w_1(N)) \\ &\qquad\qquad\qquad - \sum_{1\backslash V/1}\sum_{2\backslash W/2} n_{1,V} n_{2,W} (V\cdot W) \end{aligned}$$

(5.4) Convention: Since it takes less space, we denote the last and similiar double sums by $(1\backslash V/1, 2\backslash W/2)$. Another example for this notation is:

$$l(A/(\eta_1', \eta_2')) = (2\backslash^f V/1, 1\backslash^f W/2).$$

Finally we compute $l(A_V/w_2(N))$. Consider first a component V of type $V/^f 1$. Since $\eta_2 \notin \mathfrak{p}_V$, we get $\xi_2^{k_0} \in A\eta_2 + \mathfrak{p}_V$. This implies $\alpha_2\xi_2^N \in \mathfrak{m}\cdot\eta_2 + \mathfrak{p}_V$. Therefore if V is of type $V/^f 1$, we find $l(A_V/w_2(N)) = l(A_V/\eta_2)$. If $\eta_2 \in \mathfrak{p}_V$, i.e. V is of type $V/\cdot$, we get $l(A_V/w_2(N)) = l(A_V/\alpha_2\xi_2^N)$.

Taking this into account, we obtain:

$$
\begin{aligned}
&l(A/\mathfrak{c}(N)) \\
&\quad = \sum_{1\backslash V/\cdot} n_{1,V}(A_V/\alpha_2\xi_2^N) \\
&\qquad + \sum_{2\backslash W/\cdot} n_{2,W}l(A_W/\alpha_1\xi_1^N) + (1\backslash V/^f1, W/2) \\
&\qquad + (V/1, 2\backslash W/^f2) + (2\backslash^f V/1, 1\backslash^f W/2) - (1\backslash V/1, 2\backslash W/2).
\end{aligned}
$$

Here is the verification, that the fixed term in this formula is equal to the fixed term in (5.1):

$$
\begin{aligned}
\text{fixed term} = {} & (V/^f1, W/2) - (2\backslash^f V/^f1, W/2) \\
& + (V/^f1, 2\backslash W/^f2) + (V/\cdot, 2\backslash W/^f2) \\
& + (2\backslash^f V/^f1, 1\backslash^f W/2) + (2\backslash V/\cdot, 1\backslash^f W/2) \\
& - (V/^f1, 2\backslash W/2) - (1\backslash V/\cdot, 2\backslash W/2) \\
& + (2f\backslash V/^f1, 2\backslash W/2) \\
\\
= {} & (V/^f1, W/^f2) + (V/^f1, W/\cdot) \\
& - (V/^f1, 2\backslash W/\cdot) + (V/\cdot, 2\backslash W/^f2) \\
& + (2\backslash V/\cdot, 1f\backslash W/2) - (1\backslash V/\cdot, 2\backslash W/2) \\
\\
= {} & (V/^f1, W/^f2) + (V/^f1, 1\backslash W/\cdot) \\
& + (2\backslash V/\cdot, 2\backslash W/^f2) - (1\backslash V/\cdot, 2\backslash W/\cdot) \\
& + (2\backslash V/\cdot, 1\backslash^f W/2) \\
\\
= {} & (V/^f1, W/^f2) + (V/^f1, 1\backslash W/\cdot) \\
& + (2\backslash V/\cdot, W/^f2) + (2\backslash V/\cdot, 1\backslash W/\cdot) + \\
& - (1\backslash V/\cdot, 2\backslash W/\cdot)
\end{aligned}
$$

This is the result we wanted.

6. Vanishing of the local terms

In this section we show that Deligne's conjecture holds in the situation (4.25). We start with a normal correspondence of a smooth surface of finite type $X \xleftarrow{b_1} B \xrightarrow{b_2} X$, such that b_1 is proper and b_2 is finite. We consider a compactification, with the same properties as above (4.25). Especially D is normal and $(d_1, d_2) : D \to Y \times Y$ is finite.

We assume that X is obtained by base change from a scheme $\mathbf{X}$ over a finite field $\mathbf{F}_q$ and that the compactification $X \to Y$ is also defined over $\mathbf{F}_q$. We use the symbols B_n and D_n to denote the correspondences $(\mathrm{Fr}_X^n b_1, b_2)$ and $(\mathrm{Fr}_Y^n d_1, d_2)$.

To prove the Theorem (2.7), it is enough to consider the case $L = \Lambda$, where Λ may be finite or $\mathbf{Q}_\ell$. Hence (2.7) follows from the more precise statement:

(6.1) THEOREM. *There exists a constant n_0, which depends only on d_2, such that for $n > n_0$ the local terms LT_u of the correspondence D_n defined by (4.25) vanish.*

The proof is the rest of this section.

This theorem has the advantage to be invariant under base change $\mathbf{F}_q \to \mathbf{F}_{q^r}$. Indeed, if we have proven the theorem over $\mathbf{F}_{q^r}$, it follows that it holds for all correspondences $(\mathrm{Fr}_Y^m d_1, d_2)$ with the same constant. Note that Fr_Y denotes the Frobenius over $\mathbf{F}_q$.

Therefore we may assume that the divisor at infinity $\mathbf{A} = \mathbf{Y} - \mathbf{X}$ is a union of smooth divisors over $\mathbf{F}_q$, which meet transversally.

We fix n. We denote by $u \in A$ a fixed point of the correspondence D_n, i.e. $\mathrm{Fr}_Y^n d_1(u) = d_2(u) = x$. We choose rational local parameters t_1, t_2 in the point x, i.e. local parameters on the scheme $\mathbf{Y}$ in the point $\mathbf{x}$ lying below x. If A is smooth in the point x, we may assume that $t_1 = 0$ is the local equation of A in the point x. Assume that x is a double point of A, and that A_1 and A_2 are the two components of A passing through x. Because of the base change made, we may assume that $t_i = 0$ is the local equation of A_i. Clearly t_1, t_2 are also local parameters in the point $z = d_1(u)$. Therefore the following functions are defined in u:

$$\eta_i = d_2^*(t_i), \qquad \xi_i = d_1^*(t_i).$$

We have $\xi_i^{q^n} = d_1^* \mathrm{Fr}_Y^*(t_i)$.

Let us first consider the case, where x is not a double point. Since $d_1^{-1}(A) = d_2^{-1}(A)$, we find a number s, such that $\xi_1^s \in O_{D,u}\eta_1$. Clearly it is enough to take s greater than the multiplicities appearing in

$\operatorname{div}\eta_1$. This choice depends only on d_2. Assume that $q^n > s$. We obtain:

$$\begin{aligned}(D_n \cdot \Delta_Y)_u & \\ &= l(O_{D,u}/(\xi_1^{q^n} - \eta_1, \xi_2^{q^n} - \eta_2)) = l(O_{D,u}/(\eta_1, \xi_2^{q^n} - \eta_2)) \\ &= \sum_{V/A_1} n_{1,V} l(O_{V,u}/(\xi_2^{q^n} - \eta_2)) = \sum_{V/A_1} n_{1,V}(V_n \cdot \Delta_{A_i})_u\end{aligned}$$

Hence the local term (4.25.1) vanishes in this case.

Consider now the case, where u is mapped to a double point x. Let V be a component of type $V/^f A_1$. Then $O_{V,u}/\eta_2$ is of finite length smaller than $\deg d_2$. Hence for $q^n > \deg d_2$, we get:

$$\begin{aligned}n_{1,V}(V_n \cdot \Delta_{A_i})_u & \\ &= n_{1,V} l(O_{V,u}/(\xi_2^{q^n} - \eta_2)) = n_{1,V} l(O_{V,u}/\eta_2) = n_{1,V} \deg_u V/A_1 \\ &= \mu_{V,u}\end{aligned}$$

Similiary for a component W of type $W/^f A_2$, we get for $q^n > \deg d_2$:

$$n_{1,W}(W_n \cdot \Delta_{A_2})_u = -\mu_{W,u}.$$

If we write in the following $A_1 \backslash V$ etc., this is meant with respect to the correspondence D_n. If $u \in V$, the meaning of $A_1 \backslash V$ is independent of n. We see, that the local term (4.25.2) for $q^n > \deg d_2$ may be written as follows:

$$\begin{aligned}LT_u = (D_n \cdot \Delta_Y)_u &- \sum_{i=1,2} \sum_{A_i \backslash V/x} n_{i,V}(V_n \cdot \Delta_{A_i})_u \\ &- \sum_{A_1 \backslash^f P/^f A^1} \mu_{P,u} + \sum_{A_2 \backslash^f P/^f A_2} \mu_{P,u} \\ &+ \sum_{A_1 \backslash^f P} \mu_{P,u} + \sum_{x \backslash P/A_2} \mu_{P,u}.\end{aligned}$$

From this we get easily:

(6.2) Lemma. *For $q^n > \deg d_2$ the local term (4.25.2) is:*

$$\begin{aligned}LT_u & \\ &= (D_n \cdot \Delta_Y)_u - \sum_{A_1 \backslash V/x} n_{1,V} l(O_{V,u}/\xi_2^{q^n}) - \sum_{A_2 \backslash W/x} n_{2,W} l(O_{W,u}/\xi_1^{q^n}) \\ &+ \sum_{P^f/A_2} \mu_{P,u} + \sum_{A_1 \backslash P/x} \mu_{P,u}\end{aligned}$$

We have to show that this expression is zero for $n > n_0$, where n_0 depends only on d_2.

Consider a desingularization $\pi : \tilde{D} \to D$. Let be $w_i(n) = \xi_i^{q^n} - \eta_i$. We denote a function on D and its inverse image on $\tilde{D}$ by the same letter. We put $\operatorname{div}_{\tilde{D}} \eta_1 = F + E$, where F/D is finite in u and $\pi(E) = u$. Let c be the maximal multiplicity, which appears in E. This number depends only on d_2. We obtain the inequality $E \leq \operatorname{div}_{\tilde{D}} \xi_1^c$. For $q^n > c$ we get $\operatorname{div}_{\tilde{D}} w_1(n) = F_1 + E$, where the divisor F_1 is finite over u. We find by the projection formula:

$$\begin{aligned}(D_n \cdot \Delta_Y)_u &= l(O_{D,u}/(w_1(n), w_2(n))) = (\operatorname{div}_D w_1(n) \cdot \operatorname{div}_D w_2(n))_u \\ &= \sum_{z \to u} (F_1 \cdot \operatorname{div}_{\tilde{D}} w_2(n))_z\end{aligned}$$

Denote by e_z a local equation of E in the point z. We define:

$$l_{n,z} = l\left(O_{\tilde{D},z} / \left(\frac{1}{e_z}\left(\xi_1^{q^n} - \eta_1\right), \xi_2^{q^n} - \eta_2\right)\right). \tag{6.3}$$

The elements ξ_1, ξ_2, $\eta_1' = \frac{\eta_1}{e_z}$, η_2 satisfy the assumptions of our main lemma. We have the equation:

$$(D_n \cdot \Delta_Y)_u = \sum_{z \to u} l_{n,z}.$$

For a divisor V on D let be V' its proper transform on $\tilde{D}$. We obtain:

$$l(O_{V,u}/\xi_2^{q^n}) = \sum_{z \to u} l(O_{V',z}/\xi_2^{q^n}).$$

Finally by definition of the $\mu_{P,u}$ we get:

$$\mu_{P,u} = \sum_{z \to u} \mu_{P',z}.$$

Hence it is sufficient to show, that for n large enough (depending only on d_2), we have the equation:

(6.4)

$$\begin{aligned}l_{n,z} = &\sum_{A_1 \backslash V/x} n_{1,V'} l(O_{V',z}/\xi_2^{q^n}) + \sum_{A_2 \backslash W/x} n_{1,W'} l(O_{W',z}/\xi_1^{q^n}) \\ &+ \sum_{P/^f A_1} \mu_{P',z} + \sum_{A_2 \backslash P/x} \mu_{P',z}\end{aligned}$$

The main lemma 5.1 gives an expression for $l_{n,z}$. We want to rewrite the right hand side of (6.4) in the notation used there with respect to the elements ξ_1, ξ_2, η_1', η_2. Here is a dictionary:

$$\begin{aligned}
\tilde{V}/1, &\quad \text{if } \tilde{V} \text{ is a proper transform } V', \text{ with } V/A_1.\\
\tilde{V}/^f 1, &\quad \text{if moreover } V/^f A_1.\\
\tilde{W}/2, &\quad \text{if } \tilde{W}/A_2.\\
\tilde{W}/^f/2, &\quad \text{if } \tilde{W}/^f A_2, \text{ or if } \tilde{W} \text{ is exceptional.}\\
\tilde{V}/\cdot, &\quad \text{if } \tilde{V} \text{ is a proper transform } V', \text{ with } V/x.\\
i\backslash\tilde{V}, &\quad \text{if } A_i\backslash\tilde{V}.\\
i\backslash^f\tilde{V}, &\quad \text{if } A_i\backslash^f\tilde{V}.
\end{aligned}$$

In the new notation the right hand side of (6.4) looks like this:
(6.5)

$$\sum_{1\backslash\tilde{V}/\cdot} n_{1,\tilde{V}} l(O_{\tilde{V},z}/\xi_2^{q^n}) + \sum_{2\backslash\tilde{W}/\cdot} n_{2,\tilde{W}} l(O_{\tilde{W},z}/\xi_1^{q^n}) + \sum_{\tilde{P}/^f 1} \mu_{\tilde{P},z} + \sum_{2\backslash\tilde{P}/\cdot} \mu_{\tilde{P},z}$$

Using our convention (5.4), we may write:

$$\sum_{\tilde{P}/^f 1} \mu_{\tilde{P},z} = \sum_{\tilde{P}/^f 1}\sum_{\tilde{W}/A_2} n_{1,\tilde{P}} n_{2,\tilde{W}} (\tilde{P}\cdot\tilde{W})_z = (\tilde{V}/^f 1, \tilde{W}/2).$$

For the last sum of (6.5) we get:

$$\sum_{2\backslash\tilde{P}/\cdot} = \sum_{2\backslash\tilde{P}/\cdot}\sum_{\substack{\tilde{W}/A_2\\ \tilde{W}\neq\tilde{P}}} n_{1,\tilde{P}} n_{2,\tilde{W}} (\tilde{P}\cdot\tilde{W})_z - \sum_{2\backslash\tilde{P}/\cdot}\sum_{\substack{\tilde{V}/A_1\\ \tilde{V}\neq\tilde{P}}} n_{1,\tilde{V}} n_{2,\tilde{P}} (\tilde{V}\cdot\tilde{P})_z.$$

In the last expression we can replace the conditions $\tilde{W}\neq\tilde{P}$ and $\tilde{V}\neq\tilde{P}$ by the conditions: $\tilde{W}$ is not of type $2\backslash\tilde{W}/\cdot$ respectively $\tilde{V}$ is not of type $2\backslash\tilde{V}/\cdot$. This does not change the result. But by our dictionary the condition: $\tilde{W}/A_2$ and $\tilde{W}$ is not of type $2\backslash\tilde{W}/\cdot$, is equivalent to the condition: $\tilde{W}/^f 2$ or $1\backslash\tilde{W}/\cdot$. Also the condition: $\tilde{V}/A_1$ and $\tilde{V}$ is not of type $2\backslash\tilde{V}/\cdot$, is equivalent to the condition: $\tilde{V}/^f 1$, or $1\backslash\tilde{V}/\cdot$, or $\tilde{V}$ is exceptional. We recall that $\sum_{\tilde{V} \text{ exceptional}} n_{1,\tilde{V}}\tilde{V} = E|_{\operatorname{Spec} O_{\tilde{D},z}} =$

$\operatorname{div} e_z$. Taking this into account, we may write:

$$\begin{aligned}
\sum_{2\backslash\tilde{P}/\cdot} \mu_{\tilde{P},z}
&= \sum_{2\backslash\tilde{V}/\cdot} \sum_{\substack{\tilde{W}/^f 2\\ 1\backslash\tilde{W}/\cdot}} n_{1,\tilde{V}} n_{2,\tilde{W}} (\tilde{V}\cdot\tilde{W})_z \\
&\quad - \sum_{\substack{\tilde{V}/^f/1\\ 1\backslash\tilde{V}/\cdot}} \sum_{2\backslash\tilde{W}/\cdot} n_{1,\tilde{V}} n_{2,\tilde{W}} (\tilde{V}\cdot\tilde{W})_z \\
&\qquad - \sum_{2\backslash\tilde{W}/\cdot} n_{2,\tilde{W}} l(O_{\tilde{W},z}/e_z).
\end{aligned}$$

Hence for the last two sums of (6.5) we get:

$$\begin{aligned}
&\sum_{\tilde{P}/^f 1} \mu_{\tilde{P},z} + \sum_{2\backslash\tilde{P}/\cdot} \mu_{\tilde{P},z} \\
&\quad = -\sum_{2\backslash\tilde{W}/\cdot} n_{2,\tilde{W}} l(O_{\tilde{W},z}/e_z) + (\tilde{V}/^f 1, \tilde{W}/2) + (2\backslash\tilde{V}/\cdot, \tilde{W}/^f 2) \\
&\quad = (2\backslash\tilde{V}/\cdot, 1\backslash\tilde{W}/\cdot) - (\tilde{V}/^f 1, 2\backslash\tilde{W}/\cdot) - (1\backslash\tilde{V}/\cdot, 2\backslash\tilde{W}/\cdot) \\
&\quad = -\sum_{2\backslash\tilde{W}/\cdot} n_{2,\tilde{W}} l(O_{\tilde{W},z}/e_z) + (\tilde{V}/^f 1, \tilde{W}/^f 2) + (\tilde{V}/^f 1, 1\backslash\tilde{W}/\cdot) \\
&\qquad + (2\backslash\tilde{V}/\cdot, \tilde{W}/^f 2) + (2\backslash\tilde{V}/\cdot, 1\backslash\tilde{W}/\cdot) - (1\backslash\tilde{V}/\cdot, 2\backslash\tilde{W}/\cdot)
\end{aligned}$$

Inserting this result in (6.5) we see that it is enough to verify the following equation for $n > n_0$, where n_0 depends only on d_2.

$$\begin{aligned}
l_{n,z} = \sum_{1\backslash\tilde{V}/\cdot} n_{1,\tilde{V}} l\left(O_{\tilde{V},z}/\xi_2^{q^n}\right) &+ \sum_{2\backslash\tilde{W}/\cdot} n_{2,\tilde{W}} l\left(O_{\tilde{W},z}/\frac{1}{e_z}\xi_1^{q^n}\right) \\
&+ (\tilde{V}/^f 1, \tilde{W}/^f 2) + (\tilde{V}/^f 1, 1\backslash\tilde{W}/\cdot) \\
&+ (2\backslash\tilde{V}/\cdot, \tilde{W}/^f 2) + (2\backslash\tilde{V}/\cdot, 1\backslash\tilde{W}/\cdot) \\
&- (1\backslash\tilde{V}/\cdot, 2\backslash\tilde{W}/\cdot).
\end{aligned}$$

Since we already know that the constant c depends only on d_2, this is exactly what (5.1) tells us.

REFERENCES

1. D. Alibert, *Termes locaux de la formules de Lefschetz pour les courbes*, Compositio Math. **58** (1986), 135–190.
2. J.-L. Brylinski et J.-P. Labesse, *Cohomologie d'intersection et fonction L de certaines varieties de Shimura*, Ann. scient. Ec. Norm. Sup., 4[e] serie **t. 17** (1984), 361–412.
3. P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES **52** (1980), 313–428.
4. P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. **103, 1** (1976), 103–161.
5. W. G. Drinfield, *Cohomology of the compactified moduli variety of F-sheafs of rank 2 (in Russian)*, in "Automorphic functions and number theory III," Nauka, 1987, pp. 107–158.
6. Y. Z. Flicker and D. Kazhdan, *Geometric Ramanujan conjecture and Drinfeld reciprocity law, preprint*, IHES/M/87/39 (1987).
7. U. Jannsen, *Continuous Cohomology, preprint*, Regensburg.

SGA 1 "Revetments etales et groups fondamental, par A. Grothendieck, Lecture Notes in Mathematics N° 224," Springer Verlag, 1971.

SGA 4 "Theorie des topos et cohomologie etale des schemas, par M. Artin, A. Grothendieck, J.-L. Verdier, Lecture Notes in Mathematics N°s 269, 270, 305," Spinger Verlag, 1972–1973.

SGA 5 "Cohomologie l-adique et fonctions L, par A. Grothendieck, Lecture Notes in Mathematics N° 589," Springer Verlag, 1977.

SGA $4\frac{1}{2}$ "Cohomologie etale, par P. Deligne, Lecture Notes in mathematics N° 569," Springer Verlag, 1977.

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L^2-cohomology of Shimura varieties

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Before the Ann Arbor conference, I allowed the designation of a title for my talks to slide into the hands of the organizers. Inevitably, they left it as "The proof of the Zucker conjecture". I didn't expect that a better alternative existed. Perhaps the one chosen here shows that I was mistaken.

Since I have already written a full-length survey article on the subject, to be published elsewhere [39], I will attempt here to give just a brief indication of the proof of the result, together with its relevance to the theme of the conference.

1. L^2-COHOMOLOGY AND THE STATEMENT OF THE CONJECTURE

(1.1) Let $\mathbf{G}$ be a reductive algebraic group defined over $\mathbf{Q}$. The Shimura variety associated to an open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_f)$ (see [30]) has finitely many connected components; each of them is isomorphic to an Hermitian locally symmetric space $M = \Gamma\backslash X$, where

$$X\ \cancel{\mathfrak{X}} \simeq Z_\infty\backslash G_\infty/K_\infty \quad (K_\infty \text{ maximal compact in } G_\infty = \mathbf{G}(\mathbf{R}),$$
$$Z_\infty \text{ the center of } G_\infty)$$

—here, the Lie group (i.e., classical) topology is used for G_∞—and Γ is an arithmetic group of the form $gK_fg^{-1} \cap \mathbf{G}(\mathbf{Q})$ with $g \in \mathbf{G}(\mathbb{A}_f)$. We assume throughout that K_f is sufficiently small, so that Γ is torsion-free, or even neat [5 : Ch.17], since it suffices to consider this case.

(1.2) Even if one sets aside questions of an arithmetical nature, it is still interesting to understand automorphic forms from a topological point of view. One begins with the isomorphism

$$H^{\cdot}(\Gamma, V) \simeq H^{\cdot}(M, \mathbf{V}) \tag{2}$$

for any complex, finite-dimensional representation space V of Γ, where $\mathbf{V}$ denotes the corresponding local system on M. By the deRham

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theorem, the inclusion of the constant sheaf $\mathbb{C}$ into the complex of smooth $\mathbb{C}$-valued differential forms $\mathcal{E}^{\cdot}_M$ is a quasi-isomorphism, so one obtains:

$$H^{\cdot}(M, \mathbf{V}) \simeq \mathbf{H}^{\cdot}(M, \mathcal{E}^{\cdot}_M \otimes_{\mathbb{C}} \mathbf{V}) \simeq H^{\cdot}(E^{\cdot}(M, \mathbf{V})), \tag{3}$$

—where $E^{\cdot}(M, \mathbf{V}) = H^0(M, \mathcal{E}^{\cdot}_M \otimes \mathbf{V})$—since $\mathcal{E}^{\cdot}_M$ is fine.

(1.3) Suppose that V is actually an irreducible representation of G_∞. From the observation that $\Gamma\backslash G_\infty$ is a principal K_∞-bundle over M, it follows that (see [28])

$$E^{\cdot}(M, \mathbf{V}) \simeq (E^0(\Gamma\backslash G_\infty) \otimes \Lambda^{\cdot}(\mathbf{g}/\mathbf{k})^* \otimes V)^{K_\infty}, \tag{4}$$

where $\mathbf{g}$ and $\mathbf{k}$ denote the Lie algebras of G_∞ and K_∞ respectively. Thus enters relative Lie algebra cohomology:

$$H^{\cdot}(M, \mathbf{V}) \simeq H^{\cdot}(\mathbf{g}, K_\infty; E^0(\Gamma\backslash G_\infty) \otimes V). \tag{5}$$

Remark. One has similarly that $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ is a principal $(K_f K_\infty)$-bundle over the Shimura variety.

Of course, $E^0(\Gamma\backslash G_\infty)$ is already a representation space for G_∞, but to get to the level of unitary representations, we must bring in the customary notion of square-integrability on M. Before doing so, we wish to remark that there is (for a local system on any C^∞ manifold) an intrinsically defined L^2 complex of sheaves, $\mathcal{L}^{\cdot}_{(2)}(M, \mathbf{V})$, whose sections are just the domain of the exterior derivative in the context of locally-L^2 forms, here denoted $\mathcal{L}^{\cdot}_{(2),\mathrm{loc}}(M, \mathbf{V})$. An L^2 version of the Poincaré lemma gives that $\mathcal{L}^{\cdot}_{(2)}(M, \mathbf{V})$ is quasi-isomorphic to $\mathbf{V}$ or $\mathcal{E}^{\cdot}_M \otimes \mathbf{V}$. We write

$$H^{\cdot}(M, \mathbf{V}) \simeq H^{\cdot}_{(2),\mathrm{loc}}(M, \mathbf{V}) \stackrel{\mathrm{def}}{=} H^{\cdot}(L^{\cdot}_{(2),\mathrm{loc}}(M, \mathbf{V})), \tag{6}$$

but note that formula (5) has no immediate analogue, in view of issues surrounding the domain of d.

(1.4) To get a global notion, one must metrize both M and $\mathbf{V}$. For that, one descends to M the G_∞-invariant Riemannian metric on X determined by the Killing form of $\mathbf{g}$, and uses the G_∞-equivariant metrization of $\mathbf{V}$ induced by a so-called admissible inner product (see [28 : p.375]) on V. One then defines (for a metrized local system on any Riemannian manifold) the global L^2-complex $L^{\cdot}_{(2)}(M, \mathbf{V})$ analogously to (1.3), and puts

$$H^{\cdot}_{(2)}(M, \mathbf{V}) = H^{\cdot}(L^{\cdot}_{(2)}(M, \mathbf{V})). \tag{7}$$

Borel has shown [8] that (for any arithmetic quotient of a symmetric space) one can use instead the subcomplex defined by C^∞ vectors, and write an analogue of (5);

$$H^{\cdot}_{(2)}(M, \mathbf{V}) \simeq H^{\cdot}(\mathbf{g}, K_\infty; L^2(\Gamma\backslash G_\infty)^\infty \otimes V). \tag{8}$$

The non-zero contributions on the right-hand side come from the summand of $L^2(\Gamma\backslash G_\infty)$ whose infinitesimal character matches the central character of V.

Remark. As was the case with $L^{\cdot}_{(2),\mathrm{loc}}$, $L^{\cdot}_{(2)}(M, \mathbf{V})$ has the same cohomology as its subcomplex of C^∞ elements [17 : §8].

(1.5) In the generality of metrized local systems on Riemannian manifolds, there is a version of the *Hodge theorem:*

$$H^i_{(2)}(M, \mathbf{V}) \simeq h^i_{(2)}(M, \mathbf{V}) \oplus \left(\overline{B^i}/B^i\right), \tag{9}$$

where $B^i = dL^{i-1}_{(2)}(M, \mathbf{V})$, and $h^i_{(2)}$ indicates the space of *strictly* harmonic i-forms: in the (domain and) kernel of the densely-defined unbounded composite $dd^* + d^*d$, operating on the Hilbert space of L^2 i-forms. In case M is complete, $h^i_{(2)}(M, \mathbf{V})$ is just the space of all L^2 i-forms annihilated by the second-order Laplacian operator; in any case, it is a subspace of the preceding, namely the set of closed i-forms ϕ such that $d^*\phi$ (is defined and) vanishes[1]. Because B^i is the range of an operator, there are only two possibilities for the second summand on the right-hand side of (9): it is zero, of course, when B^i is a closed subspace of the L^2 i-forms; it is an infinite-dimensional vector space otherwise.

[1] The distinction being made here becomes less mysterious if one considers Riemannian manifolds with boundary (with the metric extending across the boundary), where the Neumann boundary condition is imposed by the definition of d^*. For instance, consider the L^2 harmonic functions on the unit disc in the plane (so $i = 0$ now); this is an infinite-dimensional space, but only the constant functions satisfy the boundary conditions and have cohomological significance.

(1.6) One can analyze formula (9) in the case of an arithmetic quotient of a symmetric space. By a formula of Kuga (see [28 : (6.9)]), the harmonic forms are given, in terms of (8), by certain eigenfunctions of the Casimir element for $\mathbf{g}$. According to [13], they comprise a finite-dimensional space. Furthermore, by [11], $B^{\cdot}$ is closed whenever G_∞ and K_∞ have the same $\mathbb{C}$-rank (let's say then that X is an *equal-rank* symmetric space), in particular when X is Hermitian. One thus has:

PROPOSITION. *Let M be an arithmetic quotient of an Hermitian symmetric space, V as above. Then $H^{\cdot}_{(2)}(M, \mathbf{V})$ is finite-dimensional, and is represented by L^2 automorphic forms.*

(1.7) What the preceding proposition really says is that the story of L^2 cohomology is essentially the same whether M is compact or not. Moreover, the Hecke algebra corresponding to K_f is easily seen to act naturally on the L^2 cohomology (see [1 : p.175]). The topological understanding aspired to in (1.2) comes down to establishing the L^2-cohomology (together with its Hecke correspondences) as a sort of deRham theoretical realization of some (familiar) *motive.* Two cases of this would be considered "classical":

a) $\mathbf{G}$ is anisotropic over $\mathbb{Q}$ (i.e., of $\mathbb{Q}$-rank zero). Then M is compact, and

$$H^{\cdot}_{(2)}(M, \mathbf{V}) \xrightarrow{\sim} H^{\cdot}(M, \mathbf{V})$$

b) $\mathbf{G} = SL_2$. Though M is non-compact, it can be compactified by adjoining a finite number of cusps, yielding a compact Riemann surface M^*. Let $j : M \hookrightarrow M^*$ denote the inclusion. The metric on M is the Poincaré metric, and one has from [34]:

$$H^{\cdot}_{(2)}(M, \mathbf{V}) \cong H^{\cdot}(M^*, j_*\mathbf{V}) \tag{10}$$

(note that $j_*\mathbf{V} \approx \tau_{\leq 0} R_{j_*}\mathbf{V}$); for the critical degree $i = 1$, this can be written as:

$$H^1(M^*, j_*\mathbf{V}) \simeq \operatorname{im}\{H^1_c(M, \mathbf{V}) \to H^1(M, \mathbf{V})\} \simeq H^1_{\text{par}}(\Gamma, V). \tag{11}$$

Here, the "topological" interpretation of automorphic forms is given by the Eichler-Shimura isomorphism:

$$H^1_{(2)}(M, \mathbf{V}) \simeq S_{2+k}(\Gamma) \oplus \overline{S_{2+k}(\Gamma)}, \tag{12}$$

—where $S_{2+k}(\Gamma)$ denotes the space of holomorphic cusp forms of weight $2+k$ for Γ, when $V = \mathrm{Sym}^k(\mathbb{C}^2)$—which is just giving the Hodge decomposition of the parabolic cohomology (see [34 : §12]).

(1.8) An arithmetic quotient of an Hermitian symmetric space is often called an arithmetic (or locally symmetric) *variety* for the good reason that it can be endowed with the structure of a quasi-projective algebraic variety over $\mathbb{C}$. (Indeed, the theory of canonical models provides structure over a number field.) One can embed M in projective space by holomorphic automorphic forms, and its closure M^* is a normal projective variety; this is the Baily-Borel Satake compactification [3] of M.

(1.9) For the purposes of the conjecture, only the underlying topological space of M^* is needed. This is described in [3 : (4.11)] (see also [37 : (3.11)], [38 : (1.6)]). The boundary of M in M^* consists of arithmetic varieties of lower rank and dimension, suitably attached of course. M^* becomes a stratified pseudomanifold in the sense of [20 : (1.1)], with the number of singular strata equaling the $\mathbb{Q}$-rank of $\mathbf{G}$. Since this is a complex stratification, all strata are of even real codimension, and therefore its *middle perversity intersection (co)homology* $IH^{\cdot}(M^*, \mathbf{V})$ is defined [20 : §5] (also [21 : §6]).

On the basis of calculations for $SU(n,1)$ and the restriction of scalars of SL_2, for all V, we were led to conjecture [36 : §6]:

THEOREM [27], [33]. *If M is an arithmetic variety,*

$$H^{\cdot}_{(2)}(M, \mathbf{V}) \simeq IH^{\cdot}(M^*, \mathbf{V}).$$

Weak evidence for the above isomorphism was that both sides are finite-dimensional and share Poincaré duality. Each side is actually given as the (hyper)cohomology of a complex of fine sheaves on M^*, viz. the L^2-complex $\mathcal{L}^{\cdot}_{(2)}(M^*, \mathbf{V})$[2] and intersection complex $\mathcal{IC}^{\cdot}_{M^*}(\mathbf{V})$ respectively. Thus, the theorem follows (as was my original intention) from:

THEOREM (LOCAL VERSION). *Let M^* be the Baily-Borel Satake compactification of an arithmetic variety M. Then $\mathcal{L}^{\cdot}_{(2)}(M^*, \mathbf{V})$ and $\mathcal{IC}^{\cdot}_{M^*}(\mathbf{V})$ are quasi-isomorphic. Specifically, every point of M^* has*

[2] This is the sheaf associated to assigning the open subset U of M^* the complex $L^{\cdot}_{(2)}(U \cap M, \mathbf{V})$. The fineness of the L^2 complex requires the existence of admissible partitions of unity (see [38 : (3.6)] for the case at hand).

a fundamental system of neighborhoods U such that there is a "natural" isomorphism

$$H^{\cdot}_{(2)}(U \cap M, \mathbf{V}) \simeq IH^{\cdot}(U, \mathbf{V}). \tag{13}$$

Prior to [27] and [33], cases of low rank were worked out in [9], [12], [36], and [38] (see also [15]). For $\mathbf{G} = SL_2$, one recovers (10).

(1.10) As is remarked in [26 : p.29], the conjecture (now theorem) "allows one for many purposes to argue as though the quotients $M = \Gamma\backslash X$ were compact" (cf., i.e., compare, (1.7)). In particular, it allows the conjectural relationship between the Hasse-Weil zeta functions of Shimura varieties and the L-functions associated to automorphic forms [24], [25 : §7] to become formulated in the noncompact case (which includes most interesting examples). As before, the comparison is to be made in the associated Dirichlet series, which involves traces through the Lefschetz formula: for the action of Frobenius on local intersection homology (see [14 : (3.2)]), and for the action of the Hecke algebra on L^2-cohomology [1 : §3].

In [14], this program was carried out for Hilbert-Blumenthal varieties (associated to the restriction of scalars $R_{\mathbf{E}/\mathbf{Q}}SL_2$, where $\mathbf{E}$ is a totally real number field). There, the local version of the conjecture plays a very important role, for it nullifies the difficulties expected from the cusps.

2. Proof of the conjecture

(2.1) In order to appreciate what the proof of Theorem (1.9) entails, we start by recalling the local description (characterization) of intersection homology (middle perversity understood throughout). On a pseudomanifold, every point has a fundamental system of neighborhoods of the form

$$U = D \times \mathrm{Cone}(L), \tag{14}$$

where D is a disc in its stratum, with $D \times \{\text{vertex}\}$ lying in the stratum, and L is the *link* of the stratum. Then

$$U^{\mathrm{reg}} = U \cap M = D \times I \times L^{\mathrm{reg}}, \tag{15}$$

where I is an open interval. We are in a situation where the (real) codimension of the stratum is even, and write it as $2j$.

The basic (inductive) calculation of local intersection homology is given by:
(16)

$$IH^i(U,\mathbf{V}) \simeq \begin{cases} IH^i(L,\mathbf{V}) & \text{if } i<j, \\ 0 & \text{if } i \geq j, \text{ with the following exception,} \\ V & \text{if } i=j=0. \end{cases}$$

The above, together with the way the isomorphisms in (16) occur, characterize $\mathcal{IC}^{\cdot}_{M^*}(\mathbf{V})$ up to quasi-isomorphism; for more on this, see [21 : §3] or [10 : V,§4]. To prove the theorem, one sees that it suffices to verify the analogue of (16) for $\mathcal{L}^{\cdot}_{(2)}(M^*,\mathbf{V})$, i.e., show:
(17)

$$H^i_{(2)}(U^{\text{reg}},\mathbf{V}) \simeq \begin{cases} H^i_{(2)}(L^{\text{reg}},\mathbf{V}) & \text{if } i<j \\ 0 & \text{if } i \geq j, \text{ except for } i=j=0. \end{cases}$$

(We omit the third line, for that is trivial to verify.) Because there is sufficient self-duality around, one may use an alternative characterization of middle intersection homology [21 : (6.1)], if one wishes; it is, in fact, enough to verify just the second line of (17), i.e., the vanishing at and above the complex codimension.

(2.2) In order to do calculations to verify (17), one needs an explicit description of the links, together with a formula for the metrics (up to quasi-isometry is sufficient: see [17 : p.96]) in terms of (15).

The strata of M^* are arithmetic quotients of the so-called rational boundary components of X. These are parametrized by their normalizers in G, which are maximal $\mathbb{Q}$-parabolic subgroups P, whose rational Langlands decomposition is

$$P = MAN = M^B \cdot M^A \times \mathbf{R}^+ \ltimes N. \tag{18}$$

Here, we have broken up the Levi component such that M^A has type-A $\mathbb{Q}$-root system, and M^B has type-B (under the convention of [38 : p.342, #2]). In the notation of [2], $M^B = G_h$, essentially the automorphism group of the boundary component, and $M^A \cdot A = G_l$. The factor $A = \mathbb{R}^+$ corresponds to I in (15), and gives the correct metrical picture.

One obtains that L^{reg} is fibered over an arithmetic quotient Y^A of the symmetric space—usually not Hermitian—of M^A, with a (compact) arithmetic quotient of N as fiber. It is well-known that the

Leray spectral sequence of this fibration π degenerates at E_2, corresponding to a fairly canonical isomorphism:

$$(19)\quad H^{\cdot}(U^{\mathrm{reg}}, \mathbf{V}) \simeq H^{\cdot}(L^{\mathrm{reg}}, \mathbf{V}) \simeq H^{\cdot}(\mathbf{R}^+, \mathbf{C}) \otimes H^{\cdot}(Y^A, \mathbf{H}^{\cdot}(\mathbf{n}, V)),$$

where $\mathbf{n}$ is the Lie algebra of N. Of course, $H^{\cdot}(\mathbf{R}^+, \mathbf{C}) \simeq \mathbf{C}[0]$, but the reason for our excess will soon become apparent. Implicit in (19) is the assertion that for any $q \geq 0$, $R^q\pi_*\mathbf{V}$ is the local system on Y^A coming from the representation of M^A on the Lie algebra cohomology $H^q(\mathbf{n}, V)$.

(2.3) We next discuss the L^2-cohomology of U^{reg}. This is given by what Looyenga[3] calls the "L^2 Künneth formula" (from [38 : (3.19)]) which we now write, and explain subsequently:

$$(20)\quad H^{\cdot}_{(2)}(U^{\mathrm{reg}}, \mathbf{V}) \simeq \bigoplus_{\alpha}(H^{\cdot}_{(2)}(\mathbf{R}^+, \mathbf{C}; w_1^\alpha) \otimes H^{\cdot}_{(2)}(Y^A, \mathbf{H}^{\cdot}_\alpha(\mathbf{n}, V); w_2^\alpha)).$$

Since A acts semi-simply, one has a decomposition

$$(21)\qquad H^{\cdot}(\mathbf{n}, V) = \bigoplus_{\alpha} H^{\cdot}_\alpha(\mathbf{n}, V)$$

as a representation of M^A, according to the weights α of A. The ones that occur can be determined as a consequence of Kostant's theorem [23 : (5.14)]. The functions w_1^α and w_2^α in (20) are weights in the classical sense, i.e., extra multiplicative factors that are inserted into the integrals that define the L^2 semi-norms because of the formula for the metrics on U^{reg} ([6 : (4.3)]; see also [38 : (2.6(9))]); they are of the form:

$$(22)\qquad w_1^\alpha(r) = \mathrm{e}^{-kr} \qquad (r \in \mathbf{R}^+) \text{ for some } k \in \mathbf{Z},$$

$$(23)\qquad w_2^\alpha = w_1^\alpha \circ f,$$

for some fixed function f on Y^A (independent of α ; see [39 : (2.5)]).

We recall some elementary calculations [36 : (4.51)]:

$$(24)\qquad H^0_{(2)}(\mathbf{R}^+, \mathbf{C}; \mathrm{e}^{-kr}) \simeq \begin{cases} \mathbf{C} & \text{if } k > 0, \\ 0 & \text{if } k \leq 0; \end{cases}$$

[3] We use the Dutch identity $y = ij$ without further comment.

$$(25)\qquad H^1_{(2)}(\mathbb{R}^+,\mathbb{C};e^{-kr})=\begin{cases}0 & \text{if } k\neq 0,\\ \text{infinite-dimensional} & \text{if } k=0.\end{cases}$$

The infinite-dimensionality for the unweighted ($k=0$) L^2-cohomology in (25) comes about because there is non-closed range for d on $\mathbb{R}^+$ (recall (1.5)). It should be pointed out that the non-contribution of the weight α corresponding to $k=0$ is of course necessary for the conjecture; one even needs to know a little about this case to know that (20) is, in fact, correct (see [38 : Thm. (3.19,iii)]). If one grants this, then we can observe that if the L^2's and weights are erased from (20), one gets (19). Indeed, one sees from (24) that the natural mapping

$$(26)\qquad H^{\cdot}_{(2)}(U^{\mathrm{reg}},\mathbf{V})\to H^{\cdot}(U^{\mathrm{reg}},\mathbf{V})$$

is injective,—something that is not true in general—and the L^2-cohomology effects, a priori, a truncation by A-weight. The problem is to show that this is the same as truncating by *degree*, as is required in (17); equivalently, if $k>0$ in (22), show that

$$(27)\qquad H^r_{(2)}(Y^A,\mathbf{H}^s_\alpha(\mathbf{n},V;w_2^\alpha))=0\qquad\text{for}\qquad r+s\geq j.$$

In effect, it is this that Looyenga [27] and Saper-Stern [33] accomplish, by entirely different methods.

(2.4) The two approaches diverge already over the ground-level question: is the conjecture a problem in algebraic geometry over $\mathbb{C}$, or is it one in Lie groups and Lie algebras? Looyenga's proof takes the first point of view, while that of Saper and Stern is rooted (sorry) in the second.

Because it uses a lot of previously established algebraic geometery and Hodge theory ([2], [16]/[22], [32]), some rather recent, Looyenga's can be written down or explained fairly quickly. On the other hand, the Saper-Stern approach is technical, so is harder to explain, though the main tools ([29], [31], [38 : (3.6)]) are older, and at bottom more elementary.

(2.5) My preference for [33] may be slipping out. It had always been my feeling that the complex structure was incidental. One can find in (1.6) the hint that the natural setting for the result should be a broader class of spaces than the arithmetic varieties. Indeed, there are equal-rank symmetric spaces that are not Hermitian, and Borel has suggested at least one of the following:

CONJECTURE. *Let M^* be a Satake compactification of an arithmetic quotient of an equal-rank symmetric space whose boundary components are also equal-rank. Let $\mathbf{V}$ be the local system associated to a finite-dimensional representation of G_∞, etc. Then*

$$\mathcal{L}^{\cdot}_{(2)}(M^*, \mathbf{V}) \approx \mathcal{IC}^{\cdot}_{M^*}(\mathbf{V}).$$

CONJECTURE. *(strong form)—as above, but assume only that all* rational *boundary components are equal-rank.*

For further information on what is involved in the above, see [37 : §3], [38 : (A.2)], [39 : (5.2)]. The two main non-Hermitian cases are a Satake compactification analogous to the Baily-Borel for classification types BI ($SO(p,q)$, with $p+q$ odd) and CII ($Sp(p,q)$). It seems quite likely that the methods of [33] will be adaptable to prove this conjecture, though perhaps not in its strong form. (Recall that only the rational boundary components enter in the construction of M^*.) But [27] also has its virtues.

(2.6) Outline of Looyenga's proof. (see also [39 : §3]):

i) Argue by induction on the codimension of the stratum. Given that the quasi-isomorphism

$$\mathcal{L}^{\cdot}_{(2)}(M^*, \mathbf{V}) \approx \mathcal{IC}^{\cdot}_{M^*}(\mathbf{V})$$

has been checked on all strata of complex codimension less than j, verify (17). The inductive hypothesis gives

$$IH^{\cdot}(L, \mathbf{V}) \simeq \bigoplus_{\alpha}(H^{\cdot}(\mathbb{R}^+, \mathbb{C}) \otimes H^{\cdot}_{(2)}(Y^A, \mathbf{H}^{\cdot}_{\alpha}(\mathbf{n}, V); w_2^{\alpha})); \tag{28}$$

to which one must compare (20).

ii) Eliminate L^2-cohomology from what is to be proved, replacing it by a problem purely in intersection homology. This is achieved by recognizing (28) as the weight space decomposition of a geometrically defined endomorphism: for certain $a \in A$ left-multiplication by a induces a proper, finite-to-one stratified endomorphism Φ_a of U, whose action on

$$IH^{\cdot}((0,1) \times L, \mathbf{V}) \simeq IH^{\cdot}(L, \mathbf{V})$$

is semi-simple, and yields (28). Then show that for $i \geq j$, all eigenvalues of Φ_a on $IH^i(L, \mathbf{V})$ are of the form a^k with $k > j$; we abbreviate this by writing "the weights are $> j$". By (16) and duality, this is a consequence of:

PROPOSITION. *The weights of* Φ_a *on* $IH^i(U, \mathbf{V})$ *are* $\leq i$.

iii) One recognizes that there is a similar result for variations of Hodge structure near normal crossings singularities, namely the "purity theorem" from [16 : (1.13)] or [22 : (4.0.1)]. It turns out to be possible to reduce the proposition from (ii) to this. Take a toroidal resolution of singularities

$$\pi : \tilde{U} \to U,$$

for which $\pi^{-1}(U^{\mathrm{sing}})$ is a divisor with normal crossings, such that π is a projective morphism. The existence of such is proved in [2]. Moreover, the action of Φ_a lifts to $\tilde{U}$. Because $\mathbf{V}$ underlies a polarizable variation of Hodge structure [35 : §4], the decomposition theorem of Saito [32] can be applied to $R\pi_*\mathcal{IC}^{\cdot}_{\tilde{U}}(\mathbf{V})$, giving the existence of embeddings

$$IH^{\cdot}(U, \mathbf{V}) \hookrightarrow IH^{\cdot}(\tilde{U}, \mathbf{V})$$

compatible with Φ_a. It thus suffices to prove the proposition with U replaced by $\tilde{U}$.

iv) From its construction, $\tilde{U}$ is covered by $(\Delta^*)^\nu$-bundles, and these intersect in specific ways. Use various spectral sequences to make the reduction to the Hodge theoretic result, as described at the beginning of (iii).

(2.7) Outline of the proof by Saper and Stern. (see also [39 : §4]):

i) Show directly (i.e., without induction) the requisite vanishing (17) for $H^{\cdot}_{(2)}(U^{\mathrm{reg}}, \mathbf{V})$. The vanishing of $H^i_{(2)}$ is a consequence of the *a priori* estimate of L^2-norms:

$$\|\Delta\phi\| \geq c\,\|\phi\| \qquad \text{for some } c > 0 \tag{29}$$

for all i-forms ϕ in the domain of Δ (recall (1.5)). If we regard U^{reg} as the interior of a complete manifold with boundary, where the boundary is defined by $0 \in \overline{\mathbf{R}^+}$ (from (18)), then it suffices to verify (29) only for ϕ smooth to the boundary and of compact support.

(ii) Break up U^{reg} into pieces, according to parabolic subgroups Q of M^A:

$$U^{\mathrm{reg}} = \bigcup_Q U(Q). \tag{30}$$

Or, conversely, one can take $U^{\rm reg}$ to be such a union. If the pieces are carefully chosen , it is actually enough to verify (29) for forms supported in each $U(Q)$.

(iii) A procedure for verifying the estimate for $U(Q)$ for $(i \geq j)$ is carried out in [33]. It admits a cohomological reformulation, which runs parallel to their estimation process, given in [39]. Although the desired estimate on $U(Q)$ has itself no cohomological interpretation, the same for certain *larger* domains does. Thus, it suffices to show:

PROPOSITION. *Whenever $i \geq j$, there exists a set of boundary conditions* b *for Δ, such that the corresponding L^2-cohomology group ${}_bH^i_{(2)}(U(Q), \mathbf{V})$ vanishes.* (What else one needs to get the estimate follows.)

iv) The Langlands decomposition of Q gives rise to a picture of $U(Q)$ that is similar to the one given by (2.1) and (2.2) If we accordingly write (cf. (18)):

$$Q = M_Q A_Q N_Q, \tag{31}$$

$$QAN = M_Q(A_Q \times A)(N_Q \times N), \tag{32}$$

one can, with a little care (see [38 : (1.3) Remark]), write $U(Q)$ as an arithmetic quotient of a space diffeomorphic to

$$\tilde{X}_Q \times ((\mathbb{R}^+)^l \times \mathbb{R}^+) \times N' \tag{33}$$

where $\tilde{X}_Q$ is a relatively compact deformation retract of the symmetric space of M_Q. There is a corresponding L^2 Künneth formula for ${}_bH^{\cdot}_{(2)}(U(Q), \mathbf{V})$, which we write as

(34)

$${}_bH^{\cdot}_{(2)}(U(Q), \mathbf{V}) \simeq \bigoplus_{\beta}({}_bH^{\cdot}_{(2)}((\mathbb{R}^+)^l \times \mathbb{R}^+, \mathbb{C}; w^{\beta}) \otimes {}_bH^{\cdot}_{(2)}(\tilde{Y}_Q, \mathbf{H}^{\cdot}_{\beta}(\mathfrak{n}', V))),$$

where $\tilde{Y}_Q$ is some arithmetic quotient of $\tilde{X}_Q$, β runs over weights of $A_Q \times A$, etc. On the right-hand side of (34), we understand that if we have a decomposition of the $\tilde{Y}_Q$ cohomology induced by Δ-invariants, we may use different boundary conditions on the different factors, and also on the $(\mathbb{R}^+)^{l+1}$ cohomology.

v) The formulas (24) and (25) generalize without much ado (see [36 : (4.51)] again, and also [39 : (4.6.4)]). This gives an easy criterion for the vanishing of terms in (34). The point is to establish the analogue of (27), i.e. the requisite vanishing of the $\tilde{Y}_Q$ cohomology, when the easy criterion fails.

The Laplacian Δ on $\tilde{Y}_Q$, itself semi-positive, is rather naturally the sum of two other semi-positive operators, one of which, usually called Δ_ρ, is of order zero ([28 : I,§7], [29 : §5]). Therefore, the vanishing of cohomology is implied by the positivity of the eigenvalues of Δ_ρ, which is only a matter of (some very complicated) linear algebra, as has been done in [29 : §12] and [31]. The fact that X is Hermitian imposes conditions on the real and rational root systems (see [3 : (1.2),(2.9)]), which can be couched in terms of Jordan algebras ([2 : II,III]). Use this to get the combinatorics to work out correctly (see [39 : (4.10)-(4.16)]).

References

1. Arthur, J, *Characters, harmonic analysis, and an L^2-Lefschetz formula*, Proc. Symp. Pure Math **48** (1988), 167–179.
2. Ash, A., et al., "Smooth Compactification of Locally Symmetric Varieties," Math. Sci. Press, Brookline, MA, 1975.
3. Baily, W., Borel, A., *Compactifications of arithmetic quotients of bounded symmetric domains*, Ann. of Math **84** (1966), 442–528.
4. Beilinson, A., Bernstein, J., Deligne, P., "Faisceaux Pervers," Astérisque, 1982.
5. Borel, A., "Introduction aux Groupes Arithmétiques," Hermann, 1969.
6. Borel, A., *Stable real cohomology of arithmetic groups*, Ann. Sci. ENS **7** (1974), 235–272.
7. Borel, A., *Automorphic L-functions*, Proc. Symp. Pure Math **33** (1979); Part 2 27–61.
8. Borel, A., *Regularization theorems in Lie algebra cohomology. Applications*, Duke Math. J. **50** (1983), 605–624.
9. Borel, A., *L^2-cohomology and intersection cohomology of certain arithmetic varieties*,, in "Emmy Noether in Bryn Mawr," Springer, 1983, pp. 119–131.
10. Borel, A., et al., "Intersection Cohomology," Birkhäuser, 1984.
11. Borel, A., Casselman, W., *L^2-cohomology of locally symmetric manifolds of finite volume*, Duke Math. J. **50** (1983), 625–647.
12. Borel, A., Casselman, W., *Cohomologie d'intersection et L^2-cohomologie de variéties arithmétiques de rang rationnel 2*, CR Acad. Sc. Paris **301** (1985), 369–373.
13. Borel, A., Garland, H., *Laplacian and discrete spectrum of an arithmetic group*, Amer. J. Math. **105** (1983), 309–335.
14. Brylinski, J.-L., Labesse, J.-P., *Cohomologie d'intersection et fonctions L de certaines variétés de Shimura*, Ann. Sci. ENS **17** (1984), 361–412.

15. Casselman, W., *Introduction to the L^2-cohomology of arithmetic quotients of bounded symmetric domains*, in "Complex Analytic Singularities," Adv. Stud. Pure Math. **8**, 1986, pp. 69–93.
16. Cattani, E., Kaplan, A., Schmid, W., *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. **87** (1987), 217–252.
17. Cheeger, J., *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Symp. Pure Math **36** (1980), 91–146.
18. Cheeger, J., Goresky, M., MacPherson, R., *L^2-cohomology and intersection homology of singular varieties*, in "Seminar on Differential Geometry," Ann. of Math. Studies **102**, 1982, pp. 303–340.
19. Deligne, P, *Travaux de Shimura*, Sem. Bourbaki, exp. 389, "Lecture Notes in Math," Springer, 1971.
20. Goresky, M., MacPherson, R., *Intersection homology theory*, Topology **19** (1980), 135–162.
21. Goresky, M., MacPherson, R., *Intersection homology, II*, Invent. Math. **72** (1983), 77-129.
22. Kashiwara, M., Kawai, T., *The Poincaré lemma for variations of polarized Hodge structure*, Publ. RIMS **23** (1987), 345–407.
23. Kostant, B., *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math **74** (1961), 329–387.
24. Langlands, R., *Shimura varieties and the Selberg trace formula*, Can. J. Math **29** (1977), 1292–1299.
25. Langlands, R., *Automorphic representations, Shimura varieties, and motives*, Proc. Symp. Pure Math. **33,part 2** (1979), 205–246.
26. Langlands, R., *Representation theory and arithmetic*, Proc. Symp. Pure Math **48** (1988), 25–33.
27. Looijenga, E., *L^2-cohomology of locally symmetric varieties*, Compositio Math. **67** (1988), 3–20.
28. Matsushima, Y., Murakami, S., *On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds*, Ann. of Math. **78** (1963), 365–416.
29. Matsushima, Y., Murakami, S., *On certain cohomology groups attached to Hermitian symmetric spaces*, Osaka J. Math. **2** (1965), 1–35.
30. Milne, J., *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, These Proceedings, Vol I, 283-414.
31. Raghunathan, M., *Vanishing theorems for cohomology groups associated to discrete subgroups of semi-simple Lie groups*, Osaka J. Math **3** (1966), 243–256;, **16** (1979), 295–299.
32. Saito, Morihiko, *Mixed Hodge modules*, Proc. Japan Acad. **62** (1986); Ser A, 360–363; preprint RIMS-585 (1987).
33. Saper, L., Stern, M., *L^2-cohomology of arithmetic varieties*, Proc. Natl. Acad. Sci. USA **84** (1987), 5516–5519; Ann. of Math. (to appear).
34. Zucker, S., *Hodge theory with degenerating coefficients: L^2 cohomology in the Poincaré metric*, Ann. of Math. **109** (1979), 415–476.
35. Zucker, S., *Locally homogeneous variations of Hodge structure.*, L'Ens. Math. **27** (1981), 243–276.
36. Zucker, S., *L^2 cohomology of warped products and arithmetic groups*, Invent. Math. **70** (1982), 169–218.

37. Zucker, S., *Satake compactifications*, Comentarii Math. Helv. **58** (1983), 312–343.
38. Zucker, S., *L^2-cohomology and intersection homology of locally symmetric varieties, II*, Compositio Math. **59** (1986), 339–398.
39. Zucker, S., *L^2-cohomology and intersection homology of locally symmetric varieties, III*, Hodge Theory: Proc. Luminy, Astérisque 1987 (to appear).

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